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EXACT MINIMUM CODEGREE THRESHOLD FOR K_4^--FACTORS

JIE HAN, ALLAN LO, ANDREW TREGLOWN AND YI ZHAO

ABSTRACT. Given hypergraphs F and H, an F-factor in H is a set of vertex-disjoint copies of F which cover all the vertices in H. Let K_4^- denote the 3-uniform hypergraph with 4 vertices and 3 edges. We show that for sufficiently large $n \in 4\mathbb{N}$, every 3-uniform hypergraph H on n vertices with minimum codegree at least $n/2 - 1$ contains a K_4^--factor. Our bound on the minimum codegree here is best-possible. It resolves a conjecture of Lo and Markström [15] for large hypergraphs, who earlier proved an asymptotically exact version of this result. Our proof makes use of the absorbing method as well as a result of Keevash and Mycroft [11] concerning almost perfect matchings in hypergraphs.

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Keywords: Tiling, Hypergraphs, Absorbing method.

1. INTRODUCTION

Given two hypergraphs H and F, an F-*tiling* in H is a collection of vertex-disjoint copies of F in H. An F-tiling is called *perfect* if it covers all the vertices of H. Perfect F-tilings are also referred to as F-*factors* or *perfect F-packings*. Note that perfect F-tilings are generalisations of perfect matchings (which correspond to the case when F is a single edge).

Tiling problems have been widely studied for graphs. The seminal Hajnal–Szemerédi theorem [8] states that every graph G on $n \in r\mathbb{N}$ vertices and with $\delta(G) \geq (1 - 1/r)n$ contains a K_r-factor. More generally, given any graph F, Kühn and Osthus [13] determined, up to an additive constant, the minimum degree threshold that forces a F-factor in a graph. See [14] for a survey including many of the results on graph tiling.

Given a k-uniform hypergraph (k-graph for short) H with a d-element vertex set S (where $0 \leq d \leq k - 1$) we define the *degree* $\deg_H(S)$ of S in H to be the number of edges containing S. The *minimum d-degree* $\delta_d(H)$ of H is the minimum of $\deg_H(S)$ over all d-element sets of vertices in H. We also refer to $\delta_1(H)$ as the *minimum vertex degree* of H and $\delta_{k-1}(H)$ the *minimum codegree* of H.

In recent years there have been significant efforts on finding minimum d-degree conditions that force a perfect matching in a k-graph. For example, for every $k \geq 3$, Rödl, Ruciński and Szemerédi [20] determined the minimum codegree threshold that forces a sufficiently large k-graph H to contain a perfect matching. Other than the matching problems, only a few hypergraph tiling problems have been studied – most of them recently.

Given a k-graph F of order f and an integer n divisible by f, we define the threshold $\delta_d(n, F)$ as the smallest integer t such that every n-vertex k-graph H with $\delta_d(H) \geq t$ contains an F-factor. We simply write $\delta(n, F)$ for $\delta_{k-1}(n, F)$. One of the earliest results on hypergraph tiling was given by Kühn and Osthus [12], who proved that $\delta(n, C_4^3) = n/4 + o(n)$, where C_4^3 is the (unique) 3-graph with four vertices and two edges. Later Czygrinow, DeBiasio, and Nagle [3] showed that for sufficiently large $n \in \mathbb{N}$, $\delta(n, C_4^3) = n/4 + 1$ if $n \in 8\mathbb{N}$ and $\delta(n, C_4^3) = n/4$

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otherwise. Let K_4^3 denote the complete 3-graph on four vertices. Lo and Markström [16] showed that $\delta(n, K_4^3) = 3n/4 + o(n)$. Independently and simultaneously Keevash and Mycroft [11] proved that for sufficiently large $n \in 4\mathbb{N}$, $\delta(n, K_4^3) = 3n/4 - 2$ if $n \in 8\mathbb{N}$ and $\delta(n, K_4^3) = 3n/4 - 1$ otherwise. More recently Han and Zhao [10] and independently Czygrinow [2] determined $\delta_1(n, C_4^3)$ exactly for sufficiently large n. Mycroft [17] determined $\delta(n, F)$ asymptotically for many k-partite k-graphs F (including complete k-partite k-graphs and loose cycles). One of these thresholds, $\delta(n, C_6^3)$, where C_6^3 denotes the 3-uniform loose cycle on 6 vertices, was determined exactly by Gao and Han [7] very recently. Han, Zang, and Zhao [9] determined $\delta_1(n, K)$ asymptotically for all complete 3-partite 3-graphs K. See the surveys [18, 21] for detailed overviews of matching and tiling problems in hypergraphs.

Let K_4^- denote the 3-graph with 4 vertices and 3 edges. Lo and Markström [15] proved that $n/2 - 1 \leq \delta(n, K_4^-) \leq n/2 + o(n)$. Let us recall the construction that gives the lower bound. Given two disjoint vertex sets A, B, define $\mathcal{B}[A, B]$ to be the 3-graph on $A \cup B$ whose edge set consists of all those triples that contain an odd number of vertices from A. Suppose that $n \equiv 0 \mod 4$. If $n \not\equiv 0 \mod 3$ and $|A| = |B| = n/2$, we have that $\delta_2(\mathcal{B}[A, B]) = n/2 - 2$ but $\mathcal{B}[A, B]$ does not contain a K_4^--factor. If $n \equiv 0 \mod 3$ and $|A| = n/2 + 1$, $|B| = n/2 - 1$, again we have that $\delta_2(\mathcal{B}[A, B]) = n/2 - 2$ but $\mathcal{B}[A, B]$ does not contain a K_4^--factor. (See Proposition 1 in [15] for details.)

In this paper we determine $\delta(n, K_4^-)$ exactly for sufficiently large n, thereby resolving a conjecture of Lo and Markström [15] for large hypergraphs.

Theorem 1.1. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-graph on $n \geq n_0$ vertices where n is divisible by 4. If $\delta_2(H) \geq n/2 - 1$ then H contains a K_4^--factor. Thus, $\delta(n, K_4^-) = n/2 - 1$ for $n \in 4\mathbb{N}$ and $n \geq n_0$.*

The proof of Theorem 1.1 makes use of the *absorbing method* – a technique that was first used by Rödl, Ruciński and Szemerédi in [19] and has subsequently been applied to numerous embedding problems in extremal graph theory. We also apply a result of Keevash and Mycroft [11] concerning almost perfect matchings in hypergraphs. As is common with proofs in the area, our argument splits into two cases; the *extremal case* (i.e. when H is 'close' to the extremal example $\mathcal{B}[A, B]$, see Section 2.1 for a precise definition) and the *non-extremal case*.

The paper is organised as follows. In the next section we derive Theorem 1.1 from three main lemmas, preceded by an overview of the proof and a comparison with the proof in [15]. We give some useful tools in Section 3. We prove an almost perfect tiling lemma in Section 4 and an absorbing lemma in Section 5. The extremal case is tackled in Section 6.

2. Notation and proof of Theorem 1.1

2.1. **Notation.** Given a set X and $r \in \mathbb{N}$, we write $\binom{X}{r}$ for the set of all r-element subsets of X. For simplicity, given vertices $x_1, \ldots, x_t$ and a set of vertices S, we often write $x_1 \cdots x_t$ for $\{x_1, \ldots, x_t\}$ and $S \cup x_1$ for $S \cup \{x_1\}$.

Let H be a 3-graph. We write $V(H)$ for the vertex set and $E(H)$ for the edge set of H. Sometimes we will write $(V(H), E(H))$ to denote H. Define $e(H) := |E(H)|$. We denote the *complement of H* by $\overline{H}$. That is, $\overline{H} := (V(H), \binom{V(H)}{3} \setminus E(H))$. Given $x, y \in V(H)$, we write $N_H(xy)$ to denote the *neighborhood of xy*, that is, the family of those vertices in $V(H)$ which, together with x, y, form an edge in H. If $X \subseteq V(H)$ we write $N_H(xy, X) := N_H(xy) \cap X$, and $\deg_H(xy, X) := |N_H(xy, X)|$. For this and similar notation, we often omit the subscript if the underlying hypergraph is clear from the context.

Given $X \subseteq V(H)$, we write $H[X]$ for the *subhypergraph of H induced by X*, namely, $H[X] := (X, E(H) \cap \binom{X}{3})$. We write $e_H(X)$ or simply $e(X)$ for $e(H[X])$. In addition, we let $H \setminus X :=$

$H[V(H) \setminus X]$. If $H[X]$ contains a spanning copy of K then we say that X *spans a copy of K in H*. In particular, this does not necessarily mean that X *induces* a copy of K in H. When counting the number of copies of K in H, we only count the number of subsets of $V(H)$ that span copies of K in H. For example, we say that K_4^3 only contains one copy of K_4^- (instead of four copies).

Let $\gamma > 0$ and H and H' be two 3-graphs on the same vertex set V. We say that H *γ-contains H'* if $|E(H') \setminus E(H)| \le \gamma |V|^3$, that is, H misses at most $\gamma|V|^3$ edges from H'. Given $\gamma > 0$, we call a 3-graph $H = (V, E)$ on n vertices *γ-extremal* if there is a partition of $V = A \cup B$ such that $|A| = \lfloor n/2 \rfloor$, $|B| = \lceil n/2 \rceil$ and H γ-contains $\mathcal{B}[A, B]$.

For any $x \in V(H)$, we define the *link graph L_x* to be the graph with vertex set $V(H) \setminus \{x\}$ and where $yz \in E(L_x)$ if and only if $xyz \in E(H)$. Let G be a graph, $X \subseteq V(G)$ and $x \in V(G)$. We define $N_G(x), e_G(X), G[X], N_G(x, X), \deg_G(x, X)$ analogously to the 3-graph case. We write $\delta(G)$ for the *minimum degree of G* and $\Delta(G)$ for the *maximum degree of G*. Given disjoint $X, Y \subseteq V(G)$, we write $e_G(X, Y)$ for the number of edges in G with one endpoint in X and the other endpoint in Y.

Throughout the paper, we write the hierarchy $0 < \alpha \ll \beta \ll \gamma$ to mean that we can choose the constants α, β, γ from right to left. More precisely, there are increasing functions f and g such that, given γ, whenever we choose $\beta \le f(\gamma)$ and $\alpha \le g(\beta)$, all the calculations needed in our proof are valid. Hierarchies of other lengths are defined in the obvious way.

2.2. Overview of the proof of Theorem 1.1.
In the next subsection we will combine the three main lemmas of the paper to prove Theorem 1.1. Before this we give an overview of the proof. It is instructive to first describe the strategy used by Lo and Markström in [15] to prove the asymptotic version of Theorem 1.1.

Let $0 < \varepsilon \ll \gamma \ll \eta$ and n be sufficiently large. Suppose that H is a 3-graph on n vertices where $\delta_2(H) \ge (1/2 + \eta)n$. The proof in [15] splits into two main tasks.

- **Step 1 (Absorbing set):** Find an *absorbing* set $W \subseteq V(H)$ such that $|W| \le \gamma n$. W has the property that given *any* set $U \subseteq V(H) \setminus W$ where $U \in 4\mathbb{N}$ and $|U| \le \varepsilon n$, both $H[W]$ and $H[W \cup U]$ contain K_4^--factors.
- **Step 2 (Almost tiling):** Let $H' := H \setminus W$. Find a K_4^--tiling $\mathcal{K}$ in H' that covers all but at most εn vertices.

Note that after Steps 1 and 2 one immediately obtains a K_4^--factor in H. Indeed, let $U := V(H') \setminus V(\mathcal{K})$. Then $H[W \cup U]$ contains a K_4^--factor $\mathcal{K}'$ and so $\mathcal{K} \cup \mathcal{K}'$ is a K_4^--factor in H.

To show that H contains the desired absorbing set W, Lemma 1.1 in [15] implies that it suffices to show that H is *closed*. Roughly speaking, H is closed if, for any $x, y \in V(H)$, there are many small sets $S \subseteq V(H)$ such that both $H[S \cup x]$ and $H[S \cup y]$ contain K_4^--factors (see Section 5 for the formal definition). Using that $\delta_2(H) \ge (1/2 + \eta)n$, it is not too difficult to show that there is a partition of $V(H)$ into at most three parts such that each of these partition classes is closed. So a key task in [15] is to 'merge' these closed classes into a single closed class. For this, it suffices to show that are many 'bridges' between the partition classes (see Lemma 5.3): An (X, Y)-bridge is a triple (x, y, S) where $x \in X$, $y \in Y$ and $S \subseteq V(H)$ such that $H[S \cup x]$ and $H[S \cup y]$ contain K_4^--factors. This is precisely the strategy used in [15] to prove that H is closed, and thus contains an absorbing set W. A short argument then shows that, since $\delta_2(H') \ge (1/2 + \eta/2)n$, H' contains an almost perfect K_4^--tiling, as desired.

We now turn to our proof of Theorem 1.1. Let H be a sufficiently large 3-graph on n vertices where $\delta_2(H) \ge n/2 - 1$. If H is close to the extremal example $\mathcal{B}[A, B]$ then it is not clear whether one can find an absorbing set in H. Indeed, let $H^* := \mathcal{B}[A, B]$ where $|A| = |B| = n/2$. Suppose that $U \subseteq B$ where $|U| = 4$. Consider any $W \subseteq V(H^*) \setminus U$ such that $H^*[W]$ contains a K_4^--factor. Then it is easy to see that $|W \cap B| \equiv 0 \mod 3$. However, for any such set W, $H^*[W \cup U]$ does not contain a K_4^--factor as $|(W \cup U) \cap B| \equiv 1 \mod 3$.

Thus, in the case when H is close to the extremal example $\mathcal{B}[A, B]$ we do not use the absorbing method. Instead, in Section 6, we give a direct argument to show that H contains a K_4^--factor. In the case when H is non-extremal we follow Steps 1 and 2 as above. However, since we now only have that $\delta_2(H) \geq n/2 - 1$, the argument becomes significantly more involved.

To find an absorbing set when H is non-extremal we again show that H is closed. Suppose that there exists $x \in V(H)$ such that there are very few edges $abc \in E(H)$ so that $abcx$ spans a copy of K_4^- in H. In this case we give a direct argument to show that H contains an absorbing set (see Lemma 5.6). Otherwise, we show that our minimum codegree ensures that $V(H)$ can be partitioned into at most *four* sets such that each is closed in H (see Lemma 5.9). We again merge these sets into a single closed class by finding many bridges between the sets. For this, we use that if H is non-extremal then in any partition A, B of $V(H)$ with $|A|, |B| \geq n/5$, we have many edges that intersect A in precisely 1 vertex and many edges that intersect A at precisely 2 vertices (see Lemma 3.3). The process of proving that non-extremal 3-graphs H are closed is quite involved and forms the heart of the paper (most of Section 5 is devoted to this task).

