# On perfect packings in dense graphs

Treglown, Andrew; Balogh, Jozsef; Kostochka, Alexandr

# On perfect packings in dense graphs

József Balogh,* Alexandr V. Kostochka† and Andrew Treglown‡

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**Abstract**

We say that a graph $G$ has a perfect $H$-packing if there exists a set of vertex-disjoint copies of $H$ which cover all the vertices in $G$. We consider various problems concerning perfect $H$-packings: Given $n, r, D \in \mathbb{N}$, we characterise the edge density threshold that ensures a perfect $K_r$-packing in any graph $G$ on $n$ vertices and with minimum degree $\delta(G) \geq D$. We also give two conjectures concerning degree sequence conditions which force a graph to contain a perfect $H$-packing. Other related embedding problems are also considered. Indeed, we give a structural result concerning $K_r$-free graphs that satisfy a certain degree sequence condition.

## 1 Introduction

Given two graphs $H$ and $G$, a *perfect $H$-packing* in $G$ is a collection of vertex-disjoint copies of $H$ which cover all the vertices in $G$. Perfect $H$-packings are also referred to as *$H$-factors* or *perfect $H$-tilings*. Hell and Kirkpatrick [10] showed that the decision problem whether a graph $G$ has a perfect $H$-packing is NP-complete precisely when $H$ has a component consisting of at least 3 vertices. So for such graphs $H$, it is unlikely that there is a complete characterisation of those graphs containing a perfect $H$-packing. Thus, there has been significant attention on obtaining sufficient conditions that ensure a graph $G$ contains a perfect $H$-packing.

A seminal result in the area is the Hajnal-Szemerédi theorem [9] which states that a graph $G$ whose order $n$ is divisible by $r$ has a perfect $K_r$-packing provided that $\delta(G) \geq (r-1)n/r$. Kühn and Osthus [15, 16] characterised, up to an additive constant, the minimum degree which ensures a graph $G$ contains a perfect $H$-packing for an arbitrary graph $H$.

It is easy to see that the minimum degree condition in the Hajnal-Szemerédi theorem cannot be lowered. Of course, this does not mean that one cannot strengthen this result. *Ore-type* degree conditions consider the sum of the degrees of non-adjacent vertices in a graph. The following Ore-type result of Kierstead and Kostochka [12] implies the Hajnal-Szemerédi theorem.

*University of Illinois, Urbana-Champaign, USA and University of California, San Diego, USA, jobal@math.uiuc.edu. This author is supported by NSF CAREER Grant DMS-0745185, UIUC Campus Research Board Grant 11067, and OTKA Grant K76099.

†University of Illinois, Urbana-Champaign, USA and Institute of Mathematics, Novosibirsk, Russia, kostochk@math.uiuc.edu. This author is supported in part by NSF grant DMS-0965587 and by grant 12-01-00631 of the Russian Foundation for Basic Research.

‡Queen Mary, University of London, United Kingdom, treglown@maths.qmul.ac.uk

**Theorem 1 (Kierstead and Kostochka [12])** *Let $n, r \in \mathbb{N}$ such that $r$ divides $n$. Suppose that $G$ is a graph on $n$ vertices such that for all non-adjacent $x \neq y \in V(G)$,*

$$d(x) + d(y) \geq 2(1 - 1/r)n - 1.$$

*Then $G$ contains a perfect $K_r$-packing.*

Kühn, Osthus and Treglown [17] characterised, asymptotically, the Ore-type degree condition which ensures a graph $G$ contains a perfect $H$-packing for an arbitrary graph $H$.

### 1.1 Perfect packings in dense graphs of low minimum degree

It is easy to characterise the edge density that forces a graph $G$ to contain a perfect $K_r$-packing when there are no other restrictions. Indeed, given $n, r \in \mathbb{N}$ such that $r \geq 2$ divides $n$, if $G$ is a graph on $n$ vertices and $e(G) \geq \binom{n}{2} - n + r$ then $G$ contains a perfect $K_r$-packing. Moreover, if $G$ is a copy $K$ of $K_{n-1}$ together with a vertex which sends precisely $r - 2$ edges to $K$, then $e(G) = \binom{n}{2} - n + r - 1$ and $G$ does not contain a perfect $K_r$-packing. The following result of Akiyama and Frankl [1] refines this observation.

**Theorem 2 (Akiyama and Frankl [1])** *Let $n, r \in \mathbb{N}$ such that $r$ divides $n$. Suppose $G$ is a graph on $n$ vertices and $e(\overline{G}) \leq \min\{\binom{n/r+1}{2}, n - r + 1\}$. Then $G$ has a perfect $K_r$-packing unless $\overline{G}$ is isomorphic to one of the following graphs:*

*(i) A copy of $K_{n/r+1}$ together with $(1 - 1/r)n - 1$ isolated vertices;*

*(ii) The disjoint union of $K_{1,n-r-j+1}$, $j$ edges and $r-j-2$ isolated vertices, for some $1 \leq j \leq r-2$.*

When (for example) $n \geq r^3$, $\binom{n/r+1}{2} > n - r + 1$. Hence, in this case Theorem 2 is equivalent to the following: If $G$ is a graph on $n$ vertices and $e(G) \geq \binom{n}{2} - n + r - 1$ then either $G$ contains a perfect $K_r$-packing or $\overline{G}$ is isomorphic to a graph as in (ii).

In Sections 2 and 3 we consider the following natural problem: Let $n, r \in \mathbb{N}$ such that $r$ divides $n$. Given some $D \in \mathbb{N}$, what edge density condition ensures that any graph $G$ on $n$ vertices and of minimum degree $\delta(G) \geq D$ contains a perfect $K_r$-packing?

We fully resolve the problem, and our answers for $r = 2$ and $r \geq 3$ differ.

**Theorem 3** *For an even positive $n$ and integer $1 \leq d < n/2$, let $h(n, d) := \binom{n-d-1}{2} + d(d + 1)$ and let $f(2, n, d)$ denote the maximum integer $c$ such that some $n$-vertex graph with minimum degree at least $d$ and at least $c$ edges has no perfect matching. Then*

$$f(2, n, d) = \max\{h(n, d), h(n, 0.5n - 1)\}.$$

**Theorem 4** *Let $n, r \in \mathbb{N}$ such that $r \geq 3$ and $r$ divides $n$. Given any $D \in \mathbb{N}$ such that $r - 1 \leq D \leq (r - 1)n/r - 1$ define*

$$g(n, r, D) := \max\left\{\binom{n}{2} - \binom{n/r + 1}{2}, D(n - D) + \binom{n - 1 - D}{2} + e(T(D, r - 2))\right\}.$$

*Suppose that $G$ is a graph on $n$ vertices with $\delta(G) \geq D$ and $e(G) > g(n, r, D)$. Then $G$ contains a perfect $K_r$-packing. Moreover, there exists a graph $G'$ on $n$ vertices with $\delta(G') \geq D$ and $e(G') = g(n, r, D)$ but such that $G'$ does not contain a perfect $K_r$-packing.*

Clearly a graph $G$ of minimum degree $\delta(G) < r-1$ cannot contain a perfect $K_r$-packing. Further, regardless of edge density, every graph $G$ whose order $n$ is divisible by $r$ and with $\delta(G) \geq (r-1)n/r$ contains a perfect $K_r$-packing. Thus, Theorem 4 covers all values of $D$ where our problem was not solved previously.

