University of Birmingham

On commuting varieties of parabolic subalgebras

Goodwin, Simon; Goddard, Russell

[Link to publication on Research at Birmingham portal](#)

ON COMMUTING VARIETIES OF PARABOLIC SUBALGEBRAS

RUSSELL GODDARD AND SIMON M. GOODWIN

ABSTRACT. Let G be a connected reductive algebraic group over an algebraically closed field k, and assume that the characteristic of k is zero or a pretty good prime for G. Let P be a parabolic subgroup of G and let $\mathfrak{p}$ be the Lie algebra of P. We consider the commuting variety $\mathcal{C}(\mathfrak{p}) = \{(X,Y) \in \mathfrak{p} \times \mathfrak{p} \mid [X,Y] = 0\}$. Our main theorem gives a necessary and sufficient condition for irreducibility of $\mathcal{C}(\mathfrak{p})$ in terms of the modality of the adjoint action of P on the nilpotent variety of $\mathfrak{p}$. As a consequence, for the case $P = B$ a Borel subgroup of G, we give a classification of when $\mathcal{C}(\mathfrak{b})$ is irreducible; this builds on a partial classification given by Keeton. Further, in cases where $\mathcal{C}(\mathfrak{p})$ is irreducible, we consider whether $\mathcal{C}(\mathfrak{p})$ is a normal variety. In particular, this leads to a classification of when $\mathcal{C}(\mathfrak{b})$ is normal.

1. INTRODUCTION

Let G be a connected reductive algebraic group over an algebraically closed field k with $\operatorname{char} k = p \geq 0$, and write $\mathfrak{g} = \operatorname{Lie} G$ for the Lie algebra of G. We assume that $p = 0$ or is a pretty good prime for G; the notion of pretty good prime for G was introduced by Herpel in [He, Definition 2.11], and is discussed more later in the introduction.

In [Ri], Richardson proved that the commuting variety

$$\mathcal{C}(\mathfrak{g}) = \{(X,Y) \in \mathfrak{g} \times \mathfrak{g} \mid [X,Y] = 0\}$$

of $\mathfrak{g}$ is irreducible for $\operatorname{char} k = 0$. For $\mathfrak{g} = \mathfrak{gl}_n(k)$ this was previously proved by Motzkin and Taussky in [MT] and independently by Gerstenhaber in [Ge]. Richardson's result was subsequently extended to positive (pretty good) characteristic by Levy in [Le].

There has been much recent research interest in $\mathcal{C}(\mathfrak{g})$ and related varieties, and the theory of commuting varieties finds applications in various areas of representation theory, and geometry. We refer the reader for example to [Bu], [Gi], [GS] [Ng], [Pr] and [PY], and the references therein.

Let P be a parabolic subgroup of G with Lie algebra $\mathfrak{p} = \operatorname{Lie} P$. It is a natural generalization to consider the commuting variety

$$\mathcal{C}(\mathfrak{p}) = \{(X,Y) \in \mathfrak{p} \times \mathfrak{p} \mid [X,Y] = 0\}$$

of $\mathfrak{p}$. In case $P = B$ is a Borel subgroup and $p = 0$, the commuting variety $\mathcal{C}(\mathfrak{b})$ was considered in the PhD thesis of Keeton, [Ke].

Our main theorem gives a necessary and sufficient condition for $\mathcal{C}(\mathfrak{p})$ to be irreducible. In the statement of the theorem we use the modality of an algebraic group action, which is recalled in Section 2. Also we use the notation $\mathcal{N}(\mathfrak{k})$ for the variety of nilpotent elements in a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, and we write $\operatorname{rank} G$ for the rank of G and $\operatorname{ssrank} H$ for the semisimple rank of a Levi subgroup H of G.

Theorem 1.1. *Let G be a connected reductive algebraic group over an algebraically closed field k with* $\operatorname{char} k = 0$ *or a pretty good prime for G. Let P be a parabolic subgroup of G*

and let T be a maximal torus of G contained in P. Then the commuting variety $\mathcal{C}(\mathfrak{p})$ is irreducible if and only if $\mathrm{mod}(P \cap H : \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})) < \mathrm{ssrank}\, H$ for all Levi subgroups $H \neq T$ of G containing T. Moreover, if $\mathcal{C}(\mathfrak{p})$ is irreducible, then $\dim \mathcal{C}(\mathfrak{p}) = \dim \mathfrak{p} + \mathrm{rank}\, G$.

The assumption that $p = 0$ or is a pretty good prime for G is required in the statement of Theorem 1.1 is to ensure separability all orbit maps for the adjoint action of P on $\mathfrak{p}$, or in other words that the scheme theoretic centralizer is smooth. This does not appear to be in the literature, and is stated below and proved in Section 4.

Theorem 1.2. *Let G be a connected reductive algebraic group over an algebraically closed field k with $\mathrm{char}\, k = 0$ or a pretty good prime for G. Let P be a parabolic subgroup of G and let $X \in \mathfrak{p}$. Then the adjoint orbit map $P \to P \cdot X$ is separable.*

As mentioned above $\mathcal{C}(\mathfrak{b})$ was investigated by Keeton in [Ke]. In particular, [Ke, Theorem 6.1] includes an equivalent statement to Theorem 1.1 for the case $P = B$ and $p = 0$. Our methods build on those used in Keeton, but we require a significantly different approach to deal with all parabolic subgroups, thus we include all details.

Keeton proceeds to give a partial classification of irreducibility of $\mathcal{C}(\mathfrak{b})$ in [Ke, Sections 6.3 and 6.4]. In this case, we note that $\mathcal{N}(\mathfrak{b}) = \mathfrak{u}$ is the nilradical of $\mathfrak{b}$. We make use of recent results in [GMR] and [PS], which allow us to determine $\mathrm{mod}(B : \mathfrak{u})$ for G of sufficiently large rank in order to give a complete classification of when $\mathcal{C}(\mathfrak{b})$ is irreducible.

Theorem 1.3. *Let G be a connected reductive algebraic group over an algebraically closed field k with $\mathrm{char}\, k = 0$ or a pretty good prime for G. Then $\mathcal{C}(\mathfrak{b})$ is irreducible if and only if the type of each simple component of G is one of the following.*

- *A_l for $l \leq 15$;*
- *B_l for $l \leq 6$;*
- *C_l for $l \leq 6$;*
- *D_l for $l \leq 7$;*
- *G_2 or E_6.*

For $P \neq B$, less is known about the modalities $\mathrm{mod}(P : \mathcal{N}(\mathfrak{p}))$, and we only briefly discuss cases where we can determine whether $\mathcal{C}(\mathfrak{p})$ is irreducible or reducible at the end of this paper. Here we just remark that from results of Röhrle in [Rö1], we can deduce that for a fixed value of $\mathrm{ssrank}\, L$ where L is a Levi factor of P, we have that $\mathcal{C}(\mathfrak{p})$ is reducible if $\mathrm{rank}\, G$ is sufficiently large. Also in very recent work for the case $G = \mathrm{GL}_n(k)$, Bulois–Boos in [BB, Main theorem] gives a classification of cases where $\mathrm{mod}(P : \mathcal{N}(\mathfrak{p})) = 0$, which gives cases where $\mathcal{C}(\mathfrak{p})$ is irreducible.

For cases where $\mathcal{C}(\mathfrak{p})$ is irreducible, we also consider the question of whether $\mathcal{C}(\mathfrak{p})$ is normal. Our results in this direction are contained in Section 6. In order to outline these results we recall some background on commuting schemes.

The commuting scheme $\mathcal{CS}(\mathfrak{g})$ of $\mathfrak{g}$ is the subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined by the ideal $I_\mathfrak{g}$ of $k[\mathfrak{g} \times \mathfrak{g}]$ generated by the regular functions $(X, Y) \mapsto f([X, Y])$ for $f \in \mathfrak{g}^*$; so $\mathcal{C}(\mathfrak{g})$ is the underlying variety of $\mathcal{CS}(\mathfrak{g})$. The question of whether $\mathcal{CS}(\mathfrak{g})$ is reduced and normal is a long-standing problem. For $p = 0$, it was proved by Popov in [Po] that the singular locus of $\mathcal{CS}(\mathfrak{g})$ has codimension 2, which reduces the problem to showing that $\mathcal{CS}(\mathfrak{g})$ is Cohen-Macaulay. We also mention that Ginzburg proved that the normalization of $\mathcal{C}(\mathfrak{g})$ is Cohen-Macaulay in [Gi].

Under the assumption that $\mathcal{CS}(\mathfrak{l})$ is Cohen–Macaulay for L the Levi factor of P containing T, and that $\mathrm{mod}(P \cap H : \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})) < \mathrm{ssrank}\, H - 1$ for all Levi subgroups $H \neq T$ of G containing T with $\mathrm{ssrank}\, H > 1$, we prove normality of $\mathcal{C}(\mathfrak{p})$ in Theorem 6.3. Our methods build on those of Keeton in [Ke, Section 6.2], and we show that the singular locus of the commuting scheme of $\mathfrak{p}$ has codimension 2. We note that this statement about the singular locus holds without the Cohen–Macaulay assumption on $\mathcal{CS}(\mathfrak{l})$.

For the case $P = B$, the commuting scheme of $\mathfrak{l} = \mathfrak{t}$ is certainly Cohen–Macaulay, and we show that the modality condition is also sufficient in Theorem 6.4. This leads a classification of when $\mathcal{C}(\mathfrak{b})$ is normal in Theorem 7.1.

We mention that another motivation for investigating $\mathcal{C}(\mathfrak{p})$ is that one can hope to understand $\mathcal{C}(\mathfrak{g})$ through the fibre bundle $G \times^P \mathcal{C}(\mathfrak{p})$, where P acts on $\mathcal{C}(\mathfrak{p})$ diagonally by the adjoint action. To recall the definition of $G \times^P \mathcal{C}(\mathfrak{p})$, we consider the action of P on $G \times \mathcal{C}(\mathfrak{p})$ by $x \cdot (g, (X, Y)) = (gx^{-1}, (x \cdot X, x \cdot Y))$ for $x \in P$, $g \in G$, and $X, Y \in \mathfrak{p}$. Then the set of P-orbits in $G \times \mathcal{C}(\mathfrak{p})$ has the structure of a variety, this can be explained in direct analogy with the construction given in [Ja, §8.11]. Further, there is a surjective morphism $G \times^P \mathcal{C}(\mathfrak{p}) \to \mathcal{C}(\mathfrak{g})$ given by $[(g, (X, Y))] \mapsto (g \cdot X, g \cdot Y)$. This construction is considered for the case $P = B$ in [Ke, §5.6].

We end the introduction by mentioning some related recent results, and then giving some remarks about our main results.

In [BE], Bulois and Evain investigated the irreducibility and equidimensionality of $\mathcal{C}(\mathcal{N}(\mathfrak{p}))$ for certain parabolic subgroups of $\mathfrak{g} = \mathfrak{gl}_n(k)$. In [GR], Röhrle and the second author classified when $\mathcal{C}(\mathfrak{u})$ is irreducible, where $\mathfrak{u} = \mathcal{N}(\mathfrak{b})$ is the nilradical of $\mathfrak{b}$, and also determined the irreducible components of $\mathcal{C}(\mathfrak{u})$ in minimal cases where it is not irreducible.

We note that Theorem 1.1 can be reduced to simple G, though this is not required for our proof. To see this we write G as a central product $G = Z(G)G_1 \cdots G_m$, where $G_1, \ldots, G_m$ are the simple components of G, and $P_i = P \cap G_i$. Then we have $\mathfrak{p} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$, and it is straightforward to prove that $\mathcal{C}(\mathfrak{p}) = \mathcal{C}(\mathfrak{z}(\mathfrak{g})) \times \mathcal{C}(\mathfrak{p}_1) \times \cdots \times \mathcal{C}(\mathfrak{p}_m)$. Thus $\mathcal{C}(\mathfrak{p})$ is irreducible if and only if $\mathcal{C}(\mathfrak{p}_i)$ is irreducible for each i. Further, we can easily see that $\mathrm{mod}(P \cap H : \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})) = \mathrm{mod}(P_1 \cap H : \mathcal{N}(\mathfrak{p}_1 \cap \mathfrak{h})) + \cdots + \mathrm{mod}(P_m \cap H : \mathcal{N}(\mathfrak{p}_m \cap \mathfrak{h}))$ for H a Levi subgroup of G containing T. This implies that the modality condition on P in Theorem 1.1 holds if and only if it holds for each P_i.

