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Automorphisms of K-groups I

Paul Flavell

The School of Mathematics
University of Birmingham
Birmingham B15 2TT
Great Britain

Abstract

This is the first in a sequence of papers that will develop the theory of automorphisms of nonsolvable finite groups. The sequence will culminate in a new proof of McBride's Nonsolvable Signalizer Functor Theorem, which is one of the fundamental results required for the proof of the Classification of the Finite Simple Groups.

Keywords: Automorphisms of finite groups, signalizer functor
2010 MSC: 20D45, 20D05, 20E34

1. Introduction

The theory of automorphisms of finite solvable groups is very well developed. A high point of that theory is Glauberman's Solvable Signalizer Functor Theorem [9]. This is the first in a sequence of papers that will develop the theory of automorphisms of arbitrary finite groups and will culminate in a new proof of McBride's Nonsolvable Signalizer Functor Theorem [16, 17]. This proof will differ significantly from McBride's. It will be modelled on the author's proof of the Solvable Signalizer Functor Theorem [5].

The Signalizer Functor Theorems played a crucial role in the first generation proof of the Classification of the Finite Simple Groups. They are also background results needed for the new proof of the Classification in the Gorenstein-Lyons-Solomon book series [10].

It is not however the sole aim of this sequence of papers to prove the Nonsolvable Signalizer Functor Theorem. Many ideas are explored in much greater depth than is required for that purpose and a more general theory ensues. Consequently the results proved will be applicable in situations where Signalizer

Email address: `P.J.Flavell@bham.ac.uk` (Paul Flavell)

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Functor Theory is not. Once this sequence of papers is complete, it is the intention to prepare a monograph whose main focus will be a proof of the Nonsolvable Signalizer Functor Theorem.

The results of this paper require the so-called K-group hypothesis. Recall that a K-group is a finite group all of whose simple sections are isomorphic to a cyclic group, an alternating group, a group of Lie type or one of the 26 sporadic simple groups. The Classification asserts that every finite group is a K-group. Thus, given the Classification, the K-group hypothesis is superfluous. The main application of the Nonsolvable Signalizer Functor Theorem is to analyze a minimal counterexample to the Classification. In such a group, all proper subgroups are K-groups whence the K-group hypothesis causes no difficulty. In §4 we will state explicitly the properties of simple K-groups that we use.

Let A be a group that acts as a group of automorphisms on the group G. Assume that A and G are finite with coprime orders. The main issue that will be addressed in this paper is:

> *Consider the collection of $AC_G(A)$-invariant subgroups of G. How do these subgroups relate to one another and to the global structure of G?*

In the case that G is solvable, much is known. A typical result is the following:

Theorem 1.1 (see [1, §36] or [4]). *Assume that A has prime order r, that G is solvable and that H is an $AC_G(A)$-invariant subgroup of G with $H = [H, A]$.*

(a) Let p be a prime. If $p = 2$ and r is a Fermat prime assume that the Sylow 2-subgroups of G are abelian. Then

$$O_p(H) \leq O_p(G).$$

(b) If $H = O^2(H)$ then

$$O_2(H) \leq O_2(G).$$

Thus, nearly always, the Fitting subgroup of H is contained in the Fitting subgroup of G. This result is central to the author's proof of the Solvable Signalizer Functor Theorem.

In the theory of arbitrary finite groups, attention is focussed on the generalized Fitting subgroup and components. We shall introduce the notions of A-quasisimple group, A-component and (A, sol)-component. The theory developed will revolve around these notions. Basic properties of A-quasisimple groups will be established and the main results will be stated and proved in §9. This paper concludes with an application to the study of nonsolvable signalizer functors. A precursor to this work is [6] where the author began the development of the theory, but without a K-group hypothesis.

One issue that appears to be fundamental is the following: let R be a group of prime order r that acts on the r'-group G and let V be a faithful completely reducible RG-module over a field. Then $C_V(R)$ is a module for $C_G(R)$. Let

$$K = \ker(C_G(R) \text{ on } C_V(R)).$$

In [4] this situation is analyzed completely in the case that G is solvable. In a precisely defined sense, it is shown that K is almost subnormal in G. We shall partially extend this result to arbitrary G. In §7 it will be shown that every component of K is in fact a component of G.

The K-group hypothesis is somewhat of a departure from the previous work of the author and deserves some comment. Firstly, when the new proof of the Solvable Signalizer Functor Theorem was discovered, the challenge of extending that work to the nonsolvable case proved irresistible. Secondly, and looking towards the future, this work highlights issues that are fundamental to the theory and gives direction to a more abstract study of automorphisms. Hence continuing the work begun in [6, 7, 8] for example.

Finally it must be emphasized that this work would not have been possible without the prior work of McBride [16, 17]. For example the material in §6 on A-quasisimple groups is a partial reworking of some of his results. Moreover McBride's work provided clues to the general theory developed in §9 and §10.

2. Definitions

Let G be a finite group. The reader is assumed to be familiar with the notions of the Fitting subgroup, the set of components, the layer and the generalized Fitting subgroup of G denoted by $F(G), \mathrm{comp}(G), E(G)$ and $F^*(G)$ respectively. See for example [13]. The notation $\mathrm{sol}(G)$ is used to denote the largest normal solvable subgroup of G. We define a number of variations on the notion of component.

Definition 2.1. *A* sol-component *of G is a perfect subnormal subgroup of G that maps onto a component of $G/\mathrm{sol}(G)$. The set of sol-components of G is denoted by*

$$\mathrm{comp}_{\mathrm{sol}}(G)$$

and we define

$$E_{\mathrm{sol}}(G) = \langle\ \mathrm{comp}_{\mathrm{sol}}(G)\ \rangle.$$

The sol-components of G are characterized as being the minimal nonsolvable subnormal subgroups of G.

The following lemma collects together the basic properties of sol-components.

Lemma 2.2. *Let G be a finite group.*

(a) $\mathrm{comp}(G) \subseteq \mathrm{comp}_{\mathrm{sol}}(G)$ *and* $E(G) \trianglelefteq E_{\mathrm{sol}}(G)$.
(b) $K \in \mathrm{comp}_{\mathrm{sol}}(G)$ *if and only if* $K \trianglelefteq\trianglelefteq G$, *$K$ is perfect and* $K/\mathrm{sol}(K)$ *is simple.*
(c) Let $K \in \mathrm{comp}_{\mathrm{sol}}(G)$ *and* $S \trianglelefteq\trianglelefteq G$. *Then*
  (i) $K \leq S$; *or*
  (ii) $[K, S] \leq K \cap S \leq \mathrm{sol}(K)$ *and* $S \leq N_G(K)$.
(d) $\mathrm{sol}(G)$ *normalizes every sol-component of G.*

(e) *Suppose that K and L are distinct sol-components of G. Then K and L normalize each other and $[K, L] \leq \mathrm{sol}(K) \cap \mathrm{sol}(L) \trianglelefteq\trianglelefteq \mathrm{sol}(G)$.*

(f) *Set $\overline{G} = G/\mathrm{sol}(G)$. The map $K \mapsto \overline{K}$ defines a bijection $\mathrm{comp}_{\mathrm{sol}}(G) \longrightarrow \mathrm{comp}(\overline{G})$. The inverse is given as follows: if $\overline{K} \in \mathrm{comp}(\overline{G})$, let L be the full inverse image of $\overline{K}$ in G and consider $L^{(\infty)}$.*

The proof is left as an exercise for the reader. See for example Lemma 3.2.

Definition 2.3.

- *G is* constrained *if $E(G) = 1$.*
- *G is* semisimple *if $G = E(G)$.*

Recall that $F^*(G) = F(G)E(G)$ and that $C_G(F^*(G)) = Z(F(G))$. Thus G is constrained if and only if $F^*(G) = F(G)$ if and only if $C_G(F(G)) \leq F(G)$. It is straightforward to show that any sol-component of G is either constrained or semisimple.

Next we bring into play a group A that acts as a group of automorphisms on G. It is convenient to use the language of groups with operators. Thus G is A-simple if G is nonabelian and the only A-invariant normal subgroups of G are 1 and G. This implies that G is a direct product of simple groups that are permuted transitively by A.

Recall that G is quasisimple if G is perfect and $G/Z(G)$ is simple.

Definition 2.4. *G is A-*quasisimple *if G is perfect and $G/Z(G)$ is A-simple.*

It is straightforward to show that G is A-quasisimple if and only if G is the central product of quasisimple groups that are permuted transitively by A. Equivalently, $G = E(G)$ and A is transitive on $\mathrm{comp}(G)$.

Trivially, A acts on the sets $\mathrm{comp}(G)$ and $\mathrm{comp}_{\mathrm{sol}}(G)$.

Definition 2.5.

- *An A-*component *of G is the subgroup generated by an orbit of A on $\mathrm{comp}(G)$.*
- *An (A, sol)-*component *of G is the subgroup generated by an orbit of A of $\mathrm{comp}_{\mathrm{sol}}(G)$.*

The sets of A-components and (A, sol)-components of G are denoted by

$$\mathrm{comp}_A(G) \text{ and } \mathrm{comp}_{A,sol}(G)$$

respectively.

The A-components of G are the A-quasisimple subnormal subgroups of G. The (A, sol)-components of G are the minimal A-invariant nonsolvable subnormal subgroups of G. A result entirely analogous to Lemma 2.2 holds but for (A, sol)-components instead of sol-components.

3. Preliminaries

Definition 3.1. *Suppose the group G acts on the set Ω.*

(a) The action is semiregular *if whenever $\alpha \in \Omega, g \in G$ and $\alpha g = \alpha$ then $g = 1$.*
(b) The action is regular *if it is semiregular and transitive.*

Lemma 3.2. *Let G be a group.*

(a) Let $K \in \mathrm{comp}(G)$ and $S \trianglelefteq\trianglelefteq G$. Then either $K \le S$ or $[K, S] = 1$.
(b) Suppose K is a perfect subnormal subgroup of G and that S is a solvable subgroup of G that is normalized by K. Then $S \le N_G(K)$. If in addition $\mathrm{sol}(K) = Z(K)$ then $[S, K] = 1$.

Proof. (a). This is [13, 6.5.2, p.142].

(b). Without loss, $G = KS$. If $G = K$ the result is clear so assume $G \neq K$. Set $L = \langle K^G \rangle$, so $L \neq G$ as $K \trianglelefteq\trianglelefteq G$. Now $L = K(L \cap S)$ so by induction, $K \trianglelefteq L$. Since $L \cap S$ is solvable and K is perfect it follows that $K = L^{(\infty)} \mathrm{char} L \trianglelefteq G$, so $K \trianglelefteq G$.

Suppose also that $\mathrm{sol}(K) = Z(K)$. Then $[K, S] \le K \cap S \le \mathrm{sol}(K) = Z(K)$ whence $[K, S, K] = 1$. It follows from the Three Subgroups Lemma that $[S, K] = 1$.

Definition 3.3. *The group A* acts coprimely *on the group G if A acts on G; the orders of A and G are coprime; and A or G is solvable.*

Theorem 3.4 (Coprime Action). *Suppose the group A acts coprimely on the group G.*

(a) $G = C_G(A)[G, A]$ and $[G, A] = [G, A, A]$.
(b) If G is abelian then $G = C_G(A) \times [G, A]$.
(c) Suppose N is an A-invariant normal subgroup of G. Set $\overline{G} = G/N$. Then $C_{\overline{G}}(A) = \overline{C_G(A)}$.
(d) For each prime p there exists an A-invariant Sylow p-subgroup of G. Every A-invariant p-subgroup is contained in an A-invariant Sylow p-subgroup of G. Moreover, $C_G(A)$ acts transitively by conjugation on the collection of A-invariant Sylow p-subgroups of G.
(e) Suppose $G = XY$ where X and Y are A-invariant subgroup of G. Then $C_G(A) = C_X(A)C_Y(A)$.
(f) If $[F^*(G), A] = 1$ then $[G, A] = 1$.
(g) Suppose that N is an A-invariant normal Hall-subgroup of G and that N or G/N is solvable. Then G possesses an A-invariant complement to N. All such complements are conjugate under the action of $C_G(A)$.

Proof. For (a),...,(e) see [13, p.184–188].

(f). We have $[G, A] \le C_G(F^*(G)) \le F^*(G)$ so $[G, A, A] = 1$. Apply (a).

