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REDUCED FUSION SYSTEMS OVER p-GROUPS WITH ABELIAN SUBGROUP OF INDEX p: II

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Abstract. Let p be an odd prime, and let S be a p-group with a unique elementary abelian subgroup A of index p. We classify the simple fusion systems over all such groups S in which A is essential. The resulting list, which depends on the classification of finite simple groups, includes a large variety of new, exotic simple fusion systems.

A saturated fusion system $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, whose morphisms are monomorphisms between subgroups, and whose morphism sets satisfy certain axioms, formulated originally by Puig and motivated by properties of conjugacy relations between p-subgroups of a finite group. For example, to each finite group G and each Sylow p-subgroup S of G, one associates the saturated fusion system $\mathcal{F}_S(G)$ over S whose morphisms are those homomorphisms induced by conjugation in G. We refer to [AKO] or [Cr1] for a detailed introduction to the theory of saturated fusion systems, and to the beginning of Section 1 for a little more detail about these definitions.

A saturated fusion system $\mathcal{F}$ is *simple* if it contains no non-trivial proper normal subsystems (see [AKO, Definition I.6.1] or [Cr1, Sections 5.4 and 8.1] for the precise definition of a normal subsystem). In this paper, we continue the study, started in [Ol], of simple fusion systems $\mathcal{F}$ over non-abelian p-groups S which have an abelian subgroup of index p. For $p = 2$, this was handled in [AOV, Proposition 5.2(a)]: S must be dihedral, semidihedral or wreathed, and $\mathcal{F}$ must be the 2-fusion system of $PSL_2(q)$ for $q \equiv \pm 1 \pmod 8$ or of $PSL_3(q)$ for q odd. So assume p is an odd prime. If S has more than one such subgroup, then $|S| = p^3$ by [Ol, Theorem 2.1], and this case was dealt with earlier in [RV]. Thus we assume that S has a *unique* abelian subgroup of index p, which we denote A. If A is not $\mathcal{F}$-essential (see Definition 1.1), then we are in the situation of [Ol], and [Ol, Theorem 2.8] gives a complete characterization of simple fusion systems on S. In contrast to the situation when $p = 2$, most of the fusion systems found in [Ol] are *exotic*; i.e., they are not fusion systems of finite groups.

In this paper, we handle the case when A is an $\mathcal{F}$-essential subgroup and has exponent p, and again find a very large variety of exotic fusion systems. Our main tool is Theorem 2.8, which gives precise details concerning the way in which the structure of $\mathcal{F}$ is controlled by the action of $\mathrm{Aut}_{\mathcal{F}}(A)$ on A. Indeed, Theorem 2.8 and its Corollary 2.10 reduce the problem of classifying fusion systems on S to that of determining all pairs (G, A), where G is a finite group (the candidate for $\mathrm{Aut}_{\mathcal{F}}(A)$) and A is an $\mathbb{F}_pG$-module that satisfy a certain list of conditions.

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The bulk of our analysis is thus centred on classifying modules satisfying the required conditions. After some preliminary results in Section 3, the main results are summarized in Theorem 4.1 and Table 4.1. Certain cases are then covered in more detail in Propositions 4.2, 4.3, and 4.4. By combining these results with Theorem 2.8, one can get a complete list of all simple fusion systems of the type described. (A few explicit examples are worked out at the end of Section 4.) Most of the results involving lists of modules depend on the classification of finite simple groups (CFSG), which is thus assumed throughout Sections 4 and 5 and also in Lemma 1.7.

Our strategy for listing modules is based on Aschbacher's classification [A1] of subgroups $G < GL_n(p)$. This splits into two cases, according to whether or not the image of G in $PGL_n(p)$ is almost simple. If it is not almost simple, Aschbacher gives a short list of possibilities, which we make more explicit (in our situation) in Section 5 (Propositions 5.3 and 5.4). When $G/Z(G)$ is almost simple, we use CFSG to check each of the possibilities for G in Sections 6–11.

We are still left with the case where S has a unique abelian subgroup of index p which is $\mathcal{F}$-essential and of exponent greater than p. The second author plans to handle this in a later paper with Albert Ruiz.

Our notation is mostly standard. For example, $p^{1+2k}_{\pm}$ denotes an extraspecial group of order p^{2k+1}, where (for odd p) p^{1+2k}_{+} has exponent p and p^{1+2k}_{-} exponent p^2. Also, ${}^{x}g = xgx^{-1}$, $g^x = x^{-1}gx$, G' is the derived (commutator) subgroup of G, and $A \circ B$ denotes a central product of groups A and B.

As usual, $F(G)$ denotes the Fitting subgroup of the finite group G: the largest normal nilpotent subgroup of G (i.e., the product of the subgroups $O_p(G)$ for all p). Also, $E(G)$ denotes the layer: the central product of the components of G (the subnormal quasisimple subgroups). Thus $F^*(G) = F(G)E(G)$ is the generalized Fitting subgroup.

The following definition will be useful in Sections 6–11.

Definition 0.1. Let H be a finite simple group. A finite group G is *of type* H if $Z(G)$ is cyclic and $F^*(G)/Z(G) \cong H$.

1. Background

We begin by recalling some definitions. When G is a finite group and $S \in \mathrm{Syl}_p(G)$, the *p-fusion system of* G is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S, and where for $P, Q \leq S$,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) &= \mathrm{Hom}_G(P,Q)\\ &\overset{\mathrm{def}}{=} \bigl\{\varphi \in \mathrm{Hom}(P,Q) \bigm| \varphi = c_g = (x \mapsto {}^{g}x) \text{ for some } g \in G \text{ such that } {}^{g}P \leq Q\bigr\}.\end{aligned}$$

More generally, a *saturated fusion system* over a p-group S is a category $\mathcal{F}$ whose objects are the subgroups of S, where $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ is a set of injective homomorphisms from P to Q for each $P, Q \leq S$, and which satisfies certain axioms, due originally to Puig, and listed (in the form we use them) in [AKO, Definition I.2.2] and [Cr1, § 4.1].

The following terminology is used to describe certain subgroups in a fusion system.

Definition 1.1. Fix a prime p, a finite p-group S, and a saturated fusion system $\mathcal{F}$ over S. Let $P \leq S$ be any subgroup.

- $P^{\mathcal{F}}$ denotes the set of subgroups of S which are $\mathcal{F}$-conjugate (isomorphic in $\mathcal{F}$) to P. Also, $g^{\mathcal{F}}$ denotes the $\mathcal{F}$-conjugacy class of an element $g \in S$ (the set of images of g under morphisms in $\mathcal{F}$).

- P is *fully normalized* in $\mathcal{F}$ (*fully centralized* in $\mathcal{F}$) if $|N_S(P)| \geq |N_S(Q)|$ ($|C_S(P)| \geq |C_S(Q)|$) for each $Q \in P^{\mathcal{F}}$.
- P is *$\mathcal{F}$-centric* if $C_S(Q) = Z(Q)$ for each $Q \in P^{\mathcal{F}}$.
- P is *$\mathcal{F}$-essential* if $P < S$, P is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and $\mathrm{Out}_{\mathcal{F}}(P) \stackrel{\text{def}}{=} \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ contains a strongly p-embedded subgroup. Here, a proper subgroup $H < G$ of a finite group G is *strongly p-embedded* if $p \bigm| |H|$, and $p \nmid |H \cap gHg^{-1}|$ for each $g \in G \smallsetminus H$.
  Let $\mathbf{E}_{\mathcal{F}}$ denote the set of all $\mathcal{F}$-essential subgroups of S.
- P is *normal in $\mathcal{F}$* if each morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ in $\mathcal{F}$ extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, PR)$ such that $\overline{\varphi}(P) = P$. The maximal normal p-subgroup of a saturated fusion system $\mathcal{F}$ is denoted $O_p(\mathcal{F})$.
- P is *strongly closed in $\mathcal{F}$* if for each $g \in P$, $g^{\mathcal{F}} \subseteq P$.
- $\mathfrak{foc}(\mathcal{F}) = \bigl\langle gh^{-1} \bigm| g \in S,\ h \in g^{\mathcal{F}} \bigr\rangle$.

The above definition of $\mathcal{F}$-essential subgroups is motivated by the following version for fusion systems, due to Puig, of the Alperin–Goldschmidt fusion theorem.

Theorem 1.2 ([AKO, Theorem I.3.5], [Cr1, Theorem 4.51])**.** *For each saturated fusion system $\mathcal{F}$ over a finite p-group S, each morphism in $\mathcal{F}$ is a composite of restrictions of $\mathcal{F}$-automorphisms of S and of $\mathcal{F}$-essential subgroups of S.*

Let $O^p(\mathcal{F})$ and $O^{p'}(\mathcal{F})$ denote the smallest normal fusion subsystems of p-power index, and of index prime to p, respectively. Such normal subsystems are defined by analogy with finite groups, and we refer to [AKO, § I.7] or [Cr1, § 7.5] for their precise definitions and properties.

Definition 1.3. For any saturated fusion system $\mathcal{F}$,

- $\mathcal{F}$ is *reduced* if $O_p(\mathcal{F}) = 1$, $O^p(\mathcal{F}) = \mathcal{F}$, and $O^{p'}(\mathcal{F}) = \mathcal{F}$;
- $\mathcal{F}$ is *simple* if it contains no non-trivial proper normal fusion subsystems, in the sense of [AKO, Definition I.6.1] or [Cr1, §§ 5.4 & 8.1]; and
- $\mathcal{F}$ is *realizable* if $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with Sylow p-subgroup S.

For any saturated fusion system $\mathcal{F}$ over S, $O^p(\mathcal{F})$, $O^{p'}(\mathcal{F})$, and $\mathcal{F}_{O_p(\mathcal{F})}(O_p(\mathcal{F}))$ are all normal subsystems of $\mathcal{F}$. Hence $\mathcal{F}$ is reduced if it is simple. If $\mathcal{E} \trianglelefteq \mathcal{F}$ is a normal subsystem over the subgroup $T \trianglelefteq S$, then by definition of normality, T is strongly closed in $\mathcal{F}$. Thus a reduced fusion system is simple if it has no proper non-trivial strongly closed subgroups.

The next proposition gives some very general conditions for a fusion system to be reduced.

Proposition 1.4. *The following hold for a saturated fusion system $\mathcal{F}$ over a finite p-group S.*

(a) *For each $Q \trianglelefteq S$, $Q \trianglelefteq \mathcal{F}$ if and only if for each $P \in \mathbf{E}_{\mathcal{F}} \cup \{S\}$, $Q \leq P$ and Q is $\mathrm{Aut}_{\mathcal{F}}(P)$-invariant.*

(b) *We have $\mathfrak{foc}(\mathcal{F}) = \bigl\langle [P, \mathrm{Aut}_{\mathcal{F}}(P)] \bigm| P \in \mathbf{E}_{\mathcal{F}} \cup \{S\} \bigr\rangle$.*

(c) *In all cases, $O^p(\mathcal{F}) = \mathcal{F}$ if and only if $\mathfrak{foc}(\mathcal{F}) = S$.*

(d) *If each $P \in \mathbf{E}_{\mathcal{F}}$ is minimal in the set of all $\mathcal{F}$-centric subgroups, then $O^{p'}(\mathcal{F}) = \mathcal{F}$ if and only if* $\mathrm{Aut}_{\mathcal{F}}(S) = \langle \mathrm{Inn}(S), \mathrm{Aut}_{\mathcal{F}}^{(P)}(S) \,|\, P \in \mathbf{E}_{\mathcal{F}} \rangle$, *where for $P \leq S$,*

$$\mathrm{Aut}_{\mathcal{F}}^{(P)}(S) = \left\{ \alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \,\middle|\, \alpha(P) = P,\ \alpha|_P \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) \right\}.$$

Proof. Point (a) is shown in [AKO, Proposition I.4.5], and point (c) in [AKO, Corollary I.7.5]. Point (b) follows from the definition and Alperin's fusion theorem (Theorem 1.2). Point (d) is shown in [Ol, Lemma 1.4]. □

In order to be able to identify which of the fusion systems we construct are realizable, it will be helpful to know that when realizable, they can be realized by finite simple groups.

Lemma 1.5 ([DRV], [Ol, Lemma 1.5])**.** *Let $\mathcal{F}$ be a reduced fusion system over a p-group S. Assume, for each strongly $\mathcal{F}$-closed subgroup $1 \neq P \trianglelefteq S$, that P is centric in S, is not elementary abelian, and does not factor as a product of two or more subgroups which are permuted transitively by* $\mathrm{Aut}_{\mathcal{F}}(P)$*. Under these conditions, if $\mathcal{F}$ is realizable, then it is the fusion system of a finite simple group.*

The next lemma gives a very simple, necessary condition for a p-group S to have an abelian subgroup of index p.

Lemma 1.6. *Let p be any prime, and let S be a non-abelian p-group which contains an abelian subgroup of index p. Then $|Z(S)|\cdot|[S,S]| = \frac{1}{p}|S|$.*

Proof. Let $A \trianglelefteq S$ be an abelian subgroup of index p, fix $x \in S \smallsetminus A$, and let $\varphi \in \mathrm{End}(A)$ be the homomorphism $\varphi(a) = [a, x]$. Then $Z(S) = C_A(x) = \mathrm{Ker}(\varphi)$, and $[S,S] = [x, A] = \mathrm{Im}(\varphi)$. □

Lemma 1.5 motivates the next lemma: a list of all finite simple groups whose fusion systems are of the type we are studying.

Lemma 1.7. *Fix an odd prime p. Let G be a known finite simple group such that $S \in \mathrm{Syl}_p(G)$ is nonabelian, and contains a unique abelian subgroup A of index p. Set $\mathcal{F} = \mathcal{F}_S(G)$, and assume $\mathcal{F}$ is reduced. Then $\mathcal{F}$ is isomorphic to the fusion system of one of the following simple groups:*

(a) A_{pn}, *where $p \leq n < 2p$;*

(b) $Sp_4(p)$*;*

(c) $PSL_n(q)$, *where $p|(q-1)$ and $p \leq n < 2p$;*

(d) $P\Omega_{2n}^+(q)$, *where $p|(q-1)$ and $p \leq n < 2p$;*

(e) ${}^3D_4(q)$ *or* ${}^2F_4(q)$, *where $p = 3$ and q is prime to 3;*

(f) $E_n(q)$, *where $p|(q-1)$, $p = 5$ if $n = 6, 7$, and $p = 7$ if $n = 7, 8$;*

(g) $E_8(q)$, *where $p = 5$ and $q \equiv \pm 2$ (mod 5); or*

(h) Co_1, *where $p = 5$.*

Proof. Since A is the unique abelian subgroup of index p, $|S| \geq p^4$.

Case 1: If G is an alternating group, then $G \cong A_n$ for some $p^2 \leq n < 2p^2$. Since $\mathcal{F} = \mathcal{F}_S(G)$ is reduced, it is also the fusion system of A_m, where $m = \sup\{k \in p\mathbb{Z} \,|\, k \leq n\}$ (see [A2, 16.5]). So we can choose $n = m$ to be a multiple of p.

Case 2: Assume that G is of Lie type in characteristic p, and fix $S \in \mathrm{Syl}_p(G)$. We prove that the only possibility is $G = PSp_4(p)$, by showing that in all other cases, either S is abelian, or it has more than one abelian subgroup of index p, or it has none at all.

Assume first that G is one of the following groups:

$$PSL_4(p) \cong \Omega_6^+(p), \quad PSU_4(p) \cong \Omega_6^-(p), \quad PSp_6(p), \quad G_2(p), \quad {}^2G_2(3^k). \tag{1}$$

We will show that S has no abelian group of index p in any of these cases. If $H < PSL_4(p)$ is the stabilizer of a projective plane and a point in the plane, then $O_p(H) \cong p_+^{1+4}$. If $G = PSU_4(p)$ or $PSp_6(p)$, and $H < G$ is the stabilizer of an isotropic point or a point, respectively, then $O_p(H) \cong p_+^{1+4}$. Thus if G is one of the groups $PSL_4(p)$, $PSU_4(p)$, or $PSp_6(p)$, then S contains an extraspecial subgroup of order p^5, and hence contains no abelian subgroup of index p.

For $p \geq 5$, $G_2(p)$ also contains an extraspecial p-group of order p^5 (see [Wi, p.127]). If $G \cong G_2(3)$, then its parabolic subgroups have the form $(C_3^2 \times 3_+^{1+2}) \rtimes GL_2(3)$. If S is contained in this group, then each abelian subgroup of S must intersect the subgroup $(C_3^2 \times 3_+^{1+2})$ with index at least 3, so that $SL_2(3)$ must act trivially in particular on the C_3^2 factor, while it actually acts as on the natural module (see [Wi, p.125]). So we eliminate these cases.

Now assume that $G \cong {}^2G_2(3^k)$ for $k \geq 3$. By the main theorem in [Wa], $|S| = (p^k)^3 = |Z(S)|\cdot|[S,S]|$. So by Lemma 1.6, S does not contain an abelian subgroup of index p.

Thus S has no abelian subgroup of index p if G is one of the groups in (1), or any group which contains one of them. In this way, we can eliminate all larger classical groups, as well as ${}^3D_4(p) > G_2(p)$, $E_n(p) > F_4(p) > \mathrm{Spin}_9(p)$ ($n = 6, 7, 8$), and ${}^2E_6(p) > F_4(p)$ (see, e.g., [Wi, Chapter 4] for descriptions of these inclusions), and also the groups of the same type over larger fields of characteristic p.

Since $PSL_2(p^k)$ has abelian Sylow p-subgroups, it remains to consider the groups

$$PSL_3(p^k), \quad PSU_3(p^k), \quad PSp_4(p^k) \cong \Omega_5(p^k), \tag{2}$$

for $k \geq 1$. The Sylow p-subgroups of $PSL_3(p)$ and $PSU_3(p)$ are extraspecial of order p^3, hence have more than one abelian subgroup of index p, while those of $PSp_4(p)$ do have a unique such subgroup. By [GLS3, Theorem 3.3.1.a], if G is one of the groups in (2), then $|S|$, $|Z(S)|$, and $|[S,S]|$ are all powers of p^k, and so if $k > 1$, S contains no abelian subgroup of index p by Lemma 1.6.

Case 3: Assume that $G = {}^r\mathbb{G}(q)$ is a group of Lie type in characteristic different from p. By [BMO2, Lemma 6.9], and since the Sylow p-subgroups of G are non-abelian, G has a p-fusion system isomorphic to that of one of the following groups:

(i) $PSL_n(q)$ for some $n \geq p$; or

(ii) $P\Omega_{2n}^{\varepsilon}(q)$, where $n \geq p$, $\varepsilon = \pm 1$, $q^n \equiv \varepsilon \pmod p$, and $\varepsilon = +1$ if n is odd; or

(iii) ${}^3D_4(q)$ or ${}^2F_4(q)$, where $p = 3$ and q is a power of 2; or

(iv) $G_2(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$, or $E_8(q)$ where $p|(q-1)$; or

(v) $E_8(q)$ where $p = 5$ and $q \equiv \pm 2 \pmod 5$.

Assume that $G = PSL_n(q)$. If $e = \mathrm{ord}_p(q) > 1$, then by [Ru, Theorem B], the p-fusion system of G has a proper normal subsystem of index e, and hence is not reduced. Thus $p|(q-1)$, and $p \leq n < 2p$ since the Sylow p-subgroups have abelian subgroups of index p. We are thus in the situation of (c).

Assume that $G = P\Omega_{2n}^{\varepsilon}(q)$ is as in (ii). Set $e = \mathrm{ord}_p(q)$. If e is even, then by [BMO1, Proposition A.3], the p-fusion system of $GO_{2n}^{\varepsilon}(q)$ is isomorphic to that of $SL_{2n}(q)$. So $\mathcal{F}$ is normal of index 2 in the fusion system of $SL_{2n}(q)$; this contains a normal subsystem of index e by [Ru, Theorem B] again, and hence $\mathcal{F}$ has a normal subsystem of index $e/2$. So $\mathcal{F}$ is reduced only if $e = 2$. If e is odd, then by [BMO1, Theorem A(a,b)], we can assume that q is a square, $GO_{2n}^{\varepsilon}(q)$ and $SL_{2n}(\sqrt{q})$ have isomorphic p-fusion systems by [BMO1, Proposition A.3] again, and so $\mathcal{F}$ is reduced only if $e = 1$.

We can thus assume that $q \equiv \pm 1 \pmod p$, and $p|(q-1)$ if n is odd (since $\varepsilon = +1$). If n is even and $q \equiv -1 \pmod p$, then by [BMO1, Theorem A(b)], G has the same p-fusion system as $P\Omega_{2n}^{\varepsilon}(q^*)$ for some $q^* \equiv 1 \pmod p$. So we can assume that $p|(q-1)$ in all cases, and are in the situation of (d).

Cases (iii) and (v) correspond to (e) and (g), respectively. In case (iv), if $G = \mathbb{G}(q)$ and $p|(q-1)$, then the order of the Weyl group of $\mathbb{G}$ must be a multiple of p but not of p^2, and so $(\mathbb{G}, p)$ is one of the pairs $(G_2, 3)$, $(E_6, 5)$, $(E_7, 5)$, $(E_7, 7)$, or $(E_8, 7)$. The 3-fusion system of $G_2(q)$ is not reduced, since it is the fusion system of $PSL_3(q){:}2$ or $PSU_3(q){:}2$ [A2, (16.11)]. So we are in the situation of (f).

Case 4: If G is a sporadic group, then by the tables in [GL, § 1.5] or [GLS3, § 5.3], in almost all cases, either $|S| \le p^3$, or S is abelian, or S contains an extraspecial group of type p^{1+2k} for $k \ge 2$, or S contains a special group of type 3^{2+4}. The exceptions are $(G, p) = (J_3, 3)$, $(Co_1, 5)$, and $(Th, 3)$. When $S \in \mathrm{Syl}_3(J_3)$, $|S| = 3^5$, $Z(S) \cong C_3^2$ and $[S, S] \cong C_3^3$ (see [FR, § 3]), so by Lemma 1.6, S does not contain an abelian subgroup of index 3. Since Th contains a subgroup isomorphic to $G_2(3)$ [Pa, (3.12)], whose Sylow 3-subgroups were already shown not to have abelian subgroups of index 3, the same holds for Th. Thus $p = 5$ and $G = Co_1$. □

We finish the section with some miscellaneous group-theoretic results that will be needed later.

Lemma 1.8. *Fix a prime p, a finite p-group P, and a group $G \le \mathrm{Aut}(P)$ of automorphisms of P. Let $\mathrm{Fr}(P) = P_0 \trianglelefteq P_1 \trianglelefteq \cdots \trianglelefteq P_m = P$ be a sequence of subgroups, all normal in P and normalized by G. Let $H \le G$ be the subgroup of those $g \in G$ which act via the identity on P_i/P_{i-1} for each $1 \le i \le m$. Then H is a normal p-subgroup of G, and hence $H \le O_p(G)$.*

Proof. See, e.g., [Go, Theorems 5.3.2 & 5.1.4]. □

Lemma 1.9. *Fix a finite abelian p-group A and a subgroup $G \le \mathrm{Aut}(A)$, and choose $\mathbf{U} \in \mathrm{Syl}_p(G)$. Then*

$$C_A(\mathbf{U}) \le [G, A] \iff C_A(G) \le [G, A] \iff C_A(G) \le [\mathbf{U}, A]\,.$$

Proof. Each of the first and third inequalities clearly implies the second, so it suffices to show that the second implies each of the other two. Assume $C_A(G) \le [G, A]$.

If $C_A(\mathbf{U}) \nleq [G, A]$, then choose $z \in C_A(\mathbf{U}) \smallsetminus [G, A]$. Let X be the G-orbit of z and set $m = |X|$; $p \nmid m$ since z is fixed by $\mathbf{U}$. Let $\widehat{z}$ be the product of the elements in X. Then $\widehat{z} \in C_A(G)$, and $\widehat{z} \in z^m{\cdot}[G, A]$. Thus $\widehat{z} \in C_A(G) \smallsetminus [G, A]$, contradicting our assumption.

Now assume $C_A(G) \nleq [\mathbf{U}, A]$. Since $\mathbb{Z}/p^\infty$ is injective as an abelian group, there is a homomorphism $\varphi \in \mathrm{Hom}(A, \mathbb{Z}/p^\infty)$ such that $[\mathbf{U}, A] \le \mathrm{Ker}(\varphi)$ but $C_A(G) \nleq \mathrm{Ker}(\varphi)$. Let X be the G-orbit of φ under the action of G on $\mathrm{Hom}(A, \mathbb{Z}/p^\infty)$, and set $m = |X|$. Then $p \nmid m$ since φ is fixed by $\mathbf{U}$. Let $\widehat{\varphi}$ be the product of the elements of X. Then $\widehat{\varphi}$ is G-invariant, so $[G, A] \le \mathrm{Ker}(\widehat{\varphi})$. Fix $z \in C_A(G)$ such that $\varphi(z) \ne 0$; then $\widehat{\varphi}(z) = m{\cdot}\varphi(z) \ne 0$. Thus $z \in C_A(G) \smallsetminus [G, A]$, again contradicting our assumption. □

Lemma 1.10 ([Ol, Lemma 1.11]). *Fix a finite abelian p-group A and a subgroup $G \le \mathrm{Aut}(A)$. Assume the following.*

(i) *Each Sylow p-subgroup of G has order p and is not normal in G.*

(ii) *For each $x \in G$ of order p, $[x, A]$ has order p, and hence $C_A(x)$ has index p.*

Set $A_1 = C_A(O^{p'}(G))$ and $A_2 = [O^{p'}(G), A]$. Then G normalizes A_1 and A_2, $A = A_1 \times A_2$, and $O^{p'}(G) \cong SL_2(p)$ acts faithfully on $A_2 \cong C_p^2$.

Lemma 1.11. *Let V be a finite abelian p-group (written additively), and fix a subgroup $G \le \mathrm{Aut}(V)$ with $\mathbf{U} \in \mathrm{Syl}_p(G)$ of order p. Assume also that $\dim\bigl(C_V(\mathbf{U}) \cap [\mathbf{U}, V]\bigr) = 1$. Set $V_0 = C_V(\mathbf{U})[\mathbf{U}, V]$. Define inductively $W_1 > W_2 > \cdots$ by setting $W_1 = [\mathbf{U}, V]$, and $W_{n+1} = [\mathbf{U}, W_n]$ for $n \ge 1$. Let m be the smallest integer such that $W_m = 0$. Then the following hold.*

(a) *For each $1 \le i \le m-1$, $|W_i/W_{i+1}| = p = |V/V_0|$.*

(b) *Fix $g \in N_G(\mathbf{U})$. Let $r, t \in (\mathbb{Z}/p)^\times$ be such that ${}^g u = u^r$ for all $u \in \mathbf{U}$, and g induces multiplication by t on V/V_0. Then for each $1 \le i \le m-1$, g induces multiplication by tr^i on W_i/W_{i+1}.*

Proof. **(a)** Fix a generator $u \in \mathbf{U}$, and let $\varphi\colon V \longrightarrow V$ be the homomorphism $\varphi(v) = [u, v]$. For each $0 \ne W \le V$ normalized by $\mathbf{U}$, $\mathrm{Ker}(\varphi|_W) = C_W(\mathbf{U})$, $\mathrm{Im}(\varphi|_W) = [\mathbf{U}, W]$, and thus $|W/[\mathbf{U}, W]| = |C_W(\mathbf{U})| > 1$. In particular, if $0 \ne W \le W_1 = [\mathbf{U}, V]$, then $0 \ne C_W(\mathbf{U}) \le C_{W_1}(\mathbf{U})$, with equality since $C_{W_1}(\mathbf{U}) = C_V(\mathbf{U}) \cap [\mathbf{U}, V]$ has order p by assumption. Hence $[\mathbf{U}, W]$ has index p in W.

Thus $|W_i/W_{i+1}| = p$ for each $1 \le i < m$. Also, since $|C_V(\mathbf{U})|\cdot|[\mathbf{U}, V]| = |V|$, $|V/V_0| = |C_V(\mathbf{U}) \cap [\mathbf{U}, V]| = p$.

(b) Fix $g \in N_G(\mathbf{U})$, and let $r, t \in (\mathbb{Z}/p)^\times$ be as above. For each $x = [u, v] \in W_1$, $g(x) = [{}^g u, g(v)] \equiv [u^r, tv] \equiv rt[u, v]$ modulo $[\mathbf{U}, V_0] = [\mathbf{U}, W_1] = W_2$. This proves the result when $i = 1$, and the other cases follow inductively. $\square$

2. Reduced fusion systems over non-abelian p-groups with index p abelian subgroup

Throughout this section, p is an odd prime. We want to describe all reduced fusion systems over non-abelian p-groups which contain an abelian subgroup of index p. If S has more than one abelian subgroup of index p, then by [Ol, Theorem 2.1], either S is extraspecial of order p^3 and exponent p (the case already handled by Ruiz and Viruel in [RV]), or there are no reduced fusion systems over S. In [Ol, Theorem 2.8], the second author handled the case where S contains a unique abelian subgroup of index p and that subgroup is not essential. We now look at the more complicated case: that where S contains a unique abelian subgroup of index p and it is essential (Theorem 2.8 below).

Notation 2.1. *Fix a p-group S with unique abelian subgroup A of index p, and a saturated fusion system $\mathcal{F}$ over S. Define*

$$S' = [S, S], \quad Z = Z(S), \quad Z_0 = Z \cap S', \quad Z_2 = Z_2(S), \quad A_0 = Z{\cdot}S'.$$

Thus $Z_0 \le Z \le A_0 \le A$ and $Z_0 \le S' \le A_0$. Also, set

$$\mathcal{H} = \bigl\{ Z\langle x\rangle \bigm| x \in S \smallsetminus A \bigr\} \qquad \text{and} \qquad \mathcal{B} = \bigl\{ Z_2\langle x\rangle \bigm| x \in S \smallsetminus A \bigr\}.$$

Lemma 2.2. *Assume the notation and hypotheses of 2.1. Then the following hold.*

(a) *For each* $P \in \mathbf{E}_{\mathcal{F}}$*, either* $P = A$*, or* P *is abelian and* $P \in \mathcal{H}$*, or* P *is non-abelian and* $P \in \mathcal{B}$*. In all cases,* $|N_S(P)/P| = p$*.*

(b) *If* $Z_2\langle x\rangle \in \mathbf{E}_{\mathcal{F}}$ *for some* $x \in S \smallsetminus A$*, then* $Z\langle x\rangle$ *is not* $\mathcal{F}$*-centric and* $Z\langle x\rangle \notin \mathbf{E}_{\mathcal{F}}$*.*

(c) *If* $A \ntrianglelefteq \mathcal{F}$ *(equivalently, if* $\mathbf{E}_{\mathcal{F}} \nsubseteq \{A\}$*), then* $|Z_0| = p$*,* $Z_2 \le A$*, and* S/Z *is nonabelian.*

(d) *If* $|Z_0| = p$*, then* $|A/A_0| = p$*,* $|Z_2/Z| = p$*,* $Z_2 \le A_0$*, and* $Z_2 \cap S' \cong C_p^2$*. Also, there are elements* $\mathbf{x} \in S \smallsetminus A$ *and* $\mathbf{a} \in A \smallsetminus A_0$ *such that* $A_0\langle \mathbf{x}\rangle$ *and* $S'\langle \mathbf{a}\rangle$ *are normalized by* $\mathrm{Aut}_{\mathcal{F}}(S)$*. If some element of* $S \smallsetminus A$ *has order* p*, then we can choose* $\mathbf{x}$ *to have order* p*.*

(e) *For each* $P \in \mathbf{E}_{\mathcal{F}}$ *and each* $\alpha \in N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$*,* α *extends to some* $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(S)$*.*

(f) *For each* $x \in S \smallsetminus A$ *and each* $g \in A_0$*,* $Z\langle x\rangle$ *is* S*-conjugate to* $Z\langle gx\rangle$*, and* $Z_2\langle x\rangle$ *is* S*-conjugate to* $Z_2\langle gx\rangle$*.*

Proof. **(a,b,c,e)** See points (a), (c), (b), and (e), respectively, in [Ol, Lemma 2.3]. Note, in (c), that S/Z is nonabelian since $Z(S/Z) = Z_2/Z \le A/Z$.

(d) Assume that $|Z_0| = p$. Fix a generator $\alpha \in \mathrm{Aut}_S(A) \cong C_p$. Let $f\colon A \longrightarrow A$ be the homomorphism $f(x) = x^{-1}\alpha(x)$. Then $\mathrm{Ker}(f) = C_A(\alpha) = Z$ and $\mathrm{Im}(f) = [\alpha, A] = S'$, so $|Z|{\cdot}|S'| = |A|$, and $|A/A_0| = |A/ZS'| = |Z \cap S'| = p$. Also, $|S'| > p$ since A is the unique abelian subgroup of index p in S, so S/Z is non-abelian, and $Z_2/Z = Z(S/Z) = C_{A/Z}(\alpha)$. Let $\bar{f}\colon A/Z \longrightarrow A/Z$ be the homomorphism induced by f on the quotient; then $|Z_2/Z| = |\mathrm{Ker}(\bar{f})| = |(A/Z)/\mathrm{Im}(\bar{f})| = |A/ZS'| = p$.

Since $Z_2/Z = Z(S/Z)$ has order p and S/Z is nonabelian, Z_2/Z must be contained in $[S/Z, S/Z] = S'Z/Z = A_0/Z$. Thus $Z_2 \le A_0$, so $Z_2S' = ZS'$, and hence $|Z_2 \cap S'| = |Z_2/Z|{\cdot}|Z \cap S'| = p^2$. It remains to show that $Z_2 \cap S'$ is not cyclic.

For each $x \in S' = [\alpha, A]$, $x = y^{-1}\alpha(y)$ for some $y \in A$, so $\prod_{i=0}^{p-1}\alpha^i(x) = 1$. Hence if $Z_2 \cap S' \cong C_{p^2}$ is generated by x, then $\alpha(x) = x^{1+kp}$ for some k such that $p \nmid k$, and $\sum_{i=0}^{p-1}(1+kp)^i = \bigl((1+kp)^p - 1\bigr)/kp \equiv 0 \pmod{p^2}$. Since $(1+kp)^p \equiv 1 + kp^2 \pmod{p^3}$, this is impossible.

Let $B \le A$ be minimal among subgroups which are normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$ such that $B \ge S'$ and $BA_0 = A$. The natural surjection $B/S'\mathrm{Fr}(B) \twoheadrightarrow A/A_0$ of $\mathbb{F}_p[\mathrm{Out}_{\mathcal{F}}(S)]$-modules is split since $p \nmid |\mathrm{Out}_{\mathcal{F}}(S)|$. Hence $B/S'\mathrm{Fr}(B) \cong A/A_0$ by the minimality of B, and B/S' is cyclic since $B/S'\mathrm{Fr}(B)$ is cyclic. For any generator $\mathbf{a}S'$ of B/S', $\mathbf{a} \in A \smallsetminus A_0$, and $B = S'\langle \mathbf{a}\rangle$ is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$.

For each $x \in S \smallsetminus A$, $x^p \in C_A(x) = Z \le A_0$. Hence $S/A_0 \cong C_p^2$. Since A/A_0 is an $\mathbb{F}_p[\mathrm{Out}_{\mathcal{F}}(S)]$-submodule of S/A_0 (and since $p \nmid |\mathrm{Out}_{\mathcal{F}}(S)|$), S/A_0 splits as a product $S/A_0 = (A/A_0) \times (R/A_0)$ where R is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant. Let $\mathbf{x}$ be any element of $R \smallsetminus A \subseteq S \smallsetminus A$; then $R = A_0\langle \mathbf{x}\rangle$.

Assume that there is $y \in S \smallsetminus A$ such that $y^p = 1$. Upon replacing y by some other generator of $\langle y\rangle$, we can assume that $y \in \mathbf{x}A$. If $A_0\langle y\rangle$ is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$, we are done. Otherwise, there is $y' \in yA$ such that $y'^p = 1$ and $y^{-1}y' \notin A_0$. Set $g = y^{-1}y'$; then

$$1 = y'^p = (yg)^p = (ygy^{-1})(y^2gy^{-2})\cdots(y^pgy^{-p})y^p\,,$$

so $\prod_{i=1}^{p}(y^igy^{-i}) = 1$. Then $(yg^j)^p = \prod_{i=1}^{p}(y^ig^jy^{-i})y^p = 1$ for all $j \in \mathbb{Z}$ by a similar computation. Since $g \in A \smallsetminus A_0$, there is j such that $yg^j \in \mathbf{x}A_0$. Upon replacing $\mathbf{x}$ by yg^j, we can arrange that $\mathbf{x}^p = 1$.