In Section 4 we tackle Step 2 for non-extremal 3-graphs H. Our lower minimum codegree condition means that we cannot use the argument from [15] here. Instead, we translate the problem to one on almost perfect matchings in hypergraphs. We then (somewhat carefully) apply a result of Keevash and Mycroft [11] to obtain an almost perfect matching in some auxiliary hypergraph whose 4-edges correspond to copies of K_4^- in H'. Thus, we obtain an almost perfect K_4^--tiling in H'.

2.3. **Proof of Theorem 1.1.** As outlined in the previous subsection, the proof of Theorem 1.1 consists of three main parts: the extremal case; obtaining an absorbing set in the non-extremal case; and finding an almost perfect tiling in the non-extremal case.

Our first lemma deals with the last part. In fact, it implies that H has an almost perfect K_4^--tiling even if $\delta_2(H)$ is (slightly) less than $n/2$.

Lemma 2.1. *Let $1/n \ll \phi \ll \gamma \ll 1$. Let H be a 3-graph on n vertices with $\delta_2(H) \geq (1/2 - \gamma)n$. Then H contains a K_4^--tiling covering all but at most ϕn vertices.*

The next result yields the absorbing set in the non-extremal case.

Lemma 2.2. *Let $1/n \ll \phi \ll \varepsilon \ll \gamma \ll 1$. Let H be a 3-graph of order n with $\delta_2(H) \geq (1/2 - \gamma)n$. Suppose that H is not 3γ-extremal. Then there exists an absorbing set $W \subseteq V(H)$ of order at most εn so that for any $U \subseteq V(H) \setminus W$ with $|U| \leq \phi n$ and $|U| \in 4\mathbb{N}$, both $H[W]$ and $H[U \cup W]$ contain K_4^--factors.*

If H is extremal, then we will find a K_4^--factor using the following lemma.

Lemma 2.3. *There exist $\gamma > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-graph on $n \geq n_0$ vertices where n is divisible by 4. If $\delta_2(H) \geq n/2 - 1$ and H is γ-extremal, then H contains a K_4^--factor.*

Theorem 1.1 now follows easily from Lemmas 2.1–2.3.

Proof of Theorem 1.1. Let $1/n \ll \phi \ll \varepsilon \ll \gamma \ll 1$ with $n \in 4\mathbb{N}$. Let H be a 3-graph of order n with $\delta_2(H) \geq n/2 - 1$. If H is 3γ-extremal, then by Lemma 2.3 H contains a K_4^--factor.

Therefore, we may assume that H is not 3γ-extremal. By Lemma 2.2, there exists an absorbing set $W \subseteq V(H)$ of order at most εn so that for any $U \subseteq V(H) \setminus W$ with $|U| \leq \phi n$ and $|U| \in 4\mathbb{N}$, both $H[W]$ and $H[U \cup W]$ contain K_4^--factors. Let $H' := H \setminus W$. Note that $n' := |H'| \geq (1 - \varepsilon)n$ and $\delta_2(H') \geq n/2 - 1 - \varepsilon n \geq (1/2 - 2\varepsilon)n'$. By Lemma 2.1, H' contains a K_4^--tiling $\mathcal{M}_1$ covering all but at most $\phi n'$ vertices. Let $U := V(H') \setminus V(\mathcal{M}_1)$. Since $|U| \leq \phi n' \leq \phi n$, $H[U \cup W]$ contains a K_4^--factor $\mathcal{M}_2$. Then $\mathcal{M}_1 \cup \mathcal{M}_2$ is a K_4^--factor in H, as desired. $\square$

3. Useful results

We will need the following result, which follows immediately from a theorem of Baber and Talbot [1, Theorem 2.2] and the *supersaturation* phenomenon discovered by Erdős and Simonovits [6].

Proposition 3.1. *There exist a constant $c' > 0$ and an integer n' such that every 3-graph H of order $n \geq n'$ with $e(H) > 0.3\binom{n}{3}$ contains at least $c'n^4$ copies of K_4^-.*

Let H be a 3-graph of order n. For any 3-set $T \subseteq V(H)$, let $L_H(T)$, or simply $L(T)$, be the set of vertices v such that $H[T \cup v]$ contains a copy of K_4^-. If $T = xyz \in E(H)$, then $L(T) = (N(xy) \cap N(yz)) \cup (N(xy) \cap N(xz)) \cup (N(yz) \cap N(xz))$. The following proposition gives a lower bound on the size of such $L(T)$.

Proposition 3.2. [15, Proposition 2.1] *Let H be a 3-graph of order n. Then for every edge $e = xyz$ and any $U \subseteq V(H)$, $|L(e) \cap U| \geq (\deg(xy, U) + \deg(yz, U) + \deg(xz, U) - |U|)/2$.*

Let H be a 3-graph and let $V_1, V_2, V_3 \subseteq V(H)$. We say that an edge $v_1v_2v_3 \in E(H)$ is an $V_1V_2V_3$-edge if $v_i \in V_i$ for all $i \in [3]$. We denote by $e_H(V_1V_2V_3)$ the number of $V_1V_2V_3$-edges. The following simple result will be applied in the proof of the non-extremal case of Theorem 1.1. We remark that the property guaranteed by Lemma 3.3 is in fact the only property of non-extremalness that will be used in the proof of Lemma 2.2 (and thus in the entire proof of Theorem 1.1).

Lemma 3.3. *Let $0 < 1/n \ll \gamma < 1/100$. Suppose that H is a 3-graph of order n where $\delta_2(H) \geq (1/2 - \gamma)n$. Let X, Y be any bipartition of $V(H)$ where $|X|, |Y| \geq n/5$. If H is not 3γ-extremal, then there exist at least γ^2n^3 XXY-edges and at least γ^2n^3 XYY-edges.*

Proof. Suppose that H contains fewer than γ^2n^3 XXY-edges. We will show that H is 3γ-extremal. (The case when H contains fewer than γ^2n^3 XYY-edges is analogous.) We have $\sum_{x,x' \in X} |N(xx')| = \sum_{x,x' \in X} |N(xx', X)| + \sum_{x,x' \in X} |N(xx', Y)|$ and $\sum_{x,x' \in X} |N(xx', Y)| \leq \gamma^2n^3$. Since $\delta_2(H) \geq (1/2 - \gamma)n$, we have $\sum_{x,x' \in X} |N(xx')| \geq \binom{|X|}{2}(1/2 - \gamma)n$. So we get $3e(X) = \sum_{x,x' \in X} |N(xx', X)| \geq \binom{|X|}{2}(1/2 - \gamma)n - \gamma^2n^3$. Therefore, $e(X) \geq \frac{1}{3}\binom{|X|}{2}(1/2 - 2\gamma)n$ and in particular, $|X| \geq (1/2 - 2\gamma)n$.

Since $\sum_{x \in X, y \in Y} |N(xy)| = 2e(XXY) + 2e(XYY)$, we derive that $e(XYY) \geq \frac{1}{2}|X||Y|(1/2 - \gamma)n - \gamma^2n^3$ similarly. Since $|X|, |Y| \geq n/5$, we get $e(XYY) \geq \frac{1}{2}|X||Y|(1/2 - 2\gamma)n$ and in particular, $|Y| \geq (1/2 - 2\gamma)n$.

Therefore we have $(1/2 - 2\gamma)n \leq |X|, |Y| \leq (1/2 + 2\gamma)n$. This implies that

$$e(X) \geq \frac{1}{3}\binom{|X|}{2}(1/2 - 2\gamma)n \geq \frac{1}{3}\binom{|X|}{2}\frac{(1/2 - 2\gamma)}{(1/2 + 2\gamma)}|X| \geq (1 - 8\gamma)\binom{|X|}{3}.$$

A similar calculation shows $e(XYY) \geq (1 - 8\gamma)|X|\binom{|Y|}{2}$. This implies that $|E(\mathcal{B}[X, Y]) \setminus E(H)| \leq 8\gamma\binom{n}{3}$. After moving at most $2\gamma n$ vertices from X to Y or from Y to X, we obtain a bipartition X', Y' of $V(H)$ such that $|X'| = \lfloor n/2 \rfloor$, $|Y'| = \lceil n/2 \rceil$. Then $|E(\mathcal{B}[X', Y']) \setminus E(\mathcal{B}[X, Y])| \leq 2\gamma n\binom{n-1}{2}$. Consequently

$$|E(\mathcal{B}[X', Y']) \setminus E(H)| \leq |E(\mathcal{B}[X', Y']) \setminus E(\mathcal{B}[X, Y])| + |E(\mathcal{B}[X, Y]) \setminus E(H)|$$

$$\leq 2\gamma n\binom{n-1}{2} + 8\gamma\binom{n}{3} < 3\gamma n^3.$$

Therefore H is 3γ-extremal. $\qquad\square$

The following technical result will be applied in the proof of Lemma 2.2.

Proposition 3.4. *Let $\beta > 0$ and H be a 3-graph of order n. Let X, Y be a partition of $V(H)$. Suppose that there are at most $\beta\binom{|X|}{2}\binom{|Y|}{2}$ copies of K_4^- with two vertices in X and two vertices in Y. Then*

$$(|Y| - 1)e(XXY) + (|X| - 1)e(XYY) \leq 2(1 + \beta)\binom{|X|}{2}\binom{|Y|}{2}.$$

Proof. Observe that $\sum e(S) = (|Y| - 1)e(XXY) + (|X| - 1)e(XYY)$, where the sum is over all 4-sets S such that $|S \cap X| = |S \cap Y| = 2$. Our assumption is that there are at most $\beta\binom{|X|}{2}\binom{|Y|}{2}$ 4-sets S such that $|S \cap X| = |S \cap Y|$ and $e(S) \geq 3$. So we have $\sum e(S) \leq 2\binom{|X|}{2}\binom{|Y|}{2} + 2\beta\binom{|X|}{2}\binom{|Y|}{2} = 2(1 + \beta)\binom{|X|}{2}\binom{|Y|}{2}$. $\qquad\square$

4. Almost K_4^--tiling

In this section, we prove Lemma 2.1 which implies that any 3-graph H with minimum codegree slightly less than that in Theorem 1.1 must contain an almost perfect K_4^--tiling.

The key tool in the proof of Lemma 2.1 is a result of Keevash and Mycroft [11] on almost perfect matchings in hypergraphs. Before we can state this result, we need the following terminology. For an integer k, a *k-system* is a hypergraph J in which every edge of J has size at most k and $\emptyset \in E(J)$. We call an edge of size s in J an *s-edge*. Let J_s be the *s-graph* on $V(J)$ induced by all s-edges of J. The *minimum r-degree* of J, denoted by $\hat{\delta}_r(J)$, is the minimum $\deg_{J_{r+1}}(e)$ among all $e \in E(J_r)$. (Note that this is different from $\delta_r(J_{r+1})$, which is the minimum $\deg_{J_{r+1}}(S)$ among all r-sets $S \subseteq V(J)$.) The *degree sequence of J* is $\hat{\delta}(J) = (\hat{\delta}_0(J), \hat{\delta}_1(J), \ldots, \hat{\delta}_{k-1}(J))$. Given $a_0, a_1, \ldots, a_{k-1} \geq 0$ we write $\hat{\delta}(J) \geq (a_0, a_1, \ldots, a_{k-1})$ to mean that $\hat{\delta}_i(J) \geq a_i$ for all i.

We will apply the following special case of Lemma 7.6 from [11].

Lemma 4.1. *Suppose that $1/n \ll \phi \ll \gamma \ll \beta, 1/k$. Let V be a set of size n. Suppose that J is a k-system on V such that*

(i) $\hat{\delta}(J) \geq (n, (\frac{k-1}{k} - \gamma)n, (\frac{k-2}{k} - \gamma)n, \ldots, (\frac{1}{k} - \gamma)n)$ *and*
(ii) *for any $p \in [k-1]$ and set $S \subseteq V$ with $S = \lfloor pn/k \rfloor$, we have $e(J_{p+1}[S]) \geq \beta n^{p+1}$.*

Then J_k contains a matching M which covers all but at most ϕn vertices of J.

We now prove Lemma 2.1 by defining a 4-system J such that

$$(4.1) \qquad E(J_0) = \{\emptyset\}, E(J_1) = V(H), E(J_2) = \binom{V(H)}{2}, E(J_3) = E(H), \text{ and } E(J_4) = \mathcal{K}_4^-(H),$$

where $\mathcal{K}_4^-(H)$ denotes the set of 4-tuples in $V(H)$ that span a copy of K_4^- in H. If J satisfies the hypothesis of Lemma 4.1, then we are done. Otherwise, we deduce some structural properties of H, update J appropriately and apply Lemma 4.1 again.

Proof of Lemma 2.1. Define $1/n \ll \phi \ll \gamma \ll \beta \ll c'$ where c' is the constant from Proposition 3.1. Let H be as in the statement of the lemma and $V := V(H)$. Define a 4-system J as in (4.1). By Proposition 3.2, $\hat{\delta}_3(J) \geq (1/4 - 3\gamma/2)n$. Hence $\hat{\delta}(J) \geq (n, n-1, (1/2 - \gamma)n, (1/4 - 3\gamma/2)n)$. If Lemma 4.1(ii) holds, then J_4 contains a matching M which covers all but at most ϕn vertices of J thereby proving the lemma.

Thus we may assume that there exist some $p \in [3]$ and a set $S \subseteq V$ with $S = \lfloor pn/4 \rfloor$ where $e(J_{p+1}[S]) < \beta n^{p+1}$. We note that $p \neq 1, 3$. Indeed, for any S of size $\lfloor n/4 \rfloor$, we have $e(J_2[S]) = \binom{|S|}{2} > \beta n^2$. Now consider any $S \subseteq V$ such that $|S| = \lfloor 3n/4 \rfloor$. Note that $\delta_2(H[S]) \geq \delta_2(H) - \lceil n/4 \rceil \geq (1/4 - 2\gamma)n \gg 0.3|S|$. Proposition 3.1 therefore implies that $e(J_4[S]) \geq c'|S|^4 \geq \beta n^4$. So we must have that $p = 2$.

Let $S \subseteq V$ be such that $|S| = \lfloor n/2 \rfloor$ and $e(J_3[S]) = e(H[S]) < \beta n^3$. In general, for any set $U \subseteq V$ with $|U| = \lfloor n/2 \rfloor$ and $e(H[U]) < \beta n^3$, we call a pair $xy \in \binom{U}{2}$ U-good if $\deg_{H[U]}(xy) \leq 3\beta^{1/2}n$ (and so $\deg_H(xy, V \setminus U) \geq (1/2 - \gamma - 3\beta^{1/2})n$). Then at most $\beta^{1/2}n^2$ pairs $xy \in \binom{U}{2}$ are *not* U-good. We call a triple $xyz \in \binom{U}{3}$ U-*good* if every pair in xyz is U-good. Note that

(a) for any $W \subseteq U$, at least $\binom{|W|}{3} - \beta^{1/2}n^2(|W| - 2)$ triples of W are U-good;
(b) for any U-good triple $T = xyz$, $|L(T)| \geq |(N_H(xy) \cap N_H(xz) \cap N_H(yz)) \cap (V \setminus U)| \geq (1/2 - 10\beta^{1/2})n$.

