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TRANSITIVE TRIANGLE TILINGS IN ORIENTED GRAPHS

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Abstract. In this paper, we prove an analogue of Corrádi and Hajnal's classical theorem. There exists n_0 such that for every $n \in 3\mathbb{Z}$ when $n \geq n_0$ the following holds. If G is an oriented graph on n vertices and every vertex has both indegree and outdegree at least $7n/18$, then G contains a perfect transitive triangle tiling, which is a collection of vertex-disjoint transitive triangles covering every vertex of G. This result is best possible, as, for every $n \in 3\mathbb{Z}$, there exists an oriented graph G on n vertices without a perfect transitive triangle tiling in which every vertex has both indegree and outdegree at least $\lceil 7n/18 \rceil - 1$.

1. Introduction

Let G be an oriented graph, that is a directed graph without loops such that between every two vertices there is at most one edge. We write xy for an edge directed from x to y. The *outdegree* $d^+_G(x)$ of a vertex x is the number of vertices y such that $xy \in E(G)$. Similarly, the *indegree* $d^-_G(x)$ of a vertex x is the number of vertices y such that $yx \in E(G)$. Define the *minimum outdegree* $\delta^+(G)$ of G to be the minimal $d^+_G(x)$ over all vertices x of G, and define the *minimum indegree* $\delta^-(G)$ of G similarly. Define the *minimum semidegree* $\delta^0(G)$ of G to be $\min\{\delta^+(G), \delta^-(G)\}$.

The oriented graph on $\{v_1, \dots, v_n\}$ with edge set $\{v_n v_1\} \cup \{v_i v_{i+1} : i \in \{1, \dots, n-1\}\}$ is the *directed cycle* of length n. An oriented graph in which there is exactly one edge between every pair of vertices is called a *tournament*. A tournament that does not contain a directed cycle is *transitive*. Up to isomorphism, there are two tournaments on 3 vertices: The directed cycle of length 3, which we refer to as the *cyclic triangle*, and the transitive tournament on 3 vertices, which we refer to as the *transitive triangle* or as TT_3.

A *tiling* of G is a collection of vertex-disjoint subgraphs called *tiles*. If every tile is isomorphic to some oriented graph H, then the tiling is an *H-tiling*. If every vertex in G is contained in a tile, then the tiling is *perfect*. The same definitions are applied to graphs and directed graphs.

In [5], Hajnal and Szemerédi proved that for any $k, r \in \mathbb{N}$ and for any graph G on kr vertices if the minimum degree of G is at least $(r-1)k$, then G has a perfect K_r-tiling. The case when $r = 3$ was proved earlier by Corrádi and Hajnal [1].

The problem of finding cyclic triangle tilings in an oriented graph was considered by Keevash and Sudakov [7], who proved a nearly optimal result: For some $\varepsilon > 0$ there exists n_0 such that if G is an oriented graph on $n \geq n_0$ vertices and $\delta^0(G) \geq (1/2 - \varepsilon)n$, then G contains a cyclic triangle tiling that covers all but at most 3 vertices. Furthermore, if $n \equiv 3 \pmod{18}$, then there is a tournament T such that $\delta^0(T) \geq (n-1)/2 - 1$ which does not have a perfect cyclic triangle tiling. They repeated the following question which was asked by both Cuckler [2] and Yuster [13].

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n	$\lvert W_1\rvert$	$\lvert W_2\rvert$	$\lvert W_3\rvert$	$\lvert U_1\rvert$	$\lvert U_2\rvert$	$\delta^0(G)$	$\lceil 7n/18\rceil$
$18m$	$4m$	$4m$	$4m+1$	$3m-1$	$3m$	$7m-1$	$7m$
$18m+3$	$4m+1$	$4m+1$	$4m+1$	$3m$	$3m$	$7m+1$	$7m+2$
$18m+6$	$4m+1$	$4m+2$	$4m+2$	$3m$	$3m+1$	$7m+2$	$7m+3$
$18m+9$	$4m+2$	$4m+2$	$4m+3$	$3m+1$	$3m+1$	$7m+3$	$7m+4$
$18m+12$	$4m+3$	$4m+3$	$4m+3$	$3m+1$	$3m+2$	$7m+4$	$7m+5$
$18m+15$	$4m+3$	$4m+4$	$4m+4$	$3m+2$	$3m+2$	$7m+5$	$7m+6$.

TABLE 1. The order of the sets W_1, W_2, W_3, U_1 and U_2, $\delta^0(G)$ and $\lceil 7n/8\rceil$ in the extremal graph for all values of $n \pmod{18}$.

Question 1.1. *Does every tournament T on $n \equiv 3 \pmod 6$ vertices with $\delta^0(T) = (n-1)/2$ have a perfect cyclic triangle tiling?*

In this paper, we consider the problem of finding a perfect transitive triangle tiling, proving an analogue to Corrádi and Hajnal's result for oriented graphs.

Theorem 1.2. *There exists n_0 such that for every $n \in 3\mathbb{Z}$ when $n \geq n_0$ the following holds. If G is an oriented graph on n vertices and $\delta^0(G) \geq 7n/18$, then G contains a perfect TT_3-tiling.*

Treglown [10] conjectured that Theorem 1.2 is true for every $n \in 3\mathbb{Z}$.

The related problems for directed graphs have been considered (see [12], [4], [3] and [11]).

The following family of examples, based on the example of Treglown in [10], shows that Theorem 1.2 is tight. For every positive $n \in 3\mathbb{Z}$, let G be an oriented graph on n vertices and W_1, W_2, W_3, U_1, U_2 be a partition of $V(G)$ such that

$$|W_i| = \left\lfloor \frac{2n/3 + i}{3} \right\rfloor \text{ for } i \leq 3, \qquad |U_1| = \left\lfloor \frac{n-3}{6} \right\rfloor, \qquad |U_2| = \left\lceil \frac{n-3}{6} \right\rceil.$$

Let the edges of G be all possible directed edges from W_1 to W_2, from W_2 to W_3, from W_3 to W_1, from U_1 to U_2, from $W_1 \cup W_2$ to U_1, from U_1 to W_3, from U_2 to $W_1 \cup W_2$ and from W_3 to U_2, see Figure 1. Note that, for every $v \in V(G)$, because $|W_1| \geq |U_1|$ and $|W_2| \geq |U_2|$,

$$d^+(v), d^-(v) \geq \min\{|W_1| + |U_2|, |W_2| + |U_1|\},$$

so, for $w \in W_3$,

$$\min\{d^+(w), d^-(w)\} = \min\{|W_1| + |U_2|, |W_2| + |U_1|\} = \delta^0(G) = \lceil 7n/8\rceil - 1,$$

see Table 1. Note that $G[W_1 \cup W_2 \cup W_3]$ does not contain a transitive triangle, so every transitive triangle in G contains a vertex in $U_1 \cup U_2$. Therefore, the fact that $|U_1 \cup U_2| < n/3$ implies that G does not contain a perfect TT_3-tiling.

1.1. **Outline of the paper.** We prove Theorem 1.2 using a stability approach and the absorption technique. We say that an oriented graph G on n vertices is *α-extremal* if there exists $W \subseteq V(G)$ such that $|W| \geq (2/3 - \alpha)n$ and $G[W]$ does not contain a transitive triangle.

In Section 2, we handle the case when G is not α-extremal, i.e. we prove the following lemma.

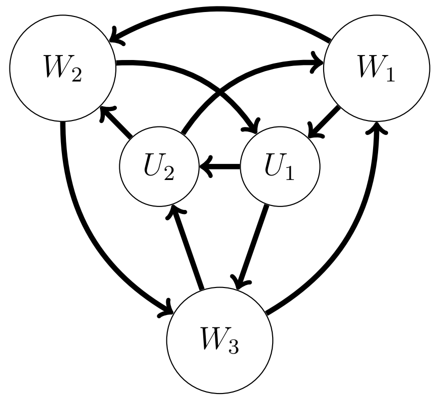

FIGURE 1. The extremal graph.

Lemma 1.3. *For every $\alpha > 0$ there exists $\varepsilon = \varepsilon(\alpha) > 0$ and $n_0 = n_0(\alpha)$ such that when G is an oriented graph on $n \in 3\mathbb{Z}$ vertices and $n \geq n_0$ the following holds. If $\delta^0(G) \geq (7/18-\varepsilon)n$, then G has a perfect TT_3-tiling or G is α-extremal.*

In Section 3, we prove Theorem 1.2 for oriented graphs G which are α-extremal.

Lemma 1.4. *There exists $\alpha > 0$ and n_0 such that when G is an oriented graph on $n \in 3\mathbb{Z}$ vertices and $n \geq n_0$ the following holds. If $\delta^0(G) \geq 7n/18$ and G is α-extremal, then there exists a perfect TT_3-tiling of G.*

Lemma 1.3 and Lemma 1.4 together clearly prove Theorem 1.2.

While proving Lemma 1.3 we prove the following result which may be of some interest because it applies for all n. Furthermore, it might be possible to extend the proof of this theorem to prove the main theorem for all n.

Theorem 1.5. *If G is an oriented graph on n vertices and $\delta^0(G) \geq 7n/18$, then there exists a TT_3-tiling of G that covers all but at most 11 vertices.*

1.2. **Notation.** Given a graph or digraph G, we write $V(G)$ for its vertex set, $E(G)$ for its edge set, and $e(G) = |E(G)|$ for the number of its edges. Given a collection $\mathcal{T}$ of subgraphs, we write $V(\mathcal{T})$ for $\bigcup_{T\in\mathcal{T}} V(T)$. When $\mathcal{W}$ is a collection of vertex subsets we will also use the notation $V(\mathcal{W})$ to denote $\bigcup_{W\in\mathcal{W}} W$.