(f) Fix $x \in S \smallsetminus A$ and $g \in A_0 = S'Z$, and choose $z \in Z$ and $g' \in S'$ such that $g = g'z$. Then there is $h \in A$ such that $g' = h^{-1}({}^{x}h)$, $gx = zh^{-1}xh$, and so $Z\langle gx\rangle = Z\langle x^h\rangle = Z\langle x\rangle^h$ and $Z_2\langle gx\rangle = Z_2\langle x^h\rangle = Z_2\langle x\rangle^h$. □

Lemma 2.3. *Let $A \trianglelefteq S$, $\mathcal{F}$, $\mathcal{H}$, $\mathcal{B}$, etc., be as in Notation 2.1. Assume that $A \ntrianglelefteq \mathcal{F}$.*

(a) *If $P \in \mathcal{B} \cap \mathbf{E}_{\mathcal{F}}$, then $P = ZP^*$ for some unique $\mathrm{Aut}_{\mathcal{F}}(P)$-invariant extraspecial subgroup P^* of order p^3 and exponent p, with centre $Z(P^*) = Z_0 = Z \cap P^*$. Also, $O^{p'}(\mathrm{Out}_{\mathcal{F}}(P)) \cong SL_2(p)$, and this group acts faithfully on $P^*/Z_0 \cong P/Z \cong C_p^2$ and acts trivially on Z.*

(b) *If $P \in \mathcal{H} \cap \mathbf{E}_{\mathcal{F}}$, then there is a unique subgroup $Z^* < Z$ which is normalized by $\mathrm{Aut}_{\mathcal{F}}(Z)$ and such that $Z = Z_0 \times Z^*$. Also, Z^* is normalized by $\mathrm{Aut}_{\mathcal{F}}(P)$, $P = Z^* \times P^*$ for some unique $\mathrm{Aut}_{\mathcal{F}}(P)$-invariant subgroup $P^* \cong C_p^2$ which contains Z_0, $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) \cong SL_2(p)$, and this group acts faithfully on P^* and acts trivially on Z^*.*

(c) *There is $x \in S \smallsetminus A$ of order p; i.e., S splits over A.*

Proof. **(a)** Assume that $P \in \mathcal{B} \cap \mathbf{E}_{\mathcal{F}}$. Then $|P/Z| = |P/Z_2| \cdot |Z_2/Z| = p^2$ (Lemma 2.2(c,d)), so P/Z is abelian, and $[P,P] \le Z \cap S' = Z_0$. Thus $[P,P] = Z_0$ since $|Z_0| = p$ and P is non-abelian.

By Lemma 2.2(a), $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$ has order p. Also, $[N_S(P), P] \le P \cap S' = Z_2 \cap S'$, and $Z_2 \cap S' \cong C_p^2$ by Lemma 2.2(c,d). Thus $[\mathrm{Aut}_S(P), P/Z_0] \le (Z_2 \cap S')/Z_0 \cong C_p$, with equality since P is essential and hence $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ cannot act trivially on $P/[P,P]$ (recall $[P,P] \le Z_0$).

By Lemma 1.10 applied to the $\mathrm{Out}_{\mathcal{F}}(P)$-action on P/Z_0, $P = P_1P_2$, where $\mathrm{Aut}_{\mathcal{F}}(P)$ normalizes P_1 and P_2, $P_1/Z_0 = C_{P/Z_0}(O^{p'}(\mathrm{Out}_{\mathcal{F}}(P)))$, $P_1 \cap P_2 = Z_0$, and $|P_2/Z_0| = p^2$. Thus P_1/Z_0 is the intersection of the subgroups in the $\mathrm{Aut}_{\mathcal{F}}(P)$-orbit of $C_{P/Z_0}(\mathrm{Aut}_S(P)) = Z_2/Z_0$, hence contains $Z(P)/Z_0 = Z/Z_0$, with equality since $|P/Z| = p^2 = |P_2/Z_0|$. Thus $P_1 = Z$, and we set $P^* = P_2$. Since $P^*/Z_0 \cong C_p^2$ by Lemma 1.10 again, P^* is extraspecial of order p^3, and has exponent p since otherwise its automorphism group would be a p-group. The last statement follows immediately from Lemma 1.10.

(b) Assume that $P \in \mathcal{H} \cap \mathbf{E}_{\mathcal{F}}$. Thus P is abelian. By Lemma 2.2(a), $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ has order p. Also, for each $g \in N_S(P) \smallsetminus P$, $1 \ne [g,P] \le P \cap S' = Z_0$, so $[g,P] = Z_0$ since $|Z_0| = p$ by Lemma 2.2(c). Hence by Lemma 1.10, $P = Z^* \times P^*$, where $Z^* \le Z$ and $P^* \ge Z(P) \cap S' = Z_0$ are both $\mathrm{Aut}_{\mathcal{F}}(P)$-invariant. Also, by the same lemma, $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) \cong SL_2(p)$, and this subgroup acts faithfully on $P^* \cong C_p^2$ and trivially on Z^*.

In particular, there is a subgroup $H \le N_{O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))}(\mathrm{Aut}_S(P))$ of order $p-1$ which acts as the full group of automorphisms of Z_0 and of P^*/Z_0, and acts trivially on Z^*. Since H restricts to a subgroup of $\mathrm{Aut}_{\mathcal{F}}(Z)$, this shows that $Z = Z^* \times Z_0$ is the unique $\mathrm{Aut}_{\mathcal{F}}(Z)$-invariant splitting of Z with one factor Z_0.

(c) Since $A \ntrianglelefteq \mathcal{F}$, there must be $P \in \mathbf{E}_{\mathcal{F}}$ in $\mathcal{H} \cup \mathcal{B}$. So there is $x \in S \smallsetminus A$ of order p by the descriptions of P in (a) and (b). □

We now need to fix some more notation.

Notation 2.4. *Assume Notation 2.1. Assume also that $|Z_0| = p$, and hence that $|A/A_0| = p$. Fix $\mathbf{a} \in A \smallsetminus A_0$ and $\mathbf{x} \in S \smallsetminus A$, chosen such that $A_0\langle \mathbf{x}\rangle$ and $S'\langle \mathbf{a}\rangle$ are each normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$, and such that $\mathbf{x}^p = 1$ if any element of $S \smallsetminus A$ has order p (Lemma 2.2(d)). For each $i = 0, 1, \ldots, p-1$, define*

$$H_i = Z\langle \mathbf{x}\mathbf{a}^i\rangle \in \mathcal{H} \qquad \text{and} \qquad B_i = Z_2\langle \mathbf{x}\mathbf{a}^i\rangle \in \mathcal{B}.$$

Let $\mathcal{H}_i$ and $\mathcal{B}_i$ denote the S-conjugacy classes of H_i and B_i, respectively, and set

$$\mathcal{H}_* = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_{p-1} \quad \text{and} \quad \mathcal{B}_* = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{p-1}.$$

Thus $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_*$ and $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_*$ by Lemma 2.2(f) and since $|A/A_0| = p$.

Set

$$\Delta = (\mathbb{Z}/p)^\times \times (\mathbb{Z}/p)^\times, \qquad \text{and} \qquad \Delta_i = \{(r, r^i) \,|\, r \in (\mathbb{Z}/p)^\times\} \le \Delta \quad (\text{for } i \in \mathbb{Z}).$$

Define

$$\mu\colon \mathrm{Aut}(S) \longrightarrow \Delta \qquad \text{and} \qquad \widehat{\mu}\colon \mathrm{Out}(S) \longrightarrow \Delta$$

by setting, for $\alpha \in \mathrm{Aut}(S)$,

$$\mu(\alpha) = \widehat{\mu}([\alpha]) = (r,s) \quad \text{if} \quad \begin{cases} \alpha(x) \in x^r A & \text{for } x \in S \smallsetminus A \\ \alpha(g) = g^s & \text{for } g \in Z_0\,. \end{cases}$$

Finally, set

$$\begin{aligned} \mathrm{Aut}_{\mathcal{F}}^{\vee}(S) &= \big\{\alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \,\big|\, [\alpha, Z] \le Z_0\big\}, \\ \mathrm{Out}_{\mathcal{F}}^{\vee}(S) &= \mathrm{Aut}_{\mathcal{F}}^{\vee}(S)/\mathrm{Inn}(S), \\ \mathrm{Aut}_{\mathcal{F}}^{\vee}(A) &= \big\{\alpha|_A \,\big|\, \alpha \in \mathrm{Aut}_{\mathcal{F}}^{\vee}(S)\big\} = \big\{\beta \in N_{\mathrm{Aut}_{\mathcal{F}}(A)}(\mathrm{Aut}_S(A)) \,\big|\, [\beta, Z] \le Z_0\big\}, \\ \mathrm{Aut}_{\mathcal{F}}^{(P)}(S) &= \big\{\alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \,\big|\, \alpha(P) = P,\ \alpha|_P \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))\big\} \qquad (\text{all } P \le S). \end{aligned}$$

Lemma 2.5. *Let S be a finite p-group with a unique abelian subgroup $A \trianglelefteq S$ of index p, and let $\mathcal{F}$ be a saturated fusion system over S. Assume that $|Z_0| = p$, and use Notation 2.1 and 2.4. Let $m \ge 3$ be such that $|A/Z| = p^{m-1}$. Then the following hold.*

(a) *$\widehat{\mu}|_{\mathrm{Out}_{\mathcal{F}}^{\vee}(S)}$ is injective.*

(b) *Fix $\alpha \in \mathrm{Aut}(S)$, set $(r,s) = \mu(\alpha)$, and let t be such that $\alpha(g) \in g^t A_0$ for each $g \in A \smallsetminus A_0$. Then $s \equiv tr^{m-1}$ (mod p).*

(c) *For $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$, either $\mu(\alpha) \in \Delta_m$, and α normalizes each of the S-conjugacy classes $\mathcal{H}_i$ and $\mathcal{B}_i$ ($0 \le i \le p-1$); or $\mu(\alpha) \notin \Delta_m$, and α normalizes only the classes $\mathcal{H}_0$ and $\mathcal{B}_0$. Also, α acts via the identity on A/A_0 if and only if $\mu(\alpha) \in \Delta_{m-1}$.*

(d) *Assume that $\mathbf{x}^p = 1$, and set $\sigma = \prod_{i=0}^{p-1} {}^{\mathbf{x}^i}(\mathbf{a}) = (\mathbf{ax})^p \mathbf{x}^{-p}$. For each $P \in \mathcal{H}_0 \cup \mathcal{B}_0$, P splits over $P \cap A$. For each $P \in \mathcal{H}_* \cup \mathcal{B}_*$, P splits over $P \cap A$ if and only if $\sigma \in \mathrm{Fr}(Z)$.*

Proof. **(b)** This follows from Lemma 1.11(b), applied with A, A_0, S', and Z_0 in the role of V, V_0, W_1, and W_{m-1}. Note that $p^{m-1} = |A/Z| = |S'|$, $|W_1| = p^{m-1}$ by Lemma 1.11(a), and thus m plays the same role here as in Lemma 1.11.

(a) Assume that $\alpha \in \mathrm{Aut}_{\mathcal{F}}^{\vee}(S)$ has order prime to p, and $\mu(\alpha) = 1$. Then α induces the identity on S/A, on A/A_0 by (b), and on A_0/S' since $[\alpha, ZS'] \le Z_0 S' = S'$ by definition of $\mathrm{Aut}_{\mathcal{F}}^{\vee}(S)$. So $\alpha = \mathrm{Id}_S$ by Lemma 1.8. Thus $\mathrm{Aut}_{\mathcal{F}}^{\vee}(S) \cap \mathrm{Ker}(\mu)$ is a p-group, and hence equal to $\mathrm{Inn}(S)$.

(c) Fix $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$, and let r, s, t be as in (b). Then α acts via the identity on A/A_0 if and only if $t = 1$; equivalently (by (b)) if and only if $\widehat{\mu}([\alpha]) = (r,s) \in \Delta_{m-1}$.

Since $\alpha(A_0\langle \mathbf{x}\rangle) = A_0\langle \mathbf{x}\rangle$ by assumption (see Notation 2.4), $\alpha(\mathbf{x}) \in A_0\mathbf{x}^r$. Hence for each $0 \le i < p$, $\alpha(A_0\langle \mathbf{x}\mathbf{a}^i\rangle) = A_0\langle \mathbf{x}^r\mathbf{a}^{it}\rangle$. Thus $\alpha(\mathcal{H}_0) = \mathcal{H}_0$ and $\alpha(\mathcal{B}_0) = \mathcal{B}_0$ in all cases, while for $0 < j < p$, $\alpha(\mathcal{H}_j) = \mathcal{H}_j$ or $\alpha(\mathcal{B}_j) = \mathcal{B}_j$ only if $r \equiv t$ (mod $p-1$). By (b), this holds if and only if $s \equiv r^m$ (mod $p-1$); i.e., if and only if $\mu(\alpha) \in \Delta_m$.

(d) Since $\mathbf{x}^p = 1$, P splits over $P \cap A$ for all $P \in \mathcal{H}_0 \cup \mathcal{B}_0$. For each $1 \le i \le p-1$, $(\mathbf{a}^i\mathbf{x})^p = \sigma^i$, so there is $z \in Z$ with $\mathbf{a}^i\mathbf{x}z$ of order p exactly when $\sigma \in \mathrm{Fr}(Z)$. This proves the claim for $P \in \mathcal{H}_*$.

Assume that $P \in \mathcal{B}_i$ for $1 \le i \le p-1$. If P splits over $P \cap A$, then $Z_2\langle \mathbf{a}^i\mathbf{x}\rangle$ splits (it is S-conjugate to P), and hence $(\mathbf{a}^i\mathbf{x}y)^p = 1$ for some $y \in Z_2$. Also, $[\mathbf{a}^i\mathbf{x}, y] = [\mathbf{x}, y] \in [S, Z_2] = Z_0$, so $1 = (\mathbf{a}^i\mathbf{x}y)^p = (\mathbf{a}^i\mathbf{x})^p y^p = \sigma^i y^p$, and hence $\sigma \in \mathrm{Fr}(Z_2)$. Also, since $Z_2 = Z(Z_2 \cap S')$, where $Z_2 \cap S' \cong C_p^2$ by Lemma 2.2(d), we have $Z_2 \cong Z \times C_p$, and hence $\mathrm{Fr}(Z_2) = \mathrm{Fr}(Z)$. The converse is clear: if $\sigma = z^p$ for some $z \in Z$, then $Z_2\langle \mathbf{a}^i\mathbf{x}\rangle$ is split over Z_2 by $\langle \mathbf{a}^i\mathbf{x}z^{-i}\rangle$. □

Lemma 2.6. *Let S be a finite p-group with a unique abelian subgroup $A \trianglelefteq S$ of index p, and let $\mathcal{F}$ be a saturated fusion system over S. We use Notation 2.1, and let m be such that $|A/Z| = p^{m-1}$. Fix $P \in \mathcal{H} \cup \mathcal{B}$. Set $t = -1$ if $P \in \mathcal{H}$ or set $t = 0$ if $P \in \mathcal{B}$.*

(a) *If $P \in \mathbf{E}_{\mathcal{F}}$, then in the notation of 2.4, $\mathrm{Aut}_{\mathcal{F}}^{(P)}(S) \le \mathrm{Aut}_{\mathcal{F}}^{\vee}(S)$ and $\mu(\mathrm{Aut}_{\mathcal{F}}^{(P)}(S)) = \Delta_t$. If in addition, $P \in \mathcal{H}_* \cup \mathcal{B}_*$, then $m \equiv t \pmod{p-1}$.*

(b) *Conversely, assume that $\mu\bigl(N_{\mathrm{Aut}_{\mathcal{F}}^{\vee}(S)}(P)\bigr) \ge \Delta_t$, and also that P splits over $P \cap A$. Then there is a unique subgroup $\Theta \le \mathrm{Aut}(P)$ such that*
   - (i) $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\Theta)$,
   - (ii) $\Theta \ge \mathrm{Inn}(P)$ *and* $O^{p'}(\Theta)/\mathrm{Inn}(P) \cong SL_2(p)$,
   - (iii) $[\alpha, Z] \le Z_0$ *for each* $\alpha \in N_{O^{p'}(\Theta)}(\mathrm{Aut}_S(P))$, *and*
   - (iv) $N_\Theta(\mathrm{Aut}_S(P)) = \bigl\{\alpha|_P \bigm| \alpha \in N_{\mathrm{Aut}_{\mathcal{F}}(S)}(P)\bigr\}$.

Proof. **(a)** Assume that $P \in \mathbf{E}_{\mathcal{F}}$. By Lemma 2.3(a,b), whether $P \in \mathcal{H}$ or $P \in \mathcal{B}$, $O^{p'}(\mathrm{Out}_{\mathcal{F}}(P)) \cong SL_2(p)$, and acts trivially on Z/Z_0. Thus $\mathrm{Aut}_{\mathcal{F}}^{(P)}(S) \le \mathrm{Aut}_{\mathcal{F}}^{\vee}(S)$.

If $P \in \mathcal{H}$, then by Lemma 2.3(b), there is $P^* \le P$ such that $P^* \cap A = Z_0$, and $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) \cong SL_2(p)$ acts faithfully on $P^* \cong C_p^2$. Each element of the normalizer $N_{O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))}(\mathrm{Aut}_S(P)) \cong C_p \rtimes C_{p-1}$ extends to an element of $\mathrm{Aut}_{\mathcal{F}}^{(P)}(S)$ by Lemma 2.2(e); and since this normalizer contains all diagonal matrices $\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1}\end{smallmatrix}\right)$ for $u \in (\mathbb{Z}/p)^\times$, $\mu(\mathrm{Aut}_{\mathcal{F}}^{(P)}(S))$ is the set $\{(u, u^{-1}) \mid u \in (\mathbb{Z}/p)^\times\} = \Delta_{-1}$.

If $P \in \mathcal{B}$, then by Lemma 2.3(a), there is $P^* \le P$ such that $Z_0 \le P^* \cap A \le Z_2$, P^* is extraspecial of order p^3 and exponent p, and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(P)) \cong SL_2(p)$ acts faithfully on $P^*/Z_0 \cong C_p^2$. Each element of $N_{O^{p'}(\mathrm{Out}_{\mathcal{F}}(P))}(\mathrm{Out}_S(P)) \cong C_p \rtimes C_{p-1}$ extends to an element of $\mathrm{Aut}_{\mathcal{F}}^{(P)}(S)$ by Lemma 2.2(e); this normalizer acts trivially on Z_0, and hence $\mu(\mathrm{Aut}_{\mathcal{F}}^{(P)}(S)) = \{(u, 1) \mid u \in (\mathbb{Z}/p)^\times\} = \Delta_0$.

If $P \in \mathcal{H}_* \cup \mathcal{B}_*$, then $\mu(\mathrm{Aut}_{\mathcal{F}}^{(P)}(S)) \le \Delta_m$ by Lemma 2.5(c), since each element in $\mathrm{Aut}_{\mathcal{F}}^{(P)}(S)$ normalizes P. So $\Delta_m = \Delta_t$, and $m \equiv t \pmod{p-1}$.

(b) Set

$$\Lambda_P = \bigl\{\alpha|_P \bigm| \alpha \in N_{\mathrm{Aut}_{\mathcal{F}}(S)}(P)\bigr\} \qquad \text{and} \qquad \widehat{P} = P \cap S' = \begin{cases} Z \cap S' = Z_0 & \text{if } P \in \mathcal{H} \\ Z_2 \cap S' \cong C_p^2 & \text{if } P \in \mathcal{B}. \end{cases}$$

Then Λ_P normalizes $\widehat{P}$, and the induced action of Λ_P on $P/\widehat{P}$ factors through the quotient group $\Lambda_P/\mathrm{Aut}_S(P)$ of order prime to p and normalizes $(A \cap P)/\widehat{P}$. Since P splits over $P \cap A$, there is a subgroup $P^*/\widehat{P} \le P/\widehat{P}$ of order p, also normalized by Λ_P, such that $P/\widehat{P} = ((P \cap A)/\widehat{P}) \times (P^*/\widehat{P})$. Since $Z_2 \le A_0 = ZS'$ by Lemma 2.2(d), $P = P^*Z$ and $P^* \cap Z = Z_0$. Also, $|P^*| = p^2$ if $P \in \mathcal{H}$, and P^* is extraspecial of order p^3 if $P \in \mathcal{B}$.

Assume first that $P \in \mathcal{H}$. Choose $\alpha \in N_{\mathrm{Aut}^\vee_{\mathcal{F}}(S)}(P)$ such that $\mu(\alpha)$ generates Δ_{-1}. Then $\alpha(P^*) = P^*$ since P^* is normalized by Λ_P, α acts non-trivially on Z_0 and on P^*/Z_0, and induces the identity on Z/Z_0. Set $Z^* = C_Z(\alpha)$; thus $Z = Z_0 \times Z^*$. Also, $P^* = [\alpha, P]$, and $P^* \cong C_p^2$ since $P = Z^* \times P^*$ splits over Z. Since $\mathrm{Aut}_{\mathcal{F}}(Z)$ has order prime to p, and normalizes Z_0 which is a direct factor in Z, there is an $\mathrm{Aut}_{\mathcal{F}}(Z)$-invariant subgroup of Z complementary to Z_0, and this can only be $Z^* = C_Z(\alpha)$. In particular, Z^* is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$, and hence by Λ_P.

Now assume that $P \in \mathcal{B}$. Choose $\alpha \in N_{\mathrm{Aut}^\vee_{\mathcal{F}}(S)}(P)$ such that $\mu(\alpha)$ generates Δ_0. Then $\alpha(P^*) = P^*$ since P^* is normalized by Λ_P, α acts non-trivially on $P^*/(P^* \cap A)$ and on $(P^* \cap A)/Z_0$ and trivially on Z_0, and hence P^* has exponent p. Also, since $P^*/Z_0 = [\alpha, P/Z_0]$, the choice of P^* was unique.

We have now shown that $O^{p'}(\mathrm{Out}(P^*)) \cong SL_2(p)$ in both cases ($P \in \mathcal{H}$ or $P \in \mathcal{B}$). Define

$$\Theta_0 = \begin{cases} \{\alpha \in \mathrm{Aut}(P) \mid \alpha|_{P^*} \in O^{p'}(\mathrm{Aut}(P^*)),\ \alpha|_{Z^*} = \mathrm{Id}\} & \text{if } P \in \mathcal{H} \\ \{\alpha \in \mathrm{Aut}(P) \mid \alpha|_{P^*} \in O^{p'}(\mathrm{Aut}(P^*)),\ \alpha|_{Z} = \mathrm{Id}\} & \text{if } P \in \mathcal{B} \end{cases}$$

and set $\Theta = \Lambda_P \Theta_0$. Since Λ_P normalizes Θ_0, Θ is a subgroup of $\mathrm{Aut}(P)$. Also, $\Theta \geq \Theta_0 \geq \mathrm{Inn}(P)$, and $\Theta_0/\mathrm{Inn}(P) \cong SL_2(p)$.

Proof of (i)–(iii) Since $\Theta = \Lambda_P \Theta_0$ where Λ_P normalizes Θ_0, $|\Theta/\Theta_0| = |\Lambda_P/(\Lambda_P \cap \Theta_0)|$, and this is prime to p since $\Lambda_P \cap \Theta_0 \geq \mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\Lambda_P)$. Thus $\Theta_0 = O^{p'}(\Theta)$, and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\Theta)$. So (i) and (ii) hold. Also, (iii) holds since $[\alpha, Z] \leq Z_0$ for all $\alpha \in N_{\Theta_0}(\mathrm{Aut}_S(P))$ and $\Theta_0 = O^{p'}(\Theta)$.

Proof of (iv) By construction, $\Lambda_P \leq N_\Theta(\mathrm{Aut}_S(P))$. So to prove (iv), it suffices to show that $\Lambda_P \geq N_{\Theta_0}(\mathrm{Aut}_S(P))$. Fix a generator $u \in (\mathbb{Z}/p)^\times$. By assumption, there is $\alpha \in N_{\mathrm{Aut}^\vee_{\mathcal{F}}(S)}(P)$ such that $\mu(\alpha) = (u, u^t)$. Thus $\alpha|_P \in \Lambda_P$, and $[\alpha, Z] \leq Z_0$. Upon replacing α by an appropriate power, if necessary, we can assume that $|\alpha|$ is prime to p.

If $P \in \mathcal{H}$, then $\alpha|_{Z^*} = \mathrm{Id}$ (recall that $\alpha|_Z \in \mathrm{Aut}_{\mathcal{F}}(Z)$ normalizes Z^*), $t = -1$, so α acts on $P^* \cong C_p^2$ via $\mathrm{diag}(u, u^{-1})$. If $P \in \mathcal{B}$, then $t = 0$, $\alpha|_Z = \mathrm{Id}$ since α induces the identity on Z_0 and on Z/Z_0, and α acts on $P^*/Z_0 \cong C_p^2$ via $\mathrm{diag}(u, u^{-1})$. Thus $\alpha|_P \in \Theta_0$, and $N_{\Theta_0}(\mathrm{Aut}_S(P)) = \mathrm{Aut}_S(P)\langle\alpha\rangle \leq \Lambda_P$. This finishes the proof of (iv).

Proof of uniqueness Let $\Theta^* \leq \mathrm{Aut}(P)$ be another subgroup which satisfies (i)–(iv). Since $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\Theta^*)$ by (i), $\Theta^* = N_{\Theta^*}(\mathrm{Aut}_S(P))O^{p'}(\Theta^*)$ by the Frattini argument, and so $\Theta^* = \Lambda_P O^{p'}(\Theta^*)$ by (iv).

It remains to prove that $O^{p'}(\Theta^*) = \Theta_0$. Since $O^{p'}(\Theta^*)/\mathrm{Inn}(P) \cong SL_2(p) \cong \Theta_0/\mathrm{Inn}(P)$ by (ii), it suffices to show that $O^{p'}(\Theta^*) \leq \Theta_0$.

If $P \in \mathcal{B}$, then set $P^+ = [O^{p'}(\Theta^*), P]$. By Lemma 1.10, applied with P/Z_0 and Θ^* in the roles of A and G, $P^+/Z_0 \cong C_p^2$ and is complementary to $C_{P/Z_0}(O^{p'}(\Theta^*)) = Z/Z_0$. Hence $P^+ \geq \widehat{P}$, $P/\widehat{P} = (Z_2/\widehat{P}) \times (P^+/\widehat{P})$, and P^+ is normalized by the action of $\Lambda_P \leq \Theta^*$; so $P^+ = P^*$ by the uniqueness of P^* shown above. Also, $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\Theta^*)$ acts trivially on Z and $Z = Z(P)$ is characteristic in P, so $O^{p'}(\Theta^*)$ also acts trivially on Z. Since Θ^* normalizes $P^+ = [O^{p'}(\Theta^*), P]$, we have $O^{p'}(\Theta^*) \leq \Theta_0$ by definition of Θ_0.

If $P \in \mathcal{H}$, then by Lemma 1.10, $P = Z^\vee \times P^\vee$ where $O^{p'}(\Theta^*) \cong SL_2(p)$ acts trivially on $Z^\vee$ and acts faithfully on $P^\vee \cong C_p^2$. We showed above that there is $\alpha \in N_{\mathrm{Aut}^\vee_{\mathcal{F}}(S)}(P)$ such that $Z^* = C_P(\alpha)$ and $P^* = [\alpha, P]$, and $\alpha|_P \in \Lambda_P \leq \Theta^*$ by (iv). Hence $Z^* = Z^\vee$ and $P^* = P^\vee$, and so $O^{p'}(\Theta^*) \leq \Theta_0$. □

In the next lemma, we describe the conditions for a saturated fusion system $\mathcal{F}$ over S to be reduced. This requires some more precise information about the $\mathcal{F}$-essential subgroups and their automorphisms in this situation.

In the proofs of the next lemma and theorem, we refer several times to the *extension axiom* for saturated fusion systems. This axiom states that in a saturated fusion system $\mathcal{F}$ over a p-group S, if $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ is such that $Q=\varphi(P)$ is fully centralized in $\mathcal{F}$, and if $\overline{P} \geq P$ is such that $P \trianglelefteq \overline{P}$ and $\varphi \mathrm{Aut}_{\overline{P}}(P)\varphi^{-1} \leq \mathrm{Aut}_S(Q)$, then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(\overline{P},S)$ which extends φ. We refer to [AKO, Proposition I.2.5] for more detail, including a description of how this axiom can be used to characterize saturated fusion systems.

Lemma 2.7. *Fix an odd prime p, and a p-group S which contains a unique abelian subgroup $A \trianglelefteq S$ of index p. Let $\mathcal{F}$ be a saturated fusion system over S, and assume that $A \ntrianglelefteq \mathcal{F}$. We use the notation of 2.1 and 2.4. Assume also that $A \in \mathbf{E}_{\mathcal{F}}$, and set $G = \mathrm{Aut}_{\mathcal{F}}(A)$ and $\mathbf{U} = \mathrm{Aut}_S(A) \in \mathrm{Syl}_p(G)$. Then the following hold.*

(a) *$O_p(\mathcal{F}) = 1$ if and only if either*
   - (i) *there are no non-trivial G-invariant subgroups of Z; or*
   - (ii) *$\mathbf{E}_{\mathcal{F}} \cap \mathcal{H} \neq \varnothing$ and Z_0 is the only non-trivial G-invariant subgroup of Z.*

(b) *If $O_p(\mathcal{F}) = 1$, then $O^p(\mathcal{F}) = \mathcal{F}$ if and only if $[G,A]=A$.*

(c) *$O^{p'}(\mathcal{F}) = \mathcal{F}$ if and only if one of the following holds: either*
   - (i) *$\mathbf{E}_{\mathcal{F}} \smallsetminus \{A\} = \mathcal{H}_0 \cup \mathcal{B}_*$, and $\mathrm{Aut}_{\mathcal{F}}(S) = \langle \mathrm{Aut}^{\vee}_{\mathcal{F}}(S), \mathrm{Aut}^{(A)}_{\mathcal{F}}(S)\rangle$; or*
   - (ii) *$\mathbf{E}_{\mathcal{F}} \smallsetminus \{A\} = \mathcal{B}_0 \cup \mathcal{H}_*$, and $\mathrm{Aut}_{\mathcal{F}}(S) = \langle \mathrm{Aut}^{\vee}_{\mathcal{F}}(S), \mathrm{Aut}^{(A)}_{\mathcal{F}}(S)\rangle$; or*
   - (iii) *$\mathbf{E}_{\mathcal{F}} \smallsetminus \{A\} \subseteq \mathcal{H}$ and $\mathrm{Aut}_{\mathcal{F}}(S) = \langle (\mathrm{Aut}^{\vee}_{\mathcal{F}}(S) \cap \mu^{-1}(\Delta_{-1})), \mathrm{Aut}^{(A)}_{\mathcal{F}}(S)\rangle$; or*
   - (iv) *$\mathbf{E}_{\mathcal{F}} \smallsetminus \{A\} \subseteq \mathcal{B}$ and $\mathrm{Aut}_{\mathcal{F}}(S) = \langle (\mathrm{Aut}^{\vee}_{\mathcal{F}}(S) \cap \mu^{-1}(\Delta_{0})), \mathrm{Aut}^{(A)}_{\mathcal{F}}(S)\rangle$.*

Proof. **(a)** Upon replacing the statements by their negatives, we must show that $O_p(\mathcal{F}) \neq 1$ if and only if

> there is a non-trivial subgroup $1 \neq Q \leq Z$ normalized by G, such that either $Q \neq Z_0$ or $\mathbf{E}_{\mathcal{F}} \cap \mathcal{H} = \varnothing$. (1)

Assume first that $O_p(\mathcal{F}) \neq 1$, and set $Q = O_p(\mathcal{F})$. Since $A \in \mathbf{E}_{\mathcal{F}}$, $Q \leq A$ and is G-invariant by Proposition 1.4(a). Since $A \ntrianglelefteq \mathcal{F}$, there is some $P \in \mathbf{E}_{\mathcal{F}} \smallsetminus \{A\} \subseteq \mathcal{B} \cup \mathcal{H}$. By Proposition 1.4(a) again, if $P \in \mathcal{H}$, then $Q \leq P \cap A = Z$, while if $P \in \mathcal{B}$, then $Q \leq \bigcap_{\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)} \alpha(P \cap A) = Z$ (see Lemma 2.3(a)). Thus Q is a non-trivial G-invariant subgroup of Z. If $Q = Z_0$, then $\mathbf{E}_{\mathcal{F}} \cap \mathcal{H} = \varnothing$, since for $P \in \mathbf{E}_{\mathcal{F}} \cap \mathcal{H}$, Z_0 is not normalized by $\mathrm{Aut}_{\mathcal{F}}(P)$ ($Z_0 < P^* \cong C_p^2$ in the notation of Lemma 2.3(b)). Thus (1) holds in this case.

Conversely, assume that (1) holds. In particular, $1 \neq Q \leq Z$ is G-invariant. For each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$, $\alpha(A) = A$ since A is the unique abelian subgroup of index p, so $\alpha|_A \in G$, and thus $\alpha(Q) = Q$. Since each element of $\mathrm{Aut}_{\mathcal{F}}(Z)$ extends to S by the extension axiom, Q is also normalized by $\mathrm{Aut}_{\mathcal{F}}(Z)$. Also, for each $P \in \mathbf{E}_{\mathcal{F}} \cap \mathcal{B}$, $Z = Z(P)$ is characteristic in P and so Q is also normalized by $\mathrm{Aut}_{\mathcal{F}}(P)$. In particular, if $\mathbf{E}_{\mathcal{F}} \cap \mathcal{H} = \varnothing$, then $Q \trianglelefteq \mathcal{F}$ by Proposition 1.4(a), so $O_p(\mathcal{F}) \neq 1$.

Now assume that $\mathbf{E}_{\mathcal{F}} \cap \mathcal{H} \neq \varnothing$, and hence by assumption that $Q \neq Z_0$. By Lemma 2.3(b), there is a unique $\mathrm{Aut}_{\mathcal{F}}(Z)$-invariant splitting $Z = Z_0 \times Z^*$. Set $Q^* = Q \cap Z^*$. If $Q \geq Z_0$, then $Q = Q^* \times Z_0$. Otherwise, $Q \cap Z_0 = 1$ (recall $|Z_0| = p$), and since Q is $\mathrm{Aut}_{\mathcal{F}}(Z)$-invariant, the uniqueness of the splitting implies that $Q \le Z^*$ and hence $Q = Q^*$. Since $Q \ne Z_0$, we have $Q^* \ne 1$ in either case.

For each $\varphi \in \mathrm{Aut}_{\mathcal{F}}(A) = G$, $\varphi(Q^*) \le Q \le Z$, so by the extension axiom, $\varphi|_{Q^*}$ extends to some $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(S)$, and $\varphi(Q^*) = \overline{\varphi}(Q^*) = Q^*$ since Q^* is $\mathrm{Aut}_{\mathcal{F}}(Z)$-invariant. So by the same arguments as those applied above to Q, Q^* is normalized by $\mathrm{Aut}_{\mathcal{F}}(P)$ for each $P \in (\{S\} \cup \mathbf{E}_{\mathcal{F}}) \smallsetminus \mathcal{H}$. If $P \in \mathbf{E}_{\mathcal{F}} \cap \mathcal{H}$, then for each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$, $\alpha(Z^*) = Z^*$ by Lemma 2.3(b), so $\alpha|_{Z^*}$ extends to an element of $\mathrm{Aut}_{\mathcal{F}}(S)$ and hence of $\mathrm{Aut}_{\mathcal{F}}(Z)$, and in particular, $\alpha(Q^*) = Q^*$. Thus $1 \ne Q^* \trianglelefteq \mathcal{F}$, and hence $O_p(\mathcal{F}) \ne 1$.

