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# CRITICAL PERCOLATION ON RANDOM REGULAR GRAPHS 

FELIX JOOS AND GUILLEM PERARNAU


#### Abstract

We show that for all $d \in\{3, \ldots, n-1\}$ the size of the largest component of a random $d$-regular graph on $n$ vertices around the percolation threshold $p=1 /(d-1)$ is $\Theta\left(n^{2 / 3}\right)$, with high probability. This extends known results for fixed $d \geq 3$ and for $d=n-1$, confirming a prediction of Nachmias and Peres on a question of Benjamini. As a corollary, for the largest component of the percolated random $d$-regular graph, we also determine the diameter and the mixing time of the lazy random walk. In contrast to previous approaches, our proof is based on a simple application of the switching method.


## 1. Introduction

For every $d \in\{3, \ldots, n-1\}$, let $\mathcal{G}_{n, d}$ be the set of all simple and vertex-labelled $d$-regular graphs on $n$ vertices and let $G_{n, d}$ be a graph chosen uniformly at random from $\mathcal{G}_{n, d}$. For $p \in[0,1]$, let $G_{n, d, p}$ be a graph obtained from $G_{n, d}$ by retaining each edge independently with probability $p$. The goal of this paper is to study the order of the largest component of $G_{n, d, p}$, denoted by $L_{1}\left(G_{n, d, p}\right)$, in terms of $n, d$ and $p$.

Most of the literature in the area focuses either on fixed $d \geq 3$ or on $d=n-1$. Goerdt [8] showed the existence of a critical probability, $p_{\text {crit }}:=1 /(d-1)$, such that for every fixed $d \geq 3$ and every $\epsilon>0$ the following holds with probability $1-o(1)$ : if $p \leq(1-\epsilon) p_{\text {crit }}$, then $L_{1}\left(G_{n, d, p}\right)=O(\log n)$, while if $p \geq(1+\epsilon) p_{c r i t}$, then $L_{1}\left(G_{n, d, p}\right)=\Theta(n)$. Similar results were also obtained in a more general setting by Alon, Benjamini and Stacey [1]. For $d=n-1$, the random graph $G_{n, d, p}$ corresponds to the classic Erdős-Rényi random graph $G_{n, p}$. In their seminal paper [5], Erdős and Rényi proved that for every $\epsilon>0$, the following holds with probability $1-o(1)$ : if $p \leq(1-\epsilon) / n$, then the largest component of $G_{n, p}$ has order $O(\log n)$, if $p=1 / n$ (critical probability), then it has order $\Theta\left(n^{2 / 3}\right)$, while if $p \geq(1+\epsilon) / n$, then it has linear order.

Both for fixed $d \geq 3$ and for $d=n-1$, the behaviour around the critical probability has attracted a lot of interest. It is well established that the critical window in $G_{n, p}$ around $p=1 / n$ is of order $n^{-1 / 3}$ (see e.g. [21]). More precise estimates can be found in [14]. Benjamini posed the problem of determining the width of the critical window in $G_{n, d, p}$ around $p_{\text {crit }}=1 /(d-$ 1) (see [20, 22]). Nachmias and Peres [20] and Pittel [22], independently showed that the critical window exhibits mean-field behaviour for fixed $d \geq 3$, namely, the following holds with probability $1-o(1)$ : for every fixed $\lambda \in \mathbb{R}$, if $p=\frac{1+\lambda n^{-1 / 3}}{d-1}$, then $L_{1}\left(G_{n, d, p}\right)=\Theta\left(n^{2 / 3}\right)$. See also Riordan [23] for more precise results on $L_{1}\left(G_{n, d, p}\right)$ in the critical window.

The case when $d$ is an arbitrary function of $n$ is much less understood. It follows from existing results in the literature ${ }^{1}$ that for every $d \in\{3, \ldots, n-1\}$, the critical probability for the existence of a linear order component in $G_{n, d, p}$ is $1 /(d-1)$. Results inside the critical window for given $d$-regular graphs have also been obtained in the context of transitive graphs under the finite triangle condition [4] or under certain expansion conditions [18].

Finally, similar results have been obtained for irregular degree sequences whenever the average degree is bounded by a constant $[3,6,7,10]$.

In view that both the sparse regime (fixed $d \geq 3$ ) and the densest one ( $d=n-1$ ) exhibit similar properties, Nachmias and Peres [20] suggested that the mean-field behaviour extends to

[^0]every $d \in\{3, \ldots, n-1\}$. In this paper we confirm this prediction in the critical window and thus answer the question posed by Benjamini for all $d \in\{3, \ldots, n-1\}$.

Theorem 1. Suppose $\lambda \in \mathbb{R}$ and $d, n \in \mathbb{N}$ such that $3 \leq d \leq n-1$ and $n$ is sufficiently large. Let $p=\frac{1+\lambda n^{-1 / 3}}{d-1}$. Then for every sufficiently large $A=A(\lambda)$, we have

$$
\mathbb{P}\left[L_{1}\left(G_{n, d, p}\right) \notin\left[A^{-1} n^{2 / 3}, A n^{2 / 3}\right]\right] \leq 20 A^{-1 / 2}
$$

The upper bound in Theorem 1 directly follows from the upper bound for $d$-regular graphs in Proposition 1 in [20]. The proof of the lower bound is more intricate and we devote the rest of the paper to it.

Most of the previous work on the component structure of $G_{n, d, p}$ uses the configuration model introduced by Bollobás in [2]. The configuration model, denoted by $G_{n, d}^{*}$, is a model of random $d$-regular multigraphs on $n$ vertices. Conditional on $G_{n, d}^{*}$ being simple, one obtains the uniform distribution on $\mathcal{G}_{n, d}$. It is well-known (see for example [24]) that

$$
\begin{equation*}
\mathbb{P}\left[G_{n, d}^{*} \text { simple }\right]=e^{-\Omega\left(d^{2}\right)} \tag{1}
\end{equation*}
$$

While $\mathbb{P}\left[G_{n, d}^{*}\right.$ simple $]$ is constant for fixed $d \geq 3$, it quickly tends to 0 if $d$ grows with $n$, and new ideas are needed to study $G_{n, d}$. A standard tool to estimate probabilities for $G_{n, d}$ when $d$ grows with $n$ is the switching method, introduced by McKay in [16]. For instance, this method has been used to estimate (1) for $d=o(\sqrt{n})$ [17] or to determine several combinatorial properties of $G_{n, d}$ when $d$ grows with $n$ [13].

