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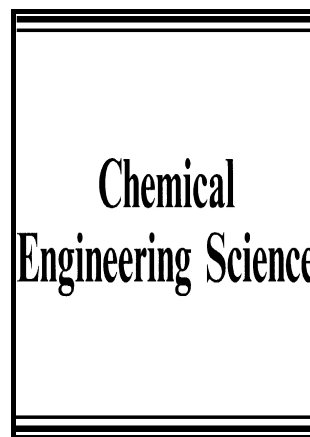
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## Author's Accepted Manuscript

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# The shape and behaviour of a granular bed in a rotating drum using Eulerian flow fields obtained from PEPT

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## Abstract

Non-invasive single-particle tracking techniques, such as positron emission particle tracking (PEPT), provide useful information about the behaviour of a representative particle moving in a bulk of similar particles in a rotating drum. The Lagrangian trajectories that they yield can be used to study, for example, particulate diffusion or granular interaction. However, often the Eulerian flow fields of the entire granular bed are more useful— they can be used to study segregation, for instance, or the evolution of the free surface of the bed. In this work, we present a technique for converting Lagrangian trajectories to Eulerian flow fields via a time-weighted residence time distribution (RTD) of the tracked particle. We then perform PEPT experiments on a mono-disperse bed of spherical particles in a cylindrical drum, rotated at various rates, and use the RTD procedure to obtain flow fields of the bed. We use these flow fields to investigate the effect of drum rotational speed on the shape and behaviour of a granular bed in a rotating drum, and the insights gained thereby to define a comprehensive set of surfaces— such as the bulk free surface— to divide the bed into regions of distinct granular behaviour. We further define scalar bed features— such as the centre of circulation of the bed— that can be used to quantitatively compare the behaviour of granular beds in rotating drums operated under various conditions.

*Keywords:* PEPT, positron emission particle tracking, granular flow, rotating drums, free surfaces, residence time distribution, RTD

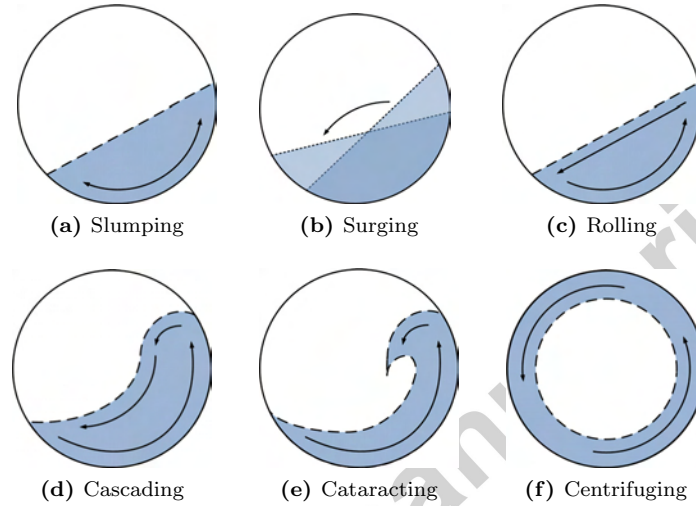
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## 1 Introduction

2 A bed of granular material in a rotating drum can experience a wide range of  
 3 phenomena– from perfect mixing to convection and size segregation– depending  
 4 on the conditions under which the drum is operated. The types of behaviour  
 5 that such a granular bed can exhibit have been classified into various categories  
 6 by Boateng and Barr (1996), Mellmann (2001) and Ding et al. (2002). These  
 7 categories, or modes, are collected and summarised in figure 1.



**Figure 1:** Six categories of granular bed motion in a rotating drum in order of increasing rotational speed, left to right and top to bottom.

8 These behaviours are exploited industrially in a range of material-handling ap-  
 9 plications, including powder mixers, rotary kilns, trommels, and grinding mills.  
 10 Of most interest for all of these applications are the cascading and cataracting  
 11 modes, which will be the focus of this paper. Which of these modes obtains in a  
 12 rotating drum is most determined by two operating variables: the drum's rota-  
 13 tional speed, and the fraction of its volume occupied by the granular material.

