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Abstract

In this paper we examine embeddings of alternating and symmetric groups into almost simple groups of exceptional type. In particular, we prove that if the alternating or symmetric group has degree equal to 5, or 8 or more, then it cannot appear as the maximal subgroup of any almost simple exceptional group of Lie type. Furthermore, in the remaining open cases of degrees 6 and 7 we give considerable information about the possible embeddings. Note that no maximal alternating or symmetric subgroups are known in the remaining cases.

This is the first in a sequence of papers aiming to substantially improve the state of knowledge about the maximal subgroups of exceptional groups of Lie type.

The maximal subgroups of the finite simple groups have been the object of intense study over the last few decades. For the exceptional groups of Lie type it seems plausible – unlike for example in the classical group case – to have a complete list of all maximal subgroups. If $G = G(q)$ is an exceptional group of Lie type, then all maximal subgroups of G are known, except for an explicit, finite, list of candidates, all of which are almost simple. Indeed, all maximal subgroups are known unless G is one of F_4, E_6, ${}^2E_6$, E_7 and E_8, so we concentrate on these cases.

Techniques for finding maximal subgroups of the larger-rank exceptional groups have until now been largely geometric (for example, [22] and [2] for F_4 and E_6) or relate to algebraic groups (for example, [14, 15, 16]). Independently the author, in a series of papers of which this is the first, and Alastair Litterick [20], introduced methods from modular representation theory to attack the problem. These seem much more suited to the remaining, difficult, open cases of small almost simple groups being embedded in large exceptional groups and offer hope that a complete solution can be obtained. Representation theory is predominantly used to prove that maximal subgroups do not exist, and other methods will need to be used to prove existence and uniqueness of maximal subgroups that do, in fact, exist.

The author's rough estimation is that, from the list of possible isomorphism types of maximal subgroups that have not yet been eliminated or found, these methods can be used to solve 90% of them or more, leaving a small number of possibilities. These methods really apply when the characteristic of the field over which the group of Lie type is defined, say p, divides the order of the potential maximal subgroup H. As an example of their power, one can use them to eliminate $\mathrm{PSL}_2(7)$ as a maximal subgroup of $E_8(3^a)$ for all a in one line, using the tables in [20] as a starting point.

In keeping with this theme of vastly reducing – but not eliminating entirely – the possibilities, we do not prove that there can be no maximal alternating or symmetric subgroups in exceptional groups, but we do prove this in almost all cases. Our strongest theorems are slightly more technical, so we begin with a weaker theorem that serves for many purposes. We should state that no maximal alternating or symmetric subgroup of an exceptional group of

Lie type is known at the moment, and so the next theorem merely lists the possibilities not yet excluded.

THEOREM 1.   *Let G be an almost simple exceptional group of Lie type over a field of characteristic p, and let H be a maximal subgroup of G such that $F^*(H) = \mathrm{Alt}(n)$ for some $n \geq 5$. Then $n = 6$ or $n = 7$, and one of the following holds:*
   (i) *$n = 6$ and (G, p) is one of $(F_4, 3)$, $(F_4, \geq 7)$, $(E_6, \geq 7)$, $(E_7, 5)$, $(E_7, \geq 7)$, $(E_8, 2)$, $(E_8, 3)$, $(E_8, \geq 7)$;*
   (ii) *$n = 7$ and (G, p) is one of $(E_7, 5)$, $(E_8, 3)$, $(E_8, 5)$, $(E_8, 7)$, $(E_8, \geq 11)$.*

In this theorem, E_6 means either E_6 or $^2E_6$, and $\geq p$ means all primes at least p, i.e., the case where $p \nmid |H|$, which includes characteristic 0. This theorem is a summary of more complicated, and stronger, theorems about embeddings of alternating groups into simple exceptional groups of Lie type. In addition, because we will always prove that there exists a positive-dimensional subgroup of the algebraic group fixing certain specific subspaces that the potential maximal subgroup also fixes (normally lines on the minimal or adjoint module), using results from Section 1 the above restrictions also apply to any almost simple group whose socle is an exceptional group, giving the full statement of the theorem above.

Let $\mathbb{G}$ be an exceptional algebraic group, let $\mathscr{X}$ be the set of all maximal positive-dimensional subgroups of $\mathbb{G}$, and let $\mathscr{X}^\sigma$ denote the fixed points under the Steinberg morphism of the σ-stable members of $\mathscr{X}$, which are subgroups of the finite group $G = \mathbb{G}^\sigma$. We prove that, with the exceptions above, H is always contained inside a member of $\mathscr{X}^\sigma$. This is stronger than simply stating that H is not maximal, since there are other potential maximal subgroups of G that could contain H. For example, inside E_6 in characteristic 0, there is a copy of $^2F_4(2)'$ acting irreducibly on the minimal module, and this contains a copy of $\mathrm{PSL}_3(3)$. This subgroup is clearly not maximal, but is not contained in any member of $\mathscr{X}$. In addition, since we understand the set $\mathscr{X}^\sigma$, we can get some handle on the possible embeddings of H into G.

THEOREM 2.   *Let $G = \mathbb{G}^\sigma$ be a simple group of type F_4, E_6, $^2E_6$, E_7 or E_8. If H is a subgroup of G with $F^*(H) \cong \mathrm{Alt}(5)$ then H lies inside a member of $\mathscr{X}^\sigma$.*

For the alternating groups $\mathrm{Alt}(6)$ and $\mathrm{Alt}(7)$ much less is known. We summarize what we do know now. In what follows we denote simple modules by their dimension, distinguishing between modules of the same dimension with an index, so 4_2 is a module of dimension 4. $P(-)$ denotes the projective cover and we delineate between socle layers with a '/' character.

THEOREM 3.   *Let $G = \mathbb{G}^\sigma$ be a simple group of type F_4, E_6, $^2E_6$, E_7 or E_8. Let H be a subgroup of G with $F^*(H) \cong \mathrm{Alt}(6)$, and suppose that H does not lie inside a member of $\mathscr{X}^\sigma$. Then one of the following holds:*
   (i) *$p = 2$ and $G = E_8$, with $F^*(H)$ acting on $L(G)$ with composition factors $8_1^6, 8_2^6, 4_1^{16}, 4_2^{16}, 1^{24}$;*
   (ii) *$p = 3$ and $G = F_4, E_8$;*
   (iii) *$p = 5$ and $G = E_7$, with $F^*(H)$ acting on $V_{\min}$ and on $L(G)$ as*

$$10^{\oplus 4} \oplus 8^{\oplus 2} \quad and \quad 10^{\oplus 2} \oplus 5_1^{\oplus 3} \oplus 5_2^{\oplus 3} \oplus P(8)^{\oplus 3} \oplus 8$$

   respectively, or acting on $V_{\min}$ and $L(G)$ as

$$10_2^{\oplus 3} \oplus 10_3 \oplus (4_1/4_2) \oplus (4_2/4_1) \quad and \quad 10^{\oplus 5} \oplus P(8)^{\oplus 3} \oplus 8$$

respectively;
(iv) $p > 5$ *and* $G = F_4, E_6, {}^2E_6, E_7, E_8$.

THEOREM 4. *Let* $G = \mathbb{G}^\sigma$ *be a simple group of type* F_4, E_6, ${}^2E_6$, E_7 *or* E_8. *Let* H *be a subgroup of* G *with* $F^*(H) \cong \mathrm{Alt}(7)$, *and suppose that* H *does not lie inside a member of* $\mathscr{X}^\sigma$. *Then one of the following holds:*
(i) $p = 3$ *and* $G = E_8$, *with* $F^*(H)$ *acting on* $L(G)$ *with composition factors* $15^4, 13^6, (10, 10^*)^5, 1^{10}$;
(ii) $p = 5$ *and* $G = E_7$, *with* $F^*(H)$ *acting on* $V_{\min}$ *as either* $(10 \oplus 10^*)^{\oplus 2} \oplus 8^{\oplus 2}$ *or* $20 \oplus (4/14) \oplus (14/4^*)$, *or* $G = E_8$, *with* $F^*(H)$ *acting on* $L(G)$ *as*

$$35^{\oplus 4} \oplus 15^{\oplus 4} \oplus 10 \oplus 10^* \oplus (8/6) \oplus (6/8);$$

(iii) $p = 7$ *and* $G = E_8$, *with* $F^*(H)$ *acting on* $L(G)$ *as*

$$35^{\oplus 3} \oplus 21 \oplus 14_1 \oplus 14_2^{\oplus 2} \oplus P(10)^{\oplus 2} \oplus 10 \quad or \quad 35^{\oplus 4} \oplus 14_1^{\oplus 2} \oplus 10^{\oplus 6} \oplus 5^{\oplus 4};$$

(iv) $p > 7$ *and* $G = E_8$, *acting on* $L(G)$ *with composition factors* $35^4, 15^4, 14_1^2, 10, 10^*$.

Once we move past $\mathrm{Alt}(7)$, however, we have complete theorems. For $\mathrm{Alt}(8)$ and above we can prove that every copy lies inside a positive-dimensional subgroup. This includes the difficult case of $\mathrm{Alt}(8) = \mathrm{GL}_4(2)$ inside $E_8(2^n)$, which starts to extend to E_8 the results on defining-characteristic embeddings of groups of rank at least 2 considered for F_4, E_6 and E_7 in work in preparation by the author and collaborators. Extending this theorem to include E_8 is ongoing work of the author.

THEOREM 5. *Let* $G = \mathbb{G}^\sigma$ *be a simple group of type* F_4, E_6, ${}^2E_6$, E_7 *or* E_8. *If* H *is a subgroup with* $F^*(H) = \mathrm{Alt}(n)$ *for* $n \geq 8$, *then* H *lies inside a member of* $\mathscr{X}^\sigma$.

In [20, Section 3.3], Litterick proves that, for $n \geq 10$, if $F^*(H) \cong \mathrm{Alt}(n)$ then H is never maximal in an almost simple exceptional group. We include an alternative proof of the case $n = 10$, $p = 2$, $G = E_8$ in the last section, as we show that H fixes a line on $L(G)$ more easily with a theoretical argument involving Frobenius reciprocity.

In addition, Litterick has proved many other cases for alternating groups, in his more wide-ranging project that included all simple groups other than Lie type in defining characteristic. We refer to [20], as yet not in publication but available on the arXiv, for full details as to the cases proved there, which are more than in his PhD thesis. We have however mostly maintained our own proofs here both for completeness and as our methods are slightly different.

These results include and extend known results on maximal subgroups of exceptional groups. For $p \geq 5$ and $G = F_4$ we recover results of Magaard in [22]. For $G = E_6$ we extend results of Aschbacher in [2], where we remove the following possibilities for maximal subgroups:
– $\mathrm{Alt}(6)$, $p = 2$;
– $\mathrm{Alt}(6)$, $p = 5$;
– $\mathrm{Alt}(7)$, $p = 5$;
– $\mathrm{Alt}(8)$, $p = 2$.
We leave $\mathrm{Alt}(6)$, $p \geq 7$ (or $p = 0$) unresolved.

The method of proof here is as follows: we firstly use traces of semisimple elements to restrict the possible sets of composition factors for the action of H on both the minimal and adjoint modules. This task was already accomplished in [20] when H is not isomorphic to a Lie type group in the same characteristic, so we only have to consider those cases.

Once we have these data, we then use the information about the action of unipotent elements on the minimal and adjoint modules in [11] to build up a picture of the possible restrictions of the minimal and adjoint modules to the subgroup. Often this is enough to prove something like the subgroup fixes a line on one of these two modules, or that the set of composition factors cannot yield an embedding of the subgroup.

We also use a result from [15] that states that for certain unipotent classes, which we call 'generic' classes in the next section, merely containing an element from them and not acting irreducibly on the minimal or adjoint module is enough to guarantee that the subgroup lies in the set $\mathscr{X}$ above. This is where many embeddings do lie, and so it is useful to exclude large numbers of embeddings from consideration early on.

If the group has a cyclic Sylow p-subgroup then we have complete information about the module category for the group, and so we can get very good information on the possible embeddings. Potential embeddings that seem not to be attackable with using the techniques here are listed in Theorems 3 and 4. Other ideas such as using structure constants, which have been used before to success, might be one avenue for these, and of course the characteristic 0 possibilities, as they have been successful before, for example for $\mathrm{Alt}(5)$ [21].

Many of the arguments for $p = 2, 3$ from later sections require use of a computer, however. Our choice of computer program for these tasks is Magma [3]. This is firstly to prove the existence or non-existence of certain modules, and secondly to determine the actions of unipotent elements on these modules. While it might be possible in many cases to perform this analysis without a computer, doing so would substantially increase both the size of the paper and the likelihood of mistakes appearing. The author believes that the probability of there being bugs in Magma that invalidate the results here is lower than the chances of making an error in hundreds of pages of module computations.

Having said that, despite the author's use of a computer to produce various modules, many claims can be checked using the structures of projective indecomposable modules for the small alternating groups in characteristics 2 and 3, that have appeared in the literature.

The structure of this paper is as follows: we begin with a section collating the notation we need, and the preliminary results, including a result on passing from maximal subgroups of the simple group to those of an almost simple group. We then briefly summarize those aspects of the theory of blocks with cyclic defect groups that we need, particularly how the structure of the Brauer tree informs the structure of the projective indecomposable modules. In Section 3 we give information about simple modules for alternating groups, in particular their dimensions and Ext^1, the Brauer trees of $\mathrm{Alt}(n)$ for $p = 5, 7$ and $n \leq 8$, and some useful lemmas for specific groups. The succeeding section classifies the composition factors of large-degree alternating groups on the adjoint module for E_8 in characteristic 2.

After these preliminary sections, Section n considers $\mathrm{Alt}(n)$, for $5 \leq n \leq 10$.

1. Notation and preliminary results

Throughout this paper, to avoid confusion with algebraic groups, we denote by $\mathrm{Alt}(n)$ the alternating group on n letters, and so to remain consistent use $\mathrm{Sym}(n)$ for the symmetric group on n letters.

Although we will remind the reader often, we establish a consistent naming convention for the various permutations we will need in Table 1

Let $\mathbb{G}$ be a simple algebraic group, and let σ be a Steinberg endomorphism of $\mathbb{G}$. The finite group $G = \mathbb{G}^\sigma$ is a finite group of Lie type. We can also think of G as $G(q^\delta)$, a finite group of Lie type defined over the field $\mathbb{F}_{q^\delta}$. In our case $G = \mathbb{G}^\sigma = G(q^\delta)$ is one of ${}^2B_2(q^2)$, ${}^3D_4(q^3)$, $E_6(q)$, ${}^2E_6(q^2)$, $E_7(q)$, $E_8(q)$, $F_4(q)$, ${}^2F_4(q^2)$, $G_2(q)$ and ${}^2G_2(q^2)$, or their universal central extensions.

Since we are examining maximal subgroups, and the maximal subgroups are known for all of the above groups except for those of twisted rank at least four, we assume that G is one of $F_4(q)$, $E_6(q)$, $^2E_6(q^2)$, $E_7(q)$ and $E_8(q)$. (The maximal subgroups of the remaining groups are given in [**25**], which includes references to the original papers.) We will assume that by E_6 and E_7 we mean the simply connected version, so $Z(E_6)$ and $Z(E_7)$ have orders $\gcd(3, p^2 - 1)$ and $\gcd(2, p - 1)$ respectively. (We will remind the reader of this regularly.)

Let $p \mid q$ be a prime, and let k be a field of characteristic p. We specifically do not take k to be algebraically closed, because embeddings of (say) H into $F_4(2)$ produce $\mathbb{F}_2 H$-modules, not modules over the algebraically closed field. This is not important when proving that a subgroup fixes a line on a module, since this is true regardless of the field over which the module is considered, but is important when asking whether, for example, $\mathrm{SL}_2(4)$ stabilizes a 2-space on a module.

If M is a module for a group, $\mathrm{soc}(M)$, the *socle* of M, is the largest semisimple submodule of M, and $\mathrm{top}(M)$ is the largest semisimple quotient of M. Write $\uparrow$ and $\downarrow$ for induction and restriction, and write k_H or 1_H for the trivial module for the group H. If the group H is obvious we will simply write k or 1. The kernel of the map $M \to \mathrm{top}(M)$ is denoted by $\mathrm{rad}(M)$, the *Jacobson radical* of M. Write $P(M)$ for the projective cover of M, the smallest projective module that has M as a quotient. If I is a set of simple modules, the I-*radical* of M is the largest submodule whose composition factors lie in I, and the I-*residual* is the smallest submodule for which the quotient only has composition factors lying in I. By M^* we refer to the dual of M. The I-radical and I-residual are related by the following: the I-radical of M^* is the dual of the quotient by the I-residual of M. In analogy with the notation of a p'-subgroup, I'-radical and I'-residual mean $(\mathrm{Irr}(H) \setminus I)$-radical and $(\mathrm{Irr}(H) \setminus I)$-residual.

We will often use the actions of p-elements in this work, and in particular their Jordan normal form. We use the same notation as in [**11**], so if a p-element g acts with two block of size 2 and one of size 1, we write $2^2, 1$. Note that if g has order p^a then all blocks have size at most p^a. As shorthand, we say that g *acts projectively* on M if all blocks of the action of g on M are of size p^a.

Write $V_{\min}$ for one of the minimal modules for G, namely $L(\lambda_4)$ for F_4, either $L(\lambda_1)$ or $L(\lambda_6)$ for E_6 and $^2E_6$, $L(\lambda_7)$ for E_7 and $L(\lambda_1)$ for E_8. We write $L(G)$ for the simple, non-trivial constituent of the Lie algebra module, which is $L(\lambda_1)$, $L(\lambda_2)$, $L(\lambda_1)$ and $L(\lambda_1)$ respectively. These have the dimensions given in Table 1.

We will remind the reader when $L(G)$ does not have the usual dimension.

The actions of certain reductive subgroups, and all Levi subgroups, on these modules is helpfully tabulated in [**24**]. There are many such subgroups and it is not necessary to reproduce the full list of tables, but we describe some of the more commonly used subgroups in Table 3.

The actions of unipotent elements on these modules are given in [**11**]. From that we can extract much information. For example, we have Tables 4 and 6 of certain actions of elements of orders 3 and 5 on $L(G)$ for E_8. In these we only see the *non-generic* classes: we should

Label	Permutation
t	$(1, 2)(3, 4)$
u	$(1, 2, 3, 4, 5)$
v	$(1, 2, 3, 4)(5, 6)$
w	$(1, 2, 3, 4, 5, 6, 7)$
x	$(1, 2, 3)$
y	$(1, 2, 3)(4, 5, 6)$
z	$(1, 2, 3, 4, 5, 6, 7, 8, 9)$

TABLE 1. *Labels for permutations in alternating groups*

define what this means now. We use the Bala–Carter–Pommerening notation, in particular the precise notation used in [**11**] which will be our main reference for unipotent actions, and this allows us to compare the actions of a unipotent class in different characteristics.

DEFINITION 1.1.   Let p be a prime, let G be an algebraic group in characteristic p, and let C be a unipotent class in G. Let $L(\lambda)$ be a highest-weight module for G. If the Jordan blocks of the action of $u \in C$ on $L(\lambda)$ are the same size as for the same class C in G for all primes $q >> 0$, then C is said to be *generic* at the prime p. Otherwise, C is *non-generic*.

As an example, as we see in [**11**, Table 7], the class $E_7(a_2)$ in E_7 acts on $V_{\min}$ with blocks $18, 16, 10, 8, 4$ for all $p \neq 2, 3, 13, 17$, but for example with blocks $13^4, 4$ for $p = 13$. This means that this class is generic for $V_{\min}$ whenever $p \neq 2, 3, 13, 17$, and non-generic for these primes. Notice that, on $L(G)$, [**11**, Table 8] shows that $E_7(a_2)$ is non-generic for $p = 2, 3, 5, 13, 17, 19$, so being non-generic does depend on the module under consideration.

Our next two results mean that, if H is a subgroup of an exceptional algebraic group G, and H contains a unipotent element of order p from a generic class, then there exists a positive-dimensional subgroup X of G containing H and stabilizing the same subspaces of $V_{\min}$ as H. In particular, if H does not act irreducibly on $V_{\min}$ then H is contained in a known maximal subgroup of G, since the set of maximal positive-dimensional subgroups $\mathscr{X}$ of G are known [**17**], and since $X \neq G$.

This result appears in [**15**, Lemma 1.14].

LEMMA 1.2.   *Let H be a subgroup of an algebraic group $\mathbb{G}$, and let $u \in H$ be a unipotent element of order p. If u is contained in an A_1 subgroup that acts with p-restricted composition*

Group	dim($V_{\min}$)	dim($L(G)$)
F_4	$26 - \delta_{q,3}$	52
E_6	27	$78 - \delta_{q,3}$
E_7	56	$133 - \delta_{q,2}$
E_8	248	248

TABLE 2. *Dimensions of some simple modules for exceptional groups*

Group	Subgroup	Factors on $V_{\min}$	Factors on $L(G)$
F_4	B_4	1000,0001,0000	0100,0001
	$A_2\tilde{A}_2$	(10,10),(01,01),(00,11)	(11,00),(00,11),(10,02),(01,20)
E_6	F_4	0001,0000	1000,0001
	A_1A_5	$(1,\lambda_1),(0,\lambda_4)$	$(2,0),(0,\lambda_1+\lambda_5),(1,\lambda_3)$
E_7	E_6	$\lambda_1,\lambda_6,0^2$	$\lambda_2,\lambda_1,\lambda_6,0$
	A_7	λ_2,λ_6	$\lambda_1+\lambda_7,\lambda_4$
	D_6A_1	$(\lambda_1,1),(\lambda_5,0)$	$(\lambda_6,1),(\lambda_2,0),(0,2)$
E_8	A_1E_7	N/A	$(2,0),(0,\lambda_1),(1,\lambda_7)$
	D_8	N/A	λ_2, λ_7
	A_8	N/A	$\lambda_1+\lambda_8, \lambda_3, \lambda_5$

TABLE 3. *Actions of some common subgroups on $V_{\min}$ and $L(G)$. 'Factors' means the same composition factors as the Weyl modules of these highest weights.*

factors on a module M for G, then there exists a positive-dimensional subgroup $\mathbb{X}$ of $\mathbb{G}$ such that $H \leq \mathbb{X}$, and H and $\mathbb{X}$ stabilize the same subspaces of M. In particular, if $\mathbb{G}$ acts irreducibly on M and H acts reducibly on M, then H is contained in a member of $\mathscr{X}$, and if H is contained in $G = \mathbb{G}^{\sigma}$ then it is contained in a member of $\mathscr{X}^{\sigma}$.

(To get from [**15**, Lemma 1.14] to this result, let X be the subgroup generated by H and the unipotent subgroup U containing $u \in H$ that is constructed in [**15**, Lemma 1.14].)

Now we need to know that such A_1s actually exist for the minimal and adjoint modules. Together with the previous lemma, this means that we can discount generic unipotent classes from consideration.

LEMMA 1.3. *Let p be an odd prime and let G be one of F_4, E_6, E_7 and E_8. Let u be a unipotent element in G of order p. If u belongs to a generic class for either the minimal or adjoint module, then u lies inside an A_1 subgroup whose composition factors on that module are p-restricted.*

Proof. In the case of the adjoint module, we simply use the tables in [**13**], which construct A_1 subgroups above unipotent classes, and whenever u comes from a generic class the composition factors are p-restricted. It remains to consider the minimal module (except for E_8, where the two coincide).

We must construct, for each unipotent class, an algebraic A_1 with p-restricted composition factors, whenever u is in the generic case. Using the embeddings $F_4 < E_6 < E_7$ we can reduce our work by embedding an A_1 inside F_4 into both E_6 and E_7.

We use the tables in [**11**] both to determine for which primes we are in the generic case, and to give a list of the unipotent classes that we will use here. As mentioned earlier this section, if the unipotent class contains blocks of size greater than p then its order is not p, and so we can ignore that class. We also consult [**24**] for the actions of Levi subgroups and irreducible A_1s.

Let $G = F_4$. For classes A_1 and $\tilde{A}_1$ we use the A_1 Levi subgroups, which act as needed. For $A_1 + \tilde{A}_1$ we use the diagonally embedded A_1 inside the $A_1 \tilde{A}_1$ Levi subgroup. For A_2 we use the irreducible A_1 inside the A_2 Levi subgroup.

For the rest of the classes we need $p \geq 5$. For $\tilde{A}_2$ and B_2 we can use the irreducible A_1s inside the $\tilde{A}_2$ and B_2 Levi subgroups. For $A_2 + \tilde{A}_1$ we use the A_1 embedded in the Levi subgroup $A_2 \tilde{A}_1$ acting as $L(2)$ on the A_2 and as $L(1)$ on the A_1 factor. The same subgroup of $\tilde{A}_2 A_1$ deals with u lying in $\tilde{A}_2 + A_1$.

For $C_3(a_1)$, we consider the C_3 Levi subgroup, which acts on the minimal module as the sum of two copies of the natural module $L(100)$, and one copy of the adjoint $L(010)$, which is the exterior square of the natural minus a trivial. If A_1 is embedded as $L(1) \oplus L(3)$ then the exterior square of this is

$$\Lambda^2(L(1)) \oplus \Lambda^2(L(3)) \oplus (L(1) \otimes L(3)) = L(0) \oplus (L(0) \oplus L(4)) \oplus (L(2) \oplus L(4)),$$

yielding the correct action. (Remember that $\Lambda^2(A \oplus B) = \Lambda^2(A) \oplus \Lambda^2(B) \oplus (A \otimes B)$.)

For u lying in $F_4(a_3)$, [**13**, Table 2] suggests we can look inside $C_3 A_1$: this acts as the sum of $010 \otimes 0$ (the exterior square of the natural minus a trivial, tensored by the trivial) and $100 \otimes 1$ (the natural tensored by the natural). Inside here, we take a diagonal A_1 which projects non-trivially along the C_3 and acts as $L(1) \oplus L(3)$ on the natural, as in the previous case of $C_3(a_1)$, but also as $L(1)$ on the A_1 factor. This then acts as $L(1)^{\otimes 2} \oplus (L(3) \otimes L(1)) = L(0) \oplus L(2)^{\oplus 2} \oplus L(4)$ on $100 \otimes 1$ and as $\Lambda^2(L(1) \oplus L(3)) = L(0)^{\oplus 2} \oplus L(2) \oplus L(4)^{\oplus 2}$ minus a copy of $L(0)$ on $010 \otimes 0$, as needed.