Suppose that G is an oriented graph. If x is a vertex of G, then $N^+_G(x)$ denotes the *out-neighborhood* of x, i.e. the set of all those vertices y for which $xy \in E(G)$. Similarly, $N^-_G(x)$ denotes the *in-neighborhood* of x, i.e. the set of all those vertices y for which $yx \in E(G)$. Note that $d^+_G(x) = |N^+_G(x)|$ and $d^-_G(x) = |N^-_G(x)|$. We write $N_G(x) = N^+_G(x) \cup N^-_G(x)$ and $d_G(x) = d^+_G(x) + d^-_G(x)$. We write $\delta(G)$ and $\Delta(G)$ for the minimum degree and maximum degree of the underlying undirected graph of G, respectively. Given a vertex v of G and a set $A \subseteq V(G)$, we define $N^+_G(v, A) = N^+_G(v) \cap A$ and $d^+_G(v, A) = |N^+_G(v, A)|$ and define $N^-(v, A)$, $d^-_G(v, A)$, $N_G(v, A)$ and $d_G(v, A)$ similarly. Given $A, B \subseteq V(G)$, let $\overrightarrow{E}_G(A, B)$ be the set of edges in G directed from A to B. Similarly, $E_G(A, B)$ denotes the set of edges with one endpoint in A and the other in B and let $e_G(A, B) = |E_G(A, B)|$. For a vertex v, we write $E_G(v)$ for $E_G(v, V(G))$. For a vertex set $A \subseteq V(G)$, we write $G[A]$ for the subgraph of G induced by A and let $e_G(A) = e(G[A])$. If G is known from the context, then we may omit the subscript. We let $G[A, B]$ be the bipartite graph in which $a \in A$ is adjacent to $b \in B$ if and only if $ab \in E(G)$ or $ba \in E(G)$. If $x, y, z \in V(G)$ we sometimes refer to $G[\{x, y, z\}]$ as xyz or as xe, where $e = yz$ or $e = zy$. If we refer to a directed path or a cyclic triangle as xyz, then it must contain the edge set $\{xy, yz\}$ or the edge set $\{xy, yz, zx\}$, respectively.

For $m \in \mathbb{N}$ we write $[m] = \{1, \ldots, m\}$. For any set V we let $\binom{V}{m}$ be the collection of subsets of V that are of order m. When it is clear that a variable i must remain in $[m]$ (e.g. when i is the index of $W_1, \ldots, W_m$) we let $i + 1 = 1$ when $i = m$ and $i - 1 = m$ when $i = 1$.

1.3. **Preliminary lemmas and propositions.**

Let G be an oriented graph. Note that if uw is an edge in $G[N^+(v)]$, then vuw is a transitive triangle. We get the following easy proposition.

Proposition 1.6. *Let G be a tournament on 4 vertices. Then every vertex of G is contained in a transitive triangle.*

Proposition 1.7. *Let G be an oriented graph on n vertices. Then*

(a) *every (directed) edge uv is contained in at least $3\delta^0(G) - n$ transitive triangles uvw such that $w \in N^-(v)$;*
(b) *every (directed) edge uv is contained in at least $3\delta^0(G) - n$ transitive triangles uvw such that $w \in N^+(u)$;*
(c) *for every directed path uvw on 3 vertices, there are at least $2(3\delta^0(G) - n)$ vertices x such that there exists a transitive triangle in $G[\{u, v, w, x\}]$ containing x and v.*

Proof. Let uv be an edge in G. Note that every vertex w in $N(u) \cap N^-(v)$ forms a transitive triangle with uv. Since $|N(u) \cap N^-(v)| \geq \delta(G) + \delta^-(G) - n \geq 3\delta^0(G) - n$, (a) follows. By a similar argument, (b) also holds.

Let uvw be a directed path on 3 vertices. By (a), there is a set $U \subseteq N^-(v)$ with $|U| \geq 3\delta^0(G) - n$ such that every $u' \in U$ forms a transitive triangle with uv. By (b), there is a set $W \subseteq N^+(v)$ with $|W| \geq 3\delta^0(G) - n$ such that every $w' \in W$ forms a transitive triangle with vw. Since $U \cap W = \emptyset$, (c) holds. □

2. Non-extremal case

2.1. **Absorbing structure.**

In this section, we prove Lemma 2.2. Roughly speaking, the lemma states that there exists a small vertex set $U \subseteq V(G)$ such that $G[U \cup W]$ contains a perfect TT_3-tiling for every small $W \subseteq V(G) \setminus U$. Thus, in order to find a perfect TT_3-tiling in G, it is suffices to find a TT_3-tiling covering almost all vertices in $G[V(G) \setminus U]$. This technique was introduced by Rödl, Ruciński and Szemerédi [9] to obtain results on matchings in hypergraphs.

For any $r \in \mathbb{N}$ and any collection $\mathcal{H}$ of oriented graphs on $[r]$, define $\mathcal{F}(\mathcal{H}, G)$ to be the set of functions f from $[r]$ to $V(G)$ such that f is a directed graph homomorphism from some $H \in \mathcal{H}$ to G. Let $\mathcal{K}$ be the set of oriented graphs K on $\{1, \ldots, 21\}$ such that both K and $K[\{1, \ldots, 18\}]$ have a perfect TT_3-tiling. For any ordered triple $X = (x_1, x_2, x_3)$ of vertices in G, let $\mathcal{A}'(X)$ be the set of functions $f \in \mathcal{F}(\mathcal{K}, G)$ such that $f(19) = x_1$, $f(20) = x_2$ and $f(21) = x_3$. Let $\mathcal{A}(X)$ be the set of functions in $\mathcal{A}'(X)$ restricted to $[18]$. Clearly $|\mathcal{A}(X)| = |\mathcal{A}'(X)|$. Note that we do not require the functions in $\mathcal{F}(\mathcal{H}, G)$ to be injective, but at a later stage of the proof non-injective functions will essentially be discarded. We consider non-injective functions only to make the following arguments simpler.

Lemma 2.1. *For $\varepsilon_0 = 1/250$ and $0 \le \varepsilon \le \varepsilon_0$, there exists $\tau = \tau(\varepsilon) > 0$ and $n_0 = n_0(\varepsilon)$ such that the following holds. If G is an oriented graph on $n \ge n_0$ vertices and $\delta^0(G) \ge (7/18-\varepsilon)n$, then $|\mathcal{A}(X)| \ge \tau n^{18}$ for every ordered triple $X = (x_1, x_2, x_3)$ of vertices in G.*

Proof. Let $0 < \beta < (1/249 - \varepsilon)/10$ and $\tau = \beta^{18}$. Let $\mathcal{T}$ be the set of functions from $\{1, 2, 3\}$ to $V(G)$ that are digraph homomorphisms from a transitive triangle on $\{1, 2, 3\}$ to G. In other words, $\mathcal{T}$ contains all functions from $\{1, 2, 3\}$ to $V(G)$ whose image induces a transitive triangle. If we let $f(1)$ be any vertex $a \in V(G)$ and let $f(2)$ be any $b \in N_G(a)$, by Proposition 1.7, there are $3\delta^0(G) - n \ge (1/6 - 3\varepsilon)n$ vertices we can assign to $f(3)$ so that $f \in \mathcal{T}$. This gives us that

$$|\mathcal{T}| \ge n \cdot (7/9 - 2\varepsilon)n \cdot (1/6 - 3\varepsilon)n > n^3/9 > (\beta n)^3. \tag{1}$$

For any $p \ge 1$, let $\mathcal{L}_p$ be the set of oriented graphs L on $[3p+1]$ such that both $L[\{2, \dots, 3p+1\}]$ and $L[\{1, \dots, 3p\}]$ have perfect TT_3-tilings (see Figure 2 for some examples). For any $p \ge 1$ and $x, y \in V(G)$ (we allow $x = y$), we let $\mathcal{C}_p(x, y)$ be the set of $f \in \mathcal{F}(\mathcal{L}_p, G)$ such that $f(1) = x$ and $f(3p + 1) = y$, and we say that x and y are *p-linked* if $|\mathcal{C}_p(x, y)| \ge (\beta n)^{3p-1}$.

For any $q > p \ge 1$, $f \in C_p(x, y)$ and $g_i \in \mathcal{T}$ for $i \in [q - p]$, the function

$$h(j) = \begin{cases} x & \text{if } j = 1 \\ f(j) & \text{if } 2 \le j \le 3p \\ g_i(k) & \text{if } j = 3p + 3(i-1) + k \text{ for some } i \in [q-p] \text{ and } k \in [3] \\ y & \text{if } j = 3q + 1 \end{cases}$$

is in $\mathcal{C}_q(x, y)$. Therefore,

$$|\mathcal{C}_q(x, y)| \ge |\mathcal{C}_p(x, y)||\mathcal{T}|^{q-p} \text{ for any } q > p.$$

With (1), this implies that, if x and y are p-linked, then x and y are also q-linked for any $q > p$. Recall that we do not require the functions in $\mathcal{C}_q(x, y)$ to be injective.

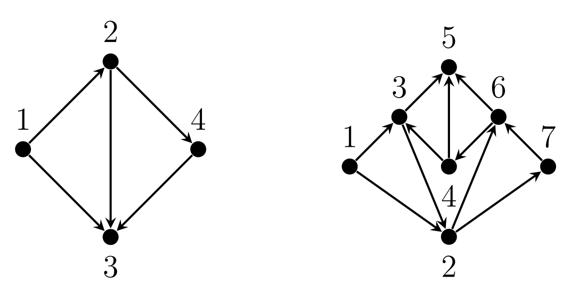

FIGURE 2. A graph in $\mathcal{L}_1$ and a graph in $\mathcal{L}_2$.

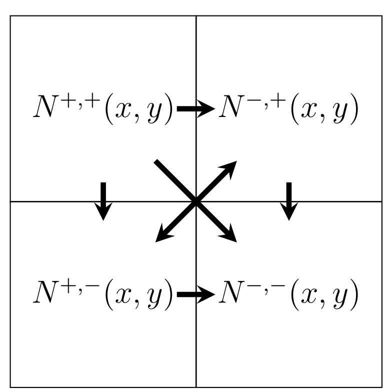

FIGURE 3. The possible orientations of edges in $G'[N(x,y)]$.