For G simple it seems plausible that the condition $\mathrm{mod}(P \cap H : \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})) < \mathrm{ssrank}\, H$ for all Levi subgroups $H \neq T$ of G containing T can be replaced by the single condition that $\mathrm{mod}(P : \mathcal{N}(\mathfrak{p})) < \mathrm{ssrank}\, G$ in Theorem 1.1. As consequence of Theorem 1.3, we can, a fortiori, weaken our condition in this way for $P = B$. The methods used in our proof do not get around the need for the inductive assumption. However, we do not consider this to be a serious limitation, because if $\mathrm{mod}(P : \mathcal{N}(\mathfrak{p}))$ has been determined, then most likely $\mathrm{mod}(P \cap H : \mathcal{N}(\mathfrak{p} \cap \mathfrak{h}))$ can be determined for all H as in the statement as well.

As already mentioned the assumption that $p = 0$ or is a pretty good prime for G is essentially required for Theorem 1.2. We recall from [He, Lemma 2.12], that p is a pretty good prime for G if and only if p is a good prime for G and there is no p-torsion in both the quotient of the character group of G by the root lattice and the quotient of the cocharacter group of G by the coroot lattice. Also we note that p is pretty good is implied by the *standard hypothesis* from [Ja, §2.9]:

(H1) The derived subgroup of G is simply connected;

(H2) char k is zero or a good prime for G; and

(H3) there is a nondegenerate G-invariant symmetric bilinear form on $\mathfrak{g}$.

In fact as shown in [He, Theorem 5.2], we have that $p > 0$ is pretty good for G if and only if we can obtain G from a reductive group G' satisfying (H1)–(H3) after a finite number of the operations:

(R1) the replacement of G' by a separably isogenous group G''; or

(R2) the replacement of $G' = G'' \times S$ by G'', where S is a torus.

(We recall that a separable isogeny $\pi : G' \to G''$ is a surjective homomorphism with finite kernel such that $d\pi$ is an isomorphism, and we say that G' and G'' are separably isogenous if there is a separable isogeny between them.)

We remark that in [Le], Levy proved that $\mathcal{C}(\mathfrak{g})$ is irreducible under the hypothesis (H1)–(H3). Using [He, Theorem 5.2], we can see that irreducibility of $\mathcal{C}(\mathfrak{g})$ holds when $p = 0$ or is a pretty good prime. Indeed if $p = 0$ or is pretty good prime for G, then there exists G' satisfying (H1)–(H3) such that $\mathfrak{g} \cong \mathfrak{g}'$.

Our methods require the assumption that p is pretty good. In particular, this is required for Theorem 1.2. In low rank examples, we can already see that the situation is different when p is not very good. For example, for $p = 2$, and $G = \mathrm{SL}_2(k)$, we have that $\mathfrak{b}$ is abelian, so that $\dim \mathcal{C}(\mathfrak{b}) > \dim B + \mathrm{rank}\, G$. Also for $p = 2$ and $G = SO_5(k)$, it is possible to show that $\mathcal{C}(\mathfrak{g})$ is not irreducible. It would be interesting to understand the general situation for p not pretty good.

We note that our methods apply also to the commuting variety

$$\mathcal{C}(P) = \{(x, y) \in P \times P \mid xy = yx\}$$

of P. Thus with some modifications, analogous results about $\mathcal{C}(P)$ can be proved.

2. Algebraic group actions, modality and commuting varieties

Let G be a linear algebraic group over an algebraically closed field k with Lie algebra $\mathfrak{g}$, and let V be a variety on which G acts morphically, in the sense of [Bo, §1.7]. Let $g \in G$, $v \in V$ and let U be a subvariety of V. We write $g \cdot v$ for the image of v under g, and $G \cdot v = \{g \cdot v \mid g \in G\}$ for the G-orbit of v. The stabilizer of v in G is denoted $C_G(v) = \{g \in G \mid g \cdot v = v\}$. We write $g \cdot U = \{g \cdot u \mid u \in U\}$ for the image of U under g, and $G \cdot U = \{g \cdot u \mid g \in G, u \in U\}$ for the G-saturation of U. The normalizer of U in G is denoted $N_G(U) = \{g \in G \mid g \cdot u \in U \text{ for all } u \in U\}$.

Below we recall the definition of the modality of G on V, and the sheets of G on V. In order to do this we set $V_j = \{v \in V \mid \dim G \cdot v = j\}$ for $j \in \mathbb{Z}_{\geq 0}$.

The *modality of G on V* is defined to be

$$\mathrm{mod}(G : V) = \max_{j \in \mathbb{Z}_{\geq 0}} (\dim V_j - j).$$

Informally, $\mathrm{mod}(G : V)$ is the maximum number of parameters on which a family of G-orbits depends. The notion of modality originates in the work of Arnold [Ar], we refer the reader also to [Vi] and [PV, Section 5.2].

4

We define $\mathcal{S}(G, V)_j$ to be the set of irreducible components of V_j, and let

$$\mathcal{S}(G, V) = \bigsqcup_{j \in \mathbb{Z}_{\geq 0}} \mathcal{S}(G, V)_j.$$

The elements of $\mathcal{S}(G, V)$ are called *sheets of G on V*. The notion of sheets was introduced by Borho and Kraft in [BK]. We remark that in the case that G acts on V with finitely many orbits, the sheets of G on V coincide with the orbits of G on V.

Next we recall that the commuting variety of $\mathfrak{g}$ is

$$\mathcal{C}(\mathfrak{g}) = \{(X, Y) \in \mathfrak{g} \times \mathfrak{g} \mid [X, Y] = 0\}.$$

The connection between commuting varieties and modalities is made clear in the Lemma 2.1. Before stating this lemma we briefly discuss separability of the orbit maps for the adjoint action of G on $\mathfrak{g}$. Let $X \in \mathfrak{g}$, write $\phi_X : G \to G \cdot X$ for the orbit map for X and $\mathfrak{c}_{\mathfrak{g}}(X) = \{Y \in \mathfrak{g} \mid [Y, X] = 0\}$ for the centralizer of X in $\mathfrak{g}$. We recall from [Bo, Proposition 6.7] that separability of ϕ_X is equivalent to $\operatorname{Lie} C_G(X) = \mathfrak{c}_{\mathfrak{g}}(X)$ (or in other words that the scheme theoretic centralizer is smooth). Therefore, given the assumption that orbit maps are separable, Lemma 2.1 can be proved with the same argument as in characteristic zero, see for example [GR, Lemma 2.1].

Lemma 2.1. *Let G be a linear algebraic group with Lie algebra $\mathfrak{g}$. Assume that for all $X \in \mathfrak{g}$, the orbit map $G \to G \cdot X$ is separable. Then*

$$\dim \mathcal{C}(\mathfrak{g}) = \dim \mathfrak{g} + \operatorname{mod}(G : \mathfrak{g}).$$

3. Notation for reductive groups

We introduce the notation used in the remainder of this paper, and recall some standard results about reductive algebraic groups.

Let G be a reductive algebraic group over an algebraically closed field k of characteristic $p \geq 0$ and let $\mathfrak{g} = \operatorname{Lie} G$ be the Lie algebra of G. We assume throughout that $p = 0$ or p is a pretty good prime for G.

For a closed subgroup K of G, we write $\mathfrak{k} = \operatorname{Lie} K$ for the Lie algebra of K. For a reductive subgroup H of G, we write $\operatorname{rank} H$ for the rank of H and $\operatorname{ssrank} H$ for the semisimple rank of H. For a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, we denote the centre of $\mathfrak{k}$ by $\mathfrak{z}(\mathfrak{k})$, and the variety of nilpotent elements of $\mathfrak{k}$ by $\mathcal{N}(\mathfrak{k})$.

Let P be a parabolic subgroup of G and let $T \subseteq P$ be a maximal torus of G. Let L be the Levi factor of P containing T and let U_P be the unipotent radical of P; the Lie algebra of U_P is denoted $\mathfrak{u}_P$. Let B be a Borel subgroup of G contained in P and containing T, and let $U = U_B$ be the unipotent radical of U.

We write Φ for the root system of G with respect to T. For $\alpha \in \Phi$, we denote the corresponding root subspace of $\mathfrak{g}$ by $\mathfrak{g}_\alpha$ and let E_α be a generator for $\mathfrak{g}_\alpha$. For a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ stable under the adjoint action of T, we write $\Phi(\mathfrak{k})$ for the set of $\alpha \in \Phi$ such that $\mathfrak{g}_\alpha \subseteq \mathfrak{k}$. Let $\Phi^+ = \Phi(\mathfrak{b})$ be system of positive roots determined by B and let $\Pi \subseteq \Phi$ be the corresponding set of simple roots.

We recall that a subgroup H of G is called a Levi subgroup if it is a Levi factor of some parabolic subgroup of G. This is equivalent to $H = C_G(S)$ where S is a torus in G, or, under the assumption that p is good, $H = C_G(X)$ where $X \in \mathfrak{g}$ is semisimple.

For $p > 0$, we note that, under our assumption that p is a pretty good prime for G, we also have that p is a pretty good prime for any Levi subgroup of G. This follows easily from [He, Definition 2.11].

Given a subset J of Π, we let Φ_J be the closed subsystem of Φ generated by J. We say that a Levi subgroup H of G is a standard Levi subgroup if H contains T and $\Phi(\mathfrak{h}) = \Phi_J$ for some $J \subseteq \Pi$, or equivalently if H is the Levi factor containing T of a parabolic subgroup containing B. In particular, L is a standard Levi subgroup of G and we let I be the subset of Π such that $\Phi(\mathfrak{l}) = \Phi_I$.

We recall that $X \in \mathfrak{g}$ is called *regular* if $\dim \mathfrak{c}_{\mathfrak{g}}(X) = \operatorname{rank} G$. We write $\mathfrak{g}^{\mathrm{reg}}$ for the set of regular elements in $\mathfrak{g}$, and for a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, we let $\mathfrak{k}^{\mathrm{reg}} = \mathfrak{k} \cap \mathfrak{g}^{\mathrm{reg}}$.

The set of regular semisimple elements in $\mathfrak{g}$ is denoted by $\mathfrak{g}_{\mathrm{ss}}^{\mathrm{reg}}$. Next we observe that under our assumption that $p = 0$ or is a pretty good prime for G, there exist regular semisimple elements in $\mathfrak{t}$; we expect this is well-known.

We view $\Phi \subseteq X(T)$, where $X(T)$ is the character group of T, and given $\alpha \in \Phi$, and write $d\alpha : \mathfrak{t} \to k$ for its differential. To check there are regular semisimple elements in $\mathfrak{t}$, we need to observe that there exists $X \in \mathfrak{t}$ such that $d\alpha(X) \neq 0$ for all $\alpha \in \Phi$. It suffices to check that for any $\alpha \in \Phi$, we have that $d\alpha$ is nonzero. Let $\alpha \in \Phi$. We may write $\alpha = m\beta$ where β is an indivisible element of $X(T)$ and $m \in \mathbb{Z}_{\geq 1}$. Then there is m-torsion in $X(T)/\mathbb{Z}\Phi$, so that $p \nmid m$, because p is pretty good for G. There exists a cocharacter $\lambda : k^{\times} \to T$ such that $\langle \beta, \lambda \rangle = 1$, and we write $X = d\lambda(1) \in \mathfrak{t}$. Then we have $d\alpha(X) = m$, where we view m as an element of k, which is nonzero in k.

Now using [Le, Theorem 1], the proof of [Hu, Theorem 2.5] can be adapted to prove that the regular semisimple elements form an open subvariety of $\mathfrak{g}$. In turn it follows that $\mathfrak{g}^{\mathrm{reg}}$ is an open subvariety, and that the minimal dimension of a centralizer in $\mathfrak{g}$ is $\operatorname{rank} G$.

4. Separability of orbit maps for the adjoint action of a parabolic subgroups

The goal of this section is to prove Theorem 1.2. The main idea of the proof is based on arguments in [SS, Chapter I §5], see also [Ja, Theorem 2.5], where the theorem is proved in the case $P = G$. Also the ideas for the proof of [He, Theorem 3.3] are used for Lemma 4.1, which gives a reduction that we use in the proof of Theorem 1.2.