(g). This follows by applying the Schur-Zassenhaus Theorem and a Frattini argument to the semidirect product AG.

Lemma 3.5. *Suppose the group A acts on the perfect group K and that A acts trivially on $K/Z(K)$. Then A acts trivially on K.*

Proof. We have $[K, A, K] \leq [Z(K), K] = 1$ and similarly $[A, K, K] = 1$. The Three Subgroups Lemma forces $[K, K, A] = 1$. Since K is perfect, the result follows.

Lemma 3.6. *Let A be a group that acts on the group G. Suppose that $G = K_1 \times \cdots \times K_n$ where $\{ K_1, \ldots, K_n \}$ is a collection of subgroups that is permuted transitively by A. For each i let $\pi_i : G \longrightarrow K_i$ be the projection map and set $B = N_A(K_1)$. Then*

$$C_G(A) \cong C_G(A)\pi_1 = C_{K_1}(B).$$

Proof. Let $c \in G$. Then there exist unique $c_i \in K_i$ such that $c = c_1 \cdots c_n$, in fact $c_i = c\pi_i$. Suppose that $c \in C_G(A)$. Uniqueness implies that $c_1 \in C_{K_1}(B)$. Then $C_G(A)\pi_1 \leq C_{K_1}(B)$. Suppose also that $c_1 = 1$. Since A acts transitively on $\{ K_1, \ldots, K_n \}$ it follows that $c_i = 1$ for all i and then that $c = 1$. We deduce that the map $c \mapsto c_1$ is an isomorphism $C_G(A) \longrightarrow C_G(A)\pi_1$.

Suppose now that we are given $c_1 \in C_{K_1}(B)$. For each i choose $a_i \in A$ with $K_i = K_1^{a_1}$, so $\{ a_1, \ldots, a_n \}$ is a right transversal to B in A. Define $c_i = c_1^{a_i} \in K_i$ and set $c = c_1 \cdots c_n$. A simple argument shows that A permutes $c_1, \ldots, c_n$, so as $[K_i, K_j] = 1$ for all $i \neq j$ we have $c \in C_G(A)$. Then $c\pi_1 = c_1$ so $C_{K_1}(B) \leq C_G(A)\pi_1$. The proof is complete.

We use the symbol $*$ to denote a central product. Thus $G = H * K$ means $G = HK$ and $[H, K] = 1$.

Lemma 3.7. *Let A be a group that acts coprimely on the group K. Suppose $K = K_1 * \cdots * K_n$ for some A-invariant collection $\{ K_1, \ldots, K_n \}$ of subgroups of K on which A acts regularly. Then $C_K(A) \cong K_1/Z$ for some subgroup $Z \leq Z(K_1) \cap Z(K_2 * \cdots * K_n)$.*

Proof. For each i let a_i be the unique member of A with $K_i = K_1^{a_i}$, so $a_1 = 1$. The map $\tau : k \mapsto k^{a_1} \cdots k^{a_n}$ is a homomorphism $K_1 \longrightarrow C_K(A)$. If $k \in \ker \tau$ then $k = k^{a_1} = (k^{a_2} \cdots k^{a_n})^{-1} \in K_1 \cap (K_2 * \cdots * K_n) \leq Z(K_1) \cap Z(K_2 * \cdots * K_n)$. In order to complete the proof, it suffices to show that τ is surjective.

Consider the external direct product $\widetilde{K} = K_1 \times \cdots \times K_n$ and the map $\sigma : \widetilde{K} \longrightarrow K$ defined by $(k_1, \ldots, k_n)\sigma = k_1 \cdots k_n$. Then A acts coprimely on $\widetilde{K}$ and σ is an A-epimorphism. By Coprime Action(c), $C_{\widetilde{K}}(A)\sigma = C_K(A)$. Visibly $C_{\widetilde{K}}(A) = \{ (k^{a_1}, \ldots, k^{a_n}) \mid k \in K_1 \}$ and the proof is complete.

Lemma 3.8. *Let A be a group that acts coprimely on the group X. Suppose that AX, the semidirect product of X with A, acts on the set Ω and that A acts transitively on Ω. Then X acts trivially on Ω.*

Proof. Choose $\alpha \in \Omega$. Let $p \in \pi(X)$. Now $AX = A\mathrm{Stab}_{AX}(\alpha)$ because A is transitive. As A is a p'-group it follows that $\mathrm{Stab}_{AX}(\alpha)$ contains a Sylow p-subgroup P of AX. Now X is a normal Hall-subgroup of AX, whence $P \leq X$. It follows that $X \leq \mathrm{Stab}_{AX}(\alpha)$. Now α was arbitrary, so X acts trivially on Ω.

Lemma 3.9. *Let $\mathbb{F}$ be a field, G a group and V an $\mathbb{F}[G]$-module.*

(a) Suppose that $\operatorname{char}\mathbb{F}$ *does not divide* $|G|$*. Then*

$$V = C_V(G) \oplus [V, G].$$

(b) Suppose V is faithful and $\operatorname{char}\mathbb{F} = p$*. Then*

$$O_p(G) = \bigcap C_G(U)$$

where U ranges over the irreducible constituents of V and $O_p(G)$ is defined to be 1 if $p = 0$.

Proof. (a). By Maschke's Theorem, V is a direct sum of irreducible submodules. Then $C_V(G)$ is the sum of those submodules that are trivial and $[V, G]$ is the sum of those modules that are nontrivial.

(b). Suppose $p = 0$. Then we may write V as a direct sum of irreducible submodules, whence the intersection acts trivially on V. Suppose $p > 0$. If U is any irreducible $\mathbb{F}[G]$-module then $C_U(O_p(G)) \neq 0$ whence $O_p(G) \leq C_G(U)$. Thus $O_p(G)$ is contained in the intersection. Let q be a prime not equal to p and let Q be a Sylow q-subgroup of the intersection. By considering a composition series for V, we have $[V, Q, \ldots, Q] = 0$ and then (a), with Q in the role of G, implies $[V, Q] = 0$. Then $Q = 1$ and we deduce that the intersection is a p-group.

Lemma 3.10. *Let R be a group of prime order r that acts on the q-group Q with $q \neq r$ and $[Q, R] \neq 1$. Let V be an $\mathbb{F}[RQ]$-module where $\mathbb{F}$ is a field with* $\operatorname{char}\mathbb{F} \neq q$*. Assume that $[Q, R]$ acts nontrivially on V. If $q = 2$ and r is a Fermat prime assume that Q is abelian. Then $\mathbb{F}[R]$ is a direct summand of V_R. In particular $C_V(R) \neq 0$.*

Proof. By Coprime Action(a) we may assume $Q = [Q, R]$. Apply [4, Theorem 5.1].

The following is an easy special case of the main result of [4].

Lemma 3.11. *Let r, t and p be primes. Suppose the group $R \times S$ acts on the group T and that V is an $\mathbb{F}[RST]$-module with $\mathbb{F}$ a field of characteristic p. Assume that:*

(i) $|R| = r$, S is an r'-group, T is a t-group and $t \neq p$.
(ii) $T = [T, S]$.
(iii) $[C_V(R), S] = 0$.
(iv) If T is nonabelian then $[C_V(R), C_T(R)] = 0$ and $t \neq 2$.

Then $[V, [T, R]] = 0$.

Proof. By [4, Lemma 2.2] we may assume that $\mathbb{F}$ is algebraically closed. Now $V = C_V([T, R]) \oplus [V, [T, R]]$ by Theorem 3.9(a) and $[T, R] \trianglelefteq RST$ so $[V, [T, R]]$ is an RST-module, hence we may suppose that $C_V([T, R]) = 0$ and moreover that T acts faithfully on V. Let $V_1, \ldots, V_n$ be the homogeneous components for $Z(T)$. Then T normalizes each V_i and RS permutes the V_i amongst themselves. Since $t \neq p$ we have $V = V_1 \oplus \cdots \oplus V_n$.

Suppose that R does not normalize each V_i. Then without loss $\{ V_1, \ldots, V_r \}$ is an orbit for the action of R. Set $W = V_1 \oplus \cdots \oplus V_r$ so $C_W(R)$ is a diagonal subspace of W. By assumption $[C_W(R), S] = 0$ so S permutes the V_i onto which $C_W(R)$ projects nontrivially. We deduce that S permutes $\{ V_1, \ldots, V_r \}$. Lemma 3.8 implies that S normalizes each V_i, $1 \leq i \leq r$. Then as $[C_W(R), S] = 0$ it follows that S centralizes V_1. But $T = [T, S]$ so T centralizes V_1, contrary to $C_V([T, R]) = 0$. We deduce that R normalizes each V_i.

Choose i with $1 \leq i \leq n$. Now V_i is a homogeneous component for $Z(T)$ and $\mathbb{F}$ is algebraically closed so $Z(T)$ acts as scalar multiplication on V_i. Thus $[Z(T), R]$ is trivial on V_i. As $V = V_1 \oplus \cdots \oplus V_n$ we deduce that $[Z(T), R] = 1$. In particular, the conclusion has been established in the case that T is abelian, hence we assume that T is nonabelian.

By assumption $[C_V(R), C_T(R)] = 0$ so $C_V(R) \leq C_V(Z(T))$. Also $t \neq 2$ so as $C_V([T, R]) = 0$, Lemma 3.10 implies $C_{V_i}(R) \neq 0$. Consequently $C_{V_i}(Z(T)) \neq 0$. Now V_i is a homogeneous component for $Z(T)$ whence $Z(T)$ is trivial on V_i. Since $V = V_1 \oplus \cdots \oplus V_n$ it follows that $Z(T) = 1$. Then $T = 1$ and the result is established in this case also.

Lemma 3.12. *Suppose the group A acts on the constrained group G. Then*

$$F(G) = \bigcap C_G(V)$$

where V ranges over the A-chief factors of G below $F(G)$.

Remark 3.13. *The A-chief factors of G below $F(G)$ are by definition the quotients X/Y where X and Y are A-invariant normal subgroups of G with $Y < X \leq F(G)$ and X/Y being the only nontrivial A-invariant normal subgroup of X/Y. In particular, X/Y is an elementary abelian p-group for some prime p and an irreducible* $\mathrm{GF}(p)[AG]$*-module.*

Proof. If $1 < N \trianglelefteq F$ with F nilpotent then $[N, F] < N$. It follows that $F(G)$ is contained in the right hand side. To prove the opposite inclusion, it suffices to show that if D is an A-invariant normal subgroup of G with $[F(G), D, \ldots, D] = 1$ then $D \leq F(G)$.

Suppose that $D' < D$. By induction, $D' \leq F(G)$ whence $[D', D, \ldots, D] = 1$. Thus D is nilpotent. As $D \trianglelefteq G$ we have $D \leq F(G)$ as desired. Hence we may assume that $D' = D$. We have $[F(G), D] = [D, F(G)]$ so $[F(G), D, D] = [D, F(G), D] \trianglelefteq G$ so $[D, D, F(G)] \leq [F(G), D, D]$ by the Three Subgroups Lemma. Now $[F(G), D] = [D, F(G)] = [D, D, F(G)] \leq [F(G), D, D]$. As $[F(G), D, \ldots, D] = 1$ this forces $[F(G), D] = 1$. Since G is constrained we have $D \leq F(G)$ and the proof is complete.

K	$\mathrm{L}_2(2^r)$	$\mathrm{L}_2(3^r)$	$\mathrm{Sz}(2^r)$	$\mathrm{U}_3(2^r)$
C	$\mathrm{L}_2(2) \cong 3:2$	$\mathrm{L}_2(3) \cong 2^2:3$	$\mathrm{Sz}(2) \cong 5:4$	$\mathrm{U}_3(2) \cong 3^2:Q_8$
N	3	2^2	5	3^2
$\lvert C:N \rvert$	2	3	4	8
S	$(2^r+1):2$	C	$(2^r + 2^{\frac{1}{2}(r-1)}\epsilon + 1):4$	C
$\mathrm{Out}(K)$	r	$2 \times r$	r	$3:2 \times r$

where $\epsilon = 1$ if $r \equiv \pm 1 \bmod 8$ and $\epsilon = -1$ if $r \equiv \pm 3 \bmod 8$.
$K:H$ indicates a Frobenius group with kernel K and complement H.