For B_3 we need $p \geq 7$. Inside the B_3 Levi subgroup we take an A_1 acting as $L(6)$, i.e., irreducibly, on the natural, and hence as $L(0) \oplus L(6)$ on the 8-dimensional module, giving the correct action.

We now need $p \geq 11$. For C_3 we take the A_1 lying in the C_3 Levi subgroup acting as $L(5)$: since $\Lambda^2(L(5)) = L(0) \oplus L(4) \oplus L(8)$ we get the correct action on the minimal module.

For $F_4(a_2)$, [**13**, Table 2] suggests that as for $F_4(a_3)$ we can look inside $C_3 A_1$: the diagonal A_1 acting as $L(5)$ on the natural for C_3 and the natural on the A_1 factor works, because the $100 \otimes 1$ restricts to $L(4) \oplus L(6)$, and the $010 \otimes 0$ restricts to $L(4) \oplus L(8)$, as needed.

For $F_4(a_1)$ and F_4, we see in [**24**, Table 10] that there are two irreducible A_1s, namely subgroups 7 and 10, that cover these two classes, completing the proof for F_4.

For E_6 we can take the A_1s from F_4 and add a trivial, dealing with many classes. We run through those that are left.

The first such class is $A_2 + A_1$, which requires $p \geq 5$, and lies in the $A_2 A_1$ Levi subgroup. We take an irreducible diagonal A_1 acting as $L(2)$ along the A_2 factor and as $L(1)$ along the A_1 factor. The Levi subgroup acts (up to duality) on $V_{\min}$ with factors $L(10) \otimes L(1)$, four copies of $L(10) \otimes L(0)$, and three copies each of $L(00) \otimes L(1)$ and $L(00) \otimes L(0)$. The restriction to the A_1 clearly yields composition factors of $L(3), L(2)^4, L(1)^4, L(0)^3$, as needed.

The next class is A_4, which requires $p \geq 7$. The A_4 Levi subgroup acts as (up to duality, which is not important for A_1s) three copies of the natural, one of its exterior square, and two trivials. If we embed as A_1 as $L(4)$ then $\Lambda^2(L(4)) = L(2) \oplus L(6)$, and so we get $L(6), L(4)^3, L(2), L(0)^2$ as the factors of the A_1 on the minimal module, as needed.

The next class is $A_4 + A_1$, also needing $p \geq 7$, for which we embed an A_1 diagonally into the $A_4 A_1$ Levi as $L(4) \otimes L(1)$. This acts on $1000 \otimes 1$ as $L(5) \oplus L(3)$, and on $0001 \otimes 0$ as $L(4)$. It acts on $0010 \otimes 0$ as $L(6) \oplus L(2)$, as in the previous case, and so we get the correct factors for covering an element from class $A_4 + A_1$.

There are only two classes left: $D_5(a_1)$ needs $p \geq 11$, and this A_1 can be found inside D_5, acting as $L(7) \oplus L(5) \oplus L(1)$ on the 16-dimensional spin module and $L(6) \oplus L(2)$ on the 10-dimensional natural module. The last one is $E_6(a_1)$, which needs $p \geq 13$, and is covered by subgroup 6 from [**24**, Table 11]. This completes the proof for $G = E_6$.

As with E_6 and F_4, we can use E_6 to exclude many classes for E_7, and deal with only those that are left. However, this still leaves twenty-four classes, and so we want to cut a few of those down before constructing individual groups.

We use the subgroup $F_4 A_1$ for this, which acts with factors $0000 \otimes 3$ and $1000 \otimes 1$. Notice that every generic class of order p in F_4 for the module 1000 is covered by an A_1 acting with p-restricted factors, as we have just proved it, and obviously the same is true for the A_1 factor; therefore at least for $V_{\min}$ we know that every unipotent class of $F_4 A_1$ that is generic is covered by an A_1 subgroup acting with p-restricted composition factors also. (If one is worried that there is a composition factor $L(p-1) \otimes L(1)$ in $V_{\min}$, note that the class is not generic for p, and only for primes larger than p.)

Consulting the first and third columns of Table 38 in [**12**], we get the unipotent classes contained in $F_4 A_1$: the third column contains the classes $(3A_1)''$, $4A_1$, $A_2 + 3A_1$, $(A_3 + A_1)''$, $A_3 + 2A_1$, $D_4(a_1) + A_1$, $A_3 + A_2$, $A_3 + A_2 + A_1$, $D_5(a_1) + A_1$, $D_6(a_2)$, $E_7(a_5)$, $E_7(a_4)$ and $E_7(a_2)$.

This leaves only eleven unipotent classes to cover with A_1s with p-restricted composition factors, which is enough that we can construct the subgroups explicitly. The list of classes left to cover is

$$(A_5)'', \ D_4 + A_1, \ A_4 + A_2, \ A_5 + A_1, \ D_6(a_1), \ A_6, \ D_5 + A_1, \ D_6, \ E_7(a_3), \ E_7(a_1), \ E_7.$$

In the A_5 Levi subgroup one sees $(A_5)''$ as the regular class, and the regular A_1 covering it acts as $L(3) \oplus L(5) \oplus L(9)$ on the exterior cube of the natural, hence has the right properties for $p \geq 11$. Taking this class and tensoring it by an A_1 factor inside the A_5A_2 Levi subgroup provides a module acting with composition factors of dimension 6 (from $L(\lambda_1) \otimes L(0)$ and its dual), $10, 6, 4$ (from $L(\lambda_3) \otimes L(0)$) and $7, 5$ (from $L(\lambda_1) \otimes L(1)$ and its dual), hence covers $A_5 + A_1$.

Inside A_7 a multiplicity-free module for PGL_2 will have a fixed-point-free exterior square, and so we can take A_1s acting as $L(0) \oplus L(6)$ and $L(2) \oplus L(4)$, and these act on $V_{\min}$ as $L(10) \oplus L(6)^{\oplus 2} \oplus L(2)$ and $L(6)^{\oplus 2} \oplus L(4) \oplus L(2)^{\oplus 3}$, containing representatives from classes A_6 and $A_4 + A_2$.

For $E_7(a_1)$ and E_7 we need $p \geq 23$ and $p \geq 29$ respectively, and the maximal A_1s inside E_7 provide the subgroups.

For the rest we need to examine the D_6A_1 maximal-rank subgroup. Consider firstly a copy of SL_2 inside D_6 that acts on the natural as $L(10) \oplus L(0)$. This acts on the spin module as $L(15) \oplus L(9) \oplus L(5)$, at least as long as the characteristic is at least 17. This will cover the unipotent class D_6.

To cover $E_7(a_3)$ we simply take the diagonal subgroup between this A_1 and the A_1 factor. This leaves the 32 alone but tensors the $L(10) \oplus L(0)$ by $L(1)$ to get $L(11) \oplus L(9) \oplus L(1)$, finishing this case.

This also shows how to construct A_1s above $D_4 + A_1$ and $D_5 + A_1$: we take the A_1s lying above the D_4 and D_5 classes, which are known to exist and act as p-restricted composition factors whenever we are in the generic case, since they lie inside the E_6 Levi subgroup, and as with F_4A_1 we tensor with the A_1 in the second factor of D_6A_1: as this A_1 acts as $L(8) \oplus L(0)^{\oplus 3}$ on the natural, we get p-restricted composition factors, as needed. Similarly, the A_1 covering the D_4 class acts as $L(6) \oplus L(0)^{\oplus 5}$ on the natural, and so we get the right composition factors for the diagonal subgroup with the A_1 factor.

The last unipotent class left to cover is $D_6(a_1)$. Now we consider an SL_2 acting on the natural as $L(8) \oplus L(2)$, which lifts to the spin module with factors $L(11), L(9), L(5)$ and $L(3)$, at least for $p \geq 13$. This yields the final class of A_1s needed to prove the result. $\square$

LEMMA 1.4.   *Suppose that H is a subgroup of $G = G(q)$, and suppose that*
– *H stabilizes a 1-space on either $V_{\min}$ or $L(G)$,*
– *H stabilizes a 2-space on $V_{\min}$ for G of type F_4, E_6 or E_7, or*
– *H stabilizes a 3-space on $V_{\min}$ for G of type E_6.*
Then H is contained in a member of $\mathscr{X}^\sigma$.

Class	Action on $L(G)$
$3A_1$	$3^{31}, 2^{50}, 1^{55}$
A_2	$3^{57}, 1^{77}$
$4A_1$	$3^{44}, 2^{40}, 1^{36}$
$A_2 + A_1$	$3^{58}, 2^{20}, 1^{34}$
$A_2 + 2A_1$	$3^{65}, 2^{16}, 1^{21}$
$A_2 + 3A_1$	$3^{70}, 2^{14}, 1^{10}$
$2A_2$	$3^{78}, 1^{14}$
$2A_2 + A_1$	$3^{79}, 2^2, 1^7$
$2A_2 + 2A_1$	$3^{80}, 2^4$

TABLE 4. *Non-generic unipotent classes for E_8 and order 3*

Proof. If H fixes a 1-space on $L(G)$ then H is contained in either a parabolic or semisimple subgroup by Seitz [**23**, (1.3)], and thus is in a member of $\mathscr{X}^\sigma$. If H stabilizes a line on $V_{\min}$ then H is contained in a member of $\mathscr{X}^\sigma$, for example see [**18**, Lemma 2.2]. (For $G = F_4$, H is contained in B_4 or a maximal parabolic, for $G = E_6$, H is contained in F_4 or a D_5 parabolic, and for $G = E_7$, H is contained inside an E_6 or D_6 parabolic, or inside an E_6 or $^2E_6$ subgroup with an automorphism on top.)

For the others, we simply compute the dimension of the subspace stabilizer inside the algebraic group, and note that it is non-zero. For example, the dimension of the stabilizer of a 3-space in $V_{\min}$ for $G = E_6$ is at least

$$78 - (27 + 26 + 25 - 3 - 2 - 1) = 6 > 0.$$

Class	Action on $L(G)$
A_2	$4^2, 3^{54}, 1^{78}$
$A_2 + A_1$	$4^{14}, 3^{30}, 2^{34}, 1^{34}$
$A_2 + 2A_1$	$4^{22}, 3^{14}, 2^{52}, 1^{14}$
$A_2 + 3A_1$	$4^{26}, 3^6, 2^{62}, 1^2$
$2A_2$	$4^{28}, 3^{36}, 1^{28}$
$2A_2 + A_1$	$4^{40}, 3^{12}, 2^{18}, 1^{16}$
$2A_2 + 2A_1$	$4^{44}, 3^4, 2^{28}, 1^4$
A_3	$4^{46}, 2^{10}, 1^{44}$
$A_3 + A_1$	$4^{46}, 2^{24}, 1^{16}$
$A_3 + 2A_1$	$4^{46}, 2^{30}, 1^4$
$A_3 + A_2$	$4^{50}, 3^{10}, 2^6, 1^6$
$A_3 + A_2^{(2)}$	$4^{54}, 3^2, 2^{10}, 1^6$
$A_3 + A_2 + A_1$	$4^{54}, 3^2, 2^{12}, 1^2$
$D_4(a_1)$	$4^{54}, 3^2, 1^{26}$
$D_4(a_1) + A_1$	$4^{54}, 3^2, 2^{10}, 1^6$
$D_4(a_1) + A_2$	$4^{56}, 3^8$
$2A_3$	$4^{60}, 2^4$

TABLE 5. *Unipotent classes for E_8 and order 4*

Class	Action on $L(G)$
$2A_2 + A_1$	$5^{14}, 4^{14}, 3^{23}, 2^{18}, 1^{17}$
$2A_2 + 2A_1$	$5^{18}, 4^{12}, 3^{20}, 2^{20}, 1^{10}$
A_3	$5^{13}, 4^{32}, 1^{55}$
$A_3 + A_1$	$5^{21}, 4^{16}, 3^9, 2^{14}, 1^{24}$
$A_3 + 2A_1$	$5^{25}, 4^{10}, 3^{14}, 2^{14}, 1^{13}$
$A_3 + A_2$	$5^{30}, 4^8, 3^{13}, 2^8, 1^{11}$
$A_3 + A_2 + A_1$	$5^{32}, 4^8, 3^{10}, 2^{10}, 1^6$
$D_4(a_1)$	$5^{29}, 3^{25}, 1^{28}$
$D_4(a_1) + A_1$	$5^{29}, 4^6, 3^{14}, 2^{14}, 1^9$
$D_4(a_1) + A_2$	$5^{36}, 3^{20}, 1^8$
$2A_3$	$5^{38}, 4^{12}, 1^{10}$
A_4	$5^{45}, 1^{23}$
$A_4 + A_1$	$5^{45}, 3, 2^6, 1^8$
$A_4 + 2A_1$	$5^{45}, 3^4, 2^4, 1^3$
$A_4 + A_2$	$5^{46}, 3^5, 1^3$
$A_4 + A_2 + A_1$	$5^{46}, 4^2, 3^2, 2^2$
$A_4 + A_3$	$5^{48}, 4^2$

TABLE 6. *Non-generic unipotent classes for E_8 and order 5*

Having done this, the stabilizer of a subspace W of the appropriate dimension above is therefore positive dimensional. To see σ-stability, we note that W is fixed by $H \leq G$, so W is defined over $\mathbb{F}_q$ and in $V_{\min}^{\sigma}$. $\qquad\square$

This proves non-maximality of simple subgroups of simple groups, but when we move to almost simple exceptional groups we need to be more careful, as $V_{\min}$ is not always stable under outer automorphisms.

The next lemma deals with so-called novelty maximal subgroups. Suppose that G is an almost simple group with socle X, and let M be a maximal subgroup of G not containing X. If $M \cap X$ is a maximal subgroup of X then M is said to be an *ordinary* maximal subgroup, and if $X \cap M$ is not maximal in X then M is said to be a *novelty* maximal subgroup. Notice that, while ordinary maximal subgroups can be easily found from a list of maximal subgroups of X, novelties are harder, and arise when a subgroup H is stabilized by a group of outer automorphisms of X (hence extending to a larger subgroup of G) but any subgroup of X containing H is not stabilized by the outer automorphisms, so it becomes maximal.

The alternating subgroups of exceptional groups considered here never become novelty maximal subgroups.

LEMMA 1.5.   *Let G be an almost simple group with socle an exceptional group of Lie type. Suppose that H is an almost simple subgroup of G such that $F^*(H)$ has abelian outer automorphism group.*
  (i) *If the preimage of $F^*(H)$ in the simply connected form of $F^*(G)$ fixes a line on either the minimal or adjoint module for $F^*(G)$ then H is contained in a member of $\mathscr{X}^{\sigma}$. In particular, H is not a novelty maximal subgroup of G.*
  (ii) *If there exists a positive-dimensional subgroup X of $F^*(G)$ containing $F^*(H)$ and stabilizing the same subspaces of a simple module V for $F^*(G)$ that is G-stable and such that H acts reducibly on V, then H is not a novelty maximal subgroup of G.*
  (iii) *Suppose that G induces a graph automorphism θ on $F^*(G)$, and let $\bar{H}$ denote the preimage of $F^*(H)$ in the simply connected form $\bar{G}$ of $F^*(G)$. If W is an $\bar{H}$-submodule of the minimal module $V_{\min}$ for $\bar{G}$ such that W^{ϕ} is not an $\bar{H}$-submodule of $V_{\min}^{\theta}$ for any $\phi \in \operatorname{Out}(\bar{H})$, then the G-conjugacy class of $F^*(H)$ splits upon restriction to $F^*(G)$. In particular, H is not maximal in G.*

Proof.   The first part follows from the orbit-stabilizer theorem: if G is an almost simple group that does not induce a graph automorphism on $F^*(G)$ then the automorphisms of $G/F^*(G)$ stabilize both the minimal and adjoint modules, and so the dimension of a line stabilizer is positive.

If G does induce a graph automorphism then $L(G)$ is still stabilized, so that is fine, but $V_{\min}$ is swapped with another module of the same dimension: however, $V_{\min} \oplus V_{\min}^{\theta}$ has dimension at most that of G (in the case F_4 and $p = 2$) and so the line stabilizer on that module is still at least 1-dimensional. As $F^*(H)$ fixes a line, and $\operatorname{Out}(F^*(H))$ is abelian, H still fixes a line on $V_{\min} \oplus V_{\min}^{\theta}$, so lies inside a positive-dimensional subgroup of G, hence H lies inside a member of $\mathscr{X}^{\sigma}$.

(Notice that if $H \leq G^{\sigma}$ then while not every line is σ-stable, there is a σ-stable line, so H is contained in a member of $\mathscr{X}^{\sigma}$.)

The second part is easy: the subgroup $\langle X, H \rangle$ is proper in G since it acts reducibly on V and hence H cannot be a novelty maximal subgroup.

For the third part, we note that, in order for $F^*(H)$ to be stabilized by θ, we must have an automorphism $\phi \in \operatorname{Out}(F^*(H))$ such that $V_{\min}^{\theta} \downarrow_H = V_{\min} \downarrow_H^{\phi}$. By assumption this is not true,

so θ does not normalize $F^*(H)$ and $N_G(F^*(H)) = N_{G'}(F^*(H))$ for some proper subgroup G' of G. In particular, $H \leq G'$, and so is not maximal. $\qquad\square$

In order to examine embeddings of alternating groups into exceptional groups, we need to know the potential composition factors of the restrictions of $V_{\min}$ and $L(G)$ to the subgroup. One way to do this is to know the traces of semisimple elements of the algebraic group, i.e., the possible Brauer characters of the restriction. We end up using traces of elements of order up to 15 (as there are elements of order 15 in $\text{Alt}(8)$) but it would be impractical to list these here. We simply list the integral traces of elements of orders 2, 3 and 5, since this is enough for many purposes. The others can be downloaded from the author's webpage, or computed with Litterick's program in [**19**].

LEMMA 1.6.   *The traces of semisimple elements of orders 2, 3 and 5, whose entries are integers, on $V_{\min}$ and $L(G)$ respectively, are as follows.*

Group	Order	Trace on $V_{\min}$	Trace on $L(G)$
F_4	2	2, -6	-4, 20
	3	-1, 8	-2 or 7, 7
	5	1	2
E_6	2	3, -5	-2, 14
	3	0, 9	-3 or 6, 15
	5	2	3
E_7	2	8, -8	5, 5
	3	-25, -7, 2, 20	52, 7, -2 or 7, 34
	5	6	8
E_8	2	N/A	-8, 24
	3	N/A	-4, 5, 14, 77
	5	N/A	-2, 3, 23

Since we are aiming to prove that a module has a trivial submodule or quotient, we need a result that will guarantee such an outcome. A slightly weaker version of this has been considered before, for example [**15**, Lemma 1.2] or [**20**, Proposition 3.6]. We start with a definition.

DEFINITION 1.7.   Let G be a finite group, k be a field, and let M be a kG-module. Suppose that, for every composition factor V of M, $H^1(G, V) = H^1(G, V^*)$. (This is true, for example, if there exists an automorphism swapping simple modules and their duals.) Suppose further that $H^1(G, k) = 0$, i.e., $O^p(G) = G$. Write $\text{cf}(M)$ for the multiset of composition factors of M. The *pressure* of M is the quantity

$$\sum_{V \in \text{cf}(M)} \left(H^1(G, V) - \delta_{V,k} \right).$$

The pressure of a module is an indicator of how easy it is to hide trivial composition factors in the middle of it. Before we prove this we need the following lemma, which gives some indication that the pressure of a module is an interesting invariant.

LEMMA 1.8.   Let $G = O^p(G)$ be a finite group, k be a field, and let M be a kG-module. Suppose that, for every composition factor V of M, $H^1(G, V) = H^1(G, V^*)$.
   (i) If M is a module with negative pressure, then M has a trivial submodule and a trivial quotient.
   (ii) Suppose that M contains trivial composition factors, but no trivial submodules. If M has pressure n then $\dim(H^1(G, M)) \leq n$.

Proof.   (i) is known, for example [20, Proposition 3.6].
   For (ii), let M' be the module with no trivial submodules obtained from M by extending by k^m, where $m = \dim(H^1(G, M))$, and suppose that M has pressure less than m, so that M' has negative pressure. The module $(M')^*$ still has negative pressure but no trivial quotient, a contradiction.   □

Even better than modules of negative pressure, we have the following result on pressures, which helps in eliminating a variety of cases from consideration.

PROPOSITION 1.9.   Let $G = O^p(G)$ be a finite group, k be a field, and let M be a kG-module. Suppose that, for every composition factor V of M, $H^1(G, V) = H^1(G, V^*)$. Suppose that M contains trivial composition factors, but no trivial submodule or quotient.
   If M has pressure n then there is no subquotient A of M with pressure greater than n or less than $-n$. In particular, if M has a trivial composition factor but has non-positive pressure, then M has a trivial submodule or quotient, and if M has a composition factor V whose 1-cohomology has dimension greater than the pressure of M, then M has a trivial submodule or quotient.

Proof.   Let A be a subquotient of M, and let B and C be the other two factors involved in M, i.e., B is a submodule of M and C a quotient of M whose kernel contains B and such that the surjective map $M/B \to C$ has kernel isomorphic to A. If A has pressure greater than n then at least one of B and C must have negative pressure, without loss of generality (take duals) B. Since B has negative pressure, it has a trivial submodule, a contradiction. Similarly, if A has pressure less than $-n$, then one of B and C has pressure at most n, say B again. Since the pressure of C is less than $-n$, it has a trivial submodule of dimension at least $n + 1$ by repeated application of Lemma 1.8. However, $H^1(G, B)$ has dimension at most n, again by Lemma 1.8, and so we cannot fit this trivial submodule from C on top of B without getting a trivial submodule in M.   □

Modules M of pressure 1 are fairly easy to characterize: ignoring non-trivial composition factors with no 1-cohomology, the structure of M is uniserial, alternating between non-trivial and trivial modules. (Of course, M need not be uniserial, but ignoring non-trivial composition factors with no 1-cohomology it is. What this means is that there is no subquotient $k \oplus k$ and no subquotient $V \oplus W$ with $H^1(G, V), H^1(G, W) > 0$.)
   The reason for the name pressure is that modules of low pressure will have socle structures that look long and thin (close to uniserial) and as the pressure increases they tend to get squashed down.

Suppose that H is a subgroup of a finite group G, and let M be a kG-module. Frobenius reciprocity gives us information about how to relate M and $M \downarrow_H$. Recall that Frobenius reciprocity states that if N is a kH-module, then

$$\operatorname{Hom}(N, M \downarrow_H) = \operatorname{Hom}(N \uparrow^G, M).$$

The next result follows immediately from this.

LEMMA 1.10. *Let G be a finite group and let H be a subgroup of G. Let k be a field and let M be a kG-module. Write N for the permutation module of G on the cosets of H. If $\alpha \in M$ is fixed by H then there is a module homomorphism $\phi : N \to M$ whose image contains α.*

The next result finds singular subspaces of the natural module for orthogonal and symplectic groups preserved by a particular subgroup. It comes in handy when analysing copies of $\operatorname{Alt}(5)$ in the D_6 parabolic of E_7 in characteristic 3, at the end of Section 5.

LEMMA 1.11. *Let $\mathbb{F}$ be a field, let H be a finite group, and let H act on an $\mathbb{F}$-vector space V of dimension n preserving a bilinear form f. If -1 is a sum of m squares in $\mathbb{F}$ then H stabilizes a totally isotropic n-dimensional subspace of $V^{\oplus(m+1)}$.*

Consequently, if H is a subgroup of an orthogonal or symplectic group in positive characteristic, and the action of H on the natural module has a submodule isomorphic with three copies of a given module of dimension n, then H stabilizes a totally isotropic subspace of the natural module of dimension n, and hence lies inside a parabolic subgroup of the orthogonal or symplectic group.

Proof. If $\lambda_1, \dots, \lambda_m \in \mathbb{F}$, then let H act on the diagonal subspace $\{(v, \lambda_1 v, \dots, \lambda_m v) : v \in V\}$: the restriction of f to this subspace has norm $(1 + \lambda_1^2 + \cdots + \lambda_m^2)(v, w)$, for $v, w \in V$. If $\lambda_1^2 + \cdots + \lambda_m^2 = -1$ then this means the form vanishes, and we get the result.

If the characteristic of $\mathbb{F}$ is p then we claim that -1 is always the sum of two squares in $\mathbb{F}_p$: to see this, note that if non-squares are not the sum of two squares, then the squares form a subfield of $\mathbb{F}_p$.

The consequence arises because the stabilizers of totally isotropic subspaces are parabolic subgroups. $\square$

We often know that a subgroup H of G is contained in a member of $\mathscr{X}^\sigma$, and we want to find all possible embeddings. We of course can run through all members of $\mathscr{X}^\sigma$, but some of them, like the normalizer of a torus, are maximal while the quotient by the connected component of the identity could contain a copy of our simple group, i.e., X/X^0 is not soluble. The next result states that, if an alternating group H is contained inside a member of $\mathscr{X}^\sigma$ then H is contained inside a *connected* parabolic or reductive subgroup, rather than inside a subgroup like the normalizer of a torus, where H lies inside the Weyl group. In this lemma we leave out the case where $G = E_7$, and $H = \operatorname{PSU}_3(3)$ in all characteristics and $\operatorname{SL}_2(8)$ in characteristic 2, but this does not mean that these are genuine counterexamples, merely that our methods do not immediately cover it, and the author's later results do not need these specific instances of the lemma. (This improves on [**20**, Proposition 3.10].)

LEMMA 1.12. *Let H be a simple subgroup of a member X of $\mathscr{X}$. Either there exists some member Y of $\mathscr{X}$ such that $H \leq Y^0$, or $G = E_7$ and $H = \operatorname{PSU}_3(3)$, or $G = E_7$, $p = 2$, and $H = \operatorname{SL}_2(8)$.*

Proof. In [**17**] all maximal subgroups of positive dimension are found, and so it is easy to see which members of $\mathscr{X}$ have X/X^0 insoluble. If $G = G_2$ or $G = F_4$ then all subgroups X/X^0, which are subgroups of the Weyl group, are soluble, so the result holds.