An *equitable $k$-colouring* of a graph $G$ is a proper $k$-colouring of $G$ such that any two colour classes differ in size by at most one. Let $n, r \in \mathbb{N}$ such that $r$ divides $n$. Notice that a graph $G$ on $n$ vertices has a perfect $K_r$-packing if and only if the complement $\overline{G}$ of $G$ has an equitable $n/r$-colouring. So, for example, the Hajnal-Szemerédi theorem can be stated in terms of equitable colourings: Let $G$ be a graph on $n$ vertices such that $r$ divides $n$. If $\Delta(G) \leq n/r - 1$ then $G$ has an equitable $n/r$-colouring.

It is often easier to work in the language of equitable colourings compared to perfect packings. Indeed, rather than prove Theorem 1 directly, Kierstead and Kostochka proved the equivalent statement for equitable colourings. Here we also find it more convenient to work with equitable colourings. Thus, instead of proving Theorem 4 directly we prove the following equivalent result.

**Theorem 5** *Let $n, r \in \mathbb{N}$ such that $r \geq 3$ and $r$ divides $n$. Recall that $T(n, r)$ denotes the Turán graph. Given any $D \in \mathbb{N}$ such that $n/r \leq D \leq n - r$ define*

$$f(n, r, D) := \min\left\{\binom{n/r+1}{2}, D + e(\overline{T}(n-D-1, r-2))\right\}.$$

*Suppose that $G$ is a graph on $n$ vertices with $\Delta(G) \leq D$ and $e(G) < f(n, r, D)$. Then $G$ has an equitable $n/r$-colouring. Moreover, there exists a graph $G'$ on $n$ vertices with $\Delta(G') \leq D$ and $e(G') = f(n, r, D)$ but such that $G'$ does not have an equitable $n/r$-colouring.*

We prove Theorem 3 and describe extremal constructions for Theorems 4 and 5 in Section 2. That is, we show that the edge density condition in Theorem 4 is best possible for all values of $D$ such that $r - 1 \leq D \leq (r-1)n/r - 1$. Section 3 contains a proof of Theorem 5.

## 1.2 Degree sequence conditions forcing a perfect packing

Chvátal [4] gave a condition on the degree sequence of a graph which ensures Hamiltonicity: Suppose that $G$ is a graph on $n$ vertices and that the degrees of the graph are $d_1 \leq \cdots \leq d_n$. If $n \geq 3$ and $d_i \geq i + 1$ or $d_{n-i} \geq n - i$ for all $i < n/2$ then G is Hamiltonian. The following is a simple consequence of Chvátal's theorem.

**Theorem 6 (Chvátal [4])** *Suppose that $G$ is a graph on $n \geq 2$ vertices and the degrees of the graph are $d_1 \leq \cdots \leq d_n$. If*

$$d_i \geq i \quad \text{or} \quad d_{n-i+1} \geq n - i \quad \text{for all} \quad 1 \leq i \leq n/2$$

*then $G$ contains a Hamilton path.*

We propose the following conjecture on the degree sequence of a graph which forces a perfect $K_r$-packing.

**Conjecture 7** *Let $n, r \in \mathbb{N}$ such that $r$ divides $n$. Suppose that $G$ is a graph on $n$ vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that:*

*($\alpha$) $d_i \geq (r-2)n/r + i$ for all $i < n/r$;*

*($\beta$) $d_{n/r+1} \geq (r-1)n/r$.*

*Then $G$ contains a perfect $K_r$-packing.*

Note that Conjecture 7, if true, is much stronger than the Hajnal-Szemerédi theorem since the degree condition allows for $n/r$ vertices to have degree less than $(r-1)n/r$. Further, Proposition 17 in Section 4 shows that the condition on the degree sequence in Conjecture 7 is essentially "best possible". It is easy to see that Theorem 6 implies Conjecture 7 in the case when $r = 2$. We prove the conjecture in the case when $G$ is additionally $K_{r+1}$-free (see Section 5).

If one can prove Conjecture 7, it seems likely it can be used to prove the next conjecture.

**Conjecture 8** *Suppose $\gamma > 0$ and $H$ is a graph with $\chi(H) = r$. Then there exists an integer $n_0 = n_0(\gamma, H)$ such that the following holds. If $G$ is a graph whose order $n \geq n_0$ is divisible by $|H|$, and whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

- *$d_i \geq (r-2)n/r + i + \gamma n$ for all $i < n/r$,*

*then $G$ contains a perfect $H$-packing.*

Since first submitting this paper, the third author and Knox [13] have proven Conjecture 8 in the case when $r = 2$. (In fact, they have proven a much more general result concerning embedding spanning bipartite graphs of small bandwidth.)

The following result of Erdős [8] characterises those degree sequences which force a copy of $K_r$ in a graph $G$.

**Theorem 9 (Erdős [8])** *Let $G$ be a graph on $n$ vertices with degree sequence $d_1 \leq \cdots \leq d_n$. If $G$ is $K_{r+1}$-free then there is an $r$-partite graph $G'$ on $n$ vertices whose degree sequence $d'_1 \leq \cdots \leq d'_n$ satisfies*

$$d_i \leq d'_i \text{ for all } i \leq n.$$

In Section 6 we prove the following related structural theorem.

**Theorem 10** *Suppose that $n, r \in \mathbb{N}$ such that $n \geq r$ and so that $r$ divides $n$. Let $G$ be a $K_{r+1}$-free graph on $n$ vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ is such that $d_{n/r} \geq (r-1)n/r$. Then $G \subseteq T(n, r)$, where $T(n, r)$ is the complete $r$-partite* Turán graph *on $n$ vertices; so each vertex class has size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$.*

# 2 The case $r = 2$ and extremal examples for $r \geq 3$

## 2.1 Perfect matchings in dense graphs

In this section we establish the density threshold that ensures every graph $G$ on an even number $n$ of vertices and of minimum degree $\delta(G) \geq d$ contains a perfect matching. Note that we only consider values of $d$ such that $1 \leq d < n/2$, since if $\delta(G) \geq n/2$ then $G$ has a perfect matching, regardless of the edge density.

Recall that $h(n, d) := \binom{n-d-1}{2} + d(d+1)$. Note that for a fixed even $n$, $h(n, d)$ decreases with $d$ in the interval $[0, n/3 - 5/6]$ and increases with $d$ in $[n/3 - 5/6, 0.5n - 1]$.