The proof of the lower bound in Theorem 1 is based on the analysis of an exploration process in $G_{n, d, p}$ using the switching method. The central quantity that we track through the process is the number of edges between the explored and unexplored parts of the graph, denoted by $X_{t}$. Our proof relies on sharp estimations of the first and second moments of $X_{t}$.

This approach is inspired by recent developments of the switching method for the study of the component structure of random graphs with a given degree sequence [7, 11]. We take this opportunity to illustrate the use of our method with a simple proof that makes no assumptions on $d$.

The critical window. Theorem 1 shows that the critical window has width $\Omega\left(n^{-1 / 3}\right)$. Proposition 1 in [20] implies that, as $\lambda \rightarrow-\infty$, the typical order of the largest component is $o\left(n^{2 / 3}\right)$. Following analogous ideas as the ones used in the proof of Theorem 1, one obtains that, as $\lambda \rightarrow \infty$, the typical order of the largest component is $\omega\left(n^{2 / 3}\right)$. More precisely, there exist constants $c, C>0$ such that for every $3 \leq d \leq n-1$ and $\lambda>0$, if $p=\frac{1+\lambda n^{-1 / 3}}{d-1}$, then

$$
\mathbb{P}\left[L_{1}\left(G_{n, d, p}\right) \leq c \cdot \lambda n^{2 / 3}\right] \leq C \lambda^{-1} .
$$

The proof of this statement is simpler than the proof of our main theorem, since the assumption $\lambda>0$ implies that $X_{t}$ has positive drift. In particular, the first part of the exploration process can be analysed using a first moment argument only and for the entire process it suffices to control the variance of $X_{t}$ from above. It follows that the width of the critical window is $\Theta\left(n^{-1 / 3}\right)$.

In its current form, our method does not give sharp estimates for $L_{1}\left(G_{n, d, p}\right)$ in the barely subcritical and barely supercritical regimes. However, we believe that similar estimates as the ones in Lemma 6 hold in general and may be used to extend the results of Nachmias and Peres in [20] to all $d \in\{3, \ldots, n-1\}$.

Diameter and Mixing Time. We present a consequence of Theorem 1. For a component $\mathcal{C}$, let $\operatorname{diam}(\mathcal{C})$ denote its diameter and let $T_{\text {mix }}(\mathcal{C})$ denote the mixing time of the lazy random walk on $\mathcal{C}$. Theorem 1.2 in [19] implies the following corollary.

Corollary 2. Suppose $\lambda \in \mathbb{R}$ and $d, n \in \mathbb{N}$ such that $3 \leq d \leq n-1$ and $n$ is sufficiently large. Let $p=\frac{1+\lambda n^{-1 / 3}}{d-1}$. Let $\mathcal{C}$ be the largest component of $G_{n, d, p}$. Then, for every $\epsilon>0$, there exists $A=A(\lambda, \epsilon)$ such that

$$
\mathbb{P}\left[\operatorname{diam}(\mathcal{C}) \notin\left[A^{-1} n^{1 / 3}, A n^{1 / 3}\right]\right]<\epsilon
$$

and

$$
\mathbb{P}\left[T_{\text {mix }}(\mathcal{C}) \notin\left[A^{-1} n, A n\right]\right]<\epsilon
$$

Organisation of the paper. The paper is organized as follows. In Section 2, we describe our exploration process of $G_{n, d, p}$ and introduce different quantities we will track during the process. In Section 3, we present our main combinatorial tool (switching method) and prove two technical lemmas. In Section 4, we use these lemmas to study a single step of the exploration process. Finally, in Section 5, we conclude with the proof of the lower bound in Theorem 1.

## 2. The exploration process

Before describing the exploration process, we briefly introduce some notation. For a graph $G$, a subset of vertices $X$ of $G$, and a vertex $u$ of $G$, we write $d_{G}(u)$ for the number of neighbours of $u$ in $G$ and $d_{G, X}(u)$ for the number of neighbours of $u$ in $G$ that belong to $X$. We also write $\Delta(G)$ for the maximum degree of $G$. Finally, for $p \in[0,1]$, we write $G_{p}$ for the graph where each edge in $G$ is independently retained with probability $p$.

We will use an exploration process to reveal the component structure of $G_{n, d, p}$. Let us denote the vertex set by $V$, which we equip with a linear order (from now on $V$ is always a vertex set of size $n$ ). For technical reasons, we perform our exploration process not on $G_{n, d, p}$, but on what we call an input. An input is a tuple $(G, \mathfrak{S})$, where $G \in \mathcal{G}_{n, d}$ and $\mathfrak{S}=\left\{\sigma_{v}\right\}_{v \in V}$ is a collection of $n$ permutations of length $d$. For each vertex of $G$, arbitrarily label the edges incident to it with distinct elements from $\{1, \ldots, d\}$. Thus every edge receives two labels. In fact, we may think about this as a labelling of the semi-edges of $G$. Let $\mathcal{I}$ be the set of all inputs $(G, \mathfrak{S})$ where $G \in \mathcal{G}_{n, d}$ and $\mathfrak{S}$ is a collection of $n$ permutations of length $d$. Observe that every graph in $G \in \mathcal{G}_{n, d}$ gives rise to exactly $(d!)^{n}$ inputs. Thus, choosing an input uniformly at random from $\mathcal{I}$ and ignoring the edge-labels is equivalent to choosing $G_{n, d}$. Let $\mathfrak{S}_{n, d}$ be a collection of $n$ permutations of length $d$ each chosen independently and uniformly at random. Hence, if an input is chosen uniformly at random from $\mathcal{I}$, then this input is distributed as $\left(G_{n, d}, \mathfrak{S}_{n, d}\right)$.

Next, we describe our exploration process on an input ( $G, \mathfrak{S}$ ). First, for every $u v \in E(G)$, we denote by $I(u v)$ the indicator random variable that is 1 if $u v$ belongs to $G_{p}$ (it percolates) and 0 otherwise. If $I(u v)$ is revealed, we say that the edge $u v$ has been exposed. For each integer $t \geq 0$, the set $S_{t}$ consists of the vertices explored up to time $t$ (with $S_{0}=\emptyset$ ); the bipartite graph $F_{t}$, with bipartition $\left(S_{t}, V \backslash S_{t}\right)$, consists of all edges in $G$ between $S_{t}$ and $V \backslash S_{t}$ that have been exposed and have failed to percolate; and the graph $H_{t}$, with vertex set $S_{t}$, consists of all edges in $G$ within $S_{t}$, that is, $H_{t}:=G\left[S_{t}\right]$. Let $\mathcal{H}_{t}$ be the history of all random choices we make until time $t$ (which we will treat as an event).

