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Abstract

We study the minimum degree necessary to guarantee the existence of perfect and almost-perfect triangle-tilings in an n-vertex graph G with sublinear independence number. In this setting, we show that if $\delta(G) \geq n/3 + o(n)$ then G has a triangle-tiling covering all but at most four vertices. Also, for every $r \geq 5$, we asymptotically determine the minimum degree threshold for a perfect triangle-tiling under the additional assumptions that G is K_r-free and n is divisible by 3.

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1 Introduction

A *triangle-tiling* in a graph G is a collection $\mathcal{T}$ of vertex-disjoint triangles in G. We say that $\mathcal{T}$ is *perfect* if $|\mathcal{T}| = n/3$, where n is the order of G. A trivial necessary condition for the existence of a perfect triangle-tiling is that 3 divides n. We let $V(\mathcal{T}) := \bigcup_{T \in \mathcal{T}} V(T)$ and say $\mathcal{T}$ *covers* $U \subseteq V(G)$ (respectively $v \in V(G)$) when $U \subseteq V(\mathcal{T})$ (respectively $v \in V(\mathcal{T})$), so a perfect triangle-tiling covers every vertex of the host graph. Given disjoint sets A and B which partition $V(G)$, we say that a triangle T in G is an A-*triangle* if T contains two vertices of A and one vertex of B, and likewise that T is a

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B-*triangle* if T contains two vertices of B and one vertex of A. Observe that if $|A| = 1 \pmod 3$ and $|B| = 2 \pmod 3$, there are no B-triangles in G and also there is no pair of vertex-disjoint A-triangles in G, then G does not have a perfect triangle-tiling. In that case, we call the ordered pair (A, B) a *divisibility barrier* in G (note that order is important here). Similarly, if $A \subseteq V(G)$ has size $|A| \geq 2n/3 + r$ for some $r > 0$, but $G[A]$ has no triangles, then every triangle-tiling in G contains at most $n - |A| \leq n/3 - r$ triangles, and so leaves at least $3r$ vertices uncovered. We call such a set A a *space barrier*.

The classical Corrádi-Hajnal theorem [4] states that if G has minimum degree $\delta(G) \geq 2n/3$, and n is divisible by 3, then G contains a perfect triangle-tiling. The minimum degree condition of this result is easily seen to be best-possible by considering, for an arbitrary $m \in \mathbb{N}$, the complete tripartite graph $G_1(m)$ with vertex classes of size $m-1, m$ and $m+1$. Indeed, $G_1(m)$ then has $n := 3m$ vertices and $\delta(G_1(m)) \geq 2m-1 = 2n/3-1$, but $G_1(m)$ has no perfect triangle-tiling, as the union of the two largest vertex classes is a space barrier. Observe, however, that $G_1(m)$ contains large independent sets. By proving the following theorem, Balogh, Molla and Sharifzadeh [2] recently showed that the minimum degree condition can be significantly weakened if we additionally assume that G has no large independent set. Throughout this paper we write $\alpha(G)$ to denote the independence number of G.

Theorem 1.1 ([2, Theorem 1.2]). *For every $\omega > 0$ there exist $n_0, \gamma > 0$ such that the following holds for every integer $n \geq n_0$ which is divisible by 3. If G is a graph on n vertices with $\delta(G) \geq n/2 + \omega n$ and $\alpha(G) \leq \gamma n$, then G contains a perfect triangle-tiling.*

For an arbitrary $m \in \mathbb{N}$, the graph $G_2(m)$ consisting of two copies of K_{3m+2} intersecting in a single vertex has $n := 6m + 3$ vertices, minimum degree $\delta(G_2(m)) = 3m + 1 = \lfloor n/2 \rfloor$ and independence number two. Moreover, $G_2(m)$ has a divisibility barrier (A, B), where B is the vertex set of one of the copies of K_{3m+2} and $A = V(G_2(m)) \setminus B$, and so $G_2(m)$ does not contain a perfect triangle-tiling. This example demonstrates that the minimum degree condition of Theorem 1.1 is best-possible up to the ωn additive error term. Alon suggested that if one only wants a triangle-tiling that covers all but a constant number of vertices, then perhaps the condition $\delta(G) \geq (1/3 + o(1))n$ is sufficient. In this paper, we show that this is indeed the case, by proving that if $\delta(G) \geq (1/3 + o(1))n$ and $\alpha(G) = o(n)$, then G has a triangle-tiling covering all but at most four vertices. Furthermore, under the additional assumptions that G has no divisibility barrier and 3 divides n, we show that G contains a perfect triangle-tiling.

Theorem 1.2. *For every $\omega > 0$ there exist $n_0, \gamma > 0$ such that if G is a graph on $n \geq n_0$ vertices with $\delta(G) \geq n/3 + \omega n$ and $\alpha(G) \leq \gamma n$, then*

(a) G contains a triangle-tiling covering all but at most four vertices of G, and
(b) if 3 divides n and G contains no divisibility barrier, then G contains a perfect triangle-tiling.

Observe that for an arbitrary $m \in \mathbb{N}$, the graph $G_3(m)$ consisting of two disjoint copies of K_{3m+2} has $n := 6m + 4$ vertices, minimum degree $\delta(G_3(m)) = 3m + 1 = n/2 - 1$ and independence number two, but every triangle-tiling in $G_3(m)$ covers at most $n - 4$ vertices. This demonstrates that the conditions of Theorem 1.2 do not guarantee a triangle-tiling which leaves fewer than four vertices uncovered. Furthermore, a straightforward construction demonstrates that the ωn error term in the minimum degree condition of Theorem 1.2 cannot be removed completely. For this we use the existence of triangle-free graphs on n vertices with independence number $o(n)$ and minimum degree $\omega(1)$, as exhibited by Erdős in [5]; we refer to such a graph as an *Erdős graph* and denote it by $\mathrm{ER}(n)$. For an arbitrary $m \in \mathbb{N}$ we then form a graph $G_4(m)$ by taking the complete bipartite graph whose vertex classes U and V have sizes $2m+1$ and $m-1$ respectively, and then placing copies of $\mathrm{ER}(|U|)$ and $\mathrm{ER}(|V|)$ on U and V respectively. The graph $G_4(m)$ formed in this way has $n := 3m$ vertices, minimum degree $\delta(G_4(m)) \geq n/3 + \omega(1)$ and sublinear independence number. Moreover, since U is a space barrier, $G_4(m)$ has no perfect triangle-tiling.

The relationship between the results in this paper and the Corrádi-Hajnal theorem is clearly analogous to the relationship between Ramsey-Turán theory and Turán's theorem, as Ramsey-Turán theory is concerned with the maximum possible number of edges in an H-free graph on n vertices with some upper bound on $\alpha(G)$. More precisely, in classical Ramsey-Turán theory the principle object of study is the function $\mathbf{RT}(n, H, m)$, which is defined to be the maximum number of edges in an H-free, n-vertex graph with independence number at most m, whenever such a graph exists for n, H and m. The asymptotic value of $\mathbf{RT}(n, K_r, o(n))$ was established for odd r by Erdős and Sós [6] and for even r by Erdős, Hajnal, Sós and Szemerédi [7], giving the following theorem.

Theorem 1.3 ([6, Theorem 1] and [7, Theorem 1])**.** *For every $r \geq 3$, we define*

$$f_{RT}(r) := \begin{cases} \frac{r-3}{r-1} & \textit{if } r \textit{ is odd,} \\ \frac{3r-10}{3r-4} & \textit{if } r \textit{ is even.} \end{cases}$$

(a) For every $\omega > 0$, there exists $\gamma, n_0 > 0$ such that if G is a graph on $n \geq n_0$ vertices with $\alpha(G) \leq \gamma n$ and with at least $(f_{RT}(r) + \omega)\binom{n}{2}$ edges, then G contains a copy of K_r.

(b) For every $\omega > 0$ and $\gamma > 0$, there exists $n_0 > 0$ such that for every $n \geq n_0$, there exists a K_r-free graph $G := G_{RT}(n, r, \omega, \gamma)$ on n vertices such that $\delta(G) \geq (f_{RT}(r) - \omega)n$ and $\alpha(G) \leq \gamma n$.

Observe that for any $r \geq 3$, $\omega, \gamma > 0$ and each sufficiently large n divisible by 6, the graph $G_5(n)$ on n vertices consisting of the disjoint union of $G_{\mathrm{RT}}(\frac{n}{2} - 1, r, \omega, \gamma)$ and $G_{\mathrm{RT}}(\frac{n}{2} + 1, r, \omega, \gamma)$ is K_r-free, has minimum degree $\delta(G_5(n)) \geq \left(\frac{f_{\mathrm{RT}}(r)}{2} - \omega\right) n$ and independence number at most γn. However, as $G_5(n)$ contains a divisibility barrier, it has no perfect triangle-tiling. Although the construction of $G_{\mathrm{RT}}(n, r, \omega, \gamma)$ was given in [6] (when r is odd) and [7] (when r is even), for completeness, we describe $G_{\mathrm{RT}}(n, r, \omega, \gamma)$ at the end of Section 5.

By combining Theorems 1.2 and 1.3 we determine, for every $r \geq 5$, the asymptotic minimum degree threshold for a perfect triangle-tiling in a K_r-free graph with sublinear independence number; this is the following corollary.

Corollary 1.4. *For every $r \geq 5$ and $\omega > 0$ there exist $n_0, \gamma > 0$ such that the following holds for every integer $n \geq n_0$ which is divisible by 3. If G is a K_r-free graph on n vertices with*

$$\delta(G) \geq \begin{cases} \frac{f_{RT}(r)}{2} n + \omega n & \textit{if } r \geq 7 \\ \frac{n}{3} + \omega n & \textit{if } r \in \{5, 6\} \end{cases}$$

and $\alpha(G) \leq \gamma n$, then G contains a perfect triangle-tiling.

Proof. Given $\omega > 0$, choose γ small enough and n_0 large enough to apply Theorem 1.2 with the same constants there as here and so that we may apply Theorem 1.3(a) with 3γ and $n_0/3$ in place of γ and n_0 respectively. We also insist that $\gamma n_0 + 2 \leq \omega n_0/2$. Since $\frac{f_{\mathrm{RT}}(r)}{2} \geq \frac{1}{3}$ if and only if $r \geq 7$, by Theorem 1.2(b) it suffices to prove that no K_r-free graph on $n \geq n_0$ vertices with $\delta(G) \geq \frac{f_{\mathrm{RT}}(r)}{2}n + \omega n$ and $\alpha(G) \leq \gamma n$ contains a divisibility barrier. So let G be such a graph, and suppose for a contradiction that (X, Y) is a divisibility barrier in G. Let A be the smaller of X and Y, and let B be the larger, so $|A| \leq n/2$. By definition of a divisibility barrier, if $A = Y$ then there is no pair of vertex-disjoint B-triangles in G, whilst if $A = X$ then there are no B-triangles in G at all. It follows that at most one vertex $a \in A$ has more than $\gamma n + 2$ neighbours in B, as given two such vertices $a, a' \in A$ we could use the fact that $\alpha(G) \leq \gamma n$ to choose an edge bc in $N(a) \cap B$ and then an edge $b'c'$ in $(N(a') \cap B) \setminus e$ to obtain a pair of vertex-disjoint B-triangles abc and $a'b'c'$ in G. So at least $|A| - 1$ vertices of A have at least $\delta(G) - \gamma n - 2 \geq \frac{f_{\mathrm{RT}}(r)}{2}n + \frac{\omega}{2}n$ neighbours in A. So in particular $|A| \geq \frac{f_{\mathrm{RT}}(r)}{2}n \geq \frac{n}{3}$. Moreover we have

$$e(G[A]) \geq \frac{1}{2}(|A|-1)\left(\frac{f_{\mathrm{RT}}(r)}{2} + \frac{\omega}{2}\right)n = \frac{n}{2|A|}\left(f_{\mathrm{RT}}(r) + \omega\right)\binom{|A|}{2} \geq \left(f_{\mathrm{RT}}(r) + \omega\right)\binom{|A|}{2},$$

so $G[A]$ contains a copy of K_r by Theorem 1.3(a). This contradicts our assumption that G was K_r-free and so completes the proof. $\square$

Observe that the graph $G = G_4(m)$ given by the construction following Theorem 1.2 has $n = 3m$ vertices, minimum degree at least $n/3 + \omega(1)$ and independence number $o(n)$, and that G contains a space barrier (and therefore does not contain a perfect triangle-tiling). Moreover, G is K_5-free since $G[U]$ and $G[V]$ are each triangle-free. This demonstrates that the minimum degree condition in Corollary 1.4 is best-possible up to the ωn error term for $r \in \{5, 6, 7\}$ (and that the error term cannot be removed entirely in these cases). Furthermore, the graph $G_5(n)$ presented after Theorem 1.3 shows that the minimum degree condition in Corollary 1.4 is best-possible up to the ωn error term for $r \geq 8$ also.

In a K_4-free graph, we can only construct space barriers when $\delta(G) < n/6$, so it may be true that, in a K_4-free graph, the conditions $\delta(G) \geq (1/6 + o(1))n$ and $\alpha(G) = o(n)$ are sufficient to guarantee a perfect triangle-tiling when n is divisible by 3; we discuss this further in Section 5. Also in Section 5, we consider the problem of determining the minimum degree condition which guarantees a perfect K_k-tiling in a graph with sublinear independence number when $k \geq 4$.