For a positive even $n$ and an integer $0 \leq d < n/2$, let $A, B$ and $C$ be disjoint sets with $|A| = d+1$, $|B| = d$, $|C| = n - 2d - 1$. Let $H = H(n,d)$ be the graph with the vertex set $A \cup B \cup C$ such that $H[B \cup C] = K_{n-d-1}$, and each vertex in $A$ is adjacent to each vertex in $B$ and to no vertex in $C$. So $H$ does not contain a perfect matching and has exactly $h(n,d)$ edges.

The examples of $H(n,d)$ show that $f(2,n,d) \geq \max\{h(n,d), h(n, 0.5n-1)\}$. Thus to derive Theorem 3, it suffices to prove that an $n$-vertex graph $G$ with $\delta(G) \geq d$ and $e(G) > \max\{h(n,d), h(n, 0.5n-1)\}$ has a perfect matching.

Consider such a graph $G$. Let $d_1 \leq \cdots \leq d_n$ denote the degree sequence of $G$. If $d_i \geq i$ for all $1 \leq i \leq n/2$ then Theorem 6 implies that $G$ contains a perfect matching. Suppose for a contradiction that $d_{i'} \leq i' - 1$ for some $1 \leq i' \leq n/2$. Note that $i' > d$ as $\delta(G) \geq d$.

Let $A$ denote the set of $i'$ vertices in $G$ that correspond to the first $i'$ terms $d_1, \dots, d_{i'}$ of the degree sequence. Set $B := V(G) \backslash A$. Then

$$e(G[B]) \geq e(G) - i'(i'-1) > \max\{h(n,d), h(n, 0.5n-1)\} - i'(i'-1)$$

since $d(x) \leq i' - 1$ for all $x \in A$. Note that $\max\{h(n,d), h(n, 0.5n-1)\} \geq h(n, i'-1)$ since $d < i' \leq n/2$. Therefore,

$$e(G[B]) > \max\{h(n,d), h(n, 0.5n-1)\} - i'(i'-1) \geq h(n, i'-1) - i'(i'-1) = \binom{n-i'}{2},$$

a contradiction as $|B| = n - i'$. Thus, $d_i \geq i$ for all $1 \leq i \leq n/2$, as desired.

### 2.2 Examples for $r \geq 3$

We will give the extremal examples for Theorem 5. Since Theorems 4 and 5 are equivalent, the complements of the extremal graphs for Theorem 5 are the extremal graphs for Theorem 4.

**Proposition 11** *Suppose that $n, r \in \mathbb{N}$ such that $r \geq 3$ and $r$ divides $n$. Then there exists a graph $G_1$ on $n$ vertices such that $\Delta(G_1) = n/r$,*

$$e(G_1) = \binom{n/r+1}{2},$$

*but such that $G_1$ does not have an equitable $n/r$-colouring.*

**Proof.** Let $G_1$ denote the disjoint union of a clique $V$ on $n/r + 1$ vertices and an independent set $W$ of $(1 - 1/r)n - 1$ vertices. So every independent set in $G_1$ contains at most one vertex from $V$. But since $|V| = n/r + 1$, $G_1$ does not have an equitable $n/r$-colouring. Further, $\Delta(G_1) = n/r$ and $e(G_1) = \binom{n/r+1}{2}$. □

**Proposition 12** *Suppose that $n, r \in \mathbb{N}$ such that $r \geq 3$ and $n = kr$ for some $k \geq 2$. Further, let $D \in \mathbb{N}$ such that $n/(r-1) \leq D \leq n - r$. Then there exists a graph $G_2$ on $n$ vertices such that $\Delta(G_2) = D$,*

$$e(G_2) = D + e(\overline{T}(n-D-1, r-2)),$$

*but such that $G_2$ does not have an equitable $n/r$-colouring.*

**Proof.** Let $G_2$ denote the disjoint union of a copy $K$ of $K_{1,D}$ and a copy of $\overline{T}(n-D-1,r-2)$. So $|G|=n$. Let $v$ denote the vertex of degree $D$ in $K$. The largest independent set in $G_2$ that contains $v$ is of size $r-1$. Thus, $G_2$ does not have an equitable $n/r$-colouring. Further, $e(G_2)=D+e(\overline{T}(n-D-1,r-2))$.

Since $n/(r-1)\leq D$ we have that $n-1\leq (r-1)D$. Thus, every vertex in the copy of $\overline{T}(n-D-1,r-2)$ has degree at most

$$\left\lceil \frac{n-D-1}{r-2} \right\rceil - 1 \leq \frac{n-D-1}{r-2} \leq D.$$

This implies that $\Delta(G_2)=D$. □

Clearly Propositions 11 and 12 show that one cannot lower the edge density condition in Theorem 5 in the case when $n/(r-1)\leq D\leq n-r$. The following result, together with Proposition 11, shows that Theorem 5 is best possible in the case when $n/r\leq D\leq n/(r-1)$.

**Proposition 13** *Let $n,r\in\mathbb{N}$ such that $r\geq 3$ and $r$ divides $n\geq 2r$. Suppose that $D\in\mathbb{N}$ such that $n/r\leq D\leq n/(r-1)$. Then*

$$f(n,r,D)=\binom{n/r+1}{2}.$$

The following simple consequence of Turán's theorem will be used in the proof of Theorem 5.

**Fact 14** *Let $n,r\in\mathbb{N}$ such that $r\leq n$. Then*

$$e(T(n,r))\leq\left(1-\frac{1}{r}\right)\frac{n^2}{2} \quad \textit{and thus} \quad e(\overline{T}(n,r))\geq\frac{n^2}{2r}-\frac{n}{2}.$$

We will also require the following easy result.

**Lemma 15** *Let $n,r\in\mathbb{N}$ such that $r\geq 4$ and $r$ divides $n\geq 3r$. Suppose that $D\in\mathbb{N}$ such that $n/r\leq D<(n+r)/(r-1)$. Then*

$$f(n,r,D)=\binom{n/r+1}{2}.$$

# 3 Proof of Theorem 5

## 3.1 Preliminaries

Suppose for a contradiction that the result is false. Let $G$ be a counterexample with the fewest vertices. That is, $n=|V(G)|=rk$ for some $k\in\mathbb{N}$, $\Delta(G)\leq D$ for some $D\in\mathbb{N}$ such that $n/r\leq D\leq n-r$, $e(G)<f(n,r,D)$ and $G$ has no equitable $n/r$-colouring. By the Hajnal-Szemerédi theorem, $\Delta(G)\geq n/r$. Notice that given fixed $n$ and $r$, $f(n,r,D)$ is non-increasing with respect to $D$. Thus, we may assume that $\Delta(G)=D$.