Let $X = (x_1, x_2, x_3)$ be an ordered triple of vertices in G, $f \in \mathcal{T}$ and $g_i \in \mathcal{C}_2(f(i), x_i)$ for $i \in [3]$. Define $h : [21] \to V(G)$ by

$$h(j) = \begin{cases} g_i(k) & \text{if } j = 6(i-1) + k \text{ for some } i \in [3] \text{ and } k \in [6] \\ x_i & \text{if } j = 18 + i \text{ for some } i \in [3]. \end{cases}$$

By the definition of $\mathcal{C}_2$ and the fact that the image $h(\{1, 7, 13\}) = f([3])$ induces a transitive triangle in G, we have that $h \in \mathcal{A}'(X)$. Therefore,

$$|\mathcal{A}(X)| = |\mathcal{A}'(X)| \geq \sum_{f \in \mathcal{T}} \prod_{i \in [3]} |\mathcal{C}_2(f(i), x_i)|,$$

so, with (1), we can complete the proof of the lemma by showing that every pair of vertices in $V(G)$ is 2-linked.

The remainder of the proof relies on analyzing the intersection of the neighborhood of two vertices in detail. To facilitate this, we make the following definition and simple computations. For any $U \subseteq V(G)$, let $N_G(U) = \bigcap_{u \in U} N_G(u)$. If $U = \{x, y\}$ is a 2-set, we often write $N_G(x, y)$ instead of $N_G(U)$. We have the following inequality

$$N_G(U) \geq |U|\delta(G) - (|U| - 1)n \geq \left(\frac{9 - 2|U|}{9} - 2|U|\varepsilon\right) n. \tag{2}$$

For any pair $x, y \in V(G)$, let

$$N^{+,+}(x, y) = N_G^+(x) \cap N_G^+(y), \qquad N^{-,+}(x, y) = N_G^-(x) \cap N_G^+(y),$$
$$N^{+,-}(x, y) = N_G^+(x) \cap N_G^-(y), \qquad N^{-,-}(x, y) = N_G^-(x) \cap N_G^-(y)$$

and $\mathcal{N}(x, y) = \{N^{+,+}(x, y), N^{+,-}(x, y), N^{-,+}(x, y), N^{-,-}(x, y)\}$.

It will be important for us to know when two vertices are 1-linked, so we let $F(x, y)$ be the set of edges such that both xe and ye are transitive triangles, which means that every edge in $F(x, y)$ corresponds to two distinct homomorphisms in $\mathcal{C}_1(x, y)$, and

$$\text{if } |F(x, y)| \geq (\beta n)^2/2, \text{ then } x \text{ and } y \text{ are 1-linked.} \tag{3}$$

Let $uv \in E(G[N(x, y)])$. If $u \in N^{-,-}(x, y)$ or $v \in N^{+,+}(x, y)$ or both u and v are in the same set $A \in \mathcal{N}(x, y)$, then $uv \in F(x, y)$. Otherwise, $uv \notin F(x, y)$. Indeed, since $u \notin N^{-,-}(x, y)$, u is an outneighbor of one of x or y, say x. If we assume xuv is a transitive triangle, then $v \in N^+(x)$. This implies that $v \in N^-(y)$, because $v \notin N^{+,+}(x, y)$, which further implies that $u \in N^+(y)$, because u and v cannot both be in $N^{+,-}(x, y)$. Therefore, yuv is not a transitive triangle. Hence,

$$F(x, y) = \overrightarrow{E}_G(N^{-,-}(x, y), N(x, y)) \cup \overrightarrow{E}_G(N(x, y), N^{+,+}(x, y)) \cup \bigcup_{A \in \mathcal{N}(x,y)} E(G[A]). \tag{4}$$

Claim 1. *For any pair $u, w \in V(G)$, if there exists $A \in \mathcal{N}(u, w)$ such that*

$$|A| \geq (2/9 + \beta + 2\varepsilon)n,$$

then u and u are 1-linked. In particular, for any $v \in V(G)$, the pair (v, v) is 1-linked.

Proof. Since, by (4),

$$|F(u, w)| \geq |E(G[A])| \geq |A|(\delta(G) + |A| - n)/2 > (\beta n)^2/2,$$

the claim follows from (3). □

Claim 2. *If $|N(u,w)| \geq (11/18+\beta+\varepsilon)n$, then u and w are 1-linked.*

Proof. Let $B = N^{+,+}(u,w) \cup N^{-,-}(u,w)$ and let $v \in B$. By (4), if $v \in N^{+,+}(u,w)$ and $v' \in N_G(u,w) \cap N_G^-(v)$, then $v'v \in F(u,w)$, and if $v \in N^{-,-}(u,w)$ and $v' \in N_G(u,w) \cap N_G^+(v)$, then $vv' \in F(u,w)$. Therefore,

$$|E_G(v) \cap F(u,w)| \geq \delta^0(G) + |N_G(u,w)| - n \geq \beta n.$$

Hence, if $|B| \geq \beta n$, then $|F(u,w)| \geq (|B| \cdot \beta n)/2 = (\beta n)^2/2$ and, by (3), u and w are 1-linked. If $|B| < \beta n$, then there exists $C \in \{N^{+,-}(u,w), N^{-,+}(u,w)\}$ such that

$$|C| \geq (|N_G(u,w)| - |B|)/2 > n/4,$$

which, with Claim 1, implies that u and w are 1-linked. $\square$

Assume that there exists a pair $x, y \in V(G)$ that is not 1-linked. We have that $|F(x,y)| < (\beta n)^2/2$ by (3). Let $G' = G - F(x,y)$ (see Figure 3), and let

$$N^0 = \{v \in N(x,y) : |E_G(v) \cap F(x,y)| \geq \beta n\}$$

be the set of vertices in $N(x,y)$ incident to a significant number of edges in $F(x,y)$. Note that, since $(\beta n)^2/2 > |F(x,y)| \geq (|N^0| \cdot \beta n)/2$,

$$|N^0| < \beta n. \tag{5}$$

Our goal now is to show that x and y must be 2-linked. To achieve this, we will use the following two claims. Let Γ be the set of triples $(w_1, w_2, w_3) \in V(G)^3$ such that, for some ordering $\{i,j,k\} = [3]$, xw_iw_k and yw_jw_k are transitive triangles and w_i and w_j are 1-linked.

Claim 3. *If $|\Gamma| \geq (\beta n)^3$, then x and y are 2-linked.*

Proof. Let $(w_1, w_2, w_3) \in \Gamma$ with the required ordering $\{i,j,k\} = [3]$. There are at least $(\beta n)^2$ pairs $(u,v) \in V(G)^2$ such that w_iuv and w_juv are transitive triangles, and for every such pair, the 7-tuple $(x, w_1, w_2, w_3, u, v, y)$ corresponds to a function in $\mathcal{C}_2(x,y)$. Therefore, $|\mathcal{C}_2(x,y)| \geq |\Gamma| \cdot (\beta n)^2$, and the conclusion follows (see Figure 2). $\square$

Claim 4. *If $u, w \in N(x,y)$ and*

$$\min\{d^+(u, N(x,y)), d^-(u, N(x,y))\}, \min\{d^+(w, N(x,y)), d^-(w, N(x,y))\} < 4\beta n,$$

then u and w are 1-linked.

Proof. Let $\overline{N} = V(G) - N(x,y)$. By (2),

$$|\overline{N}| \leq n - (5/9 - 4\varepsilon)n = (4/9 + 4\varepsilon)n.$$

If u and w are as in the statement of the claim, then there exist $\sigma_u, \sigma_w \in \{+,-\}$ such that $d_G^{\sigma_u}(u, \overline{N}), d_G^{\sigma_w}(w, \overline{N}) > \delta^0(G) - 4\beta n$. Note that $N^{\sigma_u,\sigma_w}(u,w) \in \mathcal{N}(u,w)$ satisfies

$$\begin{aligned} |N^{\sigma_u,\sigma_w}(u,w)| &\geq d_G^{\sigma_u}(u, \overline{N}) + d_G^{\sigma_w}(w, \overline{N}) - |\overline{N}| > 2(\delta^0(G) - 4\beta n) - (4/9 + 4\varepsilon)n \\ &\geq (7/9 - 2\varepsilon - 8\beta)n - (4/9 + 4\varepsilon)n > n/4. \end{aligned}$$

Applying Claim 1 then completes the proof. $\square$

By (4), every vertex in $N^{+,+}(x,y) \setminus N^0$ has at most βn inneighbors in $N(x,y)$, and every vertex in $N^{-,-}(x,y) \setminus N^0$ has at most βn outneighbors in $N(x,y)$. This and Claim 4 imply that

$$\text{every pair of vertices in } \big(N^{+,+}(x,y) \cup N^{-,-}(x,y)\big) \setminus N^0 \text{ is 1-linked.} \tag{6}$$

Suppose that $|N^{+,+}(x,y)|, |N^{-,-}(x,y)| \geq 2\beta n$, so there are $(\beta n)^2$ ways to select $a \in N^{+,+}(x,y) \setminus N^0$ and $b \in N^{-,-}(x,y) \setminus N^0$. By (6), any such a and b are 1-linked. Let $c \in N_G(\{a,b,x,y\})$. If $c \notin N^{+,-}(x,y) \cup N^{-,+}(x,y)$, then, by (4), either $a, c \in N^{+,+}(x,y)$ and the edge between a and c is in $F(x,y)$ or $b, c \in N^{-,-}(x,y)$ and the edge between b and c is in $F(x,y)$. Recall that since both a and b are not in N^0, they both are incident to at most βn edges in $F(x,y)$. With (2), this gives us that

$$|N_G(a,b) \cap (N^{+,-}(x,y) \cup N^{-,+}(x,y))| \geq |N_G(\{a,b,x,y\})| - 2\beta n > \beta n,$$

and we can pick $c \in N_G(a,b) \cap (N^{+,-}(x,y) \cup N^{-,+}(x,y))$ in one of βn ways. If $c \in N^{+,-}(x,y)$, then xac and ybc are transitive triangles, and if $c \in N^{-,+}(x,y)$, then xbc and yac are transitive triangles. Therefore, in any case, $(a,b,c) \in \Gamma$, which implies $|\Gamma| \geq (\beta n)^3$, so x and y are 2-linked by Claim 3.