Table 1: Exceptional centralizers

4. Properties of K-groups

The following result collects together all the specific properties of K-groups that we shall use.

Theorem 4.1. *Let K be a simple K-group and suppose r is a prime that does not divide $\lvert K \rvert$.*

(a) The Sylow r-subgroups of $\mathrm{Aut}(K)$ are cyclic.

Suppose $R \leq \mathrm{Aut}(K)$ has order r. Set $C = C_K(R)$.

(b) C possesses a unique minimal normal subgroup N. Except for the cases listed in Tables 1 and 2, $C = N$ and C is simple. Either $F^*(C)$ is simple or C is solvable. If C is solvable then the possibilities for C are listed in Table 1.

(c) K possesses a unique maximal RC-invariant solvable subgroup S. Suppose $S \neq 1$. The possibilities for K are listed in Table 1; C is solvable; $C \leq S$; and S is maximal subject to being an RC-invariant proper subgroup of K.

(d) C is contained in a unique maximal R-invariant subgroup M. If $M \neq C$ then M is solvable and $K \cong \mathrm{L}_2(2^r)$ or $\mathrm{Sz}(2^r)$.

(e) Suppose that X is an R-invariant r'-subgroup of $\mathrm{Aut}(K)$ and that $[C, X] = 1$. Then $X = 1$.

(f) Suppose that $\widetilde{K}$ is quasisimple with $\widetilde{K}/Z(\widetilde{K}) \cong K$, that $\widetilde{R} \leq \mathrm{Aut}(\widetilde{K})$ has order r and that V is a faithful $\mathbb{F}[\widetilde{R}\widetilde{K}]$-module for some field $\mathbb{F}$.

(i) $\mathbb{F}[\widetilde{R}]$ is a direct summand of $V_{\widetilde{R}}$. In particular $C_V(\widetilde{R}) \neq 0$.

(ii) Suppose V is irreducible. Then $E(C_{\widetilde{K}}(\widetilde{R}))$ acts faithfully on $C_V(\widetilde{R})$.

Proof (Proof of Theorem 4.1(a),...,(e)). (a). This is [11, Theorem 7.1.2, p.336].

(b). This is [11, Theorem 2.2.7, p.38].

(c). This is [11, Theorem 7.1.9, p.340].

(d). This is the main result of [2].

(e). This is established in the third paragraph of the proof of [11, Theorem 7.1.4, p.337].

K	$\mathrm{Sp}_4(2^r)$	${}^2G_2(3^r)$	$G_2(2^r)$	${}^2F_4(2^r)$
C	$\mathrm{Sp}_4(2)$	${}^2G_2(3)$	$G_2(2)$	${}^2F_4(2)$
N	$\mathrm{Sp}_4(2)' \cong \mathrm{Alt}(6) \cong \mathrm{L}_2(9)$	${}^2G_2(3)' \cong \mathrm{L}_2(8)$	$G_2(2)' \cong \mathrm{U}_3(3)$	${}^2F_4(2)'$
$\lvert C : N \rvert$	2	3	2	2

Table 2: Exceptional centralizers

The author is indebted to Richard Lyons for the proof of the following lemma.

Lemma 4.2. *Let R be a group of prime order r that acts nontrivially and coprimely on the simple K-group K. Then there exists a prime power q and R-invariant subgroups $L_1, \ldots, L_n$ such that*

$$K = \langle\, L_1, \ldots, L_n \,\rangle$$

and for each i, the action of R on L_i is nontrivial and $L_i \cong \mathrm{L}_2(q^r), \mathrm{SL}_2(q^{mr}), m = 1, 2, 3$ or $\mathrm{Sz}(q^r)$.

Proof. Since R acts nontrivially and coprimely on K it follows that $K \in \mathrm{chev}(p)$ for some prime p and that R is generated by a field automorphism, by [11, 7.1.2]. Then $K = {}^d\mathcal{L}(q^r)$ where $q = p^k$ for some k. Since the Sylow r-subgroups of $\mathrm{Aut}(K)$ are cyclic, the image of R in $\mathrm{Aut}(K)$ is determined up to conjugacy. Then replacing R by a conjugate if necessary, we may assume that R has a generator ρ which is a field automorphism in the sense of [14, Sec. 10] (cf. [11, 2.5.1]). That is ρ transforms a set of Chevalley generators $x_\alpha(t)$ or $x_\alpha(t,u)$, etc. by taking them to $x_\alpha(t^\psi), x_\alpha(t^\psi, u^\psi)$, etc., where ψ is an automorphism of $\mathrm{GF}(q)$. Thus for each root α, R normalizes the (twisted) rank one group $\langle\, X_\alpha, X_{-\alpha} \,\rangle$. Such rank one groups generate K so we may assume that K has rank one. If $K \cong A_1(q^{mr})$ or $\mathrm{Sz}(q^r)$ there is nothing to prove. If $K \cong {}^2G_2(q^r)$ then R centralizes some $S \in \mathrm{Syl}_2(K)$, so R normalizes each $C_K(t) \cong \langle\, t \,\rangle \times \mathrm{L}_2(q^r), t \in S^\#$, and $K = \langle\, E(C_K(t)) \mid t \in S^\# \,\rangle$ since otherwise the right hand side would be strongly embedded in K. If $K \cong \mathrm{U}_3(q^r)$, then we may take the sesquilinear form to have matrix the 3×3 identity matrix, and ρ to be the automorphism $t \mapsto t^q$ on all matrix entries. Then $K = \langle\, K_{12}, K_{23} \,\rangle$ where K_{12} and K_{23} are block-diagonal copies of $\mathrm{SU}_2(q^r)$. As K_{12} and K_{23} are ρ-invariant, the proof is complete.

Proof of Theorem 4.1(f)(i). Let $p = \mathrm{char}\,\mathbb{F}$. By Lemma 3.10, it suffices to show that $\widetilde{K}$ possesses an $\widetilde{R}$-invariant abelian p'-subgroup on which $\widetilde{R}$ acts nontrivially. The inverse image in $\widetilde{K}$ of any cyclic subgroup of K is abelian. Hence it suffices to show that K possesses an R-invariant cyclic p'-subgroup on which R acts nontrivially.

By Lemma 4.2 we may suppose that $K = \mathrm{L}_2(q^r)$ or $\mathrm{Sz}(q^r)$ for some prime power q. Suppose $K = \mathrm{L}_2(q^r)$. Set $d = (2, q-1)$. Then K possesses R-invariant cyclic subgroups of orders $(q^r-1)/d$ and $(q^r+1)/d$ on which R acts nontrivially. These orders are coprime, so one will be coprime to p. Suppose $K = \mathrm{Sz}(q^r)$. Then $q = 2^n$ for some n. By [18], K possesses R-invariant cyclic subgroups of orders $2^{nr} + 2^{(nr+1)/2} + 1$ and $2^{nr} - 2^{(nr+1)/2} + 1$ on which R acts nontrivially. Again, one of these numbers is coprime to p.

Lemma 4.3. *Let R be a group of prime order r that acts nontrivially and coprimely on the simple K-group K. Let p be a prime. Then there exists a prime $t \notin \{2, p\}$ and an R-invariant dihedral group $D \leq K$ of order $2t$ such that R is nontrivial on $O_t(D)$ and $C_K(R)$ contains an involution of D. If $K \ncong \mathrm{L}_2(2^r)$ and $\mathrm{Sz}(2^r)$ then D may be chosen such that $C_K(R)'$ contains an involution of D.*

Proof. We begin by considering the special cases $K \cong \mathrm{L}_2(q^r)$ or $\mathrm{Sz}(q^r)$ for some prime power q. Suppose that $K \cong \mathrm{L}_2(q^r)$. Choose $\epsilon \in \{-1, 1\}$, set $\delta = 1$ if q is even and $\delta = 1/2$ if q is odd. Now $C_K(R) \cong \mathrm{L}_2(q)$ and K possesses an R-invariant cyclic subgroup X with order $\delta(q^r - \epsilon)$ that is inverted by an involution $z \in C_K(R)$ and satisfies $|\,C_X(R)\,| = \delta(q - \epsilon)$. Now

$$\delta(q^r - \epsilon) = \delta(q - \epsilon)\left((\epsilon q)^{r-1} + \cdots + 1\right)$$

and X possesses a subgroup Y of order $(\epsilon q)^{r-1} + \cdots + 1$. Then Y is R-invariant, inverted by z, has odd order and $C_Y(R) = 1$. The two choices for Y, depending on the choice of ϵ, have coprime orders. Hence we may choose ϵ such that $p \notin \pi(Y)$. Choose a prime $t \in \pi(Y)$ and let T be the subgroup of Y with order t. Set $D = T\langle z \rangle$. Recall that $C_K(R) \cong \mathrm{L}_2(q)$. If $q > 3$ then $C_K(R)$ is simple, whence $z \in C_K(R)'$. If $q = 3$ then $\mathrm{L}_2(q) \cong 2^2 : 3$ and again $z \in C_K(R)'$.

Suppose that $K \cong \mathrm{Sz}(q^r)$. Then $q = 2^n$ for some odd n. Again choose $\epsilon \in \{-1, 1\}$. Now $C_K(R) \cong \mathrm{Sz}(q)$ and by [18], K contains an R-invariant cyclic Hall-subgroup X of order $2^{nr} + \epsilon 2^{(nr+1)/2} + 1$ that is inverted by an involution $z \in C_K(R)$. Note that X has odd order and is not centralized by R. Set $Y = [X, R] \neq 1$. Then Y is inverted by z. As previously, we may choose ϵ such that Y is a p'-group. Choose $t \in \pi(Y)$ and let T be the subgroup of Y with order t. Set $D = T\langle z \rangle$. If $q > 2$ then $C_K(R)$ is simple so $z \in C_K(R)'$.

We now consider the general case. Using Lemma 4.2 and what we have just done, there exists an R-invariant dihedral subgroup $D \leq K$ with order $2t$ for some $t \notin \{2, p\}$, R is nontrivial on $O_t(D)$ and $C_K(R)$ contains an involution of D. It remains to prove the final assertion. If $C_K(R)$ is simple then there is nothing further to prove. Hence we may assume that K is one of the eight groups listed in Tables 1 and 2 of Theorem 4.1. The cases $\mathrm{L}_2(2^r)$ and $\mathrm{Sz}(2^r)$ are excluded by hypothesis. The case $\mathrm{L}_2(3^r)$ has been dealt with. If $K \cong \mathrm{U}_3(2^r)$ then $C_K(R) \cong 3^2 : Q_8$ so $C_K(R)'$ contains every involution of $C_K(R)$. If $K \cong {}^2G_2(3^r)$ then $C_K(R)'$ has odd index in $C_K(R)$ so again $C_K(R)'$ contains every involution of $C_K(R)$. The remaining three cases require a little more work.

Suppose $K \cong \mathrm{Sp}_4(2^r)$ or $G_2(2^r)$. Then K contains an R-invariant subgroup $H \cong \mathrm{L}_2(2^r) \times \mathrm{L}_2(2^r)$ with R acting nontrivially on each component. This is clear in the case $K \cong \mathrm{Sp}_4(2^r)$ and follows from [3] in the case $K \cong G_2(2^r)$. By what we have done previously, H contains an R-invariant subgroup $D = D_1 \times D_2$ with each D_i dihedral of order $2t$ for some prime $t \notin \{2, p\}$, each D_i is R-invariant and R acts nontrivially on $O_t(D_i)$. From Table 2 in Theorem 4.1 we have $|\, C_K(R) : C_K(R)' \,| = 2$ so $C_K(R)'$ contains an involution $u \in D$. Choose i such that u inverts $O_t(D_i)$. Then $O_t(D_i)\langle\, u \,\rangle$ is the desired dihedral subgroup.

Suppose $K \cong {}^2F_4(2^r)$. By [15], K contains an R-invariant subgroup $H \cong \mathrm{Sp}_4(2^r)$ on which R acts nontrivially. Apply the previously considered case.

Proof of Theorem 4.1(f)(ii). Let $\widetilde{E} = E(C_{\widetilde{K}}(\widetilde{R}))$, $\widetilde{X} = \ker(\widetilde{E}$ on $C_V(\widetilde{R}))$ and let E be the image of $\widetilde{E}$ in $C_K(R)$. Since $\widetilde{K}/Z(\widetilde{K}) = K$ we have $E = E(C_K(R))$.