14 For ease of comparison across drum sizes, rotational speed can be expressed  
 15 as a fraction of critical speed,  $N_C$ , which is the rotational speed at which the  
 16 outermost layer of granular material in a rotating drum begins to centrifuge. It  
 17 is defined by Rose and Sullivan (1957), in revolutions per second, as:

$$N_c = \frac{1}{2\pi} \sqrt{\frac{2g}{D - 2r}} \quad (1)$$

18 where  $D$  is the inner diameter of the rotating drum,  $g$  is the acceleration due to  
 19 gravity, and  $r$  is the radius of a representative grain in the bed. Many industrial

20 rotating drums satisfy the additional condition that  $r \ll D$ , which allows the  
 21 critical speed to be expressed more simply, in revolutions per minute, as:

$$N_c \approx \frac{42.3}{\sqrt{D}}. \quad (2)$$

22 Although equation 2 is an imperfect approximation, even for mono-disperse  
 23 granular materials, and although the critical speed of a rotating drum has been  
 24 shown to be more complicated than that shown in equation 1 (Watanabe, 1999),  
 25 we employ the definition in equation 2 for ease of comparison with industry and  
 26 industrial data.

27 Also for comparison with industrial data, the volume of granular material in the  
 28 drum is not expressed as a volume fraction, but as a load fraction,  $f$ . The load  
 29 is that volume of material that would completely fill the drum. Thus the load  
 30 fraction differs from the volume fraction by a factor of the packing fraction,  $\eta$ .  
 31 The load fraction can be expressed as:

$$f = \frac{V_m}{\eta V_d} \quad (3)$$

32 where  $V_m$  is the volume of the granular material, and  $V_d$  the internal volume of  
 33 the drum. Defining the load fraction in this way has the effect that the occupied  
 34 fraction of the drum can increase as the dynamic granular bed dilates, and thus  
 35 the bed packing fraction decreases.

36 Due to the breadth of applications of rotating drums in granular handling oper-  
 37 ations, knowing how the behaviour of a granular bed will change in response  
 38 to varying operating conditions, such as drum rotational speed and filling, is of  
 39 significant interest to industry. This work will go some way towards providing  
 40 an unambiguous means of characterising the type of motion undergone by a  
 41 granular bed from either Lagrangian or Eulerian experimental data.

42 The rest of the paper is organised as follows: first, various sources of the ne-  
 43 cessary experimental data are discussed generally, and the technique employed  
 44 here, positron emission particle tracking (PEPT), in detail. Next, a means  
 45 of interpreting such Lagrangian trajectory data as a Eulerian probability and  
 46 kinematic fields is introduced. Numerical schemes are then presented to char-  
 47 acterise and delineate various features and regions of a dynamic granular bed  
 48 undergoing cascading and cataracting motion in a rotating drum. Finally, these  
 49 numerical schemes are used to compare the shape and behaviour of the granular  
 50 bed in a rotating drum operated at three different rotational speeds, using data  
 51 obtained with the PEPT technique.

52 **The experimental study of granular motion in rotating drums**

53 Various experimental techniques have been used to study granular flows in ro-  
54 tating drums, predominantly in the context of rotary kilns and of tumbling mills.  
55 These techniques can be divided into two categories– visual and radiological.

56 The visual experiments generally consist of a laboratory-scale rotating drum  
57 fitted with a transparent end window through which the dynamic granular bed  
58 is observed. The dynamic bed is then either filmed to obtain a dynamic picture  
59 of the motion of the bed, or photographed. Such techniques were used by, for  
60 instance Rogovin and Herbst (1989), Morrell (1992), Santomaso et al. (2003) and  
61 Tordesillas and Arber (2005). Another well-developed method is Particle Image  
62 Velocimetry (PIV), which involves comparing the positions of tracer particles  
63 in sequences of high-speed photographs (Jain et al., 2002).