Therefore, we can assume that $\min\{|N^{+,+}(x,y)|, |N^{-,-}(x,y)|\} < 2\beta n$, and, by considering the graph in which all of the edge orientations are reversed, we can further assume that

$$|N^{-,-}(x,y)| < 2\beta n.$$

Note that, by (2), Claim 1 and the fact that x and y are not 1-linked,

$$|N^{+,-}(x,y)|, |N^{-,+}(x,y)| \geq |N(x,y)| - |N^{-,-}(x,y)| - 2(2/9 + \beta + 2\varepsilon)n > n/12. \tag{7}$$

Let $A \in \{N^{+,-}(x,y), N^{-,+}(x,y)\}$, and let $u, w \in A \setminus N^0$. Since both u and w have all but less than $2\beta n$ of their neighbors in $V \setminus A$, using (7), we have that

$$|N_G(u,w)| \geq (14/9 - 4\varepsilon)n - 4\beta n - (n - |A|) > (23/36 - 4(\beta + \varepsilon))n.$$

Claim 2 and the fact that $5(\beta + \varepsilon) < 1/36$ then imply that

$$\text{if } u, w \notin N^0 \text{ are both in } N^{+,-}(x,y) \text{ or both in } N^{-,+}(x,y), \text{ then } u \text{ and } w \text{ are 1-linked.} \tag{8}$$

By (7), we can pick $a \in N^{+,-}(x,y) \setminus N^0$ in at least βn ways. By (2), we have that $|N_G(x,y,a)| \geq (1/3 - 6\varepsilon)n$, so since $a \notin N^0$ and $|N^{-,-}(x,y)| < 2\beta n$, a has at least $n/4$ neighbors in $N^{+,+}(x,y) \cup N^{-,+}(x,y)$. Therefore, Claim 1 and our assumption that x and y are not 1-linked imply that

$$a \text{ has at least } 2\beta n \text{ neighbors in both } N^{+,+}(x,y) \text{ and } N^{-,+}(x,y). \tag{9}$$

Using (9), we can then select $b \in (N(a) \cap N^{-,+}(x,y)) \setminus N^0$ in at least βn ways.

Assume $ab \in E$, which implies that yab is a transitive triangle. We will show that either there are at least βn vertices c such that c and a are 1-linked and xbc is a transitive triangle, or there are at least βn vertices c such that c and b are 1-linked and xac is a transitive triangle, so, in either case, $(a,b,c) \in \Gamma$. By the symmetry of x and y, the same argument shows that if $ba \in E$, then there are βn vertices c such that $(a,b,c) \in \Gamma$. This will complete the proof as $|\Gamma| \geq (\beta n)^3$ and Claim 3 imply that x and y are 2-linked.

Suppose that b has at least $4\beta n$ outneighbors in $N(x,y)$. This, with the fact that $b \notin N^0$, implies that $|N^+_{G'}(b, N(x,y))| \geq 3\beta n$. By (4), $b \in N^{-,+}(x,y)$ implies that $N^+_{G'}(b, N(x,y))$ is contained in $N^{-,-}(x,y) \cup N^{+,-}(x,y)$, so since $|N^{-,-}(x,y)| < 2\beta n$, there exist at least βn

outneighbors c of b in $N^{+,-}(x,y) \setminus N^0$. This completes the case, as for every such c, xbc is a transitive triangle, and, since $a, c \in N^{+,-}(x,y) \setminus N^0$, c and a are 1-linked by (8).

Otherwise, b has less than $4\beta n$ outneighbors in $N(x,y)$. Since, by (4), every vertex in $N^{+,+}(x,y) \setminus N^0$ has at most βn inneighbors in $N(x,y)$, Claim 4 implies that b is 1-linked with every vertex in $N^{+,+}(x,y) \setminus N^0$, and, by (9), a has βn neighbors in $c \in N^{+,+}(x,y) \setminus N^0$. For every such c, xac is a transitive triangle and b and c are 1-linked, which completes the case and the proof. □

Lemma 2.2 (Absorbing Lemma). *For every $0 \le \varepsilon \le 1/250$, there exists $\sigma_0 = \sigma_0(\varepsilon)$ such that for every $0 < \sigma < \sigma_0$, there exists $n_0 = n_0(\varepsilon, \sigma)$ such that the following holds. If G is an oriented graph on $n \ge n_0$ vertices with $\delta^0(G) \ge (7/18 - \varepsilon)n$, then G contains a vertex set $U \subseteq V(G)$ with $|U| \le 3\sigma n$ and $|U| \in 3\mathbb{Z}$ such that, for every $W \subseteq V(G)\setminus U$ with $|W| \le 3\sigma^2 n$ and $|W| \in 3\mathbb{Z}$, $G[U \cup W]$ contains a perfect TT_3-tiling.*

Proof. Let $\tau = \tau(\varepsilon)$ be the constant given by Lemma 2.1 and let $\sigma_0 = \tau/(72^2 + 1)$ and let $0 < \sigma < \sigma_0$. Let G be sufficiently large oriented graph with $\delta^0(G) \ge (7/18 - \varepsilon)n$. Let $\mathcal{F}$ be the set of functions from [18] to $V(G)$. Call a map $f \in \mathcal{F}$ *absorbing* if there exists an ordered triple X of vertices such that $f \in \mathcal{A}(X)$.

Choose $\mathcal{U}' \subseteq \mathcal{F}$ by selecting each $f \in \mathcal{F}$ independently at random with probability $p = 2\sigma n^{-17}$. Call a pair $f, g \in \mathcal{F}$ *bad* if either f or g is not injective or the images of f and g intersect and note that there are less than $n \cdot \binom{36}{2} \cdot n^{34}$ bad pairs in $\mathcal{F}$. Therefore, the expected number of bad pairs in $\mathcal{U}'$ is less than $18 \cdot 35 \cdot 4\sigma^2 n$. Thus, using Markov's inequality, we derive that, with probability more than $1/2$, $\mathcal{U}'$ contains at most $(72\sigma)^2 n$ bad pairs.

By Chernoff's bound, the union bound and Lemma 2.1, with positive probability the set $\mathcal{U}'$ also satisfies $|\mathcal{U}'| \le 3\sigma n$ and $|\mathcal{A}(X) \cap \mathcal{U}'| \ge \tau\sigma n$ for each ordered triple X of vertices. By deleting every bad pair from $\mathcal{U}'$ and any $f \in \mathcal{U}'$ for which f is not absorbing, we get $\mathcal{U} \subseteq \mathcal{U}'$ consisting of injective homomorphisms with pairwise disjoint images. Moreover, for each ordered triple X of vertices, there are at least $\tau\sigma n - (72\sigma)^2 n > \sigma^2 n$ functions in $\mathcal{A}(X) \cap \mathcal{U}$. Let U be the union of the images of every $f \in \mathcal{U}$. Since $\mathcal{U}$ consists only of absorbing functions, $G[U]$ has a perfect TT_3-tiling, so $|U| \in 3\mathbb{Z}$. For any set $W \subseteq V\setminus U$ of size $|W| \le 3\sigma^2 n$ and $|W| \in 3\mathbb{Z}$, W can be partitioned into at most $\sigma^2 n$ sets of size 3. Each such set can be arbitrarily ordered to give a triple X which then can be successively paired up with a different absorbing homomorphism $f \in \mathcal{A}(X) \cap \mathcal{U}$. Therefore, $G[U \cup W]$ contains a perfect TT_3-tiling. □

2.2. **Almost TT_3-tilings.** Theorem 1.5 and Lemma 2.3, which we prove simultaneously, show that if $\delta^0(G) \ge 7n/18$ then there is a TT_3-tiling on all but at most 11 vertices of G, and if $\delta^0(G)$ is slightly less than $7n/18$ then there is a TT_3-tiling on all but at most 14 vertices or G is α-extremal for some small $\alpha > 0$, respectively.

Lemma 2.3. *For any $\alpha > 0$ there exists $\varepsilon = \varepsilon(\alpha)$ such that the following holds. If G is an oriented graph on n vertices such that $\delta^0(G) \ge (7/18 - \varepsilon)n$, then either G has a TT_3-tiling on all but at most 14 vertices or G is α-extremal.*

Proof of Theorem 1.5 and Lemma 2.3. For the proof of Lemma 2.3, let $\varepsilon < \alpha/50$. For the proof of Theorem 1.5, let $\varepsilon = 0$. So, in either case, we have that $\delta^0(G) \ge (7/18 - \varepsilon)n$.

Let $\mathcal{M} = \mathcal{T} \cup \mathcal{P} \cup F \cup I$ be a collection of vertex-disjoint subgraphs of G such that every vertex in G is contained in a subgraph of $\mathcal{M}$, every $T \in \mathcal{T}$ is a transitive triangle, every

$P \in \mathcal{P}$ is a directed path on 3 vertices, every $e \in F$ is an edge and every $v \in I$ is a single vertex. Clearly such a set $\mathcal{M}$ exists. Assume that $\mathcal{M}$ is selected to maximize $(|\mathcal{T}|, |\mathcal{P}|, |F|)$ lexicographically. Let $X = V(\mathcal{T})$, $Y = V(\mathcal{P})$ and $Z = V(G) \setminus (X \cup Y)$.

We will show that if $\varepsilon = 0$, then $|\mathcal{P}| \leq 2$ and if $\varepsilon > 0$ and $|\mathcal{P}| \geq 4$, then G is α-extremal. We will also show that $|F| \leq 1$ and $|I| \leq 3$, and this will prove the theorem.

