Anelasticity in Flexure Strips Revisited

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Abstract

This note reviews previous analyses by the author of the damping produced by the anelasticity of a simple flexure element that is loaded in tension by an extended object such as a beam balance. The correct calculation of the anelasticity of a simple flexure appeared in an appendix in Quinn et al (1995)[10] where the change in the gravitational potential energy due to the shortening of the flexure was calculated enabling expressions for the elastic energy and its associated losses to be derived. Publications prior to this paper did not include this lossless term which led to incorrect predictions of the anelastic losses in flexure pivots in Quinn et al (1987) [7]. In this current paper the derivation of the result is given in such a way that it can be easily contrasted with the expressions in these earlier papers. I also extend the methodology to calculate the elastic and gravitational energy associated with the motion of a suspended object whose dimensions are significantly smaller than the length of the flexure.

PACS:

1   Introduction

Many physical measurements are derived from mechanical oscillators. For example, the measurement of mass relies on the precision achievable in common balances and thermal noise in mechanical suspensions plays a key role in the sensitivity of gravitational wave detectors. It has been known for some time that anelasticity in material suspensions produces non-linearity and is also a source of thermal noise ([7],[13],[11]). Derivation of the key equations describing the elastic behaviour of a simple flexure-strip supporting a beam balance has been given in previous papers [7, 12, 9, 8]. However, looking over these papers, that were published now more than 20 years ago, it appears appropriate to collect

the key results together and present them in a coherent way. In particular a treatment of anelasticity that included the change in gravitation potential energy due to the shortening of the bending flexure was given in an appendix in reference [10]. In this current paper I will revisit the results of these earlier papers and present them in the context of the work reported in the 1995 paper to provide confirmation of the correct result. I will also use the same methodology as used in reference [10]to derive expressions for the anelastic losses in a simple pendulum suspension.

2   Key Equations

I will derive the general equations governing the quasi-static stiffness of a simple flexure element of uniform cross section. As shown in Figure 1, a flexure-strip, of second moment of area H and length L, supports a load of weight W. We will assume that the flexure has a rectangular cross-section with width b and thickness t and in this case we have $H = \frac{bt^3}{12}$. A torque, τ, and a horizontal force, F, are applied to the free end of the flexure at $x = L$. This results in a reaction force $-F$ and a reaction torque τ_0 at the upper end of the flexure. As described in reference [9] and basic undergraduate texts, provided that the radius of curvature of the flexure is much larger than its thickness, Hooke's law can be used to equate the moment of the forces, acting at any point, to the curvature of the flexure, which in turn can be found in terms of the stress distribution across its cross-section. This method is that adopted by previous authors (see references given in [9]), however in reference [9], we discuss limitations to this method such as those imposed due to the finite width of the flexure and its Poisson's ratio. The equation describing the bending of the flexural element is given, more generally, by a fourth order equation but, here, we ignore any shear forces due to a distributed load (*ie* the mass per unit length of the flexure is considered to be negligible). Thus our approach differs from the analyses of other authors: in reference [6], for example, the problem of the dynamic excitation of the suspension elements for mirrors for gravitational wave detectors is analysed and the mass per unit length of the suspensions is key to the analysis.

We therefore seek solutions to the second order differential equation specifying the bending moment as a function of position that can be written in terms of the torque, τ, acting at the free end and the applied forces,

$$M\left(x\right) = EH\frac{d^2y}{dx^2} = \tau + F\left(L - x\right) - W\left(y(L) - y(x)\right) \tag{1}$$

where E is the Young's modulus of the flexure. A solution to this problem, that satisfies the boundary conditions given by the positions and tangents at both ends of the flexure is,

$$y = \frac{F}{\alpha W}\left(\tanh \alpha L\left(\cosh \alpha x - 1\right) + \alpha x - \sinh \alpha x\right) + \frac{\tau}{W\cosh \alpha L}\left(\cosh \alpha x - 1\right) \tag{2}$$

where $\alpha^2 = \frac{W}{EH}$. Plots of equation 2for the cases where there are either torques or forces applied to the flexure are shown in Figures 2a and 2b where the physical values of the parameters that characterise the flexure are those of the flexure described in [9]. This solution can then be used to express the tangent angle, θ_0 and transverse displacement, y_0, as shown in Figure 1, in terms of the applied torque and force. It is convenient to write these relations in the form a compliance matrix, $\underline{\underline{C}}$,

