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DOI:

[10.1002/malq.201500090](https://doi.org/10.1002/malq.201500090)

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Document Version

Peer reviewed version

Citation for published version (Harvard):

Vickers, S 2017, 'The localic compact interval is an Escardo-Simpson interval object', *Mathematical Logic Quarterly*, vol. 63, no. 6, pp. 614–629. <https://doi.org/10.1002/malq.201500090>

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The localic compact interval is an Escardó-Simpson interval object

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Key words Constructive, localic, real, analysis
MSC (2010) 54F65; 03F55, 06D22, 18D35

The locale corresponding to the real interval $[-1, 1]$ is an interval object, in the sense of Escardó and Simpson, in the category of locales. The map $c: 2^\omega \rightarrow [-1, 1]$, mapping a stream s of signs ± 1 to $\sum_{i=1}^\infty s_i 2^{-i}$, is a proper localic surjection; it is also expressed as a coequalizer.

The proofs are valid in any elementary topos with natural numbers object.

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1 Introduction

In [1], Escardó and Simpson prove a universal property for the real interval $[-1, 1]$, using a theory they develop of *midpoint algebras*: sets equipped with a binary operation that, abstractly, provides the midpoint of any two elements. In an *iterative* midpoint algebra there are also some limiting processes, and it becomes possible there to define arbitrary convex combinations of two elements. This property is expressed by saying that the interval $[-1, 1]$ is freely generated, as an iterative midpoint algebra, by its endpoints. That is the universal property, and it thus characterizes the interval in a way that does not explicitly describe the structure of reals.

It is also conjectured in [1, Section 10] that there is an analogous property for the *locale* $[-1, 1]$ of Dedekind reals, which we shall write \mathbb{I} , in the category **Loc** of locales. In this paper we confirm that conjecture. Our proof is valid in any elementary topos with natural numbers object. Moreover, we have kept the argument geometric as much as possible, with a view to possibly transporting it to formal topology in predicative type theory, or to the arithmetic universe techniques of [2].

The layout of the paper can be summarized section by section as follows.

Section 2 recalls midpoint algebras.

Section 3 develops some preliminary results on Cantor space 2^ω . Principally, we analyse its localic presentation in order to get it in a “join stable” form suitable for the preframe coverage theorem, a technical result used in Section 6.

Section 4 shows as its main result that the interval \mathbb{I} is iterative. Our proof relies on its metric structure, and its embedding as the maximal points of a “ball domain”. The result of the iteration is then got via approximations in the ball domain.

Section 5 introduces a map $c: 2^\omega \rightarrow \mathbb{I}$ that can be understood as the evaluation of infinite binary expansions. We calculate some features of its inverse image function; these results are needed in Section 6.

Section 6 shows that c is a localic surjection, exploiting the fact that, as a map between compact regular locales, c is *proper*. In essence this is a conservativity result: to reason about real numbers it suffices to reason about the infinite binary expansions, and this holds even in the absence of choice principles allowing one to choose an expansion for every (Dedekind) real. To prove it we use the preframe coverage theorem, relying on the analysis of Sections 3 and 5.

Section 7 describes c as the coequalizer of two maps from 2^* to 2^ω .

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Section 8 now completes the proof of our main result, Theorem 8.11, that $(\mathbb{I}, -1, 1)$ is a cancellative interval object in **Loc**. Suppose we are given an iterative A with two specified points as in Definition 2.4 (3), and we want to define the unique $N: \mathbb{I} \rightarrow A$. The composite $Nc = M$ (say) is easy to find, so the task is to factor M via c . The unique existence of the factorization will follow from the coequalizer property of c . It remains to show that N preserves midpoints, and for this it is convenient to introduce 3^ω , for streams of signs and zeros.

2 Iterative midpoint algebras

We recall the definitions from [1], in an arbitrary category with finite products.

Definition 2.1 A *midpoint algebra* is an object A equipped with a morphism $m: A \times A \rightarrow A$ satisfying the following conditions:

$$\begin{aligned} m(x, x) &= x \\ m(x, y) &= m(y, x) \\ m(m(x, y), m(z, w)) &= m(m(x, z), m(y, w)) \end{aligned}$$

A homomorphism of midpoint algebras is a morphism that preserves the midpoint operation. A midpoint algebra is *cancellative* if it satisfies

$$m(x, z) = m(y, z) \implies x = y.$$

Definition 2.2 A midpoint algebra A is *iterative* if, for every object X and pair of morphisms $h: X \rightarrow A$, $t: X \rightarrow X$ (*head* and *tail*), there is a unique morphism $M: X \rightarrow A$ making $M(x) = m(h(x), M(t(x)))$ – in other words, the following diagram commutes.

$$\begin{array}{ccc} A \times X & \xrightarrow{A \times M} & A \times A \\ \langle h, t \rangle \uparrow & & \downarrow m \\ X & \xrightarrow{M} & A \end{array}$$

To illustrate the “iterative” condition, a particular case would be where $X = \mathbb{N}$ and t is the successor function. Then h is a sequence $(h_i)_{i \in \mathbb{N}}$. In an affine setting, we would then have that $M(n)$ is the infinitary convex combination

$$M(n) = \sum_{i=n}^{\infty} \frac{1}{2^{i-n+1}} h_i.$$

We now specialize to the category **Loc** of locales. The closed Euclidean interval $\mathbb{I} = [-1, 1]$ is a cancellative midpoint algebra with $m(x, y) = \frac{x+y}{2}$. We shall think of the discrete two-point space 2 as $\{-, +\}$, so that Cantor space 2^ω is the space of infinite sequences (or *streams*) of signs.

We also write 2^* for the set of finite sequences of signs, ε for the empty sequence, \sqsubseteq for the prefix order and $|s|$ for the length of s . We use juxtaposition to denote concatenation.

Definition 2.3 Suppose A is an iterative midpoint algebra equipped with two points a_- and a_+ . We define $M_{a_- a_+}: 2^\omega \rightarrow A$ as the unique map such that

$$M_{a_- a_+}(\pm s) = m(a_\pm, M_{a_- a_+} s).$$

Referring to Definition 2.2, X is 2^ω and h, t are such that $\langle h, t \rangle(\pm s) = (a_\pm, s)$ (so t is the tail map in the usual sense).

Definition 2.4 An *interval object* I is a free iterative midpoint algebra over 2 . That is to say:

1. I is equipped with two points x_- and x_+ (its *endpoints*).

2. I is an iterative midpoint algebra.
3. For every iterative midpoint algebra A with points a_- and a_+ there is a unique midpoint homomorphism $N: I \rightarrow A$ that takes x_- and x_+ to a_- and a_+ respectively.

We shall prove (Theorem 8.11) that \mathbb{I} , with endpoints -1 and 1 , is a cancellative interval object.

Note that our definition of “interval object” is slightly different from that of [1]. On the one hand, we don’t assume that is cancellative; but on the other we expect it to be initial amongst *all* the doubly pointed iterative midpoint algebras, not just the cancellative ones. Since our \mathbb{I} is cancellative, we have proved a slightly stronger result than that conjectured in [1].

3 Preliminary remarks on Cantor space

We take Cantor space 2^ω to be the localic exponential of the discrete locales 2 (two points $+$ and $-$) and \mathbb{N} (natural numbers $1, 2, 3, \dots$).¹ This certainly exists, since discrete locales are locally compact. Its points can be described as the functions from \mathbb{N} to 2 , and so its frame $\Omega 2^\omega$ can be presented by generators and relations as follows, using the notation $\mathbf{Fr}\langle \text{generators} \mid \text{relations} \rangle$ from [3]:

$$\begin{aligned} \Omega 2^\omega \cong \mathbf{Fr}\langle (n, \sigma) \in \mathbb{N} \times 2 \mid (n, +) \wedge (n, -) \leq 0, \\ 1 \leq (n, +) \vee (n, -) \rangle. \end{aligned}$$

(Here, abstractly, we write 1 and 0 for the top and bottom of a frame. Where the locale has a definite name X , we shall also often write them as X and \emptyset .) Every generator (n, \pm) has a Boolean complement (n, \mp) , so the locale is Stone. Its frame is the ideal completion of the free Boolean algebra on countably many generators $(n, +)$.

A little calculation shows that

$$\begin{aligned} \Omega 2^\omega \cong \mathbf{Fr}\langle \uparrow s \ (s \in 2^*) \mid \uparrow t \leq \uparrow s \ (\text{if } s \sqsubseteq t), \\ 1 \leq \uparrow \varepsilon, \\ \uparrow s \wedge \uparrow t \leq 0 \ (\text{if } s, t \text{ incomparable}), \\ \uparrow s \leq \uparrow(s-) \vee \uparrow(s+) \rangle. \end{aligned} \tag{1}$$

The isomorphisms are given by

$$\begin{aligned} \uparrow s &\mapsto \bigwedge_{i=1}^{|s|} (i, s_i) \\ (n, \sigma) &\mapsto \bigvee_{|s|=n-1} \uparrow(s\sigma). \end{aligned}$$

The generators $\uparrow s$ form a base. $\uparrow s$ comprises those streams of which s is a prefix.