We now describe how to obtain $\mathcal{H}_{t+1}$, given $\mathcal{H}_{t}$. Suppose there exists at least one vertex $u \in S_{t}$ such that $d_{H_{t}}(u)+d_{F_{t}}(u)<d$. Among all such vertices $u$, let $v_{t+1}$ be the vertex which comes first in the linear order of $V$. Let $w_{t+1}$ be the vertex $w \in V \backslash S_{t}$ with $v_{t+1} w \in E(G) \backslash E\left(F_{t}\right)$ that minimizes $\sigma_{v_{t+1}}(\ell(w))$, where $\ell(w)$ is the label of the semi-edge incident to $v_{t+1}$ that corresponds to $v_{t+1} w$. Thereafter, we expose $v_{t+1} w_{t+1}$. If $I\left(v_{t+1} w_{t+1}\right)=0$, then we set $S_{t+1}:=S_{t}, Y_{t+1}:=0$, $Z_{t+1}:=0$ and we let $F_{t+1}$ be the graph obtained from $F_{t}$ by adding $v_{t+1} w_{t+1}$. If $I\left(v_{t+1} w_{t+1}\right)=1$, then we set

$$
S_{t+1}:=S_{t} \cup\left\{w_{t+1}\right\}, \quad Y_{t+1}:=d_{F_{t}}\left(w_{t+1}\right), \quad Z_{t+1}:=d_{G, S_{t}}\left(w_{t+1}\right)-Y_{t+1}-1
$$

and we let $F_{t+1}$ be the graph obtained from $F_{t}$ by deleting all edges incident to $w_{t+1}$ and moving $w_{t+1}$ to the other side of the bipartition. Since $H_{t+1}=G\left[S_{t+1}\right]$, we also reveal all the edges between $w_{t+1}$ and $S_{t}$. Observe that $Z_{t+1}$ counts the number of neighbours of $w_{t+1}$ in $S_{t} \backslash\left\{v_{t+1}\right\}$ whose corresponding edge has not yet been exposed.

If $d_{H_{t}}(u)+d_{F_{t}}(u)=d$ for all $u \in S_{t}$, that is, every edge incident to a vertex in $S_{t}$ has been exposed, then we pick a vertex $x \in V \backslash S_{t}$ that minimises $d_{F_{t}}(x)$ and set $w_{t+1}:=x$, $S_{t+1}:=S_{t} \cup\left\{w_{t+1}\right\}, Y_{t+1}:=d_{F_{t}}\left(w_{t+1}\right), Z_{t+1}:=0$ and we let $F_{t+1}$ be the graph obtained from $F_{t}$ by deleting all edges incident to $w_{t+1}$ and by moving $w_{t+1}$ to the other side of the bipartition. Observe that, in any of the above-mentioned cases, $\left|E\left(F_{t+1}\right)\right| \leq\left|E\left(F_{t}\right)\right|+1$ and hence $\left|E\left(F_{t}\right)\right| \leq t$.

A crucial parameter of our exploration process is the number of edges between $S_{t}$ and $V \backslash S_{t}$ which have not yet been exposed:

$$
X_{t}:=\sum_{u \in S_{t}}\left(d-d_{H_{t}}(u)-d_{F_{t}}(u)\right) .
$$

For the sake of simplicity, we define $\eta_{t+1}:=X_{t+1}-X_{t}$. If $X_{t}>0$, then

$$
\begin{equation*}
\eta_{t+1}=-\left(1-I\left(v_{t+1} w_{t+1}\right)\right)+I\left(v_{t+1} w_{t+1}\right)\left(d-2-Y_{t+1}-2 Z_{t+1}\right), \tag{2}
\end{equation*}
$$

and if $X_{t}=0$, then

$$
\begin{equation*}
\eta_{t+1}=d-Y_{t+1} \tag{3}
\end{equation*}
$$

Note that $Y_{t+1}$ and $Z_{t+1}$ are measurable random variables given $\mathcal{H}_{t}$ and thus $\eta_{t+1}$ is a predictable sequence with respect to $\mathcal{H}_{t}$.

## 3. The switching method and some applications

In this section we explain the switching method and we present two simple applications. In Lemma 3 we use the switching method to bound the probability from above that two vertices are adjacent. In Lemma 4 we provide an upper bound on the expectation of the number of neighbours of a vertex in a specified set of vertices.

Let $G$ be a graph and let $x_{1}, x_{2}, x_{3}, x_{4}$ be distinct vertices of $G$. Suppose $x_{1} x_{2}, x_{3} x_{4} \in E(G)$ and $x_{1} x_{4}, x_{2} x_{3} \notin E(G)$. A switching on the 4 -cycle $x_{1} x_{2} x_{3} x_{4}$ transforms $G$ into a graph $G^{\prime}$ by deleting $x_{1} x_{2}, x_{3} x_{4}$ and adding $x_{1} x_{4}, x_{2} x_{3}$. Observe that the degree sequence of $G$ is preserved by the switching. In particular, if $G$ is $d$-regular, then so is $G^{\prime}$. Moreover, the switching operation is reversible: if $G$ can be transformed into $G^{\prime}$ by a switching, then $G$ can be also obtained from $G^{\prime}$ by a switching on the same 4 -cycle. Finally, there is a natural way to extend the notion of a switching from graphs to inputs by simply preserving the labels on each semi-edge.

Switchings can be used to obtain bounds on the probability that $G_{n, d}$ satisfies a certain property. Suppose $\mathcal{A}, \mathcal{B}$ are disjoint subsets of $\mathcal{G}_{n, d}$. Suppose that for every graph $G \in \mathcal{A}$, there are at least $a$ switchings that transform $G$ into a graph in $\mathcal{B}$ and for every graph $G^{\prime} \in \mathcal{B}$, there are at most $b$ switchings that transform $G^{\prime}$ into a graph in $\mathcal{A}$. By double-counting the number of switchings between $\mathcal{A}$ and $\mathcal{B}$, we obtain $a|\mathcal{A}| \leq b|\mathcal{B}|$. Thus $a \mathbb{P}[\mathcal{A}] \leq b \mathbb{P}[\mathcal{B}]$, where we define $\mathbb{P}[\mathcal{S}]:=|\mathcal{S}| /\left|\mathcal{G}_{n, d}\right|$ for every $\mathcal{S} \subseteq \mathcal{G}_{n, d}$.