Assume the result is false. Then $\widetilde{X} \neq 1$ whence $\widetilde{E} \neq 1, E \neq 1$ and Theorem 4.1 implies $K \not\cong \mathrm{L}_2(2^r)$ and $\mathrm{Sz}(2^r)$. Also, E is simple whence $\widetilde{E}$ is quasisimple and $Z(\widetilde{E}) \leq Z(\widetilde{K})$. Since $\widetilde{X} \trianglelefteq \widetilde{E}$ we have $\widetilde{X} \leq Z(\widetilde{E})$ or $\widetilde{X} = \widetilde{E}$. Suppose that $\widetilde{X} \leq Z(\widetilde{E})$. By (f)(i) we have $0 \neq C_V(\widetilde{R}) \leq C_V(\widetilde{X})$. Also $C_V(\widetilde{X})$ is a submodule because $\widetilde{X} \leq Z(\widetilde{E}) \leq Z(\widetilde{K})$. This contradicts the irreducibility of V. We deduce that $\widetilde{X} = \widetilde{E}$. In particular, as $E = C_K(R)'$ it follows that $\widetilde{X}$ maps onto $C_K(R)'$.

By Lemma 4.3 there exists a prime $t \notin \{2, \mathrm{char}\,\mathbb{F}\}$ and an R-invariant dihedral subgroup $D \leq K$ of order $2t$ such that R is nontrivial on $O_t(D)$ and $C_K(R)'$ contains an involution of D. Let $T = O_t(D)$ and choose $S \leq C_K(R)' \cap D$ with order 2.

Let $\widetilde{S} \leq \widetilde{X}$ be a 2-subgroup that maps onto S. Since T is cyclic, the inverse image of T in $\widetilde{K}$ is abelian. Let $\widetilde{T}$ be a Sylow t-subgroup of this inverse image. Then $\widetilde{T}$ is $\widetilde{R} \times \widetilde{S}$-invariant and $\widetilde{T}$ maps onto T. Let $\widetilde{T}_0 = [\widetilde{T}, \widetilde{S}]$. Coprime Action(a) implies $\widetilde{T}_0 = [\widetilde{T}_0, \widetilde{S}]$. Note that $\widetilde{T}_0$ is $\widetilde{R}$-invariant since $[\widetilde{R}, \widetilde{S}] = 1$. Now $T = [T, S]$ and $\widetilde{T}_0$ maps onto T whence $[\widetilde{T}_0, \widetilde{R}] \neq 1$ because $[T, R] \neq 1$. But $[C_V(\widetilde{R}), \widetilde{S}] = 0$ so Lemma 3.11 implies $[\widetilde{T}_0, \widetilde{R}] = 1$, a contradiction. The proof is complete.

We close this section with some useful consequences of Theorem 4.1.

Theorem 4.4. *Let r be a prime and suppose the elementary abelian r-group A acts coprimely on the K-group G.*

(a) If $C_G(A)$ is nilpotent or has odd order then G is solvable.
(b) If $C_G(A)$ is solvable then the noncyclic composition factors of G belong to $\{\mathrm{L}_2(2^r), \mathrm{L}_2(3^r), \mathrm{U}_3(2^r), \mathrm{Sz}(2^r)\}$.
(c) Let $K \in \mathrm{comp}_A(G)$. Then $C_G(C_K(A)) = C_G(K)$.
(d) $Z(C_G(A)) \leq \mathrm{sol}(G)$.

Proof. (a),(b). Using Coprime Action(c) it follows that a minimal counterexample is A-simple. Thus $G = K_1 \times \cdots \times K_n$ where $K_1, \ldots, K_n$ are simple subgroups that are permuted transitively by A. Let $B = N_A(K_1)$. Lemma 3.6 implies that

$$C_G(A) \cong C_{K_1}(B).$$

In particular, $C_{K_1}(B)$ is solvable. Apply Theorem 4.1.

(c). Trivially $C_G(K) \leq C_G(C_K(A))$. Set $Z = C_G(C_K(A))$. Using Coprime Action(c) and Lemma 3.5 we may suppose that $Z(E(G)) = 1$. Then $E(G)$ is the direct product of the A-components of G and $C_G(A)$ permutes these A-components by conjugation. By (a), $C_K(A) \neq 1$ so as $[Z, C_K(A)] = 1$ it follows that Z normalizes K.

We have $K = K_1 \times \cdots \times K_n$ where $K_1, \ldots, K_n$ are simple subgroups that are permuted transitively by A. Lemma 3.8 implies that Z normalizes each K_i. For each i let $\pi_i : K \longrightarrow K_i$ be the projection map and set $A_i = N_A(K_i)$. Let $c \in C_K(A)$. Then $c = (c\pi_1) \cdots (c\pi_n)$. Since $[c, Z] = 1$ and Z normalizes each K_i it follows that $[c\pi_i, Z] = 1$. Lemma 3.6 implies $C_K(A)\pi_i = C_{K_i}(A_i)$ so $[C_{K_i}(A_i), Z] = 1$ and then Theorem 4.1(a),(e) imply $[K_i, Z] = 1$. Then $[K, Z] = 1$.

(d). Set $\overline{G} = G/\mathrm{sol}(G)$. Then $C_{\overline{G}}(E(\overline{G})) = 1$. Coprime Action(c) and (c) imply $[E(\overline{G}), \overline{Z(C_G(A))}] = 1$ whence $Z(C_G(A)) \leq \mathrm{sol}(G)$.

5. Direct Products

We establish some notation relating to direct products and present a lemma of McBride [17, Lemma 5.10]. Throughout this section we assume:

Hypothesis 5.1.

- *$G = K_1 \times \cdots \times K_n$ with each K_i a nonabelian simple group.*
- *For each i, π_i is the projection $G \longrightarrow K_i$.*

We remark that the subgroups K_i are the components of G and are uniquely determined, as are the projection maps.

Definition 5.2. *Let H be a subgroup of G.*

- *H is* diagonal *if for each i the projection map $H \longrightarrow K_i$ is an isomorphism.*
- *H is* overdiagonal *if for each i the projection map $H \longrightarrow K_i$ is an epimorphism.*
- *H is* underdiagonal *if for each i the projection map $H \longrightarrow K_i$ is not an epimorphism.*

Lemma 5.3. *Suppose H is an overdiagonal subgroup of G. Then there exists a unique partition $\{ \mathcal{L}_1, \ldots, \mathcal{L}_m \}$ of $\{ K_1, \ldots, K_n \}$ such that*

$$H = (H \cap \langle \mathcal{L}_1 \rangle) \times \cdots \times (H \cap \langle \mathcal{L}_m \rangle)$$

and $H \cap \langle \mathcal{L}_i \rangle$ is a diagonal subgroup of $\langle \mathcal{L}_i \rangle$ for each i.

Proof. Choose $\mathcal{L}_1 \subseteq \{ K_1, \dots, K_n \}$ minimal subject to $H \cap \langle \mathcal{L}_1 \rangle \neq 1$. Set $H_1 = H \cap \langle \mathcal{L}_1 \rangle \trianglelefteq H$. Choose $K_i \in \mathcal{L}_1$. The minimal choice of $\mathcal{L}_1$ implies $H_1 \cap \ker \pi_i = 1$. Thus $1 \neq H_1\pi_i \trianglelefteq H\pi_i = K_i$ so the simplicity of K_i forces $H_1\pi_i = H\pi_i = K_i$. Then H_1 is diagonal in $\langle \mathcal{L}_1 \rangle$. Also $H = H_1(H \cap \ker \pi_i)$ so as $H_1 \cap \ker \pi_i = 1$ we obtain

$$H = H_1 \times (H \cap \ker \pi_i).$$

As $H_1 \cong K_i$ we see that H_1 is simple and then that $H \cap \ker \pi_i = C_H(H_1)$. Set

$$G^* = \prod_{K_j \notin \mathcal{L}_1} K_j = \bigcap_{K_i \in \mathcal{L}_1} \ker \pi_i.$$

Then $H = H_1 \times (H \cap G^*)$. Now H_1 projects trivially into the direct factors of G^* so as H is overdiagonal in G it follows that $H \cap G^*$ is overdiagonal in G^*. Induction yields $\mathcal{L}_2, \dots, \mathcal{L}_m$.

Now $H = (H \cap \langle \mathcal{L}_1 \rangle) \times \cdots \times (H \cap \langle \mathcal{L}_m \rangle)$ so each $H \cap \langle \mathcal{L}_i \rangle$ is a component of H. The components of a group are uniquely determined so the uniqueness of $\{ \mathcal{L}_1, \dots, \mathcal{L}_m \}$ follows.

Lemma 5.4. *Let H be an overdiagonal subgroup of G. Then $N_G(H) = H$.*

Proof. By the previous lemma we may assume that H is diagonal. Since $H\pi_1 = K_1$ we have

$$N_G(H) = HN$$

where $N = N_G(H) \cap \ker \pi_1$. Now $[H, N] \leq H \cap \ker \pi_1 = 1$. For any i we have $1 = [H\pi_i, N\pi_i] = [K_i, N\pi_i]$ and so $N\pi_i \leq Z(K_i) = 1$. This forces $N = 1$ and completes the proof.

6. A-quasisimple groups

Throughout this section we assume:

Hypothesis 6.1.

- *r is a prime and A is an elementary abelian r-group.*
- *A acts coprimely on the K-group K.*
- *K is A-quasisimple.*

We will establish a number of basic results on the subgroup structure of K. A central theme is the study of the $AC_K(A)$-invariant subgroups of K. Of course the subgroups $C_K(B)$ for $B \leq A$ are examples. It will develop that these comprise an almost complete list. The results of this section may also be viewed as an extension of Theorem 4.1 from simple groups to A-quasisimple groups. A similar theory is also developed by McBride [16, 17] but cast in a different language.

Let $K_1, \ldots, K_n$ be the components of K. Then

$$K = K_1 * \cdots * K_n$$

and A acts transitively on $\{ K_1, \ldots, K_n \}$. In particular, K has a unique nonsolvable composition factor.

Definition 6.2. *The* type *of K is the isomorphism type of the unique nonsolvable composition factor of K.*

Let $\overline{K} = K/Z(K)$, so $\overline{K}$ is A-simple and

$$\overline{K} = \overline{K_1} \times \cdots \times \overline{K_n}$$

with each $\overline{K_i}$ being simple.

Definition 6.3. *Let H be an A-invariant subgroup of K. Then H is* underdiagonal, diagonal *or* overdiagonal *in K depending on whether $\overline{H}$ has the respective property in $\overline{K}$.*

Note that since H is A-invariant and A is transitive on $\{ K_1, \ldots, K_n \}$ it follows that H is either underdiagonal or overdiagonal.

We fix the notation

$$A_\infty = \ker(A \longrightarrow \mathrm{Sym}(K_1, \ldots, K_n)).$$

Since A is abelian and transitive on $\{ K_1, \ldots, K_n \}$ it follows that $A_\infty = N_A(K_i)$ for each i and that the action of A/A_∞ on $\{ K_1, \ldots, K_n \}$ is regular.

Lemma 6.4. *$A_\infty/C_A(K)$ acts faithfully on each K_i and $|\, A_\infty/C_A(K) \,| = 1$ or r.*

Proof. Since A is abelian and transitive on $\{ K_1, \ldots, K_n \}$ it follows that $C_{A_\infty}(K_i) = C_A(K_1 * \cdots * K_n) = C_A(K)$. Theorem 4.1 and Lemma 3.5 imply that the Sylow r-subgroups of $\mathrm{Aut}(K_i)$ are cyclic and the result follows.

Next we describe the structure of the subgroups $C_K(B)$ for $B \leq A$.

Lemma 6.5. *Let $B \leq A$.*

(a) Suppose $B \cap A_\infty \leq C_A(K)$. Then there exists $Z \leq Z(K_1) \cap Z(K_2 \cdots K_n)$ such that $C_K(B)$ is isomorphic to the central product of $|\, A : BA_\infty \,|$ copies of K_1/Z that are permuted transitively by A. In particular, $C_K(B)$ is overdiagonal and A-quasisimple with the same type as K. If K is A-simple then so is $C_K(B)$.