Later we shall need a preframe base, in other words opens of which every other open is a directed join of finite meets, and for this we shall introduce subbasics $\downarrow s$ and $\uparrow s$ that involve the lexicographic ordering. Let us first introduce some notation.

Definition 3.1 If $s, t \in 2^*$ then we write $s < t$ if there is some u such that $u- \sqsubseteq s$ and $u+ \sqsubseteq t$. We say that s and t *differ* if either $s < t$ or $t < s$: this is equivalent to their being incomparable under \sqsubseteq . The relation $<$ extends to an open $\bigvee_{u \in 2^*} (\uparrow(u-) \times \uparrow(u+))$ of $2^\omega \times 2^\omega$.

We write $s \leq t$ if either $s < t$ or $s \sqsubseteq t$. This is just the lexicographic order in which $-$ is less than $+$.

We write $s \leq t$ if either $s < t$ or $t \sqsubseteq s$: in other words, t precedes s in the dual lexicographic order with $+$ less than $-$.

Both \leq and \leq can be extended in the obvious way to the case where s or t may be infinite.

¹ There is a technical reason here for preferring to start at 1 , in that the first term in an infinite binary expansion is for 2^{-1} . For finite sequences too, the indexes will start at 1 .

If $s \in 2^*$, then we define a *right bristle* of s to be a finite sequence $t+$ such that $t- \sqsubseteq s$, in other words a u that is minimal (under \sqsubseteq) subject to $s < u$. Dually, a *left bristle* of s is a u minimal subject to $u < s$.

Definition 3.2 If $s \in 2^*$ then we define the open $\uparrow s$ of 2^ω as the finite join $\uparrow s \vee \bigvee \{\uparrow t \mid t \text{ a right bristle of } s\}$. It comprises those u in 2^ω such that $s \leq u$. Dually, we define $\downarrow s = \uparrow s \vee \bigvee \{\uparrow t \mid t \text{ a left bristle of } s\}$, comprising those u such that $u \leq s$.

Lemma 3.3 \uparrow and \downarrow have the following properties.

1. $\uparrow s = \downarrow s \wedge \uparrow s$.
2. If $s \leq t$ in 2^* then $\downarrow t \leq \downarrow s$; if $s \leq t$ then $\uparrow s \leq \uparrow t$.
3. $\downarrow(s-) = \downarrow s$; $\uparrow(s+) = \uparrow s$.
4. $\downarrow s \vee \uparrow s = 2^\omega$.
5. If $t < s$ then $\downarrow s \wedge \uparrow t = \emptyset$.
6. $\uparrow s \leq \downarrow(s+) \vee \uparrow(s-)$.

Proof. (1) Suppose t and u are right and left bristles of s . They both differ from s , but cannot differ at the same place. Thus they must differ from each other, and we deduce that $\uparrow t \wedge \uparrow u = \emptyset$.

(2) We prove only the first assertion, since the second is dual. If $s \sqsubseteq t$ then $\uparrow t \leq \uparrow s$, and any right bristle of t either is a right bristle of s or has s as a prefix. If $s < t$ then there is a unique $t' \sqsubseteq t$ such that t' is a right bristle of s . Then $\uparrow t \leq \uparrow t'$. Also, any right bristle of t either is a right bristle of $t' -$ and hence of $s -$ or has t' as a prefix.

(3) From $s \leq s-$ we deduce $\downarrow(s-) \leq \downarrow s$. For the reverse, any right bristle of s is also a right bristle of $s-$. Also, $\uparrow s = \uparrow(s-) \vee \uparrow(s+)$, and $s+$ is a right bristle of $s-$. The other assertion is dual.

(4) We use induction on the length of s ; the base case $s = \varepsilon$ is obvious. Using part (3), and also the fact that s and $s-$ have the same left bristles, we find that

$$\downarrow(s-) \vee \uparrow(s-) = \downarrow s \vee \uparrow(s-) \vee \bigvee \{\uparrow t \mid t \text{ a left bristle of } s\} = \downarrow s \vee \uparrow s = 2^\omega.$$

By symmetry the same works for $s+$.

(5) Let u be the greatest common prefix of s and t : then $u- \sqsubseteq t$ and $u+ \sqsubseteq s$. It suffices to consider the case for $\downarrow(u-) \wedge \uparrow(u+)$, which is the meet of

$$\left(\uparrow(u-) \vee \bigvee \{\uparrow u' \mid u' \text{ a left bristle of } u\} \right)$$

and

$$\left(\uparrow(u+) \vee \bigvee \{\uparrow u'' \mid u'' \text{ a right bristle of } u\} \right).$$

If u' and u'' are bristles as described, then $u- < u+$, $u- < u''$, $u' < u+$ and $u' < u < u''$ and it follows that all the meets got by redistributing the expression are 0.

(6) Because $\uparrow s = \uparrow(s-) \vee \uparrow(s+)$. □

Lemma 3.4

$$\begin{aligned}
\Omega 2^\omega \cong \mathbf{Fr}(\downarrow s, \uparrow s \ (s \in 2^*) \mid \downarrow t \leq \uparrow s \ (s \leq t), \\
\downarrow s \leq \uparrow(s-), \\
\downarrow s \leq \uparrow t \ (s \leq t), \\
\downarrow s \leq \uparrow(s+), \\
1 \leq \downarrow \varepsilon, \\
1 \leq \uparrow \varepsilon, \\
1 \leq \downarrow s \vee \uparrow s, \\
\downarrow s \wedge \uparrow t \leq 0 \ (t < s), \\
\downarrow s \wedge \uparrow s \leq \uparrow(s+) \vee \uparrow(s-))
\end{aligned}$$

Proof. The homomorphism from the frame as presented here to that in (1) takes $\downarrow s$ and $\uparrow s$ to the opens as in Definition 3.2, and then Lemma 3.3 shows that the relations are respected. In the other direction we map $\uparrow s$ to $\downarrow s \wedge \uparrow s$ and it is easily shown that all the relations are respected. In particular, for respect of the relation $\uparrow s = \uparrow(s-) \vee \uparrow(s+)$ we must have

$$\downarrow s \wedge \uparrow s = (\downarrow(s-) \wedge \uparrow(s-)) \vee (\downarrow(s+) \wedge \uparrow(s+)). \quad (2)$$

For \geq we use that $\uparrow(s\pm) \leq \downarrow s$ and similarly for \downarrow . For \leq we apply distributivity to the right hand side. For three of the conjuncts we use $\downarrow s \leq \downarrow(s-)$ and $\downarrow s \leq \uparrow(s+)$; for the other we use the final relation $\downarrow s \wedge \uparrow s \leq \uparrow(s+) \vee \uparrow(s-)$.

Now Lemma 3.3 (1) shows that one composite takes $\uparrow s$ to $\downarrow s \wedge \uparrow s$ and then back to $\uparrow s$, so is the identity. To show the other composite is the identity we need

$$\downarrow s = (\downarrow s \wedge \uparrow s) \vee \bigvee_{t \in \text{RB}(s)} (\downarrow t \wedge \uparrow t),$$

where $\text{RB}(s)$ is the set of right bristles for s , and similarly for $\downarrow s$. The \geq direction is easy, since if t is a right bristle of s then $s \leq t$ and so $\downarrow t \leq \downarrow s$.

For \leq we use induction. The base case, $s = \varepsilon$, is clear. For the induction step,

$$\begin{aligned}
\downarrow(s\pm) &= \downarrow(s\pm) \wedge \downarrow s = \downarrow(s\pm) \wedge \left((\downarrow s \wedge \uparrow s) \vee \bigvee_{t \in \text{RB}(s)} (\downarrow t \wedge \uparrow t) \right) \\
&\leq (\downarrow(s\pm) \wedge \downarrow s \wedge \uparrow s) \vee \bigvee_{t \in \text{RB}(s\pm)} (\downarrow t \wedge \uparrow t)
\end{aligned}$$

since every right bristle of s is also a right bristle of $s\pm$. Now using equation (2) we have

$$\downarrow(s-) \wedge \downarrow s \wedge \uparrow s \leq (\downarrow(s-) \wedge \uparrow(s-)) \vee \bigvee_{t \in \text{RB}(s-)} (\downarrow t \wedge \uparrow t)$$

since $s+$ is a right bristle of $s-$, and

$$\downarrow(s+) \wedge \downarrow s \wedge \uparrow s \leq \downarrow(s+) \wedge \uparrow(s+)$$

since $s- < s+$ giving $\downarrow(s+) \wedge \uparrow(s-) \leq 0$. □

4 \mathbb{I} is iterative

The main task in this section is to prove that \mathbb{I} , as a midpoint algebra, is iterative. We shall use the fact that it can be described as a localic completion [4], and then to construct the map M as in Definition 2.2 we shall use approximations in the ball domain ([5], following the ideas of [6]).