We require some preliminary discussion for the statement of Lemma 4.1. First we recall that the operations (R1)–(R2) are stated in the introduction.

Suppose $\pi : G \to G'$ is a separable isogeny, where G' be a reductive algebraic group, i.e. G is obtained from G' by an (R1) operation. Then $P' = \pi(P)$ is a parabolic subgroup of G and the proof of [Ja, Proposition 2.8] can be adapted to show that the adjoint orbit map $P \to P \cdot X$ is separable for all $X \in \mathfrak{p}$ if and only if the adjoint orbit map $P' \to P' \cdot X'$ is separable for all $X' \in \mathfrak{p}'$.

Now suppose that $G' = G \times S$, where S is a torus, i.e. G is obtained from G' by an (R2) operation. Then $P' = P \times S$ is a parabolic subgroup of G'. Further, since S is contained in the centre of G, it is clear that the adjoint orbit map $P \to P \cdot X$ is separable for all $X \in \mathfrak{p}$ if and only if the adjoint orbit map $P' \to P' \cdot X'$ is separable for all $X' \in \mathfrak{p}'$.

Iteration of the arguments in the previous two paragraphs proves the following lemma.

Lemma 4.1. *Suppose that G is obtained from the reductive algebraic group G' by a finite number of the operations* (R1)–(R2), *and let P' be the parabolic subgroup of G' corresponding*

to P. Then the adjoint orbit map $P \to P \cdot X$ is separable for all $X \in \mathfrak{p}$ if and only if the adjoint orbit map $P' \to P' \cdot X'$ is separable for all $X' \in \mathfrak{p}'$.

Armed with Lemma 4.1, we are ready to proceed with the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 4.1 and the discussion at the end of the introduction, we may assume that G satisfies (H1)–(H3). It follows from the arguments in [Ja, §2.6–2.9] that if p is pretty good for G, then G can be obtained after a finite number of the operations (R1)–(R2) from a reductive group G' with a faithful representation $\rho : G' \to \mathrm{GL}_n(k)$ such that restriction of the trace form on $\mathfrak{gl}_n(k)$ to $d\rho(\mathfrak{g}')$ is nondegenerate. Hence by Lemma 4.1, we may assume that there is a faithful representation $\rho : G \to \mathrm{GL}_n(k)$ such that $d\rho : \mathfrak{g} \to \mathfrak{gl}_n(k)$ is injective and the restriction of the trace form on $\mathfrak{gl}_n(k)$ to $d\rho(\mathfrak{g})$ is nondegenerate. We use ρ to identify G with a subgroup of $\overline{G} = \mathrm{GL}_n(k)$, and $d\rho$ to identify $\mathfrak{g}$ with a subalgebra of $\overline{\mathfrak{g}} = \mathfrak{gl}_n(k)$.

Recall that $I = \{\alpha \in \Pi \mid \mathfrak{g}_\alpha \subseteq \mathfrak{l}\}$ and let m be the index of the cocharacter group of T in the group of coweights of T. Let $\lambda : k^\times \to T$ be the unique cocharacter with image inside the derived subgroup of G satisfying

$$\langle \lambda, \alpha \rangle = \begin{cases} 0 & \text{if } \alpha \in I \\ m & \text{if } \alpha \in \Pi \setminus I, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ is the pairing between cocharacters and characters of T.

Then λ determines a grading

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(\lambda; j)$$

where $\mathfrak{g}(\lambda; j) = \{Y \in \mathfrak{g} \mid \lambda(t) \cdot Y = t^j Y\}$. We have

$$\mathfrak{p} = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \mathfrak{g}(\lambda; j) \quad \text{and} \quad \mathfrak{l} = \mathfrak{g}(\lambda; 0).$$

Similarly λ defines a grading

$$\overline{\mathfrak{g}} = \bigoplus_{j \in \mathbb{Z}} \overline{\mathfrak{g}}(\lambda; j)$$

and we define

$$\overline{\mathfrak{p}} = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \overline{\mathfrak{g}}(\lambda; j).$$

Since the restriction of the trace form on $\overline{\mathfrak{g}}$ to $\mathfrak{g}$ is nondegenerate, we may decompose $\overline{\mathfrak{g}} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$ as a G-module, where $\tilde{\mathfrak{g}}$ is the orthogonal complement to $\tilde{\mathfrak{g}}$ in $\overline{\mathfrak{g}}$. We let $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}} \cap \overline{\mathfrak{p}}$. Then we see that $\overline{\mathfrak{p}} = \mathfrak{p} \oplus \tilde{\mathfrak{p}}$ as P-modules.

We write $\overline{P}$ for the parabolic subgroup of $\overline{G}$ with Lie algebra $\overline{\mathfrak{p}}$. Then we have that $C_{\overline{P}}(X)$ consists of the invertible elements of $\mathfrak{c}_{\overline{\mathfrak{p}}}(X)$. Therefore, $\dim C_{\overline{P}}(X) = \dim \mathfrak{c}_{\overline{\mathfrak{p}}}(X)$, and thus $\mathrm{Lie}\, C_{\overline{P}}(X) = \mathfrak{c}_{\overline{\mathfrak{p}}}(X)$.

Let $f : \overline{P} \to \overline{\mathfrak{p}}$ be the map defined by $f(g) = g \cdot X - X$. The differential $(df)_1 : \overline{\mathfrak{p}} \to \overline{\mathfrak{p}}$ of f at 1 is given by $(df)_1(Y) = [Y, X]$. We have

$$\dim f(\overline{P}) = \dim \overline{P} \cdot X = \dim \overline{P} - \dim C_{\overline{P}}(X) = \dim \overline{\mathfrak{p}} - \dim \mathfrak{c}_{\overline{\mathfrak{p}}}(X) = \dim([\overline{\mathfrak{p}}, X]),$$

and it follows that $T_0(f(\overline{P})) = [\overline{\mathfrak{p}}, X]$.

We have

$$T_0(f(P)) \subseteq T_0(f(\overline{P})) \cap T_0(\mathfrak{p})$$
$$= [\overline{\mathfrak{p}}, X] \cap \mathfrak{p}$$
$$= [\mathfrak{p}, X]$$
$$\subseteq T_0(f(P)).$$

The first inclusion here is immediate, whilst the first equality follows from $T_0(f(\overline{P})) = [\overline{\mathfrak{p}}, X]$. To see the second equality holds, consider $\overline{Y} = Y + \tilde{Y} \in \overline{\mathfrak{p}}$ with $Y \in \mathfrak{p}$ and $\tilde{Y} \in \tilde{\mathfrak{p}}$. Then

$$[\overline{Y}, X] = [Y + \tilde{Y}, X] = [Y, X] + [\tilde{Y}, X]$$

where $[Y, X] \in \mathfrak{p}$ and $[\tilde{Y}, X] \in \tilde{\mathfrak{p}}$ as $\overline{\mathfrak{p}} = \mathfrak{p} \oplus \tilde{\mathfrak{p}}$ is a P-module decomposition. We deduce that $[\overline{\mathfrak{p}}, X] \cap \mathfrak{p} = [\mathfrak{p}, X]$. The last inclusion holds because $[\mathfrak{p}, X] = (df)_1(\mathfrak{p})$.

Hence, we have that $[\mathfrak{p}, X] = T_0(f(P))$. Therefore,

$$\dim C_P(X) = \dim P - \dim(P \cdot X) = \dim P - \dim f(P) = \dim \mathfrak{p} - \dim([\mathfrak{p}, X]) = \dim \mathfrak{c}_{\mathfrak{p}}(X),$$

which implies that $\operatorname{Lie} C_P(X) = \mathfrak{c}_{\mathfrak{p}}(X)$ and thus that the orbit map $P \to P \cdot X$ is separable. $\qquad \square$

Remark 4.2. The key ingredient in our proof of Theorem 1.2 is the existence of $\overline{P} \leq \overline{G}$ such that $P = \overline{P} \cap G$, the adjoint orbit maps $\overline{P} \to \overline{P} \cdot X$ are separable, and there is a P-module decomposition $\overline{\mathfrak{p}} = \mathfrak{p} \oplus \tilde{\mathfrak{p}}$. Thus the proof is applicable in other situations, for example we could replace P by any subgroup of K of G such that K is normalized by T, the restriction of the trace form to $d\rho(\mathfrak{t} \cap \mathfrak{k})$ is nondegenerate, and $\Phi(\mathfrak{k})$ is closed under addition within the set of weights of T in $\overline{\mathfrak{g}}$.

5. On irreducibility of $\mathcal{C}(\mathfrak{p})$

The goal of this section is to prove Theorem 1.1. Before getting to this proof, we give some preliminary results. For the first lemma, we define $\mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{p})$ to be the subvariety of $\mathcal{C}(\mathfrak{p})$ consisting of pairs of commuting regular semisimple elements.

Lemma 5.1.
(a) $\mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{p})$ *is open and irreducible in* $\mathcal{C}(\mathfrak{p})$,
(b) *the closure* $\overline{\mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{p})}$ *of* $\mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{p})$ *is an irreducible component of* $\mathcal{C}(\mathfrak{p})$, *and*
(c) $\dim \mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{p}) = \dim \mathfrak{p} + \operatorname{rank} G$.

Proof. Then we can deduce that the variety of regular semisimple elements $\mathfrak{p}_{\mathrm{ss}}^{\mathrm{reg}} = \mathfrak{p} \cap \mathfrak{g}_{\mathrm{ss}}^{\mathrm{reg}}$ in $\mathfrak{p}$ is open in $\mathfrak{p}$. It follows that $\mathfrak{p}_{\mathrm{ss}}^{\mathrm{reg}} \times \mathfrak{p}_{\mathrm{ss}}^{\mathrm{reg}}$ is open in $\mathfrak{p} \times \mathfrak{p}$. Thus $\mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{p}) = \mathcal{C}(\mathfrak{p}) \cap (\mathfrak{p}_{\mathrm{ss}}^{\mathrm{reg}} \times \mathfrak{p}_{\mathrm{ss}}^{\mathrm{reg}})$ is open in $\mathcal{C}(\mathfrak{p})$.

Consider the map $\mu : P \times (\mathfrak{t}^{\mathrm{reg}} \times \mathfrak{t}^{\mathrm{reg}}) \to \mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{p})$ defined by $\mu(g, X, Y) = (g \cdot X, g \cdot Y)$; we recall that $\mathfrak{t}^{\mathrm{reg}}$ is the set of regular elements in $\mathfrak{t}$. Let $(X, Y) \in \mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{p})$. There exists a maximal torus T_1 of P such that $X \in \mathfrak{t}_1$. We have $\mathfrak{c}_{\mathfrak{p}}(X) = \mathfrak{t}_1$, as X is regular semisimple, so that $Y \in \mathfrak{t}_1$ too. Since maximal tori of P are conjugate, it follows that μ is surjective. Therefore, $\mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{p})$ is irreducible.

Having proved $\mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{p})$ is both open and irreducible, we deduce that $\overline{\mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{p})}$ is an irreducible component of $\mathcal{C}(\mathfrak{p})$.

Now suppose that $\mu(g, X, Y) = \mu(1, X', Y')$ for some $g \in P$ and $X, Y, X', Y' \in \mathfrak{t}^{\mathrm{reg}}$. Then $g \cdot X = X'$, so $g \cdot \mathfrak{t} = g \cdot \mathfrak{c}_{\mathfrak{p}}(X) = \mathfrak{c}_{\mathfrak{p}}(X') = \mathfrak{t}$. Hence, $g \in N_P(\mathfrak{t})$. It follows that the dimension of each fibre of μ is $\dim N_P(\mathfrak{t}) = \dim \mathfrak{t} = \operatorname{rank} G$. Therefore,

$$\dim \mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{p}) = \dim P + \dim \mathfrak{t}^{\mathrm{reg}} + \dim \mathfrak{t}^{\mathrm{reg}} - \dim \mathfrak{t} = \dim \mathfrak{p} + \operatorname{rank} G. \qquad \square$$

We have the following immediate corollary.

Corollary 5.2. $\mathcal{C}(\mathfrak{p})$ *is irreducible if and only if* $\mathcal{C}(\mathfrak{p}) = \overline{\mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{p})}$.

We note that Richardson proved that $\mathcal{C}(\mathfrak{g})$ is irreducible by proving that $\mathcal{C}(\mathfrak{g}) = \overline{\mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{g})}$ in [Ri]; this is also the case for Levy's proof in [Le] for positive characteristic. In particular, we have $\dim \mathcal{C}(\mathfrak{g}) = \dim G + \operatorname{rank} G$.