If $G = E_6$ then the only maximal subgroup X with X/X^0 insoluble is the normalizer of the torus, and as the Weyl group has $\mathrm{PSU}_4(2)$ of index 2, we have that H is $\mathrm{PSU}_4(2)$. If $p = 2$ then H is a subgroup of $N_G(T)$, and an easy calculation shows that H acts on $V_{\min}$ as $1 \oplus (6/14/6)$, so H is contained in a line stabilizer. Outside of characteristic 2, $N_G(T)$ is a non-split extension and there is no such subgroup H of it. (This is easy to check by computer for some prime p, and the extension to all primes is formal.)

For $G = E_7$, we can have that $X = N_G(T)$ or $X = A_1^7.\mathrm{PSL}_2(7)$. In the first case, the Weyl group is $2 \times \mathrm{Sp}_6(2)$, and by [**20**, Lemma 3.9] we reduce to the case where $H = \mathrm{PSL}_2(8), \mathrm{PSU}_3(3), \mathrm{Sp}_6(2)$.

If $H = \mathrm{Sp}_6(2)$, then again there is copy of H inside $N_G(T)$ unless $p = 2$, in which case H is a subgroup of $N_G(T)$, and acts on $V_{\min}$ as

$$1/(6,1,14)/(6,6)/(1,14)/6/1,$$

so it fixes a line on $V_{\min}$ and therefore lies inside an E_6 parabolic or D_6 parabolic, as needed. (Since the 1-cohomology of the reflection representation is 0 for $p = 3, 5, 7$ by an easy computer calculation, there is only one class of complements to the torus in $N_G(T)$.) The same is not true for $\mathrm{PSU}_3(3)$ and $\mathrm{SL}_2(8)$ though, which have 1-dimensional 1-cohomology on this module in characteristic 3.

If $H = \mathrm{SL}_2(8)$ then the non-split extension of T by $\mathrm{Sp}_6(2)$ restricts to a non-split extension of T by H, so H is not a subgroup of $N_G(T)$ for p odd.

We leave $\mathrm{PSU}_3(3)$, and consider the subgroup $X = A_1^7.\mathrm{PSL}_2(7)$, which can be thought of as $A_1 \wr \mathrm{PSL}_2(7)$ with $\mathrm{PSL}_2(7)$ acting on seven points. The conjugacy classes of complements in a permutational wreath product are in bijection with the homomorphisms from the point stabilizer to the base group, in this case maps from $\mathrm{Sym}(4)$ to $\mathrm{SL}_2(q)$. Of course, there are exactly two of these, and both maps restrict to maps from $\mathrm{Sym}(4)$ to a maximal torus of $\mathrm{SL}_2(q)$, say T_1. By the description in [**10**] – but see [**9**, p.208] for a clearer description – the two conjugacy classes of complements in $A_1 \wr \mathrm{PSL}_2(7)$ intersect the two classes of complements in $T_1 \wr \mathrm{PSL}_2(7)$, and hence if $H = \mathrm{PSL}_2(7)$ then H can be chosen to lie in $N_G(T)$, which has already been dealt with. This completes the proof for E_7.

Finally, let $G = E_8$. If H is contained in $N_G(T)$ then we are done by [**20**, Lemma 3.9], so we may assume that H is contained in $A_1 \wr \mathrm{AGL}_3(2)$. There are two classes of simple subgroup of this, both $\mathrm{PSL}_2(7)$ of course, one transitive and one fixing a point. The one fixing a point centralizes an A_1 subgroup and lies in the reductive subgroup $A_1 E_7$. The transitive one has point stabilizer a Frobenius group of order 21, so again there are exactly two homomorphisms from this to A_1, both of which restrict to homomorphisms to T_1, and so we again lie inside $N_G(T)$, so inside X^0 for some $X \in \mathscr{X}$. This completes the proof. $\square$

We end this section with one obvious, but useful, result, which can be found in, for example [**8**, Proof of Lemma 4.2]. The proof simply uses the fact that the two classes of involutions in E_8 in odd characteristic have centralizers $A_1 E_7$ and D_8.

LEMMA 1.13. *Let H be a perfect subgroup of $G = E_8$ in characteristic $p \neq 2$, and suppose that $C_G(H)$ contains an involution. Either $H \leq D_8$ or $H \leq A_1 E_7$. If H has no representation of dimension 2 over $\overline{\mathbb{F}}_p$ then H is contained in either E_7 or D_8.*

2. *Blocks with cyclic defect group*

In this section we will summarize the theory of blocks with cyclic defect groups, at least as much as is needed for our purposes. More details can be found in, for example [**1**] and [**5**]. This includes the case where the Sylow p-subgroup of a finite group G is cyclic.

Throughout this section, G is a finite group and k is an algebraically closed field of characteristic p, B is a block of kG with cyclic defect group D. The number of simple B-modules is e, and $(|D| - 1)/e$, an integer, is the *exceptionality* of B. If D is a Sylow p-subgroup then $e = |N_G(D)|/|C_G(D)|$. **We will assume that $e > 1$**.

To B we may associate a planar-embedded tree, the *Brauer tree* of B (technically it is merely a ribbon graph, but a planar-embedded tree is easier to envisage), whose edges are labelled by the simple B-modules, and whose vertices are labelled by the ordinary characters of B. If the exceptionality of B is 1, then any two different ordinary irreducible characters in B have different p-modular reductions, but if the exceptionality is $n > 1$ then there is a unique p-modular reduction that occurs n times amongst all p-modular reductions of irreducible characters, with all others occurring with multiplicity 1. The n vertices of the Brauer tree that correspond to characters with the same p-modular reduction are identified and referred to as the *exceptional vertex*.

If χ is an irreducible ordinary character of B then the Brauer characters that are constituents of the p-modular reduction of χ occur with multiplicity 1. The edges incident to a given vertex of the Brauer tree label the composition factors of the p-modular reduction of the corresponding ordinary character.

We need to describe the embedding of the tree now. We order the vertices so that M and N appear next to each other in clockwise order around some mutually incident vertex if and only if $\mathrm{Ext}^1(M, N) = k$, and in all other cases $\mathrm{Ext}^1(M, N) = 0$.

In our case of alternating groups all Brauer trees are lines, and so the issue of the cyclic ordering of edges around the vertices is not important. However, for arbitrary simple groups the Brauer tree can be very complicated, and it is for this reason that we have defined the general situation.

We now explain how to use the tree to construct the projective cover of each simple module in B. If S is a simple B-module, corresponding to an edge in the tree with incident vertices labelled by χ and ϕ, then $\chi + \phi$ is the projective character of S. Furthermore, the projective cover $P(S)$ has the following structure:

The top and socle of $P(S)$ are both S. Removing both of these, the module splits as the direct sum of two (possibly one is the zero module) uniserial modules M and N, corresponding to the two incident vertices χ and ϕ respectively.

Write n for the exceptionality of χ if it is exceptional, and $n = 1$ otherwise. Starting from S, write the edges appearing in clockwise order in n complete revolutions around the vertex χ. For example, if there are four edges, $S = S_1$, S_2, S_3 and S_4 around χ, and $n = 2$, we get the list

$$S_1, S_2, S_3, S_4, S_1, S_2, S_3, S_4.$$

Delete the first copy of S from the start: these are the radical layers of the uniserial module M. The same process around ϕ produces N, and $P(S)$ has structure

$$S/(M \oplus N)/S.$$

Notice that if the Brauer tree is a star (i.e., all but one vertex has valency 1) and the central vertex is exceptional (if there is such a vertex), then all projective modules are uniserial. If G is p-soluble then the Brauer tree of B is always a star with exceptional node in the centre.

The quotients of the projective modules give many indecomposable modules in B, and in general, using the Green correspondence and the structure of the Brauer tree of blocks of

p-soluble groups, it is easy to see that there are $(|D| - 1)e$ non-isomorphic indecomposable modules in B. Hence it is possible to list them all, and in particular know when we have all of them.

If B is a block with Brauer tree a line, and with simple modules $S_1, S_2, \ldots, S_e$, listed in order along the line, and with exceptionality 1, then the structure of the projectives is

$$P(S_i) = S_i/(S_{i-1}, S_{i+1})/S_i,$$

for $1 < i < e$, and

$$P(S_1) = S_1/S_2/S_1, \qquad P(S_e) = S_e/S_{e-1}/S_e.$$

This yields obvious indecomposable modules S_i, $P(S_i)$, S_i/S_{i-1} (and its dual) and $S_i/S_{i-1}, S_{i+1}$ (and its dual) for all appropriate i. The remaining simple modules (up to duality) have two socle layers, with socle

$$S_i, S_{i+2}, S_{i+4}, \ldots, S_{i+2m}$$

and top

$$S_{i+1}, S_{i+3}, \ldots, S_{i+2m\pm1},$$

for some m so that all indices are at most e. To construct these modules, we generalize in the obvious way the construction of the module $(S_1, S_3)/(S_2, S_4)$ obtained by taking the sum of $(S_1, S_3)/S_2$ and S_3/S_4 and taking the kernel of the appropriate diagonal homomorphism onto S_3. This process constructs all non-projective indecomposable modules, up to duality.

As our modules in the future will have many composition factors, we will suppress the brackets between each socle layer to save space, and for example write $S_1, S_3/S_2, S_4$ for the module above.

We quickly count the number of modules so obtained. We have e projective modules, e simple modules, and for each $1 \le i \le e$, exactly $2(e - i)$ non-simple indecomposable modules, the above modules and their duals. This yields

$$2e + 2\sum_{i=1}^{e}(e - i) = 2e + e(e - 1) = e(e + 1) = (|D| - 1)e,$$

as needed.

If the Brauer tree of B is a line, but this time the exceptionality of B is 2 and the exceptional vertex is at the end of the Brauer tree of B, then we have exactly the same indecomposable modules as above, except that $S_i = S_{e-i+1}$ for all i.

The Brauer tree of any block of any alternating group with cyclic defect group is always a line, with exceptionality 1 or 2, and if it is 2 then the exceptional vertex lies at the end of the tree, so this covers all cases.

To see that the Brauer trees for alternating groups are lines with exceptionality 1 or 2, If B is a block of a symmetric group with cyclic defect group, then there are p ordinary characters in B, and they are all real (all characters of symmetric groups are real), so the Brauer tree of B must be a line, and the exceptionality is 1. The theory of so-called 'foldings' given by Feit in [**6**], shows that the Brauer tree of a block of an alternating group must be a folding of a line, which is a line with exceptionality 1 or 2, and with exceptional node (if it exists) at one end of the tree.

One final point to make is the following lemma.

LEMMA 2.1.   *Let B be a block with cyclic defect group, whose Brauer tree is a line, Suppose that the exceptionality of B is 1, or it is 2 and the exceptional vertex has valency 1. If S is a simple B-module and M is an indecomposable module with S as a composition factor but*

with no submodule or quotient isomorphic to S, then $M = P(T)$ for some T, and T and S label edges that share a vertex of the Brauer tree. In particular, if S is the trivial module then M is uniquely determined, and has the structure $T/1, U/T$ for some simple B-module U.

Since we often want to prove that $V_{\min}$ has a trivial submodule or quotient, such a lemma is of tremendous importance.

3.   Modules for alternating groups

In this section we summarize some information about modules for alternating groups. We begin by giving in Table 7 all simple modules in all characteristics for all alternating groups $\mathrm{Alt}(n)$ for $5 \le n \le 9$, and for $p = 2, 3$. The first row is the principal block, the second row is modules in the second block, and so on. The only exception to this is when we write 'a and b', where these are two projective simple modules (hence appearing in different blocks) and we have done so simply to save space. Here, the outer automorphism of $\mathrm{Alt}(n)$ induced by the transposition $(1, 2)$ in the symmetric group swaps all simple modules that are not self dual with their duals, and swaps i_1 and i_2 for all i apart from 4_1 and 4_2 for $\mathrm{Alt}(6)$ in characteristic 2.

We also need to describe 1-cohomology for these simple modules if we want to apply Proposition 1.9.

PROPOSITION 3.1.   Let $5 \le n \le 9$, and $p = 2, 3$. If M is a simple module for $\mathrm{Alt}(n)$ in characteristic p then $H^1(\mathrm{Alt}(n), M) = 0$ unless M appears in Table 8. Here, all 1-cohomologies are 1-dimensional unless the module is in bold, in which case it is 2-dimensional.

For $p = 5$ and $p = 7$, the Sylow p-subgroups are cyclic and so we can simply describe the Brauer trees. If there is an exceptional node we represent it by filling in the node. By χ_i we mean an ordinary character of degree i. We start with $p = 5$.

For $\mathrm{Alt}(5)$, we have a principal block and a single projective simple module of dimension 5.

Group	$p = 2$	$p = 3$
$\mathrm{Alt}(5)$	$1, 2_1, 2_2$	$1, 4$
	4	3_1 and 3_2
$\mathrm{Alt}(6)$	$1, 4_1, 4_2$	$1, 3_1, 3_2, 4$
	8_1 and 8_2	9
$\mathrm{Alt}(7)$	$1, 14, 20$	$1, 10, 10^*, 13$
	$4_1, 4_2, 6$	$6, 15$
$\mathrm{Alt}(8)$	$1, 4, 4^*, 6, 14, 20, 20^*$	$1, 7, 13, 28, 35$
	64	21
		45 and 45^*
$\mathrm{Alt}(9)$	$1, 8_1, 8_2, 20, 20^*, 26, 78$	$1, 7, 21, 35, 41$
	$8_3, 48, 160$	$27, 189$
		162

TABLE 7. Simple modules in characteristics 2 and 3 for alternating groups

$$\circ \overset{1}{\rule{2cm}{0.4pt}} \circ \overset{3}{\rule{2cm}{0.4pt}} \bullet$$
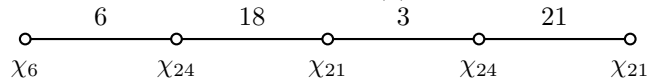$$\chi_1 \qquad\qquad \chi_4 \qquad\qquad \chi_3$$
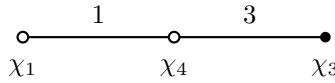
For Alt(6) there is the principal block with the following Brauer tree, and three projective simple modules, 5_1, 5_2 (not permuted by the Sym(6) outer automorphism, but are by the other two) and 10.
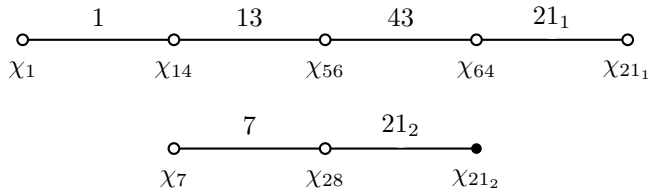
$$\circ \overset{1}{\rule{2cm}{0.4pt}} \circ \overset{8}{\rule{2cm}{0.4pt}} \bullet$$
$$\chi_1 \qquad\qquad \chi_9 \qquad\qquad \chi_8$$
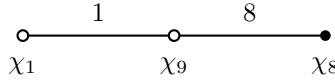
For Alt(7) there is the principal block with the following Brauer tree, and four projective simple modules, 10, 10^*, 15 and 35.

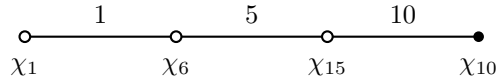$$\circ \overset{1}{\rule{1.5cm}{0.4pt}} \circ \overset{6}{\rule{1.5cm}{0.4pt}} \circ \overset{8}{\rule{1.5cm}{0.4pt}} \circ \overset{13}{\rule{1.5cm}{0.4pt}} \circ$$
$$\chi_1 \qquad\quad \chi_7 \qquad\quad \chi_{14} \qquad\quad \chi_{21} \qquad\quad \chi_{13}$$
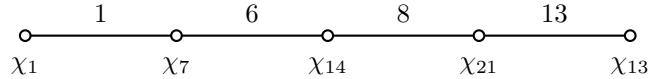
We also need the two dual faithful blocks of $3 \cdot \mathrm{Alt}(7)$, which have the same structure.

$$\circ \overset{6}{\rule{1.5cm}{0.4pt}} \circ \overset{18}{\rule{1.5cm}{0.4pt}} \circ \overset{3}{\rule{1.5cm}{0.4pt}} \circ \overset{21}{\rule{1.5cm}{0.4pt}} \circ$$
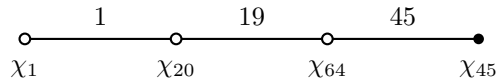$$\chi_6 \qquad\quad \chi_{24} \qquad\quad \chi_{21} \qquad\quad \chi_{24} \qquad\quad \chi_{21}$$

For Alt(8), we have the principal block and a non-principal block with the two trees below, plus five projective simple modules, 20, 35, 45, 45^* and 70.

$$\circ \overset{1}{\rule{1.5cm}{0.4pt}} \circ \overset{13}{\rule{1.5cm}{0.4pt}} \circ \overset{43}{\rule{1.5cm}{0.4pt}} \circ \overset{21_1}{\rule{1.5cm}{0.4pt}} \circ$$
$$\chi_1 \qquad\quad \chi_{14} \qquad\quad \chi_{56} \qquad\quad \chi_{64} \qquad\quad \chi_{21_1}$$

$$\circ \overset{7}{\rule{2cm}{0.4pt}} \circ \overset{21_2}{\rule{2cm}{0.4pt}} \bullet$$
$$\chi_7 \qquad\qquad \chi_{28} \qquad\qquad \chi_{21_2}$$

We now turn to $p = 7$. Of course, we only need to consider Alt(7) and Alt(8). For Alt(7), apart from the principal block we have four projective simple modules, 14_1, 14_2 (not swapped by the outer automorphism), 21 and 35.

$$\circ \overset{1}{\rule{1.5cm}{0.4pt}} \circ \overset{5}{\rule{1.5cm}{0.4pt}} \circ \overset{10}{\rule{1.5cm}{0.4pt}} \bullet$$
$$\chi_1 \qquad\quad \chi_6 \qquad\quad \chi_{15} \qquad\quad \chi_{10}$$

For Alt(8), apart from the principal block we have nine projective simple modules, namely 7, 14, 21_1, 21_2, 21_2^*, 28, 35, 56 and 70, with the outer automorphism swapping 21_2 and 21_2^*.

$$\circ \overset{1}{\rule{1.5cm}{0.4pt}} \circ \overset{19}{\rule{1.5cm}{0.4pt}} \circ \overset{45}{\rule{1.5cm}{0.4pt}} \bullet$$
$$\chi_1 \qquad\quad \chi_{20} \qquad\quad \chi_{64} \qquad\quad \chi_{45}$$

Group	$p = 2$	$p = 3$
Alt(5)	2_1, 2_2	4
Alt(6)	4_1, 4_2	**4**
Alt(7)	14, 20	10, 10^*, **13**
Alt(8)	6, 14, 20, 20^*	13, 35
Alt(9)	20, 20^*, **26**, 78	7, 35, 41

TABLE 8. *Simple modules with non-trivial 1-cohomology (bold indicates 2-dimensional)*

To end this section, we give a couple of facts about specific modules for $\mathrm{Alt}(5)$ and $\mathrm{Alt}(8)$ in characteristic 2.

LEMMA 3.2. *Let M be a module for $H \cong \mathrm{Alt}(5)$ in characteristic 2, and suppose that M has no trivial submodules or quotients. If $u \in H$ is an involution, suppose that u acts on M with exactly a Jordan blocks of size 1, and there are exactly $a + 2b$ trivial composition factors in M. There are at least $2a + 3b$ composition factors of dimension 2 in M.*

Proof. Firstly, the projective cover $P(2_i)$ has the form $2_i/1/2_{3-i}/1/2_i$, and since we may assume that H does not fix a line or hyperplane on M, it is a submodule of copies of $P(2_1)$ and $P(2_2)$, together with copies of 4, which break off as they are projective. If M has five socle layers then there must be a projective summand, and so we write M as $U \oplus W$, where U has no projective summands and W is projective. Notice that U has at most three socle layers, since it cannot have five by above, and if it had four then it would have a trivial quotient.

For U not to fix a line or hyperplane, U must have at least twice as many 2-dimensional factors as trivials, so $2a$. For W, $P(1)$ is not a summand, and so we have copies of $P(2_1)$ and $P(2_2)$, which have three 2_is for each trivial composition factor. This proves the result. □

LEMMA 3.3. *Let $G = \mathrm{Alt}(8)$ and k be a field of characteristic 2. Suppose that M is a kG-module with no composition factors of dimension 20. If M has no trivial submodules or quotients, and M has n trivial composition factors, then M also has composition factors of either $14^n, 6^{n+1}$ or $14^{n+1}, 6^n$.*

Proof. Since $\mathrm{Ext}^1(1,4) = \mathrm{Ext}^1(1,4^*) = 0$, we can remove any 4s and 4^*s from the top and bottom of M, so that M is a submodule of a sum of $P(6)$ and $P(14)$.

Of course, we can perform the same reductions to the top as well, so we can assume that this submodule has top consisting entirely of 6s and 14s. We therefore examine the largest submodules of $P(14)$ and $P(6)$ containing no 20s or 20^*s. For $P(14)$ this has a single trivial composition factor, and the smallest submodule containing it with no trivial quotient is

$$14/1, 6/14,$$

which satisfies the statement, and for $P(6)$ the module has exactly two trivial composition factors, and so we need to examine this. The smallest submodule containing both trivials and with no trivial quotient is given by the following diagram.
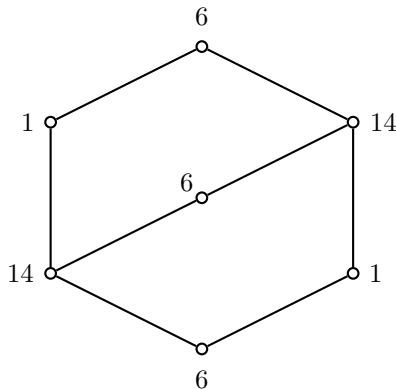
Since there are no uniserial modules $6/1/6$ or $14/1/6$, the statement clearly holds for submodules with a single trivial factor. As M contains a diagonal submodule of one of these as a submodule, the statement holds for M. $\qquad\square$

4. Alternating subgroups in positive-dimensional subgroups

In this section we classify the possible composition factors of $H \cong \mathrm{Alt}(n)$ on $V_{\min}$ and $L(G)$ for $G = F_4, E_6, E_7, E_8$ in characteristic 2, for $n = 7, 8, 9$. This is particularly useful when attacking $\mathrm{Alt}(8)$ and $\mathrm{Alt}(9)$, because there are many possible sets of composition factors (i.e., consistent with the traces of semisimple elements) that do not actually occur. We establish the base case of the induction in Theorem 4 by showing that every $\mathrm{Alt}(7)$ occurs in a positive-dimensional subgroup, hence easily classifiable. We then restrict the composition factors for $\mathrm{Alt}(8)$ and $\mathrm{Alt}(9)$ to $\mathrm{Alt}(7)$ to cut down substantially on the amount of work we have to do in later sections to prove Theorem 5.

PROPOSITION 4.1. *Let G be one of F_4, E_6, E_7 and E_8 in characteristic 2, and let $H \cong$ $\mathrm{Alt}(7)$.*

(i) *If $G = F_4$ then the possible composition factors of H on $V_{\min}$ and $L(G)$ are as follows:*

$V_{\min}$	$L(G)$
$6, (4, 4^*)^2, 1^4$	$14, 6^3, (4, 4^*)^2, 1^4$
$14, 6^2$	$14, 6^3, (4, 4^*)^2, 1^4$

(ii) *If $G = E_6$ then the possible composition factors of H on $V_{\min}$ and $L(G)$ are as follows:*

$V_{\min}$	$L(G)$
$6, (4, 4^*)^2, 1^5$	$14, 6^4, (4, 4^*)^4, 1^8$
$14, 6^2, 1$	$14^2, 6^5, (4, 4^*)^2, 1^4$
$15, 6', 6'$	$20, 14, 6^4, (4, 4^*)^2, 1^4$

(*The first two are embeddings of $\mathrm{Alt}(7)$ into E_6, the last is $3 \cdot \mathrm{Alt}(7) \le E_6$.*)

(iii) *If $G = E_7$ then the possible composition factors of H on $V_{\min}$ and $L(G)$ are as follows:*

$V_{\min}$	$L(G)$
$6^2, (4, 4^*)^4, 1^{12}$	$14, 6^6, (4, 4^*)^8, 1^{19}$
$14^2, 6^4, 1^4$	$14^4, 6^9, (4, 4^*)^2, 1^7$
$6^8, 4, 4^*$	$14^8, 6, 1^{15}$

(iv) *If $G = E_8$ then the possible composition factors of H on $L(G)$ are as follows:*

$L(G)$
$14, 6^{10}, (4, 4^*)^{16}, 1^{46}$
$14^8, 6^{17}, (4, 4^*)^2, 1^{18}$
$20^4, 14^4, 6^8, (4, 4^*)^7, 1^8$
$20^4, 14^{10}, 6^2, 4, 4^*, 1^8$

Proof. There is no embedding of $H \cong \mathrm{Alt}(7)$ into subgroups of type A_1, A_2, B_2 and G_2, and so if we see these factors in an algebraic group we can remove them. Since we only considering the sets of composition factors we may replace parabolic subgroups by their Levis, and so may assume that X, a positive-dimensional subgroup of G, is a product of simple groups not of type A_1, A_2, B_2 and G_2.

Inside F_4, we know that there are five potential sets of composition factors by [20], each of which lies inside a positive-dimensional subgroup by Theorem 4. Let X be a maximal such subgroup: if $X = B_4$ then we can embed H as $4, 4^*, 1$ or $6, 1^3$ on the natural, both of which yield $6, (4, 4^*)^2, 1^4$ on $V_{\min}$ and $14, 6^3, (4, 4^*)^2, 1^4$ on $L(G)$.

If $X = C_4$ then there are two sets of composition factors on the natural, namely $4, 4^*$ and $6, 1^2$, both of which yield $14, 6^2$ on $L(\lambda_2)$ (which is the restriction of $V_{\min}$ to X), and these factors determine the action on $L(G)$. Since $H \not\leq A_2 \tilde{A}_2$ this case need not be considered.

The only Levi subgroups that can be considered are B_3 and C_3, which lie in B_4 and C_4 respectively, and so we are done.