Recall that for the localic completion of a generalized metric space X we use the elements $(x, \varepsilon) \in X \times Q_+$, where Q_+ is the set of positive rationals, as “formal open balls” $B_\varepsilon(x)$ (centre x , radius ε). We write $\text{ball}(X)$ for $X \times Q_+$ and equip it with a transitive, interpolative “refinement” order

$$(x, \delta) \subset (y, \varepsilon) \text{ if } X(y, x) + \delta < \varepsilon.$$

Then the *ball domain* $\text{Ball}(X)$ is defined to be the continuous dequo $\text{Idl}(\text{ball}(X), \supset)$ (see [7]). Note that the small balls, the refined ones, are high in the order. We therefore think of the points of the ball domain as rounded *filters* of formal balls.

There is a *radius map* $r: \text{Ball}(X) \rightarrow \overleftarrow{[0, \infty)}$, with $r(F)$ the inf of the radii of the formal balls in F . ($\overleftarrow{[0, \infty)}$ is the locale whose points are the upper reals in that interval, namely inhabited, rounded, up-closed sets of positive rationals.)

The localic completion \overline{X} embeds in $\text{Ball}(X)$; its points are the *Cauchy filters*, those containing formal balls of arbitrarily small radius, i.e. the points of $\text{Ball}(X)$ with radius 0.

Proposition 4.1 \mathbb{I} is the localic completion of the metric space D , the set of dyadic rationals (those with denominator a power of 2) in the range $(-1, 1)$, with the usual metric.

Proof. In [4] it is shown that \mathbb{R} is the localic completion of \mathbb{Q} . We have to deal with two differences. First, \mathbb{Q} is replaced by the dyadics, which is essentially straightforward because the dyadics are dense in the rationals. Note that although the centre q of a formal ball must now be dyadic, the radius δ can be any positive rational. Second, we restrict to the closed interval. For a Dedekind section $S = (L, U)$ that is equivalent to imposing the geometric axioms $1 \notin L$ and $-1 \notin U$.

The proof in [4] sets up a geometric bijection between Dedekind sections S and Cauchy filters F of \mathbb{Q} as follows. The Dedekind section $S(F)$ has for its upper and lower sections the two sets $\{q \pm \delta \mid (q, \delta) \in F\}$. The Cauchy filter $F(S)$ comprises those (q, δ) for which $q - \delta < S < q + \delta$, where of course we now have to restrict to $q \in D$.

The main difficulty is in showing that $S = S(F(S))$. Suppose $q < S$. We can find dyadic q' with $q < q' < S$, and we know that $q' < 1$ (otherwise $1 < S$). Let $r = \frac{1}{4}(q' + 3)$, which is dyadic, with $r < 1$, and let $\delta = r - q$. Then

$$r + \delta = 2r - q = \frac{1}{2}(q' + 3) - q > \frac{1}{2}(3 - q') > 1.$$

It follows that if $r \in D$ then (r, δ) provides a ball to show $q < S(F(S))$. On the other hand, if $r \leq -1$ (so also $q < -1$) then instead we can use $(0, -q)$. The argument for $S < q$ is symmetric.

We also show that $F(S(F)) \subseteq F$. Suppose $(r, \varepsilon), (r', \varepsilon') \in F$, so that $r - \varepsilon < S(F) < r' + \varepsilon'$. This interval is the ball (q, δ) where $q = \frac{1}{2}(r - \varepsilon + r' + \varepsilon')$ and $\delta = \frac{1}{2}(r' + \varepsilon' - r + \varepsilon)$. We must show that if $q \in D$ then $(q, \delta) \in F$, but this is so because there is some common refinement in F of (r, ε) and (r', ε') , and it also refines (q, δ) . \square

We extend the midpoint map $m: \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{I}$ by allowing the second argument to be taken from a ball domain. In $\text{Ball}(D)$ we have a point with centre 0 and radius 1. As a filter, it comprises those formal balls $(q, \delta) \supset (0, 1)$. Let B be the up closure in $\text{Ball}(D)$ of this point, and write \perp for the point since it is bottom in B . Note that if $F \in \text{Ball}(D)$, then $\perp \sqsubseteq F$ iff $(0, 1 + \varepsilon) \in F$ for all $\varepsilon \in Q_+$.

Lemma 4.2 The embedding $i: \mathbb{I} \hookrightarrow \text{Ball}(D)$ factors via B .

Proof. Suppose x is a point of \mathbb{I} , i.e. a Cauchy filter for D . If $\varepsilon > 0$ then we can find $r \in D$ with $(r, \varepsilon/2) \in x$. Then $(0, 1 + \varepsilon) \supset (r, \varepsilon/2)$ and so is in x . \square

We define $m': \mathbb{I} \times B \rightarrow B$ as follows. Let x and F be in $\text{Ball}(D)$ with x Cauchy and $F \supseteq \perp$. We define

$$m'(x, F) = \sup\{(m(q, r), m(\delta, \varepsilon)) \mid (q, \delta) \in x, (r, \varepsilon) \in F\}$$

(i.e. the set of all formal balls refined by one in the set on the right). The fact that it is a filter follows from the fact that if $(q, \delta) \supset (q', \delta')$ in x and $(r, \varepsilon) \supset (r', \varepsilon')$ in F then

$$(m(q, r), m(\delta, \varepsilon)) \supset (m(q', r'), m(\delta', \varepsilon')).$$

This is because

$$\left| \frac{q+r}{2} - \frac{q'+r'}{2} \right| + \frac{\delta'+\varepsilon'}{2} \leq \frac{1}{2} (|q-q'| + \delta' + |r-r'| + \varepsilon') \leq \frac{\delta+\varepsilon}{2}.$$

To see that it is bigger than \perp , suppose $\varepsilon > 0$. Since x is Cauchy, there is some $(q, \delta) \in x$ with $\delta < \varepsilon/2$; also, $(0, 1 + \varepsilon/2) \in F$ and so $(q/2, \frac{1}{2} + \frac{\delta}{2} + \frac{\varepsilon}{4}) \in m'(x, F)$. From $|q| \leq 1$ it follows that $(0, 1 + \varepsilon) \supset (q/2, \frac{1}{2} + \frac{\delta}{2} + \frac{\varepsilon}{4})$ and so $(0, 1 + \varepsilon) \in m'(x, F)$.

Lemma 4.3 1. $m = m' \circ (\mathbb{I} \times i)$.

2. $r \circ m'(x, F) = r(F)/2$.

Proof. Both are clear. □

Theorem 4.4 *The midpoint algebra \mathbb{I} is iterative.*

Proof. Let X be a locale and $h: X \rightarrow \mathbb{I}$, $t: X \rightarrow X$ be two maps. We require a unique morphism $M: X \rightarrow \mathbb{I}$ making the following diagram commute.

$$\begin{array}{ccc} \mathbb{I} \times X & \xrightarrow{\mathbb{I} \times M} & \mathbb{I} \times \mathbb{I} \\ \langle h, t \rangle \uparrow & & \downarrow m \\ X & \xrightarrow{M} & \mathbb{I} \end{array}$$

$\mathbf{Loc}(X, B)$ is a dcpo with bottom. We define a Scott continuous endofunction T on it by $T(f) = m' \circ (\mathbb{I} \times f) \circ \langle h, t \rangle$:

$$\begin{array}{ccc} \mathbb{I} \times X & \xrightarrow{\mathbb{I} \times f} & \mathbb{I} \times B \\ \langle h, t \rangle \uparrow & & \downarrow m' \\ X & \xrightarrow{T(f)} & B \end{array}$$

Let M be its least fixpoint, $\bigsqcup_n M_n$ where M_0 is constant \perp and $M_{n+1} = T(M_n)$. Then $r \circ M = \frac{1}{2}(r \circ M)$, from which it follows that $r \circ M = 0$ and M factors via \mathbb{I} thus giving us existence of the required M .

For uniqueness, suppose M' is another such. Then $M \sqsubseteq M'$ since M is least fixpoint, but the specialization order on \mathbb{I} is discrete. □

We can calculate the inverse image function for M in the above theorem more explicitly, at least for the subbasic opens (p, α) . First of all,

$$M_0^*(p, \alpha) = \begin{cases} \top & \text{if } (p, \alpha) \supset (0, 1) \\ \perp & \text{otherwise} \end{cases}$$

(and note that the condition is decidable). Next,

$$T(f)^*(p, \alpha) = \bigvee \{ h^*(q, \delta) \wedge t^* f^*(r, \varepsilon) \mid (p, \alpha) \supset (\frac{q+r}{2}, \frac{\delta+\varepsilon}{2}) \}.$$

In particular examples this will allow us to calculate $M^*(p, \alpha) = \bigvee_n M_n^*(p, \alpha)$.