Our next lemma gives a lower bound for the dimensions of irreducible components of $\mathcal{C}(\mathfrak{p})$.

Lemma 5.3. *All irreducible components of* $\mathcal{C}(\mathfrak{p})$ *have dimension at least* $\dim \mathfrak{p} + \operatorname{rank} G$.

Proof. Consider the Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_P$. Let $X, Y \in \mathfrak{p}$, and write $X = X_1 + X_2$, $Y = Y_1 + Y_2$, where $X_1, Y_1 \in \mathfrak{l}$ and $X_2, Y_2 \in \mathfrak{u}_P$. Then $[X, Y] = [X_1, Y_1] + [X_1, Y_2] + [X_2, Y_1] + [X_2, Y_2]$; we note that $[X_1, Y_1] \in \mathfrak{l}$ and the remaining terms are in $\mathfrak{u}_P$.

Let $\mathcal{L}$ be the subvariety of $\mathfrak{p} \times \mathfrak{p}$ of pairs (X, Y) such that $(X_1, Y_1) \in \mathcal{C}(\mathfrak{l})$ and $X_2, Y_2 \in \mathfrak{u}_P$. Then we have that $\mathcal{C}(\mathfrak{p}) \subseteq \mathcal{L}$ and $\mathcal{L} \cong \mathcal{C}(\mathfrak{l}) \times (\mathfrak{u}_P \times \mathfrak{u}_P)$. Since $\mathcal{C}(\mathfrak{l})$ is irreducible of dimension $\dim \mathfrak{l} + \operatorname{rank} L$, we have that $\mathcal{L}$ is irreducible of dimension $\dim \mathfrak{l} + \operatorname{rank} G + 2 \dim \mathfrak{u}_P = \dim \mathfrak{p} + \operatorname{rank} G + \dim \mathfrak{u}_P$.

Consider the commutator map $F : \mathcal{L} \to \mathfrak{u}_P$ given by $F(X, Y) = [X, Y]$. Then $\mathcal{C}(\mathfrak{p})$ is the zero fibre of F, so the codimension in $\mathcal{L}$ of each irreducible component of $\mathcal{C}(\mathfrak{p})$ is at most $\dim \mathfrak{u}_P$. Therefore, each irreducible component of $\mathcal{C}(\mathfrak{p})$ has dimension at least

$$(\dim \mathfrak{p} + \operatorname{rank} G + \dim \mathfrak{u}_P) - \dim \mathfrak{u}_P = \dim \mathfrak{p} + \operatorname{rank} G. \qquad \square$$

We are now ready to prove our main theorem.

Proof of Theorem 1.1. In order to prove this theorem we decompose $\mathcal{C}(\mathfrak{p})$ as a disjoint union of irreducible subvarieties such that the closure of some of these subvarieties are the irreducible components of $\mathcal{C}(\mathfrak{p})$. This allows us to determine when $\mathcal{C}(\mathfrak{p})$ is irreducible and is achieved by partitioning the P-orbits in $\mathfrak{p}$ in a way that generalizes the partition of G orbits in $\mathfrak{g}$ into decomposition classes as given by Borho and Kraft in [BK]. This is similar to the approach used by Popov when considering $\mathcal{C}(\mathfrak{g})$ in [Po].

Let $\hat{\mathcal{H}}$ denote the set of all Levi subgroups H of G containing T. We note that $H \in \hat{\mathcal{H}}$ is determined by the set of $\alpha \in \Phi$ such that $\mathfrak{g}_\alpha \subseteq \mathfrak{h}$. Thus we see that $\hat{\mathcal{H}}$ is a finite set. We note that different Levi subgroups in $\hat{\mathcal{H}}$ may be conjugate under P, and we choose $\mathcal{H}$ to be a subset of $\hat{\mathcal{H}}$ containing one representative from each P-conjugacy class.

Let $H \in \mathcal{H}$. We define $\mathfrak{z}(\mathfrak{h})^{\mathrm{reg}} = \{X \in \mathfrak{z}(\mathfrak{h}) \mid \mathfrak{c}_{\mathfrak{g}}(X) = \mathfrak{h}\}$. We have that $\mathfrak{z}(\mathfrak{h})^{\mathrm{reg}}$ is open in $\mathfrak{z}(\mathfrak{h})$, so $\mathfrak{z}(\mathfrak{h})^{\mathrm{reg}}$ is irreducible and $\dim \mathfrak{z}(\mathfrak{h})^{\mathrm{reg}} = \dim \mathfrak{z}(\mathfrak{h}) = \operatorname{rank} G - \operatorname{ssrank} H$.

We have that $P \cap H$ is a parabolic subgroup of H, and we also consider $N_P(P \cap H)$. We note that $P \cap H$ has finite index in $N_P(P \cap H)$, because any coset in $N_P(P \cap H)/(P \cap H)$ can be chosen to have a representative that normalizes T. Thus $\dim N_P(P \cap H) = \dim(P \cap H)$.

We write $\mathcal{S}_H = \mathcal{S}(P \cap H, \mathfrak{p} \cap \mathfrak{h})$ for the set of sheets of $P \cap H$ on $\mathcal{N}(\mathfrak{p} \cap \mathfrak{h})$ and $\mathcal{S}_{H,j} = \mathcal{S}(P \cap H, \mathcal{N}(\mathfrak{p} \cap \mathfrak{h}))_j$. So we have $\mathcal{S}_H = \bigsqcup_{j \in \mathbb{Z}_{\geq 0}} \mathcal{S}_{H,j}$. We note that $N_P(P \cap H)$ acts of $\mathcal{N}(\mathfrak{p} \cap \mathfrak{h})$, and this gives rise to an action of $N_P(P \cap H)/(P \cap H)$ on $\mathcal{S}_H$.

Let $X \in \mathfrak{p}$ with Jordan decomposition $X = X_s + X_n$. Up to the adjoint action of P we may assume that $X_s \in \mathfrak{z}(\mathfrak{h})^{\mathrm{reg}}$ for some $H \in \mathcal{H}$. Then we have that $X_n \in \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})$, so $X_n \in S$ for some $S \in \mathcal{S}_H$.

Let $S \in \mathcal{S}_{H,j}$, where $j \in \mathbb{Z}_{\geq 0}$. We define $\mathcal{M}_{H,S} \subseteq \mathfrak{p}$ to be the variety of all $X = X_s + X_n \in \mathfrak{p}$ such that $X_s \in \mathfrak{z}(\mathfrak{h})^{\mathrm{reg}}$ and $X_n \in S$. We have $\mathcal{M}_{H,S} \cong \mathfrak{z}(\mathfrak{h})^{\mathrm{reg}} \times S$, so $\mathcal{M}_{H,S}$ is irreducible and $\dim \mathcal{M}_{H,S} = \mathrm{rank}\, G - \mathrm{ssrank}\, H + \dim S$.

Let $\mathcal{C}'_{H,S} = \{(X,Y) \mid X \in \mathcal{M}_{H,S}, Y \in \mathfrak{c}_\mathfrak{p}(X)\}$. For $X \in \mathcal{M}_{H,S}$ with Jordan decomposition $X = X_s + X_n$, we have $\mathfrak{c}_\mathfrak{p}(X) = \mathfrak{c}_\mathfrak{p}(X_s) \cap \mathfrak{c}_\mathfrak{p}(X_n) = \mathfrak{c}_{\mathfrak{p} \cap \mathfrak{h}}(X_n)$. Thus $\dim \mathfrak{c}_\mathfrak{p}(X) = \dim \mathfrak{c}_{\mathfrak{p} \cap \mathfrak{h}}(X_n) = \dim(\mathfrak{p} \cap \mathfrak{h}) - j$, where for the last equality we require Theorem 1.2. Therefore, the dimension of $\mathfrak{c}_\mathfrak{p}(X)$ does not depend on the choice of $X \in \mathcal{M}_{H,S}$.

Let $\mathcal{X}$ be an irreducible component of $\mathcal{C}'_{H,S}$ and consider the morphism $\pi : \mathcal{X} \to \mathcal{M}_{H,S}$ given by projecting on to the first component. The function taking $(X,Y) \in \mathcal{X}$ to the maximal dimension of an irreducible component of $\pi^{-1}(X,Y)$ containing (X,Y) is upper semi-continuous, see for example [Mum, §8 Corollary 3]. Thus the set of $X \in \mathcal{M}_{H,S}$ such that $\{X\} \times \mathfrak{c}_\mathfrak{p}(X) \subseteq \mathcal{X}$ is closed in $\mathcal{M}_{H,S}$; here we require that $\dim \mathfrak{c}_\mathfrak{p}(X)$ does not depend on $X \in \mathcal{M}_{H,S}$ as proved above. Combining this with the irreducibility $\mathcal{M}_{H,S}$ allows us to deduce that $\mathcal{C}'_{H,S}$ is irreducible.

Also we note that, for any $X \in \mathcal{M}_{H,S}$, we have

$$\dim \mathcal{C}'_{H,S} = \dim \mathcal{M}_{H,S} + \dim \mathfrak{c}_\mathfrak{p}(X)$$
$$= \mathrm{rank}\, G - \mathrm{ssrank}\, H + \dim S + \dim(\mathfrak{p} \cap \mathfrak{h}) - j$$
$$= (\mathrm{rank}\, G - \mathrm{ssrank}\, H) + \dim(\mathfrak{p} \cap \mathfrak{h}) + (\dim S - j). \tag{5.1}$$

We define $\mathcal{C}_{H,S} = P \cdot \mathcal{C}'_{H,S} = \{(g \cdot X, g \cdot Y) \mid g \in P, (X,Y) \in \mathcal{C}'_{H,S}\}$ to be the P-saturation of $\mathcal{C}'_{H,S}$. Then we have that $\mathcal{C}_{H,S}$ is irreducible being the image of the morphism $\phi : P \times \mathcal{C}'_S \to \mathcal{C}(\mathfrak{p})$ given by $\phi(g, (X,Y)) = (g \cdot X, g \cdot Y)$. For $S' \in \mathcal{S}_H$, we see that $\mathcal{C}_{H,S} = \mathcal{C}_{H,S'}$ if and only if S and S' lie in the same $N_P(P \cap H)/(P \cap H)$-orbit.

We claim that

$$\dim \mathcal{C}_{H,S} = (\mathrm{rank}\, G - \mathrm{ssrank}\, H) + \dim \mathfrak{p} + (\dim S - j). \tag{5.2}$$

To prove this we consider the dimension of the fibres of the morphism $\phi : P \times \mathcal{C}'_{H,S} \to \mathcal{C}_{H,S}$ defined above. We note that the dimension of these fibres is constant on P-orbits, so it suffices to determine $\dim \phi^{-1}(X,Y)$ for $(X,Y) \in \mathcal{C}'_{H,S}$.

Let $(X,Y) \in \mathcal{C}'_{H,S}$. We note that if $g \in P \cap H$, then $(g, (g^{-1} \cdot X, g^{-1} \cdot Y)) \in \phi^{-1}(X,Y)$. Now suppose that $(g, X', Y')) \in \phi^{-1}(X,Y)$, so we have $X = g \cdot X'$ and $Y = g \cdot Y'$. We have Jordan decompositions $X = X_s + X_n$ and $X' = X'_s + X'_n$, where $X_s, X'_s \in \mathfrak{z}(\mathfrak{h})^{\mathrm{reg}}$, because $X, X' \in \mathcal{M}_S$. Also we have $X_s = g \cdot X'_s$, and thus $\mathfrak{c}_\mathfrak{p}(X_s) = g \cdot \mathfrak{c}_\mathfrak{p}(X'_s)$. Since, $\mathfrak{c}_\mathfrak{p}(X_s) = \mathfrak{p} \cap \mathfrak{h} = \mathfrak{c}_\mathfrak{p}(X'_s)$, we have $g \in N_P(\mathfrak{p} \cap \mathfrak{h}) = N_P(P \cap H)$. Hence, we have shown that

$$P \cap H \subseteq \{g \in P \mid (g, (g^{-1} \cdot X, g^{-1} \cdot Y)) \in \phi^{-1}(X,Y)\} \subseteq N_P(P \cap H),$$

which implies that $\dim \phi^{-1}(X,Y) = \dim(P \cap H)$.