64 The obvious drawback of end-window techniques is that they are limited to  
65 what can be seen through the side of the drum. This makes it impossible to  
66 control for obfuscating end effects that may not be representative of the motion  
67 of the bulk of the charge. To overcome this limitation, radiological techniques  
68 were developed to image the bulk behaviour of a granular bed. Nakagawa et al.  
69 (1993), for instance, used nuclear magnetic resonance imaging (NMRI) to study  
70 the motion of mustard seeds in a rotary kiln. Similarly, Powell and Nurick  
71 (1996b) used biplanar x-ray imaging of a rotating drum to obtain, by the density-  
72 dependent transmission of x-rays through matter, a better representation of the  
73 shape of the dynamic bed in the drum. Govender et al. (2001) extended this  
74 technique to single-particle tracking of a more-attenuating particle in a bulk of  
75 similar particles as a means of obtaining a more representative velocity of the  
76 tracer particle in the bulk.

77 Subsequently, PEPT has also been successfully used to non-invasively recover  
78 the Lagrangian trajectories of a radio-labelled particle in rotating drums by  
79 Ding et al. (2001a), Ding et al. (2001b) Ding et al. (2002), Lim et al. (2003),  
80 Govender et al. (2011), and Sichelwe et al. (2011).

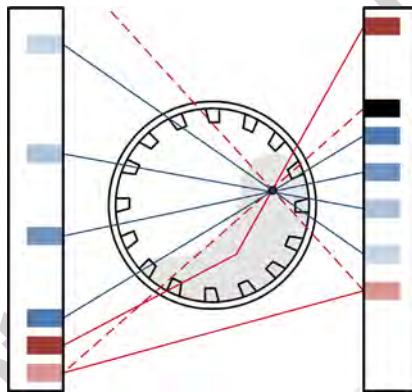
81 *The PEPT technique*

82 Positron emission particle tracking (PEPT) is a Lagrangian, single-particle  
83 tracking technique in which the trajectory of a representative tracer particle  
84 is triangulated from the decay products of the positron-emitting radioisotope  
85 with which it is labelled ( $^{18}F$  in this work). It is an extension of the medical  
86 nuclear imaging technique positron emission tomography (PET) to the tracking  
87 of a single particle in a bulk, and is performed using the same equipment– a  
88 gamma camera– as is PET.

89 Both of these techniques rely on the detection of the products of positron an-  
90 nihilation. A positron emitted by the radioactive tracer particle annihilates  
91 with an electron in the surrounding material, producing a pair of back-to-back

92 gamma rays that define a line of response (LoR) passing through (or very close  
 93 to) the tracer. However, whereas PET produces a radiation density distribution  
 94 by integrating a large number of detected LoRS over time, PEPT takes advant-  
 95 age of the *a priori* knowledge that the system of interest contains only one  
 96 radiolabelled tracer particle to triangulate its position. Since the gammas are  
 97 emitted uniformly in space, but all LoRs must pass through the tracer particle,  
 98 detecting several LoRs allows the position of the tracer particle to be trian-  
 99 gulated in space. Since each pair of gamma rays are detected coincidentally,  
 100 each detection also has a time associated with it, and thus each triangulated  
 101 tracer position can have an effective time assigned to it. The result is a series  
 102 of Lagrangian tracer locations as a function of time.

103 In addition to true LoRs, the PET scanner also detects various types of spurious  
 104 events, including false coincidences and scattered LoRs. Figure 2 shows true  
 105 LoRs in blue, and spurious LoRS in red. In particular, the figure shows a false  
 106 coincidence arising from two unassociated LoRs (dashed red) and the spurious  
 107 LoR arising from a scattered gamma ray. To exclude spurious events such as  
 108 these, Parker et al. (1993) developed a triangulation scheme that iteratively  
 109 rejects the least good LoRs until a stopping criterion, such as spatial resolution,  
 110 is met. While more sophisticated triangulation schemes— such as that of Bickell  
 111 et al. (2012)— exist, the Parker routine is used here for simplicity.