5 The map $c: 2^\omega \rightarrow \mathbb{I}$

Thinking of the signs in a point of Cantor space 2^ω as standing for 1 or -1 , such an infinite sequence can be viewed as a binary expansion, thus giving a map to \mathbb{I} .

Definition 5.1 We define a map $c: 2^\omega \rightarrow \mathbb{I}$ as $M_{-1,+1}$. It is characterized by the equation

$$c(\pm s) = \frac{1}{2}(\pm 1 + c(s)).$$

From the characterizing equation we see that, in more traditional form,

$$c((s_i)_{i=1}^\infty) = \sum_{i=1}^{\infty} \frac{s_i}{2^i}. \quad (3)$$

Definition 5.2 2^* is the discrete space of finite sequences of signs. We define $c': 2^* \rightarrow \mathbb{I}$ by the formula (3), adapted for finite sequences. Thus we think of the finite sequence s as the infinite sequence $s0^\omega$ (which is not in 2^ω , of course).

c' is an isomorphism between 2^* and D .

If s is finite of length n and t is infinite, then we see from the definition that $c(st) = c'(s) + 2^{-n}c(t)$.

We now show how to calculate the inverse image function c^* , using Theorem 4.4 and the remarks following it. Our map $h: 2^\omega \rightarrow \mathbb{I}$ is $h(\pm s) = \pm 1$. It has

$$h^*(p, \alpha) = \left\{ \begin{array}{ll} \uparrow+ & \text{if } p - \alpha < 1 < p + \alpha \\ \emptyset & \text{otherwise} \end{array} \right\} \vee \left\{ \begin{array}{ll} \uparrow- & \text{if } p - \alpha < -1 < p + \alpha \\ \emptyset & \text{otherwise} \end{array} \right\}.$$

Hence, for $f: 2^\omega \rightarrow \mathbb{I}$,

$$\begin{aligned} T(f)^*(p, \alpha) &= \bigvee \{ (\uparrow+) \wedge t^* f^*(r, \varepsilon) \mid (p, \alpha) \supset (\frac{q+r}{2}, \frac{\delta+\varepsilon}{2}), q - \delta < 1 < q + \delta \} \\ &\quad \vee \bigvee \{ (\uparrow-) \wedge t^* f^*(r, \varepsilon) \mid (p, \alpha) \supset (\frac{q+r}{2}, \frac{\delta+\varepsilon}{2}), q - \delta < -1 < q + \delta \}. \end{aligned}$$

(Keep in mind that p, q and r are all expected to be in D .)

Lemma 5.3 *In $\Omega\mathbb{R}$ we have*

$$\begin{aligned} \bigvee \{ (r, \varepsilon) \mid (p, \alpha) \supset (\frac{q+r}{2}, \frac{\delta+\varepsilon}{2}), q - \delta < -1 < q + \delta \} &= (2p + 1, 2\alpha), \\ \bigvee \{ (r, \varepsilon) \mid (p, \alpha) \supset (\frac{q+r}{2}, \frac{\delta+\varepsilon}{2}), q - \delta < 1 < q + \delta \} &= (2p - 1, 2\alpha). \end{aligned}$$

Proof. We prove only the first, since the second follows by symmetry. We have

$$\begin{aligned} (r, \varepsilon) \subset (2p + 1, 2\alpha) &\Leftrightarrow \left(\frac{-1+r}{2}, \frac{\varepsilon}{2} \right) \subset (p, \alpha) \\ &\Leftrightarrow \exists \beta > 0 \left(\frac{-1+r}{2}, \beta + \frac{\varepsilon}{2} \right) \subset (p, \alpha) \end{aligned}$$

Then the final condition is equivalent to the existence of q, δ , with $-1 < q < -1 + \delta$ and

$$\left(\frac{q+r}{2}, \frac{\delta+\varepsilon}{2} \right) \subset (p, \alpha).$$

(Note that the second condition is equivalent to this with $q = -1, \delta = 0$, and the β enables us to fatten -1 out to a positive ball.) Each $(\frac{q+r}{2}, \frac{\delta+\varepsilon}{2})$ can be refined to a $(\frac{-1+r}{2}, \beta + \frac{\varepsilon}{2})$ and vice versa. \square

In $\Omega\mathbb{I}$ the same equations hold, but we must be careful how we interpret the right-hand side. Consider the first equation. If $p < 0$ then the centre $2p + 1$ of the ball on the right is still in D . The ball is approximated from below by refinements with the same centre, and it follows in the proof that we can restrict the balls appearing in the left-hand side to those with centre in D .

Now suppose $0 \leq p$, so that $1 \leq 2p + 1$. Then the ball $(2p + 1, 2\alpha)$ is equivalent in $\Omega\mathbb{I}$ to the interval $(2p + 1 - 2\alpha, 1]$. This interval may take various forms depending on the value of $2p + 1 - 2\alpha$ – which, in particular, may be less than -1 or greater than 1 . However, in every case it is approximated by balls refining $(2p + 1, 2\alpha)$ and with centre in D . Therefore the equations in the lemma will still hold in $\Omega\mathbb{I}$.

Taking care with interpretations in $\Omega\mathbb{I}$ in that way, it follows that

$$T(f)^*(p, \alpha) = (\uparrow+) \wedge t^* f^*(2p - 1, 2\alpha) \vee (\uparrow-) \wedge t^* f^*(2p + 1, 2\alpha).$$

Although our proof of iterativity used the metric space structure and the opens balls, we shall be actually be more interested in the behaviour of the half-open intervals. In the rest of the section we shall calculate formulae for opens such as $c^*((c'(s), 1])$. First, rewriting $p - \alpha$ as p , we see, for all p , that

$$T(f)^*(p, 1] = (\uparrow+) \wedge t^* f^*(2p - 1, 1] \vee (\uparrow-) \wedge t^* f^*(2p + 1, 1]. \quad (4)$$

Now if $p = c'(s) \in D$, we have

$$(2p - 1, 1] = \begin{cases} (c'(s'), 1] & \text{if } s = +s' \\ (-1, 1] = \bigvee_k^\uparrow (c'(-k), 1] & \text{if } s = \varepsilon \\ \mathbb{I} & \text{if } s = -s' \end{cases}$$

$$(2p + 1, 1] = \begin{cases} \emptyset & \text{if } s = +s' \text{ or } s = \varepsilon \\ (c'(s'), 1] & \text{if } s = -s' \end{cases}$$

Using this we can calculate $c^*(c'(s), 1]$ by induction on the length of s , the base case requiring knowledge of $c^*(-1, 1]$.

Lemma 5.4 1. $c^*(c'(-k), 1] = \bigvee_{i=0}^{k-1} \uparrow(-i+) \vee ((\uparrow-k) \wedge (t^*)^k c^*((0, 1]))$.

2. $c^*(-1, 1] = \bigvee_{i=0}^\infty \uparrow(-i+)$.

3. $c^*(0, 1] = \bigvee_{i=0}^\infty \uparrow(+ - i+)$.

Proof. (1) is by induction on k . The base case, $k = 0$, is clear.

$$\begin{aligned} c^*(c'(-k+1), 1] &= (\uparrow+) \wedge t^* c^*(\mathbb{I}) \vee (\uparrow-) \wedge t^* c^*(c'(-k), 1] \quad (\text{equation (4)}) \\ &= (\uparrow+) \vee (\uparrow-) \wedge t^* \left(\bigvee_{i=0}^{k-1} \uparrow(-i+) \vee ((\uparrow-k) \wedge (t^*)^k c^*((0, 1])) \right) \\ &= \bigvee_{i=0}^k \uparrow(-i+) \vee ((\uparrow-k+1) \wedge (t^*)^{k+1} c^*((0, 1])) \end{aligned}$$

(2) Using part (1), and applying equation (4) to $c^*(0, 1]$, we see that

$$\begin{aligned} c^*(c'(-k), 1] &= \bigvee_{i=0}^{k-1} \uparrow(-i+) \vee ((\uparrow-k) \wedge (t^*)^k ((\uparrow+) \wedge t^* c^*((-1, 1]))) \\ &= \bigvee_{i=0}^{k-1} \uparrow(-i+) \vee ((\uparrow-k) \wedge (t^*)^{k+1} c^*((-1, 1])) \\ &\leq \bigvee_{i=0}^k \uparrow(-i+) \leq c^*(c'(-k+1), 1]. \end{aligned}$$

It follows that

$$c^*(-1, 1] = c^* \left(\bigvee_k^\uparrow (c'(-k), 1] \right) = \bigvee_k^\uparrow \bigvee_{i=0}^k \uparrow(-i+) = \bigvee_{i=0}^\infty \uparrow(-i+).$$

(3) Apply equation (4) with $p = 0$, and then use part (2). \square

In other words, $c(u) > -1$ iff u has a $+$ somewhere; and $c(u) > 0$ iff u starts with a $+$ and has at least one more.