(b) Suppose $B \cap A_\infty \not\leq C_A(K)$. Then there exists $Z \leq Z(K_1) \cap Z(K_2 \cdots K_n)$ such that $C_K(B)$ is isomorphic to the central product of $|\, A : BA_\infty \,|$ copies of $C_{K_1}(A_\infty)/Z$ that are permuted transitively by A. In particular, $C_K(B)$ is underdiagonal. Either $C_K(B)$ is solvable or $F^*(C_K(B))$ is A-quasisimple. If K is A-simple then either $C_K(B)$ is solvable or $F^*(C_K(B))$ is A-simple.

(c) If $B^* \le A$ and $C_K(B^*) = C_K(B)$ then $B^*C_A(K) = BC_A(K)$.
(d) $C_A(C_K(B)) = BC_A(K)$.

Proof. We may assume that $C_A(K) = 1$. Then $|\,A_\infty\,| = 1$ or r by Lemma 6.4.

(a). Let $m = |\,A : BA_\infty\,|$. Then B has m orbits on $\{\,K_1, \ldots, K_n\,\}$ and these orbits are permuted transitively by A. Let $L_1, \ldots, L_m$ be the subgroups of K that are generated by these orbits. Then $K = L_1 * \cdots * L_m$. Coprime Action(e) implies that $C_K(B) = C_{L_1}(B) * \cdots * C_{L_m}(B)$. The subgroups $C_{L_1}(B), \ldots, C_{L_m}(B)$ are permuted transitively by A. Without loss, $L_1 = K_1 * \cdots * K_l$. Now $B \cap A_\infty = 1$ so B is regular on $\{\,K_1, \ldots, K_l\,\}$. Apply Lemma 3.7.

(b). We have $|\,A_\infty\,| = r$ and so there exists B_0 with $B = A_\infty \times B_0$. Now

$$C_K(A_\infty) = C_{K_1}(A_\infty) * \cdots * C_{K_n}(A_\infty)$$

so applying an argument similar to that used in (a), with B_0 in place of B and the $C_{K_i}(A_\infty)$ in place of the K_i, the first assertion follows. Since $C_K(B) \le C_K(A_\infty)$, trivially $C_K(B)$ is underdiagonal. The remaining assertions follow from Theorem 4.1.

(c). We have $C_K(BB^*) = C_K(B)$. Since a subgroup cannot be both underdiagonal and overdiagonal we have $BB^* \cap A_\infty = B \cap A_\infty$. Now (a) and (b) imply $|\,A : BB^*A_\infty\,| = |\,A : BA_\infty\,|$, whence $|\,BB^*\,| = |\,B\,|$ and $B^* \le B$. Similarly, $B \le B^*$ so $B = B^*$.

(d). Apply (c) with $B^* = C_A(C_K(B))$.

The next result shows that modulo $Z(K)$, the subgroups just considered are the only $AC_K(A)$-invariant overdiagonal subgroups of K.

Lemma 6.6. *Suppose that H is an $AC_K(A)$-invariant overdiagonal subgroup of K. Then there exists $B \le A$ such that $B \cap A_\infty \le C_A(K)$ and*

$$H = C_K(B)(H \cap Z(K)).$$

In particular, if K is A-simple then $H = C_K(B)$ and H is A-simple with the same type as K.

Proof. Suppose the lemma has been established in the case that K is A-simple. Set $\overline{K} = K/Z(K)$. Coprime Action(c) implies that $C_{\overline{K}}(A) = \overline{C_K(A)}$ so $\overline{H}$ is $C_{\overline{K}}(A)$-invariant, whence $\overline{H} = C_{\overline{K}}(B)$ for some $B \le A$ with $B \cap A_\infty \le C_A(\overline{K})$. Lemma 3.5 implies that $B \cap A_\infty \le C_A(K)$. Another application of Coprime Action(c) yields

$$HZ(K) = C_K(B)Z(K).$$

Lemma 6.5(a) implies that $C_K(B)$ is A-quasisimple. In particular it is perfect. Then $H' = (HZ(K))' = (C_K(B)Z(K))' = C_K(B)$ so $C_K(B) \le H \le C_K(B)Z(K)$ and then $H = C_K(B)(H \cap Z(K))$. Hence we may suppose that K is A-simple.

Consider the case that H is diagonal. Lemma 5.4 implies $C_K(A) \leq H$. Now $H \cong K_1$ so H is simple. Set $B = C_A(H)$. Theorem 4.1 implies $|\, A : B \,| \leq r$. Observe that

$$C_K(A) \leq H \leq C_K(B).$$

Suppose $A_\infty \leq C_A(K)$. Lemma 6.5(a) implies that $C_K(A) \cong K_1$ whence $C_K(A) = H$ and we are done. Suppose $A_\infty \not\leq C_A(K)$. Now H is overdiagonal and $H \leq C_K(B)$ so $C_K(B)$ is overdiagonal. Lemma 6.5(b) implies that $B \cap A_\infty \leq C_A(K)$. As $|\, A : B \,| \leq r$ this forces $A = BA_\infty$ and then Lemma 6.5(a) implies that $H = C_K(B)$, again completing the proof in this case.

Consider now the general case. Lemma 5.3 implies there exists an A-invariant partition $\{\, \mathcal{L}_1, \ldots, \mathcal{L}_m \,\}$ of $\{\, K_1, \ldots, K_n \,\}$ such that

$$H = (H \cap \langle\, \mathcal{L}_1 \,\rangle) \times \cdots \times (H \cap \langle\, \mathcal{L}_m \,\rangle)$$

and $H \cap \langle\, \mathcal{L}_i \,\rangle$ is diagonal in $\langle\, \mathcal{L}_i \,\rangle$ for each i.

Let $A_1 = \ker(A \longrightarrow \mathrm{Sym}(\{\, \mathcal{L}_1, \ldots, \mathcal{L}_m \,\}))$. Since A is abelian and transitive on $\{\, K_1, \ldots, K_n \,\}$ it follows that A_1 is transitive on each $\mathcal{L}_i$. For each i, let $L_i = \langle\, \mathcal{L}_i \,\rangle$, so $K = L_1 \times \cdots \times L_m$ and we denote the projection map $K \longrightarrow L_i$ by λ_i. Lemma 3.6 implies that

$$C_K(A)\lambda_i = C_{L_i}(A_1).$$

In particular, $H \cap L_1$ is $A_1 C_{L_1}(A_1)$-invariant. By the diagonal case, there exists $B \leq A_1$ with

$$H \cap L_1 = C_{L_1}(B).$$

Now A is abelian and transitive on $\{\, \mathcal{L}_1, \ldots, \mathcal{L}_m \,\}$ whence $H \cap L_i = C_{L_i}(B)$ for all i and then

$$H = C_K(B).$$

Since H is overdiagonal, Lemma 6.5(b) implies that $B \cap A_\infty \leq C_A(K)$.

It remains to consider the $AC_K(A)$-invariant underdiagonal subgroups. Of particular interest is the case when there exist $AC_K(A)$-invariant solvable subgroups. These are necessarily underdiagonal.

Lemma 6.7.

(a) For each i, K_i possesses a unique maximal $A_\infty C_{K_i}(A_\infty)$-invariant solvable subgroup S_i.

Set $S = S_1 * \cdots * S_n$.

(b) S is the unique maximal $AC_K(A)$-invariant solvable subgroup of K.

(c) Suppose $S \not\leq Z(K)$. Then K is of type $\mathrm{L}_2(2^r), \mathrm{L}_2(3^r), \mathrm{U}_3(2^r)$ or $\mathrm{Sz}(2^r)$; $C_K(A) \leq C_K(A_\infty) \leq S$ and S is a maximal A-invariant subgroup of K. Moreover S is the unique maximal $AC_K(A)$-invariant underdiagonal subgroup of K.

Proof. Using Coprime Action(c) and Lemma 3.5 we may suppose that $Z(K) = 1$, so $K = K_1 \times \cdots \times K_n$. For each i let $\pi_i : K \longrightarrow K_i$ be the projection map. We may also assume $C_A(K) = 1$, so Lemma 6.4 implies $|\, A_\infty \,| = 1$ or r and A_∞ acts faithfully on each K_i.

(a). If $A_\infty = 1$ then $K_i = C_{K_i}(A_\infty)$ and K_i is simple so put $S_i = 1$. If $|\, A_\infty \,| = r$ then the existence of S_i follows from Theorem 4.1(c).

(b). Since $A_\infty \trianglelefteq A$ it follows that A permutes transitively the subgroups $A_\infty C_{K_i}(A)$ and then that A permutes the subgroups S_i. Thus S is an A-invariant solvable subgroup of K. Lemma 3.6 implies $C_K(A)\pi_i = C_{K_i}(A_\infty)$ and it follows that S is $AC_K(A)$-invariant.

Suppose H is an $AC_K(A)$-invariant solvable subgroup of K. Now $H \leq H\pi_1 \times \cdots \times H\pi_n$ and as $C_K(A)\pi_i = C_{K_i}(A_\infty)$ it follows that each $H\pi_i$ is an $A_\infty C_{K_i}(A)$-invariant solvable subgroup of K_i. Then $H\pi_i \leq S_i$ and $H \leq S$.

(c). Apply Theorem 4.1(c).

Lemma 6.8.

(a) For each i, K_i possesses a unique maximal $A_\infty C_{K_i}(A_\infty)$-invariant proper subgroup M_i.

Set $M = M_1 * \cdots * M_n$.

(b) M is the unique maximal $AC_K(A)$-invariant underdiagonal subgroup of K.

(c) Suppose $M \not\leq Z(K)$. Then $A_\infty \not\leq C_A(K)$ and $C_K(A_\infty) \leq M$. If in addition $M \neq C_K(A_\infty)Z(K)$ then K is of type $\mathrm{L}_2(2^r)$ or $\mathrm{Sz}(2^r)$ and M is solvable.

Proof. The proof is similar to the proof of Lemma 6.7 but using Theorem 4.1(d) in place of Theorem 4.1(c).

Corollary 6.9. *Let $a \in A^\#$ and suppose H is an A-invariant subgroup that satisfies $C_K(a) \leq H \leq K$ and $H^{(\infty)} \not\leq C_K(a)$. Then $H = K$.*

Proof. Using Coprime Action(c) we may assume that $Z(K) = 1$. We may also assume that $C_A(K) = 1$. Suppose that H is underdiagonal. Lemma 6.8 implies that $H \leq C_K(A_\infty)$. Now $C_K(a) < H$ so $C_K(a)$ is also underdiagonal, whence $a \in A_\infty$. As $|\, A_\infty \,| \leq r$ we obtain $\langle\, a \,\rangle = A_\infty$ whence $H = C_K(a)$, a contradiction. We deduce that H is overdiagonal.

Lemma 6.6 implies $H = C_K(B)$ for some $B \leq A$. Using Lemma 6.5(d) we have $B \leq C_A(C_K(a)) = \langle\, a \,\rangle$ whence $B = 1$ or $\langle\, a \,\rangle$. Now $C_K(a) < H = C_K(B)$ whence $B = 1$ and $H = K$.

Lemma 6.10. *Suppose that H is an $AC_K(A)$-invariant subgroup of K and that $L \in \mathrm{comp}_A(H)$. Then $L = E(H)$ and either*

(a) $C_K(A) = C_L(A)$ and L is overdiagonal; or
(b) $E(C_K(A)) = E(C_L(A)) \neq 1$ and L is underdiagonal.

Proof. Lemmas 6.5, 6.6 and 6.8 imply that $E(H)$ is trivial or A-quasisimple. Since $L \in \mathrm{comp}_A(H)$ it follows that $L = E(H)$. In particular, L is $AC_K(A)$-invariant.

Suppose that L is overdiagonal. Lemma 6.6 implies that $L = C_K(B)(L \cap Z(K))$ for some $B \leq A$ with $B \cap A_\infty \leq C_A(K)$. Lemma 6.5 implies that $C_K(B)$ is A-quasisimple. Since L is also A-quasisimple, it follows that $L = C_K(B)$. Now $B \leq A$ whence $C_L(A) = C_K(A)$ and (a) holds. Hence we may assume that L is underdiagonal.