(b) Assume that $O_p(\mathcal{F}) = 1$ and $[G,A] < A$. Since $C_A(G) \le Z_0 \le [\mathbf{U},A]$ by (a), $[G,A] \ge C_A(\mathbf{U}) = Z$ by Lemma 1.9. Hence $[G,A] \ge ZS' = A_0$, with equality since $|A/A_0| = p$. The G-action on A/A_0 is thus trivial, and hence $\widehat{\mu}(\mathrm{Out}_{\mathcal{F}}(S)) \le \Delta_{m-1}$ by Lemma 2.5(c), where $p^{m-1} = |A/Z|$.

By Lemma 2.6(a), for each $1 \le i \le p-1$, $\mathcal{H}_i \subseteq \mathbf{E}_{\mathcal{F}}$ implies $\widehat{\mu}(\mathrm{Out}_{\mathcal{F}}(S)) \ge \Delta_{-1} = \Delta_m$, while $\mathcal{B}_i \subseteq \mathbf{E}_{\mathcal{F}}$ implies $\widehat{\mu}(\mathrm{Out}_{\mathcal{F}}(S)) \ge \Delta_0 = \Delta_m$. Thus $\mathbf{E}_{\mathcal{F}} \smallsetminus \{A\} \subseteq \mathcal{H}_0 \cup \mathcal{B}_0$, so $\mathfrak{foc}(\mathcal{F}) \le \langle [G,A], \mathcal{H}_0, \mathcal{B}_0 \rangle = A_0\langle \mathbf{x} \rangle < S$ by Proposition 1.4(b), and $O^p(\mathcal{F}) \ne \mathcal{F}$ by Proposition 1.4(c).

Conversely, if $[G,A] = A$, then $\mathfrak{foc}(\mathcal{F}) \ge A$. Since $A \ntrianglelefteq \mathcal{F}$, $\mathbf{E}_{\mathcal{F}} \supsetneqq \{A\}$, and hence $\mathfrak{foc}(\mathcal{F}) = S$. So $O^p(\mathcal{F}) = \mathcal{F}$ by Proposition 1.4(c).

(c) By Proposition 1.4(d), and since $\mathrm{Aut}_{\mathcal{F}}^{(A)}(S) \ge \mathrm{Inn}(S)$, $O^{p'}(\mathcal{F}) = \mathcal{F}$ if and only if $\mathrm{Aut}_{\mathcal{F}}(S)$ is generated by the subgroups $\mathrm{Aut}_{\mathcal{F}}^{(P)}(S)$ for all $P \in \mathbf{E}_{\mathcal{F}}$. By Lemma 2.6(a), if $P \in \mathbf{E}_{\mathcal{F}} \smallsetminus \{A\}$, then $\mathrm{Aut}_{\mathcal{F}}^{(P)}(S) = \mathrm{Aut}_{\mathcal{F}}^{\vee}(S) \cap \mu^{-1}(\Delta_t)$, where $t = -1$ if $P \in \mathcal{H}$ and $t = 0$ if $P \in \mathcal{B}$. Hence the conditions in (c.iii) are necessary and sufficient if $\mathbf{E}_{\mathcal{F}} \smallsetminus \{A\} \subseteq \mathcal{H}$, and those in (c.iv) are necessary and sufficient if $\mathbf{E}_{\mathcal{F}} \smallsetminus \{A\} \subseteq \mathcal{B}$.

If $\mathbf{E}_{\mathcal{F}} \smallsetminus \{A\}$ contains subgroups in both $\mathcal{H}$ and $\mathcal{B}$, then $\mu(\mathrm{Aut}_{\mathcal{F}}^{\vee}(S)) \ge \Delta_0\Delta_{-1} = \Delta$ by Lemma 2.6(a), and $\mathrm{Aut}_{\mathcal{F}}(S) = \langle \mathrm{Aut}_{\mathcal{F}}^{(A)}(S), \mathrm{Aut}_{\mathcal{F}}^{\vee}(S) \rangle$ by the above remarks. Also, by Lemma 2.2(b), some H_i or B_i must be essential for $0 < i < p$, and by Lemma 2.6(a), $m \equiv -1 \pmod{p-1}$ if $H_i \in \mathbf{E}_{\mathcal{F}}$, while $m \equiv 0 \pmod{p-1}$ if $B_i \in \mathbf{E}_{\mathcal{F}}$. Also, all subgroups in $\mathcal{H}_*$ and in $\mathcal{B}_*$ are $\mathcal{F}$-conjugate by Lemma 2.5(c) and since $\mu(\mathrm{Aut}_{\mathcal{F}}^{\vee}(S)) = \Delta$. Hence $\mathbf{E}_{\mathcal{F}} \smallsetminus \{A\} = \mathcal{H}_0 \cup \mathcal{B}_*$ or $\mathcal{B}_0 \cup \mathcal{H}_*$, and we are in the situation of (c.i) or (c.ii). □

We are now ready to describe the reduced fusion systems over non-abelian p-groups S which contain a unique abelian subgroup of index p which is essential. Recall that we defined

$$\mathrm{Aut}_{\mathcal{F}}^{\vee}(A) = \bigl\{\alpha \in N_G(\mathbf{U}) \,\big|\, [\alpha, Z] \le Z_0\bigr\} = \bigl\{\alpha|_A \,\big|\, \alpha \in \mathrm{Aut}_{\mathcal{F}}^{\vee}(S)\bigr\}.$$

Let

$$\mu_A\colon \mathrm{Aut}_{\mathcal{F}}^{\vee}(A) \longrightarrow \Delta$$

be the homomorphism defined by setting $\mu_A(\alpha) = \mu(\overline{\alpha})$ for some $\overline{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(S)$ which extends α. Since $\alpha \in N_G(\mathbf{U}) = N_{\mathrm{Aut}_{\mathcal{F}}(A)}(\mathrm{Aut}_S(A))$, it does extend to some $\overline{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(S)$ by the extension axiom. If β is another extension, then $\beta^{-1}\overline{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(S)$ induces the identity on A and (since $C_S(A) = A$) on S/A, and hence by definition of μ lies in $\mathrm{Ker}(\mu)$. Thus $\mu_A(\alpha) = \mu(\overline{\alpha})$ is independent of the choice of $\overline{\alpha}$.

Theorem 2.8. *Fix an odd prime p, and a p-group S which contains a unique abelian subgroup $A \trianglelefteq S$ of index p. Let $\mathcal{F}$ be a reduced fusion system over S for which A is $\mathcal{F}$-essential. We use the notation of 2.1 and 2.4, and also set $\mathbf{E}_0 = \mathbf{E}_{\mathcal{F}} \smallsetminus \{A\}$ and $G = \mathrm{Aut}_{\mathcal{F}}(A)$ and $G^{\vee} = \mathrm{Aut}_{\mathcal{F}}^{\vee}(A)$, so that $\mathbf{U} = \mathrm{Aut}_S(A) \in \mathrm{Syl}_p(G)$. Let $m \ge 3$ be such that $|A/Z| = p^{m-1}$. Set $\sigma = \prod_{i=0}^{p-1} {}^{\mathbf{x}^i}(\mathbf{a}) = (\mathbf{a}\mathbf{x})^p\mathbf{x}^{-p} \in Z$. Then the following hold:*

(a) *$Z_0 = C_A(\mathbf{U}) \cap [\mathbf{U}, A]$ has order p, and hence $A_0 = C_A(\mathbf{U})[\mathbf{U}, A]$ has index p in A.*

(b) *There are no non-trivial G-invariant subgroups of $Z = C_A(\mathbf{U})$, aside (possibly) from Z_0.*

(c) $[G, A] = A$.

(d) *One of the conditions (i)–(iv) holds, described in Table 2.1, where I is always some nonempty subset of $\{0, 1, \ldots, p-1\}$.*

	$\mu_A(G^\vee)$	$G = O^{p'}(G)X$ where	$m \pmod{p-1}$	σ	$\mathbf{E}_0$
(i)	Δ	$X = G^\vee$	$\equiv 0$	$\sigma \in \mathrm{Fr}(Z)$	$\mathcal{H}_0 \cup \mathcal{B}_*$
(ii)	Δ	$X = G^\vee$	$\equiv -1$	$\sigma \in \mathrm{Fr}(Z)$	$\mathcal{B}_0 \cup \mathcal{H}_*$
(iii)	$\geq \Delta_{-1}$	$X = \mu_A^{-1}(\Delta_{-1})$	$\equiv -1$	$\sigma \in \mathrm{Fr}(Z)$	$\bigcup_{i\in I} \mathcal{H}_i$
			$-$	$-$	$\mathcal{H}_0$
(iv)	$\geq \Delta_0$	$X = \mu_A^{-1}(\Delta_0)$	$\equiv 0$	$\sigma \in \mathrm{Fr}(Z)$	$\bigcup_{i\in I} \mathcal{B}_i$
		Z_0 not G-invariant	$-$	$-$	$\mathcal{B}_0$

TABLE 2.1

Conversely, for each G, A, $\mathbf{U} \in \mathrm{Syl}_p(G)$, and $\mathbf{E}_0 \subseteq \mathcal{H} \cup \mathcal{B}$ which satisfy conditions (a)–(d), where $|\mathbf{U}| = p$, $\mathbf{U} \ntrianglelefteq G$, $G^\vee = \{\alpha \in N_G(\mathbf{U}) \,|\, [\alpha, Z] \leq Z_0\}$, and $\mathcal{B}$ and $\mathcal{H}$ are defined as in Notation 2.1 for $S = A \rtimes \mathbf{U}$, there is a reduced fusion system $\mathcal{F}$ over $A \rtimes \mathbf{U}$ with $\mathrm{Aut}_{\mathcal{F}}(A) = G$ and $\mathbf{E}_{\mathcal{F}} = \mathbf{E}_0 \cup \{A\}$, unique up to isomorphism. All such fusion systems are simple. All such fusion systems are exotic, except for the fusion systems of the simple groups listed in Table 2.2.
Such a fusion system $\mathcal{F}$ has a proper strongly closed subgroup if and only if $A_0 = C_A(\mathbf{U})[\mathbf{U}, A]$ is G-invariant, and $\mathbf{E}_0 = \mathcal{H}_i$ or $\mathcal{B}_i$ for some $i = 0, \ldots, p-1$, in which case $A_0H_i = A_0B_i$ is strongly closed.

In Table 2.2, e is such that p^e is the exponent of A. In all cases except when $\Gamma \cong PSL_p(q)$ and $e > 1$, A is homocyclic.

Proof. We prove in Step 1 that conditions (a)–(d) are necessary, and in Step 2 that they are sufficient for the existence of a reduced fusion system. We also prove the uniqueness of the fusion system in Step 2. In Step 3, we list all strongly closed subgroups for the fusion systems constructed in Step 2, and then prove in Step 4 that they are all simple. In Step 5, we handle the question of which of these fusion systems are realizable.

Step 1: Assume that $\mathcal{F}$ is a reduced fusion system over S. We must show that conditions (a)–(d) hold.

(a) By Lemma 2.2(c,d), $|Z_0| = p = |A/A_0|$.

(b,c) Since $\mathcal{F}$ is reduced, $O_p(\mathcal{F}) = 1$ and $O^p(\mathcal{F}) = \mathcal{F}$. So these claims follow from points (a) and (b), respectively, in Lemma 2.7.

(d) Cases (i)–(iv) here correspond exactly to cases (i)–(iv) of Lemma 2.7(c). The conditions on $\mathbf{E}_0$ follow immediately from that lemma, while the conditions on $\mu_A(\mathrm{Aut}_{\mathcal{F}}^\vee(A)) = \mu(\mathrm{Aut}_{\mathcal{F}}^\vee(S))$ and those on $m \pmod{p-1}$ follow from Lemma 2.6(a). By Lemma 2.5(d), for $P \in \mathcal{H}_* \cup \mathcal{B}_*$, $\sigma \in \mathrm{Fr}(Z)$ if and only if P splits over $P \cap A$, and this is a necessary condition to have $P \in \mathbf{E}_{\mathcal{F}}$ by Lemma 2.3(a,b).

Γ	p	conditions	$\mathrm{rk}(A)$	e	m	$G=\mathrm{Aut}_\Gamma(A)$	$\mathbf{E}_0$
A_{pn}	p	$p\le n<2p$	n	1	p	$\frac12 C_{p-1}\wr S_n$	$\mathcal{H}_0$
$PSp_4(p)$	p	———————	3	1	3	$GL_2(p)/\{\pm I\}$	$\mathcal{B}_0$
$PSL_p(q)$	p	$v_p(q{-}1)=1,\ p>3$	$p{-}2$	1	$p-2$	S_p	$\mathcal{H}_0\cup\mathcal{H}_*$
$PSL_p(q)$	p	$p^2\vert(q{-}1),\ p>3$	$p{-}1$	$v_p(q{-}1)$	$e(p{-}1)-1$	S_p	$\mathcal{H}_0\cup\mathcal{H}_*$
$PSL_n(q)$	p	$p\vert(q{-}1),\ p{<}n{<}2p$	$n{-}1$	$v_p(q{-}1)$	$e(p{-}1)+1$	S_n	$\mathcal{B}_0$
$P\Omega_{2n}^+(q)$	p	$p\vert(q{-}1),\ p{\le}n{<}2p$	n	$v_p(q{-}1)$	$e(p{-}1)+1$	$C_2^{n-1}\rtimes S_n$	$\mathcal{B}_0$
${}^2F_4(q)$	3	$q\ge 8$	2	$v_3(q{+}1)$	$2e$	$GL_2(3)$	$\mathcal{B}_0\cup\mathcal{B}_*$
$E_n(q)$	5	$n=6,7,\ p\vert(q{-}1)$	n	$v_p(q{-}1)$	$4e+1$	$W(E_n)$	$\mathcal{B}_0$
$E_n(q)$	7	$n=7,8,\ p\vert(q{-}1)$	n	$v_p(q{-}1)$	$6e+1$	$W(E_n)$	$\mathcal{B}_0$
$E_8(q)$	5	$q\equiv\pm2 \pmod 5$	4	$v_5(q^4{-}1)$	$4e$	$(C_4\circ 2^{1+4}).S_6$	$\mathcal{H}_0\cup\mathcal{B}_*$
Co_1	5	———————	3	1	3	$4\times S_5$	$\mathcal{B}_0\cup\mathcal{H}_*$

TABLE 2.2

To see why the conditions on G hold, let
$$R\colon \mathrm{Aut}_{\mathcal{F}}(S) \longrightarrow N_{\mathrm{Aut}_{\mathcal{F}}(A)}(\mathrm{Aut}_S(A)) = N_G(\mathbf{U})$$
be the homomorphism induced by restriction. By the extension axiom and by definition, R sends subgroups of $\mathrm{Aut}_{\mathcal{F}}(S)$ as follows:

X	$R^{-1}(Y)$	$\mathrm{Aut}_{\mathcal{F}}(S)$	$\mathrm{Aut}_{\mathcal{F}}^\vee(S)$	$\mathrm{Aut}_{\mathcal{F}}^{(A)}(S)$	$\mathrm{Aut}_{\mathcal{F}}^\vee(S)\cap\mu^{-1}(\Delta_r)$
$R(X)$	Y	$N_G(\mathbf{U})$	$\mathrm{Aut}_{\mathcal{F}}^\vee(A)$	$N_{O^{p'}(G)}(\mathbf{U})$	$\mu_A^{-1}(\Delta_r)$

(1)

In other words, the groups in the second row are the images of those in the first, and those in the first are the inverse images of those in the second. By the Frattini argument, $G=O^{p'}(G){\cdot}N_G(\mathbf{U})=O^{p'}(G){\cdot}R(\mathrm{Aut}_{\mathcal{F}}(S))$. The claims in (d) about generators for G now follow from the formulas for $\mathrm{Aut}_{\mathcal{F}}(S)$ in Lemma 2.7(c) (and since $R(\mathrm{Aut}_{\mathcal{F}}^{(A)}(S))\le O^{p'}(G)$).

Step 2: Now assume that A, G, and $\mathbf{E}_0$ are as above and satisfy (a)–(d). We will show that they are realized by a unique reduced fusion system $\mathcal{F}$.

Set $\Gamma=A\rtimes G$, and identify $S=A\rtimes\mathbf{U}\in\mathrm{Syl}_p(\Gamma)$. Choose a generator $\mathbf{x}\in\mathbf{U}<S$. Choose $\mathbf{a}\in A\smallsetminus A_0$ so that $S'\langle\mathbf{a}\rangle$ is normalized by $N_G(\mathbf{U})$. Set $Z=Z(S)$, $Z_2=Z_2(S)$, $H_i=Z\langle\mathbf{xa}^i\rangle$, and $B_i=Z_2\langle\mathbf{xa}^i\rangle$, as in Notation 2.1 and 2.4.

Set $\mathcal{F}_0=\mathcal{F}_S(\Gamma)$. We will apply Lemmas 2.5 and 2.6(b) here with $\mathcal{F}_0$ in the role of $\mathcal{F}$. Note that $\mathrm{Aut}_{\mathcal{F}_0}^\vee(S)$ is the group of all $\alpha\in\mathrm{Aut}_\Gamma(S)$ that induce the identity on Z/Z_0.

Let $Q_1,\dots,Q_k\in\mathbf{E}_0\cap\{B_i,H_i\,|\,0\le i\le p-1\}$ be a set of representatives for the Γ-conjugacy classes containing subgroups in $\mathbf{E}_0$. For each $i\le k$, set $K_i=\mathrm{Aut}_\Gamma(Q_i)$. Set $t_i=-1$ if $Q_i\in\mathcal{H}$, and $t_i=0$ if $Q_i\in\mathcal{B}$. Since $\sigma\in\mathrm{Fr}(Z)$ whenever $Q_i\in\mathcal{H}_*\cup\mathcal{B}_*$, Q_i splits over $Q_i\cap A$ in all cases by Lemma 2.5(d). By the assumptions in (d), there is $\alpha\in\mathrm{Aut}_{\mathcal{F}_0}^\vee(S)$ such that $\mu(\alpha)$ generates Δ_t. By Lemma 2.5(c) and the assumptions on m, $\alpha(Q_i)$ is S-conjugate to Q_i. So upon composing α with an inner automorphism (hence in $\mathrm{Ker}(\mu)$), we can arrange that $\alpha\in N_{\mathrm{Aut}_{\mathcal{F}_0}^\vee(S)}(Q_i)$.

Each element of $K_i = \mathrm{Aut}_\Gamma(Q_i)$ extends to an element of $\mathrm{Aut}_\Gamma(S)$ since $N_\Gamma(Q_i) \le N_\Gamma(AQ_i)$, and so $K_i = \{\beta|_{Q_i} \mid \beta \in N_{\mathrm{Aut}_\Gamma(S)}(Q_i)\}$. By Lemma 2.6(b), there is a unique subgroup $\Theta_i \le \mathrm{Aut}(Q_i)$ such that $\Theta_i \ge \mathrm{Inn}(Q_i)$, $\mathrm{Aut}_S(Q_i) \in \mathrm{Syl}_p(\Theta_i)$, $N_{\Theta_i}(\mathrm{Aut}_S(Q_i)) = K_i$, $[\beta, Z] \le Z_0$ for $\beta \in N_{O^{p'}(\Theta_i)}(\mathrm{Aut}_S(Q_i))$, and $O^{p'}(\Theta_i)/\mathrm{Inn}(Q_i) \cong SL_2(p)$.

Set $\mathcal{F} = \langle \mathcal{F}_0, \Theta_1, \dots, \Theta_k \rangle$: the smallest fusion system over S which contains $\mathcal{F}_0$, and such that $\mathrm{Aut}_\mathcal{F}(Q_i) \ge \Theta_i$ for each i. Note in particular that $\mathrm{Aut}_\mathcal{F}(S) = \mathrm{Aut}_{\mathcal{F}_0}(S) = \mathrm{Aut}_\Gamma(S)$. By [BLO4, Proposition 5.1], to see that $\mathcal{F}$ is saturated, it suffices to check that the following conditions hold.

(1) **For $\boldsymbol{i \ne j}$, $\boldsymbol{Q_i}$ is not $\boldsymbol{\Gamma}$-conjugate to a subgroup of $\boldsymbol{Q_j}$.** By assumption, Q_i and Q_j are not Γ-conjugate, and hence are not $\mathcal{F}$-conjugate since there is no larger group in the generating set which could conjugate the one into the other. If $Q_i = H_k$ and $Q_j = B_\ell$ for some $k, \ell \in \{0, 1, \dots, p-1\}$, then by (d), either $k = 0$ and $\ell \ne 0$ or vice versa, so B_k and B_ℓ are not $\mathrm{Aut}_\Gamma(S)$-conjugate (hence not $\mathcal{F}_0$-conjugate, hence not $\mathcal{F}$-conjugate) by Lemma 2.5(c), applied with $\mathcal{F}_0$ in the role of $\mathcal{F}$.

(2) **For each $\boldsymbol{i}$, $\boldsymbol{Q_i}$ is $\boldsymbol{p}$-centric in $\boldsymbol{\Gamma}$, but no proper subgroup $\boldsymbol{P < Q_i}$ is $\boldsymbol{\mathcal{F}}$-centric nor an essential $\boldsymbol{p}$-subgroup of $\boldsymbol{\Gamma}$.** Here, a p-subgroup $Q \le \Gamma$ is essential if it is p-centric in Γ and $\mathrm{Out}_\Gamma(Q)$ has a strongly p-embedded subgroup [BLO4, Definition 3.2(b)]. In all cases, $C_\Gamma(Q_i) \cap A = Z = Z(Q_i) \cap A$. Also, $Q_i \in \mathcal{H}$ implies $C_\Gamma(Q_i) \le A \rtimes C_G(Z)$ where $\mathbf{U} \in \mathrm{Syl}_p(C_G(Z))$, while $Q_i \in \mathcal{B}$ implies $C_\Gamma(Q_i) \le A \rtimes C_G(Z_2)$ where $C_G(Z_2)$ has order prime to p. Thus $Z(Q_i) \in \mathrm{Syl}_p(C_\Gamma(Q_i))$ in both cases, so Q_i is p-centric in Γ by definition.

Each proper subgroup of Q_i either does not contain Z or is in the Θ_i-orbit of (hence $\mathcal{F}$-conjugate to) a proper subgroup of A, and in either case, is not $\mathcal{F}$-centric. Since $A \trianglelefteq \Gamma$ (so $A \trianglelefteq \mathcal{F}_0 = \mathcal{F}_S(\Gamma)$), each essential p-subgroup of Γ contains A by Proposition 1.4(a).

(3) **For each $\boldsymbol{i}$, $\boldsymbol{p \nmid [\Theta_i{:}K_i]}$ and $\boldsymbol{K_i/\mathrm{Inn}(Q_i)}$ is strongly $\boldsymbol{p}$-embedded in $\boldsymbol{\Theta_i/\mathrm{Inn}(Q_i)}$.** By assumption, $\mathrm{Aut}_S(Q_i) \in \mathrm{Syl}_p(\Theta_i)$, and $K_i = N_{\Theta_i}(\mathrm{Aut}_S(Q_i))$. Thus $\mathrm{Out}_S(Q_i) \in \mathrm{Syl}_p(\Theta_i/\mathrm{Inn}(Q_i))$ has order p (and is not normal), and so its normalizer has index prime to p and is strongly p-embedded in $\Theta_i/\mathrm{Inn}(Q_i)$.

It remains to prove that $\mathcal{F}$ is reduced. By (b), there are no non-trivial G-invariant subgroups of Z except possibly for Z_0, $\mathbf{E}_\mathcal{F} \cap \mathcal{H} \ne \varnothing$ in cases (d.i)–(d.iii), and Z_0 is not G-invariant in case (d.iv). Hence $O_p(\mathcal{F}) = 1$ by Lemma 2.7(a). Also, $O^p(\mathcal{F}) = \mathcal{F}$ by Lemma 2.7(b), and since $[G, A] = A$ by (c).

As for showing that $O^{p'}(\mathcal{F}) = \mathcal{F}$, we claim that the necessary conditions in Lemma 2.7(c.i–c.iv) follow from (d.i–d.iv). The conditions on $\mathbf{E}_0 = \mathbf{E}_\mathcal{F} \smallsetminus \{A\}$ are clear. As for the conditions on $\mathrm{Aut}_\mathcal{F}(S)$, these follow since if $G = O^p(G)X$ by (d) for $X \le N_{\mathrm{Aut}(A)}(\mathbf{U})$, and $R\colon \mathrm{Aut}(S) \longrightarrow N_{\mathrm{Aut}(A)}(\mathbf{U})$ is as in Step 1, then

$$\mathrm{Aut}_\mathcal{F}(S) = R^{-1}(N_G(\mathbf{U})) = R^{-1}(N_{O^{p'}(G)}(\mathbf{U}))\cdot R^{-1}(X) = \mathrm{Aut}_\mathcal{F}^{(A)}(S)\cdot R^{-1}(X),$$

where $R^{-1}(X)$ is described in (1).

The uniqueness of $\mathcal{F}$ follows from the uniqueness of the Θ_i (Lemma 2.6(b)).

Step 3: We next list the proper non-trivial strongly closed subgroups in the reduced fusion system $\mathcal{F}$ constructed in Step 2.

Assume that $1 \ne Q < S$ is strongly closed in $\mathcal{F}$. If $Q \le Z$, then Q is contained in all $\mathcal{F}$-essential subgroups, so $Q \trianglelefteq \mathcal{F}$ by Proposition 1.4(a), which is impossible since $O_p(\mathcal{F}) = 1$. Thus $Q \nleq Z$.

Now, $(QZ/Z) \cap Z(S/Z) \neq 1$ since $Q \trianglelefteq S$, so $Q \cap Z_2 \nleq Z$. Fix $g \in (Q \cap Z_2) \smallsetminus Z$. Then $Q \geq [g, S] = Z_0$ since $Q \trianglelefteq S$.

Since $A \ntrianglelefteq \mathcal{F}$, there is $P \in \mathbf{E}_0$. If $P \in \mathcal{B}$, then the $\mathrm{Aut}_{\mathcal{F}}(P)$-orbit of $g \in Z_2 \leq P$ is not contained in A. If $P \in \mathcal{H}$, then the $\mathrm{Aut}_{\mathcal{F}}(P)$-orbit of $Z_0 \leq P$ is not contained in A. So in either case, $Q \nleq A$. Hence $Q \geq [Q, S] \geq [\mathbf{U}, A] = S'$.

Set $G_0 = O^{p'}(G)$. Since $Q \cap A$ is normalized by the action of $G = \mathrm{Aut}_{\mathcal{F}}(A)$, and contains $[\mathbf{U}, A]$ where $\mathbf{U} \in \mathrm{Syl}_p(G)$, $Q \geq [G_0, A]$. Since $[G, A] = A$ by (c), the group G/G_0 of order prime to p acts on $A/[G_0, A]$ with $C_{A/[G_0,A]}(G/G_0) = 1$. Also, $G = G_0\mathrm{Aut}_{\mathcal{F}}^{\vee}(A)$ by (d), so $C_{A/[G_0,A]}(\mathrm{Aut}_{\mathcal{F}}^{\vee}(A)) = 1$. Since $\mathrm{Aut}_{\mathcal{F}}^{\vee}(A)$ acts trivially on Z/Z_0 (and $Z_0 \leq S' \leq [G_0, A]$), the natural homomorphism $Z/Z_0 \longrightarrow A/[G_0, A]$ has trivial image. Hence $A_0 = ZS' \leq [G_0, A] \leq Q$.

Since $Q < S$ by assumption, this proves that $Q = A_0\langle \mathbf{xa}^i \rangle = A_0H_i$ for some i, and has index p in S. If A_0H_i is strongly closed, then neither H_j nor B_j can be $\mathcal{F}$-essential for any $j \neq i$ $(0 \leq j \leq p-1)$, so $\mathbf{E}_0 = \mathcal{H}_i$ or $\mathcal{B}_i$.

Conversely, if $\mathbf{E}_0 = \mathcal{H}_i$ or $\mathcal{B}_i$ for some $0 \leq i \leq p-1$ and A_0 is normalized by G, then A_0H_i is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$ and $\mathrm{Aut}_{\mathcal{F}}(A)$ and contains all $\mathcal{F}$-essential subgroups other than A, hence is strongly closed in $\mathcal{F}$.

Step 4: We now prove that $\mathcal{F}$ is simple. Assume otherwise. Then there is a proper normal subsystem $\mathcal{E} \trianglelefteq \mathcal{F}$ over a nontrivial strongly closed subgroup $1 \neq T \leq S$, and $T < S$ since $O^{p'}(\mathcal{F}) = \mathcal{F}$. We just saw that this implies $T = A_0H_i$ for some i, and $\mathbf{E}_0 = \mathcal{H}_i$ or $\mathcal{B}_i$.

Assume first that $m \geq 4$. Thus $|S'| = |A/Z| = p^{m-1} \geq p^3$. For $P \in \mathbf{E}_0$, we have $P < T$, so $N_S(P) \leq A_0P = T$. Thus $\mathrm{Aut}_S(P) \leq \mathrm{Aut}_{\mathcal{E}}(P)$ and $\mathrm{Aut}_{\mathcal{E}}(P) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(P)$, which imply that $P \in \mathbf{E}_{\mathcal{E}}$. (Since $\mathrm{Aut}_{\mathcal{E}}(P)$ has Sylow p-subgroups which are non-normal and have order p, it has strongly p-embedded subgroups.) Also, $A_0 = A \cap T$ is the unique abelian subgroup of index p in T since $|[T, T]| = \frac{1}{p}|S'| \geq p^2$, so by Steps 1 and 2 applied to $\mathcal{E}$, we have $\mathbf{E}_{\mathcal{E}} = \{A_0, \mathcal{H}_i\}$ or $\{A_0, \mathcal{B}_i\}$. Here, $\mathcal{H}_i$ or $\mathcal{B}_i$ plays the role for $\mathcal{E}$ that $\mathcal{H}$ or $\mathcal{B}$ plays for $\mathcal{F}$. So by Lemma 2.6(a) (or by Table 2.1 case (iii) or (iv)), $m - 1 \equiv t \pmod{p-1}$, where as usual, $t = -1$ if $\mathbf{E}_0 = \mathcal{H}_i$ and $t = 0$ if $\mathbf{E}_0 = \mathcal{B}_i$. Also, $G = O^{p'}(G)\mu_A^{-1}(\Delta_t)$, and by Lemma 2.5(c), elements in $\mu_A^{-1}(\Delta_t) = \mu_A^{-1}(\Delta_{m-1})$ act trivially on A/A_0. Since T is strongly closed, $A_0 = A \cap T$ is G-invariant, and so $O^{p'}(G)$ acts trivially on $A/A_0 \cong C_p$. Thus $[G, A] \leq A_0$, which contradicts (c). We conclude that $\mathcal{F}$ is simple.

Now assume that $m = 3$. Thus $|[\mathbf{U}, A_0]| = |A_0/Z| = p^{m-2} = p$, and A_0 is G-invariant. By Lemma 1.10 applied to the G-action on A_0, $A_0 = C_{A_0}(O^{p'}(G)) \times [O^{p'}(G), A_0]$, the first factor is G-invariant and contained in Z, and hence is trivial by (b). So by the same lemma, $A_0 \cong C_p^2$ and $O^{p'}(G) \cong SL_2(p)$. If $\alpha \in N_{O^{p'}(G)}(\mathbf{U})$ has order $p-1$, then for some generator $u \in (\mathbb{Z}/p)^\times$, $\alpha(z) = z^u$ for $z \in Z_0$, and $\alpha(z) \in z^{u^{-1}}Z_0$ for $z \in Z_2 \smallsetminus Z_0 = A_0 \smallsetminus Z_0$. In the notation of Lemma 1.11(b) (applied with A in the role of V), $tr = u^{-1}$ and $tr^2 = u$. So $t = u^{-3}$, and thus $\alpha(a) \in a^{u^{-3}}A_0$ for $a \in A \smallsetminus A_0$. Since $O^{p'}(G)$ acts trivially on A/A_0, this implies that $u^{-3} \equiv 1 \pmod{p}$, so $(p-1)|3$, which is impossible. Thus there are no strongly closed subgroups in this case, and so $\mathcal{F}$ is simple.

Step 5: By Lemma 1.5, if $\mathcal{F}$ is realizable, it is isomorphic to the fusion system of a finite simple group. (The hypotheses of Lemma 1.5 hold by the description in Step 3 of the strongly $\mathcal{F}$-closed subgroups.) Hence by Lemma 1.7, it is isomorphic to the fusion system of one of the simple groups listed in Table 2.2. □

Recall that when $\mathbf{U} \cong C_p$, then $\mathbb{F}_p\mathbf{U} \cong \mathbb{F}_p[X]/(X^p)$, and hence the indecomposable $\mathbb{F}_p\mathbf{U}$-modules are those of the form $\mathbb{F}_p[X]/(X^i)$ for $1 \le i \le p$. These are the "Jordan blocks" of an $\mathbb{F}_p\mathbf{U}$-module.

Notation 2.9. *Assume that G is a finite group, with $\mathbf{U} \in \mathrm{Syl}_p(G)$ of order p. Let M be an $\mathbb{F}_pG$-module, and set $Z = C_M(\mathbf{U})$ and $Z_0 = Z \cap [\mathbf{U}, M]$. Assume that $\dim(Z_0) = 1$; equivalently, that $M|_{\mathbf{U}}$ has just one non-trivial Jordan block. In this situation, we set*

$$G^\vee = \{\alpha \in N_G(\mathbf{U}) \,|\, [\alpha, Z] \le Z_0\},$$

and define $\mu_A \colon G^\vee \longrightarrow \Delta$ by setting $\mu_A(g) = (r, s)$ if ${}^g u = u^r$ and $g(z) = z^s$ for all $u \in \mathbf{U}$ and $z \in Z_0$.

Using this notation, we now get as an immediate consequence of Theorem 2.8 the following list of necessary and sufficient conditions for an $\mathbb{F}_pG$-representation A to give rise to a simple fusion system in the way described by the theorem.

Corollary 2.10. *Fix an odd prime p, a finite group G, and a finite dimensional, faithful $\mathbb{F}_pG$-module A. Fix $\mathbf{U} \in \mathrm{Syl}_p(G)$, and assume that $|\mathbf{U}| = p$ and $\mathbf{U} \ntrianglelefteq G$. Set $Z = C_A(\mathbf{U})$, $Z_0 = Z \cap [\mathbf{U}, A]$, and $m = \dim(A/Z) + 1$, and assume that $m \ge 3$. Then there is a simple fusion system $\mathcal{F}$ over a finite p-group S which contains A as its unique abelian subgroup of index p, where $G = \mathrm{Aut}_{\mathcal{F}}(A)$, if and only if the following conditions hold:*

(a) $|Z_0| = p$;

(b) *there are no non-trivial G-invariant subgroups of Z, aside (possibly) from Z_0;*

(c) $[G, A] = A$; *and*

(d) *one of the following holds: either*

  (d.1) $\mu_A(G^\vee) = \Delta$, $G = O^{p'}(G)G^\vee$, $m \equiv 0, -1 \pmod{p-1}$, *and* $\dim(A) \le p - 1$; *or*

  (d.2) $\mu_A(G^\vee) \ge \Delta_{-1}$ *and* $G = O^{p'}(G){\cdot}\mu_A^{-1}(\Delta_{-1})$; *or*

  (d.3) $\mu_A(G^\vee) \ge \Delta_0$, $G = O^{p'}(G){\cdot}\mu_A^{-1}(\Delta_0)$, *and Z_0 is not G-invariant.*

Condition (d.1) in Corollary 2.10 corresponds to the cases in Table 2.1 where $\sigma \in \mathrm{Fr}(Z)$. Since A has exponent p, this means that $\sigma = 1$, and hence that $A|_{\mathbf{U}}$ has no indecomposable summand of dimension p. (See the definition of σ in Theorem 2.8.) So by Lemma 3.7(a), $\dim(A) \le p - 1$ in these cases.

We will see in Section 3 (Propositions 3.5 and 3.7) that $m = \min(\mathrm{rk}(A), p)$ in the situation of Corollary 2.10.

We finish the section with some examples which show that proper strongly closed subgroups can be found in simple fusion systems of the type constructed in Theorem 2.8.