Let B be a $(\mathcal{P}, V(G))$ bipartite graph in which there is an edge between $P \in \mathcal{P}$ and $v \in V(G)$ if and only if $G[P \cup v]$ contains a transitive triangle. By Proposition 1.7, $d_B(P) \geq 2(3\delta^0(G) - n) \geq (1/3 - 6\varepsilon)n$. For every $P \in \mathcal{P}$, by the maximality of $|\mathcal{T}|$, $d_B(P, Y \cup Z) = 0$. Also by the maximality of $|\mathcal{T}|$, for every $T \in \mathcal{T}$ if there exists $x \in V(T)$ such that $d_B(x) \geq 2$, then $d_B(y) = 0$ for every $y \in V(T) - x$. Assume $|\mathcal{P}| \geq 3$ and note that we then have that $e_B(\mathcal{P}, V(T)) \leq |\mathcal{P}|$ for every $T \in \mathcal{T}$. Let

$$\mathcal{T}' = \{T \in \mathcal{T} : e_B(\mathcal{P}, V(T)) > 3\}.$$

We have that

$$n + (|\mathcal{P}| - 3)|\mathcal{T}'| > 3|\mathcal{T}| + (|\mathcal{P}| - 3)|\mathcal{T}'| \geq e_B(\mathcal{P}, V(G)) \geq (1/3 - 6\varepsilon)n|\mathcal{P}|.$$

Which, since $|\mathcal{T}'| < n/3$, is a contradiction when $\varepsilon = 0$, so in this case, we must have that $|\mathcal{P}| \leq 2$. If $\varepsilon > 0$ and $|\mathcal{P}| \geq 4$, then

$$|\mathcal{T}'| \geq \left(\frac{|\mathcal{P}|}{3(|\mathcal{P}| - 3)} - \frac{1}{|\mathcal{P}| - 3} - \frac{6|\mathcal{P}|\varepsilon}{|\mathcal{P}| - 3} \right) n \geq \left(\frac{1}{3} - 24\varepsilon \right) n.$$

For every $T \in \mathcal{T}'$, since $e_B(\mathcal{P}, T) \geq 4$, there exists $x_T \in V(T)$ such that $d_B(x_T) \geq 2$. Therefore, by the maximality of $|\mathcal{T}|$, $d_B(x_T) = e_B(\mathcal{P}, V(T)) \geq 4$. Let

$$W = Y \cup Z \cup \bigcup_{T \in \mathcal{T}'} (V(T) - x_T),$$

and note that $|W| > (2/3 - 48\varepsilon)\, n$. The graph $G[W]$ does not contain a transitive triangle. Indeed, if such a triangle T exists and we define $\mathcal{T}'' = \{T' \in \mathcal{T}' : V(T) \cap V(T') \neq \emptyset\}$ and $B' = B - \{P \in \mathcal{P} : V(P) \cap V(T) \neq \emptyset\}$, then for every $T' \in \mathcal{T}''$, we have that

$$d_{B'}(x_{T'}) \geq d_B(x_{T'}) - |Y \cap V(T)| \geq 4 - |Y \cap V(T)| > |X \cap V(T)| \geq |\mathcal{T}''|.$$

Therefore, there is a matching covering $\mathcal{T}''$ in B'. The edges in this matching correspond to $|\mathcal{T}''|$ disjoint transitive triangles in the graph induced by $(V(\mathcal{T}'') \cup Y \cup Z) \setminus V(T)$, contradicting the maximality of $|\mathcal{T}|$. Hence, G is α-extremal.

Assume that there exist two distinct edges ab and cd in F. For any set $U \subseteq V(G)$, define

$$w(U) = d^+_G(a, U) + \sum_{v \in \{b,c,d\}} d_G(v, U).$$

Note that by the maximality of $|\mathcal{P}|$, there are no triangles in $G[Z]$, so for every $z \in Z$, $w(z) \leq 2$. For any $P \in \mathcal{P}$, the maximality of $|\mathcal{T}|$ implies that there is no transitive triangle in the graph induced by $\{a, b, c, d\} \cup V(P)$. It is not hard to see that, with Proposition 1.6, this gives us that $d^+_G(a, V(P)) + d_G(b, V(P)) \leq 3$ and $e_G(cd, V(P)) \leq 4$, so $w(P) \leq 7$.

Claim. *For every $T \in \mathcal{T}$, $w(T) \leq 8$.*

Proof. Assume that there exists $T \in \mathcal{T}$ such that $w(T) \geq 9$. We will show that there exists a disjoint directed path on 3 vertices and a transitive triangle in the graph induced by $\{a,b,c,d\} \cup V(T)$, contradicting the maximality of $|\mathcal{P}|$.

Remove the edges into a from G to form G'. Note that this implies that for every $x \in V(T)$, if $e_{G'}(ab,x) = 2$, then abx is a transitive triangle. If, in addition to this, there exists $y \in V(T) - x$ such that $e_{G'}(cd,y) = 2$, then we have the desired directed path on 3 vertices. This is the case when for one of the edges $e \in \{ab, cd\}$, $d_{G'}(e, V(T)) = 5$. Indeed, if $f \in \{ab, cd\} - e$, then $e_{G'}(f, V(T)) \geq 4$, so we can pick $x \in V(T)$ such that $e_{G'}(f,x) = 2$. Since then $e_{G'}(e, V(T) - x) \geq 3$, we can pick $y \in V(T) - x$ such that $e_{G'}(e,y) \geq 2$. Therefore, we are only left to consider the cases when one of ab or cd and $V(T)$ induce a tournament on 5 vertices in G'.

If $e_{G'}(ab, V(T)) = 6$, then $e_{G'}(cd, V(T)) \geq 3$ and for one of c or d, say c, $e_{G'}(c, V(T)) \geq 2$, so if x and y are the neighbors of c in $V(T)$, cxy is a triangle and therefore a directed path on 3 vertices. If $z \in V(T) - x - y$, then zab is a transitive triangle.

If $e_{G'}(cd, V(T)) = 6$, then $e_{G'}(ab, V(T)) \geq 3$. We can assume that T is the unique transitive triangle in $G'[\{a,b\} \cup V(T)]$, because, if it is not, then the graph induced by the vertices of $V(T)$ not in this triangle and $\{c,d\}$ would contain a triangle, which contains a directed path on 3 vertices. This implies that $e_{G'}(ab, v) = 1$ for every $v \in V(T)$, and that $e_{G'}(a, V(T)) \leq 1$. This further implies that b has an outneighbor $x \in V(T)$, so abx is a directed path on 3 vertices. Since $G'[\{c,d\} \cup V(T) - x]$ is a tournament on 4 vertices, it contains a transitive triangle by Proposition 1.6. $\square$

Therefore,

$$\left(\frac{49}{18} - 7\varepsilon\right) n \leq 7\delta^0(G) \leq w(V(G)) \leq 2|Z| + 7|Y|/3 + 8|X|/3 \leq 8n/3,$$

a contradiction. Hence $|F| \leq 1$.

By the maximality of $|F|$, I is an independent set. Since there are no triangles in $G[Z]$, $e_G(I,e) \leq |I|$ for every $e \in F$. Let $T \in \mathcal{P} \cup \mathcal{T}$. If $e_G(I, V(T)) > 2|I| + 1$, then there exist vertices $v_1, v_2 \in I$ such that $e_G(v_1, V(T)) = e_G(v_2, V(T)) = 3$. Furthermore, in the graph induced by $\{v_1, v_2\} \cup V(T)$, if $T \in \mathcal{P}$, then there is a triangle and a disjoint edge, and if $T \in \mathcal{T}$, then there is a transitive triangle and a disjoint edge. Since both cases violate the maximality of $|F|$, we have that

$$\begin{aligned} |I|\left(\frac{7}{9} - 2\varepsilon\right) n &\leq |I|\delta(G) \leq e_G(I, V(G) \setminus I) \\ &\leq |I||F| + (2|I|+1)|\mathcal{P} \cup \mathcal{T}| \leq \frac{|I||Z|}{2} + \frac{(2|I|+1)|X \cup Y|}{3} \leq |I|\frac{2n}{3} + \frac{n}{3}. \end{aligned}$$

Hence, $|I| \leq 3 + 18\varepsilon$. $\square$

Using Lemmas 2.2 and 2.3, we can prove Lemma 1.3.

Proof of Lemma 1.3. Let $\varepsilon = \min\{\varepsilon(\alpha/3)/2, 1/250\}$ where $\varepsilon(\alpha/3)$ is as in Lemma 2.3 and let $\sigma_0 = \sigma_0(\varepsilon)$ be as in Lemma 2.2. Assume that n is sufficiently large. So, by Lemma 2.2, there exists $\sigma < \min\{\sigma_0, \varepsilon/3, \alpha/3\}$ for which there exists $U \subseteq V(G)$ such that $|U| \leq 3\sigma n < \varepsilon n$ and the conclusion of Lemma 2.2 holds. As $G[W \cup U]$ has a perfect $TT3$-tiling when $W = \emptyset$, we can conclude that $|U|$ is divisible by 3. Let $G' = G - U$ and $n' = |V(G')|$ and note that $\delta^0(G') \geq (7/18 - 2\varepsilon)n \geq (7/18 - 2\varepsilon)n'$ and n' is divisible by 3. Assume that G is not α-extremal. Because $|U| \leq 3\sigma n < \alpha n$, we have that

$$(2/3 - \alpha/3)n' = (2/3 - \alpha/3)(n - |U|) > (2/3 - \alpha)n,$$

so G' is not $(\alpha/3)$-extremal. Therefore, by Lemma 2.3 there exists a TT_3-tiling on all of $V(G')$ except a set W of size at most 12. Since $|W| < 3\sigma^2 n$ there exists a perfect TT_3-tiling of $G[U \cup W]$ completing the proof. □

3. The α-extremal case

In this section we prove Lemma 1.4. We start with some well-known and simple propositions regarding matchings in graphs.

Proposition 3.1. *Every graph G on n vertices has a matching of size at least* $\min\{\lfloor n/2 \rfloor, \delta(G)\}$.

Proof. Let M be a maximum matching in G and assume $|M| < \min\{\lfloor n/2 \rfloor, \delta(G)\}$. Let U be the set of vertices that are incident to an edge in M. Because $|M| \leq n/2 - 1$, there exist distinct $x, y \in V(G) \setminus U$. Since M is a maximum matching, $e_G(\{x,y\}, V(G) \setminus U) = 0$ which implies

$$e_G(\{x,y\}, U) \geq 2\delta(G) > 2|M|.$$

So there exists $e \in M$ such that $e_G(\{x,y\}, e) \geq 3$. This contradicts the maximality of M. □

Proposition 3.2. *Let G be an (X,Y)-bipartite graph with $d_G(x) \geq a$ for every $x \in X$ and $d_G(y) \geq b$ for every $y \in Y$. If $|X| = |Y|$ and $a + b \geq |X|$, then G contains a perfect matching.*

Proof. We show that G satisfies Hall's condition. Let $X' \subseteq X$ be non-empty, let $x \in X'$ and let Y' be the set of vertices in Y that are adjacent to a vertex in X'. Clearly, $|Y'| \geq d_G(x) \geq a$, so assume $|X'| > a$. We have that $d_G(y) \geq b > |X \setminus X'|$ for every $y \in Y$. Hence, $y \in Y'$ and $|Y'| = |Y| \geq |X'|$. □

Let G be a (V_1, V_2)-bipartite graph. For $X_1 \subseteq V_1$ and $X_2 \subseteq V_2$ both non-empty, define $d_G(X_1, X_2) := \frac{e_G(X_1,X_2)}{|X_1||X_2|}$ to be the *density* of G. For constants $0 < \varepsilon, d < 1$, we say that G is *(d,ε)-regular* if

$$(1-\varepsilon)d \leq d_G(X_1, X_2) \leq (1+\varepsilon)d$$

whenever $|X_i| \geq \varepsilon |V_i|$ for $i = 1, 2$. We say that G is *(d,ε)-superregular* if G is (d,ε)-regular and $(1-\varepsilon)d|V_i| \leq d_G(v, V_i) \leq (1+\varepsilon)d|V_i|$ for every $v \in V_{3-i}$ and $i \in \{1,2\}$.