1.1 Proof outline

To illustrate the proof ideas of this paper, we here outline the proof of Theorem 1.2(b). Let G be a graph on n vertices with sublinear independence number and minimum degree somewhat greater than $n/3$, where n is large and divisible by 3.

Our proof makes extensive use of the notion of a *regular pair* in G. Loosely speaking, this is a pair (A, B) of vertex-disjoint subsets of $V(G)$ such that the edges between A

and B are distributed in a 'randomlike' manner (see Section 2.1 for formal definitions). Now suppose that (A, B) is a regular pair in G of density d (*i.e.* there are $d|A||B|$ edges between A and B), for some not-too-small d and sets A and B of linear size. Most vertices $v \in A$ then have approximately $d|B|$ neighbours in B. Since G has sublinear independence number, there must be an edge in the neighbourhood of v, and this creates a triangle in G whose vertices are v and two neighbours of v in B. The same argument with A and B reversed allows us to find triangles with two vertices in A and one in B. It is not hard to see that, provided $|A|$ and $|B|$ differ by at most a factor of two, then we can construct a triangle-tiling covering almost all of the vertices of $A \cup B$ by greedily choosing and deleting triangles in this way (this is the first part of Lemma 3.1). Moreover, if (A, B) has density greater than $1/2$ and is super-regular, meaning that every vertex has neighbourhood of typical size, and $|A|$ and $|B|$ differ by at most a little less than a factor of two, then Lemma 3.1 shows that we can in fact construct a triangle-tiling covering every vertex of $A \cup B$ (so long as 3 divides $|A \cup B|$). The ability to find a spanning triangle-tiling in this setup is one way we may complete a perfect triangle-tiling in G at the end of the proof.

Another setup in which we can find a spanning triangle-tiling is where we have pairwise vertex-disjoint sets $A, B, C \subseteq V(G)$ whose sizes are linear and approximately equal to each other such that (A, B), (B, C) and (A, C) are each super-regular pairs of not-too-small density and 3 divides $|A \cup B \cup C|$. Indeed, we first greedily find and remove triangles by the method described above so that equally many vertices remain in each of A, B and C, and then apply the Blow-up lemma to find a triangle-tiling covering all remaining vertices of A, B and C by triangles each using one vertex from each set. This argument is formalised by Lemma 3.2.

We begin the proof by a standard application of the Szemerédi regularity lemma to find a partition of G into a bounded number of *clusters* $V_1, \dots, V_k$ of equal size and a small exceptional set V_0, and define a reduced graph R whose vertices are the clusters of G and whose edges correspond to pairs of clusters which form regular pairs of not-too-small density in G. Then a straightforward counting argument shows that either

(a) there is an edge V_iV_j of R for which the pair (V_i, V_j) has density somewhat more than $1/2$, or
(b) R has minimum degree at least $2k/3$. In particular, certainly there are clusters V_i, V_j and V_k which form a triangle in R.

In case (a), by removing a small number of vertices from V_i and V_j (and adding these to the exceptional set) we can make the pair (V_i, V_j) super-regular with density more than $1/2$, achieving the first setup described above. Similarly in case (b) we can remove a small number of vertices from each of V_i, V_j and V_k to achieve the second setup described above. These two or three clusters (according to which case we are in) form the 'core' of G. Our proof then proceeds by iteratively removing vertex-disjoint triangles so as to cover every vertex outside the core and only a small number of vertices within the core; we can then complete a perfect triangle-tiling in G by finding a triangle-tiling spanning the remaining vertices of the core as described above.

A key step in achieving this is the use of perfect fractional weighted matchings. The theory for these is developed in Section 2.4, with the key conclusion being that since R has minimum degree somewhat greater than $k/3$, we can partition all clusters outside the core into *subclusters* of linear size, so that the subclusters form regular pairs (A_i, B_i) of not-too-small density and the sizes of A_i and B_i differ by at most a little less than a factor of two (the crucial ratio for being able to find a triangle-tiling covering almost all vertices of $A_i \cup B_i$ as described above). We then define an auxiliary reduced graph R^* with a vertex v_i corresponding to each pair (A_i, B_i) and a final vertex v^* corresponding to the core of G, with an edge of R^* indicating that the corresponding pairs (or perhaps triple, in the case of the core) include subsets of clusters of an edge of R.

Suppose for simplicity that the reduced graph R is connected; it follows that R^* is connected, and using a theorem of Win (Theorem 2.7) we find a spanning tree T in R^* of bounded maximum degree. We take v^* to be the root of T, and iteratively 'work inwards' from the leaves of T to v^* to construct a perfect triangle-tiling in G, as follows. First we choose a leaf v_i of T, and remove a triangle-tiling in the corresponding pair (A_i, B_i) covering almost all vertices of this pair. Writing v_j for the parent of v_i in T, we then remove a few more triangles to cover all uncovered vertices of $A_i \cup B_i$ as well as a small number of vertices in the pair (A_j, B_j) corresponding to v_j. We then delete the leaf v_i from T, and iterate. At the end of this iteration only the root v^* of T remains, at which point we have constructed a triangle-tiling covering all vertices of T outside the core as well as a small number of vertices of the core. We then find a perfect triangle-tiling within the remaining vertices of the core (recall that the core was chosen so as to permit this step) to complete the desired perfect triangle-tiling in G.

If instead R is not connected, then R has precisely two components (since $\delta(R) > k/3$). After allocating exceptional vertices appropriately, these components yield a partition of $V(G)$ into two parts, say X and Y. We may then use the fact that G contains no divisibility barrier to find and remove at most two triangles from G so that following these deletions both $|X|$ and $|Y|$ are divisible by 3. We then proceed exactly as above within each of $G[X]$ and $G[Y]$ (and the corresponding components of R) to obtain perfect triangle-tilings in each of these subgraphs; together with the removed triangles these form a perfect triangle-tiling in G, completing the proof.

2 Notation and preliminary results

In this section we introduce various results which we will use in the proof of Theorem 1.2, beginning with helpful notation. Given a graph G, we write $|G|$ and $e(G)$ for the number of vertices and edges of G respectively. We write $x = y \pm z$ to mean $y - z \leq x \leq y + z$, and $[n]$ to denote the set of integers from 1 to n. We omit floors and ceilings throughout this paper wherever they do not affect the argument. We write $x \ll y$ to mean that for every $y > 0$ there exists $x_0 > 0$ such that the subsequent statements hold for x and y whenever $0 < x \leq x_0$. Similar statements with more variables are defined similarly.

2.1 Regularity

In a graph G, for each pair of disjoint non-empty sets $A, B \subseteq V(G)$ we write $G[A, B]$ for the bipartite subgraph of G with vertex classes A and B and whose edges are all edges of G with one endvertex in A and the other in B, and denote the *density* of $G[A, B]$ by $d_G(A, B) := \frac{e(G[A,B])}{|A||B|}$. We say that $G[A, B]$ is (d, ε)-*regular* if $d_G(X, Y) = d \pm \varepsilon$ for every $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$, and we write that $G[A, B]$ is $(\geq d, \varepsilon)$-*regular* to mean that $G[A, B]$ is (d', ε)-regular for some $d' \geq d$. Also, we say that $G[A, B]$ is (d, ε)-*super-regular* if $G[A, B]$ is $(\geq d, \varepsilon)$-regular, every vertex of A has at least $(d - \varepsilon)|B|$ neighbours in B and every vertex of B has at least $(d - \varepsilon)|A|$ neighbours in A. The following well-known results are elementary consequences of the definitions.

Lemma 2.1 (Slicing Lemma). *For every $d, \varepsilon, \beta > 0$, if $G[A, B]$ is (d, ε)-regular, and $X \subseteq A$ and $Y \subseteq B$ have sizes $|X| \geq \beta|A|$ and $|Y| \geq \beta|B|$, then $G[X, Y]$ is $(d, \varepsilon/\beta)$-regular.*

Lemma 2.2. *For every $d, \varepsilon > 0$ with $\varepsilon < \frac{1}{2}$, if $G[A, B]$ is $(\geq d, \varepsilon)$-regular, then there are sets $X \subseteq A$ and $Y \subseteq B$ with sizes $|X| \geq (1 - \varepsilon)|A|$, and $|Y| \geq (1 - \varepsilon)|B|$ such that $G[X, Y]$ is $(d, 2\varepsilon)$-super-regular.*

We make use of Chernoff bounds on the concentration of binomial and hypergeometric distributions in the following form.

Theorem 2.3 ([8, Corollary 2.3 and Theorem 2.10]). *Suppose X has binomial or hypergeometric distribution and $0 < a < 3/2$. Then $\mathbb{P}(|X - \mathbb{E}X| \geq a\mathbb{E}X) \leq 2e^{-\frac{a^2}{3}\mathbb{E}X}$.*

The following lemma is similar to lemmas of Csaba and Mydlarz [3, Lemma 14] and Martin and Skokan [14, Lemma 10]. It states that if we randomly select a collection of disjoint subsets from each of the vertex classes of a super-regular pair, every pair of sets from different classes is super-regular with high probability.

Lemma 2.4 (Random Slicing Lemma). *Suppose that $1/n \ll \beta, \varepsilon \ll d$. Let $G[A, B]$ be (d, ε)-super-regular (respectively (d, ε)-regular) where $|A|, |B| \leq n$. Also let $x_1, \ldots, x_s$ and $y_1, \ldots, y_t$ be positive integers each of size at least βn such that $\sum_{i \in [s]} x_i \leq |A|$ and $\sum_{j \in [t]} y_j \leq |B|$. If $\{X_1, \ldots, X_s\}$ is a collection of disjoint subsets of A and $\{Y_1, \ldots, Y_t\}$ is a collection of disjoint subsets of B such that $|X_i| = x_i$ and $|Y_j| = y_j$ for all $i \in [s]$ and $j \in [t]$ selected uniformly at random from all such collections, then, with probability at least $1 - e^{-\Omega(n)}$, $G[X_i, Y_j]$ is (d, ε')-super-regular (respectively (d, ε')-regular) for all $i \in [s]$ and $j \in [t]$, where $\varepsilon' := (33\varepsilon)^{1/5}$.*

For completeness we present a proof of Lemma 2.4 in the Appendix. To make use of regularity properties, we apply the degree form of Szemerédi's Regularity Lemma (see [12, Theorem 1.10]).

Theorem 2.5 (Degree form of Szemerédi's Regularity Lemma). *For every $\varepsilon > 0$, real number $d \in [0, 1]$ and integers t and q there exists integers n_0 and T such that the following statement holds. Let G be a graph on $n \geq n_0$ vertices, and let $U_1, \ldots, U_q$ be a partition of $V(G)$ into q parts. Then there is a partition of $V(G)$ into an exceptional set V_0 and k clusters $V_1, \ldots, V_k$, and a spanning subgraph $G' \subseteq G$ such that*

(a) $t \leq k \leq T$,
(b) $|V_1| = |V_2| = \ldots = |V_k|$ *and* $|V_0| \leq \varepsilon n$,
(c) *for every* $i \in [k]$ *there exists* $j \in [q]$ *such that* $V_i \subseteq U_j$,
(d) $d_{G'}(v) \geq d_G(v) - (\varepsilon + d)n$ *for all* $v \in V(G)$,
(e) $e(G'[V_i]) = 0$ *for all* $i \in [k]$, *and*

(f) for each distinct $i, j \in [k]$ either $G'[V_i, V_j]$ is $(\geq d, \varepsilon)$-regular or $G'[V_i, V_j]$ is empty.

Theorem 2.5 as stated above is stronger than the form given in [12] in that it allows us to specify an initial partition of $V(G)$ and to insist that the clusters $V_1, V_2, \ldots, V_k$ are each a subset of some part of this partition (property (c) above). However, this statement follows from the same proof, which proceeds iteratively by alternately refining a partition of $V(G)$ and deleting some vertices of $V(G)$ (which are then placed in the exceptional set V_0). So to prove Theorem 2.5 we take our specified partition as the initial partition of this process.

2.2 Robustly-matchable sets

The following application of the regularity lemma is critical to the entire proof. Given a graph G, a small $A \subseteq V(G)$ and a small matching $B \subseteq E(G)$, we form an auxiliary bipartite graph F with vertex set $A \cup B$ in which there is an edge between $a \in A$ and $bc \in B$ if and only if abc is a triangle in G. So matchings in F correspond to triangle-tilings in G. In this setting, Lemma 2.6 allows us to choose subsets $X \subseteq A$ and $Y \subseteq B$ such that if we can find a triangle-tiling in G that covers every vertex of G except for the vertices incident to edges in Y and exactly $|Y|$ of the vertices in X, then we obtain a perfect triangle-tiling in G.