Proposition 5.5 *If $s \in 2^*$ then*

1. $c^*((c'(s), 1]) = \bigvee_k^\uparrow \uparrow(s+^{-k}+)$, and
2. $c^*([-1, c'(s))) = \bigvee_k^\uparrow \uparrow(s-+^k-)$.

Proof. We prove only the first assertion, since the second is dual. We use induction on the length of s .

For $s = \varepsilon$, we use Lemma 5.4 (3) together with $\uparrow(+^{-k}+) = \bigvee_{i=0}^k \uparrow(+^{-i}+)$. Now we can use the previous calculations and see

$$\begin{aligned} c^*((c'(s), 1]) &= (\uparrow+) \wedge t^* c^*((c'(s), 1]) \\ &= (\uparrow+) \wedge t^* \left(\bigvee_k^\uparrow \uparrow(s+^{-k}+) \right) \\ &= \bigvee_k^\uparrow \uparrow(s+^{-k}+) \\ c^*((c'(-s), 1]) &= (\uparrow+) \wedge t^* 2^\omega \vee (\uparrow-) \wedge t^* c^*((c'(s), 1]) \\ &= (\uparrow+) \vee (\uparrow-) \wedge t^* \left(\bigvee_k^\uparrow \uparrow(s+^{-k}+) \right) \\ &= \bigvee_k^\uparrow \uparrow(-s+^{-k}+). \end{aligned}$$

\square

6 c is a proper surjection

Our main aim in this section is to show that c is a surjection, and in proving this we will be helped by the fact that it is proper in the sense of Vermeulen [8]: the right adjoint $\forall_c: \Omega 2^\omega \rightarrow \Omega \mathbb{I}$ of c^* preserves directed joins and satisfies a Frobenius condition $\forall_c(a \vee c^*b) = \forall_c a \vee b$. This is equivalent to saying that c is “fibrewise compact” as bundle over \mathbb{I} , by which we mean that it is compact as an internal locale in the topos of sheaves over \mathbb{I} .

As might be predicted from classical results², any locale map $f: X \rightarrow Y$ between compact regular locales is proper; in fact it is enough to assume X is compact and Y regular. The regularity of Y implies that the diagonal in $Y \times Y$ is closed: in other words, there is a map ne from $Y \times Y$ to the Sierpinski locale \mathbb{S} inducing the diagonal map $\Delta: Y \rightarrow Y \times Y$ as the corresponding closed sublocale (fibre over the bottom point of \mathbb{S}). By calculating pullbacks over Y

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xlongequal{\quad} & Y \\ \langle X, f \rangle \downarrow & & \Delta \downarrow & & \downarrow \langle \perp, Y \rangle \\ X \times Y & \xrightarrow{f \times Y} & Y \times Y & \xrightarrow{\langle \text{ne}, \pi_2 \rangle} & \mathbb{S} \times Y \end{array}$$

² I am grateful to an anonymous referee for making this prediction. He or she also suggested it as a way to shorten this section, although I have not found a way to make that work beyond simplifying the proof of Theorem 6.7.

we see that X is a closed sublocale of $X \times Y$ over Y . But, over Y , $X \times Y$ is compact, so X too is compact, in other words f is proper.

The geometric techniques of this section rely on analysing $\Omega 2^\omega$ and $\Omega \mathbb{I}$ not as frames (finite meets and arbitrary joins, all preserved by frame homomorphisms) but as *preframes*: a preframe has finite meets and directed joins, and they are what are preserved by preframe homomorphisms, and binary meets distribute over directed joins. Thus for a proper surjection c , \forall_c is a preframe homomorphism. We first use the preframe coverage theorem of [9] to present $\Omega 2^\omega$ as a preframe, and define \forall_c by its action on a preframe base, and then we show that this function is right adjoint to c^* and has the Frobenius condition.

Any open of 2^ω is a directed join of finite joins of basic opens $\uparrow s = \uparrow s \wedge \uparrow s$, hence a directed join of finite meets of finite joins of opens of the form $\uparrow s$ and $\uparrow s$. But since \leq and \leqslant are total orders, by Lemma 3.3 we get a preframe base from opens of the form $\uparrow s$, $\uparrow s$ or $\uparrow s \vee \uparrow t$. Our strategy now is to calculate \forall_c for these and to rely on preservation of finite limits and directed joins to get the rest.

Definition 6.1 The distributive lattice S_\uparrow is defined as $2^* \cup \{\perp\}$, with 2^* ordered by the reverse of \leq and \perp an adjoined bottom. Since it is totally ordered it has binary meets and joins, and also top ε and bottom \perp .

Similarly we define $S_\downarrow = 2^* \cup \{\perp\}$, with 2^* ordered by \leqslant .

We write S for $S_\uparrow \times S_\downarrow$.

Lemma 6.2

$$\begin{aligned} \Omega 2^\omega &\cong \mathbf{Fr}\langle S \text{ (qua } \vee\text{-semilattice)} \mid \\ &\quad (s, t) \leq (s, t-) \text{ (} t \in 2^* \text{)}, \\ &\quad (s, t) \leq (s+, t) \text{ (} s \in 2^* \text{)}, \\ &\quad 1 \leq (s, \varepsilon) \text{ (} s \in S_\uparrow \text{)}, \\ &\quad 1 \leq (\varepsilon, s) \text{ (} s \in S_\downarrow \text{)}, \\ &\quad 1 \leq (s, t) \text{ (} s, t \in 2^*, t \leqslant s \text{ or } t \leq s \text{)}, \\ &\quad (u, s) \wedge (t, v) \leq (u, v) \text{ (if } t < s \text{ in } 2^* \text{ and } (u, v) \leq (t, s) \text{)}, \\ &\quad (u, s) \wedge (s, v) \leq (s-, s+) \text{ (if } s \in 2^* \text{ and } (u, v) \leq (s-, s+) \text{)} \end{aligned}$$

and

$$\Omega 2^\omega \cong \mathbf{PreFr}\langle S \text{ (qua poset)} \mid \dots \text{ same relations as above } \dots \rangle$$

Proof. To map from the presentation of Lemma 3.4 to this one we map $\uparrow s$ and $\uparrow s$ to (\perp, s) and (s, \perp) . This respects all the relations and so gives a frame homomorphism. For the inverse we map (\perp, \perp) to 0; (\perp, s) and (s, \perp) to $\uparrow s$ and $\uparrow s$; and (s, t) to $\uparrow s \vee \uparrow t$. Again this respects the relations and so gives a frame homomorphism. As can be tested on generators, the two composites are both identities.

The final part is now an application of the preframe coverage theorem [9], once it is checked that the relations are all join-stable. This is mostly straightforward, but we have cheated slightly in the last two relations. In join-stabilizing the relation $\uparrow s \wedge \uparrow t \leq 0$ ($t < s$) from Lemma 3.4, we get $(u, s \vee v) \wedge (t \vee u, v) \leq (u, v)$ for all u, v . However, if $t \leq u$ or $s \leq v$ then one of the two conjuncts is (u, v) and the relation holds automatically in the preframe presented. Hence it suffices to consider only the case where $u \leq t$ and $v \leq s$. The last relation in Lemma 3.4 is similar. \square

Our strategy now is to calculate \forall_c for the opens (s, t) and to rely on preservation of finite meets and directed joins to get the rest. Using Definition 6.5 we define a preframe homomorphism that we subsequently show to be \forall_c . Let us explain roughly how the definition arises. (We don't need a rigorous definition yet, since the definition is checked in Theorem 6.7.) First consider $\forall_c(\perp, s)$, the biggest open $U \in \Omega \mathbb{I}$ such that $c^*U \leq \uparrow s$. If $c(t) < c(u)$ then $t < u$ (it is much more complicated for \leq), and it follows that if $c(s-\omega) < c(u)$ then u is in $\uparrow s$. Hence $(c(s-\omega), 1) \leq \forall_c(\perp, s)$. If s contains a $+$ then $\forall_c(\perp, s)$ cannot be any bigger, for it would then contain $c(s-\omega)$ itself. By looking at the last $+$ in s we can

replace $+^{-\omega}$ by $-^{+\omega}$ and find a u in $c^*(\forall_c(\perp, s))$ but not in $\uparrow s$. Hence $\forall_c(\perp, s) = (c(s^{-\omega}), 1]$. If s has no $+$ then the argument is slightly different. $\uparrow s = \uparrow \varepsilon = 2^\omega$, so we know $\forall_c(\perp, s) = \mathbb{I}$. Similarly, $\forall_c(s, \perp)$ is either $[-1, c(s^{+\omega})]$ or \mathbb{I} .