Let $\overline{K} = K/Z(K)$. Suppose $C_{\overline{K}}(A)$ is solvable. Lemma 6.7 implies that K is of type $\mathrm{L}_2(2^r), \mathrm{L}_2(3^r), \mathrm{U}_3(2^r)$ or $\mathrm{Sz}(2^r)$. Lemma 6.7(c) implies that any $AC_K(A)$-invariant proper subgroup of $\overline{K}$ is solvable. Then $\overline{L} = \overline{K}$ contrary to L being underdiagonal. Hence $C_{\overline{K}}(A)$ is nonsolvable. Lemma 6.5 implies that $F^*(C_{\overline{K}}(A))$ is simple. Now $F^*(C_{\overline{L}}(A)) \trianglelefteq F^*(C_{\overline{K}}(A))$ whence $E(C_{\overline{L}}(A)) = E(C_{\overline{K}}(A)) \neq 1$. Coprime Action(c) implies that $E(\overline{C_L(A)}) = E(\overline{C_K(A)})$. Since $\overline{K} = K/Z(K)$ it follows that $E(C_L(A))Z(K) = E(C_K(A))Z(K)$. Then (b) follows on taking the derived subgroup of both sides.

We record the following triviality.

Lemma 6.11. *Let $a \in A$. Then $[K, a] = 1$ or K.*

Proof. Suppose $[K, a] \neq 1$. Set $\overline{K} = K/Z(K)$. Lemma 3.5 implies that $[\overline{K}, a] \neq 1$. Now A is abelian so $[\overline{K}, a]$ is an A-invariant normal subgroup of $\overline{K}$. Then $\overline{K} = [\overline{K}, a]$ because $\overline{K}$ is A-simple. Consequently $K = [K, a]Z(K)$ so as K is perfect, we obtain $K = [K, a]$.

We close with a lemma of generation. Recall that $\mathrm{Hyp}(A)$ denotes the set of subgroups of index r in A.

Lemma 6.12.

(a) Let $B \leq A$ and suppose $C_K(B)$ is overdiagonal. Then $C_K(C)$ is overdiagonal and A-quasisimple for all $C \leq B$.

(b) Suppose $1 \neq A^* \leq A$. Then

$$K = \langle\, C_K(B) \mid B \in \mathrm{Hyp}(A^*) \text{ and } C_K(B) \text{ is overdiagonal} \,\rangle.$$

Proof. (a). Lemma 6.5(b) implies $B \cap A_\infty \leq C_A(K)$. Then for each $C \leq B$ we have $C \cap A_\infty \leq C_A(K)$. The conclusion follows from Lemma 6.5(a).

(b). If $A^* \leq C_A(K)$ then $K = C_K(B)$ for any $B \in \mathrm{Hyp}(A^*)$. Hence we may assume that $A^* \not\leq C_A(K)$. Then A^* has nontrivial image in $A/C_A(K)$ and we may replace A by $A/C_A(K)$ to assume that $C_A(K) = 1$.

Let $\mathcal{H}$ be the set of hyperplanes of A^* that intersect A_∞ trivially. Lemma 6.4 implies $|\, A^* \cap A_\infty \,| = 1$ or r. Note that if $A^* \cap A_\infty = A^*$ then $|\, A^* \,| = r$ and $\mathcal{H} = \{\, 1 \,\}$. It follows that $\cap\mathcal{H} = 1$. Let

$$L = \langle\, C_K(B) \mid B \in \mathcal{H} \,\rangle.$$

Now $C_K(B)$ is overdiagonal and perfect for each $B \in \mathcal{H}$ by Lemma 6.5(a). It follows that L is overdiagonal and perfect. Lemma 6.6 implies $L = C_K(C)$ for some $C \leq A$. Then, using Lemma 6.5(d), we have

$$C \leq C_A(L) = \bigcap_{B \in \mathcal{H}} C_A(C_K(B)) = \bigcap_{B \in \mathcal{H}} B = 1.$$

Then $L = C_K(C) = K$, completing the proof.

7. Modules

Two results on modules, which are central to the theory being developed in this paper, will be established. The first result has previously been proved by the author [7]. The proof presented here is much shorter. However, it requires the K-group hypothesis whereas the proof in [7] does not.

Theorem 7.1. *Let R be a group of prime order r that acts on the r'-group G. Assume that G is a K-group. Let V be a faithful completely reducible $\mathbb{F}[RG]$-module over a field $\mathbb{F}$ of characteristic p. Assume that $\mathbb{F}[R]$ is not a direct summand of V_R. Then either:*

- $[G, R] = 1$ *or*
- *r is a Fermat prime and $[G, R]$ is a special 2-group.*

Proof. Assume false and let G be a minimal counterexample. By [4, Theorem 5.1], G is nonsolvable. Now $R[G, R] \trianglelefteq RG$ so $V_{R[G,R]}$ is completely reducible by Clifford's Theorem. Then Coprime Action(a) and the minimality of G imply $G = [G, R]$. If $p = 0$ then define $O_p(H) = 1$ for any group H. Since V is completely reducible we have $O_p(G) = 1$.

Claim 1. *Let H be a proper R-invariant subgroup of G.*

(a) Suppose $H = [H, R]$. Then $H/O_p(H)$ is either trivial or a nonabelian 2-group.
(b) Suppose H is a q-group for some prime $q \neq p$. If $q = 2$ assume H is abelian. Then $[H, R] = 1$.

Proof. (a). Let U be an irreducible constituent of V_{RH}. Irreducibility implies $O_p(H/C_H(U)) = 1$ and then the minimality of G implies $H/C_H(U)$ is either trivial or a nonabelian 2-group. Lemma 3.9(b) implies $\cap C_H(U) = O_p(H)$ where U ranges over the irreducible constituents of V_{RH}, so the result follows.

(b). This follows from Coprime Action(a) and (a).

Claim 2. *$F(G) \leq Z(G) \leq C_G(R)$.*

Proof. Complete reducibility implies $O_p(G) = 1$. Then Claim 1(b) implies $Z(G) \leq C_G(R)$. Assume that $F(G) \not\leq Z(G)$. Since $G = [G, R]$ it follows that $[F(G), R] \neq 1$. Claim 1(b) implies $[O_q(G), R] = 1$ for each prime $q \notin \{2, p\}$. As $O_p(G) = 1$ we deduce that $[O_2(G), R] \neq 1$ and that $p \neq 2$.

Let $C = C_G(O_2(G))$. Now $[C, R] \trianglelefteq C \trianglelefteq G$ so $O_p([C, R]) \leq O_p(C) \leq O_p(G) = 1$. Claim 1(a) implies $[C, R]$ is a 2-group, whence $[C, R] \leq O_2(C) \leq O_2(G)$. As $C = C_G(O_2(G))$ we obtain $[C, R] \leq Z(O_2(G))$. Using Coprime Action(a) and Claim 1(b) we have $[C, R] = [C, R, R] \leq [Z(O_2(G)), R] = 1$. As $C \trianglelefteq RG$ and $G = [G, R]$ we deduce that $C \leq Z(G)$.

Let $t \neq 2$ be a prime. By Coprime Action(d) there exists an R-invariant Sylow t-subgroup T of G. Set $H = TO_2(G)$ and $H_0 = [H, R] \trianglelefteq H$. Then H is solvable so $H \neq G$. Now $O_p(H_0) \leq O_p(H) \leq C_G(O_2(G)) \leq Z(G)$ so as $O_p(G) = 1$ we deduce that $O_p(H_0) = 1$. Claim 1(a) implies that H_0 is a 2-group. Since $[T, R] \leq H_0$ and $t \neq 2$ we deduce that $C_G(R)$ contains a Sylow t-subgroup of G for each prime $t \neq 2$.

Let S be an R-invariant Sylow 2-subgroup of G. The previous paragraph implies $G = C_G(R)S$. Then $G = [G, R] \leq S$, contrary to G being nonsolvable. We deduce that $F(G) \leq Z(G)$.

Coprime Action(f) implies $[F^*(G), R] \neq 1$ so as $[F(G), R] = 1$ there exists $K \in \mathrm{comp}(G)$ with $[K, R] \neq 1$.

Claim 3. *R normalizes K.*

Proof. Assume false. Let $K_1, \ldots, K_r$ be the R-conjugates of K. Define $L = \langle K_1, \ldots, K_r \rangle$ and $\overline{L} = L/Z(L)$. Then $L = K_1 * \cdots * K_r$ and $\overline{L} = \overline{K_1} \times \cdots \times \overline{K_r}$. Choose $q \in \pi(\overline{K_1})$ with $q \neq p$ and let $\overline{Q_1} \leq \overline{K_1}$ have order q. Let $Q_1 \leq K_1$ be a q-subgroup that maps onto $\overline{Q_1}$. Then Q_1 is abelian. Let $Q_1, \ldots, Q_r$ be the R-conjugates of Q_1 and set $Q = Q_1 * \cdots * Q_r$. Then $[Q, R] \neq 1$ since R does not normalize $\overline{Q_1}$. But Q is abelian so Claim 1(b) implies $[Q, R] = 1$, a contradiction. The claim is established.

Now K is quasisimple and $[K, R] \neq 1$ so $K = [K, R]$. Claim 1(b) forces $G = K$. Theorem 4.1(f) supplies a contradiction.

The next result is a partial extension of the main result of [4] to nonsolvable groups.

Theorem 7.2. *Let R be a group of prime order r that acts coprimely on the K-group G. Let V be an RG-module, possibly of mixed characteristic, with $V_{[G,R]}$ faithful and completely reducible. Suppose that*

$$K \in \mathrm{comp}(\ker(C_G(R) \text{ on } C_V(R))).$$

Then $K \in \mathrm{comp}(G)$.

Proof. Let $F = F(G)$ and $M = KF$. Now $K \in \mathrm{comp}(C_G(R))$ whence $[K, C_F(R)] = 1$ and so $K \trianglelefteq C_M(R)$. Also $[M, R] = [F, R] \trianglelefteq F \cap [G, R] \trianglelefteq [G, R]$

so as $V_{[G,R]}$ is completely reducible, Clifford's Theorem implies that $V_{[M,R]}$ is also.

Let L be the subnormal closure of K in M. Then $L = \langle K^L \rangle$. Now $[M,R]$ is solvable so [4, Theorem A] implies that $L = K(S \times P)$ with S a 2-group; $S = [S,R]$; $S' = C_S(R)$; $C_K(S') = C_K(S)$; P a p-group for some odd prime p and $K/C_K(P)$ a 2-group. Since $S = [S,R] \leq [M,R] \leq F$ and $S' = C_S(R)$ we have $[K,S'] = 1$. Then $[K,S] = 1$. Also, K is perfect so as $K/C_K(P)$ is a 2-group it follows that $K = C_K(P)$. Then $K \trianglelefteq L = \langle K^L \rangle$ whence $K = L \trianglelefteq\trianglelefteq M$ and K is a component of M. Since $M = KF$ and F is nilpotent, we obtain $[K,F] = 1$.

Since $C_G(F^*(G)) = Z(F(G))$, there exists $X \in \mathrm{comp}_R(G)$ with $[K,X] \neq 1$. Now $C_X(R) \trianglelefteq\trianglelefteq C_G(R)$ and $K \in \mathrm{comp}(C_G(R))$ so $K \leq C_X(R)$ or $[K, C_X(R)] = 1$ by Lemma 3.2(a). Theorem 4.4(c) rules out the second possibility, whence $K \leq C_X(R)$. In particular, $K \in \mathrm{comp}(C_X(R))$. If $X = C_X(R)$ then $K \trianglelefteq\trianglelefteq X \trianglelefteq\trianglelefteq G$ whence $K \in \mathrm{comp}(G)$. Hence we may assume, for a contradiction, that $[X,R] \neq 1$. Lemma 6.11 implies that $X = [X,R]$.

We have $K \leq X = [X,R] \trianglelefteq\trianglelefteq [G,R]$. Clifford's Theorem implies that V_X is completely reducible. Hence we may assume that $G = X$, so G is R-quasisimple and $G = [G,R]$. In particular, V_G is completely reducible and so

$$V = C_V(G) \oplus [V,G].$$

Let U be an irreducible RG-submodule contained in $[V,G]$. Now G is R-quasisimple so either $C_G(U) \leq Z(G)$ or $C_G(U) = G$. The second possibility does not hold since $C_V(G) \cap [V,G] = 0$. Thus $C_G(U) \leq Z(G)$. Set $\overline{G} = G/C_G(U)$. Then $\overline{G}$ is R-quasisimple, $\overline{G} = [\overline{G},R]$ and $\overline{K} \neq 1$. Suppose that $U \neq V$. By induction, $\overline{K} \in \mathrm{comp}(\overline{G})$, whence $\overline{K} = \overline{G}$. This is a contradiction since $[\overline{K},R] = 1$ but $[\overline{G},R] = \overline{G}$. We deduce that V is an irreducible $\mathbb{F}[RG]$-module for some field $\mathbb{F}$.