Lemma 2.6. *Suppose that $1/n \ll \phi \ll \varepsilon \ll d$. Let F be a bipartite graph with vertex classes A and B such that $n/10 \leq |A|, |B| \leq n$ and $d_F(A, B) \geq d$. Then there exist subsets $X \subseteq A$ and $Y \subseteq B$ of sizes $|X| = \phi n$ and $|Y| = (1 - \varepsilon)\phi n$ such that $F[X', Y]$ contains a perfect matching for every subset $X' \subseteq X$ with $|X'| = |Y|$.*

Proof. Let n_0 and T be the integers returned by Theorem 2.5 given inputs ε, $d' := d/200$ and $t = q = 2$. We may assume that $\phi \leq 1/4T$. We use Theorem 2.5 with initial partition $U_1 = A$ and $U_2 = B$ to obtain a spanning subgraph $F' \subseteq F$ and a partition of $V(F)$ into sets $V_0, V_1, \ldots, V_k$ which satisfy properties (a)–(f) of Theorem 2.5. In particular, by Theorem 2.5(d) at most $(\varepsilon + d/200)n^2$ edges of F are not edges of F'. Also, by Theorem 2.5(e) there are no edges in $F'[V_i]$ for any $i \in [k]$, and since $|V_0| \leq \varepsilon n$ by Theorem 2.5(b), at most εn^2 edges of F contain a vertex of V_0. Since

$$e(F) = d_F(A, B)|A||B| \geq d\left(\frac{n}{10}\right)^2 > \left(\varepsilon + \frac{d}{200}\right)n^2 + \varepsilon n^2,$$

there must exist distinct $i, j \in [k]$ such that $F'[V_i, V_j]$ is non-empty, and since F is bipartite, by Theorem 2.5(c) we may assume without loss of generality that $V_i \subseteq A$ and $V_j \subseteq B$. Observe that $F'[V_i, V_j]$ is $(\geq d', \varepsilon)$-regular by Theorem 2.5(f). Write m for the common size of V_i and V_j, so $m = |V(F) \setminus V_0|/k \geq n/2T \geq 2\phi n$ by Theorem 2.5(a) and (b). By Lemma 2.2 we may delete at most εm vertices from each of V_i' and V_j' to obtain subsets $V_i' \subseteq V_i$ and $V_j' \subseteq V_j$ such that $F[V_i', V_j']$ is $(d', 2\varepsilon)$-super-regular. Having done so, choose $X \subseteq V_i'$ and $Y \subseteq V_j'$ uniformly at random with sizes ϕn and $(1 - \varepsilon)\phi n$ respectively (this is possible since $|V_i'|, |V_j'| \geq (1 - \varepsilon)m \geq \phi n$). Then Lemma 2.4 tells us that $F'[X, Y]$ is (d', ε')-super-regular with high probability, where $\varepsilon' := (66\varepsilon)^{1/5}$, so we may fix sets X and Y with this property. It then follows that every vertex of X has at least $(d' - \varepsilon')|Y| \geq \varepsilon'|X|$ neighbours in Y, whilst every set of at least $\varepsilon'|X|$ vertices of

X has at least $(1-\varepsilon')|Y| \geq (1-2\varepsilon')|X|$ neighbours in Y (where we say that a vertex y is a neighbour of a set X' if y is a neighbour of some element of X'). Finally, since every vertex of Y has at least $(d'-\varepsilon')|X| > 2\varepsilon'|X|$ neighbours in X, every set of at least $(1-2\varepsilon')|X|$ vertices of X has every vertex of Y as a neighbour. So Hall's criterion is satisfied for every $X' \subseteq X$ of size $|X'| \leq |Y|$, so for every $X' \subseteq X$ with $|X'| = |Y|$ there is a perfect matching in $F'[X', Y]$. □

2.3 Spanning bounded degree trees

Our proof requires us to find a spanning tree of bounded maximum degree in the reduced graph R of G. For this, we use the following theorem of Win [16].

Theorem 2.7. *If $k \geq 2$ and R is a connected graph such that*

$$\sum_{v \in S} d(v) \geq |R| - 1 \text{ for every independent set } S \text{ of size } k,$$

then R contains a spanning tree T such that $\Delta(T) \leq k$. In particular, if R is a connected graph with $\delta(R) \geq (|R|-1)/k$, then R contains a spanning tree T with maximum degree at most k.

2.4 Fractional weighted matchings via linear programming

Recall from the proof outline that we will consider regular pairs of clusters of vertices of G and use the regularity of each pair to find a triangle-tiling covering a given proportion of vertices from each cluster. We want to choose these proportions so that collectively these triangle-tilings cover (almost) all of the vertices of G. To do this we look for a generalized form of weighted matching in the reduced graph; the proportion of vertices to be covered by a triangle-tiling within a pair of clusters then corresponds to the weight in this matching of the corresponding edge of the reduced graph.

A *fractional matching* w in a graph G assigns a weight $w_e \geq 0$ to each edge $e \in E(G)$ such that for every vertex $u \in V(G)$ we have $\sum_{e \ni u} w_e \leq 1$. In other words, if we consider each edge uv to place weight w_{uv} at each of u and v, then the the combined weight placed at each vertex is at most one. This is a relaxation of an integer matching M, in which we insist that for each $e \in E(G)$ we have $w_e = 1$ (meaning that $e \in M$) or $w_e = 0$ (meaning that $e \notin M$). Here we work with a more general notion of an (η, ξ)*-weighted fractional matching*, in which we consider each edge to place different weights at each end, subject to the restriction that the ratio of these weights is at most $\eta : \xi$. It is most natural to express these matchings in terms of directed graphs, as we can then consider a directed edge $\overrightarrow{uv}$ of weight $w_{\overrightarrow{uv}}$ to place weight $\eta w_{\overrightarrow{uv}}$ on its tail u and weight $\xi w_{\overrightarrow{uv}}$ on its head v; as before, we insist that the combined weight placed at each vertex is at most one.

Definition 2.8. Let Γ be a directed graph on n vertices and let η and ξ be positive real numbers. An (η, ξ)*-weighted fractional matching* w in Γ is an assignment of a weight $w_{\overrightarrow{uv}} \geq 0$ to each edge $\overrightarrow{uv}$ of Γ such that for every vertex $u \in V(\Gamma)$ we have

$$\sum_{v \in N^+_\Gamma(u)} \eta w_{\overrightarrow{uv}} + \sum_{v \in N^-_\Gamma(u)} \xi w_{\overrightarrow{vu}} \leq 1. \tag{1}$$

The *total weight* of w is defined to be $W := \sum_{\overrightarrow{uv} \in E(\Gamma)} (\eta + \xi) w_{\overrightarrow{uv}}$. By (1) we have $W \leq n$; we say that w is *perfect* if $W = n$. Note that in this case we have equality in (1) for every vertex.

Given an undirected graph G, we consider (η, ξ)-weighted fractional matchings in the directed graph Γ formed by replacing every edge uv of G with both a directed edge $\overrightarrow{uv}$ from u to v and a directed edge $\overrightarrow{vu}$ from v to u. In particular, a $(\frac{1}{2}, \frac{1}{2})$-weighted fractional matching w in Γ then corresponds to a fractional matching w' in G (in the standard notion of fractional matching as defined above). Indeed, given w, for each edge $e = uv \in E(G)$ we may take $w'_e = w_{\overrightarrow{uv}} + w_{\overrightarrow{vu}}$. In our proof we will instead consider (η, ξ)-weighted fractional matchings in Γ where ξ is close to twice as large as η. The advantage of this is shown by Lemma 2.10, which states that the minimum degree condition on G needed to guarantee the existence of a perfect (η, ξ)-weighted fractional matching in Γ is then approximately $n/3$, well below the $n/2$ threshold needed to guarantee the existence of a perfect fractional matching in G.

Let Γ be a directed graph on n vertices $v_1, \ldots, v_n$, and fix $\eta, \xi > 0$. Then we define the (η, ξ)-*weighted characteristic vector* of an edge $\overrightarrow{uv} \in E(\Gamma)$ to be the vector $\chi_{\eta,\xi}(\overrightarrow{v_i v_j}) \in \mathbb{R}^n$ whose ith coordinate is equal to η, whose jth coordinate is equal to ξ, and in which all other coordinates are equal to zero. So an assignment w of non-negative weights to edges of Γ is an (η, ξ)-weighted fractional matching in Γ if and only if

$$\sum_{\overrightarrow{v_i v_j} \in E(\Gamma)} w_{\overrightarrow{v_i v_j}} \chi_{\eta,\xi}(\overrightarrow{v_i v_j}) \leq \mathbf{1}, \tag{2}$$

where $\mathbf{1}$ is the vector in $\mathbb{R}^n$ with each coordinate equal to 1 and the inequality is treated pointwise. As before, w is perfect if and only if we have equality for each coordinate.

To prove the existence of a (η, ξ)-weighted fractional matching in a directed graph of high minimum indegree, we use the following version of Farkas' Lemma, for which we need the following definition; a vertex $\boldsymbol{v} \in \mathbb{R}^n$ is a *weighted sum* of vectors in $\mathcal{X} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\} \subseteq \mathbb{R}^n$ if

$$\boldsymbol{v} \in \left\{ \sum_{i=1}^{m} \lambda_i \boldsymbol{x}_i : \lambda_i \geq 0 \text{ for every } i \in [m] \right\},$$

otherwise $\boldsymbol{v}$ is not a weighted sum of the vectors in $\mathcal{X}$.

Lemma 2.9 (Farkas' Lemma)**.** *For every $\boldsymbol{v} \in \mathbb{R}^n$ and every finite $\mathcal{X} \subseteq \mathbb{R}^n$, if $\boldsymbol{v}$ is not a weighted sum of the vectors in $\mathcal{X}$, then there exists $\boldsymbol{y} \in \mathbb{R}^n$ such that $\boldsymbol{y} \cdot \boldsymbol{x} \geq 0$ for every $\boldsymbol{x} \in \mathcal{X}$ and $\boldsymbol{y} \cdot \boldsymbol{v} < 0$.*

We now give the main result of this section.

Lemma 2.10. *For every $\eta > 0$, every directed graph Γ on n vertices with $\delta^-(\Gamma) \geq \eta n$ admits a perfect fractional $(\eta, 1 - \eta)$-matching. Furthermore, if $\eta = p/q$ for positive integers p and q, then we can assume that the weights of the matching are rational numbers with common denominator D bounded above by some function of p, q and n.*

Proof. Let $v_1, \dots, v_n$ be an arbitrary ordering of the vertices of Γ. Then by (2), a perfect $(\eta, 1-\eta)$-weighted fractional matching in Γ corresponds to a weighted sum of the vectors in

$$\mathcal{X} := \{\chi_{\eta,1-\eta}(\overrightarrow{v_iv_j}) : \overrightarrow{v_iv_j} \in E(\Gamma)\}$$

that equals $\mathbf{1}$.

If we assume that Γ does not have a perfect $(\eta, 1-\eta)$-weighted fractional matching, then, by Farkas' lemma (Lemma 2.9), as $\mathbf{1}$ is not a weighted sum of the vectors in $\mathcal{X}$, there exists a vector $\boldsymbol{y} \in \mathbb{R}^n$ such that $\boldsymbol{y} \cdot \mathbf{1} < 0$ but $\boldsymbol{y} \cdot \chi_{\eta,1-\eta}(\overrightarrow{v_iv_j}) \geq 0$ for every $\overrightarrow{v_iv_j} \in E(\Gamma)$. By reordering the vertices if necessary, we may assume that $y_1 \geq \dots \geq y_n$.

Let i be maximal such that $\overrightarrow{v_iv_n} \in E(\Gamma)$, so $i \geq \delta^-(\Gamma) \geq \eta n$. Then,

$$0 > \boldsymbol{y} \cdot \mathbf{1} = \sum_{j=1}^{i} y_j + \sum_{j=i+1}^{n} y_j \geq iy_i + (n-i)y_n \geq \eta n y_i + (1-\eta)ny_n = n\boldsymbol{y} \cdot \chi_{\eta,1-\eta}(\overrightarrow{v_iv_n}) \geq 0,$$

a contradiction.

The second statement is implied by basic linear programming theory, if we take the perfect fractional $(\eta, 1-\eta)$-matching to be one with the smallest possible number of non-zero weights, as then w is a basic feasible solution. $\square$

Note that if a directed graph Γ admits a perfect (η, ξ)-weighted fractional matching w with $\eta \leq \xi$ and $\eta + \xi = 1$, then $\alpha(\Gamma) \leq \xi n$, because for every independent set A in Γ we have

$$|A| = \sum_{a \in A} \left(\sum_{b \in N^+(a)} \eta w_{\overrightarrow{ab}} + \sum_{b \in N^-(a)} \xi w_{\overrightarrow{ba}} \right) \leq \xi \sum_{a \in A} \left(\sum_{b \in N^+(a)} w_{\overrightarrow{ab}} + \sum_{b \in N^-(a)} w_{\overrightarrow{ba}} \right) \leq \xi W \leq \xi n,$$

where the initial equality holds since we have equality in (1), and the penultimate inequality holds because (since A is an independent set) every edge of Γ contributes at most once to the sum. This shows that the minimum indegree condition of Lemma 2.10 is best possible for $\eta \leq 1/2$, since weaker conditions do not preclude the existence of independent sets of size greater than $(1-\eta)n$.

3 Triangle-tilings in regular pairs and triples

As described in the proof outline, the proof of Theorem 1.2 proceeds by iteratively constructing a triangle-tiling in G which covers all of the vertices outside of a small 'core' subset of vertices but leaves most vertices inside this 'core' uncovered. This gives a perfect triangle-tiling in G, because the 'core' is robust in the sense that it has a perfect triangle-tiling after the removal of any sufficiently small set of vertices (provided that the number of vertices remaining is divisible by 3). Depending on the structure of the graph G, this 'core' will either consist of sets A and B which form a super-regular pair with density greater than $\frac{1}{2}$, or of sets A, B and C which form three super-regular pairs each with density bounded below by a small constant.