Example 2.11. Fix a prime $p \ge 5$. Set $\Gamma = S_p \times \mathbb{F}_p^\times$, and let $M \cong \mathbb{F}_p^p$ be the $\mathbb{F}_p\Gamma$-module where S_p acts by permuting the coordinates, and where $a \in \mathbb{F}_p^\times$ acts via $a{\cdot}\mathrm{Id}_M$. Fix $\mathbf{U} \in \mathrm{Syl}_p(\Gamma)$.

(a) Set $A = M$ and $S = A \rtimes \mathbf{U}$. Let $\mu_A \colon N_\Gamma(\mathbf{U}) \longrightarrow \Delta$ be as in Notation 2.9. Set $G = O^p(\Gamma)\mu^{-1}(\Delta_{-1})$. By Theorem 2.8 case (d.iii), there is a simple fusion system $\mathcal{F}$ over S such that $\mathrm{Aut}_{\mathcal{F}}(A) = \mathrm{Aut}_G(A)$, and such that $\mathbf{E}_{\mathcal{F}} = \{A\} \cup \mathcal{H}_0$ in the notation of 2.4. Furthermore, the subgroup $A_0\mathbf{U}$, where $A_0 = [\mathbf{U}, A]$, is strongly closed in $\mathcal{F}$.

(b) Set $A = M/C_M(O^{p'}(\Gamma))$ and $S = A \rtimes \mathbf{U}$. Thus $|A| = p^{p-1}$ and $|S| = p^p$. Let $\mu_A \colon N_\Gamma(\mathbf{U}) \longrightarrow \Delta$ be as in Notation 2.9. Set $G = O^{p'}(\Gamma)\mu^{-1}(\Delta_0)$. By Theorem

2.8 case (d.iv), there is a simple fusion system $\mathcal{F}$ over S such that $\mathrm{Aut}_{\mathcal{F}}(A) = \mathrm{Aut}_G(A)$, and such that $\mathbf{E}_{\mathcal{F}} = \{A\} \cup \mathcal{B}_0$ in the notation of 2.4. Furthermore, the subgroup $A_0\mathbf{U}$, where $A_0 = [\mathbf{U}, A]$, is strongly closed in $\mathcal{F}$.

(c) Let A, S, μ_A, and G be as in (b). Fix $I \subseteq \{0, 1, \ldots, p-1\}$ with $|I| \geq 2$. By Theorem 2.8 case (d.iv), there is a simple fusion system $\mathcal{F}$ over S such that $\mathrm{Aut}_{\mathcal{F}}(A) = \mathrm{Aut}_G(A)$ and $\mathbf{E}_{\mathcal{F}} = \{A\} \cup \bigcup_{i \in I} \mathcal{B}_i$, and no proper non-trivial subgroup of S is strongly closed in $\mathcal{F}$.

In all of these cases, G has index 2 in Γ, and $\mathcal{F}$ is exotic by Table 2.2 and the theorem.

3. Representation-theoretic preliminaries

From now on, we restrict attention to the case where the abelian group A in Section 2 has exponent p. In other words, we are looking at $\mathbb{F}_pG$-modules, for certain finite groups G, for which the conditions in Corollary 2.10 are satisfied. We begin with some representation theory that we will need, in particular the representation theory of groups with a Sylow p-subgroup of order p, which is very well understood. Throughout this section, p is an odd prime.

Definition 3.1. Let $\mathscr{G}_p$ be the class of finite groups whose Sylow p-subgroups are not normal and have order p. Let $\mathscr{G}_p^\wedge$ be the class of all $G \in \mathscr{G}_p$ such that $|\mathrm{Aut}_G(\mathbf{U})| = p-1$ for $\mathbf{U} \in \mathrm{Syl}_p(G)$.

The following notation will be used throughout this section, and in much of the rest of the paper.

Notation 3.2. *When p is odd and $G \in \mathscr{G}_p$, we*

- *fix an element x of order p in G and set $\mathbf{U} = \langle x \rangle \in \mathrm{Syl}_p(G)$; and*
- *set $N = N_G(\mathbf{U})$, $C = C_G(\mathbf{U})$, and $C' = C'_G(\mathbf{U}) = O_{p'}(C_G(\mathbf{U}))$.*

For background to the following discussion about vertices, sources, and Green correspondents of $\mathbb{F}_pG$-modules, we refer to [Be, Chapter 3], and especially to Sections 3.10–3.12.

Let V be an indecomposable $\mathbb{F}_pG$-module. A *vertex* for V is a minimal subgroup $P \leq G$ such that V is relatively P-projective; i.e., such that each surjection $W \to V$ which splits $\mathbb{F}_pP$-linearly is also $\mathbb{F}_pG$-linearly split. This is always a p-subgroup of G, and is uniquely determined up to conjugacy. If $G \in \mathscr{G}_p$, then since $\mathbf{U} \in \mathrm{Syl}_p(G)$ has order p, either V is $\mathbb{F}_pG$-projective and has trivial vertex, or V is non-projective and $\mathbf{U}$ is a vertex of V. Note that V is projective if and only if $V|_{\mathbf{U}}$ is projective, equivalently, if $\mathbf{U}$ (or x) acts on V with Jordan blocks all of size p.

In general, if P is a vertex of V then the restriction $V|_{N_G(P)}$ of V to $N_G(P)$ is the direct sum of an indecomposable $\mathbb{F}_pN_G(P)$-module W with vertex P, the *Green correspondent* of V, and other indecomposable modules with vertices that are contained in intersections $P^g \cap N_G(P)$ for $g \notin N_G(P)$, so in particular not equal to P and of order at most $|P|$. Thus when $\mathbf{U} \in \mathrm{Syl}_p(G)$ has order p and V is not projective, $V|_N \cong W \oplus X$ (recall $N = N_G(\mathbf{U})$), where W (the Green correspondent) is indecomposable and non-projective, and where X is projective.

The restriction of any non-projective, indecomposable $\mathbb{F}_pG$-module V to $\mathbf{U}$ is a sum of free modules and of copies of a fixed $\mathbb{F}_p\mathbf{U}$-module T, called the *source* of V. (In general, U is a sum of conjugates of the source, but for cyclic groups conjugate modules are isomorphic.) If the source of V is the trivial module (i.e., $V|_{\mathbf{U}}$ is a sum of free modules and trivial modules) then V is said to be *trivial source*. The source of an indecomposable $\mathbb{F}_pG$-module and the source of its Green correspondent are the same.

As C is normal in N, the simple $\mathbb{F}_pN$-modules restrict to C as a sum of simple modules. However, as $C = \mathbf{U} \times C'$, we see that the simple $\mathbb{F}_pC$-modules are just the simple $\mathbb{F}_pC'$-modules, which are irreducible ordinary characters as C' is a p'-group. In particular, all simple $\mathbb{F}_pN$-modules are trivial source, and indeed these are the only trivial-source $\mathbb{F}_pN$-modules. Hence if V is a simple, trivial-source $\mathbb{F}_pG$-module then its Green correspondent is also simple.

The following definition will be useful in our discussion.

Definition 3.3. When $G \in \mathscr{G}_p$, an $\mathbb{F}_pG$-module M is *minimally active* if the action of x on M has at most one non-trivial Jordan block (i.e., at most one Jordan block with non-trivial action).

Lemma 3.4. *Fix a field k of characteristic p, assume that $G \in \mathscr{G}_p$, and let M be a minimally active kG-module upon which $\mathbf{U}$ acts non-trivially.*

(a) *If $M = M_1 \oplus M_2$, where the M_i are kG-submodules, then $O^{p'}(G)$ acts trivially on exactly one of the M_i.*

(b) *M is indecomposable if and only if $C_M(O^{p'}(G)) \leq [O^{p'}(G), M]$.*

(c) *If M is indecomposable, then M is absolutely indecomposable. If M is simple then M is absolutely simple.*

(d) *We can decompose $M|_N = M_0 \oplus M_1$, where $\mathbf{U}$ acts trivially on M_0 and $M_1|_{\mathbf{U}}$ is indecomposable.*

Proof. **(a)** Since M is minimally active, a generator $x \in \mathbf{U}$ acts with at most one non-trivial Jordan block. So $\mathbf{U}$ must act trivially on at least one of the M_i, and hence $O^{p'}(G)$ acts trivially on it.

(b) Set $H = O^{p'}(G)$ for short. If M is decomposable, then $C_M(H) \nleq [H, M]$ by (a).

Assume, conversely, that $C_M(H) \nleq [H, M]$. Since $k[G/H]$ is semisimple, there is a kG-submodule $V_0 \leq C_M(H)$ such that $V_0 \neq 0$ and $C_M(H) = V_0 \oplus (C_M(H) \cap [H, M])$. For the same reason, there is $V_1 \leq M$ such that $V_1 \geq [H, M]$ and $M/[H, M] = (V_0 + [H, M])/[H, M] \oplus (V_1/[H, M])$. Thus $M = V_0 \oplus V_1$, $V_1 \neq 0$ since $[H, M] \neq 0$ (since $\mathbf{U}$ acts non-trivially), and thus M is decomposable.

(c) Set $\overline{M} = \bar{k} \otimes_k M$ for short. If M is indecomposable, then $C_M(O^{p'}(G)) \leq [O^{p'}(G), M]$ by (b), so $C_{\overline{M}}(O^{p'}(G)) \leq [O^{p'}(G), \overline{M}]$, and $\overline{M}$ is indecomposable by (b) again.

If M is simple, then $\mathrm{End}_{kG}(M) \cong \mathbb{F}_{p^m}$ for some $m \geq 1$. Then $m \mid \dim(C_M(\mathbf{U}) \cap [\mathbf{U}, M]) = 1$, so $m = 1$, and M is absolutely simple.

(d) If $C_M(\mathbf{U}) \leq [\mathbf{U}, M]$, then $M|_{\mathbf{U}}$ is indecomposable by (b), applied with $\mathbf{U}$ in the role of G, and we take $(M_0, M_1) = (0, M)$. Otherwise, $M|_N$ is decomposable by (b), this time applied with N in the role of G, and $\mathbf{U} = O^{p'}(N)$ acts trivially on all but one of its indecomposable summands by (a). □

Minimally active modules are what is needed in the situation of Theorem 2.8 and Corollary 2.10. This is made more precise in the next proposition.

Proposition 3.5. *If A is an $\mathbb{F}_pG$-module that satisfies the hypotheses of Corollary 2.10, then $G \in \mathscr{G}_p^\wedge$, and A is minimally active and indecomposable.*

Proof. By Corollary 2.10(a), $\dim(C_A(\mathbf{U}) \cap [\mathbf{U}, A]) = 1$, so the action of x has only one non-trivial Jordan block. Thus A is minimally active. Also, $\mathbf{U} \in \mathrm{Syl}_p(G)$ has order p and $|\mathrm{Aut}_G(\mathbf{U})| = p - 1$ by the conditions in Corollary 2.10(d), so $G \in \mathscr{G}_p^\wedge$.

If $A = A_1 \oplus A_2$, where the A_i are $\mathbb{F}_pG$-submodules, then by Lemma 3.4, at least one of its direct factors lies in $Z = C_A(\mathbf{U})$ and intersects trivially with $[\mathbf{U}, A]$. In the terminology of Corollary 2.10, this is a non-trivial G-invariant subgroup of Z which is not equal to Z_0, contradicting point (b) in the corollary. Thus A is indecomposable. □

We note the following easy lemma.

Lemma 3.6. *The property of being minimally active is preserved under taking submodules, quotients, dual, tensoring by a 1-dimensional module, and restricting to subgroups in $\mathscr{G}_p$.*

We will show that of the almost simple groups which lie in $\mathscr{G}_p^\wedge$, very few possess minimally active modules. Lemma 3.6 shows that if M is a minimally active $\mathbb{F}_pG$-module and $H \leq G$, then $M|_H$ is also minimally active. This means that we can argue inductively and embed, say, $SL_{n-1}(q)$ into $SL_n(q)$. Here the automizers of cyclic subgroups are the same, and so if $SL_{n-1}(q)$ is not in $\mathscr{G}_p^\wedge$ then neither is $SL_n(q)$.

The next result describes some of the basic properties of minimally active modules.

Proposition 3.7. *Let $G \in \mathscr{G}_p$, and assume Notation 3.2. Then the following hold for each indecomposable, minimally active $\mathbb{F}_pG$-module M on which $\mathbf{U}$ acts faithfully.*

(a) *Either*
- $\dim(M) < p$ *and* $M|_\mathbf{U}$ *is indecomposable, or*
- $\dim(M) = p$, $M|_\mathbf{U}$ *is indecomposable and free, and* M *is projective, or*
- $\dim(M) > p$, M *is a trivial-source module, and* $M|_\mathbf{U}$ *is the sum of one copy of* $\mathbb{F}_p\mathbf{U}$ *and a module with fixed action.*

(b) M *is a trivial-source module if and only if* $\dim(M) \geq p$.

(c) *If* $\dim(M) \geq p + 2$ *then* M *is simple and absolutely simple.*

(d) *If M is simple and has trivial source, then M is the reduction modulo p of a $\mathbb{Z}_p$-lattice in a simple $\mathbb{Q}_pG$-module. Moreover, χ_M extends to an absolutely irreducible ordinary character.*

(e) *If* $\dim(M) > p$, *then the Green correspondent V of M is simple.*

Proof. **(a)** By Lemma 3.4(d), we can write $M|_N = V_0 \oplus V_1$, where $V_1|_\mathbf{U}$ is a non-trivial Jordan block and V_0 has fixed action of $\mathbf{U}$. (Recall that $N = N_G(\mathbf{U})$.) If $V_0 = 0$, then $\dim(M) \leq p$, as p is the largest size of a Jordan block, and $M|_\mathbf{U} \cong V_1|_\mathbf{U}$ is indecomposable. If in addition, $\dim(V_1) = p$, then $V_1 \cong M|_\mathbf{U}$ is projective, and hence M is projective since $\mathbf{U} \in \mathrm{Syl}_p(G)$ (cf. [Be, Corollary 3.6.9]).

If $V_0 \neq 0$, then M is not projective, so $M|_N$ is the sum of a projective module and the Green correspondent of M. Thus V_1 is projective, hence of rank p, and so $\dim(M) > p$. Also, M is a trivial source module since $\mathbf{U}$ acts trivially on V_0.

(b) If $\dim(M) > p$, then we are done by (a). If $\dim(M) = p$, then M is projective, and hence has trivial vertex and trivial source. If $\dim(M) < p$, then $M|_\mathbf{U}$ is indecomposable, hence is the source of M, and has non-trivial action since $M|_\mathbf{U}$ is assumed to be faithful.

(c) Assume that $\dim(M) \geq p+2$. By Lemma 3.4(d), $M|_N = V_0 \oplus V_1$, where V_1 is $\mathbb{F}_pN$-projective (and $V_1|_{\mathbf{U}} \cong \mathbb{F}_p\mathbf{U}$), and V_0 is the Green correspondent to M and is an indecomposable (hence simple) $\mathbb{F}_p[N/\mathbf{U}]$-module. Note that by the Frattini argument, $G = O^{p'}(G)N$.

Assume first that there is a non-trivial submodule $0 \neq M_0 < M$ on which $\mathbf{U}$ acts trivially. Then $O^{p'}(G)$ acts trivially on M_0, and $M_0 \leq C_M(\mathbf{U}) = V_0 \oplus C_{V_1}(\mathbf{U})$. Since V_0 is $\mathbb{F}_pN$-irreducible, $\dim(V_0) = \dim(M) - p \geq 2$, and $\dim(C_{V_1}(\mathbf{U})) = 1$, either $M_0 \geq V_0$ or $M_0 = C_{V_1}(\mathbf{U})$. If $M_0 \geq V_0$, then V_0 is an $\mathbb{F}_pG$-submodule of M (recall $G = O^{p'}(G)N$), hence a direct summand of M since M/V_0 is $\mathbb{F}_pG$-projective, which contradicts the indecomposability of M.

Thus $M_0 = C_{V_1}(\mathbf{U})$. As M/M_0 does not satisfy any of the conditions in (a), (since $\mathbf{U}$ acts faithfully) it must be decomposable. By the Krull–Schmidt theorem, each proper direct sum decomposition of $(M/M_0)|_N$ has a summand isomorphic to V_0, so M/M_0 contains a direct summand whose restriction to N is isomorphic to V_0, and which (by an argument similar to that in the last paragraph) must be equal to the image of V_0 in M/M_0. Hence $V_0 \oplus C_{V_1}(\mathbf{U}) = C_M(\mathbf{U})$ is an $\mathbb{F}_pG$-submodule, and we just showed that this is impossible.

Now assume that $1 \neq M_0 < M$ is an arbitrary non-trivial proper submodule. We just showed that $\mathbf{U}$ acts non-trivially on M_0, and by a similar argument applied to the dual M^*, $\mathbf{U}$ also acts non-trivially on M/M_0. If either of M_0 or M/M_0 is decomposable, then it has a direct factor on which $\mathbf{U}$ acts trivially (Lemma 3.4), which contradicts the fact that M has no submodules or quotients on which $\mathbf{U}$ acts trivially. So each of M_0 and M/M_0 has one of the forms listed in (a). Since M is minimally active, $M|_{\mathbf{U}}$ and $(M/M_0)|_{\mathbf{U}}$ must both be indecomposable, so $\dim(C_M(\mathbf{U})) \leq \dim(C_{M_0}(\mathbf{U})) + \dim(C_{M/M_0}(\mathbf{U})) = 2$. But we already saw that $\dim(C_M(\mathbf{U})) = \dim(V_0) + 1 = \dim(M) - p + 1 \geq 3$, so this is impossible. Absolute simplicity now comes from Lemma 3.4(c).

(d) That M is the mod p reduction of a $\mathbb{Z}_p$-lattice $\widehat{M}$ is a general property of all trivial-source modules (see [Be, Corollary 3.11.4(i)]). If $\mathbb{Q}_p\widehat{M}$ is not simple, then it contains a non-trivial proper submodule $0 \neq W < \mathbb{Q}_p\widehat{M}$, and the mod p reduction of $W \cap \widehat{M}$ is a proper $\mathbb{F}_pG$-submodule of M, contradicting the assumption that M is simple. Since this also holds for $\mathbb{F}_{p^n}$ for $n \geq 1$, we get absolute irreducibility since M is absolutely simple by Lemma 3.4(c).

By Lemma 3.4(c), $\bar{\mathbb{F}}_p \otimes_{\mathbb{F}_p} M$ is a simple $\bar{\mathbb{F}}_pG$-module. Hence by a similar argument, $K \otimes_{\mathbb{Q}_p} (\mathbb{Q}_p\widehat{M})$ is a simple KG-module for each finite extension $K \supset \mathbb{Q}_p$ by roots of unity. So $\bar{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} (\mathbb{Q}_p\widehat{M})$ is simple, and the character of $\mathbb{Q}_p\widehat{M}$ is irreducible when regarded as a complex character of G.

(e) By (a), M has trivial source, so its Green correspondent is an indecomposable $\mathbb{F}_p[N/\mathbf{U}]$-module, hence irreducible since $N/\mathbf{U}$ has order prime to p. □

The next lemma will be useful when showing that certain extensions of minimally active modules are again minimally active.

Lemma 3.8. *Fix $G \in \mathscr{G}_p$. Assume that V is an indecomposable $\mathbb{F}_pG$-module of dimension at most $p+1$, and that V has a nonzero submodule or quotient module which is minimally active. Then V is also minimally active.*

Proof. Assume V is not minimally active. Then $V|_N$ is the Green correspondent of V, and is thus indecomposable. By [Al, p. 42] and since $\mathbf{U}$ is normal and cyclic in N, $V|_N$ is uniserial in the sense of [Al, p. 26]. (Alperin always assumes we are working over an algebraically closed field, but this proof does not use that.) In particular, the socle $C_V(\mathbf{U})$ and the top $V/[\mathbf{U},V]$ are both irreducible $N/\mathbf{U}$-modules, and they have rank at least 2 since V is not minimally active. But this is impossible: if $W<V$ is a submodule, then $C_W(\mathbf{U})<C_V(\mathbf{U})$ has rank 1 if W is minimally active, while the image of W in $V/[\mathbf{U},V]$ has corank 1 if V/W is minimally active. □

As a consequence of Proposition 3.7, if M is a simple, minimally active module, then either $\dim(M)\le p$, or M has as Green correspondent the reduction modulo p of an irreducible ordinary character of G, whose minimal degrees are known in the case where G is quasisimple. The next result will help us to classify such modules.

Proposition 3.9. *The following hold for each faithful, indecomposable, minimally active $\mathbb{F}_pG$-module M.*

(a) *Suppose that $\dim(M)>p$, and set $a=\dim(M)-p$. Then*
- *a divides $|N/\mathbf{U}|$;*
- *if $N/\mathbf{U}$ is abelian, then $a=1$;*
- *if C is abelian, then a divides $|N/C|$; and*
- *if $C>\mathbf{U}$, then $a\le|C/\mathbf{U}|-1$.*

(b) *If $O^{p'}(G)=\langle x,y\rangle$ for some $x,y\in G$, where $|x|=|y|=p$ or $|x|=2$ and $|y|=p$, then for each central extension $\widetilde{G}$ of G of degree prime to p, the dimension of each minimally active faithful indecomposable $\mathbb{F}_p\widetilde{G}$-module is at most $2p-2$.*

Proof. **(a)** If $\dim(M)=p+1$, then all four statements hold. So we may assume that $\dim(M)>p+1$. In particular, by Proposition 3.7(c), M is absolutely simple, and the Green correspondent W of M is an absolutely simple module for N, and for the p'-group $N/\mathbf{U}$ since $\mathbf{U}$ acts trivially on W. Thus χ_W is an ordinary irreducible character for $N/\mathbf{U}$. Also, $\dim(W)=\dim(M)-p=a$.

In particular, $\dim(W)$ divides $|N/\mathbf{U}|$ (see, e.g., [Is, Theorem 3.11]), and $\dim(W)=1$ if $N/\mathbf{U}$ is abelian. If C is abelian, then we apply a theorem of Ito (see [Is, Theorem 6.15]) to get that $\dim(W)$ divides $|N/C|$. This proves the first three statements.

Set $\overline{W}=\bar{\mathbb{F}}_p\otimes_{\mathbb{F}_p}W$, where $\bar{\mathbb{F}}_p\supseteq\mathbb{F}_p$ is the algebraic closure. By Clifford theory (see [Fe, Theorem III.2.12]), $\overline{W}|_C\cong e{\cdot}[\bigoplus_{i=1}^k W_i]$, where $e\ge1$, and where $W_1,\dots,W_k$ are pairwise distinct irreducible $\bar{\mathbb{F}}_p[C/\mathbf{U}]$-modules which form one orbit under the N/C-action on the set of all irreducible representations. Also, $e=1$ since N/C is cyclic (see [Fe, Theorem III.2.14]), so $\overline{W}|_C$ is a sum of distinct irreducible representations. Since $C/\mathbf{U}\ne1$, N/C cannot act transitively on the set $\mathrm{Irr}(C/\mathbf{U})$, and hence $\dim_{\mathbb{F}_p}(W)<\dim_{\bar{\mathbb{F}}_p}(\bar{\mathbb{F}}_p[C/\mathbf{U}])=|C/\mathbf{U}|$.

(b) By Lemma 3.4(b), M is $\mathbb{F}_pO^{p'}(G)$-indecomposable if it is $\mathbb{F}_pG$-indecomposable. So we can assume $G=O^{p'}(G)$.

If $G=\langle x,y\rangle$ where x and y have order p, then since a minimally active module M has at most one non-trivial Jordan block, $C_M(x)$ and $C_M(y)$ have codimension at most $p-1$. This means that $C_M(x)\cap C_M(y)=C_M(O^{p'}(G))$ has codimension at most $2p-2$. Since M is simple, $\dim(M)\le2p-2$, as needed.

If $G=\langle t,y\rangle$ where $|t|=2$ and $|y|=p$, then $\langle {}^t y,y\rangle$ is normal of index at most 2 in G, hence equal to G since $G=O^{p'}(G)$. If $\widetilde{G}$ is a central extension of G of degree prime to p, where

$G = \langle x, y\rangle$ and $|x| = |y| = p$, then x and y lift to $\widetilde{x}, \widetilde{y} \in \widetilde{G}$ of order p, and $\langle \widetilde{x}, \widetilde{y}\rangle \geq O^{p'}(\widetilde{G})$. So in both cases, we are back in the first situation. □

Proposition 3.9(b) is useful because both sporadic and alternating groups are known to be generated by two elements of order p whenever the Sylow p-subgroup is cyclic (see [Cr2] for sporadic groups), and this can be checked for any individual groups we might encounter. In general, it appears that with few exceptions this is always true for finite simple groups, and such a statement is currently under investigation by the first author.

As a generalization of part of Proposition 3.9, we get the following condition, which is useful for bounding the size of minimally active modules for a group in terms of minimally active modules for a subgroup.

Proposition 3.10. *Let H be a subgroup of G such that $O^{p'}(H) = H$. Suppose that $O^{p'}(G) \leq \langle H_1, \ldots, H_n\rangle$ for some set $H_1, \ldots, H_n$ of n conjugates of H. Let s be the maximal dimension of an indecomposable, faithful, minimally active $\mathbb{F}_pH$-module. If M is an indecomposable, faithful, minimally active $\mathbb{F}_pG$-module such that $C_M(O^{p'}(G)) = 0$, then* $\dim(M) \leq ns$. *In particular, if $C_H(\mathbf{U})$ is abelian, then* $\dim(M) < 2np$.

Proof. The proof is similar to that of Proposition 3.9(b). If M is an indecomposable, faithful, minimally active $\mathbb{F}_pG$-module, then by Lemma 3.4(a), for each $1 \leq i \leq n$, the restriction of M to H_i must have a summand N_i of codimension at most s on which H_i acts trivially. The intersection of the N_i has codimension at most ns, and is contained in the proper submodule $C_M(O^{p'}(G))$. Since $C_M(O^{p'}(G)) = 0$ we have $\dim(M) \leq ns$, as claimed. □

Having found minimally active, simple modules, we would like to know whether there are minimally active, indecomposable modules built from them. In almost every case we will see that if V is minimally active then $\dim(V) > (p+1)/2$, so that Proposition 3.7(c) eliminates any extensions between non-trivial modules. However, this leaves open the possibility that V is minimally active for G and $H^1(G, V) \neq 0$, so that V has a minimally active extension with the trivial module; for example, the permutation module for the symmetric group S_p is minimally active and has a trivial submodule and a trivial quotient. Of course, by Proposition 3.7(c) again, $\dim(V)$ must be at most p for this to work. The next lemma deals with this case, when V is self dual.

Lemma 3.11. *Let G be a group in $\mathscr{G}_p^\wedge$, and let V be a self-dual, simple, minimally active module with* $\dim(V) \leq p$. *If $H^1(G, V) \neq 0$ then* $\dim(V) = p - 2$.

Proof. Set $m = \dim(V)$. If $m = p$, then V is projective and $H^1(G, V) = 0$, so $m < p$. Since $0 \neq H^1(G; V) \cong \mathrm{Ext}^1_{\mathbb{F}_pG}(\mathbb{F}_p, V)$, there is an indecomposable extension M of $\mathbb{F}_p$ by V (with V as submodule). Since restriction sends $H^1(G; V)$ injectively into $H^1(\mathbf{U}; V)$, $M|_{\mathbf{U}}$ is indecomposable and consists of one Jordan block. Set $M_0 = [\mathbf{U}, M] = V$, and $M_i = [\mathbf{U}, M_{i-1}]$ for each $i \geq 1$; then $\dim(M_i) = m - i$ for each $0 \leq i \leq m$.

Fix $g \in N$ such that ${}^gu = u^r$ where r generates $(\mathbb{Z}/p)^\times$. By Lemma 1.11(b), and since g acts trivially on $M/M_0 = M/V$, g acts on each M_{i-1}/M_i by multiplication by r^i. In particular, it acts on M_0/M_1 by multiplication by r and on M_{m-1} by multiplication by r^m, and since V is self-dual, $r^m \equiv r^{-1} \pmod{p}$. Thus $m+1 \equiv 0 \pmod{p-1}$, and since $0 < m < p$, we have $m = p - 2$. □

In the next lemma, we give some tools for handling some of the other properties listed in Corollary 2.10, especially those involving the homomorphism $\mu_V \colon G^\vee \longrightarrow \Delta$ of Notation 2.9. For any finite abelian group M, let $\mathrm{Aut}_{\mathrm{sc}}(M)$ be the group of scalar automorphisms: those of the form $(x \mapsto kx)$ for k prime to $|M|$.

Lemma 3.12. *Fix $G \in \mathscr{G}_p$, and let V be a faithful, minimally active $\mathbb{F}_pG$-module. Let $\mathbf{U} \in \mathrm{Syl}_p(G)$, $G^\vee \le N_G(\mathbf{U})$, and $\mu_V \colon G^\vee \longrightarrow \Delta$ be as in Notation 2.9.*

(a) *The homomorphism μ_V sends $G^\vee/\mathbf{U}$ injectively into Δ.*

(b) *If $G \in \mathscr{G}_p^\wedge$, $\dim(V) \le p$, and $G \ge \mathrm{Aut}_{\mathrm{sc}}(V)$, then $G^\vee = N_G(\mathbf{U})$ and $\mu_V(G^\vee) = \Delta$.*

(c) *Assume that $\dim(V) \ge p$, and set $Z = C_V(\mathbf{U})$. Then for each $g \in N_G(\mathbf{U}) \smallsetminus C_G(\mathbf{U})$, $\chi_V(g) = \chi_Z(g)$.*

Proof. Let $V_0 = C_V(\mathbf{U})[\mathbf{U}, V]$ and $[\mathbf{U}, V] = W_1 > W_2 > \cdots > W_m = 0$ be as in Lemma 1.11.

(a) This is essentially Lemma 2.5(a), but we give another proof here. Assume that $g \in G^\vee$ has order prime to p, and $\mu_V(g) = (1, 1)$. By Lemma 1.11(b), g acts via the identity on V/V_0, and on W_i/W_{i+1} for each $1 \le i \le m-1$ ($r = t = 1$ in the notation of the lemma). By definition of $G^\vee$, g acts via the identity on $C_V(\mathbf{U})/W_{m-1}$, and hence also acts via the identity on $C_V(\mathbf{U})[\mathbf{U}, V]/[\mathbf{U}, V] = V_0/W_1$. So by Lemma 1.8, g acts trivially on V, and hence $g = 1$ since G acts faithfully.

Thus $\mathrm{Ker}(\mu_V|_{G^\vee}) \le \mathbf{U}$, and the opposite inclusion is clear.

(b) Since $\dim(V) \le p$, $V|_\mathbf{U}$ contains only one Jordan block. So $\dim(C_V(\mathbf{U})) = 1$, $Z = Z_0$ in the notation of Corollary 2.10, and $G^\vee = N_G(\mathbf{U})$.

Let $r \in (\mathbb{Z}/p)^\times$ be a generator, and let $\psi_r \in \mathrm{Aut}_{\mathrm{sc}}(V) \le N_G(\mathbf{U})$ be the automorphism $(a \mapsto a^r)$. Then $\mu_V(\psi_r) = (1, r)$.

By assumption, there is $g \in N_G(\mathbf{U})$ such that ${}^g u = u^r$ for each $u \in \mathbf{U}$. Then $\mu_V(g) = (r, s)$ for some s, and thus $\mu_V(G^\vee) \ge \langle \mu_V(g), \mu_V(\psi_r) \rangle = \big\langle (1, r), (r, s) \big\rangle = \Delta$.

(c) Let $W_0 \le V$ and $V_0 \le Z$ be such that W_0 is a non-trivial Jordan block for the action of $\mathbf{U}$ and $V = W_0 \oplus V_0$. Since $\dim(V) \ge p$, $\dim(W_0) = p$ by Proposition 3.7(a). Hence $\dim(V/Z) = \dim(W_0/C_{W_0}(\mathbf{U})) = p - 1$.

Let $[\mathbf{U}, V] = [\mathbf{U}, W_0] = W_1 > W_2 > \cdots > W_p = 0$ be as defined above. By Lemma 1.11(a), $|W_i/W_{i+1}| = p$ for each $1 \le i \le p-1$, and also for $i = 0$ since $W_0/W_1 \cong W_0Z/W_1Z = V/V_0$.

Fix $g \in N_G(\mathbf{U})$. Let $r, t \in (\mathbb{Z}/p)^\times$ be such that ${}^g u = u^r$ for each $u \in \mathbf{U}$, and g acts on $W_0/W_1 \cong V/V_0$ via multiplication by t. Then by Lemma 1.11(b), for each $1 \le i \le p-1$, g acts on W_i/W_{i+1} via multiplication by tr^i. The action of g on $W_0/W_{p-1} \cong V/Z$ thus has eigenvalues $t, tr, tr^2, \ldots, tr^{p-2}$. Hence $\chi_V(g) - \chi_Z(g) = \sum_{i=0}^{p-2} \psi(tr^i)$ for some embedding $\psi \colon (\mathbb{Z}/p)^\times \longrightarrow \mathbb{C}^\times$. If $g \notin C_G(\mathbf{U})$, then $\psi(r) \ne 1$, the sum of this geometric series is zero since $\psi(r)^{p-1} = 1$, and hence $\chi_V(g) = \chi_Z(g)$. □

We now summarize the tools which will be used to compute $\mu_V(G^\vee)$ in later sections.

Proposition 3.13. *Fix $G \in \mathscr{G}_p^\wedge$, and let V be a faithful, minimally active $\mathbb{F}_pG$-module such that $G \ge \mathrm{Aut}_{\mathrm{sc}}(V)$. Let $\mathbf{U} \in \mathrm{Syl}_p(G)$, $G^\vee$, and μ_V be as in Notation 2.9.*

(a) *If $\dim(V) \le p$, then $\mu_V(G^\vee) = \Delta$ and $C_G(\mathbf{U}) = \mathbf{U} \times \mathrm{Aut}_{\mathrm{sc}}(V)$.*

(b) *If $\dim(V) > p$, then $|\mu_V(G^\vee)| = |G^\vee/\mathbf{U}| \le \frac{|N_G(\mathbf{U})/\mathbf{U}|}{p-1}$, with equality if $\dim(V) = p + 1$.*

(c) *If $\dim(V) = p + 1$, and $g \in N_G(\mathbf{U})$ has order prime to p and is such that c_g generates $\mathrm{Aut}(\mathbf{U})$, then the following hold.*

  (i) *If $|\chi_V(g)| = 2$, then $\mu_V(G^\vee) \ge \Delta_0$.*

  (ii) *If $\chi_V(g) = 0$, then $\mu_V(G^\vee) \ge \Delta_{(p-1)/2}$.*

  (iii) *If $|\chi_V(g)| = 1$, then $\mu_V(G^\vee) \ge \Delta_{\varepsilon(p-1)/3}$ for some $\varepsilon = \pm 1$.*

Proof. **(a)** The first statement was shown in Lemma 3.12(b). Hence $|N_G(\mathbf{U})/\mathbf{U}| = |G^\vee/\mathbf{U}| = |\Delta| = (p-1)^2$ by Lemma 3.12(b,a), so $|C_G(\mathbf{U})| = p(p-1)$, and the centralizer is as described.

(b) The first equality holds since $\mathrm{Ker}(\mu_V) = \mathbf{U}$ by Lemma 3.12(a). Also, $\mathrm{Aut}_{\mathrm{sc}}(V) \le C_G(\mathbf{U})$ and intersects trivially with $G^\vee$, so $G^\vee$ has index at least $p-1$ in $N_G(\mathbf{U})$. If $\dim(V) = p+1$, then $\dim(Z/Z_0) = 1$ (in the notation of 2.9), so for each $g \in N_G(\mathbf{U})$, the coset $g\mathrm{Aut}_{\mathrm{sc}}(V)$ contains a unique element in $G^\vee$. Hence $G^\vee$ has index exactly $p-1$ in $N_G(\mathbf{U})$ in this case.

(c) Now assume that $\dim(V) = p+1$, and fix $g \in N_G(\mathbf{U})$ of order $p-1$ such that c_g generates $\mathrm{Aut}(\mathbf{U})$. Let $g' \in g\mathrm{Aut}_{\mathrm{sc}}(V)$ be the unique element in $G^\vee$. Let $r \in (\mathbb{Z}/p)^\times$ be such that ${}^g u = {}^{g'} u = u^r$ for each $u \in \mathbf{U}$.