For $G = E_6$, since H must fix a line on $V_{\min}$ we have that $H \leq F_4$ or $H \leq D_5$. If $H \leq F_4$ then we know the answer, and if $H \leq D_5$ then the factors on the natural are $6, 1^4$ or $4, 4^*, 1^2$; in either case, H fixes a line on the natural, so that $H \leq B_4$, but this is contained in F_4 and we are done.

We could also have $H \cong 3 \cdot \mathrm{Alt}(7)$ contained in E_6 with centres coinciding. By the appropriate table in [20] we see that the composition factors on $V_{\min}$ have dimensions $15, 6, 6$, and this means that H must lie in the A_5 Levi subgroup, acting irreducibly on the natural module. The composition factors of this embedding on $L(G)$ are easily calculable (as they are two copies of $\Lambda^3(M)$ and one of $M \otimes M^*$, where M is the natural module for the A_5), and are as appear in the table above.

For $G = E_7$, $H \cong \mathrm{Alt}(7)$ either fixes a line on $V_{\min}$ or has composition factors $6^8, 4, 4^*$, by [20]. The line stabilizers are contained in an E_6 parabolic or are subgroups $q^{1+32}(q-1)B_5(q)$, which acts on $V_{\min}$ with composition factors $32, 11^2, 1^2$. Since the non-trivial irreducibles for H have even dimension, we get in the latter case that H fixes a line on the 11 so lies inside $D_5 \leq E_6$ anyway. We therefore get those inherited from E_6, which are the first two rows of the table. For the final row, this is the case stated above: if H is embedded as $4 \oplus 4$ inside the A_7 maximal-rank subgroup then it acts on $V_{\min}$ as the sum of $\Lambda^2(4)^{\oplus 2} \oplus 4 \otimes 4$ and its dual, but this is $6^{\oplus 2} \oplus 6/4/6$, as needed.

For $G = E_8$, we know that H lies inside a positive-dimensional subgroup, so we consider the maximal such subgroups X, which are given in [17]. Using Lemma 1.12 we can exclude those subgroups $X \in \mathscr{X}$ with $X^0 \leq Y$ for some other $Y \in \mathscr{X}$, for example the normalizer of a torus.

We can remove any quotient from X that does not contain a copy of $\mathrm{Alt}(7)$, so for example replace $(D_4^2).(\mathrm{Sym}(3) \times 2)$ by D_4^2.

If H lies in $X = E_7$ then we get the first two cases, as the second and third case for E_7 yield the same factors for E_8. Thus we may assume that H does not lie inside E_7. This eliminates the $E_7 A_1$, $E_6 A_2$ and $F_4 G_2$ reductive subgroups.

For $X = A_8$, we must have $H \leq A_8$ since the faithful irreducible modules for $3 \cdot \mathrm{Alt}(7)$ have dimensions 6, 15 and 24. We can embed $\mathrm{Alt}(7)$ inside A_8 with factors

$$4^2, 1 \qquad 4, 4^*, 1 \qquad 6, 1^3, \qquad 4, 1^5.$$

These result in composition factors on $L(G)$ of

$$20^4, 14^4, 6^8, (4, 4^*)^7, 1^8, \quad 20^4, 14^4, 6^8, (4, 4^*)^7, 1^8, \quad 14^8, 6^{17}, (4, 4^*)^2, 1^{18}, \quad 14, 6^{10}, (4, 4^*)^{16}, 1^{46},$$

respectively.

For $X = D_8$, we run through the possible actions of H on the natural 16, noting that it cannot have more than three trivial composition factors on this module as else it would lie inside $D_6 \le E_7$. There are only three possibilities, namely

$$(4, 4^*)^2, \qquad 6, 4, 4^*, 1^2, \qquad 14, 1^2.$$

Using the traces of elements of orders 3, 5 and 7, and comparing them with the nine classes of elements of order 3, the 53 classes of elements of order 5, and the 209 classes of elements of order 7 from D_8, we find that the composition factors on $L(\lambda_7)$ are determined uniquely for the second and third embeddings, but there are two possibilities for the first embedding.

Factors on $L(\lambda_1)$	Factors on $L(\lambda_2)$	Factors on $L(\lambda_7)$
$(4, 4^*)^2$	$14^4, 6^8, 4, 4^*, 1^8$	$20^4, (4, 4^*)^6$ or $14^4, 6^9, 4, 4^*, 1^{10}$
$6, 4, 4^*, 1^2$	$20^2, 14^2, 6^4, (4, 4^*)^3, 1^4$	$20^2, 14^2, 6^4, (4, 4^*)^4, 1^4$
$14, 1^2$	$20^2, 14^4, 6^2, 4, 4^*, 1^4$	$20^2, 14^6, 1^4$

The sum of the second and third columns give the action on $L(G)$, which contributes the last line to the table in the result.

For $X = A_4 A_4$, the only action of H on a 5-space is as $4 \oplus 1$ (or $4^* \oplus 1$), and so $H \le A_3 A_3 \le A_8$, so we have already done this case.

For the maximal Levi subgroups, they are

$$D_7, \quad A_7, \quad A_1 A_6, \quad A_1 A_2 A_4, \quad A_4 A_3, \quad D_5 A_2, \quad E_6 A_1, \quad E_7,$$

and after removing A_1s and A_2s, these are contained within $D_8, A_8, A_8, A_8, A_4 A_4, E_7, E_7$ and E_7 respectively, so we have gone through the complete list of maximal positive-dimensional subgroups of G. $\qquad\square$

PROPOSITION 4.2. *Let G be one of F_4, E_6, E_7 and E_8 in characteristic 2, and let $H \cong$ Alt(8).*
  (i) *If $G = F_4$ then the possible composition factors of H on $V_{\min}$ and $L(G)$ are as follows:*

$V_{\min}$	$L(G)$
$6, (4, 4^*)^2, 1^4$	$14, 6^3, (4, 4^*)^2, 1^4$
$14, 6^2$	$14, 6^3, (4, 4^*)^2, 1^4$

  (ii) *If $G = E_6$ then the possible composition factors of H on $V_{\min}$ and $L(G)$ are as follows:*

$V_{\min}$	$L(G)$
$6, (4, 4^*)^2, 1^5$	$14, 6^4, (4, 4^*)^4, 1^8$
$14, 6^2, 1$	$14^2, 6^5, (4, 4^*)^2, 1^4$

  (iii) *If $G = E_7$ then the possible composition factors of H on $V_{\min}$ and $L(G)$ are as follows:*

$V_{\min}$	$L(G)$
$6^2, (4, 4^*)^4, 1^{12}$	$14, 6^6, (4, 4^*)^8, 1^{19}$
$14^2, 6^4, 1^4$	$14^4, 6^9, (4, 4^*)^2, 1^7$
$6^8, 4, 4^*$	$14^8, 6, 1^{15}$

(iv) *If $G = E_8$ then the possible composition factors of H on $L(G)$ are as follows:*

$L(G)$
$14, 6^{10}, (4, 4^*)^{16}, 1^{46}$
$14^8, 6^{17}, (4, 4^*)^2, 1^{18}$
$(20, 20^*)^2, 14^4, 6^8, (4, 4^*)^7, 1^8$
$64^2, 20, 20^*, 14^4, 6^2, 4, 4^*, 1^4$

Proof. By the construction of the Alt(7)s inside $G = F_4$, E_6, E_7 and E_8, for each set of composition factors there is an Alt(7) representing these factors that extends to an Alt(8) inside G. We need to determine the composition factors of that extension.

Notice that the restrictions to Alt(7) of all but one of the simple Alt(8)-modules are simple, and are unique except that 20 and 20^* both restrict to the same module. The last module is 64, and the restriction to Alt(7) has factors $20, 14^3, 1^2$.

Thus if Alt(7) extends to $H \cong$ Alt(8) and does not have 20 as a composition factor on a module M, then the composition factors of the Alt(8) on M are uniquely determined. This completes the list for $G \neq E_8$, and for two entries for $G = E_8$.

Notice that, since $L(G)$ is self dual, we cannot have an odd number of 64s as then we would be left with an odd number of 20s in $L(G) \downarrow_H$. This deals with the third possibility for the action of Alt(7) on $L(G)$ given in Proposition 4.1. We are left with Alt(7) acting as $20^4, 14^{10}, 6^2, 4, 4^*, 1^8$, which can act as either

$$(20, 20^*)^2, 14^{10}, 6^2, 4, 4^*, 1^8 \quad \text{or} \quad 64^2, 20, 20^*, 14^4, 6^2, 4, 4^*, 1^4.$$

Using the traces of semisimple elements of order up to 15, the potential set of composition factors for Alt(8)s inside E_8 is

$$14^8, 6^{17}, (4, 4^*)^2, 1^{18}, \quad 20, 20^*, 14^6, 6^{10}, (4, 4^*)^7, 1^8, \quad (20, 20^*)^2, 14^4, 6^8, (4, 4^*)^7, 1^8,$$

$$64, 20, 20^*, 14^5, 6^6, (4, 4^*)^4, 1^6, \quad 64, (20, 20^*)^2, 14^3, 6^4, (4, 4^*)^4, 1^6, \quad 64^2, 20, 20^*, 14^4, 6^2, 4, 4^*, 1^4,$$

$$64^2, (20, 20^*)^2, 14^2, 4, 4^*, 1^4, \quad 14, 6^{10}, (4, 4^*)^{16}, 1^{46}.$$

Only one of the two possibilities we generated above is on this list, and so we are done.　□

This proposition *does not require* the case $n = 8$ from Theorem 5 to prove it and so it can be used in the proof of Theorem 5.

Similarly, the next proposition does not require the $n = 9$ case of Theorem 5.

PROPOSITION 4.3. *Let G be one of F_4, E_6, E_7 and E_8 in characteristic 2, and let $H \cong$ Alt(9).*

(i) *If $G = F_4$ then the possible composition factors of H on $V_{\min}$ and $L(G)$ are as follows:*

$V_{\min}$	$L(G)$
$8_1, 8_2, 8_3, 1^2$	$26, 8_1, 8_2, 8_3, 1^2$
26	$26, 8_1, 8_2, 8_3, 1^2$

(ii) *If $G = E_6$ then the possible composition factors of H on $V_{\min}$ and $L(G)$ are as follows:*

$V_{\min}$	$L(G)$
$8_1, 8_2, 8_3, 1^3$	$26, 8_1^2, 8_2^2, 8_3^2, 1^4$
$26, 1$	$26^2, 8_1, 8_2, 8_3, 1^2$

(iii) *If $G = E_7$ then the possible composition factors of H on $V_{\min}$ and $L(G)$ are as follows:*

$V_{\min}$	$L(G)$
$8_1^2, 8_2^2, 8_3^2, 1^8$	$26, 8_1^4, 8_2^4, 8_3^4, 1^{11}$
$26^2, 1^4$	$26^4, 8_1, 8_2, 8_3, 1^5$

(iv) *If $G = E_8$ then the possible composition factors of H on $L(G)$ are as follows:*

$L(G)$
$26, 8_1^8, 8_2^8, 8_3^8, 1^{30}$
$26^8, 8_1, 8_2, 8_3, 1^{16}$
$26^4, (20, 20^*)^2, 8_1^5, 8_2^2, 1^8$
$48^2, 26^4, 8_3^5, 1^8$
$48, 26^2, (20, 20^*)^2, 8_1^3, 8_2^3, 8_3^2, 1^4$

(Here, 8_1 and 8_2 are swapped under the outer automorphism of $\mathrm{Alt}(9)$.)

Proof. Since $\mathrm{Alt}(9) \leq F_4(2)$ by, for example, [16] (see also [20]), we consult the tables in [20] to find its composition factors, and see that there are two classes swapped by the graph automorphism. These classes propagate through E_6 and E_7, and using [20] we see that there are no more possible sets of composition factors, completing the proof for these groups. Hence $G = E_8$.

If $H \leq X = E_7$ then we get the first two lines of the tables in the proposition, so we may assume that H is not contained in E_7.

If $H \leq A_8$ then as 8_1 and 8_2 are swapped by the outer automorphism of H, so we get that H acts on the natural module with factors either $8_1, 1$ or $8_3, 1$: these yield

$$26^4, (20, 20^*)^2, 8_1^5, 8_2^2, 1^8, \qquad 48^2, 26^4, 8_3^5, 1^8$$

on $L(G)$, yielding the next two rows.

The only other positive-dimensional subgroup in which H can lie is D_8. If H acts on the 16 with eight trivial factors then $H \leq D_6 \leq E_7$, so we may assume that the factors on the 16 are $8_i, 8_j$.

Examining the table in [20], and using Proposition 4.2, we can determine the potential sets of composition factors that can occur, given their restrictions to $\mathrm{Alt}(8)$ must exist. As well as those already found, there are two more:

$$78^2, 48, 20, 20^*, 1^4, \qquad 48, 26^2, (20, 20^*)^2, 8_1^3, 8_2^3, 8_3^2, 1^4.$$

The exterior square of $8_i^{\oplus 2}$ is $26^4, 8_i, 1^8$, and so cannot be either of the two remaining sets of composition factors. The other cases are $8_1 \oplus 8_2$ and $8_1 \oplus 8_3$: their exterior squares are

$$\Lambda^2(8_1 \oplus 8_2) = 48, 26^2, 8_3^2, 1^4, \qquad \Lambda^2(8_1 \oplus 8_3) = 26^2, 20, 20^*, 8_1, 8_2^2, 1^4.$$

This means the first case cannot occur in any positive-dimensional subgroup. To see whether the second can occur we simply compute its Brauer character and compare it to that produced by $8_1 \oplus 8_2$ and $8_1 \oplus 8_3$, computing traces using the program for semisimple elements in [19].

There are 650 classes of semisimple elements of order 9 and 9375 of order 15 in D_8. Using the traces of these elements it is possible to construct the Brauer character of H on $L(G)$. If H acts as $8_1 \oplus 8_2$ on the natural then there are two options for the Brauer character on $L(G)$, arising from the modules $48^2, 26^4, 8_3^5, 1^8$ and $48, 26^2, (20, 20^*)^2, 8_1^3, 8_2^3, 8_3^2, 1^4$. For $8_1 \oplus 8_3$ we have again two possible characters on $L(G)$, arising from $48, 26^2, (20, 20^*)^2, 8_1^3, 8_2^3, 8_3^2, 1^4$ again and also $26^4, (20, 20^*)^2, 8_1^5, 8_2^2, 1^8$ this time.

Now suppose that H does not lie in a positive-dimensional subgroup of G. We are still yet to find the single case from [**20**] allowed by Proposition 4.2 but not in any positive-dimensional subgroup. Since $H^1(H, M)$ has dimension 2 for $M = 26$, dimension 1 for $M = 20, 20^*, 78$, and 0 otherwise, the first of the two options above has a trivial submodule Proposition 1.9, and so lies inside a positive-dimensional subgroup of G, a contradiction. □

5. Alt(5)

There are four possibilities for primes p when H is Alt(5): $p = 2, 3, 5$ and primes larger than 5. Recall our assumption that E_6 and E_7 are used to denote the simply connected forms, so that $Z(E_6(q))$ and $Z(E_7(q))$ have orders $\gcd(3, q - 1)$ and $\gcd(2, q - 1)$ respectively.

PROPOSITION 5.1.   *Suppose that* $p \neq 2, 3, 5$.

  (i) *If* $G = F_4$ *then there is a unique conjugacy class of subgroups isomorphic to* Alt(5) *that fixes a line on neither* $V_{\min}$ *nor* $L(G)$. *This is contained in the* $A_2\tilde{A}_2$ *subgroup, and hence is not maximal.*

  (ii) *If* $G = E_6, E_7$, *and* $H \cong$ Alt(5), *then* H *fixes a line on either* $V_{\min}$ *or* $L(G)$, *and hence is not maximal.*

  (iii) *If* $G = E_7$ *and* $H \cong 2 \cdot$ Alt(5) *with* $Z(H) = Z(G)$, *then either* H *fixes a line on* $L(G)$ *or then* H *fixes a 2-space on* $V_{\min}$ *and* $V_{\min} \downarrow_H$ *is not* Out(H)-*stable, and hence* H *is not maximal.*

  (iv) *If* $G = E_8$ *then there is a unique conjugacy class of subgroups isomorphic to* Alt(5) *that does not fix a line on* $L(G)$. *It is contained in a* D_8 *subgroup, and hence is not maximal.*

Proof.   In [**20**] we find tables of all possible sets of composition factors for embeddings of $H \cong$ Alt(5) into $G = F_4$, E_6, E_7 and E_8 for $p > 5$, which immediately proves (ii) by Lemma 1.4 (as H must act semisimply on $L(G)$ and $V_{\min}$). If we embed $2 \cdot$ Alt(5) into the simply connected version of E_7 with centres coinciding, then there is a single possibility that has no trivial composition factors on $L(G)$ from [**20**, Table 6.137], and that has two isomorphic 2-dimensional composition factors, but not their Frobenius twists, so that H stabilizes a 2-space inside $V_{\min}$ and so is not maximal by Lemma 1.4; this proves (iii). (The condition of Out(H)-stability is there to prove that in this case there is no almost simple subgroup with socle H being maximal in an almost simple group of type E_7, via Lemma 1.5(iii).)

For (i) and (iv), so G is one of F_4 and E_8, we need to consider those possible embeddings of H into G with no trivial composition factors on either $V_{\min}$ or $L(G)$. In this case, there are only two cases to consider: one in E_8 and one in F_4.

For $G = E_8$, the fixed-point-free embedding was proved to be unique by Lusztig [**21**] and contained in D_8 by [**7**, Table 7.6] (it is pattern 844). For $G = F_4$, this is proved to not be maximal by Lemma 16.4 of [**22**], and indeed is unique by [**7**, Table 4.18]. It can be found inside the $A_2\tilde{A}_2$ subgroup, acting irreducibly along both factors as different 3-dimensional modules. This completes the proof. □

PROPOSITION 5.2.   *Suppose that $p = 5$. If G is one of F_4, E_6, E_7 and E_8, and H is $\mathrm{Alt}(5)$, or H is $2 \cdot \mathrm{Alt}(5)$ for $G = E_7$ with $Z(H) = Z(G)$, then H fixes a line on either $V_{\min}$ or $L(G)$, and hence is not maximal.*

Proof.   For $p = 5$ we have the Brauer tree in Section 3, and Lemma 2.1 states that the only indecomposable module for H that has trivial composition factors but no trivial submodules or quotients is $P(3)$, which has composition factors $3^3, 1$. Thus if M is any module for H in characteristic 5 with fewer than three times as many 3s as 1s, then M has a trivial submodule or quotient. We now prove that this always occurs for $V_{\min}$ or $L(G)$. (Since this is a defining-characteristic embedding of $\mathrm{Alt}(5) = \mathrm{PSL}_2(5)$ the possible composition factors are not tabulated in [**20**].)

The traces of $t = (1,2)(3,4)$ and $x = (1,2,3)$ on the three irreducible modules are as follows:

Dimension of M	Trace on t	Trace on x
1	1	1
3	-1	0
5	1	-1

For F_4 the traces of t and x are given in Lemma 1.6, and this yields the following possible sets of composition factors for H acting on $V_{\min}$:

$$3^6, 1^8, \qquad 5, 3^7, \qquad 5^3, 3^3, 1^2.$$

Since we need at least three times as many 3s as 1s, else we fix a line, we see that the first and third case clearly fix a line on $V_{\min}$, leaving us with the second case. In this case the trace of t on $V_{\min}$ is -6, so the trace of t on $L(G)$ is 20, whence H has at least twelve trivial composition factors, so we need at least thirty-six 3-dimensional factors to avoid fixing a line, but this is more than $52 = \dim(L(G))$, so that H fixes a line on $L(G)$.

For E_6 the traces of the elements on $V_{\min}$ are 1 more than for F_4 (as the conjugacy classes of $\mathrm{Alt}(5)$ are real and therefore lie inside F_4), so we end up with the composition factors above, or no trivial composition factors at all, but this is impossible. The first and third case fall as before, and in the second case we switch to $L(G)$, on which t has trace 14. Since the trace of x on $V_{\min}$ is 0, on $L(G)$ it is -3 or 6. This gives composition factors of H on $L(G)$ as either $5^{11}, 3^5, 1^8$ or $5^8, 3^8, 1^{14}$, and we again see that H fixes a line on $L(G)$.

For $G = E_7$ the trace of an involution is ± 8 and the trace of an element of order 3 is one of -25, -7, 2 and 20. This yields four possible sets of composition factors for $V_{\min} \downarrow_H$, namely

$$3^{12}, 1^{20}, \qquad 5^2, 3^{14}, 1^4, \qquad 5^9, 3^3, 1^2, \qquad 5^6, 3^6, 1^8.$$

Of these, only the second could potentially not fix a line on $V_{\min}$, and in this case the trace of an involution on $L(G)$ must be 5, and the trace of an element of order 3 on $L(G)$ is either -2 or 7, whence the composition factors of $L(G) \downarrow_H$ are

$$5^{13}, 3^{19}, 1^{11} \qquad \text{or} \qquad 5^{10}, 3^{22}, 1^{17}.$$

In either case, H clearly fixes a line on $L(G)$.

The remaining possibility is that $H \cong 2 \cdot \mathrm{Alt}(5)$ embeds inside $G = E_7$ with $Z(H) = Z(G)$. The faithful simple modules for H are of dimension 2 and 4, on which x acts with trace -1 and 1 respectively. Since the trace of x on $V_{\min}$ is one of $-25, -7, 2, 20$, we cannot have the last case, but the rest yield

$$4, 2^{26}, \qquad 4^7, 2^{14}, \qquad 4^{10}, 2^8.$$

As with the case of Alt(5), only a projective module can have a 2-dimensional composition factor but no 2-dimensional submodule or quotient, and since $P(4)$ has structure $4/2/4$, we need at least twice as many 4s as 2s not to fix a 2-space on $V_{\min}$. This clearly never happens, and so we are never maximal for $p = 5$ and G of type E_7.

For E_8 the trace of an involution on $L(G)$ is 24 or -8 and the trace of an element of order 3 on $L(G)$ is one of -4, 5, 14 and 77. This leads to seven possible sets of composition factors (since we cannot simultaneously have traces of -8 and 77) and we have

$$5^{28}, 3^{28}, 1^{24}, \quad 5^{25}, 3^{31}, 1^{30}, \quad 5^{22}, 3^{34}, 1^{36}, \quad 5^{20}, 3^{44}, 1^{16},$$

$$5^{17}, 3^{47}, 1^{22}, \quad 5^{14}, 3^{50}, 1^{28}, \quad 5, 3^{55}, 1^{78}.$$

Although the fourth possibility gets close to the limit, it still must fix a line, and hence H always fixes a line on $L(G)$, as needed. □

PROPOSITION 5.3. *Suppose that $p = 3$. If $G = F_4, E_6, E_7, E_8$ and $H \cong \mathrm{Alt}(5)$ or $G = E_7$ and $H \cong 2 \cdot \mathrm{Alt}(5)$ with $Z(H) = Z(G)$, then H fixes a line on the simple module $L(G)$.*

Proof. Proposition 3.1 and Lemma 2.1 can be used (together with the fact that the Sylow 3-subgroup is cyclic) to see that the only indecomposable module with a trivial composition factor but no trivial submodule or quotient is the projective module $P(4)$, which has composition factors $4^2, 1$. Thus if M is any module for H in characteristic 3 with fewer than twice as many 4s as 1s, then M has a trivial submodule or quotient. We can now consult the tables in [20] to see whether, for $G = F_4, E_6, E_7, E_8$, there are any such sets of composition factors. We see that H always fixes a line on $L(G)$ for F_4, E_6 (remember to remove a single trivial as $L(G)$ is not simple) and E_7 (and in most cases fixes a line on $V_{\min}$ as well), and always fixes a line on $L(G)$ for $G = E_8$. We must also consider $2 \cdot \mathrm{Alt}(5)$ inside E_7 with $Z(H) = Z(G)$, and here we fix a line on $L(G)$ also. □

PROPOSITION 5.4. *Suppose that $p = 2$.*
  (i) *If $G = F_4, E_6, E_7$ and $H \cong \mathrm{Alt}(5)$ then H fixes either a line or a 2-space on $V_{\min}$, and hence is not maximal.*
  (ii) *If $G = F_4, E_6, E_7$ and $H \cong \mathrm{Alt}(5)$ with $V_{\min} \downarrow_H^\phi = V_{\min} \downarrow_H^*$ for ϕ a generator of $\mathrm{Out}(H)$, then H fixes a line on $V_{\min}$.*
  (iii) *If $G = E_8$ and $H \cong \mathrm{Alt}(5)$ then H fixes a line on $L(G)$.*

Proof. The modules for H have dimension 1, 2, 2 and 4, and there are semisimple elements x of order 3 and y of order 5. The two modules of dimension 2 are swapped by the outer automorphism of H.

If V is a module for H whose Brauer character is $\mathrm{Out}(G)$-stable, then V has factors 1, 4 and $2_1 \oplus 2_2 = 4_2$. Each of these has a rational trace for an element of order 5, and there is a unique rational class of semisimple elements of order 5 in each of F_4, E_6 and E_7. These have traces 1, 2 and 6 respectively on $V_{\min}$, and together with the possible traces of elements of order 3, yields very few possible sets of composition factors for $V_{\min} \downarrow_H$, namely

$$4^4, 4_2, 1^7, \qquad \text{and} \qquad 4, 4_2^4, 1^7$$

for E_6 and these with one trivial removed for F_4, and

$$4^8, 4_2^2, 1^{16}, \qquad \text{and} \qquad 4_2^8, 4^2, 1^{16}$$

for E_7. If $G = E_7$ then these have non-positive pressure and so H always fixes a line on $V_{\min}$. If $G = F_4, E_6$ then we have six and seven trivial modules respectively, and from [11] we see that an involution acts with at least two and three blocks of size 1 respectively, so from Lemma 3.2 we see that we need at least ten and twelve composition factors of dimension 2 not to fix a line on $V_{\min}$, and we have at most eight. This completes the proof of (ii).

For (i), this is easy: as 4 is projective it breaks off, and so the only way that H does not stabilize a 1- or 2-space on a module V is if $V \cong 4^{\oplus m}$ for some m. But then the trace of an element of order 5 on V is $-m < 0$, so this is not true for $V_{\min}$ as we stated above the trace of $u = (1, 2, 3, 4, 5)$ if it is rational.