Proposition 3.3. *For any $0 < \varepsilon < 1$, if G is a (V_1,V_2)-bipartite graph such that $|V_1| = |V_2| = n$ and $\delta(G) \geq (1-\varepsilon)n$ then G is $(1, \varepsilon^{1/2})$-superregular.*

Proof. It is clear that we only need to show that G is $(1, \varepsilon^{1/2})$-regular. Let $X_i \subseteq V_i$ such that $|X_i| \geq \varepsilon^{1/2} n$ for $i \in \{1,2\}$. We have that

$$1 \geq d(X_1, X_2) \geq \frac{|X_1|(|X_2| - \varepsilon n)}{|X_1||X_2|} = 1 - \frac{\varepsilon n}{|X_2|} \geq 1 - \varepsilon^{1/2}.$$ □

The following lemma follows immediately from the Chernoff type bounds on the hypergeometric distribution (see Theorem 2.10 in [6]).

Lemma 3.4. *For every $0 < \eta < 1$ there exists $k = k(\eta) > 0$ such when V is a set, $X \subseteq V$ and m is a positive integer such that $m \leq |V|$ the following holds. If U is selected uniformly at random from $\binom{V}{m}$, then with probability at least $1 - e^{-km}$*

$$\frac{|X|}{|V|} - \eta \leq \frac{|X \cap U|}{|U|} \leq \frac{|X|}{|V|} + \eta.$$

A partition of a set is *equitable* if any two parts differ in size by at most 1.

Proposition 3.5. *For every $\eta > 0$ there exist integers $k = k(\eta) > 0$ and $n_0 = n_0(\eta)$ such that when F is an (A, B)-bipartite graph with $|A| = |B| = n$ for $n \geq n_0$ the following holds. If an equitable partition $\{A_1, A_2\}$ of A and an equitable partition $\{B_1, B_2\}$ of B are both chosen uniformly at random from all such partitions, then with probability at least $1 - e^{-kn}$ we have*

$$d_F(A, B) - \eta \leq d_F(A_i, B_j) \leq d_F(A, B) + \eta$$

for every $1 \leq i, j \leq 2$.

Proof. Choose partitions $\{A_1, A_2\}$ and $\{B_1, B_2\}$ as in the proposition and let $k = 5k(\eta/2)$, where $k(\eta/2)$ is as in in Theorem 3.4, and assume that n is sufficiently large. Let $1 \leq i, j \leq 2$. By Theorem 3.4, for any $v \in A$ with probability at least $1 - e^{-5k|B_j|} \geq 1 - e^{-2kn}$

$$\frac{d_F(v, B)}{|B|} - \frac{\eta}{2} \leq \frac{d_F(v, B_j)}{|B_j|} \leq \frac{d_F(v, B)}{|B|} + \frac{\eta}{2}, \tag{10}$$

and the analogous statement holds for every $v \in B$. So with probability at least

$$1 - 2ne^{-2kn} \geq 1 - e^{-kn}$$

(10) holds for every $v \in V(G)$. Therefore,

$$\begin{aligned} d_F(A_i, B_j) &= \frac{\sum_{v \in A_i} d_F(v, B_j)}{|A_i||B_j|} \geq \frac{\sum_{v \in A_i} d_F(v, B)}{|A_i||B|} - \frac{\eta}{2} = \frac{\sum_{v \in B} d_F(v, A_i)}{|A_i||B|} - \frac{\eta}{2} \\ &\geq \frac{\sum_{v \in B} d_F(v, A)}{|A||B|} - \eta = d_F(A, B) - \eta. \end{aligned}$$

By a similar computation, the upper bound also holds. □

Theorem 3.6 (Kühn and Osthus [8])**.** *For all positive constants $d, \xi_0, \eta \leq 1$ there is a positive $\varepsilon = \varepsilon(d, \xi_0, \eta)$ and an integer $n_0 = n_0(d, \xi_0, \eta)$ such that the following holds for all $n \geq n_0$ and all $\xi \geq \xi_0$. Let G be a (d, ε)-superregular bipartite graph whose vertex classes both have size n and let F be a subgraph of G with $e(F) = \xi e(G)$. Choose a perfect matching M uniformly at random in G. Then with probability at least $1 - e^{-\varepsilon n}$ we have*

$$\xi - \eta \leq \frac{|M \cap E(F)|}{|M|} \leq \xi + \eta.$$

Proposition 3.7. *Let G be an oriented graph and let $x \in V(G)$ and let $a, b, c \in N_G(x)$. If abc is a cyclic triangle in G, then xe is a transitive triangle for at least two edges $e \in \{ab, bc, ca\}$.*

Proof. Let $i = d^+_G(x, \{a, b, c\})$. By symmetry, there are four cases depending on the value of i. Furthermore, by reversing the edges of G it is easy to see that the cases when $i = j$ are equivalent to the cases when $i = 3 - j$. It is easy to verify the statement when $i = 3$ and when $i = 2$, we omit the details. □

We will use following lemma when finishing the proof of Lemma 1.4. Lemma 3.8 essentially states that if a graph looks very similar to the graph depicted in Figure 1 except that $|W_1| + |W_2| + |W_3| = 2(|U_1| + |U_2|)$, then there exists a perfect TT_3-tiling of G.

Lemma 3.8. *There exists a constant $\beta > 0$ and an integer n_0 such that for any $m \in \mathbb{N}$ where $n = 18m \geq n_0$ and any oriented graph G on n vertices the following holds. If there exists a partition $\{W_1, W_2, W_3, U\}$ of $V(G)$ such that $|W_1| = |W_2| = |W_3| = 4m$, $|U| = 6m$ and for every $i \in [3]$ and $w \in W_i$,*

$$d^+(w, W_{i+1}), d^-(w, W_{i-1}) \geq (4 - \beta)m \text{ and} \tag{11}$$

$$d^+(w, U), d^-(w, U) \geq (3 - \beta)m; \tag{12}$$

and, for every $u \in U$ and $i \in [3]$,

$$d(u, W_i) \geq (4 - \beta)m, \tag{13}$$

then G contains a perfect TT_3-tiling.

Proof. Let $\eta = 1/12$, $\varepsilon = \varepsilon(1, \eta/2, \eta/2)$, $\beta = \min\{\varepsilon^2, 1/24\}$ where $\varepsilon(1, \eta/2, \eta/2)$ is as in Theorem 3.6. Assume m is sufficiently large. Let $W = W_1 \cup W_2 \cup W_3$.

Let $F = E(G[W_1, W_2] \cup G[W_2, W_3] \cup G[W_3, W_1])$ and define a bipartite graph B with classes U, F as follows. A vertex $u \in U$ and an edge $xy \in F$ form an edge in B if uxy is a transitive triangle in G. Clearly $|F| \leq 3(4m)^2 = 48m^2$.

Claim. *For every $u \in U$, $d_B(u) \geq (2/3 - \beta)48m^2$.*

Proof. Let $P(u)$ be the set of pairs of the form (e, abc) where $a \in W_1 \cap N_G(u)$, $b \in W_2 \cap N_G(u)$ and $c \in W_3 \cap N_G(u)$, abc is a cyclic triangle and $e \in \{ab, bc, ca\} \cap N_B(u)$. By Proposition 3.7, for every $(e, abc) \in P(u)$ the cyclic triangle abc appears at least twice as the second element of a pair in $P(u)$. Therefore, by (11) and (13),

$$|P(u)| \geq 2 \cdot (4 - \beta)m \cdot (4 - 2\beta)m \cdot (4 - 3\beta)m > (1 - 3\beta/2)128m^3.$$

Since any edge can appear as the first element in at most $4m$ of the pairs in $P(u)$,

$$d_B(u) \geq |P(u)|/(4m) \geq (1 - 3\beta/2)32m^2 = (2/3 - \beta)48m^2. \qquad \square$$

For every $u \in U$, let $F(u)$ be the graph on W with edge set $N_B(u)$. By Proposition 3.5 and the union bound, there exists an equitable partition $\{W_i^1, W_i^2\}$ of W_i for each $i \in [3]$, such that for every $u \in U$,

$$d_{F(u)}(W_i^k, W_j^\ell) \geq d_{F(u)}(W_i, W_j) - \eta/2$$

for all $1 \leq k, \ell \leq 2$ and $j \in \{1, 2, 3\} - i$. Let $G_1 = G[W_1^1, W_2^1]$, $G_2 = G[W_2^2, W_3^2]$ and $G_3 = G[W_3^1, W_1^2]$. Note that $\delta(G_i) \geq (2-\beta)m$ for every $i \in [3]$, so G_i is $(1, \beta^{1/2})$-superregular by Proposition 3.3. Therefore, by Theorem 3.6 and the union bound, there exists a perfect matching M_i of G_i such that

$$\frac{|M_i \cap E(F(u))|}{2m} \geq \frac{|E(G_i) \cap E(F(u))|}{e(G_i)} - \frac{\eta}{2} \geq d_{F(u)}(W_i, W_{i+1}) - \eta$$

for every $u \in U$ and $i \in [3]$. Note that, because $\beta^{1/2} \leq \varepsilon(1, \eta/2, \eta/2)$, Theorem 3.6 may not apply when $|E(G_i) \cap E(F(u))|/e(G_i) \leq \eta/2$, but in that case the inequality is vacuously true. Observe that $M = M_1 \cup M_2 \cup M_3$ is a perfect matching of $G[W]$ and for $B' = B[U, M]$, and every $u \in U$

$$\begin{aligned}\frac{d_{B'}(u)}{|M|} &= \frac{|M \cap E(F(u))|}{6m} = \frac{1}{3}\frac{\sum_{i=1}^{3}|M_i \cap E(F(u))|}{2m} \\ &\geq \frac{1}{3}\sum_{i=1}^{3}\left(d_{F(u)}(W_i, W_{i+1}) - \eta\right) = \frac{d_B(u)}{48m^2} - \eta \geq \frac{2}{3} - (\beta + \eta).\end{aligned}$$

We also have that, by (12), for every $xy \in M$,

$$d_{B'}(xy) \geq |N^+_G(x, U) \cap N_G(y, U)| \geq (3 - 3\beta)m > (1/2 - \beta)6m.$$

Note that since

$$2/3 - (\beta + \eta) + 1/2 - \beta \geq 7/6 - 2\beta - \eta \geq 1,$$

Proposition 3.2 implies that B' has a perfect matching. This perfect matching corresponds to a perfect TT_3-tiling of G. □

Proof of Lemma 1.4. Let β be as in Lemma 3.8. Let $\tau = \beta/320$ and let $\alpha = \tau^3$.