$$\begin{pmatrix} y_0 \\ \theta_0 \end{pmatrix} = \underline{\underline{C}} \begin{pmatrix} F \\ \tau \end{pmatrix}, \tag{3}$$

with

$$\underline{\underline{C}} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \tag{4}$$

where

$$C_{11} = \frac{\alpha L \cosh \alpha L - \sinh \alpha L}{\alpha W \cosh \alpha L}, \tag{5}$$

$$C_{12} = C_{21} = \frac{\cosh \alpha L - 1}{W \cosh \alpha L}, \tag{6}$$

and

$$C_{22} = \frac{\alpha}{W} \tanh \alpha L. \tag{7}$$
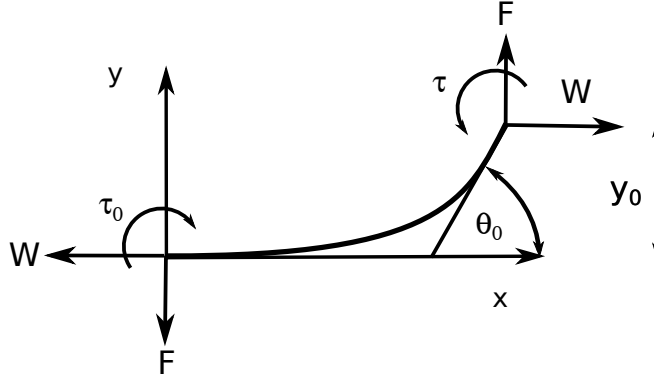
Figure 1 A schematic the flexure element. The flexure is loaded with force, W, due to the weight of the suspended object, and torques due to a horizontal force, F, and a torque, τ. This results in a transverse displacement y_0 and and an angular deflection θ_0 at the free end of the flexure. Note that Earth's gravity is vector points in the positive x direction.

The shape of a flexure with an applied torque

Figure 2a

The shape of a flexure with an applied force

Figure 2b.

Figures 2a and 2b show the shape of the flexure as given by equation 2 for the cases where a simple torque or force is applied respectively. In Figure 2a the torque has an abitrary magnitude of $10^{-3}Nm$ and in Figure 2b the force applied has an arbitrary magnitude of $10^{-3}N$. The flexure dimensions are given as $t = 50\mu m$, $b = 24mm$, $L = 20mm$ with $E = 130GPa$ and $W = 40N$.

In our previous analyses we have ignored the force, F, and the omission of the applied force is an approximation that can be understood as follows. The suspended object is assumed to be oscillating in Figure 1 and has arrived at the deflection shown there by moving throu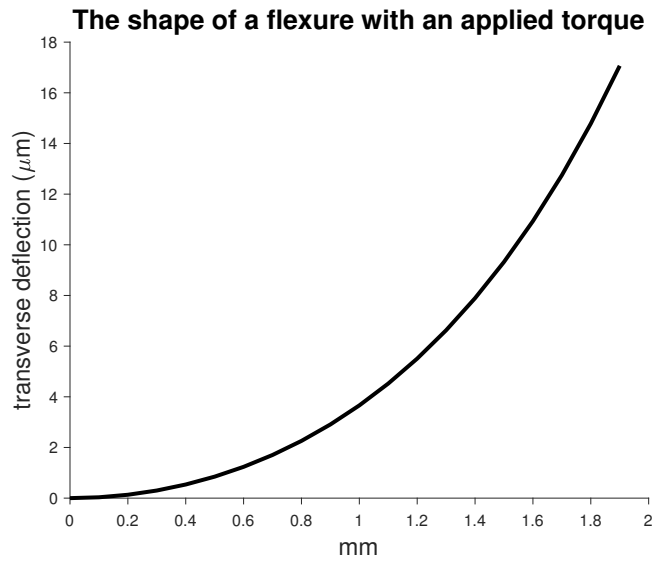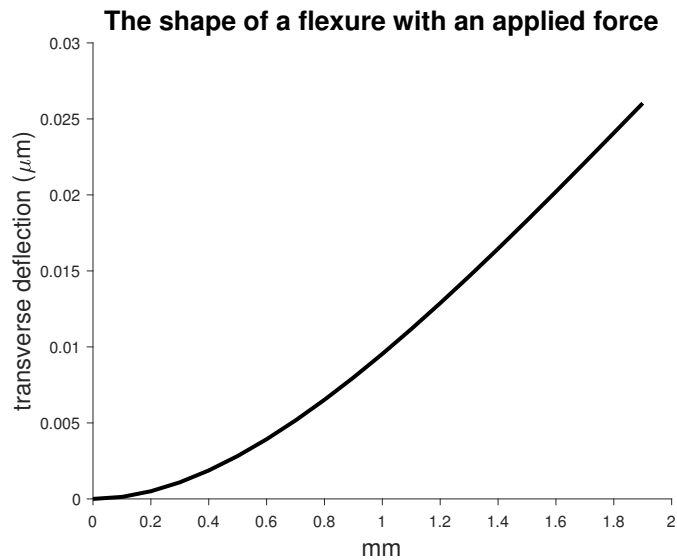gh angle θ_0. The force acting on the end of the flexure due to this motion is the inertial force proportional to the object's linear acceleration. On the other hand, the torque, τ, is due to the angular inertial acceleration and is proportional to the objects moment of inertia. In the limit that the object's radius of gyration is much larger than the distance of its centre of mass from the centre of rotation (given as R_f in the discussion below) it is reasonable to assume that the suspended object applies a pure torque to the end of the flexure and that the force is negligible. When an extended object such as a beam-balance is attached to the flexure, as was the case in our previous work, we can consider that a pure torque is being applied, on the other hand when the suspended object is a point mass we can assume that only a force acts. It is worth noting that a complete treatment of the dynamical equations of a simple flexure that is supporting a mass of a given moment of inertia has been analysed by Haag [3]. In this analysis it is shown that the inertial stability of the oscillating bob can change the shape, effective rotation axis and the energy stored in the flexure. This full analysis (that has to be treated numerically) is not necessary for quasi-dc applications as is the case here.