We begin with the case where the 'core' consists of a super-regular pair of density greater than $\frac{1}{2}$ (part (c) of Lemma 3.1). Let G be a graph whose vertex set is the disjoint union of sets A and B. Recall that a triangle T in G is an A-triangle if T contains two vertices of A and one vertex of B, and likewise that T is a B-triangle if T contains two vertices of B and one vertex of A.

Lemma 3.1. *Suppose that $1/n \ll \gamma \ll \varepsilon \ll \phi, \varepsilon' \ll d \ll \omega$. Let A and B be disjoint sets of vertices with $n/3 + \omega n \leq |A|, |B| \leq 2n/3 - \omega n$ and $|A \cup B| = n$, and let G be a graph on vertex set $V := A \cup B$ with $\alpha(G) \leq \gamma n$. Then the following statements hold.*

(a) If $G[A, B]$ is $(\geq d, \varepsilon)$-regular then G admits a triangle-tiling covering all but at most $2\varepsilon n$ vertices of G. Moreover, for every a and b with $2a + b \leq |A| - \varepsilon n$ and $a + 2b \leq |B| - \varepsilon n$ there is a triangle-tiling in G which consists of a A-triangles and b B-triangles.

(b) If $G[A, B]$ is (d, ε)-super-regular then, for every $S \subseteq A$ of size $|S| = \phi n$ for which $|A \setminus S| + |B| + \lfloor \phi\varepsilon' n \rfloor$ is divisible by 3, there is a triangle-tiling in G which covers every vertex of $G[V \setminus S]$ and which covers precisely $\lfloor \phi\varepsilon' n \rfloor$ vertices of S.

(c) If n is divisible by 3 and $G[A, B]$ is $(1/2 + d, \varepsilon)$-super-regular then G contains a perfect triangle-tiling.

Proof. For (a) the triangles may be chosen greedily. Indeed, suppose that we have already chosen a triangle-tiling $\mathcal{T}$ consisting of at most a A-triangles and at most b B-triangles, then $\mathcal{T}$ covers at most $2a + b$ vertices of A, and at most $a + 2b$ vertices of B. Taking $A' = A \setminus V(\mathcal{T})$ and $B' = B \setminus V(\mathcal{T})$, we find that $|A'|, |B'| \geq \varepsilon n$. Since $G[A, B]$ is $(\geq d, \varepsilon)$-regular it follows that $d_G(A', B') \geq d - \varepsilon$, therefore some vertex $x \in A'$ has at least $(d - \varepsilon)|B'| \geq (d - \varepsilon)\varepsilon n > \gamma n$ neighbours in B'. Since $\alpha(G) \leq \gamma n$ it follows that some two of these neighbours are adjacent, giving a B-triangle which can be added to $\mathcal{T}$. The same argument with the roles of A' and B' reversed yields instead an A-triangle which may be added to $\mathcal{T}$. This proves the second statement of (a); the first follows by setting $a = \frac{1}{3}(2|A| - |B| - \varepsilon n)$ and $b = \frac{1}{3}(2|B| - |A| - \varepsilon n)$.

Next, for (b), let $z := \lfloor \phi\varepsilon' n \rfloor$, $t_4 := \lfloor z/2 \rfloor$ and $z' := z - 2t_4 \in \{0, 1\}$, so we will construct a triangle-tiling that covers all of $(A \setminus S) \cup B$ and exactly $z = 2t_4 + z'$ vertices of S. Let $B_1' \subseteq B$ consist of all vertices in B with fewer than $(d - \frac{\varepsilon}{\phi})|S|$ neighbours in S; since $G[S, B]$ is $(\geq d, \frac{\varepsilon}{\phi})$-regular we have $|B_1'| \leq \frac{\varepsilon}{\phi} n$. Form B_1 by adding at most 2 arbitrarily selected vertices from $B \setminus B_1'$ to B_1' so that $|B \setminus B_1| - t_4$ is divisible by 3. Since $G[A, B]$ is (d, ε)-super-regular, every vertex of B_1 has at least $(d - \varepsilon)|A| - |S| \geq \frac{dn}{3} > 2|B_1| + \gamma n$ neighbours in $A \setminus S$. Since $\alpha(G) \leq \gamma n$, we may greedily form a triangle-tiling $\mathcal{T}_1$ of A-triangles in G of size $|B_1|$ which covers every vertex of B_1 and does not use any vertex from S. We now select uniformly at random a subset $B_2 \subseteq B \setminus B_1$ of size $|B_2| = t_4$. Since every vertex in A has at least $(d - \varepsilon)|B| - |B_1| \geq \frac{dn}{3}$ neighbours in $B \setminus B_1$, Theorem 2.3 implies that, with probability $1 - o(1)$, every vertex of A has at least $\frac{\phi\varepsilon' d}{7} n$ neighbours in B_2. Fix a choice of B_2 for which this event occurs. Let S' be an arbitrarily selected subset of S of size z' (so S' is either empty or a singleton) and let $A' := (A \setminus (S \cup V(\mathcal{T}_1))) \cup S'$ and $B' := B \setminus (B_1 \cup B_2)$. Recall that, by assumption, $|A \setminus S| + |B| + z$ is divisible by 3, so

$$|A'| + |B'| = |A \setminus S| + z' + |B| - |B_2| - |V(\mathcal{T}_1)| = (|A \setminus S| + |B| + z) - (3t_4 + |V(\mathcal{T}_1)|)$$

is divisible by 3. Since $|B'|$ is divisible by 3 by our selection of B_1 and B_2, it follows that $|A'|$ is divisible by 3 as well. Let $t_3 = \left\lfloor \frac{\phi\varepsilon' d}{15} n \right\rfloor$, $a := \frac{2}{3}|A'| - \frac{1}{3}|B'|$ and $b := \frac{2}{3}|B'| - \frac{1}{3}|A'| - t_3$.

Since $G[A', B']$ is $(\geq d, \frac{\varepsilon}{2})$-regular, (a) implies that there is a triangle-tiling $\mathcal{T}_2$ in $G[A' \cup B']$ such that $A'' := A' \setminus V(\mathcal{T}_2)$ and $B'' := B' \setminus V(\mathcal{T}_2)$ have sizes precisely $|A''| = |A'| - (2a+b) = t_3$ and $|B''| = |B'| - (a+2b) = 2t_3$. Since by the choice of B_2 each vertex of A'' has at least $\frac{\phi\varepsilon' d}{7} n > 2|A''| + \gamma n$ neighbours in B_2, we may greedily form a triangle-tiling $\mathcal{T}_3$ in $G[A'' \cup B_2]$ consisting of exactly t_3 B-triangles which covers every vertex of A'' and which covers precisely $2t_3$ vertices of B_2. At this point we have obtained a triangle-tiling $\mathcal{T}_1 \cup \mathcal{T}_2 \cup T_3$ in G which covers every vertex of A except for those in $S \setminus S'$ and every vertex of B except for the precisely $2t_3$ vertices in B'' and the precisely $t_4 - 2t_3$ vertices in $B_2 \setminus V(\mathcal{T}_3)$. Therefore, in total, precisely t_4 vertices of B remain uncovered, each of which has at least $(d - \frac{\varepsilon}{\phi})|S| - |S'| > 2|B_2| + \gamma n$ neighbours in $S \setminus S'$ by the choice of B_1. We may therefore greedily form a triangle-tiling $\mathcal{T}_4$ of A-triangles in G which covers all the remaining uncovered vertices in B and precisely $2t_4$ vertices of $S \setminus S'$. Then $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4$ is the claimed triangle-tiling.

Finally, since none of the assumptions for (c) involve ϕ or ε', we may assume that $\phi \ll \varepsilon'$. We also assume without loss of generality that $|B| \geq |A|$. Since $\alpha(G) \leq \gamma n$, we may greedily form a matching M of size at least $(|B| - \gamma n)/2 \geq n/10$ in $G[B]$. Fix such a matching M, and form an auxiliary bipartite graph H with vertex classes A and M where $a \in A$ and $e = xy \in M$ are adjacent if and only if xyz is a triangle in G. Note that for every edge $e = xy \in M$ we have that

$$\deg_H(e) = |N_G(x) \cap N_G(y) \cap A| \geq 2((1/2 + d) - \varepsilon)|A| - |A| \geq d|A|,$$

so H has density at least d. By Lemma 2.6, applied to H with ε' here in place of ε there, we may choose subsets $X \subseteq A$ and $M' \subseteq M$ such that $|X| = \phi n$, $|M'| = (1-\varepsilon)\phi n$ and such that $H[X', M']$ contains a perfect matching for every subset $X' \subseteq X$ with $|X'| = |M'|$. Let $B' := B \setminus V(M')$ and $n' := |A| \cup |B'|$. Then, since we assumed that $|B| \geq |A|$, we have $n'/3 + \omega n' \leq |A|, |B'| \leq 2n'/3 - \omega n'$, so we can apply (b) to $G[A \cup B']$ with A, B' and X in place of A, B and S respectively to obtain a triangle-tiling $\mathcal{T}_1$ in G which covers every vertex of G except for the vertices of $V(M')$ and precisely $(1-\varepsilon')\phi n$ vertices of X. So, taking X' to be the vertices of X not covered by $\mathcal{T}_1$, we have $|X'| = |M'|$. By the choice of X and M' it follows that $H[X', M']$ contains a perfect matching, which corresponds to a perfect triangle-tiling $\mathcal{T}_2$ in $G[X' \cup V(M')]$. This gives a perfect triangle-tiling $\mathcal{T}_1 \cup \mathcal{T}_2$ in G. □

We now turn to the case where the 'core' consists of three sets which form three super-regular pairs, for which the following lemma is analogous to Lemma 3.1.

Lemma 3.2. *Suppose that $1/n \ll \gamma, \varepsilon \ll d, \omega$, and that 3 divides n. Let V_1, V_2 and V_3 be disjoint sets of vertices with $|V_i| \geq n/6 + \omega n$ for each $i \in [3]$ such that $V := \bigcup_{i \in [3]} V_i$ has size $|V| = n$. Let G be a graph on vertex set V with $\alpha(G) \leq \gamma n$ such that $G[V_i, V_j]$ is (d, ε)-super-regular for each distinct $i, j \in [3]$. Then G contains a perfect triangle-tiling.*

To prove Lemma 3.2 we use the celebrated Blow-up Lemma of Komlós, Sárközy and Szemerédi [11] to obtain a perfect triangle-tiling. For simplicity, we state this only in the (very) special case that we use. Note that our definition of super-regularity differs slightly from theirs, but it is not hard to show that the two definitions are equivalent up to some modification of the constants involved (see, for example, [15, Fact 2]), so the validity of Theorem 3.3 is unaffected.

Theorem 3.3 (Blow-up Lemma for triangle-tilings)**.** *Suppose that $1/n \ll \varepsilon \ll d$. Let A, B and C be disjoint sets of vertices with $|A| = |B| = |C| = n$, and let G be a graph on vertex set $V := A \cup B \cup C$ such that $G[A, B]$, $G[B, C]$ and $G[C, A]$ are each (d, ε)-super-regular. Then G contains a perfect triangle-tiling.*

The proof of Lemma 3.2 proceeds by iteratively deleting triangles from G with two vertices in one cluster and one in another cluster, until the same number of vertices remain in each cluster. We complete the proof by applying the Blow-up Lemma to obtain a perfect triangle-tiling covering all remaining vertices.

Proof of Lemma 3.2. Throughout this proof we perform addition on subscripts modulo 3. For each $i \in [3]$, the fact that $G[V_i, V_{i+1}]$ is (d, ε)-super-regular implies that each vertex $v \in V_i$ has $|N(v) \cap V_{i+1}| \geq (d-\varepsilon)|V_{i+1}| \geq dn/6$. So if we choose uniformly at random a set $Z_j \subseteq V_j$ of size ωn for each $j \in [3]$, then $|N(v) \cap Z_{i+1}|$ is hypergeometrically distributed with expectation at least $d\omega n/6$. By Theorem 2.3 the probability that v has fewer than $d\omega n/7$ neighbours in $|Z_{i+1}|$ declines exponentially with n, and likewise the same is true of the probability that v has fewer than $d\omega n/7$ neighbours in $|Z_{i+2}|$. Taking a union bound, with positive probability it holds that for each $i \in [3]$ every vertex $v \in V_i$ has at least $d\omega n/7$ neighbours in each of Z_{i+1} and Z_{i+2}. We fix such an outcome of our random selection of the sets Z_j, and define $X_i^0 = V_i \setminus Z_i$ for each $i \in [3]$. Without loss of generality we may assume that $\frac{n}{6} \leq |X_1^0| \leq |X_2^0| \leq |X_3^0| \leq \frac{2n}{3} - 3\omega n$.