Lemma 3. Suppose $d, n \in \mathbb{N}$ such that $3 \leq d \leq n / 4$ and $S \subseteq V$ such that $|S| \leq n / 6$. Let $H$ be a graph with vertex set $S$ and let $F$ be a bipartite graph with vertex partition $(S, V \backslash S)$ with $\Delta(F \cup H) \leq d$. Let $u \in S$ and $v \in V \backslash S$ such that $u v \notin E(F)$. Then

$$
\mathbb{P}\left[u v \in E\left(G_{n, d}\right) \mid G_{n, d}[S]=H, F \subseteq G_{n, d}\right] \leq \frac{6\left(d-d_{H}(u)-d_{F}(u)\right)}{n} .
$$

Proof. Let $\mathcal{F}^{+}$be the set of graphs $G \in \mathcal{G}_{n, d}$ such that $G[S]=H, F \subseteq G$ and $u v \in E(G)$, and let $\mathcal{F}^{-}$be the set of graphs $G \in \mathcal{G}_{n, d}$ such that $G[S]=H, F \subseteq G$ but $u v \notin E(G)$. We will only perform switchings that involve edges and non-edges that are not contained in $E(H) \cup E(F)$. This ensures that the graph $G^{\prime}$ obtained from a switching also satisfies $G^{\prime}[S]=H$ and $F \subseteq G^{\prime}$.

Suppose $G \in \mathcal{F}^{+}$. In order to bound the number of switchings from below it suffices to switch on a cycle $u v x y$ that satisfies $x y \in E(G), u y, v x \notin E(G)$, and $x, y \in V \backslash S$. There are at least $d n-2 d|S|$ ordered edges $x y$ with both endpoints in $V \backslash S$. There are at most $d^{2}$ edges $x y$ such that $x$ is at distance at most 1 from $v$ and at most $d^{2}$ edges $x y$ such that $y$ is at distance at most 1 from $u$. Thus, there are at least $d n-2 d|S|-2 d^{2} \geq d n / 6$ switchings that transform $G$ into a graph in $\mathcal{F}^{-}$. Suppose now $G \in \mathcal{F}^{-}$. Then there are clearly at most $d \cdot\left(d-d_{H}(u)-d_{F}(u)\right)$
switchings that transform $G$ into a graph in $\mathcal{F}^{+}$. It follows that

$$
\begin{aligned}
\mathbb{P}\left[u v \in E\left(G_{n, d}\right)\right. & \left.\mid G_{n, d}[S]=H, F \subseteq G_{n, d}\right] \\
& \leq \frac{d\left(d-d_{H}(u)-d_{F}(u)\right)}{d n / 6} \cdot \mathbb{P}\left[u v \notin E\left(G_{n, d}\right) \mid G_{n, d}[S]=H, F \subseteq G_{n, d}\right] \\
& \leq \frac{6\left(d-d_{H}(u)-d_{F}(u)\right)}{n}
\end{aligned}
$$

Lemma 4. Suppose $d, n \in \mathbb{N}$ such that $3 \leq d \leq n / 4$ and $S \subseteq V$ such that $|S| \leq n / 6$. Let $H$ be a graph with vertex set $S$ and let $F$ be a bipartite graph with vertex partition $(S, V \backslash S)$ with $\Delta(F \cup H) \leq d$. Let $v \in V \backslash S$. Then

$$
\mathbb{E}\left[d_{G, S}(v)-d_{F}(v) \mid G_{n, d}[S]=H, F \subseteq G_{n, d}\right] \leq 6 d|S| / n
$$

Proof. For every $k \geq 0$, let $\mathcal{F}_{k}$ be the set of graphs $G \in \mathcal{G}_{n, d}$ such that $G[S]=H, F \subseteq G$, and $d_{G, S}(v)-d_{F}(v)=k$. As in Lemma 3, we will only perform switchings using edges and non-edges that are not contained in $E(H) \cup E(F)$.

Consider a graph in $\mathcal{F}_{k}$. There are at most $\left(d-d_{F}(v)\right) \cdot d|S| \leq d^{2}|S|$ switchings that lead to a graph in $\mathcal{F}_{k+1}$. For every graph in $\mathcal{F}_{k+1}$, we can use a switching on a cycle $u v x y$ that satisfies $u v, x y \in E(G) \backslash E(F)$, uy, vx $\notin E(G)$ and $u \in S$, and $v, x, y \in V \backslash S$. There are $k+1$ choices for $u v$ and, for any particular choice of $u v$, there are at least $d n-2 d|S|-2 d^{2} \geq d n / 6$ choices for the (ordered) edge $x y$. Hence, there are at least $(k+1) d n / 6$ switchings that lead to a graph in $\mathcal{F}_{k}$. Thus, for every $k \geq 0$, we obtain

$$
\begin{equation*}
\mathbb{P}\left[\mathcal{F}_{k+1}\right] \leq \frac{6 d|S| / n}{(k+1)} \cdot \mathbb{P}\left[\mathcal{F}_{k}\right] \tag{4}
\end{equation*}
$$

Let $X$ be a Poisson distributed random variable with mean $6 d|S| / n$. Lemma 3.4 in [15] together with (4) implies that for every $m \geq 0$

$$
\mathbb{P}\left[d_{G, S}(v)-d_{F}(v) \geq m \mid G_{n, d}[S]=H, F \subseteq G_{n, d}\right] \leq \mathbb{P}[X \geq m]
$$

which implies the statement of the lemma.

## 4. Analysis of the exploration process

In this section we show how to control the expectation of $\eta_{t}$ and $\eta_{t}^{2}$. We first use Lemmas 3 and 4 to bound the expectation of $Y_{t+1}$ and $Z_{t+1}$ from above.