There remains $\forall_c(s, t)$. If $c(s^{+\omega}) < c(t^{-\omega})$ then this turns out to be $[-1, c(s^{+\omega})] \vee (c(t^{-\omega}), 1]$ as one might expect, while if $c(s^{+\omega}) > c(t^{-\omega})$ it is \mathbb{I} . However we have to take some care where there is equality, since we then find that $\uparrow s \vee \uparrow t$ is 2^ω and so $\forall_c(s, t)$ must be \mathbb{I} – this is an instance where \forall_c does not preserve finite joins.

Definition 6.3 If $s, t \in 2^*$ we write $s \check{\ll} t$ if (i) $t < s$, or (ii) $t \sqsubseteq s$, or (iii) $s \sqsubseteq t$, or (iv) s and t are of the forms $u-+^k$ and $u+^{-l}$ respectively.

Lemma 6.4 1. $s \check{\ll} t$ iff $\uparrow s \vee \uparrow t = 2^\omega$.

2. If $s \check{\ll} t$ then $c(t^{-\omega}) \leq c(s^{+\omega})$.

3. $\check{\ll}$ is up-closed in S .

Proof. (1) \Rightarrow : In cases (i) and (ii) of the definition we have $t \leq s$, so $2^\omega = \uparrow s \vee \uparrow t \leq \uparrow s \vee \uparrow t$; similarly in case (iii). In case (iv), we have $\uparrow t = \uparrow(u+^{-l}) = \uparrow(u+)$ and similarly $\uparrow s = \uparrow(u-)$. Now

$$\begin{aligned} 2^\omega &\leq (\uparrow(u-) \vee \uparrow(u-)) \wedge (\uparrow(u+) \vee \uparrow(u+)) \\ &\leq \uparrow(u-) \vee \uparrow(u+) \vee (\uparrow(u-) \wedge \uparrow(u+)) \\ &= \uparrow(u-) \vee \uparrow(u+) \text{ because } u- < u+. \end{aligned}$$

\Leftarrow : $\check{\ll}$ is decidable. Its negation is that, first, $s < t$, so that for some u we have $u- \sqsubseteq s$ and $u+ \sqsubseteq t$, and in addition that either $u-+^k \sqsubseteq s$ or $u+^{-k} \sqsubseteq t$ for some k . Suppose the former. Then $s < u-+^\omega < t$, so $u-+^\omega$ is in neither $\uparrow s$ nor $\uparrow t$.

(2) In case (i): if $u- \sqsubseteq t$, $u+ \sqsubseteq s$, then $c(t^{-\omega}) < c'(u) < c(s^{+\omega})$. In case (ii) (and (iii) is dual), we have $t^{-k} < s+$ for some k , and can use (i). In case (iv), $c(t^{-\omega}) = c(u+^{-\omega}) = c'(u) = c(u-+^\omega) = c(s^{+\omega})$.

(3) Suppose $s \check{\ll} t$. We show that if $t' \leq t$ then $s \check{\ll} t'$. By symmetry it also follows that if $s \leq s'$ then $s' \check{\ll} t$, and the result will follow. We examine the cases of $s \check{\ll} t$. First, if $t \leq s$ then $t' \leq s$.

Second, suppose $s \sqsubseteq t$. If $t' \sqsubseteq t$ then s and t' are comparable under \sqsubseteq . Otherwise $t' < t$ and so $t' \leq s$.

Finally, suppose $s = u-+^k$, $t = u+^{-l}$.

If $t' \sqsubseteq t$ then either $t' \sqsubseteq u \sqsubseteq s$ or $u+ \sqsubseteq t' \sqsubseteq t$ and either way we get $s \check{\ll} t'$.

There remains the case $t' < t$. We have either $t' < u$, so $t' < s$, or $u- \sqsubseteq t'$. In this latter case consider whether t' has any further $-$ after $u-$. If it does then $t' \leq s$; if not then s and t' are comparable under \sqsubseteq . \square

Definition 6.5 We define a lattice homomorphism $\theta_\uparrow: S_\uparrow \rightarrow \Omega\mathbb{I}$ by

$$\theta_\uparrow(t) = \begin{cases} \mathbb{I} & \text{if } t \in 2^* \text{ and } t \text{ contains no } + \\ (c(t^{-\omega}), 1] & \text{if } t \in 2^* \text{ and } t \text{ contains at least one } + \\ \emptyset & \text{if } t = \perp \end{cases}$$

Similarly we define $\theta_\downarrow: S_\downarrow \rightarrow \Omega\mathbb{I}$ with $\theta_\downarrow(s) = [-1, c(s^{+\omega})]$ when s contains a $-$.

The monotone function $\theta: S_\uparrow \times S_\downarrow \rightarrow \Omega\mathbb{I}$ is defined by

$$\theta(s, t) = \begin{cases} \mathbb{I} & \text{if } s, t \in 2^* \text{ and } s \check{\ll} t \\ \theta_\downarrow(s) \vee \theta_\uparrow(t) & \text{otherwise} \end{cases}$$

Note that if t contains no $+$ or s contains no $-$ then $s \check{\ll} t$.

That θ_\uparrow and θ_\downarrow are lattice homomorphisms is simply to say that they are monotone and preserve top and bottom. The monotonicity of θ then follows from that and from Lemma 6.4 (3).

Lemma 6.6 We can define a preframe homomorphism $\forall_c: \Omega 2^\omega \rightarrow \Omega\mathbb{I}$ by $\forall_c(s, t) = \theta(s, t)$.

Proof. One should check that the relations in Lemma 6.2 are respected. Much of this is routine. We consider the last two in more detail.

For the last but one, suppose $t < s$ and $(u, v) \leq (t, s)$. First,

$$(\theta_1(u) \vee \theta_{\uparrow}(s)) \wedge (\theta_1(t) \vee \theta_{\uparrow}(v)) \leq \theta_1(u) \vee \theta_{\uparrow}(v) \vee (\theta_1(t) \wedge \theta_{\uparrow}(s)) = \theta_1(u) \vee \theta_{\uparrow}(v).$$

This is because, given $t < s$, t and s must contain $-$ and $+$ respectively, so

$$\theta_1(t) \wedge \theta_{\uparrow}(s) = [-1, c(t+\omega)) \wedge (c(s-\omega), 1] = \emptyset$$

because $c(t+\omega) \leq c(s-\omega)$.

We still need to examine the cases where θ takes the value \mathbb{I} . Suppose $u \not\leq s$. (The case $t \not\leq v$ is by symmetry.) We must show $\theta(t, v) \leq \theta(u, v)$. This is obvious if $t \leq u$, which is certainly the case if $s \leq u$ or $s \leq u$ (using $t < s$). It remains to check the case where $u = w-+^k$, $s = w+^-l$, combined with $u \leq t$. It is impossible to have $u < t < s$, so $t \sqsubseteq u$. It follows from $t < s$ that $t = w-+^{k'}$, with $k' \leq k$ (and we might as well assume $k' < k$), and so $\theta_1(t) = \theta_1(u)$. It remains to show that if $t \not\leq v$ then $u \not\leq v$, and this is straightforward from the various cases.

The final relation, $(u, s) \wedge (s, v) \leq (s-, s+)$, is clear since $s- \not\leq s+$. \square

Theorem 6.7 $c: 2^\omega \rightarrow \mathbb{I}$ is a proper surjection, with \forall_c right adjoint to c^* .

Proof. We show two conditions.

First, $c^* \circ \forall_c \leq \text{Id}$. For $s \not\leq t$, Lemma 6.4 tells us that $(s, t) = 2^\omega$. For the other case it remains to show that $c^*(\theta_{\uparrow}(t)) \leq \uparrow t$ (and similarly for \downarrow). If t has no $+$ then $\uparrow t = 2^\omega$, and otherwise by Proposition 5.5 we have

$$\begin{aligned} c^*(\theta_{\uparrow}(t)) &= c^*((c(t-\omega), 1]) = \bigvee_k^{\uparrow} c^*((c'(t-k), 1]) \\ &= \bigvee_{kl}^{\uparrow} \uparrow(t-k+^-l+) \leq \uparrow t. \end{aligned}$$

Second, $\forall_c \circ c^* = \text{Id}$. It suffices to check this for opens of the form $[-1, c'(s))$, $(c'(t), 1]$ and $[-1, c'(s)) \vee (c'(t), 1]$, since they form a preframe base of \mathbb{I} . We have

$$\begin{aligned} \forall_c \circ c^* ([-1, c'(s)) \vee (c'(t), 1]) &= \bigvee_{kl}^{\uparrow} \forall_c(s-+^k-, t+^-l+) \\ &\geq \bigvee_{kl}^{\uparrow} ([-1, c(s-+^k-\omega)) \vee (c(t+^-l+\omega), 1]) \\ &= \bigvee_k^{\uparrow} [-1, c'(s-+^k)) \vee \bigvee_l^{\uparrow} (c'(t+^-l), 1] \\ &= [-1, c'(s)) \vee (c'(t), 1]. \end{aligned}$$