We have seen that the dimension of each fibre of ϕ is equal to $\dim(P \cap H)$. Hence, we have $\dim \mathcal{C}_{H,S} = \dim P + \dim \mathcal{C}'_{H,S} - \dim(P \cap H)$ and substituting from (5.1) gives (5.2).

Since S is a sheet for the action of $P \cap H$ on $\mathcal{N}(\mathfrak{p} \cap \mathfrak{h})$, we deduce that

$$\dim \mathcal{C}_{H,S} \leq (\mathrm{rank}\, G - \mathrm{ssrank}\, H) + \dim \mathfrak{p} + \mathrm{mod}(P \cap H : \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})). \tag{5.3}$$

By construction we have the disjoint union

$$\mathcal{C}(\mathfrak{p}) = \bigsqcup_{\substack{H \in \mathcal{H} \\ S = \dot{\mathcal{S}}_H}} \mathcal{C}_{H,S},$$

where $\dot{\mathcal{S}}_H$ denotes a set of representatives for the $N_P(P \cap H)/(P \cap H)$-orbits in $\mathcal{S}_H$. Moreover, the closure $\overline{\mathcal{C}_{H,S}}$ of each $\mathcal{C}_{H,S}$ is closed and irreducible. Thus the irreducible components of $\mathcal{C}(\mathfrak{p})$ are given by some of the $\overline{\mathcal{C}_{H,S}}$.

We have $\mathcal{C}_{T,\{0\}}$ contains $\mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{p})$ as an open subset, so that $\overline{\mathcal{C}_{T,\{0\}}} = \overline{\mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}}(\mathfrak{p})$ is an irreducible component by Lemma 5.1.

Now suppose that $\mathrm{mod}(P \cap H : \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})) < \mathrm{ssrank}\,H$ for all $H \in \mathcal{H} \setminus \{T\}$. Then we have $\dim \overline{\mathcal{C}_{H,S}} = \dim \mathcal{C}_{H,S} < \dim \mathfrak{p} + \mathrm{rank}\,G$ for all $S \in \mathcal{S}_H$ and therefore $\overline{\mathcal{C}_{H,S}}$ is not an irreducible component of $\mathcal{C}(\mathfrak{p})$ by Lemma 5.3. Therefore, $\overline{\mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}}(\mathfrak{p})$ is the only irreducible component of $\mathcal{C}(\mathfrak{p})$ and hence $\mathcal{C}(\mathfrak{p})$ is irreducible.

Conversely, suppose that there is $H \in \mathcal{H} \setminus \{T\}$ such that $\mathrm{mod}(P \cap H : \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})) \geq \mathrm{ssrank}\,H$. Then there is $S \in \mathcal{S}_{H,j}$ for some $j \in \mathbb{Z}_{\geq 0}$ such that $\dim S - j \geq \mathrm{ssrank}\,H$. Then we have $\dim \overline{\mathcal{C}_{H,S}} = \dim \mathcal{C}_{H,S} \geq \mathrm{rank}\,G + \dim \mathfrak{p}$. Since $\mathcal{C}_{H,S} \cap \mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{p}) = \varnothing$, and $\dim \mathcal{C}_{H,S} \geq \dim \mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}(\mathfrak{p})$, we have $\overline{\mathcal{C}_{H,S}} \not\subseteq \overline{\mathcal{C}_{\mathrm{ss}}^{\mathrm{reg}}}(\mathfrak{p})$. Hence, $\mathcal{C}(\mathfrak{p})$ is not irreducible by Lemma 5.1.

Finally, it is immediate from Lemma 5.1 that $\dim \mathcal{C}(\mathfrak{p}) = \dim \mathfrak{p} + \mathrm{rank}\,G$ when $\mathcal{C}(\mathfrak{p})$ is irreducible. $\qquad\square$

We have the following immediate corollary, which gives a monotonicity for irreducibility of $\mathcal{C}(\mathfrak{p})$.

Corollary 5.4. *Suppose that $\mathcal{C}(\mathfrak{p})$ is irreducible and let H be a Levi subgroup of G containing T. Then $\mathcal{C}(\mathfrak{p} \cap \mathfrak{h})$ is irreducible.*

In fact, the commuting variety of $\mathfrak{p}$ gets much more complicated as $\mathrm{rank}\,G - \mathrm{ssrank}\,L$ gets large, as explained, in particular for the case $P = B$, in the next remark.

Remark 5.5. Let H be a Levi subgroup of G containing T and let $\mathcal{X}$ be an irreducible component of $\mathcal{C}(\mathfrak{p} \cap \mathfrak{h})$. We remark that the methods in our proof of Theorem 1.1 can be extended to show that the closure of $P \cdot \mathcal{X} = \{(g \cdot X, g \cdot Y) \mid g \in P, (X,Y) \in \mathcal{X}\}$ is an irreducible component of $\mathcal{C}(\mathfrak{p})$.

Let $P = B$ be a Borel subgroup of G. In this case $\mathcal{N}(\mathfrak{b}) = \mathfrak{u}$, and as a consequence of [Rö1, Theorem 3.1], we have that $\mathrm{mod}(B : \mathfrak{u}) \to \infty$ quadratically as $\mathrm{ssrank}\,G \to \infty$. We may choose inside G a sequence of Levi subgroup $1 = H_0 \leq \cdots \leq H_{\mathrm{ssrank}\,G} = G$ such that $\mathrm{ssrank}\,H_j = j$ for each j. Then we can see that for each j such that $j \leq \mathrm{mod}(B \cap H_j : \mathfrak{u} \cap \mathfrak{h}_j) < \mathrm{mod}(B \cap H_{j+1} : \mathfrak{u} \cap \mathfrak{h}_{j+1}) - 1$, there are more irreducible components of $\mathcal{C}(\mathfrak{b} \cap \mathfrak{h}_{j+1})$ than of $\mathcal{C}(\mathfrak{b} \cap \mathfrak{h}_j)$. Since $\mathrm{mod}(B : \mathfrak{u}) \to \infty$ quadratically as $\mathrm{ssrank}\,G \to \infty$, we can deduce that the number of irreducible components of $\mathcal{C}(\mathfrak{b})$ gets arbitrarily large as $\mathrm{ssrank}\,G \to \infty$. Indeed the number of components grows at least linearly in the rank, and the dimension of some irreducible components get much larger than $\dim \mathfrak{b} + \mathrm{rank}\,G$.

We end this section by recording a useful inductive result regarding irreducibility of $\mathcal{C}(\mathfrak{p})$.

Proposition 5.6. *Let P and Q be parabolic subgroups of G. Suppose that $P \leq Q$. Then $\mathcal{C}(\mathfrak{q}) = Q \cdot \mathcal{C}(\mathfrak{p})$. In particular, if $\mathcal{C}(\mathfrak{p})$ is irreducible, then $\mathcal{C}(\mathfrak{q})$ is irreducible.*

Proof. Let $(X, Y) \in \mathcal{C}(\mathfrak{q})$ with Jordan decompositions $X = X_s + X_n$ and $Y = Y_s + Y_n$. Then $X_n, Y_n \in \mathfrak{c}_\mathfrak{g}(X_s) \cap \mathfrak{c}_\mathfrak{g}(Y_s)$, which is the Lie algebra of the Levi subgroup $H = C_G(X_s) \cap C_G(Y_s)$ of G. By [He, Corollary 5.5], there is a Springer isomorphism $\phi : \mathcal{N}(\mathfrak{h}) \to \mathcal{U}(H)$, where $\mathcal{U}(H)$ denotes the unipotent variety of H. Since ϕ is H-equivariant we deduce that $\phi(X_n), \phi(Y_n)$ commute. Therefore, $\phi(X_n)$ and $\phi(Y_n)$ lie in a Borel subgroup B_H of $Q \cap H$. Since Borel subgroups of $Q \cap H$ are conjugate, there exists $g \in Q \cap H$ such that $gB_Hg^{-1} \subseteq P \cap H$. By H-equivariance, we have that ϕ sends $\mathcal{N}(\mathfrak{h}) \cap \mathfrak{p}$ to $\mathcal{U}(H) \cap P$ and thus that $(g \cdot X, g \cdot Y) \in \mathcal{C}(\mathfrak{p})$. Hence, $\mathcal{C}(\mathfrak{q}) = Q \cdot \mathcal{C}(\mathfrak{p})$.

Suppose $\mathcal{C}(\mathfrak{p})$ that is irreducible. Then $\mathcal{C}(\mathfrak{q})$ is the image of the irreducible variety $Q \times \mathcal{C}(\mathfrak{p})$ under the morphism $(g, (X, Y)) \mapsto (g \cdot X, g \cdot Y)$, and thus is irreducible. □

We end this section with a discussion of when the commuting variety $\mathcal{C}(\mathfrak{p})$ is equidimensional. By Lemma 5.1, if $\mathcal{C}(\mathfrak{p})$ is equidimensional then it must have dimension $\dim \mathfrak{p} + \operatorname{rank} G$. Now the proof of Theorem 1.1 can be adapted to show that $\mathcal{C}(\mathfrak{p})$ is equidimensional if and only if $\operatorname{mod}(P \cap H : \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})) \leq \operatorname{ssrank} H$ for all Levi subgroups $H \neq T$ of G containing T.

In the case $P = B$, it is now natural to consider whether $\mathcal{C}(\mathfrak{b})$ is a complete intersection, as is done in [Ke, Section 6], and the argument in the last paragraph of the proof of [Ke, Lemma 6.4] suffices to show that $\mathcal{C}(\mathfrak{b})$ is a complete intersection whenever it is equidimensional. The reason to consider this question just for $P = B$ is that in this case the image of the commutator map $\mathfrak{b} \times \mathfrak{b} \to \mathfrak{b}$ lies in $\mathfrak{u}$. Whereas for general P there is not a subspace of codimension $\operatorname{rank} G$ in $\mathfrak{p}$ in which the image of the commutator map $\mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ lies.

We remark that after this paper was completed, the paper [Ba] of Basili appeared. In *loc. cit.* the question of whether $\mathcal{C}(\mathfrak{b})$ is a complete intersection is considered in the case $G = \operatorname{GL}_n(k)$, and results similar to those in this paper are obtained.

6. On normality of $\mathcal{C}(\mathfrak{p})$

In this section we consider whether $\mathcal{C}(\mathfrak{p})$ is a normal variety in the case where it is irreducible. This question was considered by Keeton for the case $P = B$ in [Ke, Section 6.2], and we take a similar approach here.

We work with the commuting scheme $\mathcal{CS}(\mathfrak{p})$, which is the subscheme of $\mathfrak{p} \times \mathfrak{p}$ defined by the ideal $I(\mathfrak{p})$ of $k[\mathfrak{p} \times \mathfrak{p}]$ generated by the regular functions $(X, Y) \mapsto f([X, Y])$ for $f \in \mathfrak{p}^*$. We let $A(\mathfrak{p}) = k[\mathfrak{p} \times \mathfrak{p}]/I(\mathfrak{p})$ be the algebra underlying $\mathcal{CS}(\mathfrak{p})$. The commuting scheme $\mathcal{CS}(\mathfrak{l})$ of $\mathfrak{l}$ is defined similarly, as are the ideal $I(\mathfrak{l})$ of $k[\mathfrak{l} \times \mathfrak{l}]$ and the algebra $A(\mathfrak{l}) = k[\mathfrak{l} \times \mathfrak{l}]/I(\mathfrak{l})$.

Our results depend on whether $A(\mathfrak{l})$ is Cohen–Macaulay, so we include this as a hypothesis in the statements of our results. Also for the case $P = B$, we have $\mathfrak{l} = \mathfrak{t}$ and $A(\mathfrak{t}) = k[\mathfrak{t} \times \mathfrak{t}]$, which is certainly Cohen–Maucaulay. Further, we only consider cases where $\mathcal{C}(\mathfrak{p})$ is irreducible.

We start by showing that $\mathcal{CS}(\mathfrak{p})$ is a Cohen–Macaulay under these assumptions.