Lemma 5. Suppose $d, n \in \mathbb{N}$ such that $3 \leq d \leq n-1$ and $n$ is sufficiently large. Fix $p \in[0,1]$. Consider the exploration process described above on $\left(G_{n, d}, \mathfrak{S}_{n, d}\right)$ with percolation probability $p$ and suppose $t \leq d n^{2 / 3}$. Conditional on $\mathcal{H}_{t}$ satisfying $\left|S_{t}\right| \leq 5 n^{2 / 3}$, we have

$$
\mathbb{E}\left[Y_{t+1} \mid \mathcal{H}_{t}\right] \leq 20 d n^{-1 / 3} \text { and } \mathbb{E}\left[Z_{t+1} \mid \mathcal{H}_{t}\right] \leq 180 d n^{-1 / 3}
$$

Proof. If $\mathcal{H}_{t}$ satisfies $X_{t}=0$, then $Y_{t+1} \leq t /\left(n-\left|S_{t}\right|\right) \leq 2 d n^{-1 / 3}$ by our choice of $w_{t+1}$ (we always choose the vertex $x$ that minimises $\left.d_{F_{t}}(x)\right)$ and $\left|E\left(F_{t}\right)\right| \leq t$. Note that $Z_{t+1}=0$ by definition. Hence we may assume from now on that $X_{t}>0$.

Note that if $d \geq n / 4$, then the lemma follows directly from the fact that $Y_{t+1} \leq\left|S_{t}\right| \leq 5 n^{2 / 3} \leq$ $20 d n^{-1 / 3}$, and similarly for $Z_{t+1}$. Thus, in the following we assume that $d \leq n / 4$.

Given $w \in V \backslash S_{t}$ such that $v_{t+1} w \notin E\left(F_{t}\right)$, we apply Lemma 3 with $S=S_{t}, F=F_{t}, H=H_{t}$, $u=v_{t+1}$ and $v=w$ to obtain

$$
\mathbb{P}\left[v_{t+1} w \in E\left(G_{n, d}\right) \mid v_{t+1} w \notin E\left(F_{t}\right), \mathcal{H}_{t}\right] \leq \frac{6\left(d-d_{H_{t}}\left(v_{t+1}\right)-d_{F_{t}}\left(v_{t+1}\right)\right)}{n}
$$

Observe that we run our exploration process on inputs. In order to apply Lemma 3, we fix the semi-edge labelings and perform switchings on the graphs.

Since $\sigma_{v_{t+1}}$ is a random permutation, each edge incident to $v_{t+1}$ that is not contained in $E\left(F_{t}\right) \cup E\left(H_{t}\right)$ is chosen with the same probability to continue the exploration process. Hence,
given that $v_{t+1} w \in E\left(G_{n, d}\right) \backslash E\left(F_{t}\right)$, the probability that $w_{t+1}=w$ is precisely $\left(d-d_{H_{t}}\left(v_{t+1}\right)-\right.$ $\left.d_{F_{t}}\left(v_{t+1}\right)\right)^{-1}$. Therefore,

$$
\begin{aligned}
& \mathbb{P}\left[w_{t+1}=w \mid v_{t+1} w \notin E\left(F_{t}\right), \mathcal{H}_{t}\right] \\
= & \mathbb{P}\left[w_{t+1}=w \mid v_{t+1} w \in E\left(G_{n, d}\right) \backslash E\left(F_{t}\right), \mathcal{H}_{t}\right] \cdot \mathbb{P}\left[v_{t+1} w \in E\left(G_{n, d}\right) \mid v_{t+1} w \notin E\left(F_{t}\right), \mathcal{H}_{t}\right] \leq \frac{6}{n} .
\end{aligned}
$$

Since $\mathbb{P}\left[w_{t+1}=w \mid v_{t+1} w \in E\left(F_{t}\right), \mathcal{H}_{t}\right]=0$, it follows that for every $w \in V \backslash S_{t}$

$$
\begin{equation*}
\mathbb{P}\left[w_{t+1}=w \mid \mathcal{H}_{t}\right] \leq \frac{6}{n} \tag{5}
\end{equation*}
$$

Using that $\left|E\left(F_{t}\right)\right| \leq t$, we conclude

$$
\mathbb{E}\left[Y_{t+1} \mid \mathcal{H}_{t}\right]=\sum_{w \in V \backslash S_{t}} d_{F_{t}}(w) \mathbb{P}\left[w_{t+1}=w \mid \mathcal{H}_{t}\right] \stackrel{(5)}{\leq} \frac{6}{n} \sum_{w \in V \backslash S_{t}} d_{F_{t}}(w) \leq \frac{6}{n} \cdot t \leq 6 d n^{-1 / 3}
$$

We now prove the second statement. Given $w \in V \backslash S_{t}$ with $\mathbb{P}\left[w_{t+1}=w \mid \mathcal{H}_{t}\right]>0$ (that is, $v_{t+1} w \notin E\left(F_{t}\right)$ ), we apply Lemma 4 with $S=S_{t}, F$ obtained from $F_{t}$ by adding $v_{t+1} w, H=H_{t}$, and $v=w$, to obtain

$$
\begin{aligned}
\mathbb{E}\left[Z_{t+1} \mid \mathcal{H}_{t}\right] & =\sum_{w \in V \backslash S_{t}} \mathbb{E}\left[Z_{t+1} \mid w_{t+1}=w, v_{t+1} w \notin E\left(F_{t}\right), \mathcal{H}_{t}\right] \mathbb{P}\left[w_{t+1}=w \mid v_{t+1} w \notin E\left(F_{t}\right), \mathcal{H}_{t}\right] \\
& \stackrel{(5)}{\leq} \sum_{w \in V \backslash S_{t}} \frac{6 d\left|S_{t}\right|}{n} \cdot \frac{6}{n} \leq 180 d n^{-1 / 3}
\end{aligned}
$$

Lemma 6. Suppose $\mu \geq 0$ and $d, n \in \mathbb{N}$ such that $3 \leq d \leq n-1$ and $n$ is sufficiently large. Consider the exploration process described above on $\left(G_{n, d}, \mathfrak{S}_{n, d}\right)$ with $p=\frac{1-\mu n^{-1 / 3}}{d-1}$ and suppose $t \leq d n^{2 / 3}$. Conditional on $\left|S_{t}\right| \leq 5 n^{2 / 3}$, then

$$
\mathbb{E}\left[\eta_{t+1} \mid \mathcal{H}_{t}\right] \geq-(570+\mu) n^{-1 / 3} \quad \text { and } \quad \mathbb{E}\left[\eta_{t+1}^{2} \mid \mathcal{H}_{t}\right] \geq d / 4
$$