Define a 4-system J' obtained from J by adding all S-good triples that are not edges of H to the edge set. Because of (b), we have $\deg_{J'}(T) \geq (1/2 - 10\beta^{1/2})n \geq (1/4 - 3\gamma/2)n$. Thus $\hat{\delta}(J') \geq (n, n-1, (1/2 - \gamma)n, (1/4 - 3\gamma/2)n)$. Since $J \subseteq J'$, J' satisfies Lemma 4.1(ii) for $p = 1, 3$. If J' also satisfies Lemma 4.1(ii) for $p = 2$, then Lemma 4.1 gives a matching in $J'_4 = J_4$ which covers all but at most ϕn vertices, proving the lemma. Otherwise, there exists $S' \subseteq V$ such that $|S'| = \lfloor n/2 \rfloor$ and $e(J'_3[S']) < \beta n^3$. We claim that $|S' \cap S| \leq 3\beta^{1/4}n$ – otherwise by (a), the number of S-good triples in $S' \cap S$ is at least

$$\binom{3\beta^{1/4}n}{3} - \beta^{1/2}n^2 \cdot (3\beta^{1/4}n - 2) > \beta n^3,$$

implying that $e(J'_3[S']) > \beta n^3$, a contradiction.

Define a 4-system J^* obtained from J' by adding all S'-good triples that are not in J'. Once again, we have $\hat{\delta}(J^*) \geq (n, n-1, (1/2 - \gamma)n, (1/4 - 3\gamma/2)n)$ by (b). Since $J \subseteq J^*$, J^* satisfies Lemma 4.1(ii) for $p = 1, 3$. Consider a set $S^* \subseteq V$ with $|S^*| = \lfloor n/2 \rfloor$. As $|S| = |S'| = \lfloor n/2 \rfloor$ and $|S' \cap S| \leq 3\beta^{1/4}n$, we have $|S' \cup S| \geq n - 3\beta^{1/4}n - 1$. Thus S^* contains at least $n/6$ vertices from S or at least $n/6$ vertices from $S' \setminus S$. In either case, since J^* contains all S-good and S'-good triples, we have $e(J^*_3[S^*]) \geq \binom{n/6}{3} - \beta^{1/2}n^2 \cdot n/6 > \beta n^3$. So J^* satisfies Lemma 4.1(ii) for $p = 2$. Therefore $J_4 = J^*_4$ contains a matching M which covers all but at most ϕn vertices, proving the lemma. $\qquad\square$

5. The absorbing lemma

In this section we prove Lemma 2.2 which is an absorbing result for the case when H is not 3γ-extremal. For this, we need the following terminology. Let H be a 3-graph of order n. Given an integer $c \geq 1$ and vertices $x, y \in V(H)$, we say that the vertex set $S \subseteq V(H)$ is an (x, y)-*connector of length c* if $S \cap \{x, y\} = \emptyset$, $|S| = 4c - 1$ and both $H[S \cup x]$ and $H[S \cup y]$ contain K_4^--factors. Given an integer $c \geq 1$ and a constant $\eta > 0$, two vertices $x, y \in V(H)$ are (c, η)-*close* to each other if there exist at least ηn^{4c-1} (x, y)-connectors of length c in H. For $x \in V(H)$, we denote by $\tilde{N}_{c,\eta}(x)$ the set of vertices y in H that are (c, η)-close to x. A subset $U \subseteq V(H)$ is said to be (c, η)-*closed* in H if any two vertices in U are (c, η)-close to each other. If $V(H)$ is (c, η)-closed in H then we simply say that H *is (c, η)-closed*.

Given $X \subseteq V(H)$, X being (c, η)-closed in H is *not* the same notion as $H[X]$ being (c, η)-closed. Indeed, the former implies that between any $x, y \in X$ there are at least ηn^{4c-1} (x, y)-connectors of length c in H. On the other hand, the latter implies that between any $x, y \in X$ there are at least $\eta|X|^{4c-1}$ (x, y)-connectors of length c in $H[X]$.

Given an integer $c \geq 1$ and $X, Y \subseteq V(H)$, a triple (x, y, S) is an (X, Y)-*bridge of length c* if $x \in X$, $y \in Y$ and S is an (x, y)-connector of length c.

We will apply the following two results from [16]. The first, a special case of Lemma 1.1 from [16], states that if H itself is (c, η)-closed then H contains a small absorbing set.

Lemma 5.1. [16] *Let $0 < 1/n \ll \phi \ll \varepsilon \ll \eta, 1/c$. Let H be a 3-graph of order n. Suppose that H is (c, η)-closed. Then there exists an absorbing set $W \subseteq V(H)$ of order at most εn such that*

$|W| \in 4\mathbb{N}$ *and for any* $U \subseteq V(H) \setminus W$ *such that* $|U| \leq \phi n$ *and* $|U| \in 4\mathbb{N}$, $H[W]$ *and* $H[U \cup W]$ *have* K_4^-*-factors.*

The next result is a special case of Proposition 2.1 from [16].

Proposition 5.2. [16] *Let* $0 < 1/n \ll \eta' \ll \eta, \varepsilon, 1/c$ *with* $c \in \mathbb{N}$. *Suppose* H *is a 3-graph of order* n *and there exists a vertex* $x \in V(H)$ *with* $|\tilde{N}_{c,\eta}(x)| \geq \varepsilon n$. *Then* $\tilde{N}_{c,\eta}(x) \subseteq \tilde{N}_{c+1,\eta'}(x)$.

The proof of the next simple result is similar to that of Lemma 2.2 from [16] (so we omit it). It states that if one has two disjoint 'closed' sets X and Y in H and H contains many (X, Y)-bridges, then in fact $X \cup Y$ is closed.

Lemma 5.3. *Let* $0 < 1/n \ll \eta' \ll \eta, \varepsilon, 1/c, 1/p$ *with* $c, p \in \mathbb{N}$. *Let* H *be a 3-graph of order* n. *Let* $X, Y \subseteq V(H)$ *be disjoint such that both* X *and* Y *are* (c, η)*-closed in* H. *Suppose further there exist at least* εn^{4p+1} (X, Y)*-bridges of length* p. *Then* $X \cup Y$ *is* $(2c + p, \eta')$*-closed in* H.

The following result gives another condition that ensures we can 'merge' two closed sets V_1, V_2 into a larger closed set $V_1 \cup V_2$.

Lemma 5.4. *Let* $0 < 1/n \ll \eta' \ll \eta, \varepsilon, 1/c$ *with* $c \in \mathbb{N}$. *Let* H *be a 3-graph of order* n. *Let* $V_1, \ldots, V_d$ *be disjoint subsets of* $V(H)$ *such that* $2 \leq d \leq 4$ *and each* V_i *is* (c, η)*-closed in* H. *Let* $a_1, \ldots, a_d$ *be non-negative integers such that* $a_1 \geq 1$ *and* $\sum a_i = 4$. *Suppose there exist at least* εn^4 *copies* F *of* K_4^- *in* H *such that* $|V(F) \cap V_i| = a_i$ *for all* $i \in [d]$ *and there exist at least* εn^4 *copies* F' *of* K_4^- *in* H *such that* $|V(F') \cap V_1| = a_1 - 1$, $|V(F') \cap V_2| = a_2 + 1$ *and* $|V(F') \cap V_j| = a_j$ *for all* $3 \leq j \leq d$. *Then* $V_1 \cup V_2$ *is* $(5c + 1, \eta')$*-closed in* H.

Proof. Let η'' be such that $\eta' \ll \eta'' \ll \eta, \varepsilon, 1/c$. Consider any vertex-disjoint copies F, F' of K_4^- in H such that $|V(F) \cap V_i| = a_i$ for all $i \in [d]$ and $|V(F') \cap V_1| = a_1 + 1$, $|V(F') \cap V_2| = a_2 - 1$ and $|V(F') \cap V_j| = a_j$ for all $3 \leq j \leq d$. Note that there are at least $(\varepsilon n^4/2)^2$ choices for (F, F'). Let $V(F) = \{x, x_1, x_2, x_3\}$ and $V(F') = \{y, y_1, y_2, y_3\}$ such that $x \in V_1$, $y \in V_2$ and for each $j \in [3]$, $x_j, y_j \in V_{i_j}$ for some i_j. For each $j \in [3]$, V_{i_j} is (c, η)-closed. Therefore, there exist S_1, S_2, S_3 such that each S_j is an (x_j, y_j)-connector of length c and $V(F), V(F'), S_1, S_2, S_3$ are vertex-disjoint. Note that there are at least $(\eta n^{4c-1}/2)^3$ choices of (S_1, S_2, S_3). Set $S := \{x_1, x_2, x_3, y_1, y_2, y_3\} \cup S_1 \cup S_2 \cup S_3$. Note that S is a (x, y)-connector of length $3c + 1$. Indeed, $H[x \cup S] \supseteq F \cup \bigcup_{j \in [3]} H[y_j \cup S_j]$ has a K_4^--factor, and similarly $H[y \cup S]$ has a K_4^--factor. Thus (x, y, S) is a (V_1, V_2)-bridge of length $3c + 1$. Hence, we have at least

$$\frac{1}{(4(3c+1)+1)!}\left(\frac{\varepsilon n^4}{2}\right)^2\left(\frac{\eta n^{4c-1}}{2}\right)^3 \geq \eta'' n^{4(3c+1)+1}$$

(V_1, V_2)-bridges of length $3c + 1$. By Lemma 5.3, $V_1 \cup V_2$ is $(5c + 1, \eta')$-closed in H. $\qquad \square$

5.1. There exists a vertex $v \in V(H)$ **such that** $v \in L(e)$ **for very few edges** $e \in E(H)$. Let H be a 3-graph satisfying the hypothesis of Lemma 2.2. Suppose further that there exists a vertex $v \in V(H)$ such that there are at most εn^3 edges e such that $v \in L(e)$ (that is, $e \cup v$ spans at least three edges). In Lemma 5.6, we show that there exists a small set $V_0 \subseteq V(H)$ such that $H[V_0]$ contains a K_4^--factor and $H \setminus V_0$ is $(6, \eta_*)$-closed for some constant $\eta_* > 0$. First we will need the following result for graphs.

Proposition 5.5. *Let* $n \in \mathbb{N}$ *and* $0 < \gamma \leq 1/20$. *Let* G *be a graph of order* n *with* $(1/2 - \gamma)n \leq \delta(G) \leq \Delta(G) \leq 3n/5$. *Suppose that* $|N(x) \triangle N(y)| \leq \gamma n$ *for every edge* $xy \in E(G)$. *Then there exists a bipartition* X, Y *of* $V(G)$ *such that* $\delta(G[X]), \delta(G[Y]) \geq (1/2 - 5\gamma)n$ *and* $(1/2 - 5\gamma)n \leq |X|, |Y| \leq (1/2 + 5\gamma)n$.

Proof. Let X, Y be a bipartition of $V(G)$ such that $e(X,Y)$ is minimised. First we show that $e(X, Y) \leq 3\gamma n^2/5$. Consider a vertex $x_0 \in V(G)$. Let $X_0 := N(x_0) \cup x_0$ and $Y_0 := V(G) \setminus X_0$. Note that

$$(5.1) \qquad e(X, Y) \leq e(X_0, Y_0) \leq \sum_{x \in N(x_0)} |N(x) \triangle N(x_0)| \leq \deg(x_0)\gamma n \leq 3\gamma n^2/5,$$

as claimed.

Suppose there exists $v \in V(G)$ such that $\deg(v, X), \deg(v, Y) > 4\gamma n$. Without loss of generality, assume that $\deg(v, X) \geq \deg(v)/2 \geq n/5$. For each $w \in N(v, X)$, we have

$$\deg(w, Y) \geq |N(w) \cap N(v) \cap Y| \geq \deg(v, Y) - |N(v) \triangle N(w)| > 3\gamma n,$$

as $|N(v) \triangle N(w)| \leq \gamma n$. Thus $e(X, Y) > 3\deg(v, X)\gamma n \geq 3\gamma n^2/5$ contradicting (5.1). Therefore, for all $v \in V(G)$, either $\deg(v, X) \leq 4\gamma n$ or $\deg(v, Y) \leq 4\gamma n$. Since $e(X, Y)$ is minimal, we have $\delta(G[X]), \delta(G[Y]) \geq (1/2 - 5\gamma)n$. $\qquad\square$

Define $\eta_* > 0$ to be the constant η' obtained by applying Lemma 5.4 with $1/30$, $1/128$ and 1 playing the roles of η, ε and c, respectively.

Lemma 5.6. *For $0 < 1/n \ll \varepsilon \ll \gamma \ll 1$, let H be a 3-graph of order n with $\delta_2(H) \geq (1/2 - \gamma)n$. Suppose there exists a vertex $v \in V(H)$ such that there are less than εn^3 edges $e \in E(H)$ such that $v \in L(e)$. Then there exists $V_0 \subseteq V(H)$ of order at most $8\sqrt[4]{\varepsilon}n$ such that $H[V_0]$ contains a K_4^--factor and $H \setminus V_0$ is $(6, \eta_*)$-closed.*

Here is a sketch of our proof. First we show that there exists a partition X, Y, V_0 such that $H[V_0]$ contains a K_4^--factor and almost all XXY- and YYX-edges exist. We then show that both X and Y are $(1, 1/30)$-closed in $H[X \cup Y]$. Furthermore, we show that there are many copies F, F' of K_4^- such that $|V(F) \cap X| = 3$, $|V(F) \cap Y| = 1$ and $|V(F') \cap X| = 2 = |V(F') \cap Y|$. Then by Lemma 5.4, $H[X \cup Y]$ is $(6, \eta_*)$-closed.

Proof of Lemma 5.6. Set $G := L_v$. Then $\delta_2(H) \geq (1/2 - \gamma)n$ implies that $\delta(G) \geq (1/2 - \gamma)n$. Note that, from the property in the lemma, there are fewer than εn^2 edges in H which contain two edges in G. We say that an edge $uw \in E(G)$ is *good* if $|N_G(u) \cap N_H(uw)| \leq 3\sqrt{\varepsilon}n$ and $|N_G(w) \cap N_H(uw)| \leq 3\sqrt{\varepsilon}n$, otherwise we call it *bad*.

Claim 5.7.
    (i) *There are at most $\sqrt{\varepsilon}n^2$ bad edges in G.*
    (ii) *If $u \in V(G)$ is incident with a good edge, then $\deg_G(u) \leq (1/2 + \gamma + 3\sqrt{\varepsilon})n$.*

Proof of claim. For each bad edge uw, there are at least $3\sqrt{\varepsilon}n$ edges e of H such that $uw \subseteq e$ and e contains at least two edges in G. Moreover, $v \in L(e)$. Thus there are at least $3\sqrt{\varepsilon}n$ edges e of $H \setminus \{v\}$ such that $uw \subseteq e$ and $v \in L(e)$. If there are at least $\sqrt{\varepsilon}n^2$ bad edges, then there are at least $\frac{1}{3}\sqrt{\varepsilon}n^2 \cdot 3\sqrt{\varepsilon}n = \varepsilon n^3$ edges $e \in E(H)$ such that $v \in L(e)$, contradicting the assumption. Thus (i) holds.