**Figure 2:** A 2D illustration of a realistic dataset consisting of both true (blue lines)  
 and spurious events (red lines).

112 Figure 3 shows the University of Birmingham’s ADAC Forte gamma camera. It  
 113 is described in detail by Parker et al. (2002). In the field of view of the camera is  
 114 the experimental rotating drum used for this work. It has an internal diameter  
 115 of 300 mm and length of 270 mm, and is fitted with 20 lifter bars to increase  
 116 the apparent friction between the drum and the granular bed.



**Figure 3:** The experimental rotating drum in the field of view of the Birmingham ADAC Forte gamma camera.

### 117 Lagrangian-to-Eulerian particle tracking

118 Although the trajectories of a tracer particle moving in a bulk of similar particles  
 119 can be of interest, it is often more informative to employ the ergodic assumption  
 120 and to thus convert trajectory data in the Lagrangian reference frame of the  
 121 tracer into the fixed Eulerian reference frame of the system. The ergodic as-  
 122 sumption can be stated as follows: Over a sufficiently long time, the time spent  
 123 by a particle in some region of the phase space of microstates corresponding to a  
 124 fixed macrostate of a system is proportional to the volume of that region of the  
 125 phase space. That is, since the likelihood of a particle being in some microstate  
 126 at a given time is proportional, and often assumed identical, to the likelihood  
 127 of it being in that state at all, over a sufficiently long time a tracked particle  
 128 will visit all possible combinations of position, momentum and so on with the  
 129 probability of that microstate's instantiation.

130 Thus, given sufficiently many Lagrangian experiments performed on an ergodic  
 131 system, it is possible to aggregate the recovered trajectories and sample them  
 132 at fixed points in space to obtain a flow field in the Eulerian reference frame.  
 133 In general, there is no way to determine *a priori* how many experiments need  
 134 be performed, and for how long, to sample the entire phase space— for very  
 135 asymmetric phase spaces, it may be practically impossible to do so— and so a  
 136 pragmatic cut-off must be decided on. It may be possible, however, to measure  
 137 some bulk property of the system that is also calculable from the accumulated  
 138 trajectories in Eulerian form, and to use the convergence of the measured and  
 139 calculated values as a guide. For our system of interest, the rotating drum, we  
 140 use the convergence of the measured and calculated power drawn by the drum  
 141 (Bbosa et al., 2011) as an ergodic criterion.

142 If the system of interest is in, or is sufficiently close to, steady state then it should  
 143 be possible to obtain an Eulerian picture by combining (in some way) sufficient  
 144 repetitions of a Lagrangian experiment as to completely sample every microstate

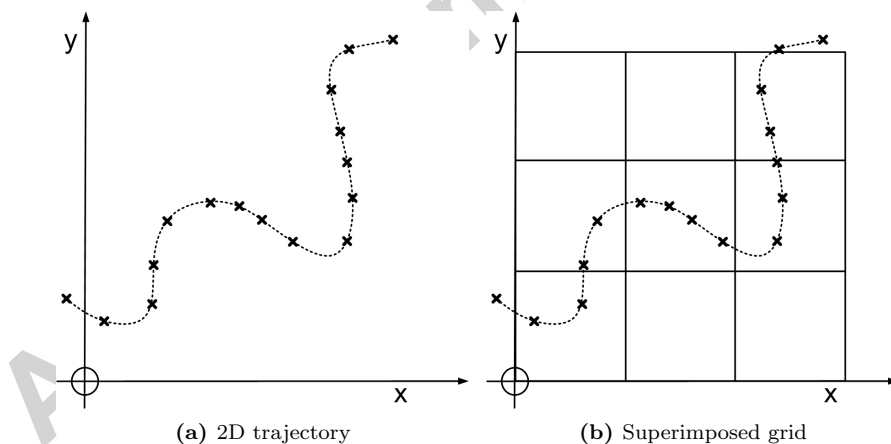


145 of that system's phase space. If the system of interest is both steady-state and  
 146 periodic, however, then a single, sufficiently long experiment could meet this  
 147 requirement. Systems must at least meet the steady-state condition to be said  
 148 to be *ergodic*.