Moreover, if $X_{t}>0$, then $\mathbb{E}\left[\eta_{t+1}^{2} \mid \mathcal{H}_{t}\right] \leq d$.
Proof. First assume that $X_{t}>0$. Recall that for any $\mathcal{H}_{t}$ and for any edge $u v$ that has not been exposed yet, we have $\mathbb{E}\left[I(u v) \mid \mathcal{H}_{t}\right]=p=\left(1-\mu n^{-1 / 3}\right) /(d-1)$. Recall that $Y_{t+1}$ and $Z_{t+1}$ are measurable with respect to $\mathcal{H}_{t}$. Taking conditional expectations on (2) and using Lemma 5 , we obtain

$$
\begin{aligned}
\mathbb{E}\left[\eta_{t+1} \mid \mathcal{H}_{t}\right] & =-\left(1-\frac{1-\mu n^{-1 / 3}}{d-1}\right)+\frac{1-\mu n^{-1 / 3}}{d-1}\left(d-2-\mathbb{E}\left[Y_{t+1} \mid \mathcal{H}_{t}\right]-2 \mathbb{E}\left[Z_{t+1} \mid \mathcal{H}_{t}\right]\right) \\
& \geq-\frac{\mathbb{E}\left[Y_{t+1} \mid \mathcal{H}_{t}\right]+2 \mathbb{E}\left[Z_{t+1} \mid \mathcal{H}_{t}\right]}{d-1}-\mu n^{-1 / 3} \\
& \geq-\frac{380 d n^{-1 / 3}}{d-1}-\mu n^{-1 / 3} \geq-(570+\mu) n^{-1 / 3}
\end{aligned}
$$

since $d \geq 3$.
Again, by Lemma 5 and (2), we obtain

$$
\begin{aligned}
\mathbb{E}\left[\eta_{t+1}^{2} \mid \mathcal{H}_{t}\right] & =\left(1-\frac{1-\mu n^{-1 / 3}}{d-1}\right)(-1)^{2}+\frac{1-\mu n^{-1 / 3}}{d-1} \mathbb{E}\left[\left(d-2-Y_{t+1}-2 Z_{t+1}\right)^{2} \mid \mathcal{H}_{t}\right] \\
& \geq \frac{d-2}{d-1}+\frac{\left(1-\mu n^{-1 / 3}\right)(d-2)^{2}}{d-1}-\frac{2(d-2)\left(\mathbb{E}\left[Y_{t+1} \mid \mathcal{H}_{t}\right]+2 \mathbb{E}\left[Z_{t+1} \mid \mathcal{H}_{t}\right]\right)}{d-1} \\
& \geq\left(1-\mu n^{-1 / 3}\right)(d-2)-2\left(\mathbb{E}\left[Y_{t+1} \mid \mathcal{H}_{t}\right]+2 \mathbb{E}\left[Z_{t+1} \mid \mathcal{H}_{t}\right]\right) \\
& \geq\left(1-\mu n^{-1 / 3}\right)(d-2)-760 d n^{-1 / 3} \\
& \geq d / 4
\end{aligned}
$$

where the last inequality holds since $d \geq 3$ and $n$ is sufficiently large. Observe that $\mathbb{E}\left[\eta_{t+1}^{2} \mid \mathcal{H}_{t}\right] \leq d$ follows from a similar argument as $\left(d-2-Y_{t+1}-2 Z_{t+1}\right)^{2} \leq(d-2)^{2}$.

If $X_{t}=0$, then clearly $\mathbb{E}\left[\eta_{t+1} \mid \mathcal{H}_{t}\right] \geq 0$ and, since $\mathbb{E}\left[\eta_{t+1}^{2} \mid \mathcal{H}_{t}\right]=\mathbb{E}\left[\left(d-Y_{t+1}\right)^{2} \mid \mathcal{H}_{t}\right]$, similarly as before, one can prove that $\mathbb{E}\left[\eta_{t+1}^{2} \mid \mathcal{H}_{t}\right] \geq d / 4$.

Lemma 7. Suppose $\mu \geq 0$ and $d, n \in \mathbb{N}$ such that $3 \leq d \leq n-1$ and $n$ is sufficiently large. Consider the exploration process described above on $\left(G_{n, d}, \overline{\mathfrak{S}}_{n, d}\right)$ with $p=\frac{1-\mu n^{-1 / 3}}{d-1}$. Then, for every fixed $\delta>0$ and all $0 \leq t_{1} \leq t_{2} \leq 5 d n^{2 / 3}$, we have

$$
\begin{aligned}
& \mathbb{P}\left[\left|S_{t_{2}} \backslash S_{t_{1}}\right|-\frac{t_{2}-t_{1}}{d-1} \geq-\delta n^{2 / 3}\right]=1-o\left(n^{-2}\right) \quad \text { and } \\
& \mathbb{P}\left[\left|S_{t_{2}} \backslash S_{t_{1}}\right|-\frac{t_{2}-t_{1}}{d-1}-\left[\frac{t_{2}}{5 d / 6}\right] \leq \delta n^{2 / 3}\right]=1-o\left(n^{-2}\right)
\end{aligned}
$$

Proof. We add a vertex to $S_{t}$ either if $I\left(v_{t+1} w_{t+1}\right)=1$ or if we start exploring a new component of $G_{n, d, p}$ at time $t+1$. Thus, $\left|S_{t_{2}} \backslash S_{t_{1}}\right|$ stochastically dominates a binomial random variable with parameters $t_{2}-t_{1}$ and $\left(1-\mu n^{-1 / 3}\right) /(d-1)$. A standard application of Chernoff's inequality implies the first statement.

Let $A_{t} \subseteq S_{t}$ be the set of vertices that start a new component in $G_{n, d, p}$. For every $0 \leq$ $t \leq 5 d n^{2 / 3}$, let $a_{t}:=\left|A_{t}\right|$, let $c_{t}:=\left|S_{t} \backslash A_{t}\right|$ and let $b_{t}:=\left|S_{t} \backslash\left(S_{t_{1}} \cup A_{t}\right)\right|$. Observe that $c_{t}$ is stochastically dominated by a binomial random variable with parameters $t$ and $1 /(d-1)$. Using Chernoff's inequality, we have $c_{t} \leq 8 n^{2 / 3}$ with probability $1-o\left(n^{-2}\right)$ for any $t \leq 5 d n^{2 / 3}$.