We now proceed by an iterative process. At time step $t \geq 0$, if we have $|X_1^t| = |X_2^t| = |X_3^t|$ then we terminate. Otherwise, we choose a triangle xyz in G with $x \in X_2^t$ and $y, z \in X_3^t$ (we shall explain shortly why this will always be possible). We then set $Y_j^{t+1} := X_j^t \setminus \{x, y, z\}$ for $j \in [3]$ and define X_1^{t+1}, X_2^{t+1} and X_3^{t+1} such that $\{X_1^{t+1}, X_2^{t+1}, X_3^{t+1}\} = \{Y_1^{t+1}, Y_2^{t+1}, Y_3^{t+1}\}$ and $|X_1^{t+1}| \leq |X_2^{t+1}| \leq |X_3^{t+1}|$, before proceeding to the next time step $t+1$.

Suppose that this procedure does not terminate prior to some time step T. Using the fact that 3 divides n it is easily checked that we must then have $|X_3^{t+2}| - |X_1^{t+2}| \leq |X_3^t| - |X_1^t| - 3$ for each $t \in [T-2]$. In other words, the size difference between the smallest and largest set decreases by at least 3 over each two time steps. Similarly we find that $|X_1^t| - |X_1^{t+2}| \leq 1$ for each $t \in [T-2]$, meaning that the smallest set size decreases by at most 1 over each two time steps. Furthermore, if at some time t we have $0 < |X_3^t| - |X_1^t| < 3$, then (since 3 divides n) we must have $|X_1^t| + 2 = |X_2^t| + 1 = |X_3^t|$, whereupon the procedure will terminate at time $t+1$. It follows that the procedure must terminate at some time T, and moreover that

$$T \leq \frac{2}{3}\left(|X_3^0| - |X_1^0|\right) \leq \frac{2}{3}\left(\left(\frac{2n}{3} - 3\omega n\right) - \frac{n}{6}\right) = \frac{n}{3} - 2\omega n.$$

This implies that at each time $t < T$ we have $|X_3^t| \geq |X_2^t| \geq |X_1^t| \geq |X_1^0| - \left\lceil \frac{t}{2} \right\rceil \geq |X_1^0| - \frac{T}{2} \geq \omega n$, and so throughout the procedure it is always possible to pick a triangle as desired. Indeed, $G[X_2^t, X_3^t]$ is $(\geq d, \varepsilon/\omega)$-regular by the Slicing Lemma (Lemma 2.1), so some vertex of X_2^t has at least $(d - \varepsilon/\omega)|X_3^t| \geq \omega dn/2$ neighbours in X_3^t. Since $\alpha(G) \leq \gamma n < \omega dn/2$ some two of these neighbours must be adjacent, giving the desired triangle.

After the procedure terminates, define $V_i' := X_i^T \cup Z_i$ for each $i \in [3]$. Then $|V_1'| = |V_2'| = |V_3'| \geq 2\omega n$, so by Lemma 2.1 and our choice of the sets Z_j it follows that $G[V_i', V_j']$

is $(d\omega/7, \varepsilon/2\omega)$-super-regular for each distinct $i, j \in [3]$. By Theorem 3.3 there is a perfect triangle-tiling in $G[\bigcup_{i\in[3]} V_i']$; together with the triangles selected by the iterative procedure this gives a perfect triangle-tiling in G. □

4 Proof of Theorem 1.2

In this section we prove Theorem 1.2. The following lemma is the central part of the proof, showing that if a graph G can be decomposed into clusters which form regular and super-regular pairs, indexed by a graph R which admits a bounded degree spanning tree, then by 'working inwards' from the leaves of the tree we can form a perfect triangle-tiling in G.

Lemma 4.1. *Suppose that $1/m \ll \gamma \ll 1/k \ll \varepsilon \ll d, \omega$. Let G be a graph whose vertex set is partitioned into k sets $V_1, \dots, V_k$, and let R be a graph with vertex set $[k]$ which admits a spanning tree T of maximum degree at most 10. Suppose also that the following statements hold.*

- *(a)* $|V_1| \geq (1-\varepsilon)m$.
- *(b)* *V_1 admits either a partition into parts A_1 and B_1 with $|A_1|, |B_1| \geq (1/3+\omega)|V_1|$ such that $G[A_1, B_1]$ is $(1/2+d, \varepsilon)$-super-regular, or a partition into parts A_1, B_1 and C_1 with $|A_1|, |B_1|, |C_1| \geq (1/6+\omega)|V_1|$ such that $G[A_1, B_1]$, $G[A_1, C_1]$ and $G[B_1, C_1]$ are each (d, ε)-super-regular.*
- *(c)* *For each $2 \leq i \leq k$, $(1-\varepsilon)m \leq |V_i| \leq m$ and V_i admits a partition into parts A_i and B_i with $|A_i|, |B_i| \geq (1/3+\omega)m$ such that $G[A_i, B_i]$ is (d, ε)-super-regular.*
- *(d)* *If $ij \in E(R)$, then at least $m/5$ vertices of V_i have at least $dm/5$ neighbours in V_j.*
- *(e)* $\alpha(G) \leq \gamma m$.

Then G contains a triangle-tiling covering all but at most two vertices of G.

Proof. Introduce new constants ϕ and ε' with $\varepsilon \ll \phi \ll \varepsilon' \ll d$ and iterate the following process. Pick a leaf of T other than vertex 1, say vertex i, and let j be the neighbour of i in T. We will show that there exists a triangle-tiling in $G[V_i \cup V_j]$ that covers every vertex of V_i and at most $2\phi m$ vertices of V_j. We then delete the vertices covered by this tiling from G and delete vertex i from T. We proceed in this way until only vertex 1 of T remains. We then arbitrarily delete at most two further vertices of V_1 so that the number of remaining vertices in V_1 is divisible by three. Since, at this point, we have removed at most $2\phi m \cdot \Delta(T) + 2 \leq 21\phi m \leq \varepsilon' m/7$ vertices from V_1, by (a), (b) and (e) there exists a bipartition or tripartition of the remaining vertices of V_1 which satisfies the conditions of Lemma 3.1(c) or Lemma 3.2 respectively (with $\omega/2$, ε' and 2γ in place of ω, ε and γ respectively). In either case there is a perfect triangle-tiling in the graph induced by the remaining vertices of V_1, which together with the deleted triangle-tilings gives a triangle-tiling in G covering every vertex except for the at most two deleted vertices.

It therefore suffices to show that we can find the desired triangle-tiling in $G[V_i \cup V_j]$ at each step of this process. To this end, let S' be the set of vertices of V_i which have at least $dm/6$ neighbours in V_j. Observe that previous deletions can have removed at most $2\phi m \cdot \Delta(T) \leq dm/30$ vertices from each of V_i and V_j, so by (d) we have $|S'| \geq m/6$, and by (c) the remaining vertices of V_i can be partitioned into parts A_i and B_i with

$|A_i|, |B_i| \geq (1/3 + \omega/2)m$ such that $G[A_i, B_i]$ is (d, ε')-super-regular. Without loss of generality we may assume that $|S' \cap A_i| \geq |S' \cap B_i|$, so $|S' \cap A_i| \geq |S'|/2 \geq m/12$ and we can arbitrarily select $S \subseteq S' \cap A_i$ of size ϕn. Now we may use Lemma 3.1(b) (again with $\omega/2$, ε' and 2γ in place of ω, ε and γ respectively) to find a triangle-tiling $\mathcal{T}$ in $G[V_i]$ which covers every vertex of $V_i \setminus S$. Since each uncovered vertex has at least $dm/6 \geq 2\phi m + \gamma m$ neighbours in V_j, we may greedily extend $\mathcal{T}$ to a triangle-tiling $\mathcal{T}'$ in G which covers every vertex of V_i and which covers at most $2\phi m$ vertices of V_j. □

It now suffices to show that for every graph G satisfying the conditions of Theorem 1.2, we can delete triangles and/or vertices from G to obtain a subgraph whose structure meets the conditions of Lemma 4.1. The following lemma shows how to do this under the additional assumption that G has no large sparse cut; this assumption is useful as it allows us to assume that the reduced graph R of G is connected, and so has spanning trees of bounded maximum degree. For this we make the following definition: given a graph G and a partition $\{A, B\}$ of $V(G)$, we say that an edge of G is (A, B)-*crossing* if it has one endvertex in A and one endvertex in B.

Lemma 4.2. *For every $\omega, \psi > 0$ there exist $n_0, \gamma > 0$ such that the following statement holds. Let G be a graph on $n \geq n_0$ vertices with $\delta(G) \geq n/3 + \omega n$ and $\alpha(G) \leq \gamma n$. Suppose additionally that for every partition $\{A, B\}$ of $V(G)$ with $|A|, |B| \geq n/3$ there are at least ψn^2-many (A, B)-crossing edges of G. Then G contains a triangle-tiling covering all but at most two vertices of G (so in particular, if 3 divides n then G contains a perfect triangle-tiling).*

Proof. Introduce new constants satisfying the following hierarchy:

$$1/n \ll \gamma \ll 1/D \ll 1/T \ll 1/t \ll \varepsilon' \ll \varepsilon \ll d \ll \omega, \psi.$$

Then we may assume that n and T are large enough to apply Theorem 2.5 with constants $\varepsilon'/2, d, t$ and $q = 1$. We also assume without loss of generality that ω^{-1} is an integer, and define $D' := 30\omega^{-1}(D!)$. Let G be as in the statement of the lemma, and apply Theorem 2.5 to G to obtain a spanning subgraph $G' \subseteq G$, an integer k' with $t \leq k' \leq T$, an exceptional set U_0 of size at most $\varepsilon' n/2$ and clusters $U_1, \ldots, U_{k'}$ of equal size. We now remove at most D' vertices from each cluster so that the number of remaining vertices in each cluster is divisible by D', and add all removed vertices to the exceptional set U_0. Since the total number of vertices moved in this way is at most $D'k' \leq 30\omega^{-1}(D!)T \leq \varepsilon' n/2$, and at most $D' \leq \varepsilon' n/2T \leq \varepsilon'/2|U_i|$ vertices are removed from each cluster U_i, by Lemma 2.1 the resulting partition of $V(G)$ into $U_0, U_1, \ldots, U_{k'}$ has the following properties.

(i) $|U_0| \leq \varepsilon' n$ and $|U_1| = |U_2| = \ldots = |U_{k'}| =: m'$, where D' divides m'.
(ii) $d_{G'}(v) \geq d_G(v) - (\varepsilon' + d)n \geq n/3 + 2\omega n/3$ for all $v \in V(G)$.
(iii) $e(G'[U_i]) = 0$ for all $i \in [k']$.
(iv) for each distinct $i, j \in [k']$ either $G'[U_i, U_j]$ is $(\geq d, \varepsilon')$-regular or $G'[U_i, U_j]$ is empty.

In particular (i) implies that $(1 - \varepsilon')n/k' \leq m' \leq n/k'$. We form the reduced graph R on vertex set $[k']$ in the usual way, that is, with $ij \in E(R)$ if and only if $e(G'[U_i, U_j]) > 0$. For each $i \in [k']$ the number of edges of G' with an endvertex in U_i is at least $m'(n/3+2\omega n/3)$ by (ii). Also, by (iii) there is no edge in $G'[U_i]$, and by (i) there are at most at most $m'\varepsilon' n$

edges in $G'[U_0, U_i]$. Since for each $j \in [k']$ there are at most $(m')^2$ edges in $G'[U_i, U_j]$, it follows that

$$\delta(R) \geq \frac{m'(n/3 + 2\omega n/3) - m'\varepsilon' n}{(m')^2} \geq \left(\frac{1}{3} + \frac{2\omega}{3} - \varepsilon'\right)\frac{n}{m'} \geq \left(\frac{1}{3} + \frac{\omega}{2}\right)k'. \tag{3}$$

Now consider a partition $\{A_R, B_R\}$ of $[k']$ with $|A_R|, |B_R| \geq \delta(R)$, and define $A := U_0 \cup \bigcup_{i \in A_R} U_i$ and $B := \bigcup_{i \in [k'] \setminus B_R} U_i$. Then

$$|A|, |B| \geq \delta(R)m' \geq \left(\frac{1}{3} + \frac{\omega}{2}\right)k' \cdot \frac{(1-\varepsilon')n}{k'} \geq \frac{n}{3},$$

so by assumption G has at least ψn^2-many (A, B)-crossing edges. By (ii) at most $(d+\varepsilon')n^2$ edges of G are not in G', and by (i) at most $\varepsilon' n^2$ edges of G intersect U_0, so G' contains at least $\psi n^2 - (d + \varepsilon')n^2 - \varepsilon' n^2 > 0$ edges which are (A, B)-crossing but do not intersect U_0. Let U_i and U_j be clusters containing the endvertices of some such edge; then ij is an (A_R, B_R)-crossing edge of R. In other words, for every partition $\{A_R, B_R\}$ of $[k']$ with $|A_R|, |B_R| \geq \delta(R)$ there is an (A_R, B_R)-crossing edge of R. Since every connected component of R has size at least $\delta(R)$, it follows that R is connected.