We thus move to $G = E_8$. There are four and fourteen classes of elements of order 3 and 5 respectively, and this yields many possibilities for the composition factors of H on $L(G)$. Using a computer we can list them all, and by Lemma 3.2 we firstly need that there are more 2-dimensional factors than trivials, else we certainly fix a line.

By [11, Table 9], we see that t in H acts with at least eight blocks of size 1, and so if there are $8 + 2b$ trivial composition factors, we use Lemma 3.2 to see that we need at least $16 + 3b$ composition factors of dimension 2. We now check in each case that there are fewer 2-dimensional factors than this, proving that H always fixes a line on $L(G)$, as needed. □

For the next section, we want to understand copies of $H \cong \mathrm{Alt}(5)$ inside $G = E_7$ in characteristic 3 with specific composition factors on the minimal module.

LEMMA 5.5.   *Let $p = 3$. If $H \cong \mathrm{Alt}(5)$ is a subgroup of $G = E_7$ with composition factors $4^6, 3_1^6, 3_2^4, 1^2$ on $V_{\min}$, then H is contained in an A_6-parabolic subgroup of G and and the action of H on $L(G)$ has $P(1)^{\oplus 3}$ as a summand.*

Proof.   Let H be embedded in G with those factors. The action of $x = (1, 2, 3)$ on $V_{\min}$ must have at least sixteen blocks of size 3, whence it is $3^{18}, 1^2$ by [11, Table 7], and the only way this can work is if H embeds as

$$(4/1/4)^{\oplus 2} \oplus 3_1^{\oplus 6} \oplus 3_2^{\oplus 4} \oplus 4^{\oplus 2}.$$

Since we know that H lies in a positive-dimensional subgroup, we run through the list of maximal connected subgroups from [17] and see how many embeddings we can find.

Suppose that H is contained in the A_6-parabolic subgroup X of G: this acts on $V_{\min}$ with composition factors $7, 7^*, 21, 21^*$, where 7 is the natural module and 21 its exterior square, and 7 is a submodule. Since 7 is a submodule of $V_{\min} \downarrow_X$, H cannot fix a line on the natural module, and so the only embedding can be as $4 \oplus 3_1$ (or $4 \oplus 3_2$). The action of X on $L(G)$ has $7 \otimes 7^*$, $\Lambda^3(7)$ and $\Lambda^3(7^*)$ as factors, and it is easy to check that all three of these have a copy of $1/4/1 = P(1)$ as a summand, so the result holds if H is contained in an A_6-parabolic.

We now check the other parabolics: since H does not fix a line on $V_{\min}$, it does not lie in the E_6 parabolic. The $D_5 A_1$ parabolic has two 2-dimensional composition factors on $V_{\min}$, so not compatible with the composition factors of H. For the $A_1 A_5$ parabolic X, notice that both $(0, \lambda_1)$ and $(1, \lambda_1)$ are composition factors of $V_{\min} \downarrow_X$, so in fact H is contained in the A_5 parabolic, which is in the A_6 parabolic, so done. If H lies inside the $A_2 A_4$ parabolic then, as it cannot act irreducibly on a 5-dimensional module it must fix a line or hyperplane, hence lie inside $A_2 A_3$, which is contained inside the A_6 parabolic again. A similar argument kills off the $A_1 A_2 A_3$ parabolic.

If X is the D_6 parabolic, then H cannot fix a line on the natural module, since the action of X on $V_{\min}$ is $12/32/12$. Computing with the involutions and the thirty-four classes of elements of order 5, one sees that $3_1^{\oplus 3} \oplus 3_2$ is the only possibility for the restriction of the 12 to H.

We can therefore apply Lemma 1.11 to find that the image of H in D_6 lies inside a parabolic subgroup, hence H lies inside a different parabolic subgroup of G, but all other parabolics have been dealt with.

Having proved the result for parabolics, we move on to reductive maximal subgroups, starting with the maximal-rank subgroups.

If X is the A_7 maximal-rank subgroup then H embeds as either $\mathrm{Alt}(5)$ or $2 \cdot \mathrm{Alt}(5)$ in the natural module for SL_8. In the former case, if H fixes a line or hyperplane if lies inside an A_6-parabolic, considered before, and the only other case if $4 \oplus 4$, but $\Lambda^2(4 \oplus 4)$ has a trivial submodule, and this is a summand of $V_{\min} \downarrow_H$. If $2 \cdot \mathrm{Alt}(5)$ embeds in SL_8 then, as the faithful simple modules for $2 \cdot \mathrm{Alt}(5)$ have dimension 2 and 6, it either has a 2-dimensional submodule or quotient, whence its exterior square has a trivial submodule or quotient, not permitted.

For $X = A_2A_5$ it must act irreducibly as 3_1 along the A_2 factor and cannot fix a line along the A_5 factor, else it would be inside the A_2A_4 parabolic. This means it must act as $3_i \oplus 3_j$ along the A_5 factor: this however means that either $i = j = 2$ or H fixes a line on $(10, 1000)$, which is a submodule of $V_{\min} \downarrow_X$. If $i = j = 2$ though, we have that $4/1/4$ appears four times in $V_{\min} \downarrow_H$, which is not allowed either.

For $X = A_1D_6$, if H embeds in the D_6 factor we are done, whence H acts irreducibly along the A_1 factor, say as 2_1 on its natural module. Since the summands of $V_{\min} \downarrow_X$ are the spin module for the D_6 factor and the product of the natural modules for A_1 and D_6, this means that H embeds as $2 \cdot \mathrm{Alt}(5)$ when acting on the natural for the D_6, with composition factors 2_1, 2_2 and 6. Since $x = (1,2,3)$ acts as $3^{18}, 1^2$ on $V_{\min}$, it must act projectively on the 24-dimensional summand of $V_{\min} \downarrow_X$, hence its projection along D_6 acts projectively on the 12. Thus the projection of H along D_6 acts with summands from the list

$$2_1/2_2/2_1, \qquad 2_2/2_1/2_2, \qquad 6,$$

i.e., the faithful projectives for $2 \cdot \mathrm{Alt}(5)$. The first is not possible as H would then fix a line on $V_{\min}$ as $2_1^{\otimes 2}$ would be a submodule. If H embeds as the sum of two isomorphic 6-spaces then since $\mathbb{F}_9$ must be a subfield of our field of definition, else $V_{\min} \downarrow_H$ is not definable, -1 is a square in our field, so Lemma 1.11 applies and the projection of H stabilizes a totally isotropic 6-space, thus lies inside one of the two A_5-parabolics of D_6; this means that H is contained inside A_7 or A_2A_5, examined before. Thus the projection of H along D_6 acts on the natural module as $2_2/2_1/2_2 \oplus 6$, but this does not fix an orthogonal form, completing the result for this group.

Now for the other maximal connected reductive subgroups: for A_1F_4, since $Z(F_4) = 1$ H cannot act along the A_1 factor and so must lie inside the F_4 factor, but this is contained in E_6, which is not allowed. For A_1G_2 we again get that H is contained in the G_2, and this time $G_2 \leq A_6$, as was seen in [**23**, pp.33–35], so this case is also done.

Finally, for $H \leq X = G_2C_3$, the summands of $V_{\min} \downarrow_X$ are $(10, 100)$ and $(00, 001)$, and one has that the exterior cube of $L(100)$ is the sum of $L(100) \oplus L(001)$. To embed H into C_3 we need to choose a 6-dimensional module: $3_1 \oplus 3_2$, $1/4/1$, $4 \oplus 1^{\oplus 2}$ and $3_i \oplus 1^{\oplus 3}$ do not support symplectic forms (the quickest way to see this is to note that they are not summands of their exterior cubes) and $3_1^{\oplus 2}$ has four trivial submodules on its exterior cube, so not correct. Thus H does not embed into X, completing the proof. $\qquad \square$

6. Alt(6)

The cases under consideration are $p = 2, 3, 5$ and primes larger than 5. We continue our assumption that E_6 and E_7 are used to denote the simply connected forms, so that $Z(E_6(q))$ and $Z(E_7(q))$ have orders $\gcd(3, q-1)$ and $\gcd(2, q-1)$ respectively.

For $p \neq 2, 3, 5$, in Theorem 3 we say nothing, so we will not give a formal proposition here, and just list those entries from [20] that are fixed-point free on both $V_{\min}$ and $L(G)$.

For F_4 there is one possibility that fixes no line on either $V_{\min}$ or $L(G)$, and this acts as $9^2, 8_1$ and $10^2, 8_1, 8_2^3$ on $V_{\min}$ and $L(G)$ respectively.

For $G = E_6$, there is a single set of factors for Alt(6)s that fix lines on neither module, and these have factors $9, 8_1, 5_1, 5_2$ and $10^2, 9^2, 8_1^2, 8_2^3$ on $V_{\min}$ and $L(G)$ respectively.

For $3 \cdot \text{Alt}(6)$ embedding into E_6 with centres coinciding, there is only one possibility whose image in $L(G)$ does not fix a line, and this acts as $9^2, 6, 3$ on $V_{\min}$, so stabilizes a 3-space on $V_{\min}$, hence not maximal in G by Lemma 1.4, although we don't consider almost simple groups with socle Alt(6) inside almost simple groups with socle E_6.

For E_7 there is a single example, acting as $10^4, 8_1^2$ on $V_{\min}$ and as $10^2, 9^3, 8_1^4, 8_2^3, 5_1^3, 5_2^3$ on $L(G)$, which exists inside the A_7 subgroup, but there might be other classes.

For $2 \cdot \text{Alt}(6)$ inside E_7 with the centres coinciding, there is (up to outer automorphism) a single possible action on $V_{\min}$ of $8_3^2, 10_2^3, 10_3$, acting on $L(G)$ as $10^5, 9^3, 8_1^4, 8_2^3$.

For Alt(6) inside E_8, there are four potential collections of composition factors for $L(G) \downarrow_H$ with no fixed points, namely (up to an outer automorphism that swaps 5_1 and 5_2)

$$10^{10}, 9^6, 8_1^4, 8_2^4, 5_1^6 \qquad 10^{10}, 9^6, 8_1^4, 8_2^4, 5_1^3, 5_2^3, \qquad 10^8, 9^8, 8_1^6, 8_2^6, \qquad 10^8, 9^4, 8_1^7, 8_2^7, 5_1^2, 5_2^2.$$

Since we prove specific things about other primes, we now switch to formal propositions, as with the previous and later sections.

PROPOSITION 6.1. *Suppose that $p = 5$.*
(i) *If $G = F_4, E_6$ and $H \cong \text{Alt}(6)$ then H fixes a line on $V_{\min}$, and hence is not maximal.*
(ii) *If $G = E_6$ and $H \cong 3 \cdot \text{Alt}(6)$ with $Z(H) = Z(G)$ then either H fixes a line on $L(G)$ or $V_{\min} \downarrow_H$ has exactly one of a semisimple submodule of dimension 12 and a semisimple quotient of dimension 12, and in particular satisfies the conditions for non-maximality of Lemma 1.5(iii) in an almost simple group with socle E_6.*
(iii) *If $G = E_7$ and $H \cong \text{Alt}(6)$ then either H fixes a line on $L(G)$ or $V_{\min} \downarrow_H$ and $L(G) \downarrow_H$ are*

$$10^{\oplus 4} \oplus 8^{\oplus 2} \qquad and \qquad 10^{\oplus 2} \oplus 5_1^{\oplus 3} \oplus 5_2^{\oplus 3} \oplus P(8)^{\oplus 3} \oplus 8.$$

(iv) *If $G = E_7$ and $H \cong 2 \cdot \text{Alt}(6)$ with $Z(H) = Z(G)$ then either H fixes a line on $L(G)$ or $V_{\min} \downarrow_H$ and $L(G) \downarrow_H$ are*

$$10_2^{\oplus 3} \oplus 10_3 \oplus 4_1/4_2 \oplus 4_2/4_1 \qquad and \qquad 10_1^{\oplus 5} \oplus P(8)^{\oplus 3} \oplus 8.$$

(v) *If $G = E_8$ and $H \cong \text{Alt}(6)$ then H fixes a line on $L(G)$.*

Proof.   From the Brauer tree in Section 3, and Lemma 2.1, the appropriate module is the projective cover of 8, which is $8/1, 8/8$, and so we need at least three times as many 8s as 1s. We can go through the tables in [20] to see if this is the case. For F_4 we are therefore done using $V_{\min}$ or $L(G)$, and for E_6 we are done using $V_{\min}$. If $H \cong 3 \cdot \text{Alt}(6)$ however, we can have the action of H on $L(G)$ having factors $10^2, 8^7, 1^2$, with all other cases fixing lines on $L(G)$. In this case we need $P(8)^{\oplus 2}$ in $L(G) \downarrow_H$, and the 10^2 are projective, leaving 8 remaining. This yields an action of H on $L(G)$ of $10^{\oplus 2} \oplus P(8)^{\oplus 2} \oplus 8$, on which the element $y = (1, 2, 3, 4, 5)$ acts with blocks $5^{15}, 3$, so class $A_4 + A_1$. This acts on $V_{\min}$ as $5^5, 2$. The composition factors of $V_{\min} \downarrow_H$ are one of

$$15, 6^2, \qquad 6^3, 3^3,$$

with neither of these modules definable over $\mathbb{F}_5$, so that $H \leq E_6(5^{2a})$ for some a. In this case, if there is a 3 in the socle of $V_{\min} \downarrow_H$ then H is contained in a member of $\mathscr{X}^\sigma$ by Lemma 1.4.

The Brauer tree of the block containing 3 has two edges with exceptional vertex at the end labelled by 3, so the projectives are

$$6/3/6, \qquad 3/3, 6/3,$$

and therefore we need twice as many 6s as 3s in order not to stabilize a 3-space; thus $6^3, 3^3$ does indeed stabilize a 3-space. Since the 15 is projective it splits off from $V_{\min}\downarrow_H$, and so we must have $15 \oplus 6 \oplus 6$, and so u cannot act on $V_{\min}$ as $5^5, 2$, a contradiction.

To deal with almost simple groups of type E_6, we need to examine the $6^3, 3^3$ case more closely. Since the action of u is $5^5, 2$, there is a unique non-projective summand of $V_{\min}\downarrow_H$, necessarily of dimension 12. This must be one of $3/3, 6$ or $3, 6/3$, with the remaining summand being $P(6)$. Therefore $V_{\min}\downarrow_H$ has $6^{\oplus 2}$ as a submodule or quotient, but not both, and so since the graph automorphism interchanges $V_{\min}$ and its dual, we may apply Lemma 1.5(iii) to get no maximal subgroups in the almost simple group case.

For E_7 there is a single possibility for the composition factors of H on $L(G)$ and $V_{\min}$ that has at least three times as many 8s as 1s on $L(G)$, and this is H acting with factors $10^4, 8^2$ for $V_{\min}$ and $10^2, 8^{10}, 5_1^3, 5_2^3, 1^3$ for $L(G)$.

The only way this could work without fixing a line or hyperplane on $L(G)$ is

$$10^{\oplus 2} \oplus 5_1^{\oplus 3} \oplus 5_2^{\oplus 3} \oplus P(8)^{\oplus 3} \oplus 8,$$

on which $u = (1, 2, 3, 4, 5)$ acts as $5^{26}, 3$. This is consistent with coming from class $A_4 + A_2$.

For $H \cong 2 \cdot \text{Alt}(6)$ in $G = E_7$ with the centres coinciding, the unique action on $L(G)$ with at least three times as many 8s as 1s is similar to above, namely $10^5, 8^{10}, 1^3$, so again has u coming from class $A_4 + A_2$. This must act on $V_{\min}$ as

$$10_2^{\oplus 3} \oplus 10_3 \oplus 4_1/4_2 \oplus 4_2/4_1,$$

and everything is again consistent.

In E_8, we are left with two sets of composition factors on $L(G)$, after examining the table in [**20**]. These are

$$10^8, 8^{18}, 5_1^2, 5_2^2, 1^4, \qquad 10^9, 8^{16}, 5_1^4, 5_2, 1^5.$$

For the first case, we need $P(8)^{\oplus 4}$ to hide the four trivial modules, and then have six 8-dimensional modules left over. Taking projectives into account, if the 8s form simply an $8^{\oplus 6}$, we would get 46 blocks of size 5 in the action of u on $L(G)$. Since the only modules constructible using only 8s are 8 and $8/8$, on which u acts as $5, 3$ and $5^3, 1$ respectively, we get the action of u to be one of

$$5^{46}, 3^6, \qquad 5^{47}, 3^4, 1, \qquad 5^{48}, 3^2, 1^2, \qquad 5^{49}, 1^3.$$

None of these appears on Table 6, and so H must fix a line on $L(G)$.

For the second one, the only way it could not have a trivial submodule is $P(8)^{\oplus 5} \oplus 8$, which means that u acts on $L(G)$ with Jordan blocks $5^{49}, 3$, which does not appear on Table 6. This proves that H always fixes a line. □

The two open cases occur in E_7, inside the A_7 and $A_2 A_5$ maximal-rank subgroups: the $\text{Alt}(6)$ acts irreducibly on the natural for A_7; the $2 \cdot \text{Alt}(6)$ is really $6 \cdot \text{Alt}(6)$ acting irreducibly on the naturals for A_2 and A_5.

For $p = 3$, Theorem 3 gave no information about F_4 and E_8: for F_4, this will be considered in a later paper of the author on $\text{SL}_2(q)$ subgroups of exceptional groups not of type E_8. For E_8 there are many possible sets of composition factors for the action of H on $L(G)$ and we do not consider this here.

PROPOSITION 6.2. *Suppose that* $p = 3$.

(i) *If $G = E_6$ and $H \cong \mathrm{Alt}(6)$ then H fixes a line on either $V_{\min}$ or $L(G)$.*

(ii) *If $G = E_7$ and $H \cong \mathrm{Alt}(6)$ then H fixes a line on $V_{\min}$ or $L(G)$.*

(iii) *If $G = E_7$ and $H \cong 2 \cdot \mathrm{Alt}(6)$ with $Z(H) = Z(G)$ then H always stabilizes a σ-stable 2-space, and if ϕ is any automorphism of H that stabilizes $V_{\min} \downarrow_H$ then $\langle H, \phi \rangle$ also stabilizes a 2-space on $V_{\min}$.*

Proof. From Section 3, we know that there are five simple modules: 1, 3_1, 3_2, 4 and the projective 9. Only the 4 has any 1-cohomology, and it is 2-dimensional. The structure of $P(4)$ is

$$4/1, 1, 3_1, 3_2/4, 4, 4/1, 1, 3_1, 3_2/4,$$

and so has five 4s for every four trivials. If an indecomposable module is not projective and has no trivial submodules or quotients, then by removing the 3_is from top and bottom we may assume that it has three socle layers, so in particular needs as many 4s as 1s. Thus if M is a module with more trivial factors then 4-dimensional factors then H fixes a line or hyperplane on M.

Let $G = E_6$. If $H \cong \mathrm{Alt}(6)$ acts on $V_{\min}$ with more trivial composition factors than 4-dimensional ones then it fixes a line by the previous paragraph, and so we may assume the contrary. Using the traces of semisimple elements, one finds only three possible sets of composition factors on $V_{\min}$, up to swapping 3_1 and 3_2, and these are

$$9, 4^3, 3_1, 1^3, \qquad 9, 3_1^3, 3_2^3, \qquad 4^3, 3_1^3, 3_2^2.$$

The restriction of the last case to the point stabilizer $\mathrm{Alt}(5)$ must be semisimple as the 3_is become projective and the 4s have no self extensions, so that $x = (1, 2, 3)$ acts on $V_{\min}$ with Jordan blocks $3^{17}, 1^5$, not present in [**11**, Table 5], and so this case cannot occur.

For the middle case we turn to the Lie algebra, where the traces of the semisimple elements on $V_{\min}$ mean that $t = (1, 2)(3, 4)$ and $v = (1, 2, 3, 4)(5, 6)$ act with trace 13, and $u = (1, 2, 3, 4, 5)$ acts with trace 2 on the 77-dimensional module $L(G)$. This is enough to uniquely determine the composition factors of $L(G) \downarrow_H$, and they are

$$9^6, 4^2, 3_1, 3_2, 1^9,$$

so that H centralizes at least a 7-space on $L(G)$.

Finally we have the first case. However, we only have three 4s here, so we cannot have two 4s both above and below the three trivials, so this case must also fix a line. (If there are an odd number $2n - 1$ of trivial composition factors then we need at least $2n$ 4-dimensional factors to avoid fixing a line, in a slight improvement to our earlier result.) Thus H always fixes a line on $V_{\min}$ or $L(G)$, as needed.

If $G = E_7$ then, as we said before, we are only interested in possible sets of composition factors on $V_{\min}$ that have at least as many 4s as 1s. Using the traces of all semisimple elements, it turns out that, up to application of an outer automorphism swapping 3_1 and 3_2, there are five such sets of composition factors, namely

$$4^{10}, 3_1^2, 1^{10}, \qquad 9, 4^9, 3_1, 1^8, \qquad 4^6, 3_1^9, 3_2, 1^2, \qquad 4^6, 3_1^6, 3_2^4, 1^2, \qquad 9, 4^5, 3_1^5, 3_2^4.$$

The first three of these yield unique sets of composition factors on $L(G)$, namely

$$9^6, 4^9, 3_1^6, 3_2^5, 1^{10}, \qquad 9^5, 4^{10}, 3_1^7, 3_2^5, 1^{12}, \qquad 4^{16}, 3_1^{10}, 3_2^6, 1^{21}.$$

Notice that each of these has more trivials than 4s, so the first three cases fix a line on $L(G)$. For the final case, we claim it cannot occur. To see this, restrict to the point stabilizer $\mathrm{Alt}(5)$ inside H: the 3_is become projective and the 4s have no self-extensions, with the 9s restricting to the projective $P(4)$, and so this module becomes

$$P(4) \oplus 3_1^{\oplus 5} \oplus 3_2^{\oplus 4} \oplus 4^{\oplus 5},$$

on which $x = (1, 2, 3)$ acts with Jordan blocks $3^{17}, 1^5$. A quick check of [**11**, Table 7] proves that this is not a valid unipotent class.

We are left with the fourth case: restricting to an $\text{Alt}(5)$ subgroup L, we notice that it has composition factors $4^6, 3_1^6, 3_2^4, 1^2$, and so Lemma 5.5 applies. Thus $L(G) \downarrow_L$ has three copies of $P(1)$ as summands. We claim that, if V is a module for H with no trivial submodules, and its restriction to L has $P(1)^{\oplus m}$ as a summand if and only if V has $P(4)^{\oplus m}$ as a summand. To see this, it suffices to prove the case $m = 1$ by induction, and notice that the restriction of $P(4)$ for H to L is $P(4)^{\oplus 2} \oplus P(1)$, so one direction is true. For the other, since V has no trivial submodules, in order to restrict to $1/4/1$ there must be five socle layers to V, but $P(4)$ only has five socle layers, so the whole projective must be a submodule of V, hence a summand.

Thus $L(G) \downarrow_H$ has $P(4)^{\oplus 3}$ as a summand, so since $P(4)$ has structure

$$4/1, 1, 3_1, 3_2/4, 4, 4/1, 1, 3_1, 3_2/4,$$

there are at least six copies of 3_2 in $L(G) \downarrow_H$. However, using the Brauer character of $V_{\min} \downarrow_H$, we get two options for the composition factors of $L(G) \downarrow_H$, and these are

$$9^3, 4^{15}, 3_1^6, 3_2^5, 1^{13} \qquad \text{and} \qquad 9^5, 4^{11}, 3_1^6, 3_2^5, 1^{11},$$

neither of which has six 3_2s. Thus H fixes a line on $L(G)$, as required.

Of course, we also have to deal with $H \cong 2 \cdot \text{Alt}(6) = \text{SL}_2(9)$ embedding in $G = E_7$ with the centres coinciding. We firstly assume that $G = G(q)$ for q an odd power of 2. In this case the irreducible modules are $4 = 2_1 \oplus 2_2$ and $12 = 6_1 \oplus 6_2$. An element of order 8 acts with trace 0 on both of these, and so this element acts with trace 0 on all of $V_{\min}$. However, no such element exists in E_7, and so H does not embed with $Z(H) = Z(G)$ unless q is an even power of 2. In particular, this means that we may assume that there is no 2-dimensional submodule of $V_{\min} \downarrow_H$. Hence $V_{\min} \downarrow_H$ is a submodule of copies of $P(6_1)$ and $P(6_2)$. The structure of $P(6_1)$ is

$$6_1/2_1/2_2/2_1/6_1,$$

and so we actually see that $V_{\min} \downarrow_H$ either has a 2-dimensional submodule or quotient, or is actually projective. Since 56 is not a multiple of 3, this is impossible, so H always fixes a 2-dimensional subspace of $V_{\min}$, as needed. □

For $p = 2$ we get a result for everything but E_8, where there is one set of composition factors that might yield an example that does not fix a line on $L(G)$.

PROPOSITION 6.3. *Suppose that $p = 2$.*
  (i) *If $G = F_4$ and $H \cong \text{Alt}(6)$ then either H or its image under the graph automorphism fixes a line on $V_{\min}$, and hence is not maximal.*
  (ii) *If $G = E_6$ and $H \cong \text{Alt}(6)$ then H fixes a line on $V_{\min}$, and hence is not maximal.*
  (iii) *If $G = E_6$ and $H \cong 3 \cdot \text{Alt}(6)$ then H fixes a line on $L(G)$ and hence is not maximal.*
  (iv) *If $G = E_7$ and $H \cong \text{Alt}(6)$ then H fixes a line on $V_{\min}$.*
  (v) *If $G = E_8$ and $H \cong \text{Alt}(6)$ then either H fixes a line on $L(G)$ or the composition factors of $L(G) \downarrow_H$ are*

$$8_1^6, 8_2^6, 4_1^{16}, 4_2^{16}, 1^{24}.$$

Proof. From Section 3 we see that there are five simple modules: 1, 4_1, 4_2, and two projectives 8_1 and 8_2. The projective cover of 4_i is given by

$$P(4_i) = 4_i/1/4_{3-i}/1/4_i/1/4_{3-i}/1/4_i,$$

so we see that for H not to fix a line on a module we must have at least five modules of dimension 4 for each four of dimension 1.