We first show that $k\geq 4$. Indeed, if $n=2r$ then $f(n,r,D)\leq\binom{3}{2}=3$. But it is easy to see that every graph $G_1$ on $2r$ vertices and with $e(G_1)\leq 2$ has an equitable 2-colouring. If $n=3r$ then $f(n,r,D)\leq\binom{4}{2}=6$. Consider any graph $G_1$ on $3r$ vertices with $e(G_1)\leq 5$ and $3\leq\Delta(G_1)\leq 5$. Let $x$ denote the vertex in $G_1$ where $d_{G_1}(x)=\Delta(G_1)$. Since $3\leq d_{G_1}(x)\leq 5$, $x$ lies in an independent set $I$ in $G_1$ of size $r$. But then $G_1 - I$ contains $2r$ vertices and at most 2 edges. So $G_1 - I$ has an equitable 2-colouring and hence $G_1$ has an equitable 3-colouring.

Let $v \in V(G)$ such that $d_G(v) = D$. Set $G^* := G - (N_G(v) \cup \{v\})$. Since $f(n, r, D) \leq D + e(\overline{T}(n - D - 1, r - 2))$ we have that $e(G^*) < e(\overline{T}(n - D - 1, r - 2))$. Thus, by Turán's theorem, $G^*$ contains an independent set of size $r - 1$. Hence, $v$ lies in an independent set in $G$ of size $r$. Amongst all such independent sets of size $r$ that contain $v$, choose a set $I = \{v, x_1, \dots, x_{r-1}\}$ such that $d_G(x_1) + \dots + d_G(x_{r-1})$ is maximised.

Set $G' := G - I$, $n' := |V(G')| = n - r$ and $D' := \Delta(G') \leq D$. Notice that $D' \geq n'/r$. (Indeed, if not, then by the Hajnal-Szemerédi theorem $G'$ contains an equitable $n'/r$-colouring. Thus, as $I$ is an independent set in $G$ this gives us an equitable $n/r$-colouring of $G$, a contradiction.) Furthermore, $D' \leq n' - r$. If not then

$$e(G) \geq D + D' \geq 2D' \geq 2(n' - r + 1) = 2n - 4r + 2$$

and further,

$$\begin{aligned} e(G) < f(n, r, D) &\leq f(n, r, n - 2r + 1) \leq (n - 2r + 1) + e(\overline{T}(2r - 2, r - 2)) \\ &\leq (n - 2r + 1) + (r + 3) = n - r + 4. \end{aligned}$$

Therefore, $2n - 4r + 2 < n - r + 4$ and so $n < 3r + 2$ a contradiction since $n = kr \geq 4r$.

Since $n'/r \leq D' \leq n' - r$, if $e(G') < f(n', r, D')$ then the minimality of $G$ implies that $G'$ has an equitable $n'/r$-colouring. This then implies that $G$ has an equitable $n/r$-colouring, a contradiction. Thus,

$$e(G') \geq f(n', r, D'). \tag{1}$$

We now split our argument into three cases.

### 3.2 Case 1: $\mathbf{f(n', r, D') = \binom{n'/r+1}{2}}$.

By (1), $e(G') \geq \binom{n'/r+1}{2} = \binom{n/r}{2}$. Since $d_G(v) = D \geq n/r$,

$$e(G) \geq \frac{n}{r} + \binom{n/r}{2} = \binom{n/r + 1}{2} \geq f(n, r, D),$$

a contradiction, as desired.

### 3.3 Case 2: $\mathbf{D' \leq D - 1}$ and $\mathbf{f(n', r, D') = D' + e(\overline{T}(n' - D' - 1, r - 2))}$.

The following claim will be useful.

**Claim 16** $D' < \frac{r-1}{2r-3}n - \frac{(r^2-r+1)}{2r-3}$.

**Proof.** Note that

$$D + D' + e(\overline{T}(n' - D' - 1, r - 2)) \overset{(1)}{\leq} e(G) < f(n, r, D) \leq D + e(\overline{T}(n - D - 1, r - 2)). \tag{2}$$

Since $D' \le D - 1$, clearly $e(\overline{T}(n' - D, r-2)) \le e(\overline{T}(n' - D' - 1, r-2))$. Thus, (2) implies that

$$D' + e(\overline{T}(n' - D, r-2)) < e(\overline{T}(n - D - 1, r-2)). \tag{3}$$

One can obtain $\overline{T}(n - D - 1, r-2)$ from $\overline{T}(n' - D, r-2)$ by adding $r-1$ vertices and at most

$$(n' - D) + \frac{n - D - 2}{r - 2} \quad \text{edges.} \tag{4}$$

Hence (3) and (4) give

$$D' < n' - D + \frac{n - D - 2}{r - 2}.$$

Rearranging, and using that $D' \le D - 1$ and $n' = n - r$ we get that

$$\left(2 + \frac{1}{r-2}\right) D' < \left(1 + \frac{1}{r-2}\right) n - \frac{(r^2 - r + 1)}{r - 2}.$$

Thus,

$$D' < \frac{r-1}{2r-3} n - \frac{(r^2 - r + 1)}{2r - 3},$$

as desired. □

Since we are in Case 2 we have that

$$D' + e(\overline{T}(n - r - D' - 1, r-2)) \le \binom{n'/r + 1}{2} = \binom{n/r}{2}. \tag{5}$$

Notice that for fixed $n$ and $r$, $D' + e(\overline{T}(n - r - D' - 1, r-2))$ is non-increasing as $D'$ increases. Hence, (5) and Claim 16 imply that

$$D'' + e(\overline{T}(n - r - D'' - 1, r-2)) \le \frac{n^2}{2r^2} - \frac{n}{2r} \tag{6}$$

where $D'' := \lfloor (r-1)n/(2r-3) - (r^2 - r + 1)/(2r-3) \rfloor$. Note that

$$n - r - \frac{r-1}{2r-3} n + \frac{(r^2 - r + 1)}{2r - 3} - 1 = \frac{r-2}{2r-3} n + \frac{4 - r^2}{2r - 3}.$$

So Fact 14 and (6) imply that

$$\left(\frac{r-1}{2r-3} n - \frac{(r^2 - r + 1)}{2r-3} - \frac{(2r-4)}{2r-3}\right) + \frac{1}{2(r-2)} \left(\frac{r-2}{2r-3} n + \frac{4 - r^2}{2r-3}\right)^2$$
$$- \frac{1}{2}\left(\frac{r-2}{2r-3} n + \frac{4 - r^2}{2r-3}\right) \le \frac{n^2}{2r^2} - \frac{n}{2r}.$$

Next we will move all terms from the previous equation to the left hand side and simplify. The coefficient of $n^2$ is

$$\frac{r-2}{2(2r-3)^2} - \frac{1}{2r^2} = \frac{r^3 - 6r^2 + 12r - 9}{2r^2(2r-3)^2}. \tag{7}$$

The coefficient of $n$ is

$$\frac{r-1}{2r-3}-\frac{(r-2)}{2(2r-3)}+\frac{1}{2r}+\frac{(4-r^2)}{(2r-3)^2}=\frac{r^2-4r+9}{2r(2r-3)^2}. \tag{8}$$

The constant term is

$$-\frac{(r^2+r-3)}{2r-3}+\frac{(r^2-4)^2}{2(r-2)(2r-3)^2}+\frac{(r^2-4)}{2(2r-3)}=\frac{-r^4+3r^3+4r^2-26r+28}{2(r-2)(2r-3)^2}. \tag{9}$$

Since $n \geq 4r$, (7)–(9) imply that

$$\frac{8(r^3-6r^2+12r-9)}{(2r-3)^2}+\frac{2(r^2-4r+9)}{(2r-3)^2}+\frac{-r^4+3r^3+4r^2-26r+28}{2(r-2)(2r-3)^2}\leq 0. \tag{10}$$

Multiplying (10) by $2(r-2)(2r-3)^2$ we get

$$15r^4-121r^3+364r^2-486r+244\leq 0$$

This yields a contradiction, since it is easy to check that

$$15r^4-121r^3+364r^2-486r+244>0$$

for all $r\in\mathbb{N}$ such that $r\geq 3$.