Proposition 6.1. *Assume that $A(\mathfrak{l})$ is Cohen–Macaulay and that $\mathcal{C}(\mathfrak{p})$ is irreducible. Then $A(\mathfrak{p})$ is Cohen–Macaulay.*

Proof. Let $J(\mathfrak{l})$ be the ideal of $A(\mathfrak{p})$ generated by $I(\mathfrak{l})$, and let $B(\mathfrak{l}) = k[\mathfrak{p} \times \mathfrak{p}]/J(\mathfrak{l})$. We have $k[\mathfrak{p} \times \mathfrak{p}] \cong k[\mathfrak{l} \times \mathfrak{l}] \otimes k[\mathfrak{u}_P \times \mathfrak{u}_P]$, and under this identification $J(\mathfrak{l}) = I(\mathfrak{l}) \otimes k[\mathfrak{u}_P \times \mathfrak{u}_P]$. Thus we have that $B(\mathfrak{l}) \cong (k[\mathfrak{l} \times \mathfrak{l}]/I(\mathfrak{l})) \otimes k[\mathfrak{u}_P \times \mathfrak{u}_P]$, which is Cohen–Macaulay by [Ei, Proposition 18.9].

For $\alpha \in \Phi(\mathfrak{u}_P)$, we define $p_\alpha \in \mathfrak{u}_P^*$ by $p_\alpha(\sum_{\beta \in \Phi(\mathfrak{u}_P)} a_\beta E_\beta) = a_\alpha$ and $f_\alpha \in k[\mathfrak{p} \times \mathfrak{p}]$ by $f_\alpha(X, Y) = p_\alpha([X, Y])$. Let $J(\mathfrak{p})$ be the image of $I(\mathfrak{p})$ in $B(\mathfrak{l})$. Since $I(\mathfrak{p})$ is generated by the set of f_α for $\alpha \in \Phi(\mathfrak{p})$, we have that $J(\mathfrak{p})$ is generated by the images of the f_α in $B(\mathfrak{l})$. Further, we have that $A(\mathfrak{p}) \cong B(\mathfrak{l})/J(\mathfrak{p})$.

The zero set of $J(\mathfrak{l})$ in $\mathfrak{p} \times \mathfrak{p}$ is $\mathcal{L}$ as defined in the proof of Lemma 5.3 and has dimension $\dim \mathfrak{l} + \operatorname{rank} G + 2 \dim \mathfrak{u}_P$. Our assumption that $\mathcal{C}(\mathfrak{p})$ is irreducible implies that it has dimension $\dim \mathfrak{p} + \operatorname{rank} G = \dim \mathcal{L} - \dim \mathfrak{u}_P$. Therefore, as $J(\mathfrak{p})$ is generated by $\dim \mathfrak{u}_P = |\Phi(\mathfrak{u}_P)|$ many elements and $B(\mathfrak{l})$ is Cohen–Macaulay, we deduce that $A(\mathfrak{p}) \cong B(\mathfrak{l})/J(\mathfrak{p})$ is Cohen–Macaulay, see [Ei, Proposition 18.13]. $\square$

We remark that the only place that we use the fact that $\mathcal{C}(\mathfrak{p})$ is irreducible in the proof above is in saying that the dimension is $\dim \mathfrak{p} + \operatorname{rank} G$. Therefore, the proof is also valid under the assumption that $\mathcal{C}(\mathfrak{p})$ is equidimensional, because in this case it has dimension $\dim \mathfrak{p} + \operatorname{rank} G$ by Lemma 5.1.

In Theorem 6.3 we give a sufficient condition for $A(\mathfrak{p})$ to be normal. Before we move on to this it is helpful to make some observations about smooth points in $\mathcal{CS}(\mathfrak{p})$, which lead to Lemma 6.2.

Let $(X, Y) \in \mathcal{C}(\mathfrak{p})$ considered as a closed point of $\mathcal{CS}(\mathfrak{p})$. The (scheme theoretic) tangent space of $\mathcal{CS}(\mathfrak{p})$ at (X, Y) is

$$T_{(X,Y)}(\mathcal{CS}(\mathfrak{p})) = \{(W, Z) \in \mathfrak{p} \times \mathfrak{p} \mid [X, Z] + [W, Y] = 0\}.$$

Suppose that $\mathcal{C}(\mathfrak{p})$ is irreducible so that $\dim \mathcal{C}(\mathfrak{p}) = \dim \mathfrak{p} + \operatorname{rank} G$. Then we have that (X, Y) is a smooth point of $\mathcal{CS}(\mathfrak{p})$ if and only if $\dim T_{(X,Y)}(\mathcal{CS}(\mathfrak{p})) = \dim \mathfrak{p} + \operatorname{rank} G$, or equivalently $\dim([\mathfrak{p}, X] + [\mathfrak{p}, Y]) = \dim \mathfrak{p} - \operatorname{rank} G$. We recall that $X \in \mathfrak{p}$ is regular if and only if $\dim \mathfrak{c}_\mathfrak{p}(X) = \operatorname{rank} G$, so that $\dim([\mathfrak{p}, X]) = \dim \mathfrak{p} - \operatorname{rank} G$. Therefore, if X is regular, then $\dim([\mathfrak{p}, X] + [\mathfrak{p}, Y]) = \dim \mathfrak{p} - \operatorname{rank} G$.

In the previous paragraph, we have proved the following lemma.

Lemma 6.2. *Assume that that $\mathcal{C}(\mathfrak{p})$ is irreducible. Let $(X, Y) \in \mathcal{C}(\mathfrak{p})$ considered as a closed point of $\mathcal{CS}(\mathfrak{p})$ and suppose that X (or Y) is regular in $\mathfrak{p}$. Then (X, Y) is a smooth point of $\mathcal{CS}(\mathfrak{p})$.*

We are now ready to proceed with Theorem 6.3. In the proof below we use the notation given in the proof of Theorem 1.1.

Theorem 6.3. *Assume that $A(\mathfrak{l})$ is Cohen–Macaulay and that $\operatorname{mod}(P \cap H : \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})) < \operatorname{ssrank} H - 1$ for all Levi subgroups $H \neq T$ of G containing T with $\operatorname{ssrank} H > 1$. Then $\mathcal{C}(\mathfrak{p})$ is irreducible and normal.*

Proof. By Theorem 1.1, we have that $\mathcal{C}(\mathfrak{p})$ is irreducible, so we just have to prove normality. To do this we prove that $A(\mathfrak{p})$ is normal, which in turn implies that $I(\mathfrak{p})$ is a prime ideal, and $k[\mathcal{C}(\mathfrak{p})] = A(\mathfrak{p})$.

Since, $A(\mathfrak{p})$ is Cohen–Macaulay, it suffices by Serre's criterion for normality, see [Ei, Theorem 11.5], to show that the singular locus of $\mathcal{CS}(\mathfrak{p})$ has codimension at least 2. We let $\mathcal{C}(\mathfrak{p})^{\mathrm{ssm}}$ be the set of points in $\mathcal{C}(\mathfrak{p})$, which are smooth as closed points in $\mathcal{CS}(\mathfrak{p})$. Then we have to show that $\mathcal{C}(\mathfrak{p}) \setminus \mathcal{C}(\mathfrak{p})^{\mathrm{ssm}}$ has codimension at least 2.

Let $(X, Y) \in \mathcal{C}_{T,\{0\}}$. Then X is regular (semisimple), so that (X, Y) is a smooth point of $\mathcal{CS}(\mathfrak{p})$ by Lemma 6.2. Thus $\mathcal{C}_{T,\{0\}} \subseteq \mathcal{C}(\mathfrak{p})^{\mathrm{ssm}}$.

Consider a Levi subgroup H of G containing T with ssrank $H = 1$. We have two possibilities for $H \cap P$, either this is equal to H or is a Borel subgroup of H; it turns out that the analysis of these two cases can be done simultaneously.

There are two $P \cap H$-orbits in $\mathcal{N}(\mathfrak{p} \cap \mathfrak{h})$ namely the regular nilpotent orbit and the zero orbit. Thus $\mathcal{S}_H$ has these two orbits as its elements.

First consider $S = \{0\} \in \mathcal{S}_H$, and look at $\mathcal{C}'_{H,S}$. We have that $\mathcal{C}'_{H,S} = \{(X,Y) \mid X \in \mathfrak{z}(\mathfrak{h})^{\mathrm{reg}}, Y \in \mathfrak{h}\}$. Let $\mathcal{Y} = \{(X,Y) \mid X \in \mathfrak{z}(\mathfrak{h})^{\mathrm{reg}}, Y \in \mathfrak{h}^{\mathrm{reg}}\} \subseteq \mathcal{C}'_S$; we recall that $\mathfrak{z}(\mathfrak{h})^{\mathrm{reg}} = \{X \in \mathfrak{z}(\mathfrak{h}) \mid \mathfrak{c}_{\mathfrak{g}}(X) = \mathfrak{h}\}$ and that $\mathfrak{h}^{\mathrm{reg}}$ denotes the set of regular elements in $\mathfrak{h}$. By Lemma 6.2 we have that $\mathcal{Y}$ and, thus $P \cdot \mathcal{Y} \subseteq \mathcal{C}(\mathfrak{p})^{\mathrm{ssm}}$. Also $P \cdot \mathcal{Y}$ is the subset of $\mathcal{C}_{H,S}$ of those (X,Y) for which Y is regular, and is thus open in $\mathcal{C}_{H,S}$. Therefore, the complement of $P \cdot \mathcal{Y}$ has codimension at least 1 in $\mathcal{C}_{H,S}$. In the proof of Theorem 1.1, we have shown that $\mathcal{C}_{H,S}$ has codimension 1 in $\mathcal{C}(\mathfrak{p})$. So we can deduce that $\mathcal{C}_{H,S} \setminus (P \cdot \mathcal{Y})$ has codimension at least 2 in $\mathcal{C}(\mathfrak{p})$.

Next consider the case S is the regular orbit of $P \cap H$ in $\mathcal{N}(\mathfrak{p} \cap \mathfrak{h})$. Here we observe that any element of $\mathcal{C}'_{H,S}$ is of the form $(X + Z, Y + aZ)$, where $X, Y \in \mathfrak{z}(\mathfrak{h})^{\mathrm{reg}}$, $Z \in \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})$ is regular in $\mathfrak{h}$, and $a \in k$. Then $X + Z$ is regular in $\mathfrak{g}$, so it follows from Lemma 6.2 that $\mathcal{C}'_{H,S} \subseteq \mathcal{C}(\mathfrak{p})^{\mathrm{ssm}}$, and thus $\mathcal{C}_{H,S} = P \cdot \mathcal{C}'_{H,S} \subseteq \mathcal{C}(\mathfrak{p})^{\mathrm{ssm}}$.

Now consider the decomposition

$$\mathcal{C}(\mathfrak{p}) = \bigsqcup_{\substack{H \in \mathcal{H} \\ S = \dot{S}_H}} \mathcal{C}_{H,S},$$

given in the proof of Theorem 1.1.

Let $H \in \mathcal{H}$ with ssrank $H > 1$. Then by assumption we have that $\mathrm{mod}(P \cap H : \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})) < \mathrm{ssrank}\, H - 1$, so $\dim \mathcal{C}_{H,S} < \dim \mathfrak{p} + \mathrm{rank} - 1$ by (5.3). Thus $\mathcal{C}_{H,S}$ has codimension at least 2 in $\mathcal{C}(\mathfrak{p})$ for each $S \in \mathcal{S}_H$.

Moreover, we have shown above that $(\mathcal{C}(\mathfrak{p}) \setminus \mathcal{C}(\mathfrak{p})^{\mathrm{ssm}}) \cap \mathcal{C}_{H,S}$ has codimension at least 2 in $\mathcal{C}(\mathfrak{p})$, for H of semisimple rank 0 or 1 and $S \in \mathcal{S}_H$.

Hence, we have proved that $(\mathcal{C}(\mathfrak{p}) \setminus \mathcal{C}(\mathfrak{p})^{\mathrm{ssm}})$ has codimension at least 2 in $\mathcal{C}(\mathfrak{p})$ as required. $\square$

In the case $P = B$ is a Borel subgroup, we can remove the assumption that $A(\mathfrak{l})$ is Cohen–Macaulay in Theorem 6.3, as $\mathfrak{l} = \mathfrak{t}$ and $A(\mathfrak{t}) = k[\mathfrak{t} \times \mathfrak{t}]$. The converse also holds as stated Theorem 6.4 below. In its proof we use the setup given in the proof of Theorem 1.1.