We now form a set V_1 from which we shall form the 'core' set of vertices mentioned in the proof overview at the beginning of Section 3. Suppose first that there exist $i, j \in [k']$ with $d(G'[U_i, U_j]) \geq 2/3$. Then $G'[U_i, U_j]$ is $(\geq 3/5, \varepsilon')$-regular by (iv). In this case we define $V_1 := U_i \cup U_j$, and for convenience of notation later we define $X_1 := U_i$ and $Y_1 := U_j$. Now suppose instead that $d(G'[U_i, U_j]) < 2/3$ for every $i, j \in [k']$, that is, that each $G'[U_i, U_j]$ has at most $2(m')^2/3$ edges. Then we have an extra factor of $2/3$ in the denominator of the second term of (3), so we have $\delta(R) \geq k'/2$, and so R contains a triangle $ij\ell$ by Mantel's theorem. In this case we take $V_1 := U_i \cup U_j \cup U_\ell$ and set $X_1 := U_i$, $Y_1 := U_j$, and $Z_1 := U_\ell$. We define an auxiliary graph R_0 to be the subgraph of R formed by deleting vertices i and j in the former case, and by deleting vertices i, j and ℓ in the latter case.

Since ω^{-1} is an integer, we may write $\eta := 1/3 + \omega/10$ as a rational number with denominator $L := 30 \cdot \omega^{-1}$. Let $\overrightarrow{R_0}$ be the directed graph formed from R_0 by replacing each edge by a pair of edges, one in each direction. Then by Lemma 2.10, we can find a perfect $(\eta, 1-\eta)$-weighted fractional matching w in $\overrightarrow{R_0}$ in which all weights are rational, and the least common denominator L' of all weights is bounded above by a function of $|V(R_0)|$ and L, that is, a function of k' and ω. Since $k' \leq 1/T$ and we assumed that $1/D \ll 1/T, \omega$, we may assume that $L' \leq D$, so L' divides $D!$, and so $D! w_{\overrightarrow{ij}}$ is an integer for every $\overrightarrow{ij} \in \overrightarrow{R_0}$. Define $m := m'/D!$, and observe that that since $D' = D!L$ divides m by (i), both m and ηm are integers.

We now partition each cluster not contained in V_1 into parts of size ηm and $(1-\eta)m$ according to the weights in w, using the following probabilistic argument. For every $i \in V(R_0)$, we select a partition $\mathcal{U}_i$ of U_i uniformly at random from all such partitions in which exactly $\sum_{j \in N^+(i)} D! w_{\overrightarrow{ij}}$ sets are of size ηm and exactly $\sum_{j \in N^-(i)} D! w_{\overrightarrow{ji}}$ sets are of size $(1-\eta)m$. Since w is a perfect fractional $(\eta, 1-\eta)$-weighted matching, by (1) we have

$$\eta m \sum_{j \in N^+(i)} D! w_{\overrightarrow{ij}} + (1-\eta)m \sum_{j \in N^-(i)} D! w_{\overrightarrow{ji}} = D! m = m' = |U_i|,$$

so we can indeed partition U_i in this way. We also consider the two or three clusters contained in V_1 to be partitioned into a single part. That is, for each $i \in [k'] \setminus V(R_0)$ we set $\mathcal{U}_i$ to be the trivial partition $\{U_i\}$ of U_i. Now consider any edge $ij \in E(R)$, and recall that $G'[U_i, U_j]$ is $(\geq d, \varepsilon')$-regular by (iv), so by Lemma 2.4[1], with probability at least $1 - e^{-\Omega(n)}$ we have that $G'[U'_i, U'_j]$ is $(\geq d, \varepsilon)$-regular for every $U'_i \in \mathcal{U}_i$ and for every $U'_j \in \mathcal{U}_j$. Taking a union bound over all of the at most $\binom{k'}{2}$ edges of R we find that with positive probability this property holds for every edge of R. Fix a choice of partitions $\mathcal{U}_i$ for $i \in [k']$ for which this is the case.

We now define another auxiliary graph R_1 with vertex set $\bigcup_{i \in [k']} \mathcal{U}_i$ in which, for each distinct $i, j \in [k']$, each $X \in \mathcal{U}_i$ and each $Y \in \mathcal{U}_j$, there is an edge XY if and only if $G'[X, Y]$ is $(\geq d, \varepsilon)$-regular. Observe that by our choice of partitions $\mathcal{U}_i$ the graph R_1 is then a blow-up of R, formed by replacing each vertex $i \in [k']$ by a set of $|\mathcal{U}_i|$ vertices and replacing each edge $ij \in E(R)$ by a complete bipartite graph between the corresponding sets. In particular, R_1 is connected. Also note that for each distinct $i, j \in [k']$ with $ij \notin E(R)$, each $X \in \mathcal{U}_i$ and each $Y \in \mathcal{U}_j$, the graph $G'[X, Y]$ is empty by (iv).

Next, for every edge $\overrightarrow{ij} \in E(\overrightarrow{R_0})$, we define $s_{ij} := D! \cdot w_{\overrightarrow{ij}}$. We then label s_{ij} of the sets in $\mathcal{U}_i$ of size ηm as $X_{ij}^1, \ldots, X_{ij}^{s_{ij}}$ and label s_{ij} of the sets in $\mathcal{U}_j$ of size $(1-\eta)m$ as $Y_{ij}^1, \ldots, Y_{ij}^{s_{ij}}$. Since $\mathcal{U}_i$ has exactly $\sum_{j \in N^+(i)} s_{ij}$ sets of size ηm and exactly $\sum_{j \in N^-(i)} s_{ji}$ sets of size $(1-\eta)m$, we may do this so that for each $i \in [k']$ each set in $\mathcal{U}_i$ is uniquely labelled. We now relabel the sets X_{ij}^ℓ and Y_{ij}^ℓ for $\overrightarrow{ij} \in E(\overrightarrow{R_0})$ and $\ell \in s_{ij}$ as $X_2, \ldots, X_k$ and $Y_2, \ldots, Y_k$ respectively, where $k - 1 := \sum_{\overrightarrow{ij} \in E(\overrightarrow{R_0})} s_{ij} = D! \sum_{\overrightarrow{ij} \in E(\overrightarrow{R_0})} w_{ij} = D!|V(R_0)|$ since w is perfect, so $k' \leq k \leq D!k'$. Then for each $2 \leq \ell \leq k$ our choice of partition implies that $G'[X_\ell, Y_\ell]$ is $(\geq d, \varepsilon)$-regular; we define $V_\ell := X_\ell \cup Y_\ell$, and observe that $|V_\ell| = m$.

We now define a final auxiliary graph R^* with vertex set $[k]$ in which ij is an edge of R^* if and only if $e(G'[V_i, V_j]) > 0$. Observe that R^* is then a contraction of R_1, in which the vertices of R_1 corresponding to the sets X_1 and Y_1 (and Z_1 if defined) are contracted to the single vertex 1 of R^*, and for $2 \leq i \leq k$ the vertices of R_1 corresponding to X_i and Y_i are contracted to the single vertex i of R^*. So, since R_1 is connected, R^* is connected also. Now suppose that ij is an edge of R^*. Since $G'[V_i, V_j]$ is nonempty there must exist sets $S \in \{X_i, Y_i, Z_i\}$ and $T \in \{X_j, Y_j, Z_j\}$ such that $G'[S, T]$ is non-empty (ignore Z_i unless $i = 1$ and Z_1 exists, and likewise for Z_j). We then have $S \in \mathcal{U}_{i'}$ and $T \in \mathcal{U}_{j'}$ for some $i', j' \in [k']$, so ST is an edge of R_1, and so $G'[S, T]$ is $(\geq d, \varepsilon)$-regular. Also, a similar calculation to (3) shows that we must have $\delta(R^*) \geq k/3$, so by Theorem 2.7 there is a spanning tree T in R^* with $\Delta(T) \leq 3$.

To recap, at this point we have a formed a partition $\{U_0, V_1, \ldots, V_k\}$ of $V(G)$ and a graph R^* with vertex set $[k]$ which contains a spanning tree of maximum degree at most 3, such that the following statements hold.

(v) V_1 admits either a partition $\{X_1, Y_1\}$ with $|X_1| = |Y_1| = m'$ such that $G'[X_1, Y_1]$ is $(\geq 3/5, \varepsilon')$-regular, or a partition $\{X_1, Y_1, Z_1\}$ with $|X_1| = |Y_1| = |Z_1| = m'$ such that $G'[X_1, Y_1]$, $G'[X_1, Z_1]$ and $G'[Y_1, Z_1]$ are each $(\geq d, \varepsilon')$-regular.

[1] Note that m is much smaller than $\varepsilon' m'$ (since D is much larger than $1/\varepsilon'$) so we must use the random slicing lemma (Lemma 2.4) here, as opposed to, say, the standard slicing lemma (Lemma 2.1).

(vi) For each $2 \le i \le k$, we have $|V_i| = m$ and V_i admits a partition $\{X_i, Y_i\}$ with $|X_i|, |Y_i| \ge \eta m = (1/3 + \omega/10)m$ such that $G'[X_i, Y_i]$ is $(\ge d, \varepsilon)$-regular.

(vii) If $ij \in E(R^*)$, then there are sets $S \subseteq V_i$ and $T \subseteq V_j$ with $|S| \ge |V_i|/3$ and $|T| \ge |V_j|/3$ such that $G'[S, T]$ is $(\ge d, \varepsilon)$-regular.

If we are in the first case of (v), then by Lemma 2.2 we may choose subsets $A_1 \subseteq X_1$ and $B_1 \subseteq Y_1$ with $|A_1|, |B_1| \ge (1-\varepsilon')m'$ such that $G'([A_1, B_1])$ is $(3/5, 2\varepsilon')$-super-regular, and we then define $W_1 := A_1 \cup B_1$. If we are instead in the second case, by three applications of Lemma 2.2 we may choose subsets $A_1 \subseteq X_1$, $B_1 \subseteq Y_1$ and $C_1 \subseteq Z_1$ with $|A_1|, |B_1|, |C_1| \ge (1-2\varepsilon')m'$ such that $G'([A_1, B_1])$, $G'([B_1, C_1])$ and $G'([C_1, A_1])$ are each $(d, 3\varepsilon')$-super-regular, and we then define $W_1 := A_1 \cup B_1 \cup C_1$.

Next, for each $2 \le \ell \le k$, by (vi) and Lemma 2.2 we may choose subsets $A_\ell \subseteq X_\ell$ and $B_\ell \subseteq Y_\ell$ with $|A_\ell| \ge (1-\varepsilon)|X_\ell|$ and $|B_\ell| \ge (1-\varepsilon)|Y_\ell|$ such that $G'[A_\ell, B_\ell]$ is $(d, 2\varepsilon)$-super-regular, and define $W_\ell := A_\ell \cup B_\ell$. Finally, define $W_0 := U_0 \cup \bigcup_{i \in [k]} V_i \setminus W_i$. Then $\{W_0, W_1, \dots, W_k\}$ is a partition of $V(G)$ and, since $|W_i| \ge (1-\varepsilon)|V_i|$ for each $i \in [k]$, we have $|W_0| \le 2\varepsilon n$.

Write $W_0 := \{x_1, \dots, x_q\}$, so $q \le 2\varepsilon n$. To complete the proof we greedily form a triangle-tiling $\mathcal{T} = \{T_1, \dots, T_q\}$ such that $x_i \in T_i$ for each $i \in [q]$ and $|V(\mathcal{T}) \cap W_j| \le 20\varepsilon|W_j|$ for each $j \in [k]$. To see that this is possible, suppose that we have already chosen triangles $T_1, \dots, T_{s-1}$ for some $s \in [q]$, let $X := \bigcup_{i \in [s-1]} V(T_i)$ be the set of vertices covered by these triangles, and let the set X' consist of all vertices in sets W_i with $|X \cap W_i| \ge 18\varepsilon|W_i|$ (that is, from which the previously-chosen triangles cover more than a 18ε-proportion of the vertices). Then we have $18\varepsilon|X'| \le |X| \le 3q \le 6\varepsilon n$, so $|X'| \le n/3$, and so x_s has at least $\delta(G) - |X| - |X'| - |W_0| \ge \omega n - 10\varepsilon n \ge \omega n/2$ neighbours not in X, X' or W_0, so (since $\alpha(G) \le \gamma n < \omega n/2$) two of these neighbours must be adjacent, giving the desired triangle T_s containing x_s. Having chosen T_s in this way for every $s \in [q]$ to obtain $\mathcal{T}$, observe that since we chose each T_s to avoid every set W_i from which at least $18\varepsilon|W_i|$ vertices were covered by previously-chosen triangles, we must have $|V(\mathcal{T}) \cap W_i| \le 20\varepsilon|W_i|$ for each $i \in [k]$, as desired.