In what follows we will use an superscript * to indicate a result that is revised in Section III. The well-known result (see for example [9]) is that the bending stiffness, κ_t, is

$$\kappa_t = \frac{\tau}{\theta_0} = \frac{1}{C_{22}} = \frac{W}{\alpha}\coth\alpha L, \tag{8}$$

This result follows from equation 3 by setting $F = 0$. It is evident that the stiffness of the flexure does not depend on its length in the limit that $\alpha L \gg 1$ and for this reason, in reference [9], the product αL was chosen to be approximately two. It is helpful to keep the length of flexure as small as possible as this increases the stiffness of higher order modes and reduces their coupling to the simple rotational mode that we are modelling. It is tempting to consider that the trajectory of the supported object is determined by an effective radius that is determined by the ratio of transverse deflection, y_0, and the angular deflection, θ_0. We could define

$$r_f^* = \frac{y_0}{\theta_0} = \frac{C_{12}}{C_{22}} = \alpha^{-1}\tanh\left(\frac{\alpha L}{2}\right). \tag{9}$$

Further we could then assume that the system behaves as a compound pendulum and possesses a gravitational potential energy determined by this length. A gravitational stiffness, κ_g^*, could then be defined as

$$\kappa_g^* = W r_f^* = \frac{W}{\alpha}\tanh\left(\frac{\alpha L}{2}\right) \tag{10}$$

The total stiffness, κ_t, then comprises the gravitational term, κ_g^*, and another

term that could be considered to be the 'intrinsic' elastic stiffness of the flexure, κ_{el}^*, where

$$\kappa_t = \kappa_{el}^* + \kappa_g^*. \tag{11}$$

Hence we find

$$\kappa_{el}^* = \frac{W}{\alpha} csh(\alpha L). \tag{12}$$

This type of analysis has also been used to derive expressions for the stiffnesses of end suspensions of beam balances in reference [9]. We will revise these equations in Section III below.