Suppose that $u \in V(G)$ is incident with a good edge uw in G. Note that

$$3\sqrt{\varepsilon}n \geq |N_G(u) \cap N_H(uw)| \geq \deg_G(u) + \deg_H(uw) - n.$$

Since $\delta_2(H) \geq (1/2 - \gamma)n$, (ii) holds. $\qquad\square$

Let V_0' be the set of vertices $u \in V(G)$ that are incident to at least $\sqrt[4]{\varepsilon}n$ bad edges in G. Since there are at most $\sqrt{\varepsilon}n^2$ bad edges, $|V_0'| \leq 2\sqrt[4]{\varepsilon}n$. By the greedy algorithm and Proposition 3.2, there exists a vertex set $V_0 \supseteq V_0'$ such that $H[V_0]$ contains a K_4^--factor and $|V_0| \leq 8\sqrt[4]{\varepsilon}n$. By pairing v with a vertex in V_0', we can ensure that $v \in V_0$.

Set $V' := V(H) \setminus V_0$ and $n' := |V'|$. Let $H' := H[V']$, so $\delta_2(H') \geq (1/2 - \gamma - 8\sqrt[4]{\varepsilon})n \geq (1/2 - 2\gamma)n'$. Let G' be the spanning subgraph of $G \setminus V_0$ induced by the good edges. Note that for all $u \in V'$,

$$\deg_{G'}(u) \geq \delta(G) - |V_0| - \sqrt[4]{\varepsilon}n \geq (1/2 - \gamma - 9\sqrt[4]{\varepsilon})n \geq (1/2 - 2\gamma)n'$$

and by Claim 5.7(ii), $\deg_{G'}(u) \leq (1/2 + \gamma + 3\sqrt{\varepsilon})n \leq (1/2 + 2\gamma)n'$. Since each edge uw in G' is good, we have

$$(5.2) \qquad |N_{G'}(u) \cap N_{H'}(uw)| \leq 3\sqrt{\varepsilon}n \leq 4\sqrt{\varepsilon}n'$$

and so $|N_{G'}(u) \cup N_{H'}(uw)| \geq 2(1/2 - 2\gamma)n' - 4\sqrt{\varepsilon}n'$. Consequently

$$\begin{aligned}
|N_{G'}(u) \setminus N_{G'}(w)| &= |(N_{G'}(u) \setminus N_{G'}(w)) \setminus N_{H'}(uw)| + |(N_{G'}(u) \setminus N_{G'}(w)) \cap N_{H'}(uw)| \\
&\leq |V(H') \setminus (N_{G'}(w) \cup N_{H'}(uw))| + |N_{G'}(u) \cap N_{H'}(uw)| \\
&\leq (n' - 2(1/2 - 2\gamma)n' + 4\sqrt{\varepsilon}n') + 4\sqrt{\varepsilon}n' \leq 5\gamma n'.
\end{aligned}$$

Similarly, we have $|N_{G'}(w) \setminus N_{G'}(u)| \leq 5\gamma n'$. Thus $|N_{G'}(u) \triangle N_{G'}(w)| \leq 10\gamma n'$.

Applying Proposition 5.5 to G' we obtain a bipartition $X \cup Y$ of V' such that $(1/2 - 50\gamma)n' \leq |X|, |Y| \leq (1/2 + 50\gamma)n'$ and $\delta(G'[X]) \geq (1/2 - 50\gamma)n' \geq (1 - 200\gamma)|X|$ and $\delta(G'[Y]) \geq (1 - 200\gamma)|Y|$.

Consider any edge uw in $G'[X]$. Observe that

$$|X \setminus N_{G'}(u)| \leq |X| - \deg_{G'}(u, X) \leq (1/2 + 50\gamma)n' - (1/2 - 50\gamma)n' \leq 100\gamma n'.$$

So by (5.2),

$$\begin{aligned}
\deg_{H'}(uw, Y) &= \deg_{H'}(uw) - \deg_{H'}(uw, X) \\
&\geq (1/2 - 2\gamma)n' - (|X \setminus N_{G'}(u)| + |N_{G'}(u) \cap N_{H'}(uw)|) \\
&\geq (1/2 - 2\gamma - 100\gamma - 4\sqrt{\varepsilon})n' \\
(5.3) \qquad &\geq (1/2 - 103\gamma)n' \geq (1 - 400\gamma)|Y|,
\end{aligned}$$

as $|Y| \leq (1/2 + 50\gamma)n'$. Therefore,

$$\begin{aligned}
e_{H'}(XXY) &\geq \sum_{uw \in E(G'[X])} \deg_{H'}(uw, Y) \geq e(G'[X])(1 - 400\gamma)|Y| \\
(5.4) \qquad &\geq (1 - 200\gamma)(1 - 400\gamma)|X|^2|Y|/2 \geq (1 - 600\gamma)|X|^2|Y|/2
\end{aligned}$$

as $\delta(G'[X]) \geq (1 - 200\gamma)|X|$.

Note that for every 3-set $T \subseteq X$ that forms a triangle in $G'[X]$, (5.3) implies that

$$(5.5) \qquad |L(T) \cap Y| \geq (1 - 1200\gamma)|Y|.$$

Recall that $\delta(G'[X]) \geq (1 - 200\gamma)|X|$. So there are at least $(1 - 200\gamma)(1 - 400\gamma)\binom{|X|}{3}(1 - 1200\gamma)|Y| \geq n^4/128$ copies of K_4^- in H' with three vertices in X and one in Y.

We now show that X is $(1, 1/30)$-closed in H'. Let x, x' be two distinct vertices in X. Since $\delta(G'[X]) \geq (1 - 200\gamma)|X|$, there are at least $(1 - 400\gamma)|X|(1 - 600\gamma)|X|/2 \geq n^2/10$ choices of $x_1, x_2 \in X$ such that both xx_1x_2 and $x'x_1x_2$ form triangles in G'. Let $Z := L_{H'}(xx_1x_2) \cap L_{H'}(x'x_1x_2) \cap Y$. Thus $|Z| \geq (1 - 2400\gamma)|Y| \geq n/3$ by (5.5). Notice that for each $z \in Z$, x_1x_2z is an (x, x')-connector of length 1 in H'. Thus, we get at least $n^3/30$ (x, x')-connectors of length 1, that is, x and x' are $(1, 1/30)$-close in H'. Therefore, X is $(1, 1/30)$-closed in H' as required.

By a similar argument, we have that Y is $(1, 1/30)$-closed in H' and $e_{H'}(XYY) \geq (1 - 600\gamma)|X||Y|^2/2$. Together with (5.4), we have

$$(|Y| - 1)e_{H'}(XXY) + (|X| - 1)e_{H'}(XYY) \geq 4(1 - 600\gamma)\binom{|X|}{2}\binom{|Y|}{2}.$$

By Proposition 3.4, there are at least $(1 - 1200\gamma)\binom{|X|}{2}\binom{|Y|}{2} \geq n^4/128$ copies of K_4^- in H' with two vertices in each of X and Y. Recall that there are at least $n^4/128$ copies of K_4^- in H' with three vertices in X and one in Y. Lemma 5.4 implies that $X \cup Y$ is $(6, \eta_*)$-closed in H', as desired. □

5.2. **Partitioning $V(H)$ into (c, η)-closed components.** Because of Lemma 5.6, we may assume that for every $v \in V(H)$, there are at least εn^3 edges e such that $v \in L(e)$. Recall that $\tilde{N}_{c,\eta}(v)$ is the set of vertices that are (c, η)-close to v in H. First we show that $\tilde{N}_{1,\eta}(v)$ is large for each $v \in V(H)$.

Proposition 5.8. *Let $n \in \mathbb{N}$ and $0 < \varepsilon, \gamma < 1$. Let H be a 3-graph of order n with $\delta_2(H) \geq (1/2 - \gamma)n$. Let $v \in V(H)$. Suppose that there are at least εn^3 edges $e \in E(H)$ such that $v \in L(e)$. Then $|\tilde{N}_{1,\gamma\varepsilon}(v)| \geq (1/4 - 3\gamma)n$.*

Proof. Let $E' \subseteq E(H)$ be the set of edges e such that $v \in L(e)$. By Proposition 3.2, for every edge $e \in E'$, $|L(e)| \geq (1/4 - 2\gamma)n$. Thus, we have

$$\sum_{e \in E'} |L(e)| \geq |E'|(1/4 - 2\gamma)n.$$

For any $e \in E'$ and any $u \in L(e) \setminus \{v\}$, e is a (u, v)-connector of length 1. Hence, if $u \neq v$ is a vertex in $V(H)$ and there are at least $\gamma\varepsilon n^3$ edges $e \in E'$ such that $u \in L(e)$, then $u \in \tilde{N}_{1,\gamma\varepsilon}(v)$. Thus

$$\sum_{e \in E'} |L(e)| \leq |\tilde{N}_{1,\gamma\varepsilon}(v)||E'| + n \cdot \gamma\varepsilon n^3 \leq (|\tilde{N}_{1,\gamma\varepsilon}(v)| + \gamma n)|E'|$$

as $|E'| \geq \varepsilon n^3$. Therefore, we have $|\tilde{N}_{1,\gamma\varepsilon}(v)| \geq (1/4 - 3\gamma)n$. □

Next we show that $V(H)$ can be partitioned into at most 4 parts such that each part is $(16, \eta)$-closed in H.

Lemma 5.9. *Let $0 < 1/n \ll \eta \ll \varepsilon, \gamma \ll 1$. Suppose H is a 3-graph of order n such that $|\tilde{N}_{1,\gamma\varepsilon}(v)| \geq (1/4 - 3\gamma)n$ for every $v \in V(H)$. Then there is a partition $\mathcal{P} = \{V_1, \ldots, V_d\}$ of $V(H)$ such that $d \leq 4$ and each V_i is $(16, \eta)$-closed in H and $|V_i| \geq (1/4 - 4\gamma)n$.*

Proof. Let $\alpha, \eta_0, \eta_1, \eta_2, \eta_3, \eta_4$ be such that $\alpha \ll \gamma$, $\eta_0 := \varepsilon\gamma$ and

$$1/n \ll \eta = \eta_4 \ll \eta_3 \ll \eta_2 \ll \eta_1 \ll \eta_0, \alpha.$$

Throughout this proof, for $v \in V(H)$ and $i \in [4]$, we write $\tilde{N}_{2^i, \eta_i}(v)$ as $\tilde{N}_{2^i}(v)$ for short. By assumption, for any $v \in V(H)$, $|\tilde{N}_{2^0}(v)| = |\tilde{N}_{1,\eta_0}(v)| \geq (1/4 - 3\gamma)n$. We also write 2^i-close (respectively 2^i-closed) for $(2^i, \eta_i)$-close (respectively $(2^i, \eta_i)$-closed). By Proposition 5.2 and the choice of the η_is, we may assume that $\tilde{N}_{2^i}(v) \subseteq \tilde{N}_{2^{i+1}}(v)$ for all $0 \leq i \leq 3$ and all $v \in V(H)$. Hence, if $W \subseteq V(H)$ is 2^i-closed in H for some $i \leq 4$, then W is 2^4-closed in H.

Since $|\tilde{N}_{2^0}(v)| \geq (1/4 - 3\gamma)n$ for any $v \in V(H)$, any set of five vertices in $V(H)$ contains two vertices u, v such that $|\tilde{N}_{2^0}(u) \cap \tilde{N}_{2^0}(v)| \geq (5(1/4 - 3\gamma)n - n)/\binom{5}{2} \geq n/50$. Thus the number of (u, v)-connectors of length 2 in H is at least

$$\frac{1}{7!} \cdot \frac{n}{50} \cdot (\eta_0 n^3 - n^2) \cdot (\eta_0 n^3 - 4n^2) \geq \eta_1 n^7,$$

which implies that u and v are 2^1-close to each other in H. Also we may assume that there are two vertices that are not 2^4-close to each other, as otherwise H is 2^4-closed and the lemma holds with $\mathcal{P} = \{V(H)\}$.

Let d be the largest integer such that there exist $v_1, \ldots, v_d \in V(H)$ such that no pair of them are 2^{6-d}-close to each other. Note that d exists by our assumption and $2 \leq d \leq 4$ by our observation.

Fix such $v_1, \ldots, v_d \in V(H)$, by Proposition 5.2, we can assume that any two of them are not 2^{5-d}-close to each other. Consider $\tilde{N}_{2^{5-d}}(v_i)$ for all $i \in [d]$. We have the following facts.

(i) Any $v \in V(H) \setminus \{v_1, \ldots, v_d\}$ must be in $\tilde{N}_{2^{5-d}}(v_i)$ for some $i \in [d]$.

(ii) For any $i \neq j$, $|\tilde{N}_{2^{5-d}}(v_i) \cap \tilde{N}_{2^{5-d}}(v_j)| < \alpha n$.

Note that if (i) fails for some $v \in V(H)$, then $v, v_1, \ldots, v_d$ contradicts the definition of d. If (ii) fails with $|\tilde{N}_{2^{5-d}}(v_1) \cap \tilde{N}_{2^{5-d}}(v_2)| \geq \alpha n$ say, then the number of (v_1, v_2)-connectors S of length 2^{6-d} of the form $S = z \cup S_1 \cup S_2$, where $z \in \tilde{N}_{2^{5-d}}(v_1) \cap \tilde{N}_{2^{5-d}}(v_2)$ and for $i = 1, 2$, S_i is a (v_i, z)-connector of length 2^{5-d}, is at least

$$\frac{1}{(2^{6-d}4 - 1)!} \alpha n \left(\frac{\eta_{5-d} n^{2^{5-d}4-1}}{2} \right)^2 \geq \eta_{6-d} n^{2^{6-d}4-1}.$$

This implies that v_1 and v_2 are 2^{6-d}-close in H, a contradiction.

For each $i \in [d]$, let $U_i := (\tilde{N}_{2^{5-d}}(v_i) \cup v_i) \setminus \bigcup_{j \in [d] \setminus \{i\}} \tilde{N}_{2^{5-d}}(v_j)$. Note that each U_i is 2^{5-d}-closed in H. Indeed, if there exist $u_1, u_2 \in U_i$ that are not 2^{5-d}-close to each other, then $\{u_1, u_2\} \cup (\{v_1, \ldots, v_d\} \setminus \{v_i\})$ contradicts the definition of d.