### 3.4 Case 3: $\mathbf{D'=D}$ and $\mathbf{f(n',r,D')=D'+e(\overline{T}(n'-D'-1,r-2))}$.

By (1) we have that

$$e(G')\geq f(n',r,D')=D'+e(\overline{T}(n'-D'-1,r-2)). \tag{11}$$

Consider any vertex $x\in V(G')$ such that $d_{G'}(x)=D'=D$. Since $\Delta(G)=D$, $x$ is not adjacent to any vertex in $I=\{v,x_1,\dots,x_{r-1}\}$. Further, $I$ was chosen such that $d_G(x_1)+\cdots+d_G(x_{r-1})$ is maximised. Thus, $d_G(x_1)=\cdots=d_G(x_{r-1})=D$. Together with (11) this implies that

$$e(G)\geq (r+1)D+e(\overline{T}(n'-D-1,r-2)). \tag{12}$$

Since $e(G)<f(n,r,D)\leq D+e(\overline{T}(n-D-1,r-2))$, (12) implies that

$$rD+e(\overline{T}(n'-D-1,r-2))<e(\overline{T}(n-D-1,r-2)). \tag{13}$$

One can obtain $\overline{T}(n-D-1,r-2)$ from $\overline{T}(n'-D-1,r-2)$ by adding $r$ vertices and at most

$$(n'-D-1)+\frac{2(n-D-3)}{r-2}+1 \text{ edges}. \tag{14}$$

Thus, (13) and (14) imply that

$$rD<n-r-D+\frac{2(n-D-3)}{r-2}$$

and so

$$\left(r+1+\frac{2}{r-2}\right)D < \left(1+\frac{2}{r-2}\right)n + \frac{(-r^2+2r-6)}{r-2} < \left(1+\frac{2}{r-2}\right)n. \qquad (15)$$

If $r = 3$ then (15) implies that

$$D < \frac{n}{2}.$$

Since $f(n', 3, D) = \min\{\binom{n'/3+1}{2}, D + \binom{n'-D-1}{2}\}$ it is easy to see that if $f(n', 3, D) = D + \binom{n'-D-1}{2}$ then $D \geq 2n'/3 + 1 = 2n/3 - 1$. Thus, $2n/3 - 1 \leq D < n/2$, a contradiction since $n \geq 4r = 12$.

If $r \geq 4$ then (15) implies that

$$D < \frac{n}{r-1} = \frac{n'}{r-1} + \frac{r}{r-1}.$$

Since $n' \geq 3r$, Lemma 15 implies that $f(n', r, D') = \binom{n'/r+1}{2}$ and so we are in Case 1, which we have already dealt with.

## 4 The extremal examples for Conjecture 7

**Proposition 17** *Suppose that $n, r, k \in \mathbb{N}$ such that $r \geq 2$ divides $n$ and $1 \leq k < n/r$. Then there exists a graph $G$ on $n$ vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

- $d_i = (r-2)n/r + k - 1$ *for all* $1 \leq i \leq k$;
- $d_i = (r-1)n/r$ *for all* $k+1 \leq i \leq (r-2)n/r + k$;
- $d_i = n - k - 1$ *for all* $(r-2)n/r + k + 1 \leq i \leq n - k + 1$;
- $d_i = n - 1$ *for all* $n - k + 2 \leq i \leq n$,

*but such that $G$ does not contain a perfect $K_r$-packing.*

**Proof.** Let $G'$ denote the complete $(r-2)$-partite graph whose vertex classes $V_1, \dots, V_{r-2}$ each have size $n/r$. Obtain $G$ from $G'$ by adding the following vertices and edges: Add a set $V_{r-1}$ of $2n/r - 2k + 1$ vertices to $G'$, a set $V_r$ of $k-1$ vertices and a set $V_0$ of $k$ vertices. Add all edges from $V_0 \cup V_{r-1} \cup V_r$ to $V_1 \cup \cdots \cup V_{r-2}$. Further, add all edges with both endpoints in $V_{r-1} \cup V_r$. Add all possible edges between $V_0$ and $V_r$.

So $V_0$ is an independent set, and there are no edges between $V_0$ and $V_{r-1}$. This implies that any copy of $K_r$ in $G$ containing a vertex from $V_0$ must also contain at least one vertex from $V_r$. But since $|V_0| > |V_r|$ this implies that $G$ does not contain a perfect $K_r$-packing. Furthermore, $G$ has our desired degree sequence. □

Notice that the graphs $G$ considered in Proposition 17 satisfy ($\beta$) from Conjecture 7 and only fail to satisfy ($\alpha$) in the case when $i = k$ (and in this case $d_k = (r-2)n/r + k - 1$).

Let $n, r \in \mathbb{N}$ such that $r$ divides $n$. Denote by $T^*(n, r)$ the complete $r$-partite graph on $n$ vertices with $r-2$ vertex classes of size $n/r$, one vertex class of size $n/r - 1$ and one vertex class of size $n/r + 1$. Then $T^*(n, r)$ does not contain a perfect $K_r$-packing. Furthermore, $T^*(n, r)$ satisfies ($\alpha$) but condition ($\beta$) fails; we have that $d_{n/r+1} = (r-1)n/r - 1$ here. Thus, together $T^*(n, r)$ and Proposition 17 show that, if true, Conjecture 7 is essentially best possible.

## 5 A special case of Conjecture 7

We now give a simple proof of Conjecture 7 in the case when $G$ is $K_{r+1}$-free.

**Theorem 18** *Let $n, r \in \mathbb{N}$ such that $r \geq 2$ divides $n$. Suppose that $G$ is a graph on $n$ vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that:*

- $d_i \geq (r-2)n/r + i$ *for all* $i < n/r$;
- $d_{n/r+1} \geq (r-1)n/r$.