If, in a module V, there are the same number of 8_1s as 8_2s then $u = (1, 2, 3, 4, 5)$ acts with rational trace, namely $+1$ on 1 and $16 = 8_1 \oplus 8_2$, and -1 on the 4_i. There is a single rational class of elements of order 5 in F_4, E_6 and E_7, with trace on $V_{\min}$ given by 1, 2 and 6 respectively. The last thing we need to note is that the minimal module for F_4 and E_6 restricts to H with at least two and three trivial composition factors, since there are at least two and three Jordan blocks of size 1 in the action of $t = (1, 2)(3, 4)$ on $V_{\min}$ by [**11**], and t has no blocks of size 1 on all non-trivial simple modules. These facts prove that $V_{\min} \downarrow_H$ always has at least as many trivials as 4-dimensionals, hence has non-positive pressure and fixes a line or hyperplane on $V_{\min}$, as needed.

Thus we may assume that there are more 8_1s than 8_2s in $V_{\min} \downarrow_H$, and in particular that there is at least one 8_1. If $G = F_4$ then the 8_1, together with the two trivials, and three 4s needed to avoid fixing a line on $V_{\min}$, gives us 22 of 26 dimensions, so our composition factors must be $8, 4^4, 1^2$. The trace of $x = (1, 2, 3)$ on this module is one of $-7, -4, -1, 2, 5$ depending on the numbers of 4_1s and 4_2s, and -1 is the only one of these that is a trace of a class in F_4, so the composition factors are $8_1, 4_1^2, 4_2^2, 1^2$. The Brauer character of its image under the graph automorphism is easy to compute, and is the character of $8_2^3, 1^2$, which clearly fixes a line on $V_{\min}$, as needed.

For E_6, we have three trivial composition factors, and hence four 4s else we fix a line, and an 8_1, so we have one more trivial, but the same composition factors as before, namely

$$8_1, 4_1^2, 4_2^2, 1^3.$$

Since this has pressure 1, and the 8_1 is projective so breaks off, by the discussion after Proposition 1.9 if we ignore the 8_1 the module must be uniserial. However, the element $t = (1, 2)(3, 4)$ acts projectively on a module of the form $1/4_i/1$, and so t would have to act on $V_{\min}$ as $2^{13}, 1$, not a valid action by [**11**, Table 5]. Thus H fixes a line on $V_{\min}$, as needed.

If $H \cong 3 \cdot \mathrm{Alt}(6)$ with $Z(H) = Z(G)$ then the composition factors for $V_{\min} \downarrow_H$ consist of 3_1, 3_2 and 9, with the 3_i being swapped by the outer automorphism of H fixing $Z(H)$.

Using the traces of elements of orders 3 and 5, there are five possible sets of composition factors for $V_{\min} \downarrow_H$, up to swapping the 3_i, and these are

$$3_1^8, 3_2, \quad 3_1^6, 3_2^3, \quad 9, 3_1^3, 3_2^3, \quad 9^2, 3_1^2, 3_2,$$

and all of these fix 3-spaces on $V_{\min}$, so H lies inside a positive-dimensional subgroup of G by Lemma 1.4. However, we need more because the graph automorphism does not stabilize $V_{\min}$, so we switch to $L(G)$. Here the third and fourth sets of composition factors for $V_{\min} \downarrow_H$ do not have corresponding factors on $L(G)$, so these do not exist, and the other two have factors

$$8_1^8, 1^{14} \qquad \text{and} \qquad 8, 3_1^7, 3_2^7, 1^{14}$$

respectively, both of which have non-positive pressure and so H fixes a line on $L(G)$.

Now consider $G = E_7$. The element $v = (1, 2, 3, 4)(5, 6)$ acts projectively on all non-trivial simple modules, so we can get a lower bound on the number of trivial composition factors by examining the action of the unipotent classes on $V_{\min}$ from [**11**, Table 7]. The maximum number of blocks of size 4 in the action of v is twelve, so there are at least eight trivials, hence either H fixes a line on $V_{\min}$ or there are nine 4s, and also an 8_1 from our assumption. This takes up 52 of our 56 dimensions, and so the composition factors of $V_{\min} \downarrow_H$ must be $8_1, 4^{10}, 1^8$, for some distribution of the 4s among 4_1 and 4_2. Since $V_{\min} \downarrow_H$ has pressure 2, this means that it is a submodule of $8_1 \oplus P(4_i) \oplus P(4_j)$ for some i, j. However, $P(4_i)$ is equal to

$$4_i/1/4_{3-i}/1/4_i/1/4_{3-i}/1/4_i,$$

which has dimension 24, so $V_{\min}$ is actually projective. But v does not act projectively on $V_{\min}$, a contradiction, so that H fixes a line on $V_{\min}$.

We lastly turn to E_8. Using a computer and the traces of elements of orders 3 and 5, we are left with two possibilities (up to applying outer automorphisms), namely

$$8_1^6, 8_2^6, 4_1^{16}, 4_2^{16}, 1^{24}, \qquad 8_1^5, 8_2^5, 4_1^{19}, 4_2^{16}, 1^{28}.$$

In these two cases there are at least fifty-five blocks of size 4 in the action of $v = (1,2,3,4)(5,6)$ on $L(G)$. From Table 5 we see that there are only two possible classes in E_8 in which v can lie, namely $D_4(a_1) + A_2$ acting as $4^{56}, 3^8$, and $2A_3$ acting as $4^{60}, 2^4$. Notice that, in order to make a block of size 4 in the action of v we need a projective factor $P(4_i)$.

Notice that the second possibility for the composition factors of $L(G) \downarrow_H$ has factors $8^{10}, 4^{35}, 1^{28}$, so that there are exactly five 4s for every four 1s. Since this is the exact ratio as for $P(4_i)$, we see that $L(G) \downarrow_H$ consists solely of projective modules, which is not allowed by the action of v above. Thus the second set of possible composition factors must fix a line.

For the first set of factors, if v acted with eight blocks of size 3 then we would need thirty-two 4s to cover these, just enough, yielding

$$(8_1 \oplus 8_2)^{\oplus 6} \oplus (4_1/1/4_2/1/4_1/1/4_2 \oplus 4_2/1/4_1/1/4_2/1/4_1)^{\oplus 4}.$$

We are also able to construct a module with action of v given by $4^{60}, 2^4$, for example

$$(8_1 \oplus 8_2)^{\oplus 6} \oplus (P(4_1) \oplus P(4_2))^{\oplus 2} \oplus (4_1/1/4_2/1/4_1 \oplus 4_2/1/4_1/1/4_2)^{\oplus 2}.$$

The restriction of this to $L = \mathrm{Alt}(5)$ has factors $1^{48}, (2_1, 2_2)^{34}, 4^{16}$, which cannot be distributed amongst the factors of $L(G) \downarrow_{E_7}$, so that L does not lie in the E_7 parabolic. However, it can lie inside the A_8 maximal-rank subgroup and inside the A_7 parabolic, with composition factors $2_1, 2_1, 2_2, 2_2$ on the natural module for the A_7. $\qquad\square$

In order to be of use in the next section, just as in the last section, we include a lemma. This is deducible from [**8**], but we produce a quick proof here.

LEMMA 6.4. *There is no copy of* $H \cong \mathrm{Alt}(6)$ *in* $G = E_8$ *with composition factors* $10^{11}, 9^6, 8_1^3, 8_2^3, 5_1^2, 5_2^5, 1$ *on* $L(G)$, *in characteristic at least 7 or 0.*

Proof. Firstly, $\mathrm{Alt}(6)$ does not embed, even projectively, in A_1, nor in B_2 or G_2. Secondly, as H is a reductive subgroup of G, so is $C_G(H)$ (because the characteristic of the field does not divide the order of H), and because H has a trivial submodule on $L(G)$, $C_G(H)$ has dimension 1 (again because of the restriction on the characteristic), and in particular has an involution in it, hence we may apply Lemma 1.13 to get that we need only consider E_7 and D_8.

Since E_7 centralizes a 3-space on $L(G)$, clearly H cannot lie in it, so $H \leq D_8$.

Element	λ_1	λ_2	λ_7
Possible traces for x	7	21	-16
	-2	3	2
Possible traces for y	-5	15	-1
	4	6	8
	13	78	-64

We thus search for 16-dimensional modules for $\mathrm{Alt}(6)$, or $2 \cdot \mathrm{Alt}(6)$, with $x = (1, 2, 3)$ acting with trace 7 or -2, and $y = (1, 2, 3)(4, 5, 6)$ acting with trace -5, 4 or 13.

Since $L(\lambda_2) = \Lambda^2(L(\lambda_1))$, it is easy to compute the traces of the elements of order 3 in potential embeddings of H into D_8. Firstly, we cannot have more than two trivial factors on the 16-dimensional because then we would get more than one trivial on $L(\lambda_2)$, which is not allowed. This leaves

$$10, 5_i, 1, \qquad 9, 5_i, 1^2, \qquad 5_i^2, 5_{3-i}, 1, \qquad 8_1^2, \qquad 8_1, 8_2$$

The element x acts on these with trace 4 or 1, 4 or 1, 4 or 1, -2 and -2 respectively, so only the one with 8s works, but y acts on these with trace -2 as well, which is not allowed.

For embedding $2 \cdot \mathrm{Alt}(6)$ into Ω_{16}^+, we have modules 4_1, 4_2, 8_1 and 8_2, on which x acts with trace 1, -2, -1 and -1 respectively, and y acts with trace -2, 1, -1 and -1 respectively. This means we need a different number of 4_1s to 4_2s, and which of the 8_i we use does not matter, so either $4_i^2, 8_1$, 4_i^4 or $4_i^3, 4_{3-i}$, none of which works with the traces above. Hence H cannot lie in D_8 either, and we are done. $\qquad \square$

7. Alt(7)

We consider the cases $p > 7$, $p = 7$, $p = 5$, $p = 3$ and $p = 2$ in turn. We continue our assumption that E_6 and E_7 are used to denote the simply connected forms, so that $Z(E_6(q))$ and $Z(E_7(q))$ have orders $\gcd(3, q - 1)$ and $\gcd(2, q - 1)$ respectively.

PROPOSITION 7.1.   *Suppose that $p \neq 2, 3, 5, 7$.*
  (i) *If $G = F_4$ and $H \cong \mathrm{Alt}(7)$ then H does not embed into G.*
 (ii) *If $G = E_6$ and $H \cong \mathrm{Alt}(7)$, or $H \cong 3 \cdot \mathrm{Alt}(7)$ with $Z(H) = Z(G)$, then H fixes a line on $L(G)$ and hence is not maximal.*
(iii) *If $G = E_7$ and $H \cong \mathrm{Alt}(7)$ then H fixes a line on $L(G)$ and hence is not maximal.*
 (iv) *There is no embedding of $H \cong 2 \cdot \mathrm{Alt}(7)$ into $G = E_7$ with $Z(H) = Z(G)$.*
  (v) *If $G = E_8$ and $H \cong \mathrm{Alt}(7)$ then either H fixes a line on $L(G)$ and hence is not maximal, or H acts on $L(G)$ with composition factors*

$$35^4, 15^4, 14_1^2, 10, 10^*.$$

(Such a subgroup exists inside a D_8 maximal-rank subgroup acting as $15 \oplus 1$ on the natural module.)

Proof.   This follows immediately from the tables in [20] together with Lemma 1.4. $\qquad \square$

PROPOSITION 7.2.   *Suppose that $p = 7$.*
  (i) *If $G = F_4$ and $H \cong \mathrm{Alt}(7)$ then H does not embed into G.*
 (ii) *If $G = E_6$ and $H \cong \mathrm{Alt}(7)$, or $H \cong 3 \cdot \mathrm{Alt}(7)$ with $Z(H) = Z(G)$, then H fixes a line on $L(G)$ and hence is not maximal.*
(iii) *If $G = E_7$ and $H \cong \mathrm{Alt}(7)$ then H fixes a line both on $V_{\min}$ and on $L(G)$ and hence is not maximal.*
 (iv) *There is no embedding of $H \cong 2 \cdot \mathrm{Alt}(7)$ into $G = E_7$ with $Z(H) = Z(G)$.*
  (v) *If $G = E_8$ and $H \cong \mathrm{Alt}(7)$ then either H fixes a line on $L(G)$ and hence is not maximal, or H acts on $L(G)$ as*

$$35^{\oplus 3} \oplus 21 \oplus 14_1 \oplus 14_2^2 \oplus P(10)^{\oplus 2} \oplus 10 \quad or \quad 35^{\oplus 4} \oplus 14_1^{\oplus 2} \oplus 10^{\oplus 6} \oplus 5^{\oplus 4}.$$

(The second of these is the reduction modulo p of the case for $p = 0$.)

Proof. The Brauer tree in Section 3 implies that the projective modules are

$$1/5/1, \qquad 5/1, 10/5, \qquad 10/5, 10/10,$$

and the only indecomposable module with a trivial composition factor but no trivial submodule or quotient is $P(5)$ by Lemma 2.1.

The action of H on the adjoint module $L(G)$ has at least three trivials and at most two 5-dimensionals, whether $H \cong \mathrm{Alt}(7)$ or $H \cong 3 \cdot \mathrm{Alt}(7)$ inside E_6, and so H always fixes a line on $L(G)$. For $G = E_7$, the unique set of composition factors has the same number of 5s as 1s, so fixes a line on $V_{\min}$ (it also does on $L(G)$).

For $G = E_8$ however, there are three potential sets of composition factors that have no trivial composition factor at all, and others with enough 5s and 10s to cover the trivials they do have.

Those without trivials are, up to projectives,

$$10^8, \qquad 10^7, 5^2, \qquad 10^6, 5^4.$$

In the case of 10^8 we can only produce 10s and $10/10$, on which $w = (1, 2, 3, 4, 5, 6, 7)$ acts as $7, 3$ and $7^2, 6$ respectively. Thus we have blocks of size 7 and $6^i, 3^{8-2i}$, which is not allowed by [**11**, Table 9]. Hence H does not embed with these factors.

In the cases of $10^7, 5^2$ and $10^6, 5^4$, we have the following possible indecomposable modules:

$$10, \quad 5, \quad 10/10, \quad 10/5 \oplus 5/10, \quad 10, 5/10 \oplus 5, 10/10, \quad 5, 10/5, 10, \quad 10/5, 10/10.$$

The actions of w on these elements are (up to projectives) $3, 5, 6, 1^2, 4^2, 2$ and 0. For $10^7, 5^2$ we are allowed only $P(10)^{\oplus 2} \oplus 10$, and for $10^6, 5^4$ we are only allowed the semisimple case $10^{\oplus 6} \oplus 5^{\oplus 4}$, leading to the two cases listed in the proposition.

If the composition factors of $L(G) \downarrow_H$ have trivials but have twice as many 5s and as many 10s, then up to projective simple modules we have one of

$$10^9, 5^2, 1, \quad 10^8, 5^4, 1, \quad 10^{10}, 5^4, 1^2.$$

Removing copies of $10, 5^2, 1$ (i.e., $P(5)$) yields $10^8, 10^7, 5^2$ and 10^8, and so the first and last of these cannot occur without fixing a line, but the second can. However, the set of composition factors on $L(G)$ is $35^2, 14_1, 14_2^2, 10^8, 5^4, 1$, (where 14_1 is a factor of $5^{\otimes 2}$). This restricts to $\mathrm{Alt}(6)$ with composition factors $10^{11}, 9^6, 8_1^3, 8_2^3, 5_1^2, 5_2^5, 1$, which happen to be those which were proved not to exist in Lemma 6.4. Thus this embedding does not exist. $\qquad\square$

PROPOSITION 7.3. *Suppose that $p = 5$.*
  (i) *If $G = F_4$ and $H \cong \mathrm{Alt}(7)$ then H fixes a line on $V_{\min}$, and hence is not maximal.*
 (ii) *If $G = E_6$ and $H \cong \mathrm{Alt}(7)$ then either H fixes a line on $L(G)$, or is contained inside an algebraic A_2 subgroup with action of $V_{\min}$ given by (up to duality) $8/19$, with two classes of H being swapped by the graph automorphism. In particular, H is not maximal.*
(iii) *If $G = E_6$ and $H \cong 3 \cdot \mathrm{Alt}(7)$ with $Z(H) = Z(G)$, then H fixes a line on $L(G)$, and hence is not maximal.*
 (iv) *If $G = E_7$ and $H \cong \mathrm{Alt}(7)$, then H fixes a line on either $V_{\min}$ or $L(G)$, and hence is not maximal, or H acts on $V_{\min}$ and $L(G)$ as*

$$(10 \oplus 10^*)^{\oplus 2} \oplus 8^{\oplus 2} \qquad and \qquad 10 \oplus 10^* \oplus P(8)^{\oplus 3} \oplus 8.$$

  (v) *If $G = E_7$ and $H \cong 2 \cdot \mathrm{Alt}(7)$ with $Z(H) = Z(G)$, then H acts on $V_{\min}$ and $L(G)$ as*

$$20 \oplus 4/14 \oplus 14/4^* \qquad and \qquad 35^{\oplus 3} \oplus 10 \oplus 10^* \oplus 8.$$

 (vi) *If $G = E_8$ and $H \cong \mathrm{Alt}(7)$, then either H fixes a line on $L(G)$, hence is not maximal, or H acts on $L(G)$ as*

$$35^{\oplus 4} \oplus 15^{\oplus 4} \oplus 10 \oplus 10^* \oplus 8/6 \oplus 6/8.$$

(This is again the reduction modulo p of the case for p = 0.)

Proof.   Using the Brauer tree in Section 3, the projective modules are

$$1/13/1, \quad 13/1, 8/13, \quad 8/6, 13/8, \quad 6/8/6.$$

For $G = F_4$ we consult the table in [20] and see that H has two trivial and three 8-dimensional factors, so acts semisimply on $V_{\min}$ and in particular fixes a line on $V_{\min}$.

For $G = E_6$, there are three possible sets of composition factors on $V_{\min}$ according to [20]. Two of these have trivial composition factors on $L(G)$, and no 13s in either case, and hence these fix lines. The final case has composition factors 6, 8 and 13 on $V_{\min}$ and $35^2, 8$ on $L(G)$. This means that $u = (1, 2, 3, 4, 5)$ acts with factors $5^{15}, 3$ on $L(G)$ (since the 35s are projective), and hence comes from class $A_4 + A_1$, which acts with blocks $5^5, 2$ on $V_{\min}$. This means that H acts indecomposably on $V_{\min}$, hence as $8/6, 13$ or its dual.

The restriction of H to $L = \mathrm{Alt}(6)$ has structure $5_1 \oplus 5_2 \oplus 8/1, 8$, which clearly fixes a line, and so lies in the D_5-parabolic (as the other line stabilizer is F_4 which has a trivial summand on $V_{\min}$). The D_5-parabolic acts with structure $10/16/1$, and obviously the embedding of L into this acts as $8/8$ on the spin module and $5_1 \oplus 5_2$ on the natural. In particular, this proves that the image of L is uniquely determined up to conjugacy in the D_5 Levi.

To determine the number of classes in the D_5 parabolic we need to consider 1-cohomology on the spin module, which is the action of the Levi on the unipotent radical of the parabolic. The 1-cohomology of $8/8$ is 1-dimensional, and so there are q classes of $\mathrm{Alt}(6)$s inside the D_5 parabolic, where $G = E_6(q)$. We claim that all $q - 1$ of those outside the D_5 Levi subgroup are permuted by the T_1 factor of the $D_5 T_1$, at least in the adjoint form of E_6: to see this, any element of T_1 centralizes the maximal subgroup $D_5 T_1$ of the D_5-parabolic, and so if it centralizes anything else in the parabolic, for example, another class of $\mathrm{Alt}(6)$, then it centralizes the whole parabolic so lies in $Z(G)$. Thus there is a unique class in the adjoint form of E_6 and either one or three in the simply connected form, depending on whether $|Z(G)|$ has order 1 or 3.

We can proceed in two different ways here, but both are similar. The first is to note that L is contained in one of the two algebraic A_2s, say X, which act on $V_{\min}$ as $8/19$ (up to duality induced by the graph automorphism). Since the 19-space is uniquely determined in the action of both L and H, we must have that H is contained in $\langle H, X \rangle$, which is just X as X is known to be maximal. (Notice that this inclusion was known in [2] but uniqueness of H was not.)

Alternatively, if $\bar{H}$ is any other $\mathrm{Alt}(7)$ subgroup containing L, and with the same module structure on $V_{\min}$ as H, then both H and $\bar{H}$ must fix the same 19-space (with quotient 8), since L and H both fix a unique 19-space. This proves that either H is uniquely determined up to conjugacy or that H is not maximal. If H is unique up to conjugacy, however, then it is contained in the algebraic A_2 above, and hence is not maximal either.

If $3 \cdot \mathrm{Alt}(7)$ embeds in G with $Z(H) = Z(G)$, then there are three sets of composition factors for $L(G)$, two of which have at least three 1s and at most one 13, so fix a line. The remaining possibility has composition factors 21, 6 on $V_{\min}$ and $35^2, 8$ on $L(G)$. The 35 is projective, so splits off, and thus $L(G) \downarrow_H$ is semisimple, with the action of $u = (1, 2, 3, 4, 5)$ on $L(G)$ being $5^{15}, 3$, so class $A_4 + A_1$, which acts on $V_{\min}$ as $5^5, 2$. Thus $V_{\min} \downarrow_H$ cannot be semisimple, but $\mathrm{Ext}^1(21, 6) = 0$ from the Brauer tree in Section 3. Thus H cannot embed with these factors, as needed.

For $G = E_7$, the only set of composition factors embedding $\mathrm{Alt}(7)$ into G with at least twice as many 13s as 1s for both $V_{\min}$ and $L(G)$ is

$$(10, 10^*)^2, 8^2, \qquad (10, 10^*), 13^3, 8^7, 6^3,$$

and modules can be constructed so that $u = (1, 2, 3, 4, 5)$ acts on $V_{\min}$ as $5^{10}, 3^2$ and on $L(G)$ as $5^{26}, 3$. The first is semisimple, and the second is $10 \oplus 10^* \oplus P(8)^{\oplus 3} \oplus 8$. Note that this exists

as a copy of H acting irreducibly on the natural module for the A_7 maximal-rank subgroup. There is also a possible set of composition factors for $2 \cdot \mathrm{Alt}(7)$ embedding in E_7 with centres coinciding, that yields the same action of u on the two modules: it acts as

$$20 \oplus 4/14 \oplus 14/4^*, \qquad 35^{\oplus 3} \oplus 10 \oplus 10^* \oplus 8$$

on $V_{\min}$ and $L(G)$ respectively.

For E_8 and $p = 5$ we again consult the tables of possible sets of composition factors from [20], and once we apply the statement that there needs to be twice as many 13s as trivials, leaves us with five options, four with no trivial factors at all:

$$[35^4, 15^2, 10, 10^*], 13^2, 8^4, \qquad [35^4, 15^3, 10, 10^*], 13, 8^3, 6, \qquad [35^4, 15^4, 10, 10^*], 8^2, 6^2,$$

$$[35, (10, 10^*)^4], 13, 8^{12}, 6^4, \qquad 13^7, 8^{14}, 6^7, 1^3.$$

(We place brackets around the composition factors in blocks of defect zero, which simply become summands and $u = (1, 2, 3, 4, 5)$ acts projectively.)

In the first case we have at least 5^{46}, and so there are few classes to which u can belong, from Table 6. With just 13 and 8, the only indecomposable modules we can produce are 13, 8, and 13/8 and 8/13 on which u acts as projective plus a block of size either 3 or 1. Thus u acts only with blocks 5, 3 and 1, so must come from class $A_4 + A_2$ acting as $5^{46}, 3^5, 1^3$; however, this cannot work either, so H cannot embed with these factors.

In the second case, u acts with at least 47 blocks of size 5, so must come from $A_4 + A_2$ and act as $5^{48}, 4^2$. This means we can have no 13 as a summand of $L(G) \downarrow_H$ (as it would contribute a 3 to the action of u), thus it lies inside a self-dual summand, which is only $P(8)$, thus we get $P(8) \oplus 8$, another contradiction. Thus H cannot embed with these factors either.

In the fourth case, the indecomposable modules containing an 8 but no 1s are (up to duality)

$$8, \quad 8/13, \quad 8/6, \quad 8/13, 6, \quad P(8), \quad P(6).$$

With the exception of 8, each of these has at least as many factors that are not of dimension 8 as those that are, and since there are five factors that are not of dimension 8 in the principal block for $L(G) \downarrow_H$, this means we have seven summands of $L(G) \downarrow_H$ of dimension 8, thus 3^7 contributing to the action of u on $L(G)$. In addition, we have at least 5^{41} from the projective blocks in $L(G) \downarrow_{\langle u \rangle}$, and there is no such class in Table 6. Thus H cannot embed with these factors.

In the fifth case, in order not to fix a line, $L(G) \downarrow_H$ must have three copies of $P(13)$, meaning that we have at least 5^{41}, and leaves factors $13, 8^{11}, 6^7$ to understand. In addition, applying the argument before yields $8^{\oplus 3}$ as a summand of $L(G) \downarrow_H$, so we have blocks in u of $5^{41}, 3^3$; looking at Table 6, this means that u lies in one of two classes, $A_4 + 2A_1$ acting as $5^{45}, 3^4, 2^4, 1^3$, or $A_4 + A_2$ acting as $5^{46}, 3^5, 1^3$. The four indecomposable modules contributing a 1 to the action of u are 1, 6, 8/13 and 13/8, and since we have three blocks of size 1 (so we need a self-dual summand), this means we have a 6 as a summand in $L(G) \downarrow_H$, thus another 8 as a summand by the above argument. For the other 1^2, we have either $6^{\oplus 2}$, which means we have another $8^{\oplus 2}$ and at least six blocks of size 3 in u, which is not allowed, or we have $8/13 \oplus 13/8$, which is not allowed because we only have a single 13. This contradiction means that H must fix a line if it embeds with these factors.