It is generally accepted that a significant contribution to the damping of the motion of flexures comes from material anelasticity. Of particular importance to the field of mass metrology is the anelastic behaviour of a beam balance when it is subjected to a change in stress. This is referred to as the anelastic after-effect. Anelasticity, although observed by Henry Cavendish [2], is analysed in detail in connection with measurements of weak forces in reference [7]. Following reference [7], the stiffness can be expanded in terms of a frequency dependent modulus defect $\delta E(\omega)$,

$$\kappa_{el}(E) = \kappa_{el}(E_0 + \delta E(\omega)) \simeq \kappa_{el}(E_0) + \frac{d\kappa_{el}}{dE} \delta E(\omega), \tag{13}$$

where E_0 is the component of the Young's modulus that obey's Hooke's law. It is also useful to define

$$\kappa_{anel} = \left[-\frac{\alpha}{2} \frac{d\kappa_{el}}{d\alpha} \right] \frac{\delta E(\omega)}{E_0}. \tag{14}$$

The frequency dependent part of the modulus defect, $\delta E(\omega)$, has a real and imaginary part. The latter, $\delta E^i(\omega)$, is responsible for damping. In reference [7] we were only interested in the damping and the anelastic after-effect and thus we ignored the real part of the modulus defect. We assumed that the total modulus defect comprised the sum of a distribution of relaxation processes that all have the same relaxation strength, δe, where

$$\delta E^i(\omega) = \delta e \int_{\tau_0}^{\tau_\infty} f(\tau) \frac{\omega\tau}{1 + \omega^2\tau^2} d\tau. \tag{15}$$

The weighting factor, $f(\tau)$, of the dissipation mechanisms of relaxation time τ (not to be confused here with the applied torque) is normalised as follows[1],

$$\int_{\tau_0}^{\tau_\infty} f(\tau) d\tau = 1. \tag{16}$$

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[1]This distribution of relaxation processes is discussed in [5]. However Wiechert [15] (1893) appears to have been the first to have modelled an arbitrary spectrum of such processes. To the best of the knowledge of the current author, the authors of reference [7] were the first to apply a general model of anelasticity to the low frequency behaviour of metals.

Our measurements, that extended to oscillation periods of 690 s, indicated that the imaginary component of the modulus was independent of frequency even at these low frequencies. This behaviour is reproduced with

$$f(\tau) = \frac{1}{\ln(\tau_\infty/\tau_0)} \frac{1}{\tau},$$ (17)

and for $\omega\tau_0 \ll 1$ and $\omega\tau_\infty \gg 1$. Thus by performing the integrations (see [7]) we can define

$$\delta E^i(\omega) = \delta e \frac{\pi}{2\ln(\tau_\infty/\tau_0)}.$$ (18)

Using 12, we find

$$\left[-\frac{\alpha}{2}\frac{d\kappa_{el}}{d\alpha}\right] = \frac{W}{2\alpha}\frac{1}{\sinh\alpha L}(1 + \alpha L \coth \alpha L).$$ (19)

We can find the variation of κ^*_{anel} with the change in length, L, by expanding 19 around $L \approx \alpha^{-1}$, for constant load and flexure geometry. We find

$$\kappa^*_{anel} = i\frac{\delta E^i}{E_0}\frac{W}{\alpha}\left(0.98 - 1.60(\alpha L - 1) + O(\alpha L - 1)^2\right).$$ (20)

We also can show that κ^*_{anel} tends to zero as L tends to infinity. This was a result that was revised in the later publication [10]and is discussed further below.

See references [7] and [14] for more details of the calculation of the anelastic-after-effect and the damping produced by anelasticity and references to the work of other authors.

3 Revision of these equations

Section II derives the results that were developed in our papers up until reference [10] in 1995. The result of equations 12 and 20 is that the anelastic stiffness reduces exponentially as the flexure becomes longer or thinner. It makes sense physically that the damping reduces to zero as the second moment are area reduces to zero. We do not, however, expect that it reduces by simply making the flexure-strip longer as the bending moment at the end of the flexure cannot physically reduce to zero as the flexure becomes longer. This is consistent with the stiffness of the flexure becoming independent of the length of the flexure in the limit that $L \gg \alpha^{-1}$. The calculations given above were also not consistent with experimental results obtained in [10]. At this point I tried another approach where the elastic stored energy and the gravitational energy were treated separately and, importantly the change in gravitational potential energy was calculated as being due to the change in height of the load, W, as the flexure deforms. The stored elastic energy can be written (see Appendix A in [10]),

$$V_{el} = \frac{1}{2EH}\int_0^L M^2(x)\,dx,$$ (21)

where again $M(x)$ is the bending moment:

$$M(x) = EH \frac{d^2 y}{dx^2}. \tag{22}$$

If we confine ourselves to the situation where F is insignificant in equation 3 then from equation 2 and 7,

$$y(x) = \frac{\theta_0}{\alpha} \frac{(\cosh(\alpha x) - 1)}{\sinh(\alpha L)}. \tag{23}$$