*Further suppose that no vertex $x \in V(G)$ of degree less than $(r-1)n/r$ lies in a copy of $K_{r+1}$. Then $G$ contains a perfect $K_r$-packing.*

**Proof.** We prove the theorem by induction on $n$. In the case when $n = r$ then $d_{n/r+1} = d_2 \geq (r-1)r/r = r-1$. This implies that every vertex in $G$ has degree $r-1$. Hence $G = K_r$ as desired. So suppose that $n > r$ and the result holds for smaller values of $n$. Let $x_1 \in V(G)$ such that $d_G(x_1) = d_1 \geq (r-2)n/r + 1$. If $d_G(x_1) \geq (r-1)n/r$ then $\delta(G) \geq (r-1)n/r$. Thus $G$ contains a perfect $K_r$-packing by the Hajnal-Szemerédi theorem. So we may assume that $(r-2)n/r+1 \leq d_G(x_1) < (r-1)n/r$. In particular, $x_1$ does not lie in a copy of $K_{r+1}$. We first find a copy of $K_r$ containing $x_1$. If $r = 2$, $x_1$ has a neighbour and so we have our desired copy of $K_2$. So assume that $r \geq 3$. Certainly $N_G(x_1)$ contains a vertex $x_2$ such that $d_G(x_2) \geq (r-1)n/r$. Thus, $|N_G(x_1) \cap N_G(x_2)| \geq (r-3)n/r + 1 > 0$. So if $r = 3$ we obtain our desired copy of $K_r$. Otherwise, we can find a vertex $x_3 \in N_G(x_1) \cap N_G(x_2)$ such that $d_G(x_3) \geq (r-1)n/r$. We can repeat this argument until we have obtained vertices $x_1, \dots, x_r$ that together form a copy $K'_r$ of $K_r$.

Let $G' := G - V(K'_r)$ and set $n' := n - r = |V(G')|$. Since $G$ does not contain a copy of $K_{r+1}$ containing $x_1$, every vertex $x \in V(G) \backslash V(K'_r)$ sends at most $r-1$ edges to $K'_r$ in $G$. Thus, $d_{G'}(x) \geq d_G(x) - (r-1)$ for all $x \in V(G')$. So if $d_G(x) \geq (r-1)n/r$ then $d_{G'}(x) \geq (r-1)n/r - (r-1) = (r-1)n'/r$ for all $x \in V(G')$. If a vertex $y \in V(G')$ does not lie in a copy of $K_{r+1}$ in $G$ then clearly $y$ does not lie in a copy of $K_{r+1}$ in $G'$. This means that no vertex $y \in V(G')$ of degree less than $(r-1)n'/r$ lies in a copy of $K_{r+1}$.

Let $d'_1 \leq \cdots \leq d'_{n'}$ denote the degree sequence of $G'$. It is easy to check that $d'_i \geq (r-2)n'/r + i$ for all $i < n'/r$ and that $d'_{n'/r+1} \geq (r-1)n'/r$. Indeed, since $x_1 \in V(K'_r)$ where $d_G(x_1) = d_1$, we have that $d'_i \geq d_{i+1} - (r-1)$ for all $1 \leq i \leq n'$. Thus, for all $1 \leq i < n'/r = n/r - 1$, $d'_i \geq d_{i+1} - (r-1) \geq (r-2)n/r + (i+1) - (r-1) = (r-2)n'/r + i$. Similarly, $d'_{n'/r+1} = d'_{n/r} \geq d_{n/r+1} - (r-1) \geq (r-1)n/r - (r-1) = (r-1)n'/r$. Hence, by induction $G'$ contains a perfect $K_r$-packing. Together with $K'_r$ this gives us our desired perfect $K_r$-packing in $G$. □

## 6 Proof of Theorem 10

Consider any $x_1 \in V(G)$ such that $d_G(x_1) \geq (r-1)n/r$. Since $d_{n/r} \geq (r-1)n/r$ we can greedily select vertices $x_2, \dots, x_{r-1}$ such that

- $x_1, \dots, x_{r-1}$ induce a copy of $K_{r-1}$ in $G$;
- $d_G(x_i) \geq (r-1)n/r$ for all $1 \leq i \leq r-1$.

Note that since $G$ is $K_{r+1}$-free, $\cap_{i=1}^{r-1} N_G(x_i)$ is an independent set. The choice of $x_1, \dots, x_{r-1}$ implies that $|\cap_{i=1}^{r-1} N_G(x_i)| \geq n/r$. Let $V_1$ denote a subset of $\cap_{i=1}^{r-1} N_G(x_i)$ of size $n/r$. Thus $V_1$ contains a vertex $x_1^1$ of degree at least $(r-1)n/r$.

As before we can find vertices $x_2^1, \dots, x_{r-1}^1$ such that

- $x_1^1, \dots, x_{r-1}^1$ induce a copy of $K_{r-1}$ in $G$;
- $d_G(x_i^1) \geq (r-1)n/r$ for all $1 \leq i \leq r-1$.

So $\cap_{i=1}^{r-1} N_G(x_i^1)$ is an independent set of size at least $n/r$. Let $V_2$ denote a subset of $\cap_{i=1}^{r-1} N_G(x_i^1)$ of size $n/r$. Note that $N_G(x_1^1) \cap V_1 = \emptyset$ since $x_1^1 \in V_1$ and $V_1$ is an independent set. Thus as $V_2 \subseteq N_G(x_1^1)$, $V_1 \cap V_2 = \emptyset$.

Our aim is to find disjoint sets $V_1, \dots, V_r \subseteq V(G)$ of size $n/r$ and vertices $x_1^1, \dots, x_{r-1}^1$, $x_1^2, \dots, x_{r-1}^2, \dots, x_1^{r-1}, \dots, x_{r-1}^{r-1}$ with the following properties:

- $G[V_j]$ is an independent set for all $1 \leq j \leq r$;
- Given any $1 \leq j \leq r-1$, $x_k^j \in V_k$ for each $1 \leq k \leq j$;
- $d_G(x_k^j) \geq (r-1)n/r$ for all $1 \leq j \leq r-1$ and $1 \leq k \leq r-1$;
- $x_1^j, \dots, x_{r-1}^j$ induce a copy of $K_{r-1}$ in $G$ for all $1 \leq j \leq r-1$.

Clearly finding such a partition $V_1, \dots, V_r$ of $V(G)$ implies that $G \subseteq T(n, r)$.

Suppose that for some $1 < j < r$ we have defined sets $V_1, \dots, V_j$ and vertices $x_1^1, \dots, x_{r-1}^1, \dots, x_1^{j-1}, \dots, x_{r-1}^{j-1}$ with our desired properties. Since $d_{n/r} \geq (r-1)n/r$ and $V_1, \dots, V_j$ are independent sets of size $n/r$ we can choose vertices $x_1^j, \dots, x_j^j$ such that for all $1 \leq k \leq j$,

- $x_k^j \in V_k$ and $d_G(x_k^j) \geq (r-1)n/r$.

This degree condition, together with the fact that $x_1^j, \dots, x_j^j$ lie in different vertex classes, implies that these vertices form a copy of $K_j$ in $G$. We now greedily select further vertices $x_{j+1}^j, \dots, x_{r-1}^j$ such that

- $x_1^j, \dots, x_{r-1}^j$ induce a copy of $K_{r-1}$ in $G$;
- $d_G(x_k^j) \geq (r-1)n/r$ for all $j+1 \leq k \leq r-1$.