We are left with the third possibility, that H embeds with factors projective plus $8^2, 6^2$. Here there is a single possibility for the action of u, since we have at least forty-eight blocks in the action of u, namely u acts as $5^{48}, 4^2$, which means that H must act as

$$35^{\oplus 4} \oplus 15^{\oplus 4} \oplus 10 \oplus 10^* \oplus 8/6 \oplus 6/8.$$

This is the only module that satisfies the traces of elements and the action of the unipotent class, and so must be the reduction modulo 5 of the embedding into D_8 given over $\mathbb{C}$. □

As with Alt(6), the two open cases occur in E_7, inside the A_7 and A_2A_5 maximal-rank subgroups: the Alt(7) acts irreducibly on the natural for A_7; the $2 \cdot \text{Alt}(7)$ is really $6 \cdot \text{Alt}(7)$ acting irreducibly on the naturals for A_2 and A_5.

PROPOSITION 7.4.   *Suppose that $p = 3$.*
  (i) *If $G = F_4$ and $H \cong \text{Alt}(7)$ then H does not embed into G.*
  (ii) *If $G = E_6$ and $H \cong \text{Alt}(7)$ then H fixes a line on $L(G)$.*
  (iii) *If $G = E_7$ and $H \cong \text{Alt}(7)$, or $H \cong 2 \cdot \text{Alt}(7)$ with $Z(H) = Z(G)$, then H fixes a line on either $V_{\min}$ or $L(G)$.*
  (iv) *If $G = E_8$ and $H \cong \text{Alt}(7)$ then either H fixes a line on $L(G)$, and hence is not maximal, or acts on $L(G)$ with composition factors*
$$15^4, 13^6, (10, 10^*)^5, 1^{10}.$$

Proof.   From Proposition 3.1, the modules 10 and 10^* each have 1-dimensional 1-cohomology, and 13 has 2-dimensional 1-cohomology, with all other cohomology groups being zero. The projective covers $P(10)$ and $P(13)$ have structures
$$10/1/10^*, 13/1/10 \qquad \text{and} \qquad 13/1, 1/13, 10, 10^*/1, 1/13$$
respectively, and so if M is a module whose socle consists solely of 10s, 10^*s and 13s, with no trivial quotients, then it either has at most three socle layers or it contains a projective.

If H embeds in E_6 then it does so with factors $15, 13, (10, 10^*)^2, 6, 1^3$ (remember that $\dim(L(G)) = 77$, not 78, for $p = 3$), and since $L(G)$ is self dual, if 13 lies in the socle (or top) then it is a summand, and so does not need to be considered. The 6 and 15 lie in a non-principal block, and so form a summand of the form $6 \oplus 15$ and will be ignored. We therefore may assume that the socle of $L(G) \downarrow_H$ consists of 10s and 10^*s, with possibly 13 as a summand.

As $L(G)$ is self dual, if $P(10)$ is a summand of $L(G) \downarrow_H$ then so is $P(10^*)$, but this is not possible, as it uses too many 13s. This also means that the 13 is definitely a summand. However, we now do not have enough 10-dimensional modules to cover three trivials, and so H must fix a line on $L(G)$.

When $G = E_7$ we have two possible sets of composition factors for embeddings of $H \cong \text{Alt}(7)$, one of which has two trivial factors on $V_{\min}$ and no other modules from the principal block, hence centralizes a 2-space, and one of which has nine trivial factors on $L(G)$ and a single 13, with no 10s, so centralizes at least a 7-space. If $2 \cdot \text{Alt}(7)$ embeds in E_7 with centres coinciding, then there are twenty-two trivial composition factors, seven 13s and $10, 10^*$, so our module has negative pressure and fixes a line by Proposition 1.9.

In E_8 there are four possible sets of composition factors for $L(G) \downarrow_H$, but only one case has negative pressure, hence fixes a line by Proposition 1.9. We examine the other three now. If the composition factors are $15^6, 13^9, 6^5, 1^{11}$ then H fixes a line, because without any 10s we can only form $13/1, 1/13$, and so need as many 13s as 1s not to fix a line.

With factors $15^2, 13^{11}, (10, 10^*)^2, 6^3, 1^{17}$, we cannot have $P(10)$ as a factor because, together with its dual, we would need three 10s, too many. Each $P(13)$ reduces the number of 13s by three and 1s by four, and we can have at most two of them, beyond which we can only have $13/1, 1/13$, and this will not use up enough trivials. Hence we may assume that $L(G) \downarrow_H$, cut by the principal block, has three socle layers. Removing any simple summands, we have at most five 13 and two 10s in the socle, which support at most twelve trivials above, so we must fix a line again.

If the factors are $15^4, 13^6, (10, 10^*)^5, 1^{10}$, however, then we cannot yet prove that H fixes a line on $L(G)$, although it would be surprising if an embedding that has no fixed points does exist. $\square$

Although in the last case we cannot prove that H fixes a line on $L(G)$ or doesn't exist, we can prove a lot about $L(G) \downarrow_H$. Firstly, $x = (1, 2, 3)$ lies in class $2A_2$, and $y = (1, 2, 3)(4, 5, 6)$ lies in class $2A_2 + 2A_1$. Furthermore, $P(13)$ is not a summand of $L(G) \downarrow_H$, and the subgroups isomorphic to $\mathrm{Alt}(5)$ and $\mathrm{PSL}_2(7)$ both centralize a 2-space on $L(G)$, with the $\mathrm{Alt}(5)$s having two trivial summands. Finally, H fixes a line on $L(G)$ if and only if the subgroup $\mathrm{Alt}(6)$ does.

PROPOSITION 7.5.   *Suppose that $p = 2$.*
  (i) *If $G = F_4$ and $H \cong \mathrm{Alt}(7)$ then either H or the image of H under the graph automorphism fixes a line on $V_{\min}$, and hence is not maximal.*
  (ii) *If $G = E_6$ and $H \cong \mathrm{Alt}(7)$ then H fixes a line on both $V_{\min}$ and $L(G)$, and is hence not maximal.*
  (iii) *If $G = E_7, E_8$ and $H \cong \mathrm{Alt}(7)$, or $G = E_6$ and $H \cong 3 \cdot \mathrm{Alt}(7)$ with $Z(H) = Z(G)$, then H fixes a line on $L(G)$, and hence is not maximal.*

Proof.   From the table in Section 3, there are three modules in the principal block: the projective covers have structure as follows:

$$1/14, 20/1, 1/20, 14/1, \qquad 14/1/20/1, 14/14, \qquad 20/1/14/1/20.$$

For F_4, $L(G)$ always has four trivial composition factors and at most one non-trivial factor in the principal block, so certainly fixes lines. Therefore if σ denotes the graph automorphism then either H or its image H^σ under the graph automorphism fixes lines on $V_{\min}$. (This is because for $p = 2$, $L(G)$ has composition factors $V_{\min}$ and $V_{\min}^\sigma$.)

For E_6, all embeddings of H from [20] have non-positive pressure on $V_{\min}$, hence fix lines or hyperplanes. For $3 \cdot \mathrm{Alt}(7)$, the two possibilities for composition factors on $L(G)$ both have pressure -2, so again fix lines.

For $G = E_7$ there are more possibilities, but in each case there are significantly more trivial factors than non-trivial factors in the principal block (even taking into account the fact that $L(G)$ has a trivial submodule in characteristic 2), hence H fixes lines on $L(G)$ again.

For $G = E_8$, we need to be a little more precise: from the structure of the projectives, it is clear that we need at least three 14s or 20s for every two 1s. This reduces the possibilities down to three, namely

$$20^6, 14^8, 1^8, 4, 4^*, \qquad 20^5, 14^9, 1^8, 6, 4, 4^*, \qquad 20^4, 14^{10}, 1^8, 6^2, 4, 4^*.$$

The element $v = (1, 2, 3, 4)(5, 6)$ acts projectively on both 20 and 4, and acts with blocks $4, 2$ and $4^3, 2$ on 6 and 14 respectively. Thus, if the embeddings were semisimple, the action of v on these three modules has blocks

$$4^{56}, 2^8, 1^8, \qquad 4^{55}, 2^{10}, 1^8, \qquad 4^{54}, 2^{12}, 1^8.$$

Write $L(G) \downarrow_H = A \oplus B$, where A is a sum of projectives and the $\{4, 4^*, 6\}$-radical, and B is a module in the principal block with no projective summands. In particular, B has three socle layers, and examining the structure of $P(14)$, we see that B has at least as many 20s as trivials. Since the same is true for $P(20)$, we need a copy of $P(14)$ in $L(G) \downarrow_H$ for every trivial composition factor above the number of 20s, i.e., two, three and four respectively. Since there are three 14s in $P(14)$, this means the third case is not possible, and the second case cannot work either as all 14s are used up in the $P(14)^{\oplus 3}$, leaving none to cover the remaining two trivials. In the first case $P(14)^{\oplus 2}$ uses up $20^2, 14^6, 1^4$ and leaves $20^4, 14^2, 1^4, 4, 4^*$; this gives sixty blocks to the action of v already, and so we cannot have any more $P(14)$s or $P(20)$s, but this leads to a contradiction as, with three socle layers, B must have at least twice as many

non-trivial composition factors as trivial ones. Hence any embedding of H into E_8 fixes a line. □

8. Alt(8)

The cases to consider are firstly $p > 7$, and then $p = 7, 5, 3, 2$ in that order. We continue our assumption that E_6 and E_7 are used to denote the simply connected forms, so that $Z(E_6(q))$ and $Z(E_7(q))$ have orders $\gcd(3, q-1)$ and $\gcd(2, q-1)$ respectively.

PROPOSITION 8.1.   *Suppose that $p \neq 2, 3, 5, 7$.*
 (i) *If $G = F_4, E_6$ and $H \cong \mathrm{Alt}(8)$ then H does not embed into G.*
 (ii) *If $G = E_7, E_8$ and $H \cong \mathrm{Alt}(8)$ then H fixes a line on $L(G)$ and hence is not maximal.*
 (iii) *There is no embedding of $H \cong 2 \cdot \mathrm{Alt}(8)$ into $G = E_7$ with $Z(H) = Z(G)$.*

Proof.   This follows immediately from the tables in [**20**] together with Lemma 1.4.   □

PROPOSITION 8.2.   *Suppose that $p = 7$.*
 (i) *If $G = F_4, E_6$ and $H \cong \mathrm{Alt}(8)$ then H does not embed into G.*
 (ii) *If $G = E_7, E_8$ and $H \cong \mathrm{Alt}(8)$ then H fixes a line on $L(G)$ and hence is not maximal.*
 (iii) *There is no embedding of $H \cong 2 \cdot \mathrm{Alt}(8)$ into $G = E_7$ with $Z(H) = Z(G)$,*

Proof.   As before, using the Brauer tree in Section 3 and Lemma 2.1, the only projective containing the trivial module other than $P(1)$ is $P(19)$, which has the form $19/1, 45/19$. We then consult the tables in [**20**]: for E_7 there is a unique embedding of H, which has two trivial factors and no 45 on $L(G)$, so fixes lines, and for E_8 the three possible embeddings of H each have at least three trivial factors and at most one 45, so all fix lines.   □

PROPOSITION 8.3.   *Suppose that $p = 5$.*
 (i) *If $G = F_4, E_6$ and $H \cong \mathrm{Alt}(8)$ then H does not embed into G.*
 (ii) *If $G = E_7, E_8$ and $H \cong \mathrm{Alt}(8)$ then H fixes a line on $L(G)$ and hence is not maximal.*
 (iii) *There is no embedding of $H \cong 2 \cdot \mathrm{Alt}(8)$ into $G = E_7$ with $Z(H) = Z(G)$,*

Proof.   Using the Brauer trees from Section 3 the projective indecomposable modules are

$$1/13/1, \quad 13/1, 43/13, \quad 43/13, 21_1/43, \quad 21_1/43/21_1, \quad 7/21_2/7, \quad 21_2/7, 21_2/21_2.$$

Thus a module needs at least twice as many 13s as 1s in order not to fix a line. For E_7 there are three possible sets of composition factors, two of which have one trivial and at most one 13, so fix lines on $L(G)$. The remaining set of composition factors acts with factors $21_2^2, 7^2$ on $V_{\min}$ and acts as $20 \oplus 43 \oplus 70$ on $L(G)$. Clearly $u = (1, 2, 3, 4, 5)$ acts as $5^{26}, 3$ on $L(G)$, and so must come from class $A_4 + A_2$, so act as $5^{10}, 3^2$ on $V_{\min}$. This can be realized by $21 \oplus P(7)$.

However, this restricts to Alt(7) as

$$13 \oplus 8 \oplus 1/13/1 \oplus 6/8/6,$$

and so lies in the stabilizer of a unique line, either an E_6-parabolic or a B_5 type subgroup, the latter acting with composition factors $1, 1, 11, 11, 32$, clearly not possible. (To see this, the derived subgroups of the stabilizers of the other orbits from [**14**, Lemma 4.3] all lie inside an E_6-parabolic, hence stabilize more than one line.) Therefore this Alt(7) lies inside an E_6

parabolic, with composition factors $13, 8, 6$ on the minimal module W for E_6. The E_6 parabolic acts as $1/W/W^*/1$, and so there must be a non-split extension between the 6 and 8 in W, but the 13 must split off, so that up to duality we have that $\mathrm{Alt}(7)$ acts on W as $8/6 \oplus 13$. However, u acts on this module as $5^4, 4, 3$, not listed in [**11**, Table 5]. Thus this embedding of $\mathrm{Alt}(8)$ into E_7 does not exist, and H fixes a line, as needed.

For E_8 every possible set of composition factors from [**20**] has at least three trivials and at most one 13, so these all fix lines. $\qquad\square$

PROPOSITION 8.4.   *Suppose that $p = 3$.*
  (i) *If $G = F_4, E_6$ and $H \cong \mathrm{Alt}(8)$ then H does not embed into G.*
 (ii) *If $G = E_7$ and $H \cong \mathrm{Alt}(8)$ then either $L(G) \downarrow_H$ has a trivial summand or $(1, 2, 3)$ lies in a generic unipotent class for $V_{\min}$, and so there exists a positive-dimensional subgroup stabilizing the same subspaces of $V_{\min}$ as H. In either case, H is not maximal.*
(iii) *There is no embedding of $H \cong 2 \cdot \mathrm{Alt}(8)$ into $G = E_7$ with $Z(H) = Z(G)$,*
 (iv) *If $G = E_8$ and $H \cong \mathrm{Alt}(8)$ then H fixes a line on $L(G)$.*

Proof.   For E_7 there is a unique set of composition factors, both on $V_{\min}$ and on $L(G)$. On $V_{\min}$ we have that H acts with factors $21^2, 7^2$. Since the 21 lies in a separate block to the 7, and the 7 has no self extensions, $V_{\min}$ must have $7^{\oplus 2}$ as a summand. The other summand is either $21^{\oplus 2}$ or $21/21$. The element $x = (1, 2, 3)$ acts on 7 with blocks $3, 1^4$ and on 21 as $3^5, 1^6$, so on $21/21$ as $3^{10}, 2^6$. This means that x acts on $V_{\min}$ as either $3^{12}, 1^{20}$ – so lies in class A_2, which is generic, and so we are done by Lemma 1.2 – or acts like $3^{12}, 2^6, 1^8$ – so lies in class $A_2 + A_1$, and acts on $L(G)$ as $3^{37}, 2^8, 1^6$.

The composition factors of H on $L(G)$ are also uniquely determined as $35^2, 21, 13, 7^4, 1$. The 1 can be covered only by $35/1, 7/35$, on which both classes of unipotent element act as 3^{26}. Even if the rest of the module is semisimple, the 7^4 contributes four blocks of size 3, the 21 another five, and the 13 gives three, taking the total to 38, too many. Thus $35/1, 7/35$ cannot be a subquotient of $L(G) \downarrow_H$, and so $L(G) \downarrow_H$ has a trivial summand and hence H fixes a line of $L(G)$.

For E_8, there are two possible sets of composition factors for $L(G) \downarrow_H$, each with four trivial composition factors. As $H^1(H, M) = 0$ unless M is either 13 or 35, in which cases it is 1-dimensional, the fact that there are exactly three such composition factors in $L(G) \downarrow_H$ implies that it has negative pressure and so H fixes a line on $L(G)$ by Proposition 1.9, as needed. $\qquad\square$

PROPOSITION 8.5.   *Suppose that $p = 2$.*
  (i) *If $G = F_4$ and $H \cong \mathrm{Alt}(8)$ then either H or its image under the graph automorphism stabilizes a line on $V_{\min}$, and hence is not maximal.*
 (ii) *If $G = E_6$ and $H \cong \mathrm{Alt}(8)$ then H has a trivial summand on $V_{\min}$ and lies inside a conjugate of F_4 or the D_5 Levi subgroup. In particular, H is not maximal in G.*
(iii) *If $G = E_7$ and $H \cong \mathrm{Alt}(8)$ then H fixes a line on either $V_{\min}$ or $L(G)$, and hence is not maximal.*
 (iv) *If $G = E_8$ and $H \cong \mathrm{Alt}(8)$ then H fixes a line on $L(G)$, and hence is not maximal.*

Proof.   We will use Proposition 4.2 firstly, together with the dimensions of 1-cohomology from Table 8. If $G = F_4$ then we have two possible sets of composition factors, swapped by the graph automorphism, and the first one has pressure -3, so fixes a line on $V_{\min}$.

For $G = E_6$, there are again two sets of composition factors for H: the first has pressure -4 on $V_{\min}$ and -3 on $L(G)$, so fixes a line on both. Moreover, since in $V_{\min} \downarrow_H$ only one

composition factor has 1-cohomology, and there are five trivials, this means there are at least three trivial summands in $V_{\min}\downarrow_H$.

If the composition factors of $V_{\min}\downarrow_H$ are $14, 6^2, 1$, then we can check by computer that there are no modules of the form $6/1/6$ or $6/1/14$, so either H fixes a line or hyperplane on $V_{\min}$ or $V_{\min}\downarrow_H$ is indecomposable, with socle 6 up to duality. Since $\mathrm{Ext}^1(6,6)=0$, we get that $V_{\min}\downarrow_H$ has shape $6/1, 14/6$. There is a unique such module inside $P(6)$, but this has a trivial quotient, so H always fixes a line or hyperplane on $V_{\min}$. (Notice that the dimensions are incompatible with coming from the D_5 parabolic, as this acts with factors $16, 10, 1$, so in fact $H \le F_4$ and has a trivial summand.)

For $G = E_7$, Proposition 4.2 states that there are three possible sets of composition factors, with the first and third stabilizing lines on $L(G)$ (remember to remove one trivial since $\dim(L(G)) = 132$ for $p = 2$) as they have pressure -11 and -6 respectively. The second case, of $14^2, 6^4, 1^4$, has pressure 2, so assuming that H does not fix a line on $V_{\min}$, there are at most two composition factors in the socle by Proposition 1.9. The $\{1,6,14\}$-radicals of $P(6)$ and $P(14)$ have two and one trivial composition factor respectively, so the socle must therefore be $6 \oplus 2$: however, we cannot have a submodule $14/6, 6$ of $V_{\min}\downarrow_H$, since this has pressure 3. However, the $\{1,6,14\}$-radical of $P(6)$ is

$$6/14/1, 6/1, 14/6,$$

and so to support the trivial in the third socle layer we clearly need the 14 in the second socle layer, so in fact $(14/6)^{\oplus 2}$ is a submodule of $V_{\min}\downarrow_H$, a contradiction since it has pressure 4.

The remainder of this proof concerns E_8, and is very delicate and long. We will proceed in stages.

Step 1: Identifying the two difficult cases Firstly, using Proposition 4.2, we have exactly four possibilities for the set of composition factors on $L(G)$, namely

$$14, 6^{10}, (4,4^*)^{16}, 1^{46}, \quad 14^8, 6^{17}, (4,4^*)^2, 1^{18}, \quad 20^4, 14^4, 6^8, (4,4^*)^7, 1^8, \quad 20^4, 14^{10}, 6^2, 4, 4^*, 1^8.$$

Since the modules $6, 14, 20, 20^*$ each have 1-dimensional 1-cohomology by Proposition 3.1, in the first case H must have a trivial submodule by Proposition 1.9.

In the second possibility, there are no 20s at all, and so Lemma 3.3 applies to $L(G)\downarrow_H$, and we get that H fixes a line on $L(G)$ because it has fewer 6s than trivials.

We now apply Lemma 1.10 to $L(G)\downarrow_L$, where $L = \mathrm{Alt}(7)$. Since we proved in the previous section that L fixes a line on $L(G)$, this yields a map from the permutation module of H on the cosets of L to $L(G)\downarrow_H$. Since this module is $1/6/1$, and we assume that H does not fix a line on $L(G)$, we get that $1/6$ is a submodule of $L(G)\downarrow_H$.

We examine the third set of composition factors of H on $L(G)$ now, namely $20^4, 14^4, 6^8, (4,4^*)^7, 1^8$. Let $v = (1,2,3,4)(5,6)$, an element of order 4 that acts projectively on both 20 and 4, and acts with a single block of size 2 and otherwise projectively on 6 and 14. We write W for the subquotient of $L(G)\downarrow_H$ obtained by quotienting out by all submodules and taking the kernel of any quotient by factors of dimension 4, i.e., take the $\{4,4^*\}$-residual of $L(G)$, and quotient out by the $\{4,4^*\}$-radical of that to get W. This does not alter the action of v, since it acts projectively on these factors. Thus W is a submodule of copies of $P(6)$, $P(14)$ and $P(20)$.

Step 2: Finding $1/14/1$ subquotients We show that there are four disjoint subquotients of W of the form $1/14/1$. To see this, we first note that there does not exist any module M with no 14s as composition factors, with at least one trivial composition factor, and with no trivial submodule or quotient. This is proved easily: the largest submodules of $P(6)$ and $P(20)$ with no trivial quotients and no 14s are

$$6/4, 4^*/6/4, 4^*/6 \qquad \text{and} \qquad 4/20.$$

Hence we need a 14 above or below every trivial in W. However, there are four 14s and eight 1s in W, so we must have $1/14/1$ four times.

Step 3: Connecting the $1/14/1$s to 6s The socle of W has at most four 6s and two modules of dimension 20. We will prove that there is at most one $1/14/1$ lying above any 6 in the socle.

Since the dimension of $P(6)$ is 320, we cannot have $P(6)$ in $L(G) \downarrow_H$, so we may remove the top of $P(6)$, to leave its Jacobson radical. Since the top of W consists of 6s, 20s and 20^*s, we may take the $\{1, 4, 4^*, 14\}$-residual of this module, a module of dimension 290. This will contain an isomorphic copy of any submodule of $L(G) \downarrow_H$ lying above a 6 in the socle. We now take the $\{1\}'$-residual and then quotient out by the $\{1\}'$-radical to find the smallest subquotient containing all copies of $1/14/1$, and this leaves us with

$$1 \oplus 1, 1/14, 20, 20^*/1, 1, 1.$$

Thus we can have at most one $1/14/1$ lying above each 6 in the socle of W.

Performing the same calculation with $P(20)$ yields

$$1/14/1/20, 20^*/1/14/1,$$

which of course contains two $1/14/1$s, but needs $20 \oplus 20^*$ inside the module to obtain them both.

Step 4: No $6^{\oplus 4}$ in the socle Suppose that there are four 6s in $\mathrm{soc}(W)$, so that there are no 6s in $\mathrm{rad}(W)/\mathrm{soc}(W)$. We construct the largest submodule of $P(6)$ with exactly one 6, yielding

$$20, 20^*/1/1, 4, 4^*, 14/6,$$

which has no $1/14/1$ inside it, so we cannot stack enough $1/14/1$s, and hence we cannot have $6^{\oplus 4}$ inside $\mathrm{soc}(W)$. In particular, this means that we have one of the following socles:

$$6^3, 20, \qquad 6^2, 20, \qquad 6^3, 20^2, \qquad 6^2, 20^2,$$

where here 20^2 means two modules of dimension 20, either 20 or 20^*.

Step 5: Each $1/14/1$s requires a 10 or 20^* and a $6/4/6$ or $6^*/4/^*$ around it We construct a submodule M of $P(6)$ by the following process:

   (i) Start with the $\{4, 4^*, 6\}$-radical of $P(6)$ (this has dimension 40);
  (ii) Add to this all trivials possible (there is only one);
 (iii) Add to this all 14s possible (there are three);
 (iv) Add to this all trivials possible (there are three);
  (v) Add to this all 4s, 4^*s and 6s possible (this yields a module of dimension 112);
 (vi) Take the $\{6\}'$-residual of this module (this yields a module of dimension 110).

This process yields a module with five socle layers. If it were possible to construct a submodule of $P(6)$ with no 20s or 20^* and with a $1/14/1$ subquotient but no trivial submodules or quotients, it would lie inside M. It has socle layers

$$6, 6, 6/4, 4, 4^*, 4^*, 14, 14/1, 6, 6, 6/1, 4, 4^*, 14/6.$$

But this clearly does not have a $1/14/1$ inside it, and so we must need a 20 or 20^* above any such submodule to prevent the $1/14/1$ floating to the top of W.

Since we need at least two 6s in the socle of W, with $1/14/1$s above them, this means we need at least two 20s above that, so in particular they cannot be used to string together the two $1/14/1$s in the subquotient of $P(20)$ above. We therefore see that we need at least four factors in $\mathrm{soc}(W)$, so either $6^2, 20^2$ or $6^3, 20$. However, with $6^3, 20$, we would need three 20s above the 6^3, meaning the module cannot be self dual. To see this, placing a 20 above a $1/14/1$ creates a module

$$20/1/14/1,$$

and since there are four $1/14/1$s and four 20s or 20^*s, for each 20 placed above a $1/14/1$ in W there must be a 20^* placed below a $1/14/1$, to maintain self duality of W. We therefore see that there are two factors of dimension 20 in $\mathrm{soc}(W)$, and two 6s.

We now construct the same module M as in the process above, but instead of step (v), we place 20 on top of it. (We could use 20^* as well, but this would yield the image under the outer automorphism of H.) Instead of (vi) we then take the $\{20\}'$-residual. This will contain any smallest submodule of $P(6)$ with no trivial quotient or submodule and a $1/14/1$ inside it.