Let $\gamma > 0$ and let $\mathcal{W} = \{W_1, W_2, W_3\}$ be a collection of three disjoint vertex subsets. We say that $v \in V(G)$ is (i, γ)-*cyclic* for the triple $\mathcal{W}$ if

$$d^+_G(v, W_{i-1}) + d_G(v, W_i) + d^-_G(v, W_{i+1}) \leq \gamma n,$$

and that v is γ-*cyclic* for $\mathcal{W}$ if v is (i, γ)-cyclic for some i. The triple $\mathcal{W}$ is γ-*cyclic* if every vertex in W_i is (i, γ)-cyclic for every $i \in [3]$. A vertex is γ-*bad* for $\mathcal{W}$ if it is not γ-cyclic. The following claim follows from the preceding definition.

Claim 1. *For any $1 > \gamma > \gamma' \geq 0$, if a vertex v is γ-bad for $\{W_1, W_2, W_3\}$ and $|X| \leq \gamma' n$, then v is $(\gamma - \gamma')$-bad for $\{W_1 \setminus X, W_2 \setminus X, W_3 \setminus X\}$.*

For any λ, we say that $\mathcal{W}$ is λ-*equitable* if

$$||W_i| - |W_j|| \leq \lambda n \text{ for every } i, j \in [3]$$

and $|V(\mathcal{W})| \geq (2/3 - \lambda)n$. Note that this implies that $|W_i| \geq (2/9 - \lambda)n$ for every $i \in [3]$.

Let $\mathcal{W} = \{W_1, W_2, W_3\}$ be a λ-equitable triple and let $v \in V(G)$ be (i, γ)-cyclic for $\mathcal{W}$. By the degree condition,

$$\begin{aligned}d^-_G(v, W_{i-1}) + d^+_G(v, W_{i+1}) &= d_G(v, V(\mathcal{W})) - \left(d^+_G(v, W_{i-1}) + d_G(v, W_i) + d^-_G(v, W_{i+1})\right) \\ &\geq |V(\mathcal{W})| - 2n/9 - \gamma n.\end{aligned}$$

Therefore, since $|W_1|, |W_2|, |W_3| \geq (2/9 - \lambda)n$, we have the following:

$$\begin{aligned}d^-_G(v, W_{i-1}) &\geq |W_{i-1}| + |W_i| - 2n/9 - \gamma n \geq |W_{i-1}| - (\gamma + \lambda)n, \\ d^+_G(v, W_{i+1}) &\geq |W_{i+1}| - (\gamma + \lambda)n \text{ and} \\ d^-_G(v, W_{i-1}), d^+_G(v, W_{i+1}) &\geq (2/9 - 2\lambda - \gamma)n.\end{aligned} \tag{14}$$

Claim 2. *Let $0 < \gamma < 1/27$ and let $\mathcal{W} = \{W_1, W_2, W_3\}$ be such that $\mathcal{W}$ is both γ-cyclic and γ-equitable. If $v \in V(G)$ such that there are no transitive triangles in $G[\{v\} \cup W]$ that contain v, then v is 0-cyclic for $\mathcal{W}$.*

Proof. Since $|V(\mathcal{W})| \geq (2/3-\gamma)n > 11n/18$, there exists an $x \in N^+_G(v, W_{i+1})$ for some $i \in [3]$. Let $I_x = N^-_G(x, W_i)$. By (14), $|I_x| \geq (2/9-3\gamma)n$. Suppose that v is not $(i,0)$-cyclic, i.e. there exists

$$y \in N^+_G(v, W_{i-1} \cup W_i) \cup N^-_G(v, W_i \cup W_{i+1}).$$

If $y \in N^+_G(v, W_{i-1} \cup W_i)$, then let $I_y = N^-_G(y, W_{i+1} \cup W_{i-1})$ and if $y \in N^-_G(v, W_i \cup W_{i+1})$, then let $I_y = N^+_G(y, W_{i+1} \cup W_{i-1})$. Again by (14), we have that $|I_y| \geq (2/9-3\gamma)n$. Note that v has no neighbors in $I_x \cup I_y$, because any such neighbor would imply a transitive triangle containing v in $G[\{v\} \cup W]$. Also note that $I_x \subseteq W_i$ and $I_y \subseteq W_{i+1} \cup W_{i-1}$, so I_x and I_y are disjoint, and

$$|W| + \delta(G) - n \leq d_G(v, W) \leq |W| - |I_x| - |I_y| \leq |W| - (4/9-6\gamma)n < |W| - 2n/9,$$

a contradiction. $\square$

Recall, since G is α-extremal there exists $W \subseteq V(G)$ such that $|W| \geq (2/3-\alpha)n$ and $G[W]$ does not contain any transitive triangles.

Claim 3. *There exists a 0-cyclic partition $\mathcal{W} = \{W_1, W_2, W_3\}$ of W such that for every $i \in [3]$*

$$(2/9-\alpha)n \leq |W_i| \leq 2n/9.$$

Proof. Let $G' = G[W]$ and note that

$$\delta(G') \geq \delta(G) + |W| - n \geq |W| - 2n/9. \tag{15}$$

Since G' is TT_3-free, for every $v \in W$ the sets $N^+_{G'}(v)$ and $N^-_{G'}(v)$ are independent. This with (15) implies that both sets are of order at most $2n/9$ and hence that

$$\delta^0(G') \geq \delta(G') - 2n/9 \geq |W| - 4n/9. \tag{16}$$

Since G' is TT_3-free there exists a cyclic triangle $w_1w_2w_3$ in G'. This also implies that, for any $i \in [3]$, the set $\widetilde{W}_i = N^+_{G'}(w_{i-1}) \cup N^-_{G'}(w_{i+1})$ is disjoint from $N_{G'}(w_i)$. Hence, by (15), $|\widetilde{W}_i| \leq 2n/9$. Define $\widehat{W}_i = N^+_{G'}(w_{i-1}) \cap N^-_{G'}(w_{i+1})$. Then we have that $\widehat{W}_i$ is an independent set and, by (16),

$$2n/9 \geq |\widehat{W}_i| \geq d^+_{G'}(w_{i-1}) + d^-_{G'}(w_{i+1}) - |\widetilde{W}_i| \geq 2|W| - 10n/9 \geq (2/9-2\alpha)n.$$

Note that

$$\overrightarrow{E}_{G'}(\widehat{W}_{i-1}, \widehat{W}_{i+1}) \subseteq \overrightarrow{E}_{G'}(N^-_{G'}(w_i), N^+_{G'}(w_i)) = \emptyset.$$

This gives us that $\widehat{\mathcal{W}} = (\widehat{W}_1, \widehat{W}_2, \widehat{W}_3)$ is 0-cyclic.

Let $X = W \setminus V(\widehat{\mathcal{W}})$. By repeatedly applying Claim 2, we can iteratively add each $x \in X$ to a set $\widehat{W}_i$ for which x is $(i,0)$-cyclic for $\widehat{\mathcal{W}}$. Let $\mathcal{W} = \{W_1, W_2, W_3\}$ be the resulting collection. For every $i \in [3]$, the set W_i is independent, so $|W_i| \leq 2n/9$ by (15) and moreover, because $|W| \geq (2/3-\alpha)n$, $|W_i| \geq (2/9-\alpha)n$, So $\mathcal{W}$ is the desired partition of W. $\square$

Let $U = V(G) \setminus W$. If $v \in V(G)$ is (i,γ)-cyclic for $\mathcal{W}$, then

$$\begin{aligned} d^+_G(v,U), d^-_G(v,U) &\geq \delta^0(G) - \max\{|W_{i+1}|, |W_{i-1}|\} - \gamma n \\ &\geq (1/6-\gamma)n \geq |U|/2 - (\alpha/2+\gamma)n. \end{aligned} \tag{17}$$

We also have that,

$$d_G(v,U) \geq \delta(G) - (|W \setminus W_i| + d_G(v, W_i)) \geq 7n/9 - 4n/9 - \gamma n = |U| - (\alpha+\gamma)n. \tag{18}$$

By Claim 3, we can apply (18) with $\gamma = 0$ to give us that

$$e_G(W, U) \geq (|U| - \alpha n)|W| > |U||W| - \alpha n^2.$$

Defining $Z = \{u \in U : d_G(u, W) < |W| - \tau n\}$, we have that, since $\tau^3 = \alpha$,

$$|Z| < \tau^2 n. \tag{19}$$

Let $Z(i)$ be the set of vertices in Z that are (i, τ)-cyclic for $\mathcal{W}$. Clearly $Z(1), Z(2)$ and $Z(3)$ are disjoint. Let $Z'' = \bigcup_{i=1}^{3} Z(i)$, $W_i' = W_i \cup Z(i)$, $\mathcal{W}' = (W_1', W_2', W_3')$, $W' = V(\mathcal{W}') = W \cup Z''$, $U' = U \setminus Z''$ and $Z' = Z \setminus Z''$. Note that, for every $i \in [3]$, $(2/9 - \alpha)n \leq |W_i'| \leq 2n/9 + |Z|$ so $\mathcal{W}'$ is $(2\tau^2)$-equitable and that every vertex in W_i' is (i, τ)-cyclic for $\mathcal{W}$. Since $|W' \setminus W| \leq |Z|$, this implies that $\mathcal{W}'$ is (2τ)-cyclic. We also have that for every $u \in U' \setminus Z'$,

$$d_G(u, W') \geq |W| - \tau n \geq |W'| - |Z| - \tau n \geq |W'| - 2\tau n. \tag{20}$$

We will now find three collections $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ of disjoint transitive triangles. We define $X_i = V(\mathcal{T}_i)$ and $Y_i = \bigcup_{j=1}^{i} X_j$. The collections will be constructed so that the sets X_1, X_2, and X_3 are disjoint. The collections will also have the following properties:

- (P1) $|W' \setminus Y_i| = 2|U' \setminus Y_i|$ for $i \in \{1, 2, 3\}$,
- (P2) $Z' \subseteq X_2$,
- (P3) $|Y_3| \leq \tau n$,
- (P4) $|W_1' \setminus Y_3| = |W_2' \setminus Y_3| = |W_3' \setminus Y_3|$ and
- (P5) $|V(G) \setminus Y_3|$ is divisible by 18.