Let $U_0 := V(H) \setminus (U_1 \cup \cdots \cup U_d)$. By (i) and (ii), we have $|U_0| \leq \binom{d}{2}\alpha n$. We partition U_0 into $U_{0,1}, \ldots, U_{0,d}$ as follows. For each $v \in U_0$, since $|\tilde{N}_{2^0}(v) \setminus U_0| \geq (1/4 - 3\gamma)n - |U_0| \geq d\alpha n$, there exists $i \in [d]$ such that $|\tilde{N}_{2^0}(v) \cap U_i| \geq \alpha n$. In this case we add v to $U_{0,i}$ (we add v to an arbitrary $U_{0,i}$ if there are more than one such i).

Let $\mathcal{P} = \{V_1, \ldots, V_d\}$ be the partition of $V(H)$ such that $V_i = U_i \cup U_{0,i}$ for all $i \in [d]$. Let $\eta_{6-d} \ll \eta'' \ll \eta' \ll \eta_{5-d}$. Consider any $i \in [d]$ and any $v \in U_{0,i}$. Since $|\tilde{N}_{2^0}(v) \cap U_i| \geq \alpha n$, it is easy to see that $U_i \cup v$ is $(2^{5-d} + 1, \eta')$-closed. Similarly, $U_i \cup \{v, v'\}$ is $(2^{5-d} + 2, \eta'')$-closed for any $v, v' \in U_{0,i}$. Thus each V_i is $(2^{5-d} + 2, \eta'')$-closed, so 2^4-closed by Proposition 5.2. Further, $|V_i| \geq |U_i| \geq (1/4 - 3\gamma)n - \binom{d}{2}\alpha n \geq (1/4 - 4\gamma)n$. $\square$

5.3. **Proof of Lemma 2.2.** By Lemma 5.9, there is a partition $\mathcal{P} = \{V_1, \ldots, V_d\}$ of $V(H)$ such that $d \leq 4$ and each V_i is $(16, \eta)$-closed in H and $|V_i| \geq (1/4 - 4\gamma)n$. Our new goal is to find many (V_i, V_j)-bridges for some $i \neq j$ so that we can apply Lemma 5.3 to reduce that number of (c, η')-closed components of $V(H)$. The following lemma is crucial in proving Lemma 2.2; we defer its proof to Section 5.4.

For any $U_1, U_2, U_3, U_4 \subseteq V(H)$, we say that a copy F of K_4^- is *of type* $U_1 U_2 U_3 U_4$ if there is an ordering v_1, v_2, v_3, v_4 of $V(F)$ such that $v_i \in U_i$ for all $i \in [4]$.

Lemma 5.10. *Let* $0 < 1/n \ll \eta' \ll \varepsilon, \eta \ll \gamma \leq \rho \ll 1/c$. *Let* H *be a 3-graph of order* n *with* $\delta_2(H) \geq (1/2 - \gamma)n$. *Suppose that there is a partition* $\mathcal{P} = \{V_1, \ldots, V_d\}$ *of* $V(H)$ *such that* $2 \leq d \leq 4$ *and each* V_i *is* (c, η)-*closed in* H *and* $|V_i| \geq (1/4 - 4\gamma)n$. *Suppose that there are not necessarily distinct* $i_1, i_2, i_3, i_4 \in [d]$ *such that* $\{i_1, i_2\} \cap \{i_3, i_4\} = \emptyset$ *and at least* εn^4 *copies of* K_4^- *are of type* $V_{i_1} V_{i_2} V_{i_3} V_{i_4}$. *Then there exist distinct* $i, j \in [d]$ *such that* $V_i \cup V_j$ *is* $(3c + 1, \eta')$-*closed in* H.

Proof of Lemma 2.2. Let η_* be as defined before Lemma 5.6. Since $\varepsilon \ll \gamma \ll 1$, we may assume that $\varepsilon \ll \eta_*$. Let ε_0 be such that $\phi \ll \varepsilon_0 \ll \varepsilon$. Let H be a 3-graph of order n with $\delta_2(H) \geq (1/2 - \gamma)n$ and so that H is not 3γ-extremal.

If there exists a vertex $v \in V(H)$ such that there are less than $\varepsilon_0 n^3$ edges $e \in E(H)$ such that $v \in L(e)$, then Lemma 5.6 implies that there exists $V_0 \subseteq V(H)$ of order at most $8\sqrt[4]{\varepsilon_0}n$ such that $H[V_0]$ contains a K_4^--factor and $H \setminus V_0$ is $(6, \eta_*)$-closed. Apply Lemma 5.1 to $H \setminus V_0$ and obtain an absorbing set $W' \subseteq V(H) \setminus V_0$ of order at most $\varepsilon n/2$ such that for any $U \subseteq V(H) \setminus (V_0 \cup W')$ such that $|U| \leq \phi n$ and $|U| \in 4\mathbb{N}$, both $H[W']$ and $H[U \cup W']$ have K_4^--factors. Lemma 2.2 holds by setting $W := W' \cup V_0$.

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Therefore, we may assume that for every $v \in V(H)$, there are at least $\varepsilon_0 n^3$ edges $e \in E(H)$ such that $v \in L(e)$. Let $c_4 := 16$ and for $d \in [3]$, let $c_d := 5c_{d+1} + 1$. Let $\eta, \eta_1, \ldots, \eta_4$ be such that $\phi \ll \eta_1 \ll \eta_2 \ll \eta_3 \ll \eta_4 \ll \eta \ll \varepsilon_0$. Let $d \leq 4$ be the smallest integer such that there is a partition $\mathcal{P} = \{V_1, \ldots, V_d\}$ of $V(H)$ where each V_i is (c_d, η_d)-closed and $|V_i| \geq (1/4 - 4\gamma)n$. Note that d exists by Proposition 5.8 and Lemma 5.9. If $d = 1$, then Lemma 2.2 holds by Lemma 5.1. We will now show that we obtain a contradiction if $d = 2, 3, 4$ and so $d = 1$, as required.

Case 1: $d \in \{3, 4\}$.

Without loss of generality, assume $|V_1| \leq \cdots \leq |V_d|$. Note that we have $(1/2 - 8\gamma)n \leq |V_1| + |V_2| \leq 2n/3$.

First assume that $e(V_1 V_2 V_j) \geq \eta n^3$ for some $j \geq 3$. Since $\delta_2(H) \geq (1/2 - \gamma)n$, by Proposition 3.2, each $V_1 V_2 V_j$-edge is contained in at least $(1/2 - 3\gamma)n/2$ copies of K_4^-. By averaging, there exists $i \in [d]$ such that there are at least

$$\frac{1}{4}\eta n^3 \left(\frac{1}{2} - 3\gamma\right) \frac{n}{2} \cdot \frac{1}{2} \geq \frac{\eta}{40}n^4$$

copies of K_4^- of type $V_1 V_2 V_j V_i$ ($\frac{1}{2}$ appears because a copy of K_4^- may be counted twice if $i \in \{1, 2, j\}$). It is easy to see that we can order $1, 2, j, i$ as i_1, i_2, i_3, i_4 such that $\{i_1, i_2\} \cap \{i_3, i_4\} = \emptyset$. By Lemma 5.10 and Proposition 5.2, there exist distinct $i, j \in [d]$ such that $V_i \cup V_j$ is (c_{d-1}, η_{d-1})-closed in H. Also, by Proposition 5.2, every set in $\{V_1, \ldots, V_d\} \setminus \{V_i, V_j\}$ is (c_{d-1}, η_{d-1})-closed in H. Altogether, this contradicts the minimality of d.

So we may assume that $e(V_1 V_2 V_j) < \eta n^3$ for all $j \geq 3$. Since

$$\sum_{x \in V_1, y \in V_2} |N(xy)| = 2e(V_1 V_1 V_2) + 2e(V_1 V_2 V_2) + \sum_{j=3}^{d} e(V_1 V_2 V_j),$$

it follows that $|V_1||V_2|(1/2 - \gamma)n \leq 2e(V_1 V_1 V_2) + 2e(V_1 V_2 V_2) + 2\eta n^3$. As $|V_1||V_2| \geq (\frac{1}{16} - 2\gamma)n^2$, we have $2\eta n^3 \leq 34\eta|V_1||V_2|n$. It follows that $2e(V_1 V_1 V_2) + 2e(V_1 V_2 V_2) \geq |V_1||V_2|(1/2 - \gamma - 34\eta)n$. Note that $|V_2| \leq (\frac{3}{4} + 4\gamma)n/2$ because $|V_3| \geq |V_2| \geq |V_1| \geq (\frac{1}{4} - 4\gamma)n$. Therefore

$$e(V_1 V_1 V_2) + e(V_1 V_2 V_2) \geq \frac{1}{2}|V_1||V_2|\left(\frac{1}{2} - 2\gamma\right)n \geq \frac{1}{2}|V_1||V_2|\frac{5}{4}\left(\frac{3}{8} + 2\gamma\right)n \geq \frac{5}{4}|V_1|\binom{|V_2|}{2}$$

as $\eta \ll \gamma \ll 1$. Hence

$$(|V_2| - 1)e(V_1 V_1 V_2) + (|V_1| - 1)e(V_1 V_2 V_2) \geq (|V_1| - 1)\left(e(V_1 V_1 V_2) + e(V_1 V_2 V_2)\right) \geq \frac{5}{2}\binom{|V_1|}{2}\binom{|V_2|}{2}.$$

By Proposition 3.4, there are at least $\frac{1}{4}\binom{|V_1|}{2}\binom{|V_2|}{2} \geq 4^{-7}n^4$ copies of K_4^- with two vertices in V_1 and two vertices in V_2. Lemma 5.10 and Proposition 5.2 imply that there exist distinct $i, j \in [d]$ such that $V_i \cup V_j$ is (c_{d-1}, η_{d-1})-closed in H. Also, by Proposition 5.2, every set in $\{V_1, \ldots, V_d\} \setminus \{V_i, V_j\}$ is (c_{d-1}, η_{d-1})-closed in H. Altogether this contradicts the minimality of d. So $d \notin \{3, 4\}$.

Case 2: $d = 2$.

By Lemma 3.3, since H is not 3γ-extremal, H contains at least $\gamma^2 n^3$ $V_1 V_1 V_2$-edges and at least $\gamma^2 n^3$ $V_1 V_2 V_2$-edges. By Lemma 5.10, we may assume that there are at most ηn^4 copies K_4^- with two vertices in V_1 and two vertices in V_2 – otherwise $V_1 \cup V_2$ is (c_1, η_1)-closed in H. Thus, for all but at most $2\sqrt{\eta}n^3$ $V_1 V_1 V_2$-edges e, $|L(e) \cap V_2| \leq \sqrt{\eta}n$ and so $|L(e) \cap V_1| \geq n/8$ by Proposition 3.2. Therefore, there are at least $(\gamma^2 n^3 - 2\sqrt{\eta}n^3)(n/8)/3 \geq \gamma^2 n^4/50 \geq \eta n^4$ copies of K_4^- with three vertices in V_1 and one vertex in V_2. So there are at most ηn^4 copies of K_4^- with all vertices in V_1.

(Indeed, otherwise Lemma 5.4 implies that $V_1 \cup V_2$ is (c_1, η_1)-closed in H, a contradiction to the minimality of d.) Proposition 3.1 implies that $e(H[V_1]) \leq (3/10)\binom{|V_1|}{3}$. Thus we have

$$e(V_1 V_1 V_2) \geq \left(\delta_2(H) - \frac{3}{10}|V_1|\right)\binom{|V_1|}{2} \geq \left((1/2 - \gamma)n - \frac{3}{10}|V_1|\right)\binom{|V_1|}{2}.$$

Similarly, we have $e(V_1 V_2 V_2) \geq ((1/2 - \gamma)n - 3|V_2|/10)\binom{|V_2|}{2}$. So we have

$$(|V_2| - 1)e(V_1 V_1 V_2) + (|V_1| - 1)e(V_1 V_2 V_2)$$

$$\geq \frac{1}{2}(|V_1| - 1)(|V_2| - 1)\left((1/2 - \gamma)n(|V_1| + |V_2|) - \frac{3}{10}(|V_1|^2 + |V_2|^2)\right)$$

$$= \frac{1}{2}(|V_1| - 1)(|V_2| - 1)\left((1/5 - \gamma)n^2 + \frac{3}{5}|V_1||V_2|\right).$$

Since $|V_1||V_2| \leq n^2/4$, we have $(1/5 - \gamma)n^2 \geq \frac{3}{5}|V_1||V_2|$. Thus, we get

$$(|V_2| - 1)e(V_1 V_1 V_2) + (|V_1| - 1)e(V_1 V_2 V_2) \geq \frac{12}{5}\binom{|V_1|}{2}\binom{|V_2|}{2}.$$

By Proposition 3.4, there are at least $\frac{1}{5}\binom{|X|}{2}\binom{|Y|}{2} > \eta n^4$ copies of K_4^- in H with two vertices in V_1 and two vertices in V_2, contradicting our assumption. So $d \neq 2$. This completes the proof. $\square$

5.4. **Proof of Lemma 5.10.** Suppose that H is a 3-graph satisfying the hypothesis of Lemma 5.10. Let $Y = V_{i_3} \cup V_{i_4}$ and $X = V(H) \setminus Y$. Then there are many copies of K_4^- with two vertices in X and two in Y. Our aim is to show that there are many (X, Y)-bridges so that we can apply Lemma 5.3.

Let H be a 3-graph of order n. Let X, Y be two disjoint subsets of $V(H)$. Given a constant $\rho > 0$, a set $xx'y$ of three vertices with $x, x' \in X$ and $y \in Y$ is called (ρ, X, Y)-*typical* if

(T1) $\deg(xx', Y) \geq |Y| - \rho n$,
(T2) $|N(xy, X) \cap N(x'y, X)| \leq \rho n$,
(T3) $|X| - \rho n \leq \deg(xy, X) + \deg(x'y, X)$.

In the next lemma, we show that given a copy of K_4^3 on $xx'yy'$ with $x, x' \in X$ and $y, y' \in Y$, then we can find many (X, Y)-bridges (containing this K_4^3) unless $xx'y$ is typical.