Theorem 4.1(f) implies that G is not quasisimple. The remainder of the argument is an extension of Theorem 4.1(f) to R-quasisimple groups that are not quasisimple. We remark that no K-group hypothesis is required.

We have $G = K_1 * \cdots * K_r$ where $K_1, \ldots, K_r$ are quasisimple subgroups that are permuted transitively by R. Lemma 6.5(a) implies that $C_G(R)$ is quasisimple. Since $K \in \mathrm{comp}(C_G(R))$ we deduce that

$$K = C_G(R)$$

and then that $[C_V(R), C_G(R)] = 0$.

By Burnside's $p^\alpha q^\beta$-Theorem, we may choose $t \in \pi(K_1)$ with $t \notin \{2, \mathrm{char}\,\mathbb{F}\}$. Now K_1, being quasisimple, is not t-nilpotent so Frobenius' Normal Complement Theorem implies there exists a t-subgroup $T_1 \leq K_1$ and a t'-subgroup $S_1 \leq N_{K_1}(T_1)$ with $1 \neq T_1 = [T_1, S_1]$. Set $T = \langle T_1^R \rangle = T_1 * \cdots * T_r$ and $S = \langle S_1^R \rangle = S_1 * \cdots * S_r$ where $T_1, \ldots, T_r$ and $S_1, \ldots, S_r$ are the conjugates of T_1 and S_1 under the action of R.

Considering $\overline{G} = G/Z(G)$, we see that $S_1 \leq C_S(R)(S_2 * \cdots * S_r)Z(K)$ whence $T_1 = [T_1, S_1] = [T_1, C_S(R)]$. It follows that $T = [T, C_S(R)]$. Now

$[C_V(R), C_G(R)] = 0$, so we may apply Lemma 3.11, with $C_S(R)$ in the role of S, to deduce that

$$[V, [T, R]] = 0.$$

But then $[T, R] = 1$, a contradiction since $T_1 \not\leq Z(K_1)$ and so T_1 is not normalized by R. The proof is complete.

8. General Results

The first result is the starting point for the study of how the global structure of a group that admits a group of automorphisms is influenced by its local structure. The other results are applications of the module results from §7 to composite groups.

Lemma 8.1. *Let A be an elementary abelian r-group for some prime r that acts coprimely on the K-group G. Suppose that H is an $AC_G(A)$-invariant subgroup of G and that $K \in \mathrm{comp}_A(H)$. Then there exists a unique $\widetilde{K}$ with*

$$K \leq \widetilde{K} \in \mathrm{comp}_{A,\mathrm{sol}}(G).$$

Proof. We may suppose that $C_G(A) \leq H$. Uniqueness is clear since distinct (A, sol)-components have solvable intersection. Using Coprime Action(c) and the correspondence between (A, sol)-components of G and A-components of $G/\mathrm{sol}(G)$, it suffices to assume $\mathrm{sol}(G) = 1$ and show that K is contained in an A-component of G.

Since $\mathrm{sol}(G) = 1$ we have $C_G(E(G)) = 1$ so there exists $L \in \mathrm{comp}_A(G)$ with $[L, K] \neq 1$. Now $C_L(A) \leq L \cap H \trianglelefteq\trianglelefteq H$ and $K \in \mathrm{comp}_A(H)$. Since $L \cap H$ is A-invariant it follows from Lemma 3.2(a) that either $K \leq L \cap H$ or $[K, L \cap H] = 1$. As $[L, K] \neq 1$, Theorem 4.4(c) rules out the second possibility. Thus $K \leq L$, completing the proof.

We remark that it is straightforward to construct examples where $\widetilde{K}$ is not an A-component.

Lemma 8.2. *Let R be a group of prime order r that acts coprimely on the group G. Suppose that K and S are R-invariant subgroups of G that satisfy:*

- *$K = [K, R]$ and K is a K-group.*
- *$K = O^2(K)$ or r is not a Fermat prime.*
- *S is K-invariant and solvable.*
- *$KS = \langle K^S \rangle$.*

Then

$$KS = \langle K, C_S(R) \rangle.$$

Proof. We may assume that $G = KS$, so $S \trianglelefteq G$. Let V be a minimal R-invariant normal subgroup of G contained in S. Then V is an elementary abelian p-group for some prime p and hence an irreducible $\mathrm{GF}(p)[RG]$-module. By induction and Coprime Action(c) we obtain

$$G = V\langle K, C_S(R) \rangle.$$

Suppose that $V \cap \langle K, C_S(R) \rangle \neq 1$. The choice of V forces $V \leq \langle K, C_S(R) \rangle$, whence $G = \langle K, C_S(R) \rangle$. Hence we may suppose that $V \cap \langle K, C_S(R) \rangle = 1$. Now $V \leq S$ whence $C_V(R) = 1$.

Set $\overline{G} = G/C_G(V)$. Now $K = [K,R] \leq [G,R] \trianglelefteq G$ so as $G = KS = \langle K^S \rangle$ we have $G = [G,R]$ and then $\overline{G} = [\overline{G},R]$. Similarly, if r is a Fermat prime then as $K = O^2(K)$ we have $G = O^2(G)$ and $\overline{G} = O^2(\overline{G})$. Theorem 7.1 implies $\overline{G} = 1$. Hence $V \leq Z(G)$ so

$$G = \langle K^S \rangle = \langle K^G \rangle \leq \langle K, C_S(R) \rangle$$

and the proof is complete.

Lemma 8.3. *Let R be a group of prime order r that acts coprimely on the group G. Suppose that K and S are R-invariant subgroups of G that satisfy:*

- *$K = [K,R]$ and K is a K-group.*
- *K is perfect.*
- *S is a K-invariant solvable subgroup.*
- *$C_S(R) \leq N_G(K)$.*

Then

$$S \leq N_G(K).$$

If in addition $\mathrm{sol}(K) = Z(K)$ then $[S,K] = 1$.

Proof. We may assume that $G = KS$, so $S \trianglelefteq G$. Let H be the smallest subnormal subgroup of G that contains K. Then H is R-invariant and $H = K(H \cap S) = \langle K^H \rangle = \langle K^{H\cap S} \rangle$. Now K is perfect so $K = O^2(K)$. Lemma 8.2 implies that $H = \langle K, C_{H\cap S}(R) \rangle$. Since $C_S(R) \leq N_G(K)$ we obtain $K \trianglelefteq H$ and then $K = H$, so $K \trianglelefteq\trianglelefteq G$. The conclusion follows from Lemma 3.2(b).

Lemma 8.4. *Let R be a group of prime order r that acts coprimely on the K-group G. Suppose that G is constrained and that $K \in \mathrm{comp}(C_G(R))$. Set $\overline{G} = G/F(G)$. Then $\overline{K} \in \mathrm{comp}(\overline{G})$. In particular, $KF(G) \trianglelefteq\trianglelefteq G$ and $[K, \mathrm{sol}(G)] \leq F(G)$.*

Proof. Let G_0 be the smallest subnormal subgroup of G that contains K. Note that every subnormal subgroup of a constrained group is constrained. Then G_0 is R-invariant and constrained. Suppose the result has been established for G_0. Then $KF(G_0) \trianglelefteq\trianglelefteq G_0 \trianglelefteq G$ whence $KF(G_0) \trianglelefteq\trianglelefteq G$. Now $F(G_0) \trianglelefteq\trianglelefteq G$

so $F(G_0) \leq F(G)$ whence $KF(G) \trianglelefteq\trianglelefteq G$ and the conclusion follows. Hence we may assume that $G = G_0$. In particular, $G = \langle K^G \rangle$.

Since $K \in \mathrm{comp}(C_G(R))$ we have $[K, C_{F(G)}(R)] = 1$. Let V be a chief factor of RG contained in $F(G)$. Then V is an elementary abelian p-group for some prime p. Set $G^* = G/C_G(V)$, so V is a $\mathrm{GF}(p)[RG^*]$-module. Now $K \in \mathrm{comp}(C_G(R))$ and $C_V(R) \leq F(C_G(R))$ so $[K, C_V(R)] = 1$. Coprime Action(c) implies that either

$$K^* = 1 \text{ or } K^* \in \mathrm{comp}(\ker(C_{G^*}(R) \text{ on } C_V(R))).$$

In the first case, as $G = \langle K^G \rangle$, we have $G^* = 1$. In the second case, Theorem 7.2 implies $K^* \in \mathrm{comp}(G^*)$. As $G = \langle K^G \rangle$ this implies $G^* = K^*$. In particular, $[G^*, R] = 1$.

We have shown that

$$[G, R] \leq \bigcap C_G(V)$$

where V ranges over the chief factors of RG contained in $F(G)$. Lemma 3.12 implies that $[G, R] \leq F(G)$. By Coprime Action(a) we have $G = C_G(R)[G, R]$ so as $K \in \mathrm{comp}(C_G(R))$ it follows that $KF(G) \trianglelefteq\trianglelefteq G$. This completes the proof.

9. Local to global results

Theorem 9.1. *Let r be a prime and A an elementary abelian r-group that acts coprimely on the K-group G. Let $a \in A^{\#}$ and let H be an $AC_G(a)$-invariant subgroup of G. Suppose that $K \in \mathrm{comp}_A(H)$.*

- *(a) There exists a unique $\widetilde{K}$ with $K \leq \widetilde{K} \in \mathrm{comp}_{A,\mathrm{sol}}(G)$.*
- *(b) If $[K, a] = 1$ then $K = E(C_{\widetilde{K}}(a))$.*
- *(c) If $[K, a] \neq 1$ then $K = [K, a] = \widetilde{K}$. In particular, $K \in \mathrm{comp}_A(G)$.*
- *(d) If $\widetilde{K}$ is constrained then $\widetilde{K} = KF(\widetilde{K})$ and $[K, a] = 1$. In particular, K is an A-component of G modulo $F(G)$.*
- *(e) Suppose $L \in \mathrm{comp}_{A,\mathrm{sol}}(G)$ with $\widetilde{K} \neq L$ and $L = [L, a]$. Then $[\widetilde{K}, L] = 1$.*

Before launching into the proof, a number of remarks are in order. Firstly, an important special case is when $A = \langle a \rangle$ and $H = C_G(A)$. Secondly, there are of course two quite different outcomes. Either $\widetilde{K}$ is semisimple or constrained. In some senses, the first outcome is the most desired – the A-component K of H is contained in the A-component $\widetilde{K}$ of G. What part (d) shows is that in the constrained case, the situation is quite controlled. Thirdly, turning to part (e), recall that distinct A-components of G commute. This fact plays a crucial role in many arguments. Although distinct (A, sol)-components normalize each other, they do not necessarily commute. Part (e) removes the need to be concerned about this phenomena.

Proof. Set $R = \langle a \rangle$. Now K is R-invariant so it follows from commutator identities that $[K, R] = [K, a]$. Also, $K \trianglelefteq\trianglelefteq H \trianglelefteq HC_G(a)$ so $K \in \mathrm{comp}_A(HC_G(a))$. Hence we may assume that $C_G(a) \leq H$.

(a). This follows from Lemma 8.1.

(b). Suppose $[K,a]=1$. Now $K \in \mathrm{comp}_A(H)$ so $K \in \mathrm{comp}_A(C_G(a))$. Then $K \in \mathrm{comp}_A(C_{\widetilde{K}}(a))$. Let $\widetilde{K}^* = \widetilde{K}/\mathrm{sol}(\widetilde{K})$, so $\widetilde{K}^*$ is A-simple. Lemma 6.5 implies that $C_{\widetilde{K}}(a)$ has at most one A-component, whence $K = E(C_{\widetilde{K}}(a))$.