Assume we have such collections. First, we will show that $G - Y_3$ has a TT_3-tiling by showing that $G - Y_3$ satisfies the conditions of Lemma 3.8 with $\beta = 320\tau = 16 \cdot 20\tau$ and $\{W_1' \setminus Y_3, W_2' \setminus Y_3, W_3' \setminus Y_3, U' \setminus Y_3\}$ the required partition of $V(G - Y_3)$. To see this, first note that, by (P5), there exist m such that $|G - Y_3| = 18m$, and by (P1) and (P4), $|W_i' \setminus Y_3| = 4m$ for every $i \in [3]$ and $|U' \setminus Y_3| = 6m$ and, by (P3), $m \geq n/20$. Since $\mathcal{W}'$ is (2τ)-cyclic and $(2\tau^2)$-equitable, (14) implies that for every $w \in W_i' \setminus Y_3$ and $i \in [3]$, w meets condition (11). Furthermore, (18) implies that w also meets condition (12). Finally, (P2) and (20), imply that every $u \in U' \setminus Y_3$, meets condition (13).

We begin the construction by finding a collection $\mathcal{T}_1$ such that $|W' \setminus X_1| = 2|U' \setminus X_1|$. Call a transitive triangle T *standard* if $|V(T) \cap W'| = 2$ and $|V(T) \cap U'| = 1$. Every transitive triangle $T \in \mathcal{T}_2 \cup \mathcal{T}_3$ will be standard and this will give us Property (P1).

Let $c = |W'| - 2n/3$ and note that $-\alpha n \leq c \leq |Z''| \leq \tau^2 n$, so $|c| \leq \tau^2 n$. Simple computations show that the following claim gives the desired collection $\mathcal{T}_1$. Indeed, if $c > 0$, then $|W'| - 3c = 2(n/3 - c) = 2|U'|$, and if $c < 0$, then $|W'| - |c| = 2(n/3 - |c|) = 2(|U'| - 2|c|)$.

Claim 4. *There exists a collection $\mathcal{T}_1$ of $|c|$ disjoint transitive triangles such that for every $T \in \mathcal{T}_1$,*

- *if $c > 0$, $T \subseteq G[W']$; and*
- *if $c < 0$, $|V(T) \cap W'| = 1$ and $|V(T) \cap (U \setminus Z)| = 2$.*

Proof. First assume $c > 0$ and let $I = \{i \in [3] : |W_i'| > 2n/9\}$ and $c_i = |W_i'| - \lfloor 2n/9 \rfloor$ for $i \in I$. Note that, by Claim 3, $|W_i| \leq 2n/9$, so the fact that $|Z''| \leq \tau^2 n$ implies that $c_i \leq \tau^2 n$. For every $i \in I$, by the degree condition, we have that $\delta(G[W_i']) \geq c_i$. Therefore, Proposition 3.1 implies that there exists a matching M_i of size c_i in $G[W_i']$. For every $xy \in M_i$, x and y are $(i, 2\tau)$-cyclic for $\mathcal{W}'$ and $\mathcal{W}'$ is an $(2\tau^2)$-equitable triple. So, by (14),

$$|N_G^-(x, W_{i-1}') \cap N_G^-(y, W_{i-1}')| \geq |W_{i-1}'| - 4(\tau + \tau^2)n,$$

and, similarly, $|N_G^+(x, W'_{i+1}) \cap N_G^+(y, W'_{i+1})| \geq |W'_{i+1}| - 4(\tau + \tau^2)n$. Therefore, we can easily match the edges $\bigcup_{i \in I} M_i$ to vertices in W' so that the matching corresponds to disjoint transitive triangles in G. Since $\sum_{i \in I} c_i \geq c$ we have the desired collection $\mathcal{T}_1$.

Now assume $c < 0$. Let $U'' = U \setminus Z = U' \setminus Z'$. By (19), we have that $|U''| \geq (1/3 - \tau^2)n$ so by the degree condition, $\delta(G[U'']) \geq (1/9 - \tau^2)n$ and there exists a matching M of order $|c| \leq \tau^2 n$ in $G[U'']$. By Proposition 1.7, every $e \in E(G)$ has $n/6$ vertices v such that ev is a transitive triangle. Therefore, we can match each edge $e \in M$ to a vertex $v_e \in V(G) \setminus Z$ so that the ev_e are disjoint transitive triangles. Let $\mathcal{T}'_1$ be this collection of transitive triangles. Suppose that there exists $T \in \mathcal{T}'_1$ such that $V(T) \subseteq U''$. By Proposition 1.6 and the fact that, by (20), $|\bigcap_{v \in V(T)} N_G(v, W')| \geq |W'| - 6\tau n$, it is trivial to replace T with a transitive triangle that has one vertex in W' and an edge from $E(T)$ and is also disjoint from $V(\mathcal{T}'_1 - T)$. By replacing every such triangle in this manner, we can create the desired collection $\mathcal{T}_1$. ☐

We now aim to find a collection $\mathcal{T}_2$ of standard transitive triangles that satisfies Property (P2). Note that, by the definition of Z', every vertex in Z' is τ-bad for $\mathcal{W}$ and hence is τ-bad for $\mathcal{W}'$. The following claim then follows from Claim 1 and Claim 2.

Claim 5. *There exists a collection $\mathcal{T}_2$ of $|Z' \setminus X_1|$ disjoint standard transitive triangles in $G - X_1$ such that $|T \cap Z'| = 1$ for every $T \in \mathcal{T}_2$.*

Proof. Let $\mathcal{T}_2$ be a collection of disjoint standard transitive triangles in $G - X_1$ such that for every $T \in \mathcal{T}_2$, $|V(T) \cap Z'| = 1$. Let $Y_2 = V(\mathcal{T}_2) \cup X_1$. Suppose that $|\mathcal{T}_2|$ is maximal among all such collections and that there exists $z \in Z' \setminus Y_2$. Since z is τ-bad for $\mathcal{W}'$, by Claim 1 and the fact that $|Y_2| < |X_1| + 3|Z'| < 6\tau^2 n < \tau n$, z is 0-bad for $\{W'_1 \setminus Y_2, W'_2 \setminus Y_2, W'_3 \setminus Y_2\}$. Hence, by Claim 2, there exists a transitive triangle containing z and two vertices in $V(\mathcal{W}') \setminus Y'_2$. Adding T to $\mathcal{T}_2$ contradicts the maximality of $|\mathcal{T}_2|$. ☐

Let $W''_i = W'_i \setminus Y_2$ for every $i \in [3]$. Since $\mathcal{W}'$ is $(2\tau^2)$-equitable and $|Y_2| \leq |X_1| + 3|Z'| \leq 6\tau^2$, the collection $\mathcal{W}'' = \{W''_1, W''_2, W''_3\}$ is $(8\tau^2)$-equitable.

Because $|\mathcal{T}_1 \cup \mathcal{T}_2| \leq 2|Z| \leq 2\tau^2 n$, if we we can find a collection $\mathcal{T}_3$ of at most $17\tau^2 n \leq \tau n/3 - 2\tau^2 n$ disjoint standard transitive triangles in $G - Y_2$ that satisfies (P4) and (P5), we will also satisfy Property (P3). This is quite easy to do, as we now describe.

Let π be a permutation of [3] such that $|W''_{\pi(1)}| \leq |W''_{\pi(2)}| \leq |W''_{\pi(3)}|$. Let M_1, M_2 and M_3 be disjoint edge sets such that their union is a matching and

- $|M_1| = |W''_{\pi(3)}| - |W''_{\pi(2)}|$, $|M_2| = |W''_{\pi(3)}| - |W''_{\pi(1)}|$,
- $M_1 \subseteq E_G(W''_{\pi(3)}, W''_{\pi(1)})$, $M_2 \subseteq E_G(W''_{\pi(3)}, W''_{\pi(2)})$ and
- M_3 consists of three edges, one from each of $E(W''_i, W''_{i+1})$ for $i \in [3]$.

Let $M' = M_1 \cup M_2$ and $M = M' \cup M_3$. Note that since $\mathcal{W}''$ is $(8\tau^2)$-equitable, $|M'| < |M| \leq 2(8\tau^2 n) + 3 \leq 17\tau^2 n$. Let $vv' \in M$. Since v and v' are both τ-cyclic for $\mathcal{W}$, (17) and (18) give us that the number of vertices $x \in U$ such that xvv' is a transitive triangle is at least

$$|N_G^-(v, U) \cap N_G(v', U)| \geq n/6 - \alpha n - 2\tau n.$$

Therefore, we can find the desired collection $\mathcal{T}_3$ by matching edges vv' in either M or M' to unused vertices x in U' such that $vv'u$ is a transitive triangle. We can clearly satisfy Property (P4). Note that Properties (P1) and (P4) imply that $|V(G) \setminus Y_3| \in 9\mathbb{Z}$. So we can satisfy Property (P5) by picking M or M' appropriately. ☐

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