Lemma 5.11. *Let $\gamma > 0$. Let H be a 3-graph of order n with $\delta_2(H) \geq (1/2 - \gamma)n$. Suppose that X, Y is a bipartition of $V(H)$ and that $xx'yy'$ spans a copy of K_4^3 in H with $x, x' \in X$ and $y, y' \in Y$. Then at least one of the following holds:*

  (a) *$xx'yy'$ is contained in at least $\gamma n/4$ 5-sets that span (X, Y)-bridges of length 1;*
  (b) *$(1/2 - 4\gamma)n \leq |X|, |Y| \leq (1/2 + 4\gamma)n$ and $xx'y$ is $(9\gamma, X, Y)$-typical.*

Proof. First note that each $z \in L(xx'y) \cap X$ gives an (X, Y)-bridge $(z, y', \{x, x', y\})$ of length 1 because both $zxx'y$ and $xx'yy'$ span copies of K_4^-. Thus we may assume that

(5.6) $\qquad |L(xx'y) \cap X|, |L(xx'y') \cap X|, |L(xyy') \cap Y|, |L(x'yy') \cap Y| < \gamma n/4$

or else (a) holds. Since $xx'y \in E(H)$, this implies that $|N(xy, X) \cap N(x'y, X)| \leq |L(xx'y) \cap X| \leq \gamma n/4$. Furthermore, by Proposition 3.2, we have

(5.7)
$$\deg(xx', X) + \deg(xy, X) + \deg(x'y, X) \leq |X| + \gamma n/2,$$
$$\deg(xx', X) + \deg(xy', X) + \deg(x'y', X) \leq |X| + \gamma n/2,$$
$$\deg(xy, Y) + \deg(xy', Y) + \deg(yy', Y) \leq |Y| + \gamma n/2,$$
$$\deg(x'y, Y) + \deg(x'y', Y) + \deg(yy', Y) \leq |Y| + \gamma n/2.$$

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Let $D := 2\deg(xx', X) + 2\deg(yy', Y) + \deg(xy) + \deg(xy') + \deg(x'y) + \deg(x'y')$. By summing all the inequalities in (5.7), we derive that $D \le 2|X| + 2|Y| + 2\gamma n$. Since $\delta_2(H) \ge (1/2 - \gamma)n$ and $|X| + |Y| = n$, it follows that

$$(5.8) \qquad \deg(xx', X) + \deg(yy', Y) \le 3\gamma n.$$

Note that $|X| \ge \deg(yy', X) \ge (1/2 - \gamma)n - \deg(yy', Y) \ge (1/2 - 4\gamma)n$. Similarly we have $|Y| \ge (1/2 - 4\gamma)n$. Therefore

$$(5.9) \qquad (1/2 - 4\gamma)n \le |X|, |Y| \le (1/2 + 4\gamma)n$$

and

$$\deg(xx', Y) \overset{(5.8)}{\ge} (1/2 - \gamma)n - 3\gamma n \ge |Y| - 8\gamma n.$$

If $\deg(xx', X) + \deg(xy, X) + \deg(x'y, X) < |X| - 11\gamma n/2$, then we replace the first inequality in (5.7) and, summing the inequalities again, obtain that

$$\deg(xy) + \deg(xy') + \deg(x'y) + \deg(x'y') \le D < 2|X| + 2|Y| - 4\gamma n \le 4\delta_2(H),$$

a contradiction. We thus have that $\deg(xx', X) + \deg(xy, X) + \deg(x'y, X) \ge |X| - 11\gamma n/2$. Together with (5.8), it follows that $\deg(xy, X) + \deg(x'y, X) \ge |X| - 9\gamma n$. We thus deduce that $xx'y$ is $(9\gamma, X, Y)$-typical and (b) holds. $\qquad\square$

Now we prove a similar lemma in which we assume $xx'yy'$ spans a copy of K_4^- – this requires a more careful analysis of the neighbourhoods in the proof.

Lemma 5.12. *Let $\gamma > 0$. Let H be a 3-graph of order n with $\delta_2(H) \ge (1/2 - \gamma)n$. Suppose that X, Y is a bipartition of $V(H)$. Let $x, x' \in X$ and $y_1, y_2 \in Y$ such that $xy_1y_2, xx'y_1, xx'y_2 \in E(H)$. Then at least one of the following holds:*
   (a) *$xx'y_1y_2$ is contained in at least $\gamma n/4$ 5-sets that span (X, Y)-bridges of length 1;*
   (b) *there are at least $\gamma n/4$ copies K of K_4^3 such that $|V(K) \cap X| = 2 = |V(K) \cap Y|$ and $|V(K) \cap \{x, x', y_1, y_2\}| = 3$;*
   (c) *$(1/2 - 15\gamma/4)n \le |X|, |Y| \le (1/2 + 15\gamma/4)n$ and $xx'y_1$ is $(26\gamma, X, Y)$-typical.*

Proof. We assume that neither (a) nor (b) holds. Fix $i \in \{1, 2\}$ and consider the edge $xx'y_i$. Note that each $z \in L(xx'y_i) \cap X$ gives an (X, Y)-bridge $(z, y_j, \{x, x', y_i\})$ of length 1, where $j = 3 - i$. Thus $|L(xx'y_i) \cap X| < \gamma n/4$ or else (a) holds. Therefore by Proposition 3.2, we have

$$(5.10) \qquad \deg(xx', X) + \deg(xy_i, X) + \deg(x'y_i, X) \le |X| + \gamma n/2.$$

On the other hand, each $z' \in N(xx', Y) \cap N(xy_i, Y) \cap N(x'y_i, Y)$ yields a copy of K_4^3 on $xx'y_iz'$. Thus

$$(5.11) \qquad \gamma n/4 > |N(xx', Y) \cap N(xy_i, Y) \cap N(x'y_i, Y)|$$

$$(5.12) \qquad \ge \deg(xx', Y) + \deg(xy_i, Y) + \deg(x'y_i, Y) - 2|Y|$$

or else (b) holds. By combining (5.10) and (5.12),

$$(5.13) \qquad \deg(xx') + \deg(xy_i) + \deg(x'y_i) \le |X| + 2|Y| + 3\gamma n/4 = n + |Y| + 3\gamma n/4.$$

Since $\delta_2(H) \ge (1/2 - \gamma)n$, it follows that $|Y| \ge (1/2 - 15\gamma/4)n$.

By a similar argument on the edge xy_1y_2, we deduce that $|X| \ge (1/2 - 15\gamma/4)n$. Therefore,

$$(5.14) \qquad (1/2 - 15\gamma/4)n \le |X|, |Y| \le (1/2 + 15\gamma/4)n.$$

We now bound $\deg(xx', Y) + \deg(xy_i, Y) + \deg(x'y_i, Y)$, $i \in \{1, 2\}$, from below. If $\deg(xx', Y) + \deg(xy_i, Y) + \deg(x'y_i, Y) < 2|Y| - 29\gamma n/4$, then together with (5.10) this gives that

$$\deg(xx') + \deg(xy_i) + \deg(x'y_i) < |X| + 2|Y| - 27\gamma n/4 = n + |Y| - 27\gamma n/4 \overset{(5.14)}{\le} 3(1/2 - \gamma)n,$$

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implying that $\delta_2(H) < (1/2 - \gamma)n$, a contradiction. Hence, we have

$$(5.15) \qquad 2|Y| - 29\gamma n/4 \leq \deg(xx', Y) + \deg(xy_i, Y) + \deg(x'y_i, Y).$$

For $0 \leq \ell \leq 3$, let n_ℓ be the number of vertices of Y that belong to ℓ of the sets $N(xx')$, $N(xy_i)$, $N(x'y_i)$. Clearly, $n_0 + n_1 + n_2 + n_3 = |Y|$ and $n_1 + 2n_2 + 3n_3 = \deg(xx', Y) + \deg(xy_i, Y) + \deg(x'y_i, Y)$. We may assume that $n_3 \leq \gamma n/4$ otherwise (b) holds. By (5.15),

$$2|Y| - 29\gamma n/4 \leq n_1 + 2n_2 + 3n_3 = 2(n_0 + n_1 + n_2 + n_3) - 2n_0 - n_1 + n_3$$
$$\leq 2|Y| - 2n_0 - n_1 + \gamma n/4,$$

which implies that $2n_0 + n_1 \leq 15\gamma n/2$. In particular,

$$(5.16) \qquad |\overline{N}(xx', Y) \setminus (N(xy_i, Y) \cap N(x'y_i, Y))| \leq n_0 + n_1 \leq 15\gamma n/2,$$

where $\overline{N}(xx', Y) := Y \setminus N(xx', Y)$. Applying (5.16) twice with $i = 1, 2$, we obtain

$$|\overline{N}(xx', Y) \setminus (N(xy_1, Y) \cap N(x'y_1, Y))| + |\overline{N}(xx', Y) \setminus (N(xy_2, Y) \cap N(x'y_2, Y))| \leq 15\gamma n.$$

If $|\overline{N}(xx', Y)| \geq 16\gamma n$, then we get

$$|N(xy_1, Y) \cap N(x'y_1, Y) \cap N(xy_2, Y) \cap N(x'y_2, Y)| \geq \gamma n.$$

In particular, $|N(xy_1, Y) \cap N(xy_2, Y)| \geq \gamma n$. Since $N(xy_1, Y) \cap N(xy_2, Y) \subseteq L(xy_1 y_2) \cap Y$, this implies (a), a contradiction. Thus we assume that

$$(5.17) \qquad |\overline{N}(xx', Y)| < 16\gamma n.$$

We further have

$$\deg(xy_1, Y) + \deg(x'y_1, Y) \leq |Y| + |N(xy_1) \cap N(x'y_1) \cap Y|$$
$$\leq |Y| + |N(xy_1) \cap N(x'y_1) \cap N(xx') \cap Y| + |Y \setminus N(xx')|$$
$$\overset{(5.17),(5.11)}{\leq} |Y| + \gamma n/4 + 16\gamma n \overset{(5.14)}{\leq} (1/2 + 20\gamma)n.$$

Therefore

$$\deg(xy_1, X) + \deg(x'y_1, X) \geq 2(1/2 - \gamma)n - (\deg(xy_1, Y) + \deg(x'y_1, Y))$$

$$\geq (1/2 - 22\gamma)n \overset{(5.14)}{\geq} |X| - 26\gamma n.$$

So (T3) holds (with $\rho = 26\gamma$). On the other hand, we have $|N(xy_1, X) \cap N(x'y_1, X)| \leq \gamma n/4$ because $|L(xx'y_1) \cap X| \leq \gamma n/4$ and $N(xy_1, X) \cap N(x'y_1, X) \subseteq L(xx'y_1) \cap X$. So (T2) holds. Finally, (5.17) implies that $\deg(xx', Y) \geq |Y| - 16\gamma n$ so (T1) holds. We thus deduce that $xx'y_1$ is $(26\gamma, X, Y)$-typical and (c) holds. $\qquad \square$

In the next lemma, we show how to use (ρ, X, Y)-typical edges to find (X, Y)-bridges.

Lemma 5.13. *Let $0 < 1/n \ll \varepsilon' \ll \varepsilon \ll \eta \ll \gamma \leq \rho \ll 1/c$. Suppose that H is a 3-graph of order n such that $\delta_2(H) \geq (1/2 - \gamma)n$. Let X, Y be a partition $V(H)$ such that $(1/2 - \rho)n \leq |X| \leq (1/2 + \rho)n$. Further assume $X = V_{i_1} \cup V_{i_2}$ and $Y = V_{i_3} \cup V_{i_4}$ such that*

1) *either $V_{i_1} = V_{i_2}$ or $V_{i_1} \cap V_{i_2} = \emptyset$, and either $V_{i_3} = V_{i_4}$ or $V_{i_3} \cap V_{i_4} = \emptyset$;*
2) *for each $j \in [4]$, V_{i_j} is (c, η)-closed in H and has size at least $(1/4 - 4\gamma)n$.*

Suppose that there are at least εn^4 copies $xx'yy'$ of K_4^- in H such that $x, x' \in X$, $y \in V_{i_3}$, $y' \in V_{i_4}$ and $xx'y$ is (ρ, X, Y)-typical. Then at least one of the following holds:

(a) *there are at least $\varepsilon' n^{4c+5}$ (X, Y)-bridges of length $c + 1$ in H;*
(b) *$V_{i_1} \cap V_{i_2} = \emptyset$ and there are at least $\varepsilon' n^5$ (V_{i_1}, V_{i_2})-bridges of length 1 in H.*

Proof. Define ε_0 so that $\varepsilon' \ll \varepsilon_0 \ll \varepsilon$.

Let $\mathcal{Y}$ be the set of pairs (y, y') such that (i) $y \in V_{i_3}$ and $y' \in V_{i_4}$; and (ii) there are at least εn^2 pairs $x, x' \in X$ such that $xx'yy'$ spans a copy of K_4^- and $xx'y$ is (ρ, X, Y)-typical. Note that $|\mathcal{Y}| \geq \varepsilon n^2$ – otherwise there are at most $|\mathcal{Y}|\binom{n}{2} + \binom{n}{2}\varepsilon n^2 < \varepsilon n^4$ copies of such graphs, a contradiction.

Fix $(y, y') \in \mathcal{Y}$. Let G be the graph on X such that $xx' \in E(G)$ if $xx'yy'$ spans a copy of K_4^- and $xx'y$ is (ρ, X, Y)-typical. Thus $e(G) \geq \varepsilon n^2$. Applying a classical result of Erdős [5], we can find at least $\varepsilon_0 n^6$ copies of $K_{3,3}$ in G. Fix a copy of $K_{3,3}$ and label its vertices as $\{x_1, x_2, x_3, x_1', x_2', x_3'\}$, where $x_i x_j' \in E(G)$ for all $i, j \in [3]$.

Claim. Let $W := \{x_1, x_2, x_3, x_1', x_2', x_3', y, y'\}$. Then at least one of the following holds.

   (i) there are at least εn^{4c} (X, Y)-bridges S_0 of length $c + 1$ with $|V(S_0) \cap W| = 5$;

   (ii) X is not (c, η)-closed and $|L(x_1 x_2 x_3) \cap X| \geq |X| - 18\rho n$ or $|L(x_1' x_2' x_3') \cap X| \geq |X| - 18\rho n$.