Theorem 6.4. *Let B be a Borel subgroup of G and $\mathfrak{u} = \mathcal{N}(\mathfrak{b})$ the Lie algebra of the unipotent radical of B. Then $\mathcal{C}(\mathfrak{b})$ is irreducible and normal if and only if $\mathrm{mod}(B \cap H : \mathfrak{u} \cap \mathfrak{h}) < \mathrm{ssrank}\, H - 1$ for all Levi subgroups H of G containing T with $\mathrm{ssrank}\, H > 1$.*

Proof. For the case $P = B$, we have $L = T$ and $A(\mathfrak{t}) = k[\mathfrak{t} \times \mathfrak{t}]$, which is certainly Cohen–Macaulay. Thus if $\mathrm{mod}(B \cap H : \mathfrak{u} \cap \mathfrak{h}) < \mathrm{ssrank}\, H - 1$ for all H, then $\mathcal{C}(\mathfrak{b})$ is normal by Theorem 6.3. So we just have to prove the converse.

First we make the simplifying observation that, under the assumption that $\mathcal{C}(\mathfrak{b})$ is irreducible, we have that $A(\mathfrak{b})$ is reduced. To prove this we consider the functions $f_\alpha \in k[\mathfrak{b} \times \mathfrak{b}]$ for $\alpha \in \Phi(\mathfrak{u}) = \Phi^+$, as defined in the proof of Proposition 6.1.

Let $(X,Y) \in \mathcal{C}(\mathfrak{b})$. The differential $(df_\alpha)_{(X,Y)} : T_{(X,Y)}(\mathfrak{b} \times \mathfrak{b}) \to k$ can be viewed as a linear map $\mathfrak{b} \times \mathfrak{b} \to k$ and a calculation shows that this is the map $(W,Z) \mapsto f_\alpha([X,Z] + [W,Y])$. Thus the differentials $(df_\alpha)_{(X,Y)}$ for $\alpha \in \Phi^+$ are linearly independent whenever

14

$(X, Y) \in \mathcal{C}(\mathfrak{b})$ is such that $[\mathfrak{b}, X] + [\mathfrak{b}, Y] = \mathfrak{u}$; this holds for example when X is regular semisimple. Now we can deduce that $I_{\mathfrak{b}}$ is radical and thus $A(\mathfrak{b})$ is reduced, see for example [Ja, Lemma 7.1].

We just have to consider the case $\mathrm{mod}(B \cap H : \mathfrak{u} \cap \mathfrak{h}) = \mathrm{ssrank}\, H - 1$ for some Levi subgroup H of G with $\mathrm{ssrank}\, H > 1$. In this case we prove that the singular locus of $\mathcal{C}(\mathfrak{b})$ has codimension equal to 1; we can work just with the variety here, because $A(\mathfrak{b})$ is reduced. Then we can apply Serre's criterion for normality to deduce that $\mathcal{C}(\mathfrak{b})$ is not normal.

We have that H is conjugate in G to a standard Levi subgroup, so since $B \cap H$ is a Borel subgroup of H, we may as well assume that H is a standard Levi subgroup. Let $J \subseteq \Pi$ be such that $\Phi(\mathfrak{h}) = \Phi_J$.

In the notation given in the proof of Theorem 1.1, we have that there is some $S \in \mathcal{S}_H$ such that $\dim \mathcal{C}_{H,S} = \dim \mathfrak{b} + \mathrm{rank}\, G - 1$. From the proof of [GMR, Theorem 5.1] we can assume that $Z(H) C_U(U \cap H)$ has finite index in $C_{B \cap H}(X)$ for any $X \in S$; in other words the connected component of the centralizer of X in the $B \cap \mathcal{D}H$ is unipotent, where $\mathcal{D}H$ denotes the derived subgroup of H. This implies that $\mathrm{mod}(U \cap H : \mathfrak{u} \cap \mathfrak{h}) = 2\, \mathrm{ssrank}\, H - 1$. Let $j \in \mathbb{Z}_{\geq 0}$, be such that $S \in (\mathcal{S}_H)_j$. Then we have $\dim S = j + \mathrm{ssrank}\, H - 1$.

Let $X \in S$. Then we have $\dim C_{B \cap H}(X) = \dim(B \cap H) - j$ for any $X \in S$, so $\dim C_{U \cap H}(X) = \dim(U \cap H) - j - \mathrm{ssrank}\, H$. By Theorem 1.2, we have $\dim C_{B \cap H}(X) = \dim \mathfrak{c}_{\mathfrak{b} \cap \mathfrak{h}}(X)$, and also we have $\dim C_{U \cap H}(X) = \dim \mathfrak{c}_{\mathfrak{u} \cap \mathfrak{h}}(X)$, see for example [Go, Corollary 4.3]. From this we deduce that $\mathfrak{c}_{\mathfrak{b} \cap \mathfrak{h}}(X) = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{c}_{\mathfrak{u} \cap \mathfrak{h}}(X)$.

Consider the set S^{reg} of elements of $\mathfrak{u} \cap \mathfrak{h}$ of the form
$$\sum_{\alpha \in \Phi_J \cap \Phi^+} a_\alpha E_\alpha,$$
where $a_\alpha \neq 0$ for all $\alpha \in J$. The elements of S^{reg} are precisely the regular nilpotent elements in $\mathfrak{u} \cap \mathfrak{h}$, and they form a single $B \cap H$-orbit. Moreover, S^{reg} is a sheet for the action of $B \cap H$ on $\mathfrak{u} \cap \mathfrak{h}$ and $S^{\mathrm{reg}} \in (\mathcal{S}_H)_{\dim(\mathfrak{u} \cap \mathfrak{h})}$. We have $\dim S^{\mathrm{reg}} = \dim(\mathfrak{u} \cap \mathfrak{h}) < \mathrm{ssrank}\, H + \dim(\mathfrak{u} \cap \mathfrak{h}) - 1$, because $\mathrm{ssrank}\, H > 1$. Therefore, $S \neq S^{\mathrm{reg}}$.

For $\beta \in J$, we let $(\mathfrak{u} \cap \mathfrak{h})^\beta = \bigoplus_{\alpha \in \Phi(\mathfrak{u} \cap \mathfrak{h}) \setminus \{\beta\}} \mathfrak{g}_\alpha$. Then we have $(\mathfrak{u} \cap \mathfrak{h}) \setminus S^{\mathrm{reg}} = \bigcup_{\beta \in J} (\mathfrak{u} \cap \mathfrak{h})^\beta$, and it follows that $S \subseteq (\mathfrak{u} \cap \mathfrak{h})^\beta$ for some $\beta \in J$.

Next we consider $\mathcal{C}(\mathfrak{u} \cap \mathfrak{h})$, which has dimension $\dim(\mathfrak{u} \cap \mathfrak{h}) + \mathrm{mod}(U \cap H : \mathfrak{u} \cap \mathfrak{h})$ by Lemma 2.1. We define $\mathcal{C}(\mathfrak{u} \cap \mathfrak{h})_S = \mathcal{C}(\mathfrak{u} \cap \mathfrak{h}) \cap (S \times (\mathfrak{u} \cap \mathfrak{h}))$. The projection onto the first component $\mathcal{C}(\mathfrak{u} \cap \mathfrak{h})_S \to S$ is surjective and the fibre of $X \in S$ is equal to $\{X\} \times \mathfrak{c}_{\mathfrak{u} \cap \mathfrak{h}}(X)$. As seen above we have that $\mathfrak{c}_{\mathfrak{u} \cap \mathfrak{h}}(X) = \dim(U \cap H) - j - \mathrm{ssrank}\, H$ and that this is independent of $X \in S$. Therefore,

$$\begin{aligned}
\dim \mathcal{C}(\mathfrak{u} \cap \mathfrak{h})_S &= \dim S + \dim(U \cap H) - j + \mathrm{ssrank}\, H \\
&= j + \mathrm{ssrank}\, H - 1 + \dim(U \cap H) - j + \mathrm{ssrank}\, H \\
&= \dim(U \cap H) + 2\, \mathrm{ssrank}\, H - 1 \\
&= \dim(\mathfrak{u} \cap \mathfrak{h}) + \mathrm{mod}(U \cap H : \mathfrak{u} \cap \mathfrak{h}) \\
&= \dim \mathcal{C}(\mathfrak{u} \cap \mathfrak{h}).
\end{aligned}$$

Further, $\mathcal{C}(\mathfrak{u} \cap \mathfrak{h})_S$ is irreducible, so its closure in $\mathcal{C}(\mathfrak{u} \cap \mathfrak{h})$ is an irreducible component of $\mathcal{C}(\mathfrak{u} \cap \mathfrak{h})$.

Therefore, $\overline{\mathcal{C}(\mathfrak{u} \cap \mathfrak{h})_S}$ is stable under the flip $(X, Y) \mapsto (Y, X)$, see for example [GR, Lemma 2.3]. Since $S \subseteq (\mathfrak{u} \cap \mathfrak{h})^\beta$ we deduce that $\mathcal{C}(\mathfrak{u} \cap \mathfrak{h})_S \subseteq (\mathfrak{u} \cap \mathfrak{h})^\beta \times (\mathfrak{u} \cap \mathfrak{h})^\beta$, so that $\mathfrak{c}_{\mathfrak{u} \cap \mathfrak{h}}(X) \subseteq (\mathfrak{u} \cap \mathfrak{h})^\beta$ for any $X \in S$.

Now consider $\mathcal{C}'_{H,S}$ as defined in the proof of Theorem 1.1. Let $(X, Y) \in \mathcal{C}'_{H,S}$. Then $X = X_1 + X_2$, where $X_1 \in \mathfrak{z}(\mathfrak{h})^{\mathrm{reg}}$ and $X_2 \in S$. We have shown above that $\mathfrak{c}_{\mathfrak{b} \cap \mathfrak{h}}(X_2) = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{c}_{\mathfrak{u} \cap \mathfrak{h}}(X_2)$, so we can write $Y = Y_1 + Y_2$, where $Y_1 \in \mathfrak{z}(\mathfrak{h})$, $Y_2 \in \mathfrak{c}_{\mathfrak{u} \cap \mathfrak{h}}(X_2) \subseteq (\mathfrak{u} \cap \mathfrak{h})^\beta$.

Now we aim to show that (X, Y) is a singular point of $\mathcal{C}(\mathfrak{b})$. We recall from the discussion before Theorem 6.3 that (X, Y) is a smooth point of $\mathcal{C}(\mathfrak{b})$ if and only if $\dim([\mathfrak{b}, X] + [\mathfrak{b}, Y]) = \dim \mathfrak{u}$. However, we know that $X \in \mathfrak{z}(\mathfrak{h}) \oplus (\mathfrak{u} \cap \mathfrak{h})^\beta$, which implies that $[\mathfrak{b}, X] \subseteq \mathfrak{u}^\beta$. Similarly, $Y \in \mathfrak{z}(h) \oplus (\mathfrak{u} \cap \mathfrak{h})^\beta$, so that $[\mathfrak{b}, Y] \subseteq (\mathfrak{u} \cap \mathfrak{h})^\beta$. Thus, $[\mathfrak{b}, X] + [\mathfrak{b}, Y] \subseteq \mathfrak{u}^\beta$ so $\dim([\mathfrak{b}, X] + [\mathfrak{b}, Y]) < \dim \mathfrak{u}$. Hence, (X, Y) is a singular point of $\mathcal{C}(\mathfrak{b})$.

We deduce that all elements of $\mathcal{C}_{H,S} = B \cdot \mathcal{C}'_{H,S}$ are singular in $\mathcal{C}(\mathfrak{b})$. Further, $\dim \mathcal{C}_{H,S} = \dim \mathcal{C}(\mathfrak{b}) - 1$. Hence, $\mathcal{C}(\mathfrak{b})$ is not normal by Serre's criterion for normality. $\square$

7. Classification of irreducibility and normality of $\mathcal{C}(\mathfrak{b})$

We consider the case where $P = B$ is a Borel subgroup, where we can give a full classification of irreducibility and normality of $\mathcal{C}(\mathfrak{b})$. This requires recent results in [GMR] and [PS] giving $\mathrm{mod}(B : \mathfrak{u})$ for simple G of sufficiently large rank, here we recall that $\mathcal{N}(\mathfrak{b}) = \mathfrak{u}$ is the nilradical of $\mathfrak{b}$. We also use lower bounds for $\mathrm{mod}(B : \mathfrak{u})$ established in [Rö1, Theorem 3.1].