We have equality provided we have no $s-+^k- \not\leq t+^-l+$ (and also, by a similar calculation, for the opens $[-1, c'(s))$ and $(c'(t), 1]$). If $c'(t) < c'(s)$ then $[-1, c'(s)) \vee (c'(t), 1] = \mathbb{I}$, so it remains to prove that if $c'(s) \leq c'(t)$ then we have no $s-+^k- \not\leq t+^-l+$. That is to say, for all k, l we have $s-+^k- < t+^-l+$ (so for some u we have $u- \sqsubseteq s-+^k-$ and $u+ \sqsubseteq t+^-l+$), and for some m we have either $u-+^m- \sqsubseteq s-+^k-$ or $u+^-m+ \sqsubseteq t+^-l+$. (See Lemma 6.4.) From $c'(s) \leq c'(t)$ we get three cases. If $s < t$ then u is a common prefix of s and t and in fact we have m with $u-+^m- \sqsubseteq s-$. If $s \sqsubseteq t$ then from $c'(s) \leq c'(t)$ we cannot have $s- \sqsubseteq t$, so we can take $u = s$ and either $s = t$ or $s+ \sqsubseteq t$. Either way, $u+ \sqsubseteq t+$. Then we can take $m = k$. The argument for $t \sqsubseteq s$ is similar.

The two conditions together show that \forall_c is right adjoint to c^* , and the equality in the second shows that c^* is one-to-one, i.e. that c is a localic surjection. We have already remarked that *all* maps between compact regular locales are proper. By construction here \forall_c preserves directed joins, and the Frobenius condition could be checked independently. \square

7 \mathbb{I} as coequalizer of maps to Cantor space

In Section 8 we need a map from 2^ω to factor via c , and to prove this it is useful to display c as a coequalizer. In fact we already know that, as a proper surjection, c is the coequalizer of its kernel pair; in this section we prove a simpler coequalizer property. We observe that $0_- = +^{-\omega}$ and $0_+ = -^{+\omega}$ in 2^ω are both mapped by c to 0. This is the starting point for describing c as a coequalizer of two maps from 2^* .

Definition 7.1 We define two maps $u_\pm: 2^* \rightarrow 2^\omega$ by $u_\pm(s) = s0_\pm$.

Although this might appear to be a definition of functions between two sets, its geometricity implies that it defines locale maps to 2^ω from the discrete locale 2^* . The inverse image functions $u_\pm^*: \Omega(2^\omega) \rightarrow \mathcal{P}(2^*)$ are easily calculated.

Since $c(0_-) = c(0_+)$, it is clear that $c \circ u_- = c \circ u_+$. We shall show that c is in fact the coequalizer of u_- and u_+ .

For the moment, let us write C for this coequalizer. We shall describe its frame ΩC as a subframe of $\Omega 2^\omega$ – it is the equalizer of the frame homomorphisms u_\pm^* . From the Stone space structure of 2^ω we see that $\Omega 2^\omega$ can be described as the frame of subsets U of 2^* , up-closed under the prefix order, and such that if $s+, s- \in U$ then $s \in U$. If $t \in 2^*$ then $\uparrow t$ is the principal upset of t , so for s in 2^ω we have $s \vDash \uparrow t$ (by which we mean that the point s is in the open $\uparrow t$) iff $t \sqsubseteq s$.

Proposition 7.2 ΩC is the frame of those subsets $U \in \Omega 2^\omega$ satisfying the condition that for all finite sign sequences s ,

$$(\exists m)s+-^m \in U \iff (\exists n)s-+^n \in U.$$

Proof. We have

$$u_-^*(U) = \{s \mid s-+^\omega \vDash U\} = \{s \mid (\exists t \in U)t \sqsubseteq s-+^\omega\} = \{s \mid (\exists m)s-+^m \in U\}$$

and similarly for $u_+^*(U)$. The result is now immediate from the fact that $U \in \Omega 2^\omega$ is in ΩC iff $u_-^*(U) = u_+^*(U)$. \square

Having identified ΩC concretely, our task is now to show that it is isomorphic to $\Omega \mathbb{I}$. The next definition defines two decidable relations on 2^* that capture (see Proposition 7.4) properties of c' and c . For example, $s \triangleleft t$ holds if, for any stream extending t , we have $c'(s) < c(t)$.

Definition 7.3 If $s, t \in 2^*$ then we write $s \triangleleft t$ if either $s < t$, or there is some k with $s+-^k+ \sqsubseteq t$.

We write $t \triangleleft s$ if either $t < s$, or there is some k with $s-+^k- \sqsubseteq t$.

In other words, for $s \triangleleft t$ either at the first difference s has $-$ and t has $+$, or $s \sqsubseteq t$ and t has $+$ immediately after s , and at least one more $+$ somewhere further along.

Proposition 7.4 Let $s, t \in 2^*$. Then $\uparrow t \leq c^*((c'(s), 1])$ iff $s \triangleleft t$, and $\uparrow t \leq c^*([-1, c'(s)])$ iff $t \triangleleft s$.

Proof. We prove only the first part, since the second follows by interchanging $+$ and $-$. Using Proposition 5.5 and the compactness of $\uparrow t$, we see that $\uparrow t \leq c^*((c'(s), 1])$ iff $\uparrow t \leq \uparrow(s+-^k+)$ for some k , and this clearly holds iff $s \triangleleft t$. \square

Proposition 7.5 ΩC is the image of c^* .

Proof. Since c composes equally with u_+ and u_- , we know that it factors via C and so ΩC contains the image of c^* .

We show that if $U \subseteq 2^*$ satisfies the condition of Proposition 7.2, then it is a join of images under c^* of dyadic open intervals in \mathbb{I} .

Let $u \in U$. If $u = \varepsilon$ is empty then by up-closure $U = 2^* = c^*(\mathbb{I})$.

Next, suppose $u = +^n$ for some $n \geq 1$. By the condition on U , we find $s = +^{n-1}-+^m \in U$ for some m . Then $s \triangleleft u$; we show that $\{t \in 2^* \mid s \triangleleft t\} \subseteq U$. Suppose $s \triangleleft t$. If s and t disagree, it must be at the $-$ in s , so $u \sqsubseteq t$ and $t \in U$. On the other hand, if $s \sqsubseteq t$ then again $t \in U$. The case where $u = -^n$ is similar.

Now suppose u contains both $+$ and $-$. By symmetry it suffices to consider the case where U ends in $-$: so we can write $u = u' +^{-n}$ with $n \geq 1$. By the condition on U we can find $s_0 = u' - +^m \in U$ and also $s_1 = u' +^{-n-1} +^{-k} \in U$. We have $s_0 \triangleleft u \triangleleft s_1$. Suppose $s_0 \triangleleft t \triangleleft s_1$. If $s_0 \sqsubseteq t$ or $s_1 \sqsubseteq t$ then $t \in U$. Thus we assume $s_0 < t < s_1$. It cannot disagree with u' , since in its disagreement it would have to have both $+$ and $-$. Hence $u' \sqsubseteq t$. The disagreement with s_0 must therefore be at the $-$ immediately after u' . It follows that t agrees with s_1 at the first $+$ after u' , so the disagreement must be at the second. Hence $u = u' +^{-n} \sqsubseteq t$ and $t \in U$. \square

After Theorem 6.7 we can now conclude –

Theorem 7.6 $c: 2^\omega \rightarrow \mathbb{I}$ is the coequalizer of $u_\pm: 2^* \rightrightarrows 2^\omega$.

8 \mathbb{I} is an interval object in Loc

Let A be an iterative midpoint algebra equipped with points a_\pm . We shall also write

$$\begin{aligned} a_0 &= m(a_-, a_+) \\ a_{\pm/2} &= m(a_0, a_\pm). \end{aligned}$$

If $N: \mathbb{I} \rightarrow A$ can be found as in Definition 2.4, then $Nc: 2^\omega \rightarrow A$ is the map $M = M_{a_- a_+}$ (Definition 2.3), for

$$Nc(\pm s) = Nm(\pm 1, c(s)) = m(N(\pm 1), Nc(s)) = m(a_\pm, Nc(s)).$$

We can define M regardless of N , so it therefore remains to prove (i) that M factors via \mathbb{I} , as $M = Nc$ for some $N: \mathbb{I} \rightarrow A$, and (ii) that N is then a midpoint algebra homomorphism.

Lemma 8.1 $M(\pm^\omega) = a_\pm$.