So $\cap_{i=1}^{r-1} N_G(x_i^j)$ is an independent set of size at least $n/r$. Let $V_{j+1}$ denote a subset of $\cap_{i=1}^{r-1} N_G(x_i^j)$ of size $n/r$. Note that, for each $1 \leq k \leq j$, $N_G(x_k^j) \cap V_k = \emptyset$ since $x_k^j \in V_k$ and $V_k$ is an independent set. Thus as $V_{j+1} \subseteq N_G(x_k^j)$ for each $1 \leq k \leq j$, $V_{j+1}$ is disjoint from $V_1 \cup \dots \cup V_j$.

Repeating this argument we obtain our desired sets $V_1, \dots, V_r \subseteq V(G)$ and vertices $x_1^1, \dots, x_{r-1}^1$, $x_1^2, \dots, x_{r-1}^2, \dots, x_1^{r-1}, \dots, x_{r-1}^{r-1}$.

# 7 Possible extensions of Conjecture 7

We suspect that the following 'Chvátal-type' degree sequence condition forces a graph to contain a perfect $K_r$-packing.

**Question 19** *Let $n, r \in \mathbb{N}$ such that $r \geq 2$ divides $n$. Suppose that $G$ is a graph on $n$ vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that for all $i \leq n/r$:*

- *$d_i \geq (r-2)n/r + i$ or $d_{n-i(r-1)+1} \geq n - i$.*

*Does this condition imply that $G$ contains a perfect $K_r$-packing?*

Note that Theorem 6 answers this question in the affirmative when $r = 2$. The following example shows that we cannot have a lower value in the second part of the condition in Question 19.

**Proposition 20** *Suppose that $n, r, k \in \mathbb{N}$ such that $r \geq 2$ divides $n$ and $1 \leq k \leq n/r$. Then there exists a graph $G$ on $n$ vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

- *$d_{n-i(r-1)+1} \geq n - i$ for all $i \in [n/r] \backslash \{k\}$;*
- *$d_{n-k(r-1)+1} = n - k - 1$,*

*but such that $G$ does not contain a perfect $K_r$-packing.*

**Proof.** Let $G$ be the graph on $n$ vertices with vertex classes $V_1, V_2$ and $V_3$ of sizes $k$, $(r-1)k-1$ and $n - rk + 1$ respectively and with the following edges: There are all possible edges between $V_1$ and $V_2$ and between $V_2$ and $V_3$. Further add all possible edges in $V_2$ and all edges in $V_3$. Thus, $V_1$ is an independent set and there are no edges between $V_1$ and $V_3$.

The degree sequence of $G$ is

$$\underbrace{(r-1)k-1, \ldots, (r-1)k-1}_{k \text{ times}}, \underbrace{n-k-1, \ldots, n-k-1}_{n-rk+1 \text{ times}}, \underbrace{n-1, \ldots, n-1}_{(r-1)k-1 \text{ times}}.$$

Hence $G$ satisfies our desired degree sequence condition. Every copy $K'_r$ of $K_r$ in $G$ that contains a vertex from $V_1$ must contain $r-1$ vertices from $V_2$. But since $|V_1|(r-1) > |V_2|$ this implies that $G$ does not contain a perfect $K_r$-packing. □

The $r$th power of a Hamilton cycle $C$ is obtained from $C$ by adding an edge between every pair of vertices of distance at most $r$ on $C$. Seymour [18] conjectured the following strengthening of Dirac's theorem.

**Conjecture 21 (Pósa-Seymour, see [18])** *Let $G$ be a graph on $n$ vertices. If $\delta(G) \geq \frac{r}{r+1}n$ then $G$ contains the $r$th power of a Hamilton cycle.*

Pósa (see [7]) had earlier proposed the conjecture in the case of the square of a Hamilton cycle (that is, when $r = 2$). Komlós, Sárközy and Szemerédi [14] proved Conjecture 21 for graphs whose order is sufficiently larger than $r$. More recently, Châu, DeBiasio and Kierstead [3] proved Pósa's conjecture for graphs of order at least $2 \times 10^8$.

In the case when $r + 1$ divides $|G|$, a necessary condition for a graph $G$ to contain the $r$th power of a Hamilton cycle is that $G$ contains a perfect $K_{r+1}$-packing. Further, notice that the minimum degree condition in Conjecture 21 is the same as the condition in the Hajnal-Szemerédi theorem with respect to perfect $K_{r+1}$-packings. Thus an obvious question is whether the condition in Conjecture 7 forces a graph to contain the $(r-1)$th power of a Hamilton cycle. Interestingly though, when $r = 3$, this is not the case.

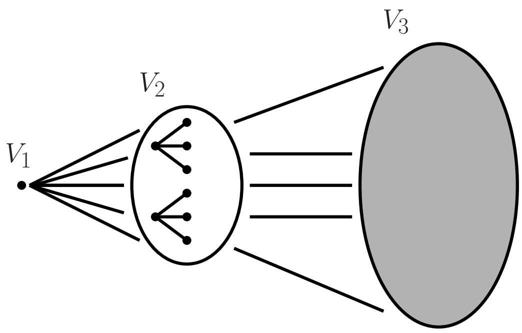


Figure 1: The example from Proposition 22 in the case when $K = 2$ and $|V_2| = 8$.

**Proposition 22** *Suppose that $C, n \in \mathbb{N}$ such that $C \ll n$ and 3 divides $n$. Then there exists a graph $G$ whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

$$d_i \geq \frac{n}{3} + C + i \quad \text{for all } 1 \leq i \leq \frac{n}{3}$$

*but such that $G$ does not contain the square of a Hamilton cycle.*

**Proof.** Choose $C, K, n \in \mathbb{N}$ so that $C \ll K \ll n$. Let $G$ denote the graph on $n$ vertices consisting of three vertex classes $V_1 = \{v\}$, $V_2$ and $V_3$ where $|V_2| = n/3 + C + 1$ and $|V_3| = 2n/3 - C - 2$ which contains the following edges:

- All edges from $v$ to $V_2$;
- All edges between $V_2$ and $V_3$ and all possible edges in $V_3$;
- There are $K$ vertex-disjoint stars in $V_2$, each of size $\lfloor |V_2|/K \rfloor$, $\lceil |V_2|/K \rceil$, which cover all of $V_2$ (see Figure 1).

Let $d_1 \leq \cdots \leq d_n$ denote the degree sequence of $G$. There are $n/3 + C - K + 1 \leq n/3 - 2C - 1$ vertices in $V_2$ of degree $2n/3 - C$. Since $C \ll K \ll n$, the remaining $K$ vertices in $V_2$ have degree at least $2n/3 - C - 2 + \lfloor |V_2|/K \rfloor \geq 2n/3 + C + 1$. Since $d_G(v) = n/3 + C + 1$ and $d_G(x) = n - 2$ for all $x \in V_3$, we have that $d_i \geq \frac{n}{3} + C + i$ for all $1 \leq i \leq \frac{n}{3}$.