We claim that for every $0 \leq t \leq 5 d n^{2 / 3}$ and conditional on $c_{t} \leq 8 n^{2 / 3}$, we have $a_{t} \leq\left\lceil\frac{t}{5 d / 6}\right\rceil$. Indeed, the claim is true for $t \in\{0,1\}$. Assume that $t \geq 2$ and that the claim holds for every $t^{\prime} \in\{0, \ldots, t-1\}$. If $X_{t-1}>0$, then $a_{t}=a_{t-1}$ and we are done. Thus, assume that $X_{t-1}=0$. Let $s$ be the largest integer $s^{\prime} \in\{0, \ldots, t-2\}$ such that $X_{s^{\prime}}=0$ (it exists since $X_{0}=0$ and $t \geq 2$ ). Recall that $w_{s+1}$ is a vertex $x \in V \backslash S_{s}$ that minimises $d_{F_{s}}(x)$. It follows that

$$
d_{F_{s}}\left(w_{s+1}\right) \leq \frac{\left|E\left(F_{s}\right)\right|}{n-\left(a_{s}+c_{s}\right)} \leq \frac{s}{n-\lceil s /(5 d / 6)\rceil-8 n^{2 / 3}} \leq \frac{d}{6}
$$

provided that $n$ is large enough. Hence, $X_{s+1} \geq 5 d / 6$ and the process will not start a new component for the next $5 d / 6$ steps. In particular, $s+5 d / 6 \leq t$. This implies $a_{t}=a_{s}+1 \leq$ $\left\lceil\frac{s}{5 d / 6}\right\rceil+1 \leq\left\lceil\frac{t}{5 d / 6}\right\rceil$.

Since $\left|S_{t_{2}} \backslash S_{t_{1}}\right| \leq a_{t_{2}}+b_{t_{2}}$, the second part of the lemma now follows from the upper bound on $a_{t_{2}}$ (which holds as we assume $c_{t} \leq 8 n^{2 / 3}$ ) and an upper bound on $b_{t_{2}}$ obtained by Chernoff's inequality.

## 5. Proof of Theorem 1

As we mentioned in the introduction, due to the result of Nachmias and Peres, we only need to prove a lower bound. Since it suffices to prove the lower bound of the statement for $\lambda \leq 0$, we use the definition $\mu:=-\lambda$. We now present a brief overview of the proof. In the first phase, we show that with probability at least $1-A^{-1 / 2}$, the process $X_{t}$ exceeds $A^{-1 / 4} d n^{1 / 3}$ in the first $d n^{2 / 3}$ steps. In the second phase and conditional on the success of the first phase, we show that $X_{t}$ stays positive for at least $2 A^{-1} d n^{2 / 3}$ steps with probability at least $1-A^{-1 / 2}$. From standard concentration inequalities, this gives the existence of a component of order at least $A^{-1} n^{2 / 3}$, concluding the proof. This proof strategy was introduced by Nachmias and Peres to prove the same statement for fixed $d \geq 3$ [20] and for $d=n-1$ [21]. We remark that, in comparison to [20], our analysis of the exploration process is simpler, as we do not need to track the number of vertices $x \in V \backslash S_{t}$ which satisfy $d_{F_{t}}(x)=k$ for $k \in\{0,1, \ldots, d\}$. If $d \geq 3$ is fixed, as in [20], almost every vertex $x$ satisfies $d_{F_{t}}(x) \in\{0,1\}$. However, this is no longer true if $d$ is an arbitrary function of $n$. We avoid the technicalities involved with this issue by averaging over the values of $d_{F_{t}}(x)$.

First phase: We start with the definition of a few parameters. Let $h:=A^{-1 / 4} d n^{1 / 3}, T_{1}:=$ $5 d n^{2 / 3} / 6$ and $T_{2}:=2 A^{-1} d n^{2 / 3}$. In addition, we define the following stopping times:

$$
\begin{aligned}
\tau_{h} & :=\min \left\{t: X_{t} \geq h\right\} \wedge T_{1} \\
\tau_{S}^{1} & :=\min \left\{t:\left|S_{t}\right| \geq 3 n^{2 / 3}\right\} \\
\tau_{1} & :=\tau_{h} \wedge \tau_{S}^{1} .
\end{aligned}
$$

Recall that $X_{t+1}=\eta_{t+1}+X_{t}$. Note also that for every $t<\tau_{1}$, we have $X_{t} \leq h$ and $\left|S_{t}\right| \leq 5 n^{2 / 3}$. Hence, Lemma 6 implies that

$$
\mathbb{E}\left[X_{t+1}^{2}-X_{t}^{2} \mid \mathcal{H}_{t}\right] \geq \mathbb{E}\left[\eta_{t+1}^{2} \mid \mathcal{H}_{t}\right]+2 \mathbb{E}\left[\eta_{t+1} X_{t} \mid \mathcal{H}_{t}\right] \geq d / 4-2 \cdot(570+\mu) n^{-1 / 3} h \geq d / 5
$$

provided that $A$ is large enough with respect to $\mu$ (and thus, with respect to $\lambda$ ). Hence $X_{t \wedge \tau_{1}}^{2}-$ $\left(t \wedge \tau_{1}\right) d / 5$ is a submartingale. By the Optional Stopping theorem for submartingales (see for example [9] p.491), $\mathbb{E}\left[X_{\tau_{1}}^{2}-\frac{d}{5} \tau_{1}\right] \geq \mathbb{E}\left[X_{0}^{2}\right]=0$, which implies that $\mathbb{E}\left[\tau_{1}\right] \leq \frac{5}{d} \mathbb{E}\left[X_{\tau_{1}}^{2}\right]$. Since $X_{\tau_{1}}^{2} \leq(h+d)^{2} \leq 2 h^{2}$, we obtain

$$
\mathbb{P}\left[\tau_{1}=T_{1}\right] \leq \frac{\mathbb{E}\left[\tau_{1}\right]}{T_{1}} \leq \frac{5 \mathbb{E}\left[X_{\tau_{1}}^{2}\right]}{d T_{1}} \leq \frac{10 h^{2}}{d T_{1}}=12 A^{-1 / 2}
$$

By Lemma 7 with $t_{1}=0$ and $t_{2}=T_{1}$, we have $\mathbb{P}\left[\tau_{S}^{1} \leq T_{1}\right]=o(1)$. Thus

$$
\begin{equation*}
\mathbb{P}\left[\left\{\tau_{h}=T_{1}\right\} \cup\left\{\tau_{S}^{1} \leq \tau_{h}\right\}\right] \leq \mathbb{P}\left[\tau_{1}=T_{1}\right]+\mathbb{P}\left[\tau_{S}^{1} \leq T_{1}\right] \leq 12 A^{-1 / 2}+o(1) \leq 13 A^{-1 / 2} \tag{6}
\end{equation*}
$$