Proof of the claim. Let $i \in [3]$. Since $x_i x_1' y$ is (ρ, X, Y)-typical, (T2) and (T3) imply that

$$(5.18) \qquad |X| - \rho n \leq \deg(x_i y, X) + \deg(x_1' y, X) \leq |X| + \rho n.$$

Since $\deg(x_1 y, X) + \deg(x_1' y, X) \leq |X| + \rho n$, by swapping x_1 and x_1' if necessary, we may assume that $\deg(x_1' y, X) \leq (|X| + \rho n)/2 \leq |X| - 11\rho n$. For any $i \in [3]$, since $x_i x_1' y$ is (ρ, X, Y)-typical, we have

$$(5.19) \qquad \deg(x_i x_1', Y) \geq |Y| - \rho n \text{ and,}$$

$$|N(x_i y, X) \cap N(x_1' y, X)| \leq \rho n.$$

Since $\deg(x_i y, X) = |N(x_i y, X) \cap N(x_1' y, X)| + |N(x_i y, X) \setminus N(x_1' y, X)|$, it follows that

$$|N(x_i y, X) \setminus N(x_1' y, X)| \geq \deg(x_i y, X) - \rho n \overset{(5.18)}{\geq} |X \setminus N(x_1' y, X)| - 2\rho n.$$

This implies that, for distinct $i, j \in [3]$,

$$|N(x_i y, X) \cap N(x_j y, X)| \geq |(N(x_i y, X) \cap N(x_j y, X)) \setminus N(x_1' y, X)|$$
$$\geq |N(x_i y, X) \setminus N(x_1' y, X)| + |N(x_j y, X) \setminus N(x_1' y, X)| - |X \setminus N(x_1' y, X)|$$
$$(5.20) \qquad \geq |X \setminus N(x_1' y, X)| - 4\rho n \geq 7\rho n.$$

Suppose that there exist $i, j \in [3]$ such that $\deg(x_i x_j, Y) \geq 4\rho n$. By applying (5.19) twice, we derive that

$$|L(x_1' x_i x_j) \cap Y| \geq |N(x_1' x_i) \cap N(x_1' x_j) \cap N(x_i x_j) \cap Y| \geq 2\rho n.$$

Without loss of generality, assume that $|L(x_1' x_i x_j) \cap V_{i_4}| \geq \rho n$. Consider any $z \in L(x_1' x_i x_j) \cap V_{i_4}$ with $z \neq y'$. Let S be a (y', z)-connector of length c such that $S \cap W = \emptyset$ (there are at least $\eta n^{4c-1}/2$ choices for S). Then $S \cup \{x_i, x_1', y', z\}$ is an (x_j, y)-connector of length $c + 1$ because both $H[S \cup \{x_i, x_1', y', z, x_j\}] = H[x_1' x_i x_j z] \cup H[S \cup y']$ and $H[S \cup \{x_1', x_i, y', z, y\}] = H[x_1' x_i y y'] \cup H[S \cup z]$ have K_4^--factors. Hence $(x_j, y, S \cup \{x_1', x_i, y', z\})$ is an (X, Y)-bridge of length $c + 1$. Since a given $4c$-set may be partitioned into a singleton and $(4c - 1)$-set in $4c$ ways, there are at least $(\rho n - 1) \cdot (\eta n^{4c-1}/2) \cdot (1/4c) \geq \varepsilon n^{4c}$ (X, Y)-bridges S_0 of length $c + 1$ such that $|V(S_0) \cap W| = 5$. This proves (i) and we are done.

Now suppose that $\deg(x_i x_j, Y) < 4\rho n$ for all distinct $i, j \in [3]$. Since $|X| \leq (1/2 + \rho)n$,

$$(5.21) \qquad \deg(x_i x_j, X) \geq (1/2 - \gamma)n - 4\rho n \geq |X| - 6\rho n.$$

Thus $|L(x_1 x_2 x_3) \cap X| \geq |X| - 18\rho n$. If X is not (c, η)-closed, then we obtain (ii); otherwise assume that X is (c, η)-closed. By (5.20) and (5.21),

$$|L(x_1 x_2 y) \cap X| \geq |N(x_1 y) \cap N(x_2 y) \cap N(x_1 x_2) \cap X| \geq \rho n.$$

Consider any $z' \in L(x_1 x_2 y) \cap X$ so that $z' \neq x_1'$. Let S' be an (x_1', z')-connector of length c such that $S' \cap W = \emptyset$ (there are $\eta n^{4c-1}/2$ choices for S). Then $S' \cup \{x_1', x_1, y, z'\}$ is an (x_2, y')-connector of length $c+1$ because both $H[S' \cup \{x_1', x_1, x_2, y, z'\}] = H[x_1 x_2 y z'] \cup H[S' \cup x_1']$ and $H[S' \cup \{x_1', x_1, y, y', z'\}] = H[x_1' x_1 y y'] \cup H[S' \cup z']$ have K_4^--factors. Thus $(x_2, y', S' \cup \{x_1', x_1, y, z'\})$ is an (X, Y)-bridge of length $c+1$. In total, there are at least $(\rho n - 1) \cdot (\eta n^{4c-1}/2) \cdot (1/4c) \geq \varepsilon n^{4c}$ (X, Y)-bridges S_0 of length $c+1$ such that $|V(S_0) \cap W| = 5$, implying (i). $\qquad \square$

Now we return to the proof of the lemma. We apply the claim for each pair $(y, y') \in \mathcal{Y}$ and each copy of $K_{3,3}$ in G.

First assume that for *some* pair $(y, y') \in \mathcal{Y}$, at least $\varepsilon_0 n^6/2$ copies of $K_{3,3}$ in G satisfy (ii) in the claim. Then X is not (c, η)-closed, so $X = V_{i_1} \cup V_{i_2}$ with $V_{i_1} \cap V_{i_2} = \emptyset$. In addition, either $|L(x_1 x_2 x_3) \cap X| \geq |X| - 18\rho n$ or $|L(x_1' x_2' x_3') \cap X| \geq |X| - 18\rho n$. This implies that at least $\varepsilon_0 n^3/2$ 3-sets $S \subset X$ satisfy $|L(S) \cap X| \geq |X| - 18\rho n$. For any $v \in L(S) \cap V_{i_1}$ and $v' \in L(S) \cap V_{i_2}$, (v, v', S) is a (V_{i_1}, V_{i_2})-bridge of length 1. Recall that $|V_i|, |V_{i'}| \geq (1/4 - 4\gamma)n$. Thus, there are at least

$$(\varepsilon_0 n^3/2)(|V_{i_1}| - 18\rho n)(|V_{i_2}| - 18\rho n) \geq \varepsilon' n^5$$

(V_{i_1}, V_{i_2})-bridges of length 1, implying (b).

The only other case to consider is when for *every* $(y, y') \in \mathcal{Y}$, at least $\varepsilon_0 n^6/2$ copies of $K_{3,3}$ in G satisfy (i) in the claim. In this case, for each $(y, y') \in \mathcal{Y}$, there exist at least $\varepsilon_0 n^6/2$ 6-sets $W' \subseteq X$ such that there are at least $\varepsilon_0 n^{4c}$ (X, Y)-bridges S_0 of length $c+1$ with $|V(S_0) \cap (W' \cup \{y, y'\})| = 5$. By averaging, for each such W', there is a 5-subset $W_0 \subset (W' \cup \{y, y'\})$ that is contained in at least $\varepsilon_0 n^{4c}/\binom{8}{5}$ (X, Y)-bridges of length $c+1$. Since there are at least $|\mathcal{Y}|(\varepsilon_0 n^6)/2\binom{n}{3}$ choices of W_0 and a $(4c+5)$-set contains at most $\binom{4c+5}{5}$ such W_0, the total number of (X, Y)-bridges of length $c+1$ is at least

$$\frac{|\mathcal{Y}|\varepsilon_0 n^6/2}{\binom{n}{3}} \cdot \frac{\varepsilon_0 n^{4c}}{\binom{8}{5}} \cdot \frac{1}{\binom{4c+5}{5}} \geq \varepsilon' n^{4c+5},$$

yielding (a). $\qquad \square$

We are ready to prove Lemma 5.10.

Proof of Lemma 5.10. Since each copy of K_4^- misses one edge, after averaging (and relabeling the indices if necessary), we assume that there are at least $\varepsilon n^4/4$ copies of K_4^- of type $V_{i_1} V_{i_2} V_{i_3} V_{i_4}$ such that all the missing edges are of type $V_{i_2} V_{i_3} V_{i_4}$. Denote by $\mathcal{F}$ the family of these copies of K_4^-. Let $Y := V_{i_3} \cup V_{i_4}$ and $X := V(H) \setminus Y$. Let $\varepsilon', \varepsilon''$ be such that

$$\eta' \ll \varepsilon' \ll \varepsilon'' \ll \varepsilon, \eta, 1/c.$$

We claim that it suffices to show that there exists $c' \leq c+1$ such that there are at least $\varepsilon' n^{4c'+1}$ (X, Y)-bridges of length c'. Indeed, there are at most four types of pairs (V_i, V_j) such that $V_i \subseteq X$ and $V_j \subseteq Y$. If there are at least $\varepsilon' n^{4c'+1}$ (X, Y)-bridges of length c', then by averaging, there exist $i \neq j \in [d]$ such that there are at least $\varepsilon' n^{4c'+1}/4$ (V_i, V_j)-bridges of length c'. By Lemma 5.3 and Proposition 5.2, we conclude that $V_i \cup V_j$ is $(3c+1, \eta')$-closed in H as $2c + c' \leq 3c+1$.

Since each $F = xx'yy' \in \mathcal{F}$ satisfies $x, x' \in X$, $y, y' \in Y$, $xx'y, xx'y', xyy' \in E(H)$, we can apply Lemma 5.12 to F. Recall that $|\mathcal{F}| \geq \varepsilon n^4/4$. First assume that there are at least $\varepsilon n^4/12$ copies F satisfying Lemma 5.12(a), that is, each $V(F)$ is contained in $\gamma n/4$ 5-sets that span (X, Y)-bridges of length 1. Then there are at least $\frac{1}{5}(\varepsilon n^4/12 \cdot \gamma n/4) = \varepsilon \gamma n^5/240 \geq \varepsilon' n^5$ (X, Y)-bridges of length 1, as desired.

Second if there exist at least $\varepsilon n^4/12$ copies $F \in \mathcal{F}$ that satisfy Lemma 5.12(c), then we are done by Lemma 5.13.

Finally, if there exist at least $\varepsilon n^4/12$ copies $F \in \mathcal{F}$ that satisfy Lemma 5.12(b), then there are $(\varepsilon n^4/12 \cdot \gamma n/4)/4n \geq \varepsilon'' n^4$ copies K of K_4^3 with $|V(K) \cap X| = |V(K) \cap Y|$. Apply Lemma 5.11 to

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each such K. First suppose there exists at least $\varepsilon'' n^4/2$ copies K that satisfy Lemma 5.11(a), that is, $V(K)$ is contained in $\gamma n/4$ 5-sets that span (X,Y)-bridges of length 1. Then there are at least $\frac{1}{5}(\varepsilon'' n^4/2 \cdot \gamma n/4) \geq \varepsilon' n^5$ (X,Y)-bridges of length 1 and we are done. Thus we may assume that at least $\varepsilon'' n^4/2$ copies $K = xx'yy'$ with $x, x' \in X$ and $y, y' \in Y$ satisfy Lemma 5.11(b). Then we are done by Lemma 5.13. $\qquad\square$

6. The extremal case

In this section we prove Lemma 2.3, that is, Theorem 1.1 in the case when H is extremal. Let H, H' be two k-graphs on the same vertex set V. Let $H' \setminus H := (V, E(H') \setminus E(H))$. Suppose that $0 \leq \alpha \leq 1$ and $|V| = n$. A vertex $v \in V$ is called α-*good in* H (otherwise α-*bad*) *with respect to* H' if $\deg_{H' \setminus H}(v) \leq \alpha n^{k-1}$. We first deal with the extremal case in the special case when every vertex in H is 'good' with respect to $\mathcal{B}[A, B]$.

Lemma 6.1. *Let* $0 < 1/m \ll \alpha < 1/10^5$ *where* $m \in \mathbb{N}$. *Suppose that* H *is a 3-graph on* $V = A \cup B$ *where* $|A| = |B| = 6m$. *Further, suppose that every vertex in* H *is* α-*good with respect to* $\mathcal{B}[A, B]$. *Then* H *contains a* K_4^--*factor.*

Proof. Write $n := 12m$. Define an auxiliary 3-graph H_B with vertex set B and where $xyz \in E(H_B)$ precisely if there are at least $|A| - 3\alpha^{1/2} n$ vertices $a \in A$ such that $xyza$ spans a copy of K_4^- in H.

Claim 6.2. H_B *contains a perfect matching* M_B.

Proof. To prove the claim, consider any vertex $x \in B$. Since x is α-good, all but at most $\alpha^{1/2} n$ vertices $y \in B \setminus \{x\}$ are such that $\deg_H(xy, A) \geq |A| - \alpha^{1/2} n$. Fix such a vertex $y \in B$. Similarly, all but at most $2\alpha^{1/2} n$ vertices $z \in B \setminus \{x, y\}$ are such that $\deg_H(xz, A), \deg_H(yz, A) \geq |A| - \alpha^{1/2} n$. Fix such a vertex $z \in B$. Notice that $xyza$ spans a copy of K_4^- for at least $|A| - 3\alpha^{1/2} n$ vertices $a \in A$. Thus, $xyz \in E(H_B)$.

There are at least $|B| - 2\alpha^{1/2} n$ choices for y and at least $|B| - 3\alpha^{1/2} n$ choices for z. Hence,

$$\deg_{H_B}(x) \geq \frac{1}{2}(|B| - 2\alpha^{1/2} n)(|B| - 3\alpha^{1/2} n),$$

and so $\delta_1(H_B) \geq \frac{1}{2}(|B| - 2\alpha^{1/2} n)(|B| - 3\alpha^{1/2} n)$. This minimum vertex degree condition forces a perfect matching in H_B (for example, by a result of Daykin and Häggkvist [4]), as required. $\qquad\square$

By definition of H_B, for each edge xyz in M_B, we can greedily pair off xyz with a distinct vertex $a \in A$ so that $xyza$ forms a copy of K_4^- in H. We therefore obtain a K_4^--tiling $\mathcal{M}_1$ in H that covers all of B and $2m$ vertices in A. Let $A' := A \setminus V(\mathcal{M}_1)$. So $|A'| = 4m$.

Set $H' := H[A']$. Further, define an auxiliary 4-graph $H_{A'}$ with vertex set A' and where $xyzw \in E(H_{A'})$ precisely if $xyzw$ spans a copy of K_4^- in H. Consider any $x \in A'$. Since x is α-good, all but at most $2\alpha^{1/2} n$ vertices $y \in A' \setminus \{x\}$ are such that $\deg_{H'}(xy) \geq |A'| - 2 - \alpha^{1/2} n$ in H'. Fix such a vertex y. Next fix a vertex $z \in N_{H'}(xy)$ where $\deg_{H'}(xz) \geq |A'| - 2 - \alpha^{1/2} n$. There are at least $|A'| - 2 - 3\alpha^{1/2} n$ choices for z. Finally, fix a vertex $w \in N_{H'}(xy) \cap N_{H'}(xz)$; there are at least $|A'| - 4 - 2\alpha^{1/2} n$ choices for w. Then $xyzw$ spans a copy of K_4^- in H' and so $xyzw \in E(H_{A'})$. There are at least $|A'| - 3\alpha^{1/2} n$ choices for y, $|A'| - 4\alpha^{1/2} n$ choices for z and $|A'| - 3\alpha^{1/2} n$ choices for w. Therefore,

$$\delta_1(H_{A'}) \geq \frac{1}{3!}(|A'| - 3\alpha^{1/2} n)(|A'| - 4\alpha^{1/2} n)(|A'| - 3\alpha^{1/2} n).$$

This implies that $H_{A'}$ contains a perfect matching (again, by the result of Daykin and Häggkvist [4]) and thus, H' contains a K_4^--factor $\mathcal{M}_2$. So $\mathcal{M}_1 \cup \mathcal{M}_2$ is a K_4^--factor in H, as desired. $\qquad\square$

We now apply Lemma 6.1 to prove Lemma 2.3.