Finally, for each $i \in [k]$ define $A'_i := A_i \setminus V(\mathcal{T})$, $B'_i := B_i \setminus V(\mathcal{T})$, $V'_i := W_i \setminus V(\mathcal{T})$. Also define $V' := V(G) \setminus V(\mathcal{T})$ and $H := G[V']$. We claim that the graphs H and R^* and the partition $\{V'_1 \dots, V'_k\}$ of $V(H)$ meet the properties of Lemma 4.1 with $\varepsilon^* := 200\varepsilon$, $\omega' := \omega/20$ and $\gamma' := 2\gamma k'(D!)$ in place of ε, ω and γ respectively and with m, d and k playing the same role there as here. Indeed, our constant hierarchy allows us to assume that $1/m \ll \gamma' \ll 1/k \ll \varepsilon^* \ll d \ll \omega'$, as required. Also observe that for each $i \in [k]$ we have $|V'_i| \ge |V_i| - 20\varepsilon|V_i| - \varepsilon|V_i| = (1-21\varepsilon)|V_i|$, so certainly $|V'_i| \ge (1-\varepsilon^*)m$ for each $i \in [k]$. So Lemma 4.1(a) holds, and Lemma 4.1(b) and (c) follow immediately from our choice of sets A_ℓ and B_ℓ (and possible C_1). Also, for each $ij \in E(R^*)$ by (vii) there exist sets $S \subseteq V'_i$ and $T \subseteq V'_j$ with $|S| \ge |V'_i|/4$ and $|T| \ge |V'_j|/4$ such that $G'[S, T]$ is $(\ge d, 2\varepsilon)$-regular, which implies that at least $(1-2\varepsilon)|S| \ge m/5$ vertices in S have at least $(d-2\varepsilon)|T| \ge dm/5$ neighbours in T, so Lemma 4.1(d) holds. Last of all, Lemma 4.1(e) holds since $\alpha(H) \le \alpha(G) \le \gamma n \le \gamma(2k'm') = \gamma' m$. So we may apply Lemma 4.1 to obtain a triangle-tiling covering all but at most two vertices of H; together with $\mathcal{T}$ this yields a triangle-tiling in G covering all but at most two vertices. $\square$

Finally, to complete the proof of Theorem 1.2 it remains only to consider graphs G which admit a large sparse cut. In this case we show that can remove a small number of vertices to obtain two vertex-disjoint subgraphs G_A and G_B of G whose vertex sets partition $V(G)$ and each of which satisfies a stronger minimum degree condition. We then apply Theorem 1.1 to obtain a perfect triangle-tiling in each of G_A and G_B (alternatively, one could note that the stronger minimum degree conditions preclude either G_A or G_B from having a large sparse cut and apply Lemma 4.2).

Proof of Theorem 1.2. Fix $\omega > 0$ and choose n_0 sufficiently large and γ sufficiently small for Lemma 4.2 with $\omega^2/40$ in place of ψ and also so that we can apply Theorem 1.1 with $\omega/2$, $n_0/3$ and 3γ in place of ω, n_0 and γ respectively. Now let G be a graph on $n \geq n_0$ vertices with $\delta(G) \geq n/3 + \omega n$ and $\alpha(G) \leq \gamma n$. If for every partition $\{A, B\}$ of $V(G)$ with $|A|, |B| \geq n/3$ there are at least $\omega^2 n^2/40$-many (A, B)-crossing edges of G, then G contains a triangle-tiling covering all but at most two vertices by Lemma 4.2, so we are done. So we may assume that for some partition $\{A, B\}$ of $V(G)$ with $|A|, |B| \geq n/3$ there are fewer than $\omega^2 n^2/40$-many (A, B)-crossing edges. Fix such a partition with the smallest number of (A, B)-crossing edges. Note that we cannot have $|A| \leq n/3 + 1$, as then there would be at least $|A|(\delta(G) - n/3 - 1) \geq (n/3) \cdot (\omega n - 1) \geq \omega n^2/4$-many (A, B)-crossing edges. It follows that every vertex $x \in A$ lies in at most $\deg(x)/2$-many (A, B)-crossing edges, as otherwise moving a from A to B would yield a partition of $V(G)$ with parts of size at least $n/3$ and with fewer (A, B)-crossing edges. So we must have $\delta(G[A]) \geq \delta(G)/2 \geq n/6 + \omega n/2$, and the same argument with B in place of A shows that $\delta(G[B]) \geq n/6 + \omega n/2$.

Our proof now diverges according to whether we are proving conclusion (a) or conclusion (b) of Theorem 1.2. For conclusion (a) we simply choose arbitrarily a set S of at most four vertices of G so that $|A \setminus S|$ and $|B \setminus S|$ are each divisible by 3. For conclusion (b) we instead use our additional assumptions that G has no divisibility barrier and that 3 divides n. Indeed, the latter implies that we must have one of the following three cases:

(a) $|A| \equiv |B| \equiv 0 \pmod 3$. In this case we take $S = \emptyset$.
(b) $|A| \equiv 1 \pmod 3$ and $|B| \equiv 2 \pmod 3$. Since (A, B) is not a divisibility barrier, either G contains an B-triangle or a pair of vertex-disjoint A-triangles, and we take S to be the vertices covered by some such triangle or pair of triangles.
(c) $|A| \equiv 2 \pmod 3$ and $|B| \equiv 1 \pmod 3$. Since (B, A) is not a divisibility barrier, either G contains an A-triangle or a pair of vertex-disjoint B-triangles, and we take S to be the vertices covered by some such triangle or pair of triangles.

Observe that in all cases we have $|S| \leq 6$ and that both $|A \setminus S|$ and $|B \setminus S|$ are divisible by 3. The remaining part of the proof is the same for both cases.

Let $X_A \subseteq A$ consist of all vertices of A with $\deg_{G[A]}(x) < n/3 + \omega n/2$. Then each vertex of X_A is contained in more than $\omega n/2$-many (A, B)-crossing edges, and since there are at most $\omega^2 n^2/40$-many (A, B)-crossing edges in total, each with one vertex in A, it follows that $|X_A| \leq \omega n/20$. Since $\alpha(G) \leq \gamma n$ and $\delta(G[A]) \geq n/6 \geq 2|X_A| + |S| + \gamma n$ we may greedily form a triangle-tiling $\mathcal{T}_A$ of size at most $|X_A|$ in $G[A]$ which covers every vertex of X_A but which does not intersect S. We then define $A' := A \setminus (V(\mathcal{T}_A) \cup S)$, $G_A := G[A']$ and $n_A := |A'|$. Then $\delta(G_A) \geq n/3 + \omega n/2 - |V(\mathcal{T}_A)| - |S| \geq n/3 + \omega n/3$, so $n/3 + \omega n/3 \leq n_A \leq 2n/3$. It follows that G_A is a graph on n_A vertices with $\delta(G_A) \geq n_A/2 + \omega n_A/2$ and $\alpha(G_A) \leq \gamma n \leq 3\gamma n_A$. Also n_A is divisible by 3 (since 3 divides each of $|A \setminus S|$ and $|V(\mathcal{T}_A)|$), so G_A contains a perfect triangle-tiling $\mathcal{T}'_A$ by Theorem 1.1.

By exactly the same argument with B in place of A we obtain a triangle-tiling $\mathcal{T}_B$ in $G[B]$ and a graph G_B on vertex set $B' := B \setminus (V(\mathcal{T}_B) \cup S)$ which contains a perfect triangle-tiling $\mathcal{T}'_B$. Finally, for conclusion (a) observe that $\mathcal{T} := \mathcal{T}_A \cup \mathcal{T}_B \cup \mathcal{T}'_A \cup \mathcal{T}'_B$ is then a triangle-tiling in G covering all vertices outside S, that is, all but at most four vertices of G, and for conclusion (b) note that adding the triangle or triangles covering S to $\mathcal{T}$ gives a perfect triangle-tiling in G. □

5 Constructions and questions

Many of the ideas of this section are due to Balogh, Molla and Sharifzadeh [2], but we include them here for completeness.

We first consider the problem of finding perfect K_k-tilings instead of perfect triangle-tilings. By slightly modifying the construction of $G_4(m)$ given in the introduction we can give lower bounds for this question.

Question 5.1. *Let $k \geq 4$ and let G be an n-vertex graph with $\alpha(G) = o(n)$. What is the best-possible minimum degree condition on G that guarantees a perfect K_k-tiling in G?*

The construction is slightly different depending on the parity of $k \geq 4$. We start with the odd case, so let $k = 2(\ell-1)+1$ for some integer $\ell \geq 3$. Consider the complete ℓ-partite graph with one part V_1 of size $n/k-1$, another part V_2 of size $2n/k+1$ and the remaining parts $V_3, \ldots, V_\ell$ each of size $2n/k$, and place the Erdős graph $\mathrm{ER}(|V_i|)$ on each of the parts V_i. When $k = 2\ell$ for some integer $\ell \geq 1$, the construction is essentially the same but we have one part of size $2n/k+1$, one part of size $2n/k-1$ and the remaining parts are each of size $2n/k$. In either case we obtain a graph G with $\delta(G) \geq \left(1 - \frac{2}{k}\right) n + \omega(1)$, sublinear independence number and no K_k-factor. It is worth noting that in the odd case the graph G is K_{k+2}-free and in the even case G contains no K_{k+1}.

We feel that the following is another interesting related question.

Question 5.2. *Let G be an n-vertex K_4-free graph with $\alpha(G) = o(n)$. What is the best-possible minimum degree condition on G that guarantees a perfect triangle-tiling in G?*

We use a modified version of the Bollobás-Erdős graph [1] to construct a K_4-free graph without a perfect triangle-tiling and with high minimum degree. For every large even n, the Bollobás-Erdős graph is an n-vertex, K_4-free graph with sublinear independence number, which we denote by $\mathrm{BE}(n)$. The vertex set of $\mathrm{BE}(n)$ is the disjoint union of two sets V_1 and V_2 of the same order such that the graphs $G[V_1]$ and $G[V_2]$ are triangle-free and every vertex in V_1 has at least $(1/4-o(1))n$ neighbors in V_2 and every vertex in V_2 has at least $(1/4-o(1))n)$ neighbors in V_1. To construct our example, start with $\mathrm{BE}(4n/3+2)$ and then remove a randomly selected subset of size $n/3+2$ from one of the two parts. Note that the two parts now have sizes $n/3-1$ and $2n/3+1$, the resulting graph clearly is K_4-free and since the larger part is a space barrier, it has no perfect triangle-factor. Furthermore, with high probability, the minimum degree is $(1/6-o(1))n$. We conjecture that $(1/6+o(1))n$ is the proper minimum degree condition.

Conjecture 5.3. *For every $\omega > 0$ there exist $\gamma, n_0 > 0$ such that every K_4-free graph on $n \geq n_0$ vertices with $\delta(G) \geq n/6 + \omega n$ and $\alpha(G) \leq \gamma n$ contains a perfect triangle-tiling.*

Using methods similar to those used in our proof of Theorem 1.2 we can show that every graph G which satisfies the conditions of Conjecture 5.3 has a triangle-tiling covering almost all of the vertices of G. More precisely, we can show that for $1/n \ll \gamma \ll \omega$, if G is a K_4-free graph on n vertices with $\delta(G) \geq (1/6+\omega)n$ and $\alpha(G) \leq \gamma n$, then G contains a triangle-tiling which covers all but at most ωn vertices. What follows is a brief sketch of the argument.

Apply Theorem 2.5 with $\gamma \ll \varepsilon \ll d \ll \omega$ to obtain a spanning subgraph $G' \subseteq G$, an exceptional set V_0 and clusters $V_1, \dots, V_k$ of equal size m. Define the corresponding reduced graph R on vertex set $[k]$ in the usual way. The fact that G is K_4-free implies the following two important facts about these clusters and the graph R. (These facts were first observed by Szemerédi in [13].)

(a) there is no pair $i, j \in [k]$ for which $G'[V_i, V_j]$ is $(1/2+d, \varepsilon)$-regular, and
(b) R is triangle-free.

Using a standard argument, it is not hard to see that (a) and the fact that $\delta(G) \geq (1/6+\omega)n$ together imply that $\delta(R) \geq k/3$. So R must be connected, as otherwise Mantel's theorem would give a triangle in the smallest connected component of R, contradicting (b). By a result of Enomoto, Kaneko and Tuza [9], the fact that R is a connected graph on k vertices with $\delta(R) \geq k/3$ implies that R contains $\lfloor |R|/3 \rfloor$ vertex-disjoint copies of P_2 (the path on three vertices). In a manner similar to the proof of Lemma 3.1, for each such path ijk we can use the fact that $\alpha(G) \leq \gamma n$ to greedily construct a triangle-tiling covering all but at most $3.1\varepsilon m$ of the vertices of $G[V_i \cup V_j \cup V_k]$, where each triangle has one vertex in V_j, the central cluster in the path, and the other two vertices either both in V_i or both in V_k. The union of these $\lfloor |R|/3 \rfloor$ triangle-tilings is then a triangle-tiling in G which covers all but at most ωn vertices.

We can generalize Question 5.2 in the following way.

Question 5.4. *Let $k \geq 3$ and let G be an n-vertex K_{k+1}-free graph with $\alpha(G) = o(n)$. What is the best-possible minimum degree condition on G that guarantees a perfect K_k-tiling in G?*

When k is even, we have previously shown that the minimum degree must be at least $\left(\frac{k-2}{k} + o(1)\right)n$. When $k = 2\ell+1 \geq 5$, we form G by starting with the complete ℓ-partite graph that has one part V_1 of size $3n/k+1$, one part V_2 of size $2n/k-1$, and the remaining parts, $V_3, \dots, V_\ell$, each of size $2n/k$. In V_1, we place $\mathrm{BE}(|V_1|)$ on V_1, and, for every $2 \leq i \leq \ell$, we place a copy of $\mathrm{ER}(|V_i|)$ on V_i. We then have $\delta(G) \geq \left(\frac{k-3}{k} + \frac{1}{4} \cdot \frac{3}{k} - o(1)\right)n = \left(\frac{4k-9}{4k} - o(1)\right)n$. Furthermore, G has sublinear independence number, is K_{k+1}-free, and has no perfect K_k-tiling, because each copy of K_k in G has at most 3 vertices in V_1.