In [GMR] a parametrization of the coadjoint orbits of U in $\mathfrak{u}^*$ is given for G of rank at most 8 apart from G of type E$_8$. It is known that $\mathrm{mod}(U : \mathfrak{u}) = \mathrm{mod}(U : \mathfrak{u}^*)$, see [Rö2, Theorem 1.4]. Also in [GMR, Theorem 5.1] it is proved that $\mathrm{mod}(U : \mathfrak{u}) = \mathrm{mod}(B : \mathfrak{u}) + \mathrm{ssrank}\, G$. Thus values of $\mathrm{mod}(B : \mathfrak{u})$ for G up to rank 8 apart from G of type E$_8$ were determined. This extended previously known values of $\mathrm{mod}(B : \mathfrak{u})$ given in [JR, Tables II and III].

The results in [PS] can be used to determine $\mathrm{mod}(B : \mathfrak{u})$ for $G = \mathrm{GL}_n(k)$ and $n \leq 16$, as we explain below. Let q be a prime power, let $U(q)$ be the subgroup of upper unitriangular matrices in $\mathrm{GL}_n(q)$ and let $\mathfrak{u}^*(q)$ be the dual space of the space $\mathfrak{u}(q)$ of strictly upper triangular matrices in $\mathfrak{gl}_n(q)$. Then $U(q)$ acts on $\mathfrak{u}^*(q)$ via the coadjoint action. The number $k(U(q), \mathfrak{u}^*(q))$ of coadjoint orbits of $U(q)$ in $\mathfrak{u}^*(q)$ is determined for $n \leq 16$ in [PS] and this number is shown to be a polynomial in q, see [PS, Theorem 1.2]. Although [PS] only deals with finite fields, the methods used can be adapted to apply for other fields. In particular, this means that the calculations carried out as part of [PS] can be used to determine $\mathrm{mod}(U : \mathfrak{u}^*)$ for $G = \mathrm{GL}_n(k)$ and $n \leq 16$; moreover, we see that $\mathrm{mod}(U : \mathfrak{u}^*)$ is equal to the degree of the polynomial in q giving $k(U(q), \mathfrak{u}^*(q))$. Combining this with the fact that $\mathrm{mod}(U : \mathfrak{u}^*) = \mathrm{mod}(U : \mathfrak{u}) = \mathrm{mod}(B : \mathfrak{u}) + \mathrm{ssrank}\, G$, we deduce the values of $\mathrm{mod}(B : \mathfrak{u})$.

Combining the results in [GMR], [PS] and [Rö1], gives Tables 1–5, containing the exact value or a lower bound for $\mathrm{mod}(B : \mathfrak{u})$ for G of low rank. We note that in higher rank cases the lower bounds from [Rö1] do give that $\mathrm{mod}(B : \mathfrak{u}) > \mathrm{rank}\, G$ in these cases.

Type of G	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
$\mathrm{mod}(B:\mathfrak{u})$	0	0	0	0	1	1	2	3	4

Type of G	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}	A_{17}
$\mathrm{mod}(B:\mathfrak{u})$	5	7	8	10	12	14	≥ 16	≥ 19

TABLE 1. Modality of the action of B on $\mathfrak{u}$ for G of type A

Type of G	B_2	B_3	B_4	B_5	B_6	B_7	B_8
$\mathrm{mod}(B:\mathfrak{u})$	0	1	2	3	5	7	9

TABLE 2. Modality of the action of B on $\mathfrak{u}$ for G of type B

Type of G	C_3	C_4	C_5	C_6	C_7	C_8
$\mathrm{mod}(B:\mathfrak{u})$	1	2	3	5	7	9

TABLE 3. Modality of the action of B on $\mathfrak{u}$ for G of type C

Type of G	D_4	D_5	D_6	D_7	D_8
$\mathrm{mod}(B:\mathfrak{u})$	1	2	4	5	8

TABLE 4. Modality of the action of B on $\mathfrak{u}$ for G of type D

Type of G	G_2	F_4	E_6	E_7	E_8
$\mathrm{mod}(B:\mathfrak{u})$	1	4	5	10	≥ 20

TABLE 5. Modality of the action of B on $\mathfrak{u}$ for G of exceptional type

From Tables 1–5 and Theorem 1.1, we immediately deduce Theorem 1.3. Also from Tables 1–5 and Theorem 6.4, we can deduce the classification of when $\mathcal{C}(\mathfrak{b})$ is normal given in the following theorem.

Theorem 7.1. *Let B be a Borel subgroup of G. Then $\mathcal{C}(\mathfrak{b})$ is irreducible and normal if and only if the type of each simple component of G is one of the following.*

- A_l *for $l \leq 14$;*
- B_l *for $l \leq 5$;*
- C_l *for $l \leq 5$; or*
- D_l *for $l \leq 7$.*

We end this paper by briefly discussing some cases where we can determine whether $\mathcal{C}(\mathfrak{p})$ is irreducible or reducible for $P \neq B$. In general little is known about the values of $\mathrm{mod}(P : \mathcal{N}(\mathfrak{p}))$, so that we can only present limited results in this direction.

First we note that by Proposition 5.6 and Theorem 1.3, we have $\mathcal{C}(\mathfrak{p})$ is irreducible whenever G satisfies the conditions in Theorem 1.3.

In [Rö1, Table 1] Röhrle lower bounds for $\mathrm{mod}(P : \mathfrak{u}_P)$ are given, where $\mathfrak{u}_P$ is the Lie algebra of the unipotent radical of P. Of course, we have $\mathrm{mod}(P : \mathfrak{u}_P) \leq \mathrm{mod}(P : \mathcal{N}(\mathfrak{p}))$. From these lower bounds, we can determine many instances where $\mathcal{C}(\mathfrak{p})$ is reducible. In

particular, these lower bounds are quadratic in $\operatorname{ssrank} G - \operatorname{ssrank} L$, so for a fixed value of $\operatorname{ssrank} L$, we have that $\mathcal{C}(\mathfrak{p})$ is reducible if $\operatorname{rank} G$ is sufficiently large.

If the number of P-orbits in $\mathcal{N}(\mathfrak{p})$ is finite, so that $\operatorname{mod}(P : \mathcal{N}(\mathfrak{p})) = 0$, then certainly $\mathcal{C}(\mathfrak{p})$ is irreducible. It follows from results of Murray in [Mur] that $\operatorname{mod}(P : \mathcal{N}(\mathfrak{p})) = 0$ for P a maximal parabolic subgroup of $\operatorname{GL}_n(k)$ such that one block has size less than 6. Very recent work of Bulois–Boos in [BB, Main theorem] gives a classification of cases where $\operatorname{mod}(P : \mathcal{N}(\mathfrak{p})) = 0$ for $G = \operatorname{GL}_n(k)$. Further, in [BB, §6], there is a discussion of cases of higher modality. In particular, it is shown that for the maximal parabolic subgroup P of $\operatorname{GL}_n(k)$ with block sizes $(200, 400)$, we have that $\mathcal{C}(\mathfrak{p})$ is reducible.

References

[Ar]   V. I. Arnold *Critical points of smooth functions*, Proc. of ICM, Vancouver, 19–41 (1974).

[Ba]   R. Basili, *On commuting varieties of upper triangular matrices*, Comm. Algebra **45** (2017), 1533–1541.

[Bo]   A. Borel, *Linear algebraic groups*, Graduate Texts in Mathematics **126**, Springer, 1991.

[BK]   W. Borho and H. Kraft, *Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen*, Comm. Math. Helv. **54** (1979), 61–104.

[Bu]   M. Bulois, *Composantes irréductibles de la variété commutante nilpotente d'une algèbre de Lie symétrique semi-simple*, Annales de l'institut Fourier, **59** (2009), 37–80.

[BB]   M. Bulois and M. Boos, *Parabolic Conjugation and Commuting Varieties*, preprint, arXiv:1606.08840 (2016).

[BE]   M. Bulois and L. Evain, *Nested punctual Hilbert schemes and commuting varieties of parabolic subalgebras*, J. Lie Theory **26** (2016), 497–533.

[Ei]   D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer, 1995.

[Ge]   M. Gerstenhaber, *Dominance over the classical groups*, Ann. of Math. **74** (1961), 532–569.

[Gi]   V. Ginzburg, *Isospectral commuting variety, the Harish-Chandra D-module, and principal nilpotent pairs*, Duke Math. J. **161** (2012), 2023–2111.

[Go]   S. M. Goodwin, *On the conjugacy classes in maximal unipotent subgroups of simple algebraic groups*, Transform. Groups **11** (2006), 51–76.

[GMR] S. M. Goodwin, P. Mosch and G. Röhrle, *On the coadjoint orbit of maximal unipotent subgroups of reductive groups*, Transform. Groups, **21** (2016), 399–426.

[GR]   S. M. Goodwin and G. Röhrle, *On commuting varieties of nilradicals of Borel subalgebras of reductive Lie algebras*, Proc. Edinb. Math. Soc. **58** (2015), 169–181.

[GS]   R. M. Guralnick and B. A. Sethuraman, *Commuting pairs and triples of matrices and related varieties*, Linear Algebra Appl., **310** (2000), 139–148.

[He]   S. Herpel, *On the smoothness of centralizers in reductive groups*, Trans. Amer. Math. Soc. **365** (2013), 3753–3774.

[Hu]   J. E. Humphreys, *Conjugacy Classes in Semisimple Algebraic Groups*, Mathematical Surveys and Monographs **43**, American Mathematical Society, Providence, RI, 1995.

[Ja]   J. C. Jantzen, *Nilpotent orbits in representation theory*, Progress in Math., vol. 228, Birkhäuser, 2004.

[JR]   U. Jürgens and G. Röhrle, *Algorithmic Modality Analysis for Parabolic Groups*, Geom. Dedicata **73** (1998), 317–337.

[Ke]   A. Keeton, *Commuting Varieties Associated with Symmetric Pairs*, PhD thesis, University of California, San Diego, 1996.

[Le]   P. Levy, *Commuting varieties of Lie algebras over fields of prime characteristic*, J. Algebra **250** (2002), 473–484.

[MT]   T. Motzkin and O. Taussky, *Pairs of matrices with property L, II*, Trans. Amer. Math. Soc. **80** (1955), 387–401.

[Mum] D. Mumford, *The red book of varieties and schemes*, Lecture Notes in Math. **1358**, Springer, 1988.

[Mur]   S. Murray, *Conjugacy classes in maximal parabolic subgroups of general linear groups*, J. Algebra **233** (2000), 135–155.

[Ng]   N. V. Ngo, *On nilpotent commuting varieties and cohomology of Frobenius kernels*, J. Algebra **425** (2015), 65–84.

[PS]   I. Pak and A. Soffer, *On Higman's $k(U_n(\mathbb{F}_q))$ conjecture*, preprint, `arXiv:1507.00411` (2015).

[PY]   D. I. Panyushev and O. Yakimova, *Symmetric pairs and associated commuting varieties*, Math. Proc. Cambridge Philos. Soc. **143** (2007), 307–321.

[Po]   V. L. Popov, *Irregular and Singular Loci of Commuting Varieties*, Transform. Groups **13** (2008), 819–837.

[PV]   V. L. Popov and E. B. Vinberg, *Invariant Theory*, in: Algebraic Geometry IV, Enc. Math. Sci., Vol. 55, Springer, Berlin, 1994, pp. 123–284.

[Pr]   A. Premet, *Nilpotent commuting varieties of reductive Lie algebras*, Invent. Math. **154** (2003), 653–683.

[Ri]   R. W. Richardson, *Commuting varieties of semisimple Lie algebras and algebraic groups*, Compos. Math. **38** (1979), 311–327.

[Rö1]   G. Röhrle, *A note on the modality of parabolic subgroups*, Indag. Math. (N.S.) **8** (1997), 549–559.

[Rö2]   ——, *On the modality of parabolic subgroups of linear algebraic groups*, Manuscripta Math. **98**, (1999), 9–20.

[SS]   T. A. Springer and R. Steinberg, *Conjugacy classes*, In *Seminar on algebraic groups and related finite groups*, Springer–Verlag, 1970.

[Vi]   È. B. Vinberg, *Complexity of actions of reductive groups*, (Russian) Funktsional. Anal. i Prilozhen. **20** (1986), 1–13.

School of Mathematics, University of Birmingham, Birmingham, B15 2TT, UK

E-mail address: `rsg276@bham.ac.uk, s.m.goodwin@bham.ac.uk`