Set $Z = C_V(\mathbf{U})$ and $Z_0 = Z \cap [\mathbf{U}, V]$. Thus $\dim(Z) = 2$, $\dim(Z_0) = 1$, and $\chi_V(g) = \chi_Z(g)$ by Lemma 3.12(c). Also, for some choice of monomorphism $\psi\colon (\mathbb{Z}/p)^\times \longrightarrow \mathbb{C}^\times$, $\chi_Z(g) = \psi(s) + \psi(t)$, where g acts on Z_0 via multiplication by s and on Z/Z_0 via multiplication by t ($s, t \in (\mathbb{Z}/p)^\times$). Since g' acts on Z/Z_0 via the identity by definition of $G^\vee$, it acts on Z_0 via multiplication by st^{-1}, and hence $\mu_V(g') = (r, st^{-1})$.

Recall that r generates $(\mathbb{Z}/p)^\times$ by the assumption on g. If $|\chi_V(g)| = |\psi(s) + \psi(t)| = 2$, then $s = t$, so $\mu_V(g') = (r, 1)$ generates Δ_0. If $\chi_V(g) = 0$, then $s = -t$, so $\mu_V(g') = (r, -1)$ generates $\Delta_{(p-1)/2}$. If $|\chi_V(g)| = 1$, then $\psi(st^{-1}) = \psi(s)/\psi(t)$ must be a primitive cube root of unity, and hence $\mu_V(g') = (r, st^{-1})$ generates $\Delta_{(p-1)/3}$ or $\Delta_{-(p-1)/3}$. $\square$

4. Representations which occur in simple fusion systems: a summary

In this section, we present a summary of the rest of paper, by outlining our classification of all possible pairs (G, A) satisfying parts (a) to (d) in Theorem 2.8 or Corollary 2.10. From now on, these will be regarded as $\mathbb{F}_pG$-modules with additive structure (as opposed to the multiplicative group structure on A), and will be denoted by V to emphasize this. Throughout this section and the next, we assume the classification of finite simple groups.

Table 4.1 is an attempt at tabulating this information, but its notation requires explanation. When the image of G in $PGL(V)$ is almost simple, the group $G_0 = F^*(G)$ is listed in the second column, and a group $\overline{G} \le N_{GL(V)}(G_0)$ such that $\overline{G}/G_0$ is a p'-subgroup of maximal order in $N_{GL(V)}(G_0)/G_0$ is listed in the fourth column. In all cases, $N_{GL(V)}(G_0)/G_0$ is solvable, so the choice of $\overline{G}$ is unique up to conjugacy, and we can assume that $G_0 \le G \le \overline{G}$. (In almost all cases, $\overline{G} = N_{GL(V)}(G_0)$.)

When the image of G in $PGL(V)$ is not almost simple, the second column is left blank, G is contained in the group $\overline{G}$ in the fourth column, and we provide more information on the possibilities for G later in this section. In all cases, the third column lists the possible dimensions of the minimally active module V that becomes the subgroup A in the saturated fusion system. The fifth and sixth columns list the images $\mu_V(G_0^\vee) \le \mu_V(\overline{G}^\vee) \le \Delta$, and the final column gives the information as to whether this representation leads to a realizable fusion system (R) and/or an exotic fusion system (E) with superscripts indicating the number of such systems. In some cases we are necessarily vague when considering whole collections of possible groups G.

Theorem 4.1. *Assume that $G \in \mathscr{G}_p^\wedge$, and let V be a minimally active, faithful, indecomposable $\mathbb{F}_pG$-module which satisfies the hypotheses of Corollary 2.10. Then either*

(a) *the image of G in $PGL(V)$ is not almost simple, and $G \le \overline{G}$ with the given action on V for one of the pairs $(\overline{G}, V)$ listed in Table 4.1 with no entry G_0; or*

(b) *the image of G in $PGL(V)$ is almost simple, and $G_0 \le G \le \overline{G}$ for one of the triples $(G_0, \overline{G}, V)$ listed in Table 4.1.*

When $G_0 \cong SL_2(p)$ or A_p for $p \ge 5$, more precise descriptions of the modules are given in Propositions 4.2 and 4.3, respectively.

p	G_0	$\dim(V)$	$\overline{G}$	$\mu_V(\overline{G}^\vee)$	$\mu_V(G_0^\vee)$	E,R
p	$SL_2(p)$ or $PSL_2(p)$ $(p \ge 5)$	$3 \le n \le p$ $^{\mathbf{(4.2)}}$ socle of dim. i	$GL_2(p)$ or $PGL_2(p) \times C_{p-1}$	Δ	$\{(u^2, u^{i-1})\}$	ER
		$2/(p{-}1)$ $^{\mathbf{(4.2)}}$		Δ_{-1}	$\frac{1}{2}\Delta_{-1}$	E
p	$A_p \quad (p \ge 5)$	$[1]/p{-}2/[1]$ $^{\mathbf{(4.3)}}$	$S_p \times (p{-}1)$	Δ	$\frac{1}{2}\Delta_0$ or $\frac{1}{2}\Delta_{-1}$	ER
p	$A_{p+1} \quad (p \ge 5)$	p	$S_{p+1} \times (p{-}1)$	Δ	$\frac{1}{2}\Delta_0$	ER
p	A_n $(p{+}2 \le n \le 2p{-}1)$	$n-1$	$S_n \times (p{-}1)$	Δ_0	$\frac{1}{2}\Delta_0$	R
p	—	n $^{\mathbf{(4.4(b))}}$	$C_{p-1} \wr S_n$ $(n \ge p)$	Δ	—	ER
3	—	$2/2$ $^{\mathbf{(4.4(c))}}$	$GL_2(3)$	Δ_1	—	E
5	$2{\cdot}A_6$	4	$4 \circ 2{\cdot}S_6$	Δ	$\Delta_{1/2}$	E
5	—	4 $^{\mathbf{(4.4(d))}}$	$(C_4 \circ 2^{1+4}).S_6$	Δ	—	ER
5	$PSp_4(3) = W(E_6)'$	6	$W(E_6) \times 4$	$\Delta_0.2$	$\frac{1}{2}\Delta_0$	R
5	$Sp_6(2) = W(E_7)'$	7	$G_0 \times 4$	Δ_0	$\frac{1}{2}\Delta_0$	R
7	$2{\cdot}A_7$	4	$2{\cdot}S_7 \times 3$	Δ	$\Delta_{3/2}$	E
7	$6{\cdot}PSL_3(4)$	6	$G_0.2_1$	Δ	$\mathbb{F}_p^{\times 2} \times \mathbb{F}_p^\times$	E
7	$6_1{\cdot}PSU_4(3)$	6	$G_0.2_2$	Δ	$\mathbb{F}_p^{\times 2} \times \mathbb{F}_p^\times$	E
7	$PSU_3(3)$	6	$G_0.2 \times 6$	Δ	$\frac{1}{2}\Delta_1$	E
7	$PSU_3(3)$	7	$G_0.2 \times 6$	Δ	$\frac{1}{2}\Delta_0$	E
7	$SL_2(8)$	7	$G_0{:}3 \times 6$	Δ	$\frac{1}{3}\Delta_1$	E
7	$Sp_6(2) = W(E_7)'$	7	$G_0 \times 6$	Δ	Δ_3	ER
7	$2{\cdot}\Omega_8^+(2) = W(E_8)'$	8	$W(E_8) \times 3$	$\Delta_0.2$	Δ_3	R
11	J_1	7	$G_0 \times 10$	Δ	Δ_3	E
11	$PSU_5(2)$	10	$G_0.2 \times 10$	Δ	$\frac{1}{2}\Delta_2$	E
11	$2{\cdot}M_{12}$	$10^{[2]}$	$G_0.2 \times 5$	Δ	$\Delta_{1/2}\,,\ \Delta_{7/2}$	E
11	$2{\cdot}M_{22}$	$10^{[2]}$	$G_0.2 \times 5$	Δ	$\Delta_{1/2}\,,\ \Delta_{7/2}$	E
13	$PSU_3(4)$	12	$G_0.4 \times 12$	Δ	$\frac{1}{3}\Delta_1$	E

In the third column, a superscript in brackets shows the number of distinct representations of G_0 of the given dimension (not counting them as distinct if they differ by an automorphism of G_0). A superscript $^{\mathbf{(4.x)}}$ means that these representations (and/or the groups) are described more precisely in Proposition 4.x. When k, ℓ are relatively prime, we set $\Delta_{k/\ell} = \{(u^\ell, u^k) \mid u \in \mathbb{F}_p^\times\}$ (always cyclic of order $p-1$).

TABLE 4.1. Groups in $\mathscr{G}_p^\wedge$ with minimally active modules of dimension at least 3 which appear in reduced fusion systems

Proof. Let V be a minimally active, faithful, indecomposable $\mathbb{F}_pG$-module with $\dim(V) \geq 3$. When the image of G in $PGL(V)$ is not almost simple, the possibilities for G and V are listed in cases (b)–(e) of Proposition 5.4. When the image of G in $PGL(V)$ is almost simple, and $G_0 = F^*(G)$, the possible pairs (G_0, V) are determined in Sections 6–11: $SL_2(p)$ is handled in Proposition 6.1 (the only groups of Lie type in defining characteristic p which appear), sporadic groups in Proposition 7.1, alternating groups in Proposition 8.1, linear, unitary and symplectic groups in Propositions 10.1, 10.2 and 10.3, and orthogonal groups in Proposition 10.4. Finally, the exceptional groups are dealt with in Proposition 11.1. Together, these show that in all cases, (G, V) appears as an entry in one of the two Tables 4.1 or 4.2, as described in (a) or (b).

If V is a simple $\mathbb{F}_pG_0$-module, then it is absolutely simple by Lemma 3.4(c), and hence $C_{GL(V)}(G_0) = \mathrm{Aut}_{\mathrm{sc}}(V)$ (the group of scalar automorphisms). So $\overline{G} = O^p(N_{GL(V)}(G_0))$ is an extension of $G_0\mathrm{Aut}_{\mathrm{sc}}(V)$ by outer automorphisms of G_0 and hence can be determined from the tables of modular characters. When V is indecomposable but not simple, $\overline{G}$ has a similar form but is not in general the full normalizer of G_0 in $GL(V)$ (see, e.g., the second-to-last sentence in Proposition 6.1.

In both tables, $\mu_V(\overline{G}^\vee) = \Delta$ by Proposition 3.13(a) whenever $\dim(V) \leq p$. When $G_0 \cong A_n$ $(n \geq p{+}2)$ or $G \leq C_{p-1}\wr S_n$ $(n > p)$, $\mu_V(\overline{G}^\vee)$ is easily calculated using the definition, and when $G_0 \cong SL_2(p)$ and $\dim(V) = p + 1$, it is calculated in Section 6. When $\overline{G} \cong (C_3 \times 2^{1+6}_+).S_8$, $\mu_V(\overline{G}^\vee)$ is determined in Lemma 5.5. In all other cases where $\dim(V) = p{+}1$, $\mu_V(\overline{G}^\vee)$ can be determined using points (b) and (c) in Proposition 3.13. This leaves only the representation where $p = 5$, $G_0 \cong Sp_6(2)$, and $\dim(V) = 7$: $|\mu_V(\overline{G}^\vee)| = p - 1$ by Proposition 3.13(b), and the structure of fusion in $E_7(q)$ (when $v_5(q - 1) = 1$) together with Table 2.1 show that it must be Δ_0.

The determination of $\mu_V(G_0^\vee)$ is slightly more delicate than that of $\mu_V(\overline{G}^\vee)$. But in almost all cases, this can either be done either directly using the definitions, or with the help of character tables and Green correspondence, or by examining $V|_H$ for some $H < G_0$ isomorphic to $SL_2(p)$ or $PSL_2(p)$.

Finally, with the help of Table 2.1, we determine which of these representations appear in simple fusion systems (i.e., satisfy the conditions in Corollary 2.10 for some choice of $G \leq \overline{G}$ containing G_0), and in those cases we determine $\mu_V(\overline{G}^\vee)$ and give $\dim(V)$. Among those fusion systems, Table 2.2 tells us which are realizable. □

The first few rows of Tables 4.1 and 4.2 require the most explanation. The next proposition concerns the first row, and is basically a restatement of Proposition 6.1.

Proposition 4.2. *Let G_0, V, and $\overline{G}$ be as in Theorem 4.1, and assume that $G_0 = SL_2(p)$. The indecomposable minimally active $\mathbb{F}_pG_0$-modules with faithful action of G_0 or of $G_0/Z(G_0)$ are described as follows (where unique always means up to isomorphism).*

(a) *A unique simple $\mathbb{F}_pG_0$-module V_i of dimension i for each $2 \leq i \leq p$. Each simple $\mathbb{F}_pG_0$-module is isomorphic to V_i for some $1 \leq i \leq p$, where $V_1 = \mathbb{F}_p$ is the trivial module, V_2 is the natural module for $G_0 = SL_2(p)$, and $V_i = \mathrm{Sym}^{i-1}(V_2)$ for $3 \leq i \leq p$ (the $(i{-}1)$-st symmetric power).*

(b) *A unique $(p - 1)$-dimensional indecomposable module $V_{j,i}$ of type V_j/V_i for each $1 \leq i, j \leq p - 2$ such that $i + j = p - 1$.*

(c) *A unique p-dimensional indecomposable module $V_{1,p-2,1}$ of type $V_1/V_{p-2}/V_1$.*

p	G_0	$\dim(V)$	$\overline{G}$	$\mu_V(\overline{G}^\vee)$
p	$(P)SL_2(p)$	type j/i **(4.2)** $(i{+}j = p{+}1,\ j \neq 2)$	$GL_2(p)$ or $PGL_2(p) \times C_{p-1}$	Δ_{i-1}
7	—	8 **(4.4(e))**	$(C_3 \times 2^{1+6}_+).S_8$	Δ_3
7	$SL_2(8)$	8	$G_0{:}3 \times 6$	Δ_3
7	$2{\cdot}Sp_6(2)$	8	$G_0 \times 3$	Δ_3
7	$2{\cdot}A_7$	4/4	$2{\cdot}S_7 \times 3$	Δ_3
7	$2{\cdot}A_8$	8	$2{\cdot}S_8 \times 3$	Δ_3
7	$2{\cdot}A_9$	8	$G_0 \times 3$	Δ_3
11	$2{\cdot}M_{12}$	12	$G_0.2 \times 5$	Δ_5
13	${}^2B_2(8)$	$14^{[2]}$	$G_0{:}3 \times 12$	$\Delta_{\pm 4}$
13	$G_2(3)$	14	$G_0{:}2 \times 12$	Δ_6
17	$Sp_4(4)$	18	$G_0.4 \times 16$	Δ_8

TABLE 4.2. Groups in $\mathscr{G}_p^\wedge$ with minimally active modules of dimension at least 3 which do not satisfy the hypotheses of Corollary 2.10

(d) *A unique $(p+1)$-dimensional indecomposable module $V_{j,i}$ of type V_j/V_i for each $2 \leq i, j \leq p-1$ such that $i+j = p+1$.*

If the simple composition factors of V are even dimensional, then $G_0 = SL_2(p)$ acts faithfully on V, and $\overline{G} \cong GL_2(p)$. If the simple composition factors of V are odd dimensional, then the action of G_0 factors through $PSL_2(p)$, and $\overline{G} \cong PGL_2(p) \times C_{p-1}$. Also, $\overline{G} = N_{\mathrm{Aut}(V)}(G_0)$ except when $V \cong V_{i,i}$ for $i = (p \pm 1)/2$ or $V \cong V_{1,p-2,1}$, in which cases $\overline{G}$ has index p in the normalizer.

We now consider the case where G_0 is A_p.

Proposition 4.3. *Let G_0, V, and $\overline{G}$ be as in Theorem 4.1, and assume that $G_0 = A_p$. Then there is a unique faithful, simple, minimally active $\mathbb{F}_pG_0$-module W, of dimension $p-2$. There are unique non-simple indecomposable minimally active modules of each of the types $\mathbb{F}_p/W$, $W/\mathbb{F}_p$, and $\mathbb{F}_p/W/\mathbb{F}_p$, where the last is the permutation module. Also, $\overline{G} \cong S_p \times C_{p-1}$ in all cases, and is equal to $N_{\mathrm{Aut}(V)}(G_0)$ except when V is the permutation module, in which case $\overline{G}$ has index p in the normalizer. All of these representations give rise to simple fusion systems via Theorem 2.8 (for some choice of G), but only W itself gives rise to simple, realizable fusion systems.*

Proof. See Proposition 8.1 for a determination of the minimally active modules. □

It remains to describe the minimally active, indecomposable $\mathbb{F}_pG$-modules, when $G \in \mathscr{G}_p^\wedge$ and the image of G in the projective group is not almost simple.

Proposition 4.4. *Assume that $G \in \mathscr{G}_p^\wedge$. Let V be a minimally active, faithful, indecomposable $\mathbb{F}_pG$-module, and set $n = \dim(V)$. Then one of the following holds.*

(a) *The image of G in $PGL(V)$ is almost simple, and p divides the order of its socle.*

(b) *$G \le C_{p-1} \wr S_n$ $(n \ge p)$ acts as a group of monomial matrices on $V \cong (\mathbb{F}_p)^n$. More precisely, if we set $K = O_{p'}(G)$, then*

$$K = \{(a_1, \dots, a_n) \in (C_{p-1})^n \mid a_1^t = a_2^t = \dots = a_n^t,\ (a_1 \cdots a_n, a_1^t) \in R\}$$

for some $1 \ne t|(p-1)$ and some $R \le C_{p-1} \times C_{p-1}$; and one of the following holds:

- *$n = p$ and $G/K \cong C_p \rtimes C_{p-1}$;*
- *$n = p+1$ and $G/K \cong PGL_2(p)$;*
- *$p \le n \le 2p-1$ and $G/K \cong S_n$;*
- *$p+2 \le n \le 2p-1$ and $G/K \cong A_n$.*

Also, $V|_K$ splits as a direct sum of pairwise non-isomorphic 1-dimensional $\mathbb{F}_pK$-modules which are permuted 2-transitively by G/K.

(c) *$p = 3$, $G \cong 2_-^{1+2}.S_3 \cong Q_8.S_3 \cong GL_2(3)$, and either $n = 2$ and V is simple, or $n = 4$ and V is non-simple of type 2/2.*

(d) *$p = 5$, $n = 4$, $O_{5'}(G) \cong C_4 \circ 2^{1+4}$ or 2_-^{1+4}, and $G/O_{5'}(G) \cong S_6 \cong Sp_4(2)$, $S_5 \cong SO_4^-(2)$, or $C_5 \rtimes C_4$. (Note that $C_4 \circ 2_+^{1+4} \cong C_4 \circ 2_-^{1+4}$. If $O_{5'}(G) \cong 2_-^{1+4}$ then $G/O_{5'}(G)$ is not S_6.)*

(e) *$p = 7$, $n = 8$, $O_{7'}(G) \cong C_3 \times 2_+^{1+6}$ or 2_+^{1+6}, and $G/O_{7'}(G) \cong S_8 \cong SO_6^+(2)$, S_7, $PGL_2(7)$, or $C_7 \rtimes C_6$.*

Proof. This will be proved in Section 5, as Propositions 5.3 and 5.4. □

In order to describe more explicitly how to get from Table 4.1 to actual fusion systems, we list a few cases in more detail in Table 4.3. The first four columns in the table correspond to information given in Table 4.1, while the last four columns consist of separate rows for the different cases (corresponding to cases (i)–(iv) in Table 2.1). In the last column, E (or E^+) means that there is one (or more than one) exotic fusion system of this type; otherwise, a group is given which realizes it.

For example, when $p \ge 5$, $G_0 \cong PSL_2(p)$, and V is the simple $(p-2)$-dimensional $\mathbb{F}_pG_0$-module, we have $m = \dim(V) \equiv -1 \pmod{p-1}$ and $\mu_V(\overline{G}^\vee) = \Delta$. Hence there are three families of simple fusion systems which arise in this way, corresponding to the three cases (ii), (iii), (iv) in Table 2.1. The fusion systems of types (ii) and (iv) are unique by Theorem 2.8, while in case (iii), $\mathbf{E}_\mathcal{F} \smallsetminus \{A\}$ can be any union of $\mathcal{H}_i$'s. In all cases, $\mathrm{Aut}_\mathcal{F}(A) = G \ge G_0$ is determined (as a subgroup of $\overline{G}$) by the third column in Table 2.1. For $P \in \mathbf{E}_\mathcal{F} \cap (\mathcal{H} \cup \mathcal{B})$, $\mathrm{Aut}_\mathcal{F}(P)$ is determined by Lemma 2.6. By Table 2.2, all of these fusion systems are exotic if $p \ge 7$. If $p = 5$, then the fusion systems of type (iii) are exotic if $\mathbf{E}_\mathcal{F} \smallsetminus \{A\} \subsetneqq \mathcal{H}$, that of type (iii) with $\mathbf{E}_\mathcal{F} = \{A\} \cup \mathcal{H}$ is realized by $PSL_5(11)$, that of type (iv) is realized by $Sp_4(5)$, and that of type (ii) is realized by Co_1.

In contrast, when $p = 7$, $G_0 \cong 2{\cdot}\Omega_8^+(2) \cong W(E_8)'$, and V is the simple 8-dimensional $\mathbb{F}_pG_0$-representation, then $m = \min(\dim(V), p) = 7$, $\mu_V(\overline{G}^\vee) = \Delta_0.2 = \Delta_0\Delta_3$, and $\mu_V(G_0^\vee) = \Delta_3$. By Table 2.1, and since $m \not\equiv 0, -1 \pmod{p}$, any simple fusion system which realizes (G, V) must be in case (iii) or (iv), and it cannot be in case (iii) since $\mu_V(\overline{G}^\vee) \not\ge \Delta_{-1}$. So there is exactly one simple fusion system $\mathcal{F}$ of this type, of type (iv) and hence with $\mathbf{E}_\mathcal{F} = \{A\} \cup \mathcal{B}_0$. Also, $\mathrm{Aut}_\mathcal{F}(A) = G$ must contain G_0 with index 2 by the condition in the third column of Table 2.1, so $G \cong W(E_8)$. By Table 2.2, $\mathcal{F}$ is realized by $E_8(q)$ for any q such that $v_7(q-1) = 1$.

p	G_0	$\dim(V)$	$\mu_V(G_0^\vee)$	G	$\mu_V(G^\vee)$	$\mathbf{E}_0$	Group/E
p	$SL_2(p)$	$4 \le n \le p{-}3$ n even	$\Delta_{\frac{n-1}{2}}$	$GL_2(p)/Z$ $\lvert Z\rvert = (p{-}1, n{-}1)$	$\Delta_0\Delta_{\frac{n-1}{2}}$	$\mathcal{B}_0$	E
				$\frac{1}{\varepsilon} GL_2(p)$ $\varepsilon = (p{-}1, n{+}1)$	$\Delta_{-1}\Delta_{\frac{n-1}{2}}$	$\mathcal{H}_0$	E
p	$PSL_2(p)$ $(p \ge 5)$	$p-2$	$\frac{1}{2}\Delta_{-1}$	$G_0.2 \times (p{-}1)$	Δ	$\mathcal{B}_0 \cup \mathcal{H}_*$	Co_1 $(p{=}5)$ E $(p{\ge}7)$
				$PGL_2(p)$	Δ_{-1}	$\bigcup \mathcal{H}_i$	E
				$G_0.2 \times \frac{p-1}{2}$	$\Delta_0.(\frac{p-1}{2})$	$\mathcal{B}_0$	$Sp_4(5)$ $(p{=}5)$ E $(p{\ge}7)$
p	A_n $(p{+}2 \le n \le 2p{-}1)$	$n-1$	$\frac{1}{2}\Delta_0$	S_n	Δ_0	$\mathcal{B}_0$	$PSL_n(q)$
7	$2{\cdot}A_7$	4	$\Delta_{3/2}$	$2{\cdot}S_7$	$\Delta_0\Delta_3$	$\mathcal{B}_0$	E
				$2{\cdot}S_7 \times 3$	Δ	$\mathcal{H}_0$	E
7	$6{\cdot}PSL_3(4)$	6	$\frac{1}{2}\Delta$	$G_0.2_1$	Δ	$\mathcal{B}_0 \cup \mathcal{H}_*$	E
				$G_0.2_1$	Δ	$\bigcup \mathcal{H}_i$	E
				$G_0.2_1$	Δ	$\mathcal{B}_0$	E
7	$Sp_6(2) = W(E_7)'$	7	Δ_3	$G_0 \times 2$	$\Delta_0\Delta_3$	$\mathcal{B}_0$	$E_7(q)$
				$G_0 \times 3$	$\Delta_{-1}\Delta_3$	$\mathcal{H}_0$	E
7	$2{\cdot}\Omega_8^+(2) \cong W(E_8)'$	8	Δ_3	$G.2 \cong W(E_8)$	$\Delta_0.2$	$\mathcal{B}_0$	$E_8(q)$
11	$2{\cdot}M_{12}$	10 $(V\vert_{SL_2(11)}$ type 8/2)	$\Delta_{1/2}$	$G_0.2 \times 5$	Δ	$\mathcal{H}_0 \cup \mathcal{B}_*$	E
				$G_0.2 \times 5$	Δ	$\mathcal{H}_0$	E
				$G_0.2 \times 5$	Δ	$\bigcup \mathcal{B}_i$	E
13	$PSU_3(4)$	12	$\frac{1}{3}\Delta_1$	$G_0.4 \times 12$	Δ	$\mathcal{H}_0 \cup \mathcal{B}_*$	E
				$G_0.4 \times 3$	$\Delta_{-1}\Delta_4$	$\mathcal{H}_0$	E
				$G_0.4 \times 3$	$\Delta_0\Delta_4$	$\bigcup \mathcal{B}_i$	E

In all cases, q is a prime power such that $v_p(q-1) = 1$, and $\Delta_{k/\ell}$ is as in Table 4.1.

TABLE 4.3. Some examples

5. Reduction to almost simple groups

In this section, we analyse the possibilities for (G, V) as in Theorem 4.1 when the image of G in $PGL(V)$ is not almost simple, by using Aschbacher's classification of the maximal subgroups of $GL_n(p)$. The almost simple cases will be handled in the later sections.

Before proving a general result, we look at representations of $G \le C_{p-1} \wr S_n$ on $\mathbb{F}_p^n$, acting via monomial matrices. Two lemmas are first needed.

Lemma 5.1. *Assume that p is an odd prime and H is a finite group with a Sylow p-subgroup $\mathbf{U}$ of order p such that $|N_H(\mathbf{U})/C_H(\mathbf{U})| = p-1$. Assume also that H acts faithfully and transitively on a set Ω in such a way that each $x \in H$ of order p acts via a p-cycle. Then H acts primitively and 2-transitively on Ω, and one of the following holds:*

(a) $|\Omega| = p$ *and* $H \cong C_p \rtimes C_{p-1}$*;*

(b) $|\Omega| = p+1$ *and* $H \cong PGL_2(p)$*;*

(c) $p \le |\Omega| \le 2p-1$ *and* $H = S_\Omega$*;*

(d) $p+2 \le |\Omega| \le 2p-1$ *and* $H = A_\Omega$*; or*

(e) $p = 7$*,* $|\Omega| = 9$*, and* $H \cong \varSigma L_2(8)$*.*

Proof. Fix $\mathbf{U} \in \mathrm{Syl}_p(H)$ and $1 \ne x \in \mathbf{U}$. We first show that H is primitive on Ω. If Σ is a block of a system of imprimitivity for the action of H on Ω (thus $|\Sigma| > 1$), then x must stabilize Σ, as otherwise x must move at least $p|\Sigma| > p$ points, contradicting our assumption. Choose Σ so that x acts non-trivially on Σ and trivially on $\Omega \smallsetminus \Sigma$. Since H acts transitively on any system of imprimitivity, there exists $h \in H$ such that x^h acts non-trivially on a different block Σ^h but trivially on Σ, $\langle x, x^h\rangle$ is a subgroup of order p^2 in H, contradicting the assumption that $\mathbf{U} \in \mathrm{Syl}_p(G)$. Thus H acts primitively on Ω.

We now appeal to the classification of primitive permutation groups containing a p-cycle, as listed, for example, in [Zi]. By that theorem, one of the following holds, where $n = |\Omega|$.

(a) $n = p$, and H is a subgroup of $AGL_1(p) = C_p \rtimes C_{p-1}$;

(b) $n = p$, $p = \frac{q^d-1}{q-1}$ for some prime power q and $d \ge 2$, and $PSL_d(q) \le H \le P\varGamma L_d(q)$;

(c) $n = p+1$, $p = 2^d - 1$ is a Mersenne prime, and $AGL_1(2^d) \le H \le AGL_d(2)$;

(d) $n = p+1$, $H \cong PSL_2(p)$ or $H \cong PGL_2(p)$;

(e) $n = p = 11$, and $H = PSL_2(11)$ or $H = M_{11}$, or $n = p = 23$ and $H = M_{23}$;

(f) $n = p+1 = 12$, and $H = M_{11}$ or $H = M_{12}$, or $n = p+1 = 24$ and $H = M_{24}$;

(g) $n = p+2$, $p = 2^d - 1$ a Mersenne prime, and $H \cong PSL_2(2^d)$ or $P\varSigma L_2(2^d)$; or

(h) H is A_n or S_n, and $p \le n \le 2p-1$.

Thus the structure of H is very tightly controlled. Since $|\mathrm{Aut}_H(\mathbf{U})| = p-1$, we can be even more restrictive: (a) can occur only with $H = C_p \rtimes C_{p-1}$, and (d) can occur only with $H \cong PGL_2(p)$. In case (b), if $q = r^k$ where r is prime, then $p-1 = |\mathrm{Aut}_H(\mathbf{U})| \le kd$, so $(q^d-1)/(q-1) \le kd+1$, which gives $(p,q,d) = (3,2,2)$ or $(5,4,2)$, and $H \cong S_3$ or S_5. In case (c), where $|\mathrm{Aut}_H(\mathbf{U})| \le d$, we get $2^d - 1 = p = d+1$ and hence $d = 2$, $p = 3$, and $H \cong S_4$. For (e) and (f), $|\mathrm{Aut}_H(\mathbf{U})| = (p-1)/2$, so these do not occur. For (g), $|\mathrm{Aut}_H(\mathbf{U})| = 2$ or $2d$, so this occurs only when $(p,d) = (3,2)$ or $(7,3)$, and $H \cong PSL_2(4) \cong A_5$, $P\varSigma L_2(4) \cong S_5$, or $P\varSigma L_2(8)$. Of course (h) can occur, but not when $H = A_p$ or $H = A_{p+1}$. By inspection, all of these actions are 2-transitive. □

Lemma 5.2. *Fix a prime p, and a subgroup $H \le S_p$ such that $H \in \mathscr{G}_p^\wedge$. Fix $m > 1$ prime to p, and regard $(\mathbb{Z}/m)^p$ as a $\mathbb{Z}/m[H]$-module via the inclusion $H \le S_p$. Let $M \le (\mathbb{Z}/m)^p$ be a submodule, and assume that $\mathbf{U}$ acts non-trivially on M/qM for each prime $q \mid m$. Then $M \ge I$, where*

$$I = \bigl\{(x_1, \ldots, x_p) \bigm| x_i \in \mathbb{Z}/m,\ x_1 + \cdots + x_p = 0\bigr\}.$$

Proof. Fix $\mathbf{U} \in \mathrm{Syl}_p(H)$. Since $(m,p) = 1$, $M|_{\mathbf{U}} = C_M(\mathbf{U}) \oplus (M \cap I)$, so $\mathbf{U}$ acts non-trivially on $(M \cap I)/q(M \cap I)$ for each prime $q \mid m$. So upon replacing M by $M \cap I$, we can assume that $M \le I$.

Assume first that m is a prime; thus $\mathbb{Z}/m \cong \mathbb{F}_m$. As an $\mathbb{F}_m\mathbf{U}$-module, $(\mathbb{F}_m)^p$ factors as a product of irreducible modules of which one is 1-dimensional with trivial action, and the others are permuted transitively by the action of $N_H(\mathbf{U})/\mathbf{U}$ (since $|\mathrm{Aut}_H(\mathbf{U})| = p-1$). Hence each $\mathbb{F}_mH$-submodule of $\mathbb{F}_m^p$ either has trivial action of $\mathbf{U}$, or contains all of the factors which have non-trivial action and thus contains their sum I. Since the action of $\mathbf{U}$ on M is non-trivial by assumption, we have $M \geq I$.

Now assume that $m = q^a$ where q is prime and $a > 1$. By assumption, M/qM has non-trivial action of $\mathbf{U}$, and we just showed that this implies that $I \leq M+(q\mathbb{Z}/m\mathbb{Z})^p$. Hence there is $\mathbf{x} = (x_1, \dots, x_p) \in M$ such that $x_1 \equiv 1 \pmod{q}$, $x_2 \equiv -1 \pmod{q}$, and $x_i \equiv 0 \pmod{q}$ for each $i \geq 2$. Then $q^{a-1}\mathbf{x}$ and its $\mathbf{U}$-translates generate $q^{a-1}I$, so $q^{a-1}I \leq M$. Since $q^{a-2}\mathbf{x} \equiv (q^{a-2}, -q^{a-2}, 0, \dots, 0) \pmod{q^{a-1}I}$ (recall that $M \leq I$), we have $(q^{a-2}, -q^{a-2}, 0, \dots, 0) \in M$, and hence $q^{a-2}I \leq M$. Upon continuing in this way, we get $I \leq M$.

If m is not a prime power, the result now follows upon splitting M and I as products of their Sylow subgroups. □

We are now ready to describe the groups and modules which appear in case (b) of Proposition 4.4.

Proposition 5.3. *Set $\overline{G} = C_{p-1} \wr S_n$ $(n \geq p)$, and let $V \cong \mathbb{F}_p^n$ be the natural $\mathbb{F}_pG$-module where $\overline{G}$ acts via monomial matrices. Let $G \leq \overline{G}$ be such that $G \in \mathscr{G}_p^\wedge$ and $V|_G$ is a simple, minimally active module. Assume also that the image of G in $PGL(V)$ is not almost simple. Let $\overline{K} \trianglelefteq \overline{G}$ be the subgroup of $\overline{G}$ acting via diagonal matrices ($\overline{K} \cong C_{p-1}^n$), and set $K = G \cap \overline{K}$. Then for some $1 < t \mid (p-1)$ and some $R \leq C_{p-1} \times C_{p-1}$,*

$$K = \bigl\{(a_1, \dots, a_n) \in (C_{p-1})^n \bigm| a_1^t = a_2^t = \cdots = a_n^t,\ (a_1 \cdots a_n, a_1^t) \in R\bigr\}\,;$$

and one of the following holds:

(a) *$n = p$ and $G/K \cong C_p \rtimes C_{p-1}$;*

(b) *$n = p+1$ and $G/K \cong PGL_2(p)$;*

(c) *$p \leq n \leq 2p-1$ and $G/K \cong S_n$;*

(d) *$p+2 \leq n \leq 2p-1$ and $G/K \cong A_n$.*

(e) *$p = 7$, $n = 9$, and $G/K \cong \Sigma L_2(8)$.*

Proof. Set $H = G/K$ for short, with its natural action on $\Omega = \{1, 2, \dots, n\}$. If H is intransitive on Ω, then $\Omega = \Omega_1 \cup \Omega_2$ with Ω_i being H-invariant. Then $V = \mathbb{F}_p\Omega_1 \oplus \mathbb{F}_p\Omega_2$, a sum of $\mathbb{F}_pG$-submodules, which is impossible since V is assumed to be indecomposable. Thus the action of H on Ω is transitive.

Fix $\mathbf{U} \in \mathrm{Syl}_p(G)$. By assumption, $|\mathbf{U}| = p$, $\mathbf{U} \ntrianglelefteq G$, and $|\mathrm{Aut}_G(\mathbf{U})| = p-1$. Each $1 \neq x \in \mathbf{U}$ acts as a single p-cycle on Ω, since x has a single non-trivial Jordan block on M. So by Lemma 5.1, H (as a subgroup of S_n) is one of the groups listed in (a)–(d). In particular, H acts 2-transitively on Ω.