Proof of Lemma 2.3. Let $0 < 1/n_0 \ll \gamma \ll \gamma_1 \ll \gamma_2 \ll \gamma_3 \ll 1$. Suppose that H is as in the statement of the lemma. In particular, since H is γ-extremal, there exists a partition A, B of $V(H)$ such that $|A| = |B|$ and H γ-contains $\mathcal{B}[A, B]$. Furthermore, this implies that all but at most $\gamma_1 n$ vertices in H are γ_1-good with respect to $\mathcal{B}[A, B]$. Let A_0 and B_0 denote the set of γ_1-bad vertices in A and B respectively.

We say that a vertex $x \in A_0 \cup B_0$ is *B-acceptable* if there are at least $n^2/40$ pairs (a, b) of vertices where $a \in A$, $b \in B$ and $abx \in E(H)$. Otherwise we say that $x \in A_0 \cup B_0$ is *A-acceptable*. Since $\delta_1(H) \geq (n-1)(n/2 - 1)/2$, if x is A-acceptable then:

- There are at least $3\binom{|A|}{2}/4$ pairs (a, a') of vertices where $a, a' \in A$ and $aa'x \in E(H)$ and;
- There are at least $3\binom{|B|}{2}/4$ pairs (b, b') of vertices where $b, b' \in B$ and $bb'x \in E(H)$.

Next we modify the partition A, B of $V(H)$ as follows. Move all A-acceptable vertices that lie in B to A and move all B-acceptable vertices that lie in A to B. Since $|A_0| + |B_0| \leq \gamma_1 n$, A, B remains a partition of $V(H)$ where now:

- $n/2 - \gamma_1 n \leq |A|, |B| \leq n/2 + \gamma_1 n$;
- H γ_2-contains $\mathcal{B}[A, B]$.

Moreover, there is a partition A_1, A_2 of A so that:

(α_1) $|A_2| \leq \gamma_1 n$;
(α_2) Every vertex in A_1 is γ_2-good with respect to $\mathcal{B}[A, B]$;
(α_3) If $x \in A_2$ then
  - there are at least $2\binom{|A|}{2}/3$ pairs (a, a') of vertices where $a, a' \in A$ and $aa'x \in E(H)$ and;
  - there are at least $2\binom{|B|}{2}/3$ pairs (b, b') of vertices where $b, b' \in B$ and $bb'x \in E(H)$.

Similarly, there is a partition B_1, B_2 of B so that:

(β_1) $|B_2| \leq \gamma_1 n$;
(β_2) Every vertex in B_1 is γ_2-good with respect to $\mathcal{B}[A, B]$;
(β_3) If $x \in B_2$ then there are at least $n^2/50$ pairs (a, b) of vertices where $a \in A$, $b \in B$ and $abx \in E(H)$.

Our aim will be to find a small K_4^--tiling which covers all the vertices in $A_2 \cup B_2$ so that the set of uncovered vertices, A^* and B^* in A and B respectively, are such that $|A^*| = |B^*| \equiv 0 \mod 6$. Then (α_2) and (β_2) will ensure that every vertex in $H[A^* \cup B^*]$ is γ_3-good with respect to $\mathcal{B}[A^*, B^*]$. Thus, Lemma 6.1 ensures a K_4^--factor in $H[A^* \cup B^*]$ and hence a K_4^--factor in H. To guarantee that $|A^*| = |B^*| \equiv 0 \mod 6$ we will require the existence of two 'parity breaking' copies of K_4^- in H. These subgraphs will be obtained in the following claim. We say a copy K of K_4^- in H is of *type* (i, j) if K contains i vertices from A and j vertices from B.

Claim 6.3. *H contains two copies K, K' of K_4^- so that one of the following conditions holds:*

(i) K and K' are not necessarily vertex-disjoint; K is of type $(2, 2)$; K' is of $(3, 1)$;
(ii) K and K' are vertex-disjoint; K and K' are of type $(3, 1)$;
(iii) K and K' are vertex-disjoint; K and K' are of type $(2, 2)$.

Proof. To prove the claim we split our argument into two cases depending on the size of B.

Case 1: $|B| \leq n/2 - 1$. Fix $a \in A$ and $b \in B$. Then there exists a vertex $a' \in N_H(ab) \cap A$ since $\delta_2(H) \geq n/2 - 1$ and $|B \setminus \{b\}| \leq n/2 - 2$. Let $x \in A \setminus \{a, a'\}$ and $y \in B \setminus \{b\}$ be arbitrary. Again there exists a vertex $x' \in N_H(xy) \cap A$. Suppose that $x' \neq a, a'$. Certainly, at least two of the sets $N_H(aa'), N_H(ab), N_H(a'b)$ have an intersection of size at least 4. Thus, H contains a copy K of K_4^- of type $(2, 2)$ or $(3, 1)$ where $a, a', b \in V(K)$ and K is disjoint from xyx'. Moreover, at least two of the sets $N_H(xx'), N_H(xy), N_H(x'y)$ have intersection at least 5. Thus, H contains a copy

K' of K_4^- of type $(2,2)$ or $(3,1)$ where $x, x', y \in V(K')$ and so that K and K' are vertex-disjoint. Hence, K and K' satisfy one of (i)–(iii) as desired.

The claim is therefore satisfied in this case, unless for every $x \in A \setminus \{a, a'\}$ and $y \in B \setminus \{b\}$ we have that $N_H(xy) \cap A \subseteq \{a, a'\}$. Suppose there are distinct $x, x_1 \in A \setminus \{a, a'\}$ and $y, y_1 \in B \setminus \{b\}$ such that $a \in N_H(xy) \cap A$ and $a' \in N_H(x_1 y_1) \cap A$. Then as above, we obtain vertex-disjoint K, K' that satisfy one of (i)–(iii). This implies we may assume that for every $x \in A \setminus \{a, a'\}$ and $y \in B \setminus \{b\}$ we have $N_H(xy) \cap A = \{a\}$. We now show that in this case, (i) is satisfied.

Choose any edge $a_1 b_1 b_2 \in E(H)$ where $a_1 \in A \setminus \{a, a'\}$ and $b_1, b_2 \in B \setminus \{b\}$; such an edge exists since H γ_2-contains $\mathcal{B}[A, B]$. Then by assumption $aa_1 b_1, aa_1 b_2 \in E(H)$. Thus, $aa_1 b_1 b_2$ spans a copy K of K_4^- in H of type $(2,2)$.

Since $a \in A$, (α_2) and (α_3) imply that there exists an edge $axx' \in E(H)$ where $x, x' \in A \setminus \{a, a'\}$. Let $y \in B \setminus \{b\}$ be arbitrary. Then by assumption $axy, ax'y \in E(H)$. Thus, $axx'y$ spans a copy K' of K_4^- in H of type $(3,1)$. Hence, the claim is satisfied in this case.

Case 2: $|B| \geq n/2$. Fix $a, a' \in A$. Then since $\delta_2(H) \geq n/2 - 1$, there exists a vertex $b \in N_H(aa') \cap B$. Let $x, y \in A \setminus \{a, a'\}$ be arbitrary. So there exists a vertex $z \in N_H(xy) \cap B$. Suppose that $b \neq z$. At least two of the sets $N_H(aa'), N_H(ab), N_H(a'b)$ have intersection at least 4. Thus, H contains a copy K of K_4^- of type $(2,2)$ or $(3,1)$ where $a, a', b \in V(K)$ and K is disjoint from xyz. Moreover, at least two of the sets $N_H(xz), N_H(xy), N_H(yz)$ have intersection at least 5. Thus, H contains a copy K' of K_4^- of type $(2,2)$ or $(3,1)$ where $x, y, z \in V(K')$ and so that K and K' are vertex-disjoint. Hence, K and K' satisfy one of (i)–(iii) as desired.

We may therefore assume that for every $x, y \in A \setminus \{a, a'\}$ we have $N_H(xy) \cap B = \{b\}$. We will show that (i) is satisfied in this case. Choose any three vertices $a_1, a_2, a_3 \in A \setminus \{a, a'\}$. Then by assumption $a_1 a_2 b, a_2 a_3 b, a_1 a_3 b \in E(H)$. Thus, $a_1 a_2 a_3 b$ spans a copy K' of K_4^- in H of type $(3,1)$.

Since $b \in B$, (β_2) and (β_3) imply that there are vertices $x, y \in A \setminus \{a, a'\}$ and $z \in B$ such that $xzb, yzb \in E(H)$. Also, by assumption we have that $N_H(xy) \cap B = \{b\}$. So $xyzb$ spans a copy K of K_4^- in H of type $(2,2)$. Hence, the claim is satisfied in this case. $\qquad \square$

The next claim will allow us to cover the vertices in $A_2 \cup B_2$ with a small K_4^--tiling.

Claim 6.4. Let $W \subseteq V(H)$ such that $|W| \leq \gamma_2 n$. Every vertex $x \in (A_2 \cup B_2) \setminus W$ lies in a copy K_x of K_4^- in H of type $(1,3)$ such that K_x is disjoint from W.

Proof. If $x \in A_2 \setminus W$ then (α_3) together with Mantel's theorem implies that the subgraph $L_x[B \setminus W]$ of the link graph L_x contains a triangle T. Note that T corresponds to a copy of K_4^- in H of type $(1,3)$ that contains x.

Suppose that $x \in B_2 \setminus W$. Since H γ_2-contains $\mathcal{B}[A, B]$, (β_3) implies that there are vertices $a \in A \setminus W$ and $b, b' \in B \setminus W$ such that $abb', abx, ab'x \in E(H)$. Therefore, x indeed lies in a copy of K_4^- in H of type $(1,3)$ that is disjoint from W. $\qquad \square$

By repeatedly applying Claim 6.4, we can obtain a K_4^--tiling $\mathcal{M}_1$ in H so that:

- $|\mathcal{M}_1| \leq 2\gamma_1 n$;
- $\mathcal{M}_1$ is vertex-disjoint from K and K';
- Every copy of K_4^- in $\mathcal{M}_1$ is of type $(1,3)$;
- Every vertex in $(A_2 \cup B_2) \setminus (V(K) \cup V(K'))$ is covered by $\mathcal{M}_1$.

Let $A' := A \setminus V(\mathcal{M}_1)$ and $B' := B \setminus V(\mathcal{M}_1)$. Since $n \equiv 0 \mod 4$ we have that $|A' \cup B'| \equiv 0 \mod 4$ and so $|A'| \equiv |B'| \mod 2$. Further, $n/2 - 7\gamma_1 n \leq |A'|, |B'| \leq n/2 + \gamma_1 n$. Since H γ_2-contains $\mathcal{B}[A, B]$, it is easy to see that we can greedily construct a K_4^--tiling $\mathcal{M}_2$ in H so that:

- $|\mathcal{M}_2| \leq 8\gamma_1 n$;
- $\mathcal{M}_2$ is vertex-disjoint from $\mathcal{M}_1, K, K'$;
- Every copy of K_4^- in $\mathcal{M}_2$ is of type $(4,0)$ or $(1,3)$;

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- $|A''| = |B''|$ where $A'' := A' \setminus V(\mathcal{M}_2)$ and $B'' := B' \setminus V(\mathcal{M}_2)$.

Note that $|A''| = |B''| \equiv 0 \mod 2$ and so (a) $|A''| = |B''| \equiv 0 \mod 6$ or; (b) $|A''| = |B''| \equiv 2 \mod 6$ or; (c) $|A''| = |B''| \equiv 4 \mod 6$. If (a) holds we set $\mathcal{M}_3 := \emptyset$. Suppose that (b) holds. If Claim 6.3(i) or (iii) holds then we set $\mathcal{M}_3 := \{K\}$. If Claim 6.3(ii) holds then we set $\mathcal{M}_3 := \{K, K', K'', K'''\}$ where K'' and K''' are two vertex-disjoint copies of K_4^- in H of type $(1,3)$ which are additionally vertex-disjoint from $\mathcal{M}_1, \mathcal{M}_2, K, K'$. (It is easy to see such K'' and K''' exist since H γ_2-contains $\mathcal{B}[A, B]$.) Finally, suppose that (c) holds. If Claim 6.3(i) or (ii) holds then we set $\mathcal{M}_3 := \{K', K''\}$ where K'' is a copy of K_4^- in H of type $(1,3)$ which is vertex-disjoint from $\mathcal{M}_1, \mathcal{M}_2, K, K'$. If Claim 6.3(iii) holds then we set $\mathcal{M}_3 := \{K, K'\}$.

In every case we have chosen $\mathcal{M}_3$ so that $|A'''| = |B'''| \equiv 0 \mod 6$ where $A''' := A'' \setminus V(\mathcal{M}_3)$ and $B''' := B'' \setminus V(\mathcal{M}_3)$. Depending on the definition of $\mathcal{M}_3$, A''' and B''' could contain vertices from K, K', and thus perhaps vertices from A_2 and B_2. However, by applying Claim 6.4 we can obtain a K_4^--tiling $\mathcal{M}_4$ in H so that:

- $|\mathcal{M}_4| \leq 24$;
- $\mathcal{M}_4$ is vertex-disjoint from $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$;
- Every copy of K_4^- in $\mathcal{M}_4$ is of type $(4,0)$ or $(1,3)$;
- $|A^*| = |B^*| \equiv 0 \mod 6$ where $A^* := A''' \setminus V(\mathcal{M}_4)$ and $B^* := B''' \setminus V(\mathcal{M}_4)$;
- $A^* \subseteq A_1$ and $B^* \subseteq B_1$.

Note that $|A^*|, |B^*| \geq n/2 - \gamma_2 n$. Set $H^* := H[A^* \cup B^*]$. By (α_2) and (β_2) we have that every vertex in H^* is γ_3-good with respect to $\mathcal{B}[A^*, B^*]$. Therefore, Lemma 6.1 implies that H^* contains a K_4^--factor, $\mathcal{M}_5$. We have that $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{M}_4 \cup \mathcal{M}_5$ is a K_4^--factor in H, as desired. □

7. Concluding remarks

In this paper we have determined the minimum codegree threshold that ensures a K_4^--factor in a 3-graph of sufficiently large order by the absorbing method. It is tempting to attempt to apply [11, Theorem 2.9] of Keevash and Mycroft to handle the non-extremal case directly. However, since the 4-system defined in (4.1) is not a *complex* (downward-closed system), we can not apply [11, Theorem 2.9] directly.

It would also be interesting to determine the minimum *vertex degree* threshold for K_4^--tiling.

Conjecture 7.1. *Let $n \in 4\mathbb{N}$ be sufficiently large. If H is a 3-graph on n vertices and*

$$\delta_1(H) > \binom{3n/4 - 1}{2}$$

then H contains a K_4^--factor.

Note that, if true, the minimum vertex degree condition in Conjecture 7.1 is best-possible. Indeed, consider the 3-graph H whose vertex set has a partition X, Y so that $|X| = n/4 + 1$, $|Y| = 3n/4 - 1$ and so that the edge set of H consists of precisely all those edges whose intersection with Y is at least 2. Since $|Y| = 3n/4 - 1 < 3(n/4)$, H does not contain a K_4^--factor. Further, $\delta_1(H) = \binom{3n/4-1}{2}$.

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