We conclude that the event $\mathcal{E}:=\left\{\tau_{h}<T_{1}, \tau_{h}<\tau_{S}^{1}\right\}$ holds with probability at least $1-13 A^{-1 / 2}$. In particular, with probability at least $1-13 A^{-1 / 2}$, the random process $X_{t}$ exceeds $h$ before time $T_{1}$.
Second phase: Write $\mathbb{P}_{*}$ and $\mathbb{E}_{*}$ for the probability and the expectation conditional on $\mathcal{E}$. We define

$$
\begin{aligned}
\tau_{0} & :=\min \left\{t: X_{\tau_{h}+t}=0\right\} \wedge T_{2} \\
\tau_{S}^{2}: & =\min \left\{t:\left|S_{\tau_{h}+t} \backslash S_{\tau_{h}}\right| \geq 2 n^{2 / 3}\right\} \\
\tau_{2} & :=\tau_{0} \wedge \tau_{S}^{2} .
\end{aligned}
$$

Consider the random variable

$$
W_{t}:=h-\min \left\{h, X_{\tau_{h}+t}\right\} .
$$

Hence

$$
\begin{aligned}
W_{t+1}^{2}-W_{t}^{2} & \leq\left(h-\min \left\{h, X_{\tau_{h}+t}\right\}-\eta_{\tau_{h}+t+1}\right)^{2}-\left(h-\min \left\{h, X_{\tau_{h}+t}\right\}\right)^{2} \\
& =\eta_{\tau_{h}+t+1}^{2}-2 \eta_{\tau_{h}+t+1}\left(h-\min \left\{h, X_{\tau_{h}+t}\right\}\right) \\
& \leq \eta_{\tau_{h}+t+1}^{2}-2 \eta_{\tau_{h}+t+1} h .
\end{aligned}
$$

If $t<\tau_{2}$ and $n$ is sufficiently large, we can apply Lemma 6 and this leads to (provided $A$ is sufficiently large with respect to $\mu$ )

$$
\mathbb{E}_{*}\left[W_{t+1}^{2}-W_{t}^{2} \mid \mathcal{H}_{\tau_{h}+t}\right] \leq d+2 \cdot(570+\mu) n^{-1 / 3} \cdot h \leq 2 d
$$

Thus, $W_{t \wedge \tau_{2}}^{2}-2 d\left(t \wedge \tau_{2}\right)$ is a supermartingale. Similar as before, we use the Optimal Stopping theorem to conclude that

$$
\mathbb{E}_{*}\left[W_{\tau_{2}}^{2}\right] \leq 2 d \mathbb{E}_{*}\left[\tau_{2}\right] \leq 2 d T_{2} .
$$

Thus

$$
\begin{aligned}
\mathbb{P}_{*}\left[\tau_{2}<T_{2}\right] & =\mathbb{P}_{*}\left[\tau_{0}<T_{2}, \tau_{S}^{2}>T_{2}\right]+\mathbb{P}_{*}\left[\tau_{S}^{2} \leq T_{2}\right] \\
& \leq \mathbb{P}_{*}\left[W_{\tau_{2}} \geq h\right]+\mathbb{P}_{*}\left[\left|S_{\tau_{h}+T_{2}} \backslash S_{\tau_{h}}\right| \geq 2 n^{2 / 3}\right] \\
& \leq \mathbb{P}_{*}\left[W_{\tau_{2}}^{2} \geq h^{2}\right]+o(1) \\
& \leq \frac{\mathbb{E}_{*}\left[W_{\tau_{2}}^{2}\right]}{h^{2}}+o(1) \leq 5 A^{-1 / 2},
\end{aligned}
$$

where we used Lemma 7 with $t_{1}=\tau_{h}$ and $t_{2}=\tau_{h}+T_{2}$ for the second inequality. (Observe that we cannot apply Lemma 7 directly, because we assume $\mathcal{E}$ holds and $\tau_{h}$ is a random time. However, as $\tau_{h} \leq T_{1}$, a simple union bound with $t_{1}=k$ and $t_{2}=k+T_{2}$ for all $k \leq T_{1}$ together with the fact that $\mathbb{P}[\mathcal{E}] \geq 1-13 A^{-1 / 2} \geq 1 / 2$, yields the desired result.) It follows that

$$
\mathbb{P}\left[\left\{\tau_{2}<T_{2}\right\} \cup\left\{\tau_{h}=T_{1}\right\} \cup\left\{\tau_{S}^{1} \leq \tau_{h}\right\}\right] \leq \mathbb{P}\left[\left\{\tau_{h}=T_{1}\right\} \cup\left\{\tau_{S}^{1} \leq \tau_{h}\right\}\right]+\mathbb{P}_{*}\left[\tau_{2}<T_{2}\right]
$$

$$
\stackrel{(6)}{\leq} 13 A^{-1 / 2}+5 A^{-1 / 2}=18 A^{-1 / 2}
$$

Since all the vertices explored from time $\tau_{h}$ to $\tau_{h}+\tau_{2}$ belong to the same component of $G_{n, d, p}$, there exists a component of size at least $\left|S_{\tau_{h}+\tau_{2}} \backslash S_{\tau_{h}}\right|$. As $\tau_{2}=T_{2}=2 A^{-1} d n^{2 / 3}$ with probability at least $1-18 A^{-1 / 2}$, by Lemma 7 with $t_{1}=\tau_{h}$ and $t_{2}=\tau_{h}+T_{2}$ (as above, strictly speaking, we apply Lemma 7 with $t_{1}=k$ and $t_{2}=k+T_{2}$ for all $k \leq T_{1}$ and use the fact that $\mathbb{P}[\mathcal{E}] \geq 1 / 2$ ) with probability at least $1-18 A^{-1 / 2}-o(1) \geq 1-19 A^{-1 / 2}$, there exists a component of size at least $A^{-1} n^{2 / 3}$.

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[^0]:    The first author was supported by the EPSRC, grant no. EP/M009408/1.
    ${ }^{1}$ The non-existence of a linear order component when $p \leq(1-\epsilon) p_{\text {crit }}$ follows from Proposition 1 in [20]. The existence of a linear order component when $p \geq(1+\epsilon) p_{c r i t}$ follows from the expansion properties of $G_{n, d}$ (see Corollary 2.8 in [13]) and the results on $(n, d, \lambda)$-graphs in [12].