Set $B = \{a_1a_2^{-1} \mid (a_1, \dots, a_n) \in K\} \leq C_{p-1}$. Since H acts 2-transitively on Ω, any other pair of distinct elements of Ω defines the same subgroup. Set

$$\overline{K}_B = \bigl\{(a_1, \dots, a_n) \in (C_{p-1})^n \bigm| a_1 \equiv \cdots \equiv a_n \pmod{B}\bigr\}\,.$$

Thus $K \leq \overline{K}_B$. Define

$$\Phi\colon \overline{K}_B \longrightarrow C_{p-1} \times (C_{p-1}/B) \quad \text{by setting} \quad \Phi(a_1, \dots, a_n) = (a_1 \cdots a_n, a_1B)\,.$$

If $B = 1$, then each element of K acts via multiplication by scalars, so either $\mathbf{U} \trianglelefteq G$ or the image of G in $PGL(V)$ is almost simple. Since we assume that neither of these holds, $B \neq 1$.

Fix a generator $u \in \mathbf{U}$. Without loss of generality, we can assume that elements in Ω are arranged so that $u = (1\,2\,\ldots\,p)$. Let H_0 be the image of $N_H(\mathbf{U})$ in S_p via restriction of its action. Choose $\mathbf{a} = (a_1, \ldots, a_n)$ such that $a_1 a_2^{-1}$ generates B, and set $\mathbf{b} = \mathbf{a}^{-1} u(\mathbf{a})$. Thus $\mathbf{b} = (b_1, \ldots, b_p, 1, \ldots, 1)$, where $b_i \in B$ for $1 \le i \le p$, $b_1 b_2 \ldots b_p = 1$, and b_2 generates B. Since $|B|$ is prime to p, the sequence $b_1, \ldots, b_p$ is not constant modulo q for any $q \bigm| |B|$. We regard $\mathbf{b}$ as an element of $B^p \cap K$, which is a subgroup invariant under the action of H_0. By Lemma 5.2, K contains the group of all $(b_1, \ldots, b_p, 1, \ldots, 1)$ such that $b_i \in B$ and $b_1 \cdots b_p = 1$. Since the action of H on Ω is 2-transitive, it now follows that

$$K \ge \bigl\{(a_1, \ldots, a_n) \in B^n \bigm| a_1 \cdots a_n = 1\bigr\} = \mathrm{Ker}(\Phi)\,.$$

Set $R = \mathrm{Im}(\Phi)$. Thus $K = \{\mathbf{a} \in \overline{K}_B \mid \Phi(\mathbf{a}) \in R\}$, and this translates to the description of K given above (where $t = |B|$). □

The possibilities in Table 4.1 where there is no G_0, corresponding to point (a) in Theorem 4.1, arise in an analysis of subgroups of $GL_n(p)$ which lie in $\mathscr{G}_p^\wedge$ and whose action on the associated module is minimally active. Using Aschbacher's classification of the maximal subgroups of $GL_n(p)$, we restrict the options for $\overline{G}$, leaving either almost simple groups or a few other possibilities. The next proposition performs that reduction.

Proposition 5.4. *Assume that $G \in \mathscr{G}_p^\wedge$, let V be a minimally active, faithful, indecomposable $\mathbb{F}_pG$-module, and set $n = \dim(V)$. Then one of the following holds.*

(a) *The image of G in $PGL(V)$ is almost simple, and $p \bigm| |F^*(G)|$.*

(b) *$G \le C_{p-1} \wr S_n$ $(n \ge p)$ and acts as a group of monomial matrices.*

(c) *$p = 3$, $G \cong 2_-^{1+2}.S_3 \cong Q_8.S_3 \cong GL_2(3)$, and either $n = 2$ and V is simple, or $n = 4$ and V is non-simple of type $2/2$.*

(d) *$p = 5$, $n = 4$, $O_{5'}(G) \cong C_4 \circ 2^{1+4}$ or 2_-^{1+4}, and $G/O_{5'}(G) \cong S_6 \cong Sp_4(2)$, $S_5 \cong SO_4^-(2)$, or $C_5 \rtimes C_4$. (Note that $C_4 \circ 2_+^{1+4} \cong C_4 \circ 2_-^{1+4}$. If $O_{5'}(G) \cong 2_-^{1+4}$ then $G/O_{5'}(G)$ is not S_6.)*

(e) *$p = 7$, $n = 8$, $O_{7'}(G) \cong C_3 \times 2_+^{1+6}$ or 2_+^{1+6}, and $G/O_{7'}(G) \cong S_8 \cong SO_6^+(2)$, S_7, $PGL_2(7)$, or $C_7 \rtimes C_6$.*

Proof. Fix $\mathbf{U} \in \mathrm{Syl}_p(G)$. By assumption, $\mathbf{U} \ntrianglelefteq G$, $|\mathbf{U}| = p$, and $|\mathrm{Aut}_G(\mathbf{U})| = p - 1$.

Case 1: Assume first that the image of G in $PGL(V)$ is almost simple, and set $G_0 = F^*(G)$ and $\Gamma = G_0/Z(G_0)$. Thus G_0 is quasisimple, Γ is simple, and we must show that $p \bigm| |\Gamma|$.

Assume otherwise; then $p \bigm| |\mathrm{Out}(G_0)|$, and hence $p \bigm| |\mathrm{Out}(\Gamma)|$. Since p is an odd prime (and clearly $\Gamma \ncong A_6$), this is impossible when Γ is an alternating or sporadic group [GL, § I.5]. Hence Γ is of Lie type. From the tables in [GL, § I.7] it follows that $p \nmid |\mathrm{Outdiag}(\Gamma)|$, and hence that $\mathrm{Out}(\Gamma)$ possesses a normal p-complement. But then $\mathrm{Aut}_G(\mathbf{U}) = 1$, which contradicts our assumptions on G.

Case 2: Assume that V is a simple $\mathbb{F}_pG$-module, and that the image of G in $PGL(V)$ is not almost simple. By the main theorem in [A1], G is contained in one of an explicit list of geometrically defined subgroups of $GL_n(p)$, which fall into eight classes $\mathcal{C}_i$ for $1 \le i \le 8$. Of these, $\mathcal{C}_8$ consists of the subgroups $Sp_n(p)$ (if n is even) and $GO_n(p)$ (for all choices of quadratic form). If $G \le GL(\mathbb{F}_p^n, \mathfrak{q})$ for some symplectic or quadratic form $\mathfrak{q}$, then G is contained in a subgroup in one of the classes $\mathcal{C}_i$ for this classical group.

Assume that $G \le \overline{G} \in \mathcal{C}_i$ for $i \le 7$. Since V is simple, $i \ne 1$. Since p is a prime, $\mathcal{C}_5 = \varnothing$. If $i = 3$, then $G \le GL_m(p^k)$ where $mk = n$ and $k > 1$, which is impossible since each Jordan block over $\mathbb{F}_{p^k}$ splits as a sum of k Jordan blocks over $\mathbb{F}_p$. If $i = 4$ or 7, then G is contained in a tensor product or wreath tensor product of representations, which again implies that no element of order p acts with exactly one non-trivial Jordan block.

Now assume that $i = 2$, so that $G \le \overline{G} \cong GL_m(p) \wr S_k$ for some m, k such that $mk = n$ and $k > 1$. If $m = 1$, then we are in the situation of (b). So assume that $m \ge 2$, set $H = G \cap (GL_m(p))^k$, and let $H_i \le GL_m(p)$ be the image of H under projection to the i-th factor ($1 \le i \le k$). The H_i are all isomorphic, since otherwise G/H would not permute them transitively, contradicting the assumption that V is simple. If $p \big| |H_i|$ for all i, then since $p^2 \nmid |G|$, the elements of order p in G act with $m \ge 2$ non-trivial Jordan blocks, a contradiction. Thus there is an element of order p which non-trivially permutes some set of the H_i, this acts on V with $k \ge 2$ non-trivial Jordan blocks, which again is a contradiction.

We are left with the case where $G \le \overline{G} \in \mathcal{C}_6$. Then for some prime $r|(p-1)$ and some $k \ge 1$, $n = r^k$ and $\overline{G} = N_{GL_n(p)}(\overline{K})$ where $\overline{K} \cong C_{p-1} \circ r_{\pm}^{1+2k}$ acts as a group of monomial matrices whose non-zero entries are r-th roots of unity. Also, $\overline{G}/\overline{K} \cong C_{\mathrm{Out}(\overline{K})}(C_{p-1})$ is isomorphic to $Sp_{2k}(r)$, or (if $r = 2$ and $4 \nmid (p-1)$) to $GO_{2k}^{\pm}(r)$. In particular, $p \bigm| |Sp_{2k}(r)|$.

Set $k = \mathbb{F}_p$, or $k = \mathbb{F}_{p^2}$ if $r = 2$. The action of the subgroup $(k^\times \circ r^{1+2k}).(Sp_{2k-2}(r) \times Sp_2(r))$ on $k \otimes_{\mathbb{F}_p} V$ factors as a tensor product $V \otimes_k W$, where $(k^\times \circ r^{1+2(k-1)}).Sp_{2k-2}(r)$ acts on V with $\dim_k(V) = r^{k-1}$, and $(k^\times \circ r^{1+2}).Sp_2(r)$ acts on W with $\dim_k(W) = r$. So if $p \big| |Sp_{2k-2}(r)|$, then the Jordan blocks for elements of order p occur in multiples of r, which is impossible since there is a unique non-trivial block. We conclude that $p|(r^{2k}-1)$, so $p|(r^k \pm 1)$, and $\dim(V) = r^k \equiv \pm 1 \pmod p$.

For each non-central element $x \in \overline{K}$, $C_{Sp_{2k}(r)}(x)$ has index $r^{2k} - 1$ in the symplectic group and hence has order prime to p. Thus $C_{\overline{K}}(\mathbf{U}) = Z(\overline{K}) = \mathrm{Aut}_{\mathrm{sc}}(V)$, and $N_{\overline{K}\mathbf{U}}(\mathbf{U}) = \mathbf{U}\mathrm{Aut}_{\mathrm{sc}}(V) \cong C_{p(p-1)}$. Also, V is irreducible as an $\mathbb{F}_p\overline{K}$-module and hence as an $\mathbb{F}_p[\overline{K}\mathbf{U}]$-module, and the $\overline{K}\mathbf{U}$-Green correspondent of V is indecomposable as an $\mathbb{F}_p[\mathbf{U}\mathrm{Aut}_{\mathrm{sc}}(V)]$-module and hence of dimension at most p. Since $\dim(V) \equiv \pm 1 \pmod p$, the Green correspondent either has dimension 1, in which case $\dim(V) = p + 1$ since V is minimally active, or it has dimension $p - 1$, in which case $\dim(V) = p - 1$.

Thus $p = \dim(V) \pm 1 = r^k \pm 1$, which is possible only if $r = 2$ and p is a Fermat or Mersenne prime. The action of $\mathbf{U} \cong C_p$ on the symplectic space $\overline{K}/\mathrm{Aut}_{\mathrm{sc}}(V) \cong (\mathbb{F}_2)^{2k}$ has at most $2k$ eigenvalues (in the algebraic closure $\overline{\mathbb{F}}_2$), and since $|\mathrm{Aut}_G(\mathbf{U})| = p-1$, they must include all $(p-1)$-th roots of unity other than 1. Thus $p - 1 \le 2k$, so $(p, k) = (3, 1)$, $(5, 2)$, or $(7, 3)$, which correspond to cases (c)–(e) listed above. Note that $7 \nmid |O_6^-(2)|$, so this case cannot occur.

Set $G_0 = G \cap \overline{K}$, and regard $\overline{K}/Z(\overline{K}) \cong \mathbb{F}_2^{2k}$ as an $\mathbb{F}_2[G/G_0]$-module. Since G contains $\mathbf{U} \cong C_p$ and $|\mathrm{Aut}_G(\mathbf{U})| = p - 1$ (and since $2k = p - 1$ in each case), $\overline{K}/Z(\overline{K})$ is a simple module. Hence either $G_0 \le Z(\overline{K})$, or $G_0Z(\overline{K}) = \overline{G}$. By Lemma 5.1 (and since $|\mathrm{Aut}_G(\mathbf{U})| = p-1$), we have $G/G_0 \cong S_3$ in case (c); $G/G_0 \cong S_6$, S_5, or $C_5 \rtimes C_4$ in case (d); or $G/G_0 \cong S_8$, S_7, $PGL_2(7)$, or $C_7 \rtimes C_6$ in case (e). So $G_0 \not\le Z(\overline{K})$, since otherwise, either the image of G in $PGL(V)$ would be almost simple or $\mathbf{U}$ would be normal in G. We are left with the possibilities listed in the proposition.

Case 3: Now assume that V is not simple. Let $0 = V_0 < V_1 < \cdots < V_k = V$ be $\mathbb{F}_pG$-submodules such that $W_i = V_i/V_{i-1}$ is simple for each $1 \le i \le k$. Set $H_i = C_G(W_i)$ for each

i. Thus G/H_i acts faithfully on W_i. Since $O_p(G) = 1$ by assumption, G acts faithfully on $W_1 \oplus \cdots \oplus W_k$ (cf. [Go, Theorem 5.3.2]), so $\bigcap_{i=1}^k H_i = 1$.

If $H \trianglelefteq G$ and $p \bigm| |H|$, then since $p^2 \nmid |G|$, $H \geq O^{p'}(G)$. Hence there is some $1 \leq \ell \leq k$ such that $p \nmid |H_\ell|$. Then $V = C_V(H_\ell) \oplus [H_\ell, V]$ as $\mathbb{F}_pG$-modules, $\dim(C_V(H_\ell)) \geq \dim(W_\ell) > 0$, and V is indecomposable, so $[H_\ell, V] = 0$ and hence $H_\ell = 1$. Thus G acts faithfully on W_ℓ, and (G, W_ℓ) is one of the pairs listed in cases (a)–(e).

Set $K = O_{p'}(G)$. Let $\mathrm{Irr}_V(K)$ be the set of irreducible $\mathbb{F}_pK$-characters which appear as summands of $V|_K$, and similarly for $\mathrm{Irr}_{W_i}(K)$. For each $\chi \in \mathrm{Irr}_V(K)$, let $V_\chi \leq V|_K$ be the submodule generated by all irreducible submodules with character χ. Thus $V|_K$ is the direct sum (as $\mathbb{F}_pK$-modules) of the V_χ for $\chi \in \mathrm{Irr}_V(K)$. Since V is $\mathbb{F}_pG$-indecomposable, the action of G on $\mathrm{Irr}_V(K)$ induced by conjugation must be transitive. In particular, since the subsets $\mathrm{Irr}_{W_i}(K)$ are non-empty and G-invariant, we have $\mathrm{Irr}_{W_i}(K) = \mathrm{Irr}_V(K)$ for each $1 \leq i \leq k$.

If the image of G in $PGL(W_\ell)$ is almost simple, then K is cyclic of order dividing $p-1$, and $\mathrm{Irr}_{W_\ell}(K) = \mathrm{Irr}_V(K)$ contains just one character. Hence K acts on V via multiplication by scalars, G/K is almost simple, and (G, V) is as in case (a).

Now assume that (G, W_ℓ) is in one of the cases (b)–(e). Let $\overline{G}$ be the maximal group listed in that case (thus $G \leq \overline{G}$), and set $\overline{K} = O_{p'}(\overline{G})$. In case (b), G/K acts 2-transitively on the set $\mathrm{Irr}_{W_\ell}(\overline{K})$ by Proposition 5.3, so after restriction to K, its elements are all distinct. (By the description of K in Proposition 5.3, they cannot be pairwise isomorphic.) Hence $\dim(V) \geq 2\dim(W_\ell) \geq 2p$, which contradicts Proposition 3.7(c) ($\dim(V) \leq p+1$).

In each of cases (c)–(e), $W_\ell|_K$ is $\mathbb{F}_pK$-simple, so $\mathrm{Irr}_V(K)$ contains only its character. Hence $\dim(V) = m\dim(W_\ell) = m(p-1)$ for some $m \geq 2$. Since $\dim(V) \leq p+1$ by Proposition 3.7(c), this is possible only when $p = 3$ and $m = 2$, as described in case (c). □

We finish the section with a computation which was used in the proof of Theorem 4.1 to show that the representations in case (e) of Proposition 5.4 cannot be used to construct simple fusion systems.

Lemma 5.5. *Set $G_0 = 2_+^{1+6}$, let V be the unique faithful, irreducible $\mathbb{F}_7[G_0]$-module, and set $G = N_{GL(V)}(G_0)$. Fix $\mathbf{U} \in \mathrm{Syl}_7(G)$. Then $G/G_0 \cong C_3 \times SO_6^+(2) \cong C_3 \times S_8$, $|N_G(\mathbf{U})/\mathbf{U}| = 6^2$, and $\mu_V(G^\vee) = \Delta_3$ (see Notation 2.9).*

Proof. Since V is the unique faithful, irreducible $\mathbb{F}_7[G_0]$-module, $\mathrm{Out}_G(G_0) = \mathrm{Out}(G_0) \cong SO_6^+(2) \cong S_8$, where $G_0/Z(G_0)$ is the natural module for S_8. Since $C_{GL(V)}(G_0) = \mathrm{Aut}_{\mathrm{sc}}(V) \cong C_6$, this proves that $G/G_0 \cong C_3 \times S_8$. Set $Z = Z(G_0)$, $\widehat{Z} = \mathrm{Aut}_{\mathrm{sc}}(V) \cong C_3 \times Z$, and $\widehat{G}_0 = G_0\widehat{Z} \cong C_3 \times G_0$.

Now, $\mathrm{Out}_G(G_0)$ is contained in the symplectic group $Sp_6(2)$, which acts transitively on the 63 involutions in $\widehat{G}_0/\widehat{Z} \cong G_0/Z$. Since $7^2 \nmid |Sp_6(2)|$, this means that $C_G(x)$ has order prime to 7 for each $x \in \widehat{G}_0 \smallsetminus \widehat{Z}$, and hence that $C_G(\mathbf{U}) = \mathbf{U} \times \widehat{Z}$ and $|N_G(\mathbf{U})/\mathbf{U}| = 6^2$. So by Proposition 3.13(b), $|G^\vee/\mathbf{U}| = |\mu_V(G^\vee)| = 6$, and hence $\mu_V(G^\vee) = \Delta_m$ for some m.

Fix $g \in G^\vee$ such that c_g has order 6 in $\mathrm{Aut}_G(\mathbf{U})$. Then $g^6 = 1$ since $|G^\vee/\mathbf{U}| = 6$.

Identify $G_0 = H \circ H \circ H$ (central product) and $V = W \otimes W \otimes W$, where $H \cong D_8$, W is a faithful, irreducible $\mathbb{F}_7H$-module ($\dim(W) = 2$), and where G_0 acts on V as the tensor power of the H-action on W. Choose $h_0 \in N_{\mathrm{Aut}(W)}(H) \smallsetminus H$ of order 2 (since $GL_2(7)$ contains an extension of $\mathbb{F}_{49}^\times$ by a field automorphism, $N_{\mathrm{Aut}(W)}(H)$ contains a dihedral subgroup of order 16). Define $h \in G = N_{GL(V)}(G_0)$ by setting $h(w_1 \otimes w_2 \otimes w_3) = h_0(w_2) \otimes h_0(w_3) \otimes h_0(w_1)$. Since the action of h on $G_0/Z \cong C_2^6$ permutes a basis transitively, the image of h in $G/\widehat{G}_0 \cong S_8$ is a 6-cycle, thus conjugate to the image of g. Also, $|h| = 6$, and $\chi_V(h) = 0$ since V has a basis permuted by h with cycles of length 6 and 2.

Since $C_{G_0/Z}(h^3) = [h^3, G_0/Z]$ (both of rank 3), $\widehat{G}_0/\widehat{Z} \cong G_0/Z \cong C_2^6$ is projective and hence cohomologically trivial as an $\mathbb{F}_2[\langle h\rangle]$-module. Hence all subgroups of $\widehat{G}_0\langle h\rangle/\widehat{Z}$ complementary to $\widehat{G}_0/\widehat{Z}$ are conjugate to each other. We already saw that g is G-conjugate to some element of order 6 in the coset $h\widehat{G}_0$, and hence g is conjugate to hz for some $z \in \widehat{Z} = \mathrm{Aut}_{\mathrm{sc}}(V)$. Thus for some 6th root of unity ζ, $\chi_V(g) = \chi_V(hz) = \zeta{\cdot}\chi_V(h) = 0$. Proposition 3.13(c.ii) now implies that $\mu_V(G^\vee) = \Delta_3$. $\square$

6. $PSL_2(p)$

Recall Definition 0.1: for a finite simple group L, we say that a finite group G is *of type L* if $Z(G)$ is cyclic and $F^*(G)/Z(G) \cong L$. This concept provides a convenient way to organize the search for all pairs (G, V), where $G \in \mathscr{G}_p^\wedge$, V is minimally active, and the image of G in $PGL(V)$ is almost simple. So in the remaining sections, we go systematically through the list of non-abelian simple groups L, and list for each prime p the groups $G \in \mathscr{G}_p^\wedge$ of type L and their indecomposable, minimally active modules.

In this section, we handle the case where L is a simple group of Lie type in defining characteristic p. If $L \in \mathscr{G}_p$, then L is isomorphic to $PSL_2(p)$, as all other groups of Lie type have Sylow p-subgroups of order greater than p. We are thus reduced to the cases where G_0 is isomorphic to $PSL_2(p)$ or $SL_2(p)$, and consider the simple and indecomposable modules for $SL_2(p)$.

There are p different simple modules $V_1, \dots, V_p$ for $SL_2(p)$, where $\dim(V_i) = i$. (One way to construct them is to let V_2 be the natural module, and set $V_i = S^{i-1}(V_2)$ for $3 \le i \le p$.) Hence by Proposition 3.7(c), the dimension of each indecomposable minimally active $\mathbb{F}_pG$-module is at most $p+1$. To determine all such modules, we describe the projective covers of the V_i, referring to [Al, pp. 75–79] for more detail. (Although Alperin's descriptions are only for an algebraically closed field, since all simple modules are defined over $\mathbb{F}_p$ the projectives must be so as well.)

The module V_p is projective of dimension p. For $2 \le i \le p-2$, the projective cover of V_i is of shape

$$V_i/(V_{p-1-i} \oplus V_{p+1-i})/V_i,$$

while the projective covers of V_1 and V_{p-1} are of shape $V_1/V_{p-2}/V_1$ and $V_{p-1}/V_2/V_{p-1}$, respectively. Here '/' delineates the radical layers, with the socle appearing on the right, so that B is the submodule and A the quotient in A/B.

This yields indecomposable modules V_i/V_{p-1-i} (for each $1 \le i \le p-2$) and V_i/V_{p+1-i} (each $2 \le i \le p-1$), of dimension $p-1$ and $p+1$ respectively (and they are all minimally active by Lemma 3.8). Any minimally active indecomposable modules not yet found must have dimension p or $p+1$. If $\dim(V) = p$, then V is projective by Proposition 3.7(a), so $V \cong V_p$ or V is the projective cover of the trivial module $V_1/V_{p-2}/V_1$ as described above.

If V has dimension $p+1$ and has at least three composition factors, then there are either three factors including a copy of V_2, or four factors including two copies of V_1. In the former case, either V or V^* has a simple socle, and so is a quotient module of one of the projectives above, which by inspection cannot occur. If there are four composition factors, then two are trivial, so at least one of the other two must have non-trivial 1-cohomology. By Lemma 3.11,

V_{p-2} is the only such module, so V would have this and three trivial modules as composition factors, which is impossible by the above discussion.

We have now shown that each minimally active indecomposable module V is of one of the following types:

- $V \cong V_i$ for $i > 1$;
- $V \cong V_i/V_{p\pm1-i}$; or
- $V \cong V_1/V_{p-2}/V_1$ the projective cover of the trivial module V_1.

Also, the action of $SL_2(p)$ on each V_i extends to one of $GL_2(p)$ (being a symmetric power of the natural module).

We claim that the action of $G_0 = SL_2(p)$ on each of the non-simple indecomposable modules listed above also extends to an action of $G = GL_2(p)$. To see this, fix i, and let $U_1, \ldots, U_{p-1}$ be the distinct simple G-modules whose restriction to G_0 is isomorphic to V_i. (These are obtained by taking one such module, and tensoring it by each of the 1-dimensional G/G_0-modules.) By Frobenius reciprocity, the induced module $V_i|^G$ is isomorphic to the direct sum of the U_j. Hence the natural projection of $P(V_i)|^G$ onto $V_i|^G$ (where $P(-)$ denotes projective cover) lifts to a homomorphism $\Phi = \bigoplus_{j=1}^{p-1} \Phi_j$ from $P(V_i)|^G$ to $\bigoplus_{j=1}^{p-1} P(U_j)$. Since by definition, the kernel of the natural projection $P(U_j) \longrightarrow U_j$ is contained in the radical of $P(U_j)$ and hence in all maximal submodules, $\mathrm{Im}(\Phi_1) = P(U_1)$, $\mathrm{Im}(\Phi_2|_{\mathrm{Ker}(\Phi_1)}) = P(U_2)$, etc. Thus Φ is onto. Also, $\dim(P(U_j)) \geq \dim(P(V_i))$ for each j since $P(U_j)|_{G_0}$ contains $P(V_i)$ as a direct summand. So by comparing dimensions, we see that Φ is an isomorphism and $P(U_j)|_{G_0} \cong P(V_i)$. Thus for each i, the action of G_0 on $P(V_i)$ extends to G, and hence the same holds for the quotient modules of these projective covers listed above.

We next determine the normalizer $N_{GL(V)}(G_0)$, when V is one of the G_0-modules just listed. We first consider the centralizer $C_{GL(V)}(G_0)$, which obviously contains the scalar matrices $\mathrm{Aut}_{\mathrm{sc}}(V)$. If V is a simple $\mathbb{F}_pG_0$-module, then it is absolutely simple by Lemma 3.4(c), and hence $C_{GL(V)}(G_0) = \mathrm{Aut}_{\mathrm{sc}}(V)$ by Schur's lemma. If V acts indecomposably with socle V_i and quotient $V_{p\pm1-i}$ (both simple), and $g \in C_{GL(V)}(G_0)$, then g stabilizes V_i, and $g|_{V_i} = u{\cdot}\mathrm{Id}$ and $g \equiv u'{\cdot}\mathrm{Id} \pmod{V_i}$ for some $u, u' \in \mathbb{F}_p^\times$. Also, $u = u'$ since V is indecomposable, and g has the form $g(x) = ux + \psi(x + V_i)$ for some $\psi \in \mathrm{Hom}_{G_0}(V/V_i, V_i)$. Thus either $C_{GL(V)}(G_0) = \mathrm{Aut}_{\mathrm{sc}}(V)$, or $V/V_i \cong V_i$ (so $i = (p \pm 1)/2$) and $C_{GL(V)}(G_0) \cong \mathrm{Aut}_{\mathrm{sc}}(V) \times C_p$. A similar argument proves that $C_{GL(V)}(G_0) = \mathrm{Aut}_{\mathrm{sc}}(V) \times C_p$ when V is the projective cover of the trivial module.

Thus $C_{GL(V)}(G_0){\cdot}G_0 = G_0 \circ \mathrm{Aut}_{\mathrm{sc}}(V)$ or $(G_0 \circ \mathrm{Aut}_{\mathrm{sc}}(V)) \times C_p$ in all cases. Also, since $N_{GL(V)}(G_0)/G_0{\cdot}C_{GL(V)}(G_0)$ is a subgroup of $\mathrm{Out}(G_0) \cong C_2$, the normalizer $N_{GL(V)}(G_0)$ contains $C_{GL(V)}(G_0){\cdot}G_0$ with index at most 2. Thus $G \leq \overline{G}$, where $\overline{G}$ (as defined in Section 4) has index 1 or p in $N_{GL(V)}(G_0)$, and in all cases has the form $(G_0 \circ \mathrm{Aut}_{\mathrm{sc}}(V)).2$. (As noted above, the action of $SL_2(p)$ or $PSL_2(p)$ on V always extends to the outer automorphism.)

By Proposition 3.13(a), it is only the $(p+1)$-dimensional modules that might not produce reduced fusion systems. To know whether they do or not, we need to understand the two modules in the socle of $V|_N$ (recall $N = N_G(\mathbf{U})$).

Assume that V is an extension of V_i (the submodule) by V_j, where $i + j = p + 1$. Let $\zeta \in \mathbb{F}_p^\times$ be a generator, and fix the following elements in $GL_2(p)$:

$$g = \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}, \qquad u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus $\mathbf{U} = \langle u \rangle$, ${}^g u = u^\zeta$, and ${}^h u = u^{\zeta^{-1}}$. On the natural module V_2, g acts with eigenvalues $\{\zeta, 1\}$ and h with eigenvalues $\{1, \zeta\}$, beginning with those of the socle.

Identify $V_i = \mathrm{Sym}^{i-1}(V_2)$ and $V_j = \mathrm{Sym}^{j-1}(V_2)$. Then g, h have eigenvalues on V_i and V_j as follows:

	V_i	V_j
g	$\zeta^{i-1}, \zeta^{i-2}, \dots, \zeta, 1$	$\zeta^{j-1}, \zeta^{j-2}, \dots, \zeta, 1$
h	$1, \zeta, \zeta^2, \dots, \zeta^{i-1}$	$1, \zeta, \zeta^2, \dots, \zeta^{j-1}$

(in each case from socle to top). In general, these actions of g and h don't extend to an action on V (while the action of $gh^{-1} \in SL_2(p)$ does extend). So let $z = \zeta{\cdot}\mathrm{Id}_{V_j}$, let $\hat{g}, \hat{h}$ have the actions of g, h on V_i, but the actions of gz^{i-1} and hz^{i-1} on V_j. Since $(i-1)+(j-1) \equiv 0 \pmod{p-1}$, we get the following eigenvalues:

	V_i	V_j
$\hat{g}$	$\zeta^{i-1}, \zeta^{i-2}, \dots, \zeta, 1$	$1, \zeta^{-1}, \zeta^{-2}, \dots, \zeta^{-j+1}$
$\hat{h}$	$1, \zeta, \zeta^2, \dots, \zeta^{i-1}$	$\zeta^{i-1}, \zeta^{i}, \zeta^{i+1}, \dots, 1$

Then $\hat{g}$ and $\hat{h}$ have the same action on the top of V_i and the socle of V_j, so they do extend to actions on V. In particular, $\hat{g} \in \overline{G}^\vee$ and $\mu_V(\hat{g}) = (\zeta, \zeta^{i-1})$, while $h^* \stackrel{\text{def}}{=} \hat{h} \circ \zeta^{1-i}{\cdot}\mathrm{Id}_V \in \overline{G}^\vee$ and $\mu_V(h^*) = (\zeta^{-1}, \zeta^{1-i})$.

We conclude that $\mu_V(\overline{G}^\vee) = \Delta_{i-1}$. Thus each of the Δ_k except Δ_0 can appear as $\mu_V(\overline{G}^\vee)$ for some choice of V. When $i = p-1$ and $j = 2$, we get $\mu_V(\overline{G}^\vee) = \Delta_{-1}$.

This yields the following proposition.

Proposition 6.1. *Let $G \in \mathscr{G}_p^\wedge$ be a group of type $PSL_2(p)$, and let V be a non-trivial, minimally active module for G. Then one of the following holds:*

(i) *$V|_{G_0} \cong V_i$ is simple for $i \geq 2$;*

(ii) *$V|_{G_0} \cong V_{p-1-i}/V_i$ is indecomposable of dimension $p-1$;*

(iii) *$V|_{G_0}$ is projective of dimension p, of the form $V_1/V_{p-2}/V_1$;*

(iv) *$V|_{G_0} \cong V_{p+1-i}/V_i$ is indecomposable of dimension $p+1$.*

Furthermore, G is contained in $(G_0 \circ Z(GL(V))).2$. In case (iv), $\mu_V(G) \leq \Delta_i$, and hence $\mu_V(G) = \Delta_{-1}$ only when $i = p-1$.

7. Sporadic groups

In this short section, we determine all minimally active modules for those sporadic groups (and their extensions) which lie in $\mathscr{G}_p^\wedge$. For the reader's convenience, we include a table listing, for each simple sporadic group G_0, all primes p for which the Sylow p-subgroup has order p. Those primes for which $G_0 \in \mathscr{G}_p^\wedge$ are in bold, and the other primes for which $\mathrm{Aut}(G_0) \in \mathscr{G}_p^\wedge$ are in italics.

Group	Primes	Group	Primes	Group	Primes
M_{11}	**5**; 11	HS	**7**; *11*	Ru	**7, 13**; 29
M_{12}	**5**; *11*	McL	*7*; 11	ON	**5, 11**; *31*; 19
M_{22}	**5**; *11*; 7	Suz	**7, 11**; *13*	Fi_{22}	**7**; *11,13*
M_{23}	**5**; 7,11,23	Co_3	**7**; 11,23	Fi_{23}	**7, 11, 17**; 13,23
M_{24}	**5, 11**; 7,23	Co_2	**7, 11**; 23	Fi'_{24}	**11, 13, 17**; *29*; 23
J_1	**3, 7, 11**; 5,19	Co_1	**11, 13**; 23	Ly	**7**; 11,31,37,67
J_2	**7**	He	*17*	B	**11, 13, 17, 19**; 23,31,47
J_3	*5,19*; 17	HN	**7, 11**; *19*	M	**17, 19, 29, 41**;23,31,47,59,71
J_4	**5, 23**;7,29,31,37,43	Th	**13, 19**; 31		

By [Cr2], if G is a sporadic simple group and $p \bigm| |G|$, then with a few exceptions when $p = 3$ (none of which are in $\mathscr{G}_3$), $G = \langle t, y \rangle$ where $|t| = 2$ and $|y| = p$. So by Proposition 3.9(b), for each central extension $\widetilde{G}$ of G of degree prime to p, each minimally active $\mathbb{F}_p\widetilde{G}$-module has dimension at most $2p - 2$.

The table [Js, Table 1] provides a helpful list of the minimal degrees for sporadic groups for each prime. This allows us to eliminate almost all cases from the above table, just by applying the bound $\dim(V) \le 2p - 2$. We are left with the following possibilities for G_0 (or for $G_0.2$ when $G_0 \notin \mathscr{G}_p^\wedge$):

Prime	Possibilities for G_0 or $G_0.2$
7	$2{\cdot}J_2$, $6{\cdot}Suz$
11	$M_{12}.2$, $2{\cdot}M_{12}.2$, J_1, $M_{22}.2$, $2{\cdot}M_{22}.2$, $[6{\cdot}Suz]$
13	$[6{\cdot}Suz.2]$, $2{\cdot}Co_1$
19	$[3{\cdot}J_3.2]$

Since our modules are defined over $\mathbb{F}_p$ and are minimally active, $Z(G_0)$ must act via multiplication by scalars, and hence $|Z(G_0)| \bigm| (p-1)$ and $Z(G_0)$ is central in G in all cases. This criterion allows us to eliminate the three entries in brackets in the above table. Note that the outer automorphisms of $6{\cdot}Suz$ and $3{\cdot}J_3$ invert the centres: as described, e.g., in the tables in [GL, § I.1.5].

When $p = 7$ and $G_0 = 2{\cdot}J_2$, the two 6-dimensional modules over the algebraically closed field amalgamate into a single 12-dimensional module over $\mathbb{F}_7$: this can be seen either by computer or from the Brauer character table on [JLPW, p.105], where we see that there are irrationalities in the Brauer character of this representation, which from [JLPW, p.289] we see require $\mathbb{F}_{49}$.

By Proposition 3.9, if an $\mathbb{F}_pG$-module V is minimally active (for $G \in \mathscr{G}_p$), and $\dim(V) > p$, then $\dim(V) - p$ divides $|N_G(\mathbf{U})/\mathbf{U}|$, and $\dim(V) = p + 1$ if $N_G(\mathbf{U})/\mathbf{U}$ is abelian. We can thus eliminate the $\mathbb{F}_pG$-modules in the following dimensions:

p	Group	$\dim(V)$	$\dim(V)-p$	$N_G(\mathbf{U})/\mathbf{U}$
7	$6{\cdot}Suz$	12	5	$6{\cdot}((3\times A_4).2)$
11	J_1	14	3	10
11	$M_{12}.2$	16	5	10
11	$M_{22}.2$	20	9	10
13	$2{\cdot}Co_1$	24	11	$2{\cdot}((6\times A_4).2)$

We are left with the following cases, all for $p = 11$. For J_1, there is one 7-dimensional module. For $2{\cdot}M_{12}.2$, there are two pairs of modules of dimension 10 and one pair of dimension 12 (where the modules in each pair are isomorphic after restriction to $2{\cdot}M_{12}$). For $2{\cdot}M_{22}.2$, there are two pairs of modules of dimension 10. All of these modules are minimally active.