A necessary condition for a graph $G$ to contain the square of a Hamilton cycle is that, for every $x \in V(G)$, $G[N(x)]$ contains a path of length 3. Note that $N(v) = V_2$ and $G[V_2]$ does not contain a path of length 3. So $G$ does not contain the square of a Hamilton cycle. □

Notice that we can set $C = o(\sqrt{n})$ in Proposition 22. We finish by raising the following question.

**Question 23** *What can be said about degree sequence conditions which force a graph to contain the $r$th power of a Hamilton cycle? In particular, can one establish a degree sequence condition that ensures a graph $G$ on $n$ vertices contains the $r$th power of a Hamilton cycle and which allows for "many" vertices of $G$ to have degree "much less" than $rn/(r+1)$?*

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# Appendix

Here we give proofs of Proposition 13 and Lemma 15. The following fact will be used in both of these proofs.

**Fact 24** *Fix $n, r \in \mathbb{N}$ such that $r \geq 3$ and $r$ divides $n \geq 2r$. Define*

$$h(x) := x + \frac{(n-x-1)^2}{2(r-2)} - \frac{1}{2}(n-x-1).$$

*Then $h(x)$ is a decreasing function for $x \in [0, n/(r-1)]$. Moreover, if $n \geq 3r$ then $h(x)$ is a decreasing function for $x \in [0, (n+r)/(r-1)]$.*

**Proof.** Notice that

$$h'(x) = \frac{3}{2} - \frac{(n-x-1)}{r-2} = \frac{x}{r-2} + \frac{1-n}{r-2} + \frac{3}{2}.$$

So for $x \leq n/(r-1)$,

$$h'(x) \leq \frac{n}{(r-1)(r-2)} + \frac{1-n}{r-2} + \frac{3}{2} = -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2}.$$

Note that $3(r-1)/2 + (r-1)/(r-2) < n$ since $n \geq 2r$ and $r \geq 3$. Thus,

$$h'(x) \leq -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2} < 0.$$

If $x \leq (n+r)/(r-1)$ then

$$h'(x) \leq \frac{n+r}{(r-1)(r-2)} + \frac{1-n}{r-2} + \frac{3}{2} = -\frac{n}{r-1} + \frac{1}{r-2} + \frac{r}{(r-1)(r-2)} + \frac{3}{2}.$$

If $n \geq 3r$ then $n > 3r/2 + 4$. So $n > 3(r-1)/2 + (2r-1)/(r-2)$. Thus,

$$h'(x) \leq -\frac{n}{r-1} + \frac{1}{r-2} + \frac{r}{(r-1)(r-2)} + \frac{3}{2} < 0,$$

as desired. □

**Proof of Proposition 13.** We need to show that, for all $D \in \mathbb{N}$ such that $n/r \leq D \leq n/(r-1)$,

$$\frac{n^2}{2r^2} + \frac{n}{2r} = \binom{n/r+1}{2} \leq D + e(\overline{T}(n-D-1, r-2)).$$

Since $D \leq n/(r-1)$, Facts 14 and 24 imply that

$$\begin{aligned} D + e(\overline{T}(n-D-1, r-2)) &\geq D + \frac{(n-D-1)^2}{2(r-2)} - \frac{(n-D-1)}{2} \\ &\geq \frac{n}{r-1} + \frac{1}{2(r-2)}\left[\frac{(r-2)}{r-1}n - 1\right]^2 - \frac{1}{2}\left[\frac{(r-2)}{r-1}n - 1\right] \\ &\geq \frac{(r-2)}{2(r-1)^2}n^2 - \frac{(r-2)}{2(r-1)}n. \end{aligned}$$

Thus, it suffices to show that

$$\frac{(r-2)}{2(r-1)^2}n - \frac{r-2}{2(r-1)} \geq \frac{n}{2r^2} + \frac{1}{2r}. \tag{16}$$

Notice that

$$\frac{r-2}{2(r-1)^2} - \frac{1}{2r^2} = \frac{(r-2)r^2 - (r-1)^2}{2r^2(r-1)^2} = \frac{r^3 - 3r^2 + 2r - 1}{2r^2(r-1)^2} \tag{17}$$

and

$$\frac{r-2}{2(r-1)} + \frac{1}{2r} = \frac{r^2 - r - 1}{2r(r-1)}.$$

Since $n \geq 2r$, (16) implies that it suffices to show that

$$\frac{r^3 - 3r^2 + 2r - 1}{r(r-1)^2} - \frac{r^2 - r - 1}{2r(r-1)} \geq 0. \tag{18}$$

Note that $r^3 \geq 4r^2 - 4r + 3$ as $r \geq 3$. Thus, $2(r^3 - 3r^2 + 2r - 1) \geq (r^2 - r - 1)(r - 1)$. So indeed (18) is satisfied, as desired. $\square$

**Proof of Lemma 15.** We need to show that, for all $D \in \mathbb{N}$ such that $n/r \leq D < (n+r)/(r-1)$,

$$\frac{n^2}{2r^2} + \frac{n}{2r} = \binom{n/r+1}{2} \leq D + e(\overline{T}(n-D-1, r-2)).$$

Since $D < (n+r)/(r-1)$ we have that $D \leq n/(r-1) + 1$. So Facts 14 and 24 imply that

$$\begin{aligned} D + e(\overline{T}(n-D-1, r-2)) &\geq D + \frac{(n-D-1)^2}{2(r-2)} - \frac{(n-D-1)}{2} \\ &\geq \frac{n}{r-1} + 1 + \frac{1}{2(r-2)}\left[\frac{(r-2)}{r-1}n - 2\right]^2 - \frac{1}{2}\left[\frac{(r-2)}{r-1}n - 2\right] \\ &\geq \frac{(r-2)}{2(r-1)^2}n^2 - \frac{(r-2)}{2(r-1)}n - \frac{n}{r-1}. \end{aligned}$$

Thus, it suffices to show that

$$\frac{(r-2)}{2(r-1)^2}n - \frac{(r-2)}{2(r-1)} - \frac{1}{r-1} \geq \frac{n}{2r^2} + \frac{1}{2r}. \tag{19}$$

Notice that

$$\frac{r-2}{2(r-1)} + \frac{1}{r-1} + \frac{1}{2r} = \frac{r^2+r-1}{2r(r-1)}.$$

Since $n \geq 3r$, (17) and (19) imply that it suffices to show that

$$\frac{3(r^3-3r^2+2r-1)}{2r(r-1)^2} - \frac{r^2+r-1}{2r(r-1)} \geq 0. \tag{20}$$

Note that $2r^3 - 9r^2 + 8r - 4 \geq 0$ as $r \geq 4$. Thus, $3(r^3 - 3r^2 + 2r - 1) \geq (r^2 + r - 1)(r - 1)$. So indeed (20) is satisfied, as desired. □