(c). Since $[K,a] \neq 1$, Lemma 6.11 implies $K = [K,a]$. Let $S = \mathrm{sol}(\widetilde{K})$. Now $S \cap H$ is a solvable normal subgroup of H and $K \in \mathrm{comp}_A(H)$ so $[K, S\cap H] = 1$. In particular, $[K, C_S(a)] = 1$. Lemma 8.3 forces $[K,S]=1$. Consequently $C_{\widetilde{K}}(F(\widetilde{K})) \nleq F(\widetilde{K})$ so $\widetilde{K}$ is not constrained. Since $\widetilde{K} \in \mathrm{comp}_{A,\mathrm{sol}}(G)$ it follows that $\widetilde{K}$ is A-quasisimple. Now $C_{\widetilde{K}}(a) \leq H \cap \widetilde{K}$. Moreover, $K^{(\infty)} = K = [K,a] \leq H$ so Corollary 6.9 forces $H \cap \widetilde{K} = \widetilde{K}$, whence $\widetilde{K} \leq H$. Now $K \leq \widetilde{K}$ and $K \in \mathrm{comp}_A(H)$ so $K \in \mathrm{comp}_A(\widetilde{K})$ and then $K = \widetilde{K}$ since $\widetilde{K}$ is A-quasisimple.

(d). Since $\widetilde{K}$ is constrained it is not equal to K so (c) implies $[K,a] = 1$. Then $K \in \mathrm{comp}_A(C_G(a))$ and so $K \in \mathrm{comp}_A(C_{\widetilde{K}}(a))$. Since K is A-quasisimple, it is the central product of its components. Let $K_0 \in \mathrm{comp}(K)$. Then $K_0 \in \mathrm{comp}(C_{\widetilde{K}}(a))$. Lemma 8.4 implies $K_0 F(\widetilde{K}) \trianglelefteq\trianglelefteq \widetilde{K}$. It follows that $KF(\widetilde{K}) \trianglelefteq\trianglelefteq \widetilde{K}$. Now $\widetilde{K}$ is minimal subject to being A-invariant, nonsolvable and subnormal in G so this forces $KF(\widetilde{K}) = \widetilde{K}$. Finally, $F(\widetilde{K}) \leq F(G)$ whence K is a component of G modulo $F(G)$.

(e). Note that $\widetilde{K}$ and L normalize each other and that $[\widetilde{K}, L] \leq \mathrm{sol}(G)$ since $\widetilde{K} \neq L$. We may assume that $G = \widetilde{K}L$ and that $\mathrm{sol}(G) \neq 1$. Let V be a minimal A-invariant normal subgroup of G that is contained in $\mathrm{sol}(G)$. Then V is an elementary abelian p-group for some prime p. Let $G^* = G/V$. Note that $\widetilde{K}^*$ and L^* are distinct since their commutator is solvable. Using Coprime Action(c) and induction, we conclude that $[\widetilde{K}^*, L^*] = 1$. Then

$$[\widetilde{K}, L] \leq V.$$

Since $K \in \mathrm{comp}_A(H)$ and $V \cap H$ is a solvable normal subgroup of H we have $[K, V\cap H] = 1$. In particular $[K, C_V(a)] = 1$, so $C_V(a) \leq C_V(K)$. Now $[\widetilde{K}, L] \leq V \leq C_G(V)$ so the images of K and L in $\mathrm{GL}(V)$ commute. In particular, $C_V(K)$ is L-invariant. Consider the action of L on $V/C_V(K)$. Now $C_V(a) \leq C_V(K)$ so Coprime Action(c) implies that a is fixed point free on $V/C_V(K)$. Since $L = [L,a]$ and L is perfect, Theorem 7.1 implies that L is trivial on $V/C_V(K)$. Thus

$$[V, L] \leq C_V(K).$$

Recall that $G = \widetilde{K}L$, so $L \trianglelefteq G$. Then $[V,L] \trianglelefteq G$ and the choice of V implies $[V,L] = 1$ or V. Suppose that $[V,L]=1$. Then $[L,\widetilde{K},L] \leq [V,L] = 1$ and $[\widetilde{K},L,L] = 1$ so the Three Subgroups Lemma forces $[L,L,\widetilde{K}] = 1$. Then $[L,\widetilde{K}] = 1$ since L is perfect. Suppose that $[V,L] = V$. Then $[V,K] = 1$. Again it follows from the Three Subgroups Lemma that $[K,L] = 1$. Then $K \leq C_{\widetilde{K}}(L)$. Since $\widetilde{K}$ is A-quasisimple and normalizes L this forces $C_{\widetilde{K}}(L) = \widetilde{K}$, whence $[\widetilde{K}, L] = 1$ in this case also.

10. An application to signalizer functors

We being by considering an elementary abelian r-group acting coprimely on a K-group and using Theorem 9.1 to analyze how various local subgroups interact with each other.

Theorem 10.1 (The Local Theorem). *Let r be a prime and A an elementary abelian r-group that acts coprimely on the K-group G. For each $a \in A^{\#}$ let*

$$\Omega_a = \{ \, K \in \mathrm{comp}_A(H) \mid H \textit{ is an } AC_G(a)\textit{-invariant subgroup of } G \, \}$$

and

$$\Omega = \bigcup_{a \in A^{\#}} \Omega_a.$$

For each $K \in \Omega$ set

$$C^*_K(A) = \begin{cases} C_K(A) & \textit{if } C_K(A) \textit{ is solvable} \\ E(C_K(A)) & \textit{if } C_K(A) \textit{ is nonsolvable.} \end{cases}$$

Let $K, L \in \Omega$, so that $K \in \Omega_a$ and $L \in \Omega_b$ for some $a, b \in A^{\#}$.

(a) Suppose $[K, L] \neq 1$. Then there exists a unique X with

$$\langle \, K, L \, \rangle \leq X \in \mathrm{comp}_{A,\mathrm{sol}}(G).$$

If X is constrained then $K = L \in \mathrm{comp}_A(C_G(\langle \, a, b \, \rangle))$.

(b) $C^*_K(A)$ is nonabelian.

(c) The following are equivalent:

(i) $[C^*_K(A), C^*_L(A)] \neq 1$.

(ii) $[K, L] \neq 1$.

(iii) $C^*_K(A) = C^*_L(A)$.

(d) "Does not commute" is an equivalence relation on Ω.

Proof. (a). Theorem 9.1 implies that there exist unique $\widetilde{K}$ and $\widetilde{L}$ with

$$K \leq \widetilde{K} \in \mathrm{comp}_{A,\mathrm{sol}}(G) \text{ and } L \leq \widetilde{L} \in \mathrm{comp}_{A,\mathrm{sol}}(G).$$

Then $[\widetilde{K}, \widetilde{L}] \neq 1$. Using Lemma 3.2 it follows that either $\widetilde{K}$ and $\widetilde{L}$ are both semisimple or both constrained. Suppose they are both semisimple. Since distinct A-components commute, we have $\widetilde{K} = \widetilde{L}$. Put $X = \widetilde{K}$. Hence we may assume that $\widetilde{K}$ and $\widetilde{L}$ are both constrained.

Theorem 9.1 implies that $[K, a] = 1$ and $\widetilde{K} = KF(\widetilde{K})$. Suppose $[\widetilde{L}, a]$ is nonsolvable. Since $\widetilde{L}$ is an (A, sol)-component it follows that $\widetilde{L} = [\widetilde{L}, a]$. Also $\widetilde{L} \neq \widetilde{K}$ as $[\widetilde{K}, a] \leq F(\widetilde{K})$. Theorem 9.1(e) implies that $[\widetilde{K}, \widetilde{L}] = 1$, a contradiction. Thus $[\widetilde{L}, a] \leq \mathrm{sol}(\widetilde{L})$. Then $[L, a] \leq \mathrm{sol}(\widetilde{L}) \cap L \leq \mathrm{sol}(L) = Z(L)$ and Lemma 3.5 implies $[L, a] = 1$. To summarize, $[K, a] = [L, a] = 1$. Similarly $[K, b] = [L, b] = 1$. Now $K \in \Omega_a$ and $[K, a] = 1$ so $K \in \mathrm{comp}_A(C_G(a))$. As

$[K, b] = 1$ we have $K \in \mathrm{comp}_A(C_G(\langle\, a, b\,\rangle))$. Similarly $L \in \mathrm{comp}_A(C_G(\langle\, a, b\,\rangle))$. As $[K, L] \neq 1$, this forces $K = L$. The uniqueness of $\widetilde{K}$ and $\widetilde{L}$ forces $\widetilde{K} = \widetilde{L}$. Put $X = \widetilde{K}$.

(b). Lemma 6.5 implies that either $C_K(A)$ is solvable or $E(C_K(A))$ is quasisimple. Theorem 4.4(a) implies that $C_K(A)$ is nonabelian. Hence the result.

(c). Trivially (i) implies (ii). Suppose (ii) holds. Choose X as in (a). If X is constrained then $K = L$ so $C_K^*(A) = C_L^*(A)$. Suppose X is semisimple. Two applications of Lemma 6.10 imply $C_K^*(A) = C_X^*(A) = C_L^*(A)$ so (iii) holds. By (b), (iii) implies (i).

(d). Trivially, $C_K^*(A) = C_L^*(A)$ defines an equivalence relation on Ω.

The reader is assumed to be familiar with elementary Signalizer Functor Theory, for example the notion of θ-subgroups. See [5]. In the following result, it is not necessary to assume G to be a K-group. It can be applied to study the θ-subgroups in a minimal counterexample to the Nonsolvable Signalizer Functor Theorem.

Theorem 10.2 (The Global Theorem). *Let r be a prime and A an elementary abelian r-group with rank at least 3. Suppose that A acts on the group G and that θ is an A-signalizer functor on G. Assume that $\theta(a)$ is a K-group for all $a \in A^\#$. For each $a \in A^\#$ let*

$$\Omega_a = \{K \in \mathrm{comp}_A(H) \mid \begin{array}{l} H \text{ is a } \theta\text{-subgroup of } G, \\ \theta(a) \leq H \text{ and} \\ H \text{ is a } K\text{-group.}\} \end{array}$$

and

$$\Omega = \bigcup_{a \in A^\#} \Omega_a.$$

For each $K \in \Omega$ set

$$C_K^*(A) = \begin{cases} C_K(A) & \text{if } C_K(A) \text{ is solvable} \\ E(C_K(A)) & \text{if } C_K(A) \text{ is nonsolvable.} \end{cases}$$

Let $K, L \in \Omega$. The following are equivalent:

(i) $[C_K^*(A), C_L^*(A)] \neq 1$.

(ii) $[K, L] \neq 1$.

(iii) $C_K^*(A) = C_L^*(A)$.

In particular, "Does not commute" is an equivalence relation on Ω.

Proof. Trivially (i) implies (ii). Also (iii) implies (i) by Theorem 10.1(b). Suppose that (ii) holds. Lemma 6.12(b), with A in the role of A^*, implies there exists $B \in \mathrm{Hyp}(A)$ with $C_K(B)$ overdiagonal and $[C_K(B), L] \neq 1$. Another application of Lemma 6.12(b), with B in the role of A^*, implies there exists $C \in \mathrm{Hyp}(B)$ with $C_L(C)$ overdiagonal and $[C_K(B), C_L(C)] \neq 1$. Then $[C_K(C), C_L(C)] \neq 1$ and Lemma 6.12(a) implies that both $C_K(C)$ and $C_L(C)$

are A-quasisimple. Now A has rank at least 3 so $C \neq 1$ and then $\theta(C)$ is a K-group. Set $M = \theta(C)$.

Since $K \in \Omega_a$ there exists a θ-subgroup H_a with $\theta(a) \leq H_a$ and $K \in \mathrm{comp}_A(H_a)$. Now K is a θ-subgroup so $C_K(C) \leq M$. In fact, $C_K(C) \trianglelefteq\trianglelefteq M \cap H_a$ since $K \trianglelefteq\trianglelefteq H_a$ so as $C_K(C)$ is A-quasisimple, we have $C_K(C) \in \mathrm{comp}_A(M \cap H_a)$. Also, $C_M(a) \leq M \cap \theta(a) \leq M \cap H_a$. Similarly, there exists a θ-subgroup H_b with $C_L(C) \in \mathrm{comp}_A(M \cap H_b)$ and $C_M(b) \leq M \cap H_b$. The Local Theorem, with M, $C_K(C)$ and $C_L(C)$ in the roles of G, K and L respectively, implies that $C_K^*(A) = C_L^*(A)$, so (iii) holds.

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