We have nearly proved the following proposition.

Proposition 7.1. *Let $G \in \mathscr{G}_p^\wedge$ be a group of sporadic type, and let V be a non-trivial, minimally active indecomposable module for G. Let $G_0 = E(G)$ be the unique quasisimple normal subgroup of G. Then $p = 11$, and one of the following holds:*

(i) *$G_0 \cong 2{\cdot}M_{12}$, $G/Z(G) \cong M_{12}{:}2$, and* $\dim(V) = 10$ *(two possible $F_{11}G_0$-modules) or* 12 *(one such module); or*

(ii) *$G_0 \cong 2{\cdot}M_{22}$, $G/Z(G) \cong M_{22}{:}2$, and* $\dim(V) = 10$ *(two possible $\mathbb{F}_{11}G_0$-modules); or*

(iii) *$G_0 \cong G/Z(G) \cong J_1$, and* $\dim(V) = 7$ *(one module).*

Proof. If V is simple, then we are done. So assume that V is indecomposable and not simple. In particular, $\dim(V) \le p + 1$ by Proposition 3.7(c). Hence V has one non-trivial composition factor W of dimension at most p and the others are trivial. Then $Z(G_0) = 1$, so $G_0 \cong J_1$, $\dim(W) = 7$, W is self-dual since it is the only module in that dimension, and hence $H^1(G_0; W) = 0$ by Lemma 3.11. So this case is impossible. □

8. Alternating groups

In this section we determine all minimally active modules for almost quasisimple groups associated to the alternating groups.

Proposition 8.1. *Let $G \in \mathscr{G}_p^\wedge$ be a group of type A_n for $n \ge 5$, and set $G_0 = E(G)$. Let V be a non-trivial, minimally active indecomposable module for G. Then one of the following holds:*

(i) *$G_0 \cong A_p$, $G/Z(G) \cong S_p$, and $V|_{G_0}$ is a subquotient of the permutation module, which has structure $1/W/1$ for W of dimension $p - 2$;*

(ii) *$G_0 \cong A_{p+1}$, $G/Z(G) \cong S_{p+1}$, and $V|_{G_0}$ is the p-dimensional non-trivial summand of the permutation module;*

(iii) *$G_0 \cong A_n$, $G/Z(G) \cong A_n$ or S_n for $p + 2 \le n \le 2p - 1$, and $V|_{G_0}$ is the $(n-1)$-dimensional summand of the permutation module;*

(iv) *$p = 5$, $G_0 \cong A_5 \cong PSL_2(5)$, $G/Z(G) \cong S_5 \cong PGL_2(5)$, and V is one of the modules described in Proposition 6.1 other than those in (i);*

(v) *$p = 5$, $G_0 \cong 2{\cdot}A_5 \cong SL_2(5)$, $G/Z(G) \cong S_5$, and V is as in Proposition 6.1;*

(vi) $p = 5$, $G_0 \cong 2{\cdot}A_6$, $G/Z(G) \cong S_6$, *and* $\dim(V) = 4$ *(two modules);*

(vii) $p = 7$, $G_0 \cong 2{\cdot}A_7$, $G/Z(G) \cong S_7$, *and* $\dim(V) = 4$*;*

(viii) $p = 7$, $G_0 \cong 2{\cdot}A_8$ *or* $2{\cdot}A_9$, $G/Z(G) \cong S_8$ *or* A_9, *and* $\dim(V) = 8$ *(one or two* $\mathbb{F}_7G_0$*-modules, respectively); or*

(ix) $p = 7$, $G_0 \cong 2{\cdot}A_7$, $G/Z(G) \cong S_7$, $\dim(V) = 8$, *and* V *has the form* W/W *where* W *is as in (vii).*

Proof. By a result of Miller [Mi, pp.29–30], for each $3 < p \le n$, A_n is generated by an element t of order 2 and an element x of order p (and this is easily seen to hold when $(p, n) = (3, 5)$). Thus by Proposition 3.9(b), each minimally active $\mathbb{F}_pA_n$- or $\mathbb{F}_p[2{\cdot}A_n]$-module has dimension at most $2p-2$. Our knowledge of small-dimensional representations of these groups is rather extensive, which makes these cases relatively easy to handle.

The smallest (faithful) simple module for S_n is the module arising from the permutation module, having dimension $n - 1 - \kappa_n$, where $\kappa_n = 0$ if $p \nmid n$ and $\kappa_n = 1$ if $p \mid n$. By [Jam, Theorem 7 and Table 1] or [BK, Lemma 1.18], if $p \ge 7$ and $n \ge 9$, or $p = 5$ and $n \ge 11$, the dimension of each larger $\mathbb{F}_pS_n$-module is strictly greater than $n(n-5)/2$, and hence that of each larger $\mathbb{F}_pA_n$-module is greater than $n(n-5)/4$. For each pair (p, n) such that $n \ge 10$ and $p \le n < 2p$, $n(n-5)/4 > 2p-2$ except when $p = n = 11$, and the smallest faithful $\mathbb{F}_{11}A_{11}$-module other than those in point (i) has dimension 36 (see [JLPW]). We are thus reduced to checking the cases $5 \le n \le 9$.

For $n \ge 12$, the smallest faithful representation of $2{\cdot}A_n$ is of dimension $2^{\lfloor (n-2-\kappa_n)/2 \rfloor}$, where κ_n is as above (see, e.g., [KT]). If V is minimally active, we have $\dim(V) \le 2p-2 \le 2n-2$. Hence

$$2n - 2 \ge 2^{\lfloor (n-2-\kappa_n)/2 \rfloor},$$

which yields $n \le 11$. (Note that since $n < 2p$, $\kappa_n = 0$ whenever $n \ne p$, in particular, when n is not prime.)

We can get yet more restrictions based on Green correspondence. Assume that G is a central extension of A_n for $p \le n \le 2p-1$, and let V be an indecomposable minimally active $\mathbb{F}_pG$-module. Recall (Proposition 3.7(c,e)) that the Green correspondent of V is an absolutely simple $N_G(\mathbf{U})/\mathbf{U}$-module.

(1) If $p \le n \le p+2$, then $N_{A_n}(\mathbf{U})/\mathbf{U}$ is cyclic of order $p-1$ or $(p-1)/2$, so $N_G(\mathbf{U})/\mathbf{U}$ is abelian, and $\dim(V) \le p+1$ by Proposition 3.9(a).

(2) If $n = p+3$, then $N_{A_n}(\mathbf{U})/\mathbf{U} \cong C_3 \rtimes C_{p-1}$. Hence $N_G(\mathbf{U})/\mathbf{U}$ has an abelian subgroup of index 2, so its irreducible representations have dimension at most 2, and $\dim(V) \le p+2$.

(3) If $n = p+4$ and $G = 2{\cdot}A_n$ (and acts faithfully on V), then $N_G(\mathbf{U})/\mathbf{U}$ contains a subgroup $H \cong B \circ 2{\cdot}A_4$ with index 2, where B is abelian. Since the absolutely irreducible representations of H on which the central involution acts non-trivially are all 2-dimensional, either $\dim(V) \le p$, or $\dim(V) = p+2$ or $p+4$.

Thus we can restrict attention to faithful representations of $2{\cdot}A_n$ for $n \le 11$ and of A_n for $n \le 9$, as well as those of $3{\cdot}A_n$ and $6{\cdot}A_n$ for $n = 6, 7$. All simple modules for all primes are known for these alternating groups, and we can simply check them one by one and prime by prime. This is done in Table 8.1, where (based on [JLPW]) we list dimensions of all irreducible $\overline{\mathbb{F}}_pG_0$-modules of dimension at most $2p-2$ when G_0 is a central extension of A_n for $p \le n \le 2p-1$, except for the natural modules for A_n described in points (i)–(iii).

For the modules listed in the table, an asterisk $(-)^*$ means that it is not realized over $\mathbb{F}_p$ (hence does not give rise to any minimally active $\mathbb{F}_pG_0$-module); a dagger $(-)^\dagger$ means

G_0	p	dimen.	p	dimen.
A_5	3	$3^*, 3^*$	5	Prop.6.1
A_6	5	$8^{(1)}$		
$3{\cdot}A_6$	5	$3 \nmid (p-1)$		
A_7	5	$8^{(1)}$	7	$10^{(1)}$
$3{\cdot}A_7$	5	$3 \nmid (p-1)$	7	$6^\dagger, 9^\dagger$
A_8	5	none	7	none
A_9	5	none	7	none
$2{\cdot}A_{10}$	7	none		

G_0	p	dimen.	p	dimen.
$2{\cdot}A_5$	3	$2^*, 2^*$	5	Prop.6.1
$2{\cdot}A_6$	5	**4,4**		
$6{\cdot}A_6$	5	$6 \nmid (p-1)$		
$2{\cdot}A_7$	5	$4^*, 4^*$	7	**4**
$6{\cdot}A_7$	5	$6 \nmid (p-1)$	7	$6^\dagger, 6^\dagger$
$2{\cdot}A_8$	5	$8^{(2)}$	7	**8**
$2{\cdot}A_9$	5	$8^{(3)}, 8^{(3)}$	7	**8,8**
$2{\cdot}A_{11}$	7	8	11	$16^{(1)}$

TABLE 8.1. Modules of dimension $\leq 2p-2$ for quasisimple alternating groups

that the module does not extend to S_n (when $n = p$ or $p+1$), and a superscript $(-)^{(i)}$ for $i = 1, 2, 3$ means that it is not minimally active by point (i) above. Note that the groups $3{\cdot}S_7$ and $6{\cdot}S_7$ are "twisted" in the sense that their outer automorphisms invert their centres, so there are no indecomposable modules for these groups over $\mathbb{F}_7$. The remaining modules (aside from the case $(n,p) = (5,5)$ of points (iv) and (v)) are shown in boldface, and are, in fact, minimally active, as can be seen by using the Magma command `IndecomposableSummands(Restriction(V,U))` to check the block sizes. These are precisely the modules listed in points (vi)–(viii).

Now assume that V is indecomposable but not simple. By Proposition 3.7(c), $\dim(V) \leq p+1$. If one or more of the simple components of V is 1-dimensional, then $G_0 \cong A_n$, and V contains a simple composition factor V_0 of dimension $p-2$ by Lemma 3.11. Hence we are in the situation of (i).

The only remaining possibility is the case where $p = 7$ and $G_0 \cong 2{\cdot}A_7$, and V is an indecomposable extension of the 4-dimensional module in Table 8.1 by itself. By [Al, Proposition 21.7], there is at most one such extension. From the tables in [JLPW], we see that the restriction to $2{\cdot}A_7$ of the simple 8-dimensional $2{\cdot}A_8$-module of case (viii) has this form. Hence there is a module of this type, and it is the restriction of a $2{\cdot}S_7$-module. $\square$

9. GROUPS OF LIE TYPE: NOTATION AND PRELIMINARIES

We continue our notation that G is a finite group, $\mathbf{U}$ is a Sylow p-subgroup of G, and $x \in \mathbf{U}$ has order p, writing $N = N_G(\mathbf{U})$ and $C = C_G(\mathbf{U})$, with $C = \mathbf{U} \times C'$.

In this section we consider groups of Lie type, and use induction to reduce the problem of classifying minimally active modules to a small set of situations, essentially where the centralizer is abelian. We start with a brief overview of the orders of finite simple groups of Lie type, using [GL] or [MT] as a reference. We assume a passing familiarity with basic concepts from algebraic groups, such as simple connectivity (see [GLS3] for a brief outline of the background assumed, and [MT] for more details). It will be useful, in many cases, to write $SL_n^{\pm}(q)$ or $E_6^{\pm}(q)$, where

$$SL_n^+(q) = SL_n(q), \quad SL_n^-(q) = SU_n(q), \quad E_6^+(q) = E_6(q), \quad E_6^-(q) = {}^2E_6(q).$$

If $G = {}^r\mathbb{G}(q)$ is a finite group of Lie type of universal type, then its order is

$$|G| = q^N \prod_d \Phi_d(q)^{a_d},$$

a power of q times a product of cyclotomic polynomials. However, to get the order of the associated simple group (if there is one), we must divide this order by $z = |Z(G)|$, an integer. The polynomials in q are given in Tables 9.1 and 9.2. We have dealt with the case where $p \mid q$ in Section 6, where we saw that the only possibility is $SL_2(p)$. So we assume for the rest of the paper that $\gcd(p, q) = 1$.

G	$\lvert G\rvert_{q'}$	$\lvert Z(G)\rvert$	d with $a_d = 1$
$SL_2(q)$	$(q-1)(q+1)$	$(2, q-1)$	$\mathbf{1, 2}$
$SL_n^\varepsilon(q)$ $(n \geq 3)$	$\prod_{i=2}^{n}((\varepsilon q)^i - 1)$	$(n, q-\varepsilon)$	$\lfloor n/2\rfloor+1, \ldots, n-2, \boldsymbol{n-1, n}$
$Sp_{2n}(q)$, $\mathrm{Spin}_{2n+1}(q)$	$\prod_{i=1}^{n}(q^{2i} - 1)$	$(2, q-1)$	$\lfloor n/2\rfloor+1, \ldots, n-1, \boldsymbol{n}$ (odd only) $n+1, \ldots, 2n-2, \mathbf{2}\boldsymbol{n}$ (even only)
$\mathrm{Spin}_{2n}^+(q)$	$(q^n - 1)\prod_{i=1}^{n-1}(q^{2i} - 1)$	$(4, q^n - 1)$ or $(2, q-1)^2$	$\lfloor n/2\rfloor+1, \ldots, \boldsymbol{n-1, n}$ (odd only) $n+1, \ldots, 2n-4, \mathbf{2}\boldsymbol{n}\mathbf{-2}$ (even only)
$\mathrm{Spin}_{2n}^-(q)$	$(q^n + 1)\prod_{i=1}^{n-1}(q^{2i} - 1)$	$(4, q^n + 1)$	$\lfloor n/2\rfloor+1, \ldots, \boldsymbol{n-1}$ (odd only) $n, \ldots, 2n-4, \mathbf{2}\boldsymbol{n}\mathbf{-2}, \mathbf{2}\boldsymbol{n}$ (even only)

TABLE 9.1. Orders of classical groups of Lie type (regular d in bold)

G	$\lvert G\rvert_{q'}$	$\lvert Z(G)\rvert$	d with $a_d = 1$
${}^2B_2(q)$	$\Phi_1\Phi_4$	1	$\mathbf{1, 4', 4''}$
${}^2G_2(q)$	$\Phi_1\Phi_2\Phi_6$	1	$\mathbf{1, 2, 6', 6''}$
${}^2F_4(q)$	$\Phi_1^2\Phi_2^2\Phi_4^2\Phi_6\Phi_{12}$	1	$\mathbf{6, 12', 12''}$
$G_2(q)$	$\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	1	$\mathbf{3, 6}$
${}^3D_4(q)$	$\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_{12}$	1	$\mathbf{12}$
$F_4(q)$	$\Phi_1^4\Phi_2^4\Phi_3^2\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}$	1	$\mathbf{8, 12}$
$E_6(q)$	$\Phi_1^6\Phi_2^4\Phi_3^3\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}$	$(3, q-1)$	$5, \mathbf{8, 9, 12}$
${}^2E_6(q)$	$\Phi_1^4\Phi_2^6\Phi_3^2\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$	$(3, q+1)$	$\mathbf{8}, 10, \mathbf{12, 18}$
$E_7(q)$	$\Phi_1^7\Phi_2^7\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_7$ $\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$(2, q-1)$	$5, \mathbf{7}, 8, \mathbf{9}, 10, 12, \mathbf{14, 18}$
$E_8(q)$	$\Phi_1^8\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4\Phi_7\Phi_8^2\Phi_9$ $\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	1	$7, 9, 14, \mathbf{15}, 18, \mathbf{20, 24, 30}$

TABLE 9.2. Orders of exceptional groups of Lie type (regular d in bold)

In Table 9.2, the values $d = 4', 4''$, etc. represent the factorizations of the cyclotomic polynomials $\Phi_d(q)$:

$$G = {}^2B_2(q) \qquad q = 2^{2m+1} \qquad d = 4 \qquad \Phi_4(q) = (q + \sqrt{2q} + 1)(q - \sqrt{2q} + 1)$$

$$G = {}^2G_2(q) \qquad q = 3^{2m+1} \qquad d = 6 \qquad \Phi_6(q) = (q + \sqrt{3q} + 1)(q - \sqrt{3q} + 1)$$

$$G = {}^2F_4(q) \qquad q = 2^{2m+1} \qquad d = 12 \qquad \Phi_{12}(q) = (q^2 + q\sqrt{2q} + q + \sqrt{2q} + 1) \cdot (q^2 - q\sqrt{2q} + q - \sqrt{2q} + 1).$$

These factors are the orders of the largest cyclic subgroups in G of order dividing $\Phi_d(q)$.

We claim that
$$G/Z(G) \in \mathscr{G}_p \quad \Longrightarrow \quad p \nmid z \text{ and hence } G \in \mathscr{G}_p. \tag{1}$$
Assume otherwise: let $G = {}^r\mathbb{G}(q)$ be a counterexample. Since p is odd and $p \mid z$, z is not a power of 2, and hence $\mathbb{G} = A_n$ or E_6. If $\mathbb{G} = E_6$, then $z = (3, q \pm 1)$, so $p = 3$, which is impossible since $3^4 \bigm| (q^2-1)^4 \bigm| |G|$ for all q prime to 3 (Table 9.2). Thus $G = SL_n^\varepsilon(q)$ and $p \mid z = (n, q-\varepsilon)$. So $n \geq 3$ and $(q-\varepsilon)^3 \nmid |G|$, which by Table 9.1 implies $n = p = 3$. But then $3 \mid (q-\varepsilon)$ implies $3^2 \mid (q^3-\varepsilon)$, so $3^3 \bigm| |G|$ and $3^2 \bigm| |G/Z(G)|$. This proves (1). In particular, since $G \in \mathscr{G}_p$,

$$\text{there exists a unique } d \text{ such that } p \mid \Phi_d(q), \text{ and for this } d,\ p^2 \nmid \Phi_d(q), \text{ and } a_d = 1. \tag{2}$$

We now want to understand the subgroups $N = N_G(\mathbf{U})$ and $C = C_G(\mathbf{U})$, which are closely related to the integer d, and not really dependent on the prime p.

We start by assuming that $\mathbb{G}$ is simply connected, e.g., SL_n. In this case, a theorem of Steinberg (see [MT, Theorem 14.16]) states that, for each semisimple element $s \in \mathbb{G}$, the centralizer $C_{\mathbb{G}}(s)$ is connected. A semisimple element $s \in G$ is *regular* if, inside the corresponding algebraic group $\mathbb{G}$, $\dim(C_{\mathbb{G}}(s))$ is minimal among all semisimple elements, so equal to the rank of $\mathbb{G}$. By [MT, Corollary 14.10], if s is regular then $C_{\mathbb{G}}(s)^0 = \mathbb{T}$ for any maximal torus $\mathbb{T}$ containing s, and hence s is contained inside a unique maximal torus $\mathbb{T}$ and $C_{\mathbb{G}}(s) = \mathbb{T}$.

Let F be a Frobenius endomorphism such that $G = \mathbb{G}^F$, and assume for the moment that G is not a Ree or Suzuki group. To continue we need one fact from the theory of d-tori. Rather than giving a formal definition of d-tori here, we instead refer to [MT, Section 25], and in particular to [MT, Definition 25.6]. The property we need is that by [MT, Theorems 25.14 and 25.19], if $a_d = 1$, then for the finite group $G = \mathbb{G}^F$, there exists a cyclic torus $T_d = \mathbb{T}_d^F$ (called a *Sylow d-torus*), where $\mathbb{T}_d$ is an F-stable torus in $\mathbb{G}$, such that $\mathbf{U} \leq T_d$ and $C_G(\mathbf{U}) = C_G(T_d)$.

Let $T_d = \langle s \rangle$. If s is regular then we know that $C_{\mathbb{G}}(s)$ is a maximal torus of $\mathbb{G}$, and so $C_G(\mathbf{U}) = C_G(s) \leq C_{\mathbb{G}}(s)$ is abelian. Note that in general, if s is regular then, although $C_G(s)$ is a torus, we have $C_G(s) \geq T_d$, with equality if and only if T_d is a maximal torus. This last statement is true if and only if $\phi(d) = \mathrm{rank}(G)$.

If G is a Suzuki or Ree group then the maximal subgroups of G are known (see for example [Wi]), so we can deduce the same result for those groups and for our particular primes p, and get $C_G(\mathbf{U})$ abelian in all cases for Table 9.2. (The d-torus theory can be extended to these groups with some complications coming from the fact that cyclotomic polynomials split, but it is easier for us to use the lists of maximal subgroups directly to prove that $C_G(\mathbf{U})$ is abelian.)

We have now shown the following:

Proposition 9.1. *Let $G = {}^r\mathbb{G}(q)$ be the universal form of a group of Lie type (so a quasisimple group as given in Tables 9.1 and 9.2), and suppose that $G \in \mathscr{G}_p$ and $p \nmid q$. Let d be the multiplicative order of q modulo p. If d is a regular number for G then $C_G(\mathbf{U})$ is abelian.*

The numbers d that yield regular elements s for a given group $\mathbb{G}$ and Frobenius endomorphism F were computed by Springer in [Sp, Section 5, 6.9–6.11]. They are given in Tables 9.1 and 9.2, where we list all d such that $a_d = 1$ (i.e., such that the Sylow Φ_d-subgroup has order p), and among them list in bold the d that are regular.

In view of Proposition 9.1, we would like to always be in the situation that d is a regular number, but from Tables 9.1 and 9.2 we see that this is not true. However, as we will see, there is always a subgroup H of G, also a group of Lie type, such that $\mathbf{U} \leq H$ and d is regular for H.

For classical groups at least, we also need to understand the structure of $N_G(\mathbf{U})$, not just of $C_G(\mathbf{U})$, although this is easy and again depends only on d. For d a positive integer, let $\bar{d}$ be defined by $\bar{d} = 2d$ if d is odd, $\bar{d} = d/2$ if d is even but $4 \nmid d$, and $\bar{d} = d$ if $4 \mid d$. Note that if d is the multiplicative order of q modulo p, then $\bar{d}$ is the order of $-q$ modulo p.

For $SL_2(q)$, the automizer is always of order 2. For $SL_n(q)$ and $SU_n(q)$ for $n \geq 3$, by [FS1, Proposition (3D)], $|N/C| = d$ or $\bar{d}$, respectively. For the other classical groups, by [FS2, p.128], $|N/C|$ equals $2d$ if d is odd or d if d is even, i.e., $\mathrm{lcm}(d, 2)$ in all cases.

The order of N/C, when $p \mid \Phi_d(q)$, is summarized for the classical groups in Table 9.3.

Group	$\lvert N/C \rvert$ when $p \mid \Phi_d(q)$
$PSL_2(q)$	2
$PSL_n(q)$ $(n \geq 3)$	d
$PSU_n(q)$	$\bar{d}$
$PSp_{2n}(q)$ or $P\Omega^{\pm}_m(q)$	$\mathrm{lcm}(2, d)$

TABLE 9.3

When we work with cyclotomic polynomials, the following relations are useful:

(i) for all $n > 1$, $\Phi_n(q) \mid (q^n - 1)/(q - 1)$;

(ii) if n is even but $4 \nmid n$, then $\Phi_n(q) \mid (q^{n/2} + 1)/(q + 1)$; and

(iii) if $4 \mid n$ then $\Phi_n(q) \mid q^{n/2} + 1$.

We are now ready to work with the individual groups: first the classical groups and then the exceptional groups.

10. Determination for classical groups

In this section, we classify minimally active modules for classical groups and their extensions in $\mathscr{G}_p^{\wedge}$ when p is not the defining characteristic. Throughout this section, $G = G(q)$ is a group of Lie type, $p \nmid q$ is a prime dividing $|G|$, $\mathbf{U}$ is a Sylow p-subgroup of G with generator x, and

$$p \text{ has order } d \text{ modulo } q, \text{ so that } p \mid \Phi_d(q).$$

Let V be a minimally active module for G. In Table 10.1, we list the minimal possible dimension $l(G)$ for a faithful $\mathbb{F}_pG$-module, as determined in [LS], [SZ], or [GT].

We begin with the linear groups.

Proposition 10.1. *Let $G \in \mathscr{G}_p^{\wedge}$ be a group of type $PSL_n(q)$, where $n \geq 2$ and $p \nmid q$, and set $G_0 = E(G)$. Let V be a non-trivial, minimally active $\mathbb{F}_pG$-module. Then one of the following holds:*

(i) *G_0 is a central extension of $PSL_2(4) \cong A_5$, $PSL_2(5) \cong A_5$, $PSL_2(9) \cong A_6$ or $PSL_4(2) \cong A_8$, and the modules are as in Proposition 8.1;*

(ii) *G_0 is a central extension of $PSL_2(7) \cong PSL_3(2)$, $p = 7$, and V is one of the modules in Proposition 6.1;*

G	Lower bound for dimension	Ref.	Exceptions
$SL_2(q)$	$(q-1)/z \quad (z=(2,q-1))$	[LS]	$l(SL_2(4))=2$ $l(SL_2(9))=3$
$SL_n(q)$ $(n\geq 3)$	$q^{n-1}-1$	[LS]	$l(SL_3(2))=2$ $l(SL_3(4))=4$
	$q(q^{n-1}-1)/(q-1)-1$	[GT]	$l(SL_4(2))=7$ $l(SL_4(3))=26$
$SU_n(q)$ $(n\geq 3)$	$(q^n-q)/(q+1) \quad (n \text{ odd})$ $(q^n-1)/(q+1) \quad (n \text{ even})$	[LS]	$l(SU_4(2))=4$ $l(SU_4(3))=6$
$Sp_{2n}(q)$	$(q^n-1)/2 \quad (2\nmid q)$ $q(q^n-1)(q^{n-1}-1)/2(q+1) \quad (2\mid q)$	[SZ]	$l(Sp_4(2)')=2$
$\mathrm{Spin}_{2n+1}(q)$ $(n\geq 3,\ 2\nmid q)$	$q^{n-1}(q^{n-1}-1)$	[LS]	$l(\mathrm{Spin}_7(3))=27$
$\mathrm{Spin}^+_{2n}(q)$ $(n\geq 4)$	$q^{n-2}(q^{n-1}-1)$	[LS]	$l(\mathrm{Spin}^+_8(2))=8$
$\mathrm{Spin}^-_{2n}(q)$ $(n\geq 4)$	$(q^{n-1}+1)(q^{n-2}-1)$	[LS]	—

TABLE 10.1. Minimal representation dimensions of classical groups

(iii) $G_0\cong SL_2(8)$, $G/Z(G)\cong SL_2(8){:}3$, $p=7$, *and* $\dim(V)=7,8$*;*

(iv) $G_0\cong 6{\cdot}PSL_3(4)$, $G/Z(G)\cong PSL_3(4).2$, $p=7$, *and* $\dim(V)=6$.

Proof. In all cases, we fix n and $q=r^t$, where r is prime and $t\geq 1$, such that G is of type $PSL_n(q)$. If $(n,q)=(2,4)$, $(2,5)$, $(2,9)$, or $(4,2)$, then $PSL_n(q)$ is an alternating group, and we are in the situation of (i). So we assume from now on that (n,q) is not one of these pairs. Since $PSL_3(2)\cong PSL_2(7)$, we can also assume that $(n,q)\neq(3,2)$.

Among the remaining cases, $SL_n(q)$ is the universal central extension of $PSL_n(q)$ with only one exception: $PSL_3(4)$ has an "exceptional cover": a central extension of the form $4^2{\cdot}SL_3(4)$ [Wi, § 3.12]. In all other cases, if $p\mid\Phi_d(q)$ for regular d, then $C_{G_0}(\mathbf{U})$ is abelian by Proposition 9.1, and hence by Proposition 3.9(a), $\dim(V)\leq 2p-1$ for each indecomposable minimally active $\mathbb{F}_pG_0$-module V.

Case 1: We start with the groups of type $PSL_2(q)$, where $q=r^t$. Thus $p\mid\Phi_d(q)$ where $d=1,2$, and the normalizer $N_{G_0}(\mathbf{U})$ is dihedral of order $\Phi_d(q)$ or quaternion of order $2{\cdot}\Phi_d(q)$, containing $C_{G_0}(\mathbf{U})$ as a cyclic subgroup of index 2. Hence if V is minimally active, then $\dim(V)\leq p+2$ by Proposition 3.9(a).

In the group $\mathrm{Aut}(SL_2(r^t))$, the automizer of $\mathbf{U}$ has order at most $2t$, so that if $G\in\mathscr{G}_p^\wedge$ then $p\leq 2t+1$. On the other hand, the smallest dimension for V is $(r^t-1)/2$ for r odd and r^t-1 for r even, so that $\dim(V)\leq p+2$ becomes

$$r^t\leq\dim(V)+1\leq p+3\leq 2t+4 \qquad \text{if } r=2$$
$$r^t\leq 2\dim(V)+1\leq 2(p+2)+1\leq 4t+7 \qquad \text{if } r>2.$$

If $r=2$, then $t\leq 3$; while if r is odd, then either $t=2$ and $r=3$, or $t=1$ and $r\leq 11$.

When $G_0\cong PSL_2(11)$ and $p=3$, there is a 5-dimensional $\mathbb{F}_3G_0$-module V, but $N_{G_0}(\mathbf{U})/\mathbf{U}\cong C_2^2$ is abelian in this case, so V is not minimally active by Proposition 3.9(a). Since we have already dealt with the alternating groups $PSL_2(4) \cong PSL_2(5) \cong A_5$ and $PSL_2(9) \cong A_6$, and do not need to deal with solvable groups, we are left with the groups $PSL_2(7)$ with $p = 3$, and $PSL_2(8)$ with $p = 7$.

For $G \cong SL_2(7)$ and $p = 3$, the modules of dimension 3 are not defined over $\mathbb{F}_3$, but only over $\mathbb{F}_9$. For $G \cong SL_2(8)$ and $p = 7$, there are four 7-dimensional modules of which only one extends to $SL_2(8){:}3 \in \mathscr{G}_7^\wedge$, and one 8-dimensional module which also extends to $SL_2(8){:}3$. Both of these are minimally active.

Case 2: We next consider the groups of type $PSL_n(q)$ for $n \geq 3$, where $p \mid \Phi_d(q)$ for some $\lfloor \frac{n}{2} \rfloor + 1 \leq d \leq n - 1$. By Table 10.1, $\dim(V) \geq q(q^{n-1} - 1)/(q - 1) - 1$ with the exceptions in Table 10.1, which (aside from cases that we already eliminated) are

$$(n, q, p) \in \{(3, 4, 5), (4, 3, 13)\}.$$

(Here we consider those $PSL_n(q)$ in the table, together with p such that $p \mid \Phi_d(q)$ with $\lfloor \frac{n}{2} \rfloor + 1 \leq d \leq n - 1$ and $p^2 \nmid |PSL_n(q)|$.) Furthermore, if V is the reduction modulo p of a complex character (this is true if $\dim(V) \geq p + 1$ by Proposition 3.7(d)) then $\dim(V) \geq q(q^{n-1} - 1)/(q - 1)$ with the exceptions above.

Suppose firstly that $d = n - 1$. If $(n, q, p) = (3, 4, 5)$, then $\mathbf{U} \cong C_5$ is self-centralizing in $PSL_3(4)$, so that $C_{G_0}(\mathbf{U})$ is abelian when G_0 is any central extension. Hence $C_G(\mathbf{U})$ is abelian and $\dim(V) \leq 2p - 1$ in all cases. With the exceptions above, we get

$$q(q^{n-1} - 1)/(q - 1) - 1 \leq \dim(V) \leq 2p - 1 \leq 2\Phi_{n-1}(q) - 1 \leq 2(q^{n-1} - 1)/(q - 1) - 1.$$

Thus $q = 2$ and $\dim(V) = 2p-1$, whence $\dim(V) > p$ and so V is the reduction of a complex character. So $\dim(V) \geq q(q^{n-1} - 1)/(q - 1)$ by Table 10.1, which is a contradiction.

If $G_0/Z(G_0) \cong PSL_3(4)$ and $p = 5$, then the smallest projective representation has dimension 6 (see [JLPW]), but this is for $6.PSL_3(4)$, and as the centre has size 6 the two 6-dimensional modules are not realizable over $\mathbb{F}_5$, so there are no minimally active modules. For G_0 a central extension of $PSL_4(3)$ and $p = 13 = \Phi_3(3)$, $\dim(V) \geq 2p$ and so there are no minimally active modules (since $d = 3$ is regular).

If $\lfloor n/2 \rfloor + 1 \leq d < n - 1$ and V is minimally active for $SL_n(q)$, then $n \geq 5$, and there is $H \cong SL_{n-1}(q)$ such that $\mathbf{U} \leq H$, where $V|_H$ is also minimally active. So by what was just shown, $H \cong SL_4(2)$ and $p = 7$, and $G_0 \cong SL_5(2)$. In this case, $N_{G_0}(\mathbf{U})/\mathbf{U} \cong 3 \times S_3$, so the Green correspondent of V has dimension at most 2, and $\dim(V) \leq p + 2 = 9$, which contradicts both bounds in Table 10.1.

Case 3: We now assume G is of type $PSL_n(q)$, where $n \geq 3$ and $p \mid \Phi_n(q)$. If $G \in \mathscr{G}_p^\wedge$, then by allowing for a possible graph automorphism and a field automorphism of order t, we have that $|\mathrm{Aut}_G(\mathbf{U})| \leq 2nt$, so $p \leq 2nt + 1$. On the other hand, by Table 10.1, with the exceptions of $SL_3(2)$ and $SL_3(4)$, the dimension of any minimally active module V is at least $r^{t(n-1)} - 1$ (as the group is $SL_n(r^t).2.t$), whence $\dim(V) < 2p$ (which also holds when G_0 is a central extension of $PSL_3(4)$ and $p = 7$) becomes

$$r^{2nt/3} \leq r^{t(n-1)} \leq \dim(V) + 1 < 2p + 1 \leq 4nt + 3.$$

Set $\ell = nt \geq 3$. Then $r^{2\ell/3} \leq 4\ell + 2$, and from this we see that $r = 2$ implies $\ell < 9$ (hence $t \leq 2$), $r = 3$ implies $\ell < 6$ (hence $t = 1$), and $r \geq 5$ is impossible. Upon returning to the inequality $r^{t(n-1)} \leq 4nt + 3$, we have $q = r^t = 4$ implies $n = 3$, $q = 3$ implies $n = 3$, and $q = 2$ implies $n \leq 5$. Since we are assuming that $(n, q) \neq (3, 2)$ or $(4, 2)$, we are left with the following possibilities:

(n, q)	$(5, 2)$	$(3, 3)$	$(3, 4)$
$\Phi_n(q)$	31	13	21

Note that this list includes the remaining exception to the Landazuri–Seitz bounds.

We can eliminate groups of type $PSL_5(2)$ for $p = 31$ since $\mathrm{Aut}(PSL_2(5)) \notin \mathscr{G}_{31}^\wedge$, and those of type $PSL_3(3)$ for $p = 13$ since $\mathrm{Aut}(PSL_3(3)) \notin \mathscr{G}_{13}^\wedge$. It remains to consider $PSL_3(4)$ and its covers when $p = 7$. In this case, since $N_{PSL_3(4)}(\mathbf{U})/\mathbf{U}$ has order 3, $N_G(\mathbf{U})/\mathbf{U}$ is abelian in all cases, and hence $\dim(V) \le p + 1 = 8$ by Proposition 3.9(a). We have modules of dimension 8 for $4_1{\cdot}PSL_3(4)$ (but these cannot be defined over $\mathbb{F}_7$ as the centre has order 4), and of dimension 6 for $6{\cdot}PSL_3(4)$, which are indecomposable on restriction to $\mathbf{U}$. □

We now turn to unitary groups.