This process produces the module M with socle layers

$$20/1/4, 14, 20/1, 6/1, 4, 4^*, 14/6$$

and its dual M^* has structure

$$6/4/1, 6/14, 14/1, 1, 4^*, 4^*/20^*, 20^*;$$

from this we easily see the submodule $6/4^*/6$ inside M (there is a unique such uniserial module). Furthermore, since the only copy of 6 not in the socle is at the top of this submodule, we can see what happens if we remove this submodule by requiring a unique 6 in our module M: even before performing stage (vi) and removing quotients, the structure is

$$20/1/1, 4, 4^*, 14/6,$$

and so does not contain a $1/14/1$. Hence for every 6 supporting a $1/14/1$ above it, we need a $6/4/6$ or $6/4^*/6$ as a submodule.

Bringing this to a conclusion, for each $1/14/1$ in our module, we need a 20 or 20^* above it, and a $6/4/6$ or a $6/4^*/6$ below it, or vice versa. In particular, we will need four submodules or quotients of the form $6/4/6$ (up to duality) in W, in order for it not to have any trivial submodules or quotients.

Step 6: Contradiction for third set of composition factors The element $v = (1, 2, 3, 4)(5, 6)$ acts projectively on this module, so since it already acts projectively on $1/14/1$, if there are i disjoint subquotients of the form $6/4/6$ (or $6/4^*/6$) then we have that v acts on $L(G)$ with blocks $4^{58+i}, 2^{8-2i}$. However, from Table 5 we see that $i = 2$. However, we have just shown that $i = 4$, a contradiction. Thus W, and hence $L(G) \downarrow_H$, has a trivial submodule or quotient, as needed.

Step 7: Elimination of all possible socles for fourth set of composition factors The last case to consider is $64^2, 20, 20^*, 14^4, 6^2, 1^4$. The socle of W is a submodule of $6 \oplus 14^{\oplus 2} \oplus 20$, and we have already shown above that it contains 6. If it is all of them then there must be a submodule

$$1, 1, 1, 1/6, 14, 14, 20,$$

and each of 6, 14 and 20^* must cover one of the trivials. However, there are two extensions of 6 by this module, both of which lie above the 14s and not the 1s, so that this cannot be the socle.

On the other extreme, if the socle is just 6 then this also fails: the largest submodule of $P(6)$ with a single 6 has two trivial factors, so cannot work.

We do a similar thing if the socle has two or three factors: compute the largest submodule M of the appropriate sum of projectives, with the proviso that the composition factors of $\mathrm{top}(M)/\mathrm{soc}(M)$ do not include 6, and do not include 20 if $20 \in \mathrm{soc}(W)$ and similarly if $14^{\oplus 2} \leq \mathrm{soc}(W)$, and remove any simple quotients from M whose duals are not isomorphic to anything in the socle.

If the socle has factors $14, 6$ then we construct the module M, add all four 6s onto M, then remove all quotients not of dimension 14 or 6 to leave a module of dimension 122 with two trivial factors, so this doesn't work either.

We end with the socle being $6 \oplus 14^{\oplus 2}$. Define A_1 to be the preimage in $P(14)$ of the $\{1, 4, 4^*\}$-radical of $P(14)/\mathrm{soc}(P(14))$. We can construct a single extension of this module by 20, yielding a module A_2. In $P(14) \oplus P(14)$, adding a single 20 to $A_1 \oplus A_1$ always makes, by choosing diagonal submodules if necessary, a copy of $A_2 \oplus A_1$, up to isomorphism. The preimage of the $\{1, 4, 4^*\}$-radical of $P(6)/\mathrm{soc}(P(6))$ is a module A_3, on which one cannot place a 20 or 20^*. For $P(14)$ we construct similar modules A_1 and A_2 as above, but this time by taking the $\{1, 4, 4^*\}$-radical. To the sum of A_1, A_2 and A_3, we then place as many 1s, 4s, 4^*s and 20^*s as we can, to make a module of dimension 108 with six trivial factors. Placing on top of this as many 14s as we can, then as many 6s as we can, still yields three trivial quotients, so we can only cover three factors, and this is the final contradiction.

Hence there is no possibility for $\mathrm{soc}(W)$, so H must always fix a line on $L(G)$, as needed. $\square$

9. Alt(9)

Characteristics other than 2 and 3 have already been solved by Litterick in [20], so we consider only $p = 3$ and then $p = 2$. We continue our assumption that E_6 and E_7 are used to denote the simply connected forms, so that $Z(E_6(q))$ and $Z(E_7(q))$ have orders $\gcd(3, q - 1)$ and $\gcd(2, q - 1)$ respectively.

PROPOSITION 9.1.   *Suppose that $p = 3$.*
  (i) *If $G = F_4, E_6$ and $H \cong \mathrm{Alt}(9)$, then H does not embed into G.*
 (ii) *If $G = E_7$ and $H \cong \mathrm{Alt}(9)$ then either $L(G) \downarrow_H$ has a trivial summand or $(1, 2, 3)$ lies in a generic unipotent class for $V_{\min}$, and so there exists a positive-dimensional subgroup stabilizing the same subspaces of $V_{\min}$ as H. In either case, H is not maximal.*
(iii) *If $G = E_7$ and $H \cong 2 \cdot \mathrm{Alt}(9)$ then there is no embedding of H into G with $Z(H) = Z(G)$.*
 (iv) *If $G = E_8$ and $H \cong \mathrm{Alt}(9)$ then H fixes a line on $L(G)$, so is not maximal.*

Proof.   Recall the simple modules and dimensions of Ext^1 from Section 3. In characteristic 3 we need to consider $G = E_7$ and $G = E_8$. In E_7, H acts on $V_{\min}$ with factors $21^2, 7^2$, and so the element $z = (1, 2, 3, 4, 5, 6, 7, 8, 9)$, which acts on 21 as $9^2, 3$ and on 7 with a single block, must come from either class A_6, acting as $9^4, 7^2, 3^2$, or from class $E_6(a_1)$, acting as $9^6, 1^2$.

In the previous section, we proved that $L = \mathrm{Alt}(8)$ either contained a unipotent element from the generic class A_2 – and hence is contained in a positive-dimensional subgroup by Lemma 1.2 – or has a trivial summand on $L(G)$ and $x = (1, 2, 3)$ comes from class $A_2 + A_1$. In addition, $y = (1, 2, 3)(4, 5, 6)$ acts on $V_{\min}$ as $3^{18}, 1^2$, and comes from class $2A_2$ (as the trivial summand on $L(G)$ means it cannot come from class $2A_2 + A_1$, which acts as $3^{43}, 2^2$).

If the socle of $V_{\min} \downarrow_H$ is $7 \oplus 21$ then the module is one of $7/21 \oplus 21/7$ or $7, 21/7, 21$, with both modules being uniquely determined. The element y acts on the former module as $3^{12}, 1^{20}$, which is not right, so this is not the correct embedding. Thus we may assume that H acts as either $7, 21/7, 21$ or as $7 \oplus 7 \oplus 21/21$. The element z acts on the first module as $9^6, 1^2$ and on the second as $9^4, 7^2, 3^2$, so acts on $L(G)$ as either $9^{11}, 7^4, 3, 1^3$ or $9^{14}, 7$.

The composition factors of H on $L(G)$ are $35^2, 27, 21, 7^2, 1$, with the 27 splitting off as a summand since it lies in a separate block. The action of z on each composition factor is

$$9^3, 7, 1, \qquad 9^3, \qquad 9^2, 3, \qquad 7, \qquad 1.$$

As L fixes a line on $L(G)$, by Lemma 1.10 there is a map from the permutation module of H on L, which is $1/7/1$, to $L(G) \downarrow_H$. As we assume that H fixes no line on $L(G)$, this means that $1/7$ is a submodule of $L(G) \downarrow_H$.

Since z acts indecomposably on $1/7$, we cannot have that the factors of z on $L(G)$ are $9^{11}, 7^4, 3, 1^3$, and so z acts as $9^{14}, 7$ on $L(G)$. In particular, the only possible simple summands of $L(G)\downarrow_H$ are 27 and 7, and in particular 35 cannot be. However, since $1/7$ is a submodule of $L(G)\downarrow_H$ (and $7/1$ is a quotient) we cannot have that 7 is a summand. Hence the socle of $L(G)\downarrow_H$ (apart from the 27) is either $7\oplus 35$ or 7.

The largest submodule of $P(7)$ with a single 7 and other factors 1, 21 and 35 is $21, 35/1, 21, 35/7$. Since this has only three socle layers, we cannot have that the 21 has an extension with the 35s (since as $L(G)$ is self dual we would need socle layers of 7, 35, 21, 35 and 7). But then the two 35s cannot lie above the 7, a contradiction. Thus the socle of $L(G)\downarrow_H$ must contain 35.

Removing the 35s from the top and bottom of $L(G)\downarrow_H$, we are left with a self-dual module with factors $7^2, 21, 1$, and with $1/7$ as a submodule. Since there is no uniserial module $7/1/7$, this must be $M = 7/1, 21/7$.

The action of y on the direct sum of the composition factors of $L(G)$ is $3^{42}, 1^7$, so we cannot have any module extension inside $L(G)\downarrow_H$ that changes the action of y. We have that $\mathrm{Ext}^1(M, 35)$ is 1-dimensional, but when this module is constructed, the element y has a block of size 2 in its action, contradicting our earlier statement. This final contradiction proves that H fixes a line on $L(G)$, as needed for $G = E_7$.

We finally consider $G = E_8$, where the composition factors of $L(G)\downarrow_H$ are uniquely determined and are $35^2, 27, 21^5, 7^6, 1^4$.

As $H^1(H, M) = 0$ unless $\dim(M) = 7, 35, 41$ (and of course we can ignore 41), by taking the $\{1, 21, 27\}$-residual of $L(G)\downarrow_H$ and then quotienting out by the $\{1, 21, 27\}$-radical, we get a subquotient W of $L(G)\downarrow_H$ whose top and socle consist of 7s and 35s. Suppose that 35 lies in the socle: this means that W is a submodule of $P(35)\oplus P(7)^{\oplus i}$ for some i, and removing the 35 from the socle and top we have no more 35s. The $\{1, 7, 21\}$-radical of $P(7)$ is $7/1, 21/7$, and the preimage in $P(35)$ of the $\{1, 7, 21\}$-radical of $P(35)/\mathrm{soc}(P(35))$ has the form $1, 7, 21/35$. As we need to hide four trivials, we clearly need the socle to be $35\oplus 7^{\oplus 3}$, whence that 7 lying above the 35 cannot lie in W. However, $\mathrm{Ext}^1(35, 1, 21/35)$ has dimension 2, but all of these extensions have a trivial quotient, whence we fix a line. Thus the socle of W consists solely of 7s.

The $\{1, 7, 21, 35\}$-radical of $P(7)$ is

$$7/1, 21, 35/7, 7, 21, 35/1, 21, 35/7,$$

and so since we need to conceal four trivial composition factors, the socle of W needs at least two 7s. However, in order to have both trivial composition factors, we need the whole of this module as a submodule of W, which contains three 35s, a contradiction. Thus we need as many 7s in the socle as 1s above them, i.e., four, which is not possible. Thus H always fixes a line on $L(G)$. $\square$

For $p = 2$, if $G \neq E_8$ then [20, Theorem 1] gives the result, but we add it for sake of completeness since the proofs are easy using Proposition 4.3.

PROPOSITION 9.2.   *Suppose that $p = 2$.*
  (i) *If $G = F_4$ and $H \cong \mathrm{Alt}(9)$, then either H or its image under the graph automorphism fixes a line on $V_{\min}$, and so is not maximal.*
  (ii) *If $G = E_6$ and $H \cong \mathrm{Alt}(9)$, then H has a trivial summand on $V_{\min}$, and hence is not maximal*
  (iii) *If $G = E_7$ and $H \cong \mathrm{Alt}(9)$, then H fixes a line on $V_{\min}$, and hence is not maximal.*
  (iv) *If $G = E_8$ and $H \cong \mathrm{Alt}(9)$, then H fixes a line on $L(G)$, and hence is not maximal.*

Proof. The dimensions of 1-cohomology for simple H-modules are given in Table 8, and the composition factors for $V_{\min} \downarrow_H$ and $L(G) \downarrow_H$ are given in Proposition 4.3. These together prove that for $G = F_4$ either H or its image under the graph automorphism has pressure -2, and for E_6 either $V_{\min} \downarrow_H$ has pressure -3 or has composition factors 26 and 1, hence fixes a line or hyperplane on $V_{\min}$. However, if the factors are $8_1, 8_2, 8_3, 1^3$ then the trivial split off and if the factors are $26, 1$ then H cannot lie in a D_5 parabolic, with factors $10, 16, 1$ on $V_{\min}$, hence H lies in F_4, which acts as $26 \oplus 1$, proving the result. For E_7 we have factors $8_1^2, 8_2^2, 8_3^2, 1^8$, hence having eight trivial summands, or $26^2, 1^4$, of pressure 0, hence H fixes a line.

We are left with the case of $G = E_8$. The proof of this is quite long, and we will break it into stages. As we saw in Section 3, the simple modules here are 1, 8_1 and 8_2, permuted by the outer automorphism, 8_3, 20, 20*, 26, 48 and 78, plus modules that do not appear in our analysis. The modules 8_3 and 48 lie in a non-principal block of H.

Step 1: Eliminating all but one set of composition factors on $L(G)$. By Proposition 4.3 we may assume that the composition factors of H acting on $L(G)$ are one of

$$26, 8_1^8, 8_2^8, 8_3^8, 1^{30}, \quad 26^8, 8_1, 8_2, 8_3, 1^{16}, \quad 26^4, (20, 20^*)^2, 8_1^5, 8_2^2, 1^8$$

$$48^2, 26^4, 8_3^5, 1^8, \quad 48, 26^2, (20, 20^*)^2, 8_1^3, 8_2^3, 8_3^2, 1^4.$$

As $H^1(H, M)$ is 2-dimensional for $M = 26$ and 1-dimensional for 78, 20 and 20*, we have that these sets of composition factors yield modules of pressure -28, 0, 4, 0 and 4, respectively, so only the third and the fifth case might not fix a line on $L(G)$.

Let $L \cong \mathrm{Alt}(8) \leq H$. Since L fixes a line on $L(G)$, we may apply Lemma 1.10 to find a map from the permutation module of H on the cosets of L, which is $1 \oplus 8_3$, to $L(G) \downarrow_H$. Thus if 8_3 is not a submodule of $L(G) \downarrow_H$ then H fixes a line on $L(G)$. This deals with the third possibility for the composition factors of $L(G) \downarrow_H$ above, so we may assume from now on that the composition factors of $L(G) \downarrow_H$ are

$$48, 26^2, (20, 20^*)^2, 8_1^3, 8_2^3, 8_3^2, 1^4,$$

and that the composition factors of $L(G) \downarrow_L$ are

$$(20, 20^*)^2, 14^4, 6^8, (4, 4^*)^7, 1^8.$$

Step 2: $8_3/48/8_3$ is a summand of $L(G) \downarrow_H$. We concentrate on the non-principal block, which has factors $48, 8_3^2$. As $\mathrm{Ext}^1(48, 8_3)$ is 1-dimensional, and there are no self extensions of 8_3 or 48, we get a unique uniserial module of structure $8_3/48/8_3$, and so this summand of $L(G) \downarrow_H$ is one of

$$8_3/48/8_3, \quad 8_3 \oplus 8_3 \oplus 48,$$

since it is self dual. We understand completely the restrictions of these modules to L, and they are

$$1/14/6, 6/4, 4^*/1, 1, 6/6, 14/1, \quad (1/6/1)^{\oplus 2} \oplus 14/6/4, 4^*/6/14.$$

Notice that, as we may assume that $L(G) \downarrow_H$ has no trivial submodules, these two modules describe the summand of $L(G) \downarrow_L$ with trivial submodules.

Suppose that we are in the second case, i.e., 8_3 is a summand of $L(G) \downarrow_H$. We now see that $14/1$ cannot be a submodule of $L(G) \downarrow_L$: to see this, let ρ_1 and ρ_2 denote the projection maps along the non-principal and principal block summands M_1 and M_2 of $L(G) \downarrow_H$. If $M = 1/14/1$ is isomorphic to a submodule of $L(G) \downarrow_L$, then the sum of the images along ρ_1 and ρ_2 must equal all of M. However, as M is uniserial, this means that one of ρ_1 and ρ_2 is an injection for M, hence M is contained in M_1, but from the actions above we see that it is not. Similarly, any module that is an extension with quotient $14/1, 6$ (indecomposable) and submodule a

module with factors 4 and 4^* cannot appear in $L(G) \downarrow_L$ either. With these facts in mind we try to understand *all* Alt(8)s inside E_8 in characteristic 2, running through positive-dimensional subgroups and using Proposition 4.2 and its proof. We are only interested in those with factors as given above, so we only need concern ourselves with those in the A_7 parabolic with factors 4^2 or $4, 4^*$ on the natural, D_7 parabolic with factors $6, 4, 4^*$ on the natural, and D_8 with factors $(4, 4^*)^2$ on the natural. Thus we may assume that L lies in one of these

If L lies inside the A_7 parabolic X and its factors are $4, 4^*$, then the action on the natural is either $4 \oplus 4^*$ or $4/4^*$ (up to outer automorphism). The filtration of the action of X on $L(G)$ is (up to duality)

$$L(\lambda_7)/L(\lambda_2)/L(\lambda_3)/L(\lambda_1 + \lambda_7)/L(\lambda_3)/L(\lambda_6)/L(\lambda_1),$$

and so we are interested in $L(\lambda_6) = \Lambda^2(L(\lambda_1))^*$. In the two cases this module is $1, 6/14/1, 6$ and $1/14/1 \oplus 6^{\oplus 2}$, so has a submodule $14/1, 6$ (indecomposable) or $14/1$. Of course, the layer below this has factors 4 and 4^*, so has no extensions with 1 or 14, so we have one of the two submodules of $L(G) \downarrow_L$ not allowed by the above arguments.

Alternatively, L acts as $4 \oplus 4$ on the natural, but then all of $L(\lambda_1), L(\lambda_2), L(\lambda_3)$ and their duals have no trivial composition factors. Hence the $\{1\}'$-residual, modulo its $\{1\}'$-radical, is a subquotient of $L(\lambda_1 + \lambda_7)$, but this module is clearly

$$(4 \otimes 4^*)^{\oplus 4} = (1/14/1)^{\oplus 4},$$

so we can never have $1/6/1$ being a submodule of $L(G) \downarrow_L$. Thus L does not lie in the A_7 parabolic.

If L lies inside $X = D_8$ with factors $(4, 4^*)^2$ on the natural module V, then $\Lambda^2(V)$ is a summand of $L(G) \downarrow_X$, and so if $4 \oplus 4^*$ is a submodule of $V \downarrow_L$ then $4 \otimes 4^* = 1/14/1$ is a submodule of $L(G) \downarrow_L$, which is not allowed. Hence $4 \oplus 4$ is the socle of $V \downarrow_L$. (The $\{4, 4^*\}$-radical of $P(4)$ is $4^*/4$.) This means that L acts on V as $(4^*/4)^{\oplus 2}$. The exterior square of this module has three summands, two of which are $\Lambda^2(4^*/4)$, which we earlier saw had $14/1, 6$ as a submodule, so not allowed in $L(G) \downarrow_L$, and $L \not\leq X$ again.

We are left with X being the D_7 parabolic, with L acting on the natural with factors $6, 4, 4^*$. Certainly L stabilizes a 4-space on the natural module, and acts as $\mathrm{GL}_4(2)$ on it, so the stabilizer in X acts irreducibly on the 4-space.

This means that the radical of the form on the 4-space is either 0 or the whole space, so that it is either non-singular or totally isotropic. These stabilizers are a maximal parabolic or a $D_2 D_5$ subgroup, the latter subgroup not containing $\mathrm{GL}_4(2)$ acting irreducibly on a 4-dimensional non-degenerate subspace of the natural module, and so we lie in a parabolic subgroup of X, hence a different parabolic subgroup of G. However, the parabolic subgroups of G contained in the D_7 parabolic are all (after A_1s and A_2s have been stripped out) in the E_7 or A_7 parabolics, and so have been dealt with.

This proves that we cannot have $8_3^{\oplus 2} \oplus 48$ as a summand of $L(G) \downarrow_H$. We note that all unipotent elements in H act projectively on $8_3/48/8_3$.

Step 3: 26 is not a submodule of $L(G) \downarrow_H$. We come back to the principal block summand, which has composition factors $26^2, (20, 20^*)^2, 8_1^3, 8_2^3, 1^4$. The element $v = (1, 2, 3, 4)(5, 6)$ acts projectively on $8_1, 8_2, 20$ and 20^*, and as $4^5, 2^3$ on the 26. Thus v has at least fifty-eight blocks of size 4 in its action on $L(G)$, so must act as $4^{60}, 2^4$ by Table 5.

Write W for the $\{8_1, 8_2\}$-residual modulo its $\{8_1, 8_2\}$-radical, so W is a submodule of a sum of $P(26)$, $P(20)$ and $P(20^*)$s. Remove also any simple summands from W.

Suppose that 26 lies in the socle of W. The $\{26, 78\}'$-radical of $P(26)/\mathrm{soc}(P(26))$, lifted back to $P(26)$, has the form

$$20, 20^*/1, 1, 8_1, 8_2/26.$$

The $\{26, 78\}'$-radical of $P(20)$ is

$$8_2/20^*/1, 8_1/20,$$

and so if 26 lies in $\mathrm{soc}(W)$, then $\mathrm{soc}(W) = 20^{\oplus 2} \oplus 26$ or $20 \oplus 20^* \oplus 26$. In both cases, the action of v on the submodule

$$1, 1, 1, 1/26, 20, 20^{\pm}$$

is projective (where $20^{\pm}$ means either 20 or 20^*), plus $3, 2^2, 1^3$. Since v acts projectively on all of 8_1, 8_2, 20 and 20^*, and acts on 26 as $4^5, 2^3$, extending this module (possibly with some 8_1s and 8_2s as well) by the $26 \oplus 20 \oplus 20^{\mp}$ (where $20^{\mp}$ means the dual of the module selected earlier) on the top of W requires us to, for the action of v, add 2^3 onto $3, 2^2, 1^3$ to make $4^2, 2^4$.

　　We claim that, for the cyclic group of order 4, there is no submodule N of $M = 4^2, 2^4$ of shape $3, 2^2, 1^3$, whose quotient is 2^3. To see this, firstly note that the socles of M and N coincide, so we may work modulo $\mathrm{soc}(M)$. Now the socle of M has dimension 6 and the socle of N has dimension 3, so the extension of N by $\mathrm{soc}(M/N)$ must be split; loosely speaking, the socle of M/N is 'contained in' the socle of N (although of course this doesn't strictly make sense). We therefore have an extension of 1^3 by $2, 1^5$ to make $3^2, 1^4$ (remember we are working modulo $\mathrm{soc}(M)$), but 1^3 is a trivial module, so that the kernel of the map from $3^2, 1^4$ to 1^3 contains the Jacobson radical of $3^2, 1^4$, namely 2^2, a contradiction.

　　What we have therefore proved is that 26 is not in $\mathrm{soc}(W)$, and hence the socle of W is one of 20, $20^{\oplus 2}$ and $20 \oplus 20^*$.

Step 4: Eliminating the remaining socle possibilities. If the socle is 20 then W is a submodule of $P(20)$, and we firstly take the $\{78\}'$-radical of $P(20)$, then take the $\{20^*\}'$-residual of that (since $\mathrm{soc}(W) = 20$, $\mathrm{top}(W) = 20^*$), to produce the self-dual module

$$20^*/1, 8_2/20, 26/1, 1, 8_1/20^*, 26/1, 8_2/20.$$

However, v acts projectively on this, and since it acts projectively on the non-principal block part of $L(G)$ as well, this means that it would act projectively on all of $L(G)$, which is not allowed by Table 5, as v is supposed to act as $4^{60}, 2^4$. Thus the socle isn't simple.

　　Thus the socle is $20 \oplus 20^{\pm}$, and in particular all $20, 20^*$s are taken up by the socle and top of W. We thus want to take the preimage in $P(20)/\mathrm{soc}(P(20))$ of the $\{1, 8_1, 8_2, 26\}$-radical M, which is

$$1, 8_1/26/1, 8_2/20.$$

In theory this could work, but we will need to place 20 or 20^* on top of this module and hide the trivial quotient. Firstly, $\mathrm{Ext}^1(20, M) = 0$, so this will not work; $\mathrm{Ext}^1(20^*, M)$ is 1-dimensional, but this module is

$$1, 8_1/26, 20^*/1, 8_2/20,$$

and so the socle cannot consist of $20^{\pm}$s either. Thus H fixes a line on $L(G)$, as needed.　　□

10. Alt(10) and above

　　For $H \cong \mathrm{Alt}(n)$ for $n \geq 10$, Litterick in [19] proved that H lies inside a σ-stable positive-dimensional subgroup for all G, except possibly a single set of composition factors for $p = 2$, $G = E_8$ and $H \cong \mathrm{Alt}(10)$, which was completed in his forthcoming Memoir [20]. Using pressure and Frobenius reciprocity we can provide a shorter proof of this result.

　　The composition factors of $L(G) \downarrow_H$ are

$$48^2, 26^4, 8^5, 1^8,$$

and since $H^1(H, 48) = 0$ and $H^1(H, 26) = H^1(H, 8)$ are 1-dimensional we have that $L(G) \downarrow_H$ has pressure 1. If we remove any 48s in the socle and top of $L(G) \downarrow_H$ to get a module W, then since W has pressure 1 by Proposition 1.9 we have that $\mathrm{soc}(W)$ is either 8 or 26, so that W is a submodule of either $P(8)$ or $P(26)$.

Using Lemma 1.10, since $L(G) \downarrow_L$ has a trivial submodule (where $L = \mathrm{Alt}(9)$), there is a non-trivial map from $1_L \uparrow^H = 1/8/1$ to $L(G) \downarrow_H$, and hence to W. Thus either H fixes a line on $L(G)$, as needed, or $1/8$ is a submodule of $L(G) \downarrow_H$, and so $W \leq P(8)$.

Furthermore, since W has pressure 1, W must have at least seventeen socle layers, since all 1s, 8s and 26s must lie in different layers. The projective $P(8)$ has exactly nineteen socle layers, and is given in [4]. The seventeenth socle layer of $P(8)$ is $26 \oplus 200$ and the eighteenth is 1, which therefore means that if $\mathrm{soc}(W) = 8$ then W cannot be self dual. Hence H fixes a line on $L(G)$, so is not maximal.

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