Proof. By the defining property of M , $M(\pm^\omega)$ is a point x_\pm such that $m(a_\pm, x_\pm) = x_\pm$. But by considering the maps $a_\pm: 1 \rightarrow A$ and $!: 1 \rightarrow 1$ as h and t in Definition 2.2, we see that there is a unique map $x_\pm: 1 \rightarrow A$ such that $m(a_\pm, x_\pm) = x_\pm$. Since a_\pm satisfies this condition, we deduce $x_\pm = a_\pm$. \square

Proposition 8.2 M composes equally with $u_\pm: 2^* \rightarrow 2^\omega$.

Proof. From Lemma 8.1 we have $M(+^{-\omega}) = m(a_+, a_-) = m(a_-, a_+) = M(-+^\omega)$, i.e. $M(u_+(\varepsilon)) = M(u_-(\varepsilon))$. It now follows by induction on the length of s that $M(u_+(s)) = M(u_-(s))$ for all $s \in 2^*$. \square

It follows that M factors via \mathbb{I} , as Nc for some unique $N: \mathbb{I} \rightarrow A$.

It remains to be shown that N preserves midpoints, i.e. that $m(N \times N) = Nm$. Since c is a proper surjection, so too is $c \times c$ and so it suffices to show that $m(Nc \times Nc) = m(M \times M) = Nm(c \times c): 2^\omega \times 2^\omega \rightarrow A$.

Definition 8.3 $\text{half}: 2^\omega \rightarrow 2^\omega$ is defined by

$$\text{half}(\pm s) = \pm \mp s.$$

Lemma 8.4

$$M_{a_- a_+} \text{half} s = m(a_0, M_{a_- a_+} s).$$

Proof.

$$\begin{aligned} M_{a_- a_+} \text{half}(\pm s) &= M_{a_- a_+}(\pm \mp s) \\ &= m(a_\pm, m(a_\mp, M_{a_- a_+} s)) \\ &= m(m(a_\pm, a_\mp), m(a_\pm, M_{a_- a_+} s)) \\ &= m(a_0, M_{a_- a_+}(\pm s)). \end{aligned}$$

\square

Lemma 8.5 *As maps from \mathbb{I} to A , we have*

1. $Nm\langle\pm 1, \mathbb{I}\rangle = m\langle a_{\pm}, A\rangle N$,
2. $Nm\langle 0, \mathbb{I}\rangle = m\langle a_0, A\rangle N$.

Proof. Since c is a surjection, it suffices to show equality when these are composed with c .
(1)

$$\begin{aligned} Nm\langle\pm 1, \mathbb{I}\rangle c(s) &= Nm(\pm 1, c(s)) = Nc(\pm s) = M(\pm s) \\ &= m(a_{\pm}, M(s)) = m(a_{\pm}, Nc(s)) = m\langle a_{\pm}, A\rangle Nc(s). \end{aligned}$$

(2)

$$\begin{aligned} Nm\langle 0, \mathbb{I}\rangle c(s) &= Nm(0, c(s)) \\ &= Nc(\text{half}(s)) \text{ (by Lemma 8.4, using } c = M_{-1,+1}\text{)} \\ &= M\text{half}(s) \\ &= m(a_0, M(s)) \text{ (by Lemma 8.4 again, using } M = M_{a_- a_+}\text{)} \\ &= m\langle a_0, A\rangle Nc(s). \end{aligned}$$

□

To analyse preservation of midpoints we shall need to define a version of the midpoint function that works entirely on sign sequences. However, it will be convenient to use sequences that may include 0: so we shall use 3^ω where we take $3 = \{+, -, 0\}$. There is an obvious inclusion $i: 2^\omega \rightarrow 3^\omega$.

We define $M_0: 3^\omega \rightarrow A$, similar to M , but with the additional condition that $M_0(0s) = m(a_0, M(s))$. In other words, in Definition 2.2 the head map $h: 3^\omega \rightarrow \mathbb{I}$ takes $0s$ to a_0 . Then clearly $M = M_0i$.

We can do the same with c instead of M , obtaining a unique map $c_0: 3^\omega \rightarrow \mathbb{I}$ such that $c_0(\pm s) = m(\pm 1, c_0(s))$, $c_0(0s) = m(0, c_0(s))$. Then $c = c_0i$.

Lemma 8.6 $M_0 = Nc_0$.

Proof.

$$Nc_0(\pm s) = Nm(\pm 1, c_0s) = m(a_{\pm}, Nc_0s) \text{ (Lemma 8.5 (1))}$$

$$Nc_0(0s) = Nm(0, c_0s) = m(a_0, Nc_0s) \text{ (Lemma 8.5 (2))}$$

It follows that Nc_0 has the characterizing property of M_0 . □

Definition 8.7 The *sequence midpoint map* $m_s: 2^\omega \times 2^\omega \rightarrow 3^\omega$ is defined by

$$\begin{aligned} m_s(\pm s_1, \pm s_2) &= \pm m_s(s_1, s_2) \\ m_s(\pm s_1, \mp s_2) &= 0m_s(s_1, s_2). \end{aligned}$$

Lemma 8.8 $m(M \times M) = M_0m_s$.

Proof. They are both the unique map $f: 2^\omega \times 2^\omega \rightarrow A$ such that $f(\pm s_1, \pm s_2) = m(a_{\pm}, f(s_1, s_2))$ and $f(\pm s_1, \mp s_2) = m(a_0, f(s_1, s_2))$. For $m(M \times M)$,

$$\begin{aligned} m(M \times M)(\pm s_1, \pm s_2) &= m(m(a_{\pm}, M(s_1)), m(a_{\pm}, M(s_2))) \\ &= m(a_{\pm}, m(M \times M)(s_1, s_2)), \\ m(M \times M)(\pm s_1, \mp s_2) &= m(m(a_{\pm}, M(s_1)), m(a_{\mp}, M(s_2))) \\ &= m(m(a_{\pm}, a_{\mp}), m(M(s_1), M(s_2))) \\ &= m(a_0, m(M \times M)(s_1, s_2)). \end{aligned}$$

For M_0m_s ,

$$\begin{aligned} M_0m_s(\pm s_1, \pm s_2) &= M_0(\pm m_s(s_1, s_2)) \\ &= m(a_{\pm}, M_0m_s(s_1, s_2)), \\ M_0m_s(\pm s_1, \mp s_2) &= M_0(0m_s(s_1, s_2)) \\ &= m(a_0, M_0m_s(s_1, s_2)). \end{aligned}$$

□

Corollary 8.9 $m(c \times c) = c_0m_s$.

Proof. Replace A by \mathbb{I} .

□

Proposition 8.10 $N: \mathbb{I} \rightarrow A$ preserves midpoints.

Proof.

$$\begin{aligned} m(N \times N)(c \times c) &= m(M \times M) = M_0m_s \text{ (Lemma 8.8)} \\ &= Nc_0m_s \text{ (Lemma 8.6)} \\ &= Nm(c \times c) \text{ (Corollary 8.9)}. \end{aligned}$$

We now use the fact that $c \times c$ is a surjection, following from the fact that c is a proper surjection. □

Putting together all the results of this section, we obtain –

Theorem 8.11 *In the category **Loc** of locales, the structure $(\mathbb{I}, -1, 1)$ is a cancellative interval object.*

9 Conclusions

The main result was about \mathbb{I} as interval object, but along the way we also showed that the map $c: 2^\omega \rightarrow \mathbb{I}$, evaluating infinite binary expansions, is a proper localic surjection that is easily expressed as a coequalizer. This result has some interest in itself. In classical point-set topology, c is a surjection because for every Dedekind section there is an infinite expansion; however, this uses choice. Essentially, the surjectivity of c , in other words the monicity of c^* , is a conservativity result, and this is known as a constructive substitute for using choice to find the existence of points. See, for example, the constructive Hahn-Banach Theorem in [10]. However, our result is unusual in using a *proper* surjection rather than an *open* one.

The proof of proper surjectivity used the preframe coverage theorem in a standard way. However, it was more intricate than I expected. I had a hope to use the metric space theory again for 2^ω , but was put off by the fact that to get 2^ω as a completion of 2^* requires each finite sequence s to be identified with an infinite sequence, either $s^{-\omega}$ or $s^{+\omega}$: this breaks symmetry. I conjecture there's a way forward using partial metrics, so that 2^* is metrized with $d(s, s) = 2^{1-|s|}$. However, we do not at present have a theory of localic completion of partial metrics. It would be easier with $c_0: 3^\omega \rightarrow \mathbb{I}$, but then that would presumably make Section 7 harder. In any case, the result with 2^ω is stronger.

The main result, on \mathbb{I} as an interval object, free on two points, suggests generalization to simplices, free on their vertices. I conjecture that similar techniques to prove this, using infinite sequences, could be developed using barycentric subdivision.

Acknowledgements Thanks are due to Martín Escardó for suggesting this investigation and for numerous helpful discussions; also to an anonymous referee for various useful comments.

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