Thus we can show that

$$V_{el} = \frac{1}{2} \frac{W}{\alpha} \theta^2 \frac{1}{2} \left(\coth(\alpha L) + \frac{\alpha L}{\sinh^2(\alpha L)} \right), \tag{24}$$

where we have deliberately separated out the two factors of one half. We can define an elastic stiffness that comes from this approach as

$$\kappa_{el} = \frac{W}{2\alpha} \left(\coth(\alpha L) + \frac{\alpha L}{\sinh^2(\alpha L)} \right). \tag{25}$$

The bending of the flexure results in its length projected onto the vertical direction being shortened. To second order in the bending angle, the change in vertical height of the suspended load can be calculated to be

$$\Delta L = \frac{1}{2} \int_0^L \left(\frac{dy(x)}{dx} \right)^2 dx. \tag{26}$$

This results in a change in the gravitational energy of

$$V_g = W \Delta L = \frac{1}{2} \frac{W}{\alpha} \theta^2 \frac{1}{2} \left(\coth(\alpha L) - \frac{\alpha L}{\sinh^2(\alpha L)} \right), \tag{27}$$

with an associated gravitational stiffness,

$$\kappa_g = \frac{W}{2\alpha} \left(\coth(\alpha L) - \frac{\alpha L}{\sinh^2 \alpha L} \right). \tag{28}$$

We can define a radius of rotation as in the previous analysis in Section 2,

$$r_f = \frac{y_0}{\theta_0} = \frac{\kappa_g}{W} = \frac{1}{2\alpha} \left(\coth \alpha L - \frac{\alpha L}{\sinh^2 \alpha L} \right). \tag{29}$$

We can see that

$$V_g + V_{el} = \frac{1}{2} \kappa_t \theta^2, \tag{30}$$

and

$$\kappa_t = \kappa_g + \kappa_{el}. \tag{31}$$

We now assume that energy loss from the oscillations of the pendulum due to non-ideality of the flexure material, such as anelasticity, originates solely in the elastic energy and that no energy loss occurs in the gravitational energy. Clearly changes in the magnitude of the Young's modulus can modulate the gravitational potential energy but this energy is not dissipated in the system via anelasticity due to the bending of the flexure. It is important to note that the equations from21 including those through to equation31, where the gravitational energy term is calculated in terms of the shortening of the flexure as it bends, appeared in the appendix of [10]. In this current paper we are reiterating the importance of these results and explaining further their interpretation.

These equations lead to the same result for the total stiffness as in equation 8 but the elastic and gravitational stiffnesses differ from the previous analysis, as given by equations 24 and 31 compared with equations 12 and 24. However the geometrical interpretation of r_f^* is a simple function of the geometry of the stressed flexure and its use to estimate the position of the effective centre of rotation of the compound pendulum remains valid. This is an important parameter when designing a device that has minimum sensitivity to horizontal ground vibrations and tilt. We will therefore refer to r_f^* as R_f, which we can consider to be the location of the effective pivot axis, as measured from the end of the flexure. The length R_f is also the parameter that enters into the calculation of the moment of inertia of the suspended object about the effective point of rotation. The importance of r_f lies in its use for calculating the gravitational potential energy following equations 28 and 29. We can calculate the ratio r_f to r_f^* and express the result numerically for the usual case when $L \approx \alpha^{-1}$,

$$\frac{r_f^*}{r_f} \approx 1.57 + 0.13(\alpha L - 1) + O\left((\alpha L - 1)^2\right) \ldots \tag{32}$$

The previous analysis therefore overerestimates the value of r_f by a factor of 1.57. Therefore using r_f^* to estimate the gravitational potential energy of the system results in an error of some 57%.