Finally, for $r \geq 3$, $\omega, \gamma > 0$ and sufficiently large n, we give the construction of $G := G_{\mathrm{RT}}(n, r, \omega, \gamma)$ from Theorem 1.3(b). For odd r the construction was first given in [6] and for even r the construction is from [7]. We say that a partition $V_1, \dots, V_\ell$ of the vertices of a graph is *equitable* if $||V_i| - |V_j|| \leq 1$ for all $1 \leq i < j \leq \ell$.

When $r = 2\ell+1$ is odd, we let $V_1, \dots, V_\ell$ be an equitable partition of $V(G)$ and form the complete ℓ-partite graph with vertex classes $V_1, \dots, V_\ell$. For every $i \in [\ell]$, we then place a copy of $\mathrm{ER}(|V_i|)$ on V_i, so

$$\delta(G) \geq n - \left\lceil \frac{n}{\ell} \right\rceil \geq \left(\frac{r-3}{r-1} - \omega \right) n.$$

We can assume that n is large enough so that for each $i \in [\ell]$ the independence number of $G[V_i]$ is at most γn, which implies that $\alpha(G) \leq \gamma n$. Note that G is K_r-free, as $G[V_i]$ is K_3-free for $i \in [\ell]$.

When $r = 2\ell$ is even, we let $U_1, \ldots, U_{3\ell-2}$ be a equitable partition of $V(G)$, so $|U_i| \in \left\{ \left\lfloor \frac{2n}{3r-4} \right\rfloor, \left\lceil \frac{2n}{3r-4} \right\rceil \right\}$ for every $i \in [3\ell - 2]$. Let

$$V_1 := U_1 \cup U_2 \cup U_3 \cup U_4 \qquad \text{and} \qquad V_i := U_{3i-1} \cup U_{3i} \cup U_{3i+1} \quad \text{for } 2 \leq i \leq \ell - 1,$$

and form the complete $(\ell-1)$-partite graph with vertex classes $V_1, \ldots, V_{\ell-1}$. On V_1, we then place a copy of $\mathrm{BE}(|V_1|)$ and assume n is large enough so that $G[V_1]$ has minimum degree at least

$$\left(\frac{1}{4} - \omega \right) |V_1| \geq |V_1| - \left(\frac{6}{3r-4} + \omega \right) n$$

and independence number at most γn. For every $2 \leq i \leq \ell - 1$, we place a copy of $\mathrm{ER}(|V_i|)$ on V_i and we ensure that n is large enough so that the independence number of $G[V_i]$ is at most γn. Because every vertex in G is adjacent to all but at most $\left(\frac{6}{3r-4} + \omega\right) n$ vertices of G, we have that

$$\delta(G) \geq \left(\frac{3r-10}{3r-4} - \omega \right) n.$$

Furthermore, $\alpha(G) \leq \gamma n$ and G is K_r-free as $G[V_1]$ is K_4-free and each of the subgraphs $G[V_2], \ldots, G[V_{\ell-1}]$ is K_3-free.

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6 Appendix

The purpose of this appendix is to prove Lemma 2.4. The lemma is essentially a corollary to the following two theorems of Kohayakawa and Rödl [10]. For this we use the following notation: let G be a bipartite graph with vertex classes A and B, and define $d := d(G[A,B])$. Then for any ε we define $D_{AB}(\varepsilon)$ to be the graph with vertex set A in which $x, x' \in A$ are adjacent if and only if

$$|N_G(x)|, |N_G(x')| > (d-\varepsilon)|B| \qquad \text{and} \qquad |N_G(x) \cap N_G(x')| < (d+\varepsilon)^2|B|.$$

Theorem 6.1 ([10, Theorem 45]). *Let $0 < \varepsilon < 1$, and let $G[A,B]$ be a bipartite graph with $|A| \geq 2/\varepsilon$. If $e(D_{AB}(\varepsilon)) > (1-5\varepsilon)|A|^2/2$, then $G[A,B]$ is $(d,(16\varepsilon)^{1/5})$-regular, where $d := d(G[A,B])$.*

Theorem 6.2 ([10, Theorem 46])**.** *Let $0 < \varepsilon < 1$, and let $G[A, B]$ be a bipartite graph with $|B| \geq 1/d$, where $d := d(G[A, B])$. If $G[A, B]$ is (d, ε)-regular, then $e(D_{AB}(\varepsilon)) \geq (1 - 8\varepsilon)|A|^2/2$.*

The following two similar lemmas do most of the remaining work required to complete the proof.

Lemma 6.3. *Suppose that $1/n \ll \xi \ll \xi'$ and that $1/n \ll \beta$. Let $G[A, B]$ be a bipartite graph such that $|A|, |B| \leq n$, and let $x_1, \ldots, x_s$ and $y_1, \ldots, y_t$ be positive integers each of size at least βn such that $\sum_{i \in [s]} x_i \leq |A|$ and $\sum_{j \in [t]} y_j \leq |B|$. If $\{X_1, \ldots, X_s\}$ is a collection of disjoint subsets of A and $\{Y_1, \ldots, Y_t\}$ is a collection of disjoint subsets of B with $|X_i| = x_i$ and $|Y_j| = y_j$ for all $i \in [s]$ and $j \in [t]$ selected uniformly at random from all such collections, then, with probability at least $1 - e^{-\Omega(n)}$, for every $i \in [s]$, $j \in [t]$, $x, x' \in A$ and $y, y' \in B$ we have*

(a) $|N_G(x) \cap Y_j|/y_j = |N_G(x)|/|B| \pm \xi$,
(b) $|N_G(y) \cap X_i|/x_i = |N_G(y)|/|A| \pm \xi$,
(c) $|N_G(x) \cap N_G(x') \cap Y_j|/y_j = |N_G(x) \cap N_G(x')|/|B| \pm \xi$,
(d) $|N_G(y) \cap N_G(y') \cap X_i|/x_i = |N_G(y) \cap N_G(y')|/|A| \pm \xi$, *and*
(e) $d(G[X_i, Y_j]) = d(G[A, B]) \pm \xi'$.

Proof. Note that the at most $t(|A|+|A|^2)+s(|B|+|B|^2) \leq 2\beta^{-1}(n+n^2)$ random variables of the form $|N_G(x) \cap Y_j|$, $|N_G(y) \cap X_i|$, $|N_G(x) \cap N_G(x') \cap Y_j|$, and $|N_G(y) \cap N_G(y') \cap X_i|$, where $i \in [s]$, $j \in [t]$, $x, x' \in A$ and $y, y' \in B$, are hypergeometrically distributed, so the fact that (a)-(d) hold with probability $1 - e^{\Omega(n)}$ follows directly from Theorem 2.3 by taking a union bound. For (e), let $\ell := \xi^{-1}/2$ and define $D_k := \{v \in A : 2(k-1)\xi \leq |N(v)|/|B| < 2k\xi\}$ for each $k \in [\ell]$. Then, with probability $1 - e^{\Omega(n)}$, for every $i \in [s]$ and $k \in [\ell]$, we have that

$$\frac{|D_k \cap X_i|}{x_i} = \frac{|D_k|}{|A|} \pm \xi^2.$$

Fix a choice of $X_1, \ldots, X_s$ and $Y_1, \ldots, Y_t$, for which (a)-(d) hold and this event occurs. Note that for every $k \in [\ell]$, $v \in D_k$, and $j \in [t]$,

$$\frac{|N_G(v)|}{|B|} = (2k-1)\xi \pm \xi \qquad \text{so} \qquad \frac{|N_G(v) \cap Y_j|}{y_j} = (2k-1)\xi \pm 2\xi.$$

We compute $d(G[A, B])$ to be

$$\frac{1}{|A|}\sum_{k \in [\ell]}\sum_{v \in D_k}\frac{|N_G(v)|}{|B|} = \sum_{k \in [\ell]}\left(((2k-1)\xi \pm \xi) \cdot \frac{|D_k|}{|A|}\right) = \left(\sum_{k \in [\ell]}(2k-1)\xi\frac{|D_k|}{|A|}\right) \pm \xi.$$

Then for any $i \in [s]$ and $j \in [t]$ we have

$$\begin{aligned} d(G[X_i, Y_j]) &= \frac{1}{x_i}\sum_{k \in [\ell]}\sum_{v \in D_k \cap X_i}\frac{|N_G(v) \cap Y_j|}{y_j} = \sum_{k \in [\ell]}\left(((2k-1)\xi \pm 2\xi) \cdot \left(\frac{|D_k|}{|A|} \pm \xi^2\right)\right) \\ &= \left(\sum_{k \in [\ell]}(2k-1)\xi\frac{|D_k|}{|A|}\right) \pm (\ell^2\xi^3 + 2\xi + 2\ell\xi^3) = d(G[A, B]) \pm \xi', \end{aligned}$$

so (e) holds. $\square$

Lemma 6.4. *Suppose that $1/n \ll \xi \ll \xi'$ and $1/n \ll \beta$, and that $x_1, \ldots, x_s$ are positive integers each of size at least βn such that $\sum_{i \in [s]} x_i \leq n$. If G is a graph on n vertices and $\{X_1, \ldots, X_s\}$ is a collection of disjoint subsets of $V(G)$ with $|X_i| = x_i$ for all $i \in [s]$ selected uniformly at random from all such collections, then, with probability at least $1 - e^{-\Omega(n)}$, for every $i \in [s]$ and $x, x' \in V(G)$ we have*

(a) $|N_G(x) \cap X_i|/x_i = |N_G(x)|/n \pm \xi$,
(b) $|N_G(x) \cap N_G(x') \cap X_i|/x_i = |N_G(x) \cap N_G(x')|/n \pm \xi$, *and*
(c) $2e(G[X_i])/x_i^2 = 2e(G)/n^2 \pm \xi'$.

Proof. It is straightforward to modify the proof of Lemma 6.3 to prove this lemma; we omit the details. □

Now we give the proof of Lemma 2.4.

Proof of Lemma 2.4. Introduce a new constant η with $1/n \ll \eta \ll \varepsilon$. Suppose that $G[A, B]$ is $(\geq d, \varepsilon)$-regular, let $d^* := d(G[A, B])$, so $d^* = d \pm \varepsilon$, and define $D := D_{AB}(\varepsilon)$. Note that, by Theorem 6.2, we have that $2e(D)/|A|^2 \geq 1 - 8\varepsilon$. We apply Lemma 6.3 to $G[A, B]$ and Lemma 6.4 to D, with ξ' replaced by η in each case, to find that with probability $1 - e^{-\Omega(n)}$ our random selection satisfies the conclusions of each of these lemmas. We fix such an outcome of our random selection, and consider any $i \in [s]$ and $j \in [t]$. Define $d_{ij} := d(X_i, Y_j)$, so $d_{ij} = d^* \pm \eta$, and

$$d_{ij} = d \pm (\varepsilon + \eta). \tag{4}$$

We also have that

$$\frac{2e(D[X_i])}{x_i^2} \geq \frac{2e(D)}{|A|^2} - \eta \geq 1 - 8\varepsilon - \eta \geq 1 - 5(2\varepsilon).$$

Recall that, if $xx' \in E(D[X_i])$, then

$$\frac{|N_G(x)|}{|B|}, \frac{|N_G(x')|}{|B|} > d^* - \varepsilon \text{ and } \frac{|N_G(x) \cap N_G(x')|}{|B|} < (d^* + \varepsilon)^2,$$

so

$$\frac{|N_G(x) \cap Y_j|}{y_j}, \frac{|N_G(x') \cap Y_j|}{y_j} > (d^* - \varepsilon) - \eta > d_{ij} - 2\varepsilon,$$

and, as we can assume η is small enough so that $\eta^{1/2} + \eta < \varepsilon$,

$$\frac{|N_G(x) \cap N_G(x') \cap Y_j|}{y_j} < (d^* + \varepsilon)^2 + \eta < (d_{ij} + \eta + \varepsilon)^2 + (\varepsilon - \eta)^2 < (d_{ij} + 2\varepsilon)^2.$$

This proves that $xx' \in E(D_{X_iY_j}(2\varepsilon))$, so D is a subgraph of $D_{X_iY_j}(2\varepsilon)$. Therefore, by Lemma 6.1 with d and ε replaced by d_{ij} and 2ε, respectively, $G[X_i, Y_j]$ is $(d_{ij}, (32\varepsilon)^{1/5})$-regular, and is therefore $(d, (32\varepsilon)^{1/5} + 2\varepsilon)$-regular, because, by (4), $d = d_{ij} \pm 2\varepsilon$. Since we can assume that ε is small enough so that $(32\varepsilon)^{1/5} + 2\varepsilon \leq (33\varepsilon)^{1/5}$, it follows that $G[X_i, Y_j]$ is $(d, (33\varepsilon)^{1/5})$-regular.

Clearly, if $G[A, B]$ is (d, ε)-super-regular, then, by (a) and (b) of Lemma 6.3, we can also ensure that $G[X_i, Y_j]$ is $(d, (33\varepsilon)^{1/5})$-super-regular for each $i \in [s]$ and $j \in [t]$. □