Proposition 10.2. *Let $G \in \mathscr{G}_p^\wedge$ be a group of type $PSU_n(q)$, where $n \ge 3$ and $p \nmid q$, and set $G_0 = E(G)$. Let V be a non-trivial, minimally active $\mathbb{F}_pG$-module. Then one of the following holds: either*

(i) $G_0 \cong PSU_3(3)$, $G/Z(G) \cong PSU_3(3).2$, $p = 7$, *and* $\dim(V) = 6, 7$; *or*

(ii) $G_0 \cong PSU_3(4)$, $G/Z(G) \cong PSU_3(4){:}4$, $p = 13$, *and* $\dim(V) = 12$; *or*

(iii) $G_0 \cong G/Z(G) \cong PSU_4(2)$, $p = 5$, *and* $\dim(V) = 6$; *or*

(iv) $G_0 \cong 6_1{\cdot}PSU_4(3)$, $G/Z(G) \cong PSU_4(3).2_2$ *(G contains the complex reflection group G_{34})*, $p = 7$, *and* $\dim(V) = 6$; *or*

(v) $G_0 \cong PSU_5(2)$, $G/Z(G) \cong PSU_5(2).2$, $p = 11$, *and* $\dim(V) = 10$.

Proof. Let G be of type $PSU_n(q)$, where $q = r^t$ and r is prime. Since $SU_3(2)$ is solvable, we assume $(n, q) \ne (3, 2)$. Among the other cases, $SU_n(q)$ is the universal central extension of $PSU_n(q)$ with exactly three exceptions: $PSU_4(2)$, $PSU_6(2)$, and $PSU_4(3)$ [Wi, § 3.12]. In all other cases, if $p \mid \Phi_d(q)$ for regular d, then $C_{G_0}(\mathbf{U})$ is abelian, and $\dim(V) \le 2p - 1$ when V is indecomposable and minimally active, by Propositions 9.1 and 3.9(a).

By Table 10.1, $\dim(V) \ge (q^n - q)/(q + 1)$ (whether n is even or odd), with just two exceptions.

Case 1: Assume that $p \mid \Phi_d(-q) = \Phi_{\bar{d}}(q)$ for some $\lfloor n/2\rfloor + 1 \le d \le n - 1$. If $d = n - 1$, then by the above remarks, either $G_0/Z(G_0) \cong PSU_4(2)$, $PSU_4(3)$, or $PSU_6(2)$, or

$$q(q^{n-1} - 1)/(q + 1) \le \dim(V) < 2p \le 2(q^{n-1} + 1)/(q + 1).$$

From this, it follows immediately that $q = 2$ and $\dim(V) = 2p-1$. Since V is minimally active and $C_{G_0}(\mathbf{U})$ is abelian, this last condition implies that $|\mathrm{Aut}_{G_0}(\mathbf{U})| = p - 1$ by Proposition 3.9(a). Hence $n - 1 \ge p - 1$, so $n \ge p > (2^{n-1} - 1)/3$, and $n \le 4$. But the group $SU_3(2)$ is solvable, and $3^2 \bigm| |PSU_4(2)|$ $(3 = \Phi_3(-2))$, so both of these are eliminated.

This leaves the two other exceptional cases. If $G_0/Z(G_0) \cong PSU_6(2)$ and $p \mid \Phi_5(-2) = 11$, then $\mathbf{U}$ is self-centralizing in $PSU_6(2)$, and so $C_{G_0}(\mathbf{U})$ is abelian for each cover $\mathbf{U}$. Thus $\dim(V) \le 2p - 1 = 21$, which contradicts Table 10.1. If $G_0/Z(G_0) \cong PSU_4(3)$ and $p \mid \Phi_3(-3) = 7$, then since $|N_{G_0}(\mathbf{U})/\mathbf{U}| = 3$, $\dim(V) \le p + 1 = 8$ when V is minimally active (Proposition 3.9(a)). By [JLPW, p.137], there is a single 6-dimensional module for $G_0 \cong 6_1{\cdot}PSU_4(3)$ over $\mathbb{F}_7$ (using ATLAS notation for the central extension), it extends to $6_1{\cdot}PSU_4(3).2_2 \in \mathscr{G}_7^\wedge$, and all other modules are of dimension larger than 8.

If $\lfloor n/2\rfloor + 1 \le d < n-1$, then $n \ge 5$, and there is $H < G_0$ such that $H/Z(H) \cong PSU_{n-1}(q)$ such that $\mathbf{U} \le H$ and $V|_H$ is still indecomposable and minimally active. Hence $(n, q) = (5, 3)$, $p = 7$, and $H \cong 6_1.SU_4(3)$. Since this central extension is not a subgroup of $SU_5(3)$ (and this group has no central extensions), there are no minimally active modules for $p = 7$ and $G = SU_5(3)$.

Case 2: Now assume that $p \mid \Phi_n(-q) = \Phi_{\bar{n}}(q)$, so that $|\mathrm{Aut}_G(\mathbf{U})| \leq 2nt$ by Table 9.3. Thus $p \leq 2nt+1$ and $\dim(V) < 2p$ imply (with the three exceptions noted above)

$$r^t(r^{t(n-1)}-1) \leq (4nt+2)(r^t+1)$$

and hence (since $n \geq 3$)

$$r^{2tn/3} \leq r^{t(n-1)} \leq 1+(4nt+2)(1+\tfrac{1}{r^t}) \leq (4nt+3)(1+\tfrac{1}{r}).$$

Set $\ell = nt$; we thus have $r^{2\ell/3} \leq \frac{r+1}{r}(4\ell+3)$. When $r=2$, this implies $\ell < 9$ and hence $t \leq 2$; when $r=3$ it implies $\ell < 6$ and hence $t=1$, and there are no solutions for $r \geq 5$ and $\ell \geq 3$. If we now go back to the original inequality, we see that the only solutions (including the exceptional cases) are the following ones:

(n,q)	$(4,2)$	$(5,2)$	$(6,2)$	$(3,3)$	$(3,4)$	$(4,3)$
$\Phi_n(-q)$	5	11	7	7	13	10

Here, we omit the pair $(n,q)=(3,2)$ since $SU_3(2)$ is solvable.

- For $PSU_4(2)$ and $p=5$, there are two 5-dimensional modules whose irrationalities in their Brauer characters [JLPW, p.62] imply that they need $\mathbb{F}_{25}$ from [JLPW, p.288], and there is a single 6-dimensional module, which is minimally active for $PSU_4(2) \in \mathscr{G}_5^\wedge$.
- For $PSU_5(2)$ and $p=11$, we see from [JLPW, p.184] that there is a module of dimension $10=p-1$ and two of dimension 11, amalgamating over $PSU_5(2).2 \in \mathscr{G}_{11}^\wedge$. So the proposition holds in these cases.
- For $PSU_6(2)$, the Landazuri–Seitz bound gives $\dim(V) \geq 21$, so there are no minimally active modules when $p=7$.
- For $PSU_3(3)$ and $p=7$, we need $PSU_3(3).2$ to be in $\mathscr{G}_7^\wedge$. By [JLPW, p.24] there is a single module for $G_0 = PSU_3(3)$ of dimension 6 and one of dimension 7 (the two other dual modules of dimension 7 amalgamate for G).
- For $PSU_3(4)$ and $p=13$, there is by [JLPW, p.73] a single module of dimension 12, extendible to $PSU_3(4).4 \in \mathscr{G}_{13}^\wedge$, and four modules of dimension 13, amalgamating into a single 52-dimensional for $PSU_3(4).4$.
- For $PSU_4(3)$ and $p=5$, there are many covers, as the Schur multiplier is of order 36. However, for our module to be definable over $\mathbb{F}_5$, the centre of G must have order dividing 4, and so G_0 is a quotient group of $SU_4(3)$, and its smallest simple module has dimension 20 [JLPW, p.128]. Thus $\dim(V) \leq 2p-1 = 9$ by Propositions 9.1 and 3.9(a), a contradiction.

This completes the proof. □

We next consider the symplectic groups.

Proposition 10.3. *Let $G \in \mathscr{G}_p^\wedge$ be a group of type $PSp_{2n}(q)$, where $n \geq 2$ and $p \nmid q$, and set $G_0 = E(G)$. Let V be a non-trivial, minimally active $\mathbb{F}_pG$-module. Then one of the following holds: either*

(i) *G is of type $Sp_4(2)' \cong A_6$ and $p=5$, and V is as in Proposition 8.1; or*

(ii) *$G_0 \cong G/Z(G) \cong PSp_4(3) \cong PSU_4(2)$, $p=5$, and $\dim(V)=6$; or*

(iii) *$G_0 \cong Sp_4(4)$, $G/Z(G) \cong Sp_4(4).4$, $p=17$, and $\dim(V)=18$; or*

(iv) *$G_0 \cong G/Z(G) \cong Sp_6(2)$, $p=5$ or $p=7$, and $\dim(V)=7$; or*

(v) $G_0 \cong 2{\cdot}Sp_6(2)$, $G/Z(G) \cong Sp_6(2)$, $p = 7$ *and* $\dim(V) = 8$.

Proof. In the first three cases, we assume that $p \mid \Phi_{2n}(q)$, or (if n is odd) that $p \mid \Phi_n(q)$. In particular, in these cases, $p \le (q^n \pm 1)/(q \pm 1)$ if n is odd, and $p \le (q^n + 1)$ if n is even. The remaining possibilities (where d is not regular) are handled inductively in Case 4.

Since the group $Sp_4(2)' \cong A_6$ has already been handled in Proposition 8.1, we assume from now on that $(2n, q) \ne (4, 2)$. The only (other) case in which $Sp_{2n}(q)$ has a proper central extension is the group $Sp_6(2)$.

Case 1: Suppose that q is even. By Table 10.1 (and since $(2n, q) \ne (4, 2)$), we have $\dim(V) \ge q(q^n - 1)(q^{n-1} - 1)/2(q + 1)$. When $(2n, q) = (6, 2)$, we have $p = 7 = \Phi_3(2)$ (since $\Phi_6(2) = 3$ and $Sp_6(2) \notin \mathscr{G}_3$), $\mathbf{U}$ is self-centralizing in $Sp_6(2)$, so $C_{G_0}(\mathbf{U})$ is abelian when G_0 is any central extension of $Sp_6(2)$, and the bound $\dim(V) < 2p$ still applies.

Suppose firstly that $p \mid \Phi_n(q)$, where $n \ge 3$ is odd. The statement $\dim(V) < 2p$ yields

$$q(q^n - 1)(q^{n-1} - 1)/2(q + 1) < 2(q^n - 1)/(q - 1).$$

If $q \ge 4$ then this can never happen, but if $q = 2$ then this reduces to $(2^{n-1} - 1) < 6$, and hence $n = 3$. Thus $(2n, q) = (6, 2)$, and $p = \Phi_3(2) = 7$. In this case, $Sp_6(2)$ has a faithful 7-dimensional module and $2{\cdot}Sp_6(2)$ a faithful 8-dimensional module, and both are minimally active.

If $p \mid \Phi_{2n}(q)$, then since $\dim(V) < 2p$, we get

$$q(q^n - 1)(q^{n-1} - 1)/2(q + 1) < 2(q^n + 1),$$

whence $q(q^{n-1} - 1)(q^n - 1) < 4(q + 1)(q^n + 1)$, and $q \le 4$. If $q = 4$, then $(4^n - 1)(4^{n-1} - 1) < 5(4^n + 1)$, which is satisfied for $n = 2$ only. If $q = 2$, we have $(2^n - 1)(2^{n-1} - 1) < 6(2^n + 1)$, which is satisfied for $n \le 3$ only, but $\Phi_6(2) = 3$, so the case $(2n, q) = (6, 2)$ can be eliminated. Since $(2n, q) \ne (4, 2)$, we are left with the case $G_0 \cong Sp_4(4)$ and $p \mid \Phi_4(4) = 17$. Then $\mathrm{Aut}(G_0) \cong G_0{:}4 \in \mathscr{G}_{17}^\wedge$, and there is a (unique) 18-dimensional simple minimally active $\mathbb{F}_{17}G_0$-module for which the action extends to $\mathrm{Aut}(G_0)$.

Case 2: If q is odd and n is odd, the smallest cross-characteristic representations are the Weil representations, of dimension $(q^n - 1)/2$ (see Table 10.1 and [GMST]). If $p \mid \Phi_n(q)$ or $\Phi_{2n}(q)$ then $p \le (q^n \pm 1)/(q \pm 1)$, whence the inequality $\dim(V) < 2p$ yields

$$(q^n - 1)/2 < 2(q^n \pm 1)/(q \pm 1),$$

and hence $q = 3$.

If V is minimally active and $G = Sp_{2n}(3)$ (n odd) lies in $\mathscr{G}_p^\wedge$, and $p \mid \Phi_n(3)$, then since there can be no graph or field automorphisms, $\mathrm{Aut}_G(\mathbf{U})$ has order $2n$, so that $p = 2n + 1$. On the other hand, $\dim(V) \ge (q^n - 1)/2$, and since q and n are both odd, we have that $(n, q) = (3, 3)$, yielding $p = 2n + 1 = 7 = \Phi_6(3)$ and V has dimension 13, larger than $p + 1$.

Case 3: Assume that $p \mid \Phi_{2n}(q)$, q is odd, and n is even. Assume that $q = r^t$ where r is prime. If $G = Sp_{2n}(q^t).t$, where t is the order of a graph automorphism, then $\mathrm{Aut}_G(\mathbf{U})$ has order at most $2nt$, so that $p \le 2nt + 1$. Thus $\dim(V) < 2p \le 4nt + 2$, and using the fact that $\dim(V) \ge (q^n - 1)/2$, we have that

$$r^{nt} - 1 < 8nt + 4.$$

As $r \ge 3$ and $n \ge 2$ is even, we have $nt = 2$, so $t = 1$ and $(n, q) = (2, 3)$. But $PSp_4(3) \cong PSU_4(2)$, so this case has already been done (Proposition 10.2).

Case 4: Now assume that $G \in \mathscr{G}_p^\wedge$ is of type $PSp_{2n}(q)$, where $p \mid \Phi_d(q)$ for odd $d < n$ or even $d < 2n$ (see Table 9.1), and that there is an indecomposable minimally active $\mathbb{F}_pG$-module

V. Then there is $H < G$ of type $PSp_{2n-2}(q)$ with $H \in \mathscr{G}_p^\wedge$, and $V|_H$ is again minimally active. So we can assume inductively that some indecomposable summand of $V|_H$ is already on our list.

Thus we next consider groups of type $Sp_6(2)$ ($p = 5$), $PSp_6(3)$ ($p = 5$), $Sp_6(4)$ ($p = 17$), and $Sp_8(2)$ ($p = 7$). Since $\mathrm{Aut}(Sp_6(4)) \notin \mathscr{G}_{17}^\wedge$, we can eliminate this case.

Assume G is of type $Sp_6(2)$ and $p = 5$. Then $N_{G_0/Z(G_0)}(\mathbf{U}) \cong 5{:}4 \times S_3$, contained in the subgroup $\Omega_6^+(2) \cong S_8$ in $Sp_6(2)$. Hence $N_{G_0/Z(G_0)}(\mathbf{U})/\mathbf{U}$ contains a cyclic subgroup of index 2, and $N_{G_0}(\mathbf{U})/\mathbf{U}$ contains an abelian subgroup of index 2. So by Green correspondence, if V is indecomposable and minimally active, then $\dim(V) \le p + 2 = 7$. By [JLPW], there is a 7-dimensional module for $Sp_6(2)$, and it is minimally active.

If G is of type $PSp_6(3)$ and $p = 5$, the smallest faithful module has dimension at least 13 by Table 10.1, more than twice that of the minimally active module for $Sp_4(3)$, so by Proposition 3.10, there are no minimally active $\mathbb{F}_pG$-modules. Similarly, if G is of type $Sp_8(2)$ and $p = 5$ or 7, then $\dim(V) \ge 28$ by Table 10.1, this is more than twice the dimension of the modules we found for $Sp_6(2)$, so this is again impossible by Proposition 3.10. □

It remains to handle the orthogonal groups.

Proposition 10.4. *Let $G \in \mathscr{G}_p^\wedge$ be a group of type $\Omega_{2n+1}(q)$ for q odd and $n \ge 3$, or of type $P\Omega_{2n}^\pm(q)$ for $n \ge 4$, where $p \nmid q$ in all cases. Set $G_0 = E(G)$. Let V be a non-trivial, minimally active $\mathbb{F}_pG$-module. Then $p = 7$, $G_0 \cong 2{\cdot}\Omega_8^+(2)$, $G/Z(G) \cong \Omega_8^+(2)$ or $\Omega_8^+(2).2$, and $\dim(V) = 8$.*

Proof. **Case 1:** Assume that $G_0/Z(G_0) \cong \Omega_{2n+1}(q)$ (where q is odd). By Table 10.1, if V is a minimally active module for G_0 or one of its covers, then $\dim(V) \ge q^{n-1}(q^{n-1} - 1)$, except possibly for $\Omega_7(3)$, which will be handled separately. As with symplectic groups, we use an inductive argument to reduce to the case where $p \mid \Phi_d(q)$ for $d = n, 2n$, starting with the base case of the induction, $\Omega_5(q) = Sp_4(q)$.

If $p \mid \Phi_n(q)$, then $p \le (q^n - 1)/(q - 1)$, and so $\dim(V) < 2p$ becomes

$$q^{n-1}(q^{n-1} - 1) < \frac{2(q^n - 1)}{q - 1} \quad \Longrightarrow \quad 2q^2 \le q^{n-1}(q - 1) < \frac{2(q^n - 1)}{q^{n-1} - 1} \le 2(1 + q),$$

which has no solutions.

If $p \mid \Phi_{2n}(q)$, then $p \le (q^n + 1)$, and so $\dim(V) < 2p$ becomes

$$q^{n-1}(q^{n-1} - 1) \le 2(q^n + 1),$$

yielding $q^{2n-2} \le 2q^n + q^{n-1} + 2$ for $n, q \ge 3$, again clearly having no solutions. Thus there are no minimally active modules when $p \mid \Phi_n(q)$ or $p \mid \Phi_{2n}(q)$. So by induction, and since the only groups of type $\Omega_5(q) \cong PSp_4(q)$ with minimally active modules are those for $q = 3$ (Proposition 10.3(ii)), the only cases left to consider are those where $G_0/Z(G_0) \cong \Omega_7(3)$ and $p = 5 \mid \Phi_4(3)$, $7 = \Phi_6(3)$, or $13 = \Phi_3(3)$.

When $p = 5$, Proposition 3.10, applied with $H < G$ of type $\Omega_5(q) = PSp_4(q)$, says that the dimension of a minimally active $\mathbb{F}_5G_0$-module V is at most twice that of a minimally active $\mathbb{F}_pH$-module (since $\Omega_7(3)$ is generated by two conjugates of $\Omega_5(3)$). Thus $\dim(V) \le 12$ by Proposition 10.3(ii), which is impossible by Table 10.1.

When $p = 7$ or 13, $C_{\Omega_7(3)}(\mathbf{U})$ is cyclic (of order 14 or 13, respectively), so the centralizer of $\mathbf{U}$ is abelian in each cover of $\Omega_7(3)$. Hence $\dim(V) < 2p$, which again contradicts Table 10.1.

Case 2: We move on to the case where $G = \Omega_{2n}^+(q)$ (for $n \geq 4$). Here we are concerned with $p \mid \Phi_d(q)$, where by Table 9.1, d is regular when $d = 2n-2$, or $d \in \{n-1, n\}$ is odd. The Landazuri–Seitz bound from Table 10.1 is $q^{n-2}(q^{n-1}-1)$, with $\Omega_8^+(2)$ the only exception. This is also the only case which has an exceptional Schur multiplier.

If $d = n-1$ or $d = n$, then $p \leq (q^n-1)/(q-1)$, and $\dim(V) < 2p$ implies

$$q^2 \leq q^{n-2}(q-1) < 2(q^n-1)/(q^{n-1}-1) \leq 2(1+q).$$

This is satisfied only for $\Omega_8^+(2)$, which we need to consider separately in any case. If $d = 2n-2$, then $p \leq (q^n+1)$, and we have

$$q^{n-2}(q^{n-1}-1) < 2(q^n+1) \quad\implies\quad q \leq q^{n-3} < 2 + q^{-2} + 2q^{-n},$$

which again is only satisfied for $\Omega_8^+(2)$.

Now, $\Omega_8^+(2) \in \mathscr{G}_p$ only for $p = 7$, $C_{\Omega_8^+(2)}(\mathbf{U}) = \mathbf{U}$ in this case, and thus $C_{G_0}(\mathbf{U})$ is abelian when G_0 is any central extension of $\Omega_8^+(2)$. Thus we can always assume that $\dim(V) < 2p = 14$. The smallest non-trivial representation of $\Omega_8^+(2)$ itself is of dimension 28, which is too large. There is an 8-dimensional representation for the exceptional cover $2{\cdot}\Omega_8^+(2)$, it is minimally active, and it extends to a module for $2{\cdot}\Omega_8^+(2).2$ (the Weyl group of E_8).

Since the only minimally active module is for the exceptional cover $2{\cdot}\Omega_8^+(2)$, it does not extend to a minimally active module over a group of type $\Omega_{10}^+(2)$.

Case 3: Finally, consider G of type $\Omega_{2n}^-(q)$ (again for $n \geq 4$). By Table 9.1, d is regular when $d = 2n$, $2n-2$, or n is even and $d = n-1$, and we first consider $p \mid \Phi_d(q)$ for such d. The Landazuri–Seitz bound gives $\dim(V) \geq (q^{n-1}+1)(q^{n-2}-1)$ (Table 10.1), and we do the same analysis as for the plus-type case.

If $d = n-1$, then $p \leq (q^{n-1}-1)/(q-1)$, and since $\dim(V) < 2p$,

$$(q^{n-1}+1)(q^{n-2}-1)(q-1) < 2(q^{n-1}-1),$$

so that $(q^{n-2}-1)(q-1) < 2$, which has no solutions when $n \geq 4$. If $d = 2n-2$ or $2n$, then $p \leq q^n+1$, and since $\dim(V) < 2p$, we have

$$(q^{n-1}+1)(q^{n-2}-1) < 2(q^n+1) \quad\implies\quad q \leq q^{n-3} < 2 + 3q^{-n} + q^{-1} - q^{-2}.$$

Thus $(2n, q) = (8, 2)$. Also, $p = \Phi_6(2) = 3$ or $p = \Phi_8(2) = 17$, and since $3^2 \bigm| |\Omega_8^-(2)|$, we have $p = 17$. However, upon checking the Brauer character table from [JLPW, p.248], we see that the smallest non-trivial module has dimension $34 = 2p$, so this case does not occur.

We now must check the base case of our induction. For $G_0 \cong \Omega_8^-(q)$, $G_0 \in \mathscr{G}_p$ only when $p \mid \Phi_d(q)$ for $d = 3, 4, 6, 8$. We have already handled $d = 3, 6, 8$, but we are left with $d = 4$, for which the centralizer of $\mathbf{U}$ might not be abelian. As with the $\Omega_7(q)$ case, we take two copies of $H = SU_4(q) \cong \Omega_6^-(q)$ inside $G_0 \cong \Omega_8^-(q)$ that generate G, and apply Proposition 3.10 to get $\dim(V) < 4p \leq 4(q^2+1)$. Thus the lower bound in Table 10.1 becomes

$$(q^3+1)(q^2-1) \leq \dim(V) < 4(q^2+1),$$

which clearly has no solutions, not even for $q = 2$. This completes the proof. □

11. Exceptional groups

In this section we treat the exceptional groups of Lie type. We maintain the notation of the previous section, so that G is of type $\mathbb{G}(q)$, an exceptional group of Lie type, $p \nmid q$ is a prime dividing G, a Sylow p-subgroup $\mathbf{U}$ has order p and is generated by x, and p has order d modulo q, so that $p \mid \Phi_d(q)$. In Table 11.1, we list the minimal possible dimensions for V, as determined in [LS] or [SZ].

G	Lower bound for $\dim(V)$	Ref.	Exceptions
${}^2B_2(q)$	$(q-1)\sqrt{q/2}$	[LS]	$l({}^2B_2(8)) = 8$
${}^2G_2(q)$	$q(q-1)$	[LS]	—
${}^2F_4(q)$	$q^4(q-1)\sqrt{q/2}$	[LS]	—
$G_2(q)$	$q(q^2-1)$	[SZ]	$l(G_2(3)) = 14$ $l(G_2(4)) = 12$
${}^3D_4(q)$	$q^3(q^2-1)$	[LS]	—
$F_4(q)$	$q^6(q^2-1) \quad (2 \nmid q)$ $q^7(q^3-1)(q-1) \quad (2 \mid q)$	[LS]	$l(F_4(2)) \geq 44$
$E_6^{\pm}(q)$	$q^9(q^2-1)$	[SZ]	—
$E_7(q)$	$q^{15}(q^2-1)$	[LS]	—
$E_8(q)$	$q^{27}(q^2-1)$	[LS]	—

TABLE 11.1. Minimal representation dimensions of exceptional groups

Proposition 11.1. *Let $G \in \mathscr{G}_p^\wedge$ be such that $E(G/Z(G))$ is an exceptional simple group of Lie type in defining characteristic different from p. Set $G_0 = E(G)$. Let V be a non-trivial, minimally active $\mathbb{F}_pG$-module. Then one of the following holds: either*

(i) *$G_0 \cong {}^2B_2(8)$, $G/Z(G) \cong {}^2B_2(8){:}3$, $p = 13$, and $\dim(V) = 14$; or*

(ii) *$G_0 \cong SL_2(8)$, $G/Z(G) \cong {}^2G_2(3) \cong SL_2(8){:}3$, $p = 7$, and $\dim(V) = 7, 8$; or*

(iii) *$G_0 \cong PSU_3(3)$, $G/Z(G) \cong G_2(2) \cong PSU_3(3){:}2$, $p = 7$, and $\dim(V) = 6, 7$; or*

(iv) *$G_0 \cong G_2(3)$, $G/Z(G) \cong G_2(3).2$, $p = 13$, and $\dim(V) = 14$.*

Proof. In all cases, $G_0/Z(G_0)$ is one of the exceptional groups listed in Tables 9.2 and 11.1, and unless stated otherwise, $p \mid \Phi_d(q)$ for one of the d listed in the first table. Thus $\mathbf{U} \in \mathrm{Syl}_p(G_0)$ has order p. As usual, V is assumed to be a minimally active $\mathbb{F}_pG_0$-module.

Case 1: Assume that G_0 is a Suzuki or Ree group or $G_0 \cong {}^3D_4(q)$. In all of these cases, d is regular by Table 9.2. Aside from ${}^2B_2(8)$, these groups have no exceptional covers [Wi, §4.2.4], so $C_{G_0}(\mathbf{U})$ is abelian in all other cases, and $\dim(V) < 2p$.

If $G_0/Z(G_0)$ is a Suzuki group ${}^2B_2(q)$, then p divides one of $q-1$, $q+\sqrt{2q}+1$ and $q-\sqrt{2q}+1$. In particular, $p \leq q+\sqrt{2q}+1$, and since $\dim(V) \geq (q-1)\sqrt{q/2}$ by Table 11.1,

$$(q-1)\sqrt{q/2} \leq \dim(V) < 2p \leq 2q + 2\sqrt{2q} + 2 \quad \Longrightarrow \quad \sqrt{q} < \sqrt{8}(1+\tfrac{1}{q}) + 5/\sqrt{q}.$$

Hence there are no minimally active modules for $q \geq 32$. For $q = 8$ we can just go through the known character tables [JLPW]. If $G_0 \cong {}^2B_2(8)$ (i.e., not an exceptional cover), then of the three primes $p = 5$, 7, and 13 for which $G_0 \in \mathscr{G}_p$, there is a non-trivial $\mathbb{F}_pG_0$-module of dimension less than $2p$ only for $p = 13$: the two 14-dimensional simple modules in this case are minimally active and each of them extends to $G_0{:}3 \in \mathscr{G}_{13}^\wedge$. If $G_0 \cong 2{\cdot}{}^2B_2(8)$, then $\mathrm{Out}(G_0) = 1$ (since $\mathrm{Out}({}^2B_2(8)) \cong C_3$ acts faithfully on the Schur multiplier C_2^2), and $G_0 \in \mathscr{G}_p^\wedge$ only for $p = 5$. In this case, there is an 8-dimensional $\mathbb{F}_5G_0$-module, but by Proposition 3.9(a) and since $C_{G_0}(\mathbf{U})$ is cyclic of order 10, it cannot be minimally active since $(8-5) \nmid (p-1) = 4$.

If G_0 is a small Ree group ${}^2G_2(q)$, then p divides one of $q-1$, $q+1$, $q-\sqrt{3q}+1$ or $q+\sqrt{3q}+1$. Together with the lower bound for $\dim(V)$ in Table 11.1, this gives

$$q(q-1) \le \dim(V) < 2p \le 2q + 2\sqrt{3q} + 2.$$

For $q \ge 27$ we therefore can have no minimally active modules. For $q = 3$, we have that ${}^2G_2(3) \cong SL_2(8){:}3$, and this case was covered in Proposition 10.1.

If G_0 is a large Ree group ${}^2F_4(q)$, then p divides $\Phi_6(q)$ or one of the two polynomials into which $\Phi_{12}(q)$ splits over $\mathbb{Z}[\sqrt{2}]$. The Landazuri–Seitz bound for $\dim(V)$ gives

$$q^4 \le q^4(q-1)\sqrt{q/2} \le \dim(V) < 2p \le 4q^2 + 4q + 2.$$

So there are no minimally active modules if $q \ge 8$. If $q = 2$, then $G_0 \in \mathscr{G}_p$ only for $p = 13$, $N_{G_0}(\mathbf{U})/\mathbf{U}$ is abelian in this case, so $\dim(V) \le p+1 = 14$, while $\dim(V) \ge 16$ by the above bound.

If G is a triality group $G = {}^3D_4(q)$, then by Table 9.2, p divides $\Phi_{12}(q) = q^4 - q^2 + 1$, and together with the bound in Table 11.1, this implies that

$$q^3(q^2-1) \le \dim(V) < 2p \le 2q^4 - 2q^2 + 2,$$

and hence $q = 2$. But in this case, $p = \Phi_{12}(2) = 13$, and the smallest non-trivial module has dimension 26 by [JLPW]. So there are no minimally active modules in any of these cases.

Case 2: We next consider the small exceptional groups $G_2(q)$ and $F_4(q)$. Again for these groups, by Table 9.2 and Proposition 9.1, $C_{G_0}(\mathbf{U})$ is abelian for all primes p such that $G_0 \in \mathscr{G}_p$ (unless possibly G_0 is one of the exceptional covers $3{\cdot}G_2(3)$, $2{\cdot}G_2(4)$, or $2{\cdot}F_4(2)$), and hence $\dim(V) < 2p$.

If $G_0 = G_2(q)$, then $p \mid \Phi_3(q)$ or $p \mid \Phi_6(q)$, so in particular $p \le q^2 + q + 1$. Together with the Seitz–Zalesskii bound in Table 11.1, this gives

$$q(q^2-1) \le \dim(V) < 2p \le 2q^2 + 2q + 2,$$

for $q \ge 5$, yielding no solutions. We have already dealt with the case $G_2(2) \cong PSU_3(3){:}2$.

If $G_0/Z(G_0) \cong G_2(3)$, then $G_0 \in \mathscr{G}_p$ implies $p = 7$ or 13, and $C_{G_0}(\mathbf{U})$ is abelian since the Sylow 7- and 13-subgroups of $G_2(3)$ are self-centralizing. There are no non-trivial $\mathbb{F}_7G_0$-modules of degree less than 14, while there are 14-dimensional simple modules for $G_2(3){:}2$ and $p = 13$ (see [JLPW]), and they are minimally active.

If $G_0/Z(G_0) \cong G_2(4)$, then $G_0 \in \mathscr{G}_p$ implies $p = 7$ or 13, and $C_{G_0}(\mathbf{U})$ is abelian since the centralizers of the Sylow 7- and 13-subgroups of $G_2(4)$ are cyclic of order 21 or 13, respectively [Atlas]. Thus $\dim(V) < 2p$, and by [JLPW], there are 12-dimensional modules for $2{\cdot}G_2(4)$ over $\mathbb{F}_7$ and over $\mathbb{F}_{13}$ to be considered. When $p = 7$, this module cannot be minimally active by Proposition 3.9(a) and since $(12-7) \nmid (p-1) = 6$. When $p = 13$, we have $|\mathrm{Aut}_{G_0}(\mathbf{U})| = 6$, so we need to extend to $G = 2{\cdot}G_2(4).2$. However, from the table on [JLPW, p.277], we see that there are irrationalities in the Brauer character of this representation, and by [JLPW, p.291], it is defined only over $\mathbb{F}_{13^2}$.

For $G_0 = F_4(q)$, the Landazuri–Seitz bounds in Table 11.1 gives the inequalities

$$q^6 \le \dim(V) < 2p \le 2{\cdot}\max\{\Phi_8(q), \Phi_{12}(q)\} = 2q^4 + 2$$

when $q > 2$, and $44 < 2{\cdot}2^4 + 2 = 34$ when $q = 2$. Since these have no solutions, G_0 has no minimally active modules.

Case 3: Assume that $G_0 = E_6^{\varepsilon}(q)$ (the universal or adjoint group). If $p \mid \Phi_d(\varepsilon q)$ for $d = 8, 9, 12$, then $C_{G_0}(\mathbf{U})$ is abelian (see Table 9.2), so $\dim(V) < 2p$, while $\dim(V) \ge q^9(q^2-1)$

by Table 11.1. Thus

$$q^9 \leq q^9(q^2-1) < 2\Phi_d(\varepsilon q) \leq 2\Phi_9(q) = q^6 + q^3 + 1,$$

which is impossible. This leaves $p \bigm| \Phi_5(\varepsilon q)$, in which case $\mathbf{U} \leq H < G$ for H of type $PSL_6^{\varepsilon}(q)$. We already saw in the proofs of Propositions 10.1 (Case 2) and 10.2 (Case 1) that $SL_6^{\varepsilon}(q)$ has no minimally active projective representations over $\mathbb{F}_p$ when $p \mid \Phi_5(\varepsilon q)$, and so G_0 has no minimally active modules.

If G_0 is an exceptional cover $2{\cdot}{}^2E_6(2)$ [Wi, § 4.11], then $G_0 \in \mathscr{G}_p$ only for $p = 11, 13, 17, 19$. In the last three cases, the Sylow p-subgroup of ${}^2E_6(2)$ is self-centralizing, so $C_{G_0}(\mathbf{U})$ is abelian, $\dim(V) < 2p$, and the above argument applies. If $p = 11 = \Phi_5(-2)$, then the above comparison with $U_6(2)$ again applies.

Assume that $G_0 = E_7(q)$ and $p \bigm| \Phi_d(q)$. If $d = 7, 9, 14, 18$, then $C_{G_0}(\mathbf{U})$ is abelian (see Table 9.2), so $\dim(V) < 2p$, while $\dim(V) \geq q^{15}(q^2-1) > q^{15}$ by Table 11.1. Thus $q^{15} < 2\Phi_d(q) \leq 2\Phi_7(q) \leq q^7 - 1$, which is impossible. If $d = 5, 8, 10, 12$, then $\mathbf{U} \leq H < G_0$ for $H \cong E_6^{\pm}(q)$, and we just showed that these groups have no minimally active modules over $\mathbb{F}_p$. So G_0 has no minimally active modules.

Finally, assume that $G_0 = E_8(q)$ and $p \bigm| \Phi_d(q)$. If $d = 15, 20, 24, 30$, then $C_{G_0}(\mathbf{U})$ is abelian (see Table 9.2), so $\dim(V) < 2p$, while $\dim(V) \geq q^{27}(q^2-1)$ by Table 11.1. Since Φ_d has degree 8 in each of these cases, one easily sees that we cannot have $\dim(V) < 2p$. If $d = 7, 9, 14, 18$, then $\mathbf{U} \leq H < G_0$ for $H \cong E_7(q)$, and we just showed that these groups have no minimally active modules over $\mathbb{F}_p$. So again in this case, G_0 has no minimally active modules. □

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