Now that we have adequately accounted for the gravitational energy involved in the flexure bending, as was already given in the 1995 publication, we can calculate the anelasticity from the stored elastic energy. We can use equation 24 to calculate the anelasticity of the flexure in terms of the elastic stiffness. We find that

$$\left[-\frac{\alpha}{2}\frac{d\kappa_{el}}{d\alpha}\right] = \frac{W}{4\alpha}\left(\coth\left(\alpha L\right) + \frac{\alpha L}{\sinh^2 \alpha L} + \frac{2\left(\alpha L\right)^2 \coth \alpha L}{\sinh^2 \alpha L}\right). \tag{33}$$

We can present this result numerically in the case of $L \approx \alpha^{-1}$, as follows,

$$\kappa_{anel} = i\frac{\delta E^i}{E}\frac{W}{\alpha}\left(0.98 - 1.04\left(\alpha L - 1\right) + O(\alpha L - 1)^2\right), \tag{34}$$

The anelastic component of the flexure stiffness behaves in similar way to the expression given for κ_{anal}^* in the previous section for the case where $L \approx \alpha^{-1}$. However, for $L \gg \alpha^{-1}$, the damping now tends to a value of 0.25 and this expression for the anelastic stiffness is physically reasonable.

9

4 Further calculations

The results above are all relevant for the case where the suspended object has a radius of gyration large compared with the radius R_f. A natural extension is to consider the case where the supported load is essentially a point mass. We can use similar methods as described above but follow the calculation through with $\tau = 0$ instead of $F = 0$.

The stiffness of the flexure can be defined as

$$K_t = \frac{F}{y_0} = \alpha W \frac{\cosh \alpha L}{\alpha L \cosh \alpha L - \sinh \alpha L}. \tag{35}$$

It should be noted that this expression states that the stiffness of the flexure against transverse forces becomes infinite as the flexure length tends to zero. This is to be expected. We can also find the length of a simple pendulum,

$$R_f = \frac{y_0}{\theta_0} = \frac{1}{\alpha} \frac{\alpha L \cosh \alpha L - \sinh \alpha L}{\cosh \alpha L - 1}. \tag{36}$$

It is interesting to note that R_f tends to $L - 1/\alpha$ as αL tends to infinity. This is consistent with the work of Lorenzini et al [4]. The elastic component of the stiffness is

$$K_{el} = \frac{\alpha W}{4} \cdot \frac{\sinh 2\alpha L - 2\alpha L}{(\alpha L \cosh \alpha L - \sinh \alpha L)^2}. \tag{37}$$

The gravitational stiffness can be written

$$K_g = \frac{\alpha W}{4} \cdot \frac{(2\alpha L (\cosh 2\alpha L + 2) - 3 \sinh 2\alpha L)}{(\alpha L \cosh \alpha L - \sinh \alpha L)^2}. \tag{38}$$

The frequency dependent component of the elastic stiffness can be calculated straightforwardly by differentiation of the K_{el} however the resulting expression is unwieldy so we will write it out as an expansion in the case of $\alpha L \approx 1$,

$$K_{anel} = i \frac{\delta E^i}{dE} \alpha W \left(2.995 - 9.00 \, (\alpha L - 1) + O \left((\alpha L - 1)^2 \right) \right). \tag{39}$$

The surprising thing here is that the leading term is not unity for this case. However this appears to be the result. Equation 39 indicates that the expansion is not reliable for values of αL that differ too much from unity. Finally, in the case that the $L \gg \alpha^{-1}$, we find that K_{anel} tends to zero. So in the case of point mass load the damping term tends to zero at the length of the flexure increases. This may seem unphysical as did the previous incorrect result for the case of the pure applied torque when we used the incorrect form of the gravitational potential energy. Clearly as the length of the flexure increases the energy stored in the flexure due to a finite transverse displacement tends to zero. However any finite load must have a finite moment of inertia and therefore there will always be some damping due to the applied torque.

At this point the quality factor of the suspension could be calculated as a ratio of the dissipated to stored energy for the different bending modes of the flexure. This straighforward but is beyond the scope of of this paper. These results agree with the paper of Cagnoli et al (2000)[1].

Conclusions

This paper has attempted to clarify some results that were aimed at understanding the quasi-static behaviour of flexure elements in their role as low-loss elastic pivots for supporting beam-balances. I have repeated the previous calculations that were published some 20 to 30 years ago and replaced them, with hindsight, with more accurate results (that are now hopefully correct!). But note that the correct calculation was previously published in an appendix in reference [10]. I have also extended the work to cover the case where the object suspended from the flexure has a negligible radius of gyration. This may be of use in understanding the damping of simple pendulum suspensions. The behaviour of the predicted damping as a function of the length of the flexure now appears to be physically reasonable in both cases.

It is worth pointing out that the above methods can be easily adapted to inverted pendulums by switching the direction of gravity in the problem and finding the solutions in terms of *sine* and *cosine* functions.

Acknowledgements

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