149 The rotating drum experiments performed for this work are in steady state, and  
 150 are performed for long enough to meet our chosen ergodic criterion, for their  
 151 calculated and measured power draw to converge. Thus we are able to convert  
 152 Lagrangian tracer particle trajectories into Eulerian flow fields via a procedure  
 153 called residence time binning.

#### 154 *Residence time binning*

155 Consider the illustrative trajectory shown in Figure 4a. It consists of a series of  
 156 sample points along a path in 2D space, shown here as a dashed line. This sample  
 157 trajectory is approximated by fitting a series of interpolating polynomials, as a  
 158 function of time, to the sample points. Separate polynomials are fitted for each  
 159 dimension—  $x$ ,  $y$  and  $z$ — for a moving window of three tracer particle locations,  
 160 corresponding to the span of a second order Lagrange polynomial. Thus, a  
 161 trajectory consisting of  $N$  tracer locations will be approximated by  $3(N - 2)$   
 162 polynomials of the form  $s_j(t) = p_j t^2 - q_j t + r_j$ , where  $s$  is the generalised spatial  
 163 coordinate,  $t$  is time, and the coefficient subscript  $j$  refers to the moving time  
 164 interval for which the polynomial is defined. These piecewise approximations  
 165 to the sample trajectory thus provide not only a means of estimating the tracer  
 166 particle's position, but also— by taking their time derivative— its instantaneous  
 167 kinematics, at any given time.



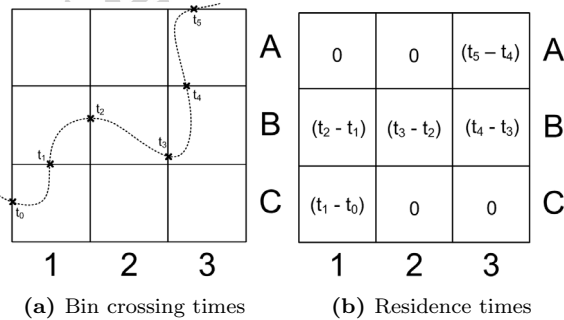
**Figure 4:** An illustrative trajectory in 2D space without (left) and with (right) a binning grid superimposed.

168 The first step in recasting this now-continuous trajectory as an Eulerian flow

169 field is to discretise the space into sample volumes (or voxels), as shown schemat-  
 170 atically in figure 4b. For clarity, the binning procedure is illustrated in 2D but,  
 171 in practice, it is performed in 3D. These voxels represent the resolution of the  
 172 flow field that will result from the binning procedure, and their minimum size  
 173 is determined with reference to both the resolution of the imaging technique,  
 174 and the chosen ergodic criterion. In this work, the voxels were cubic, and had  
 175 a side length of approximately 2 mm.

176 The first Eulerian field of interest is the probability density, the likelihood that  
 177 a particle of the sort represented by the tracer particle will be at any given  
 178 location at any given time. By the ergodic assumption, this is the same as  
 179 the residence time distribution (RTD). That is, the length of time spent by the  
 180 tracer particle in each voxel is proportional to the likelihood of such a particle  
 181 being in that voxel at any instant. The RTD of a tracer particle is obtained  
 182 from its Lagrangian trajectory as follows.

183 Once the grid has been assigned, the sample points are replaced by the sample  
 184 trajectory, defined in terms of the piecewise interpolating polynomials above.  
 185 These interpolating polynomials can be inverted and solved for the boundaries  
 186 of the voxels to determine the times at which the tracer particle entered and  
 187 left each voxel, as illustrated in figure 5a. These times bound the time interval  
 188 spent by the tracer in each voxel, as illustrated in figure 5b. Of course, the  
 189 trajectory may pass through the same voxel many times, and so each voxel will  
 190 contain a sum of all of the intervals that the tracer spent inside it. The more  
 191 likely that the tracer will be in a given voxel at any time, the more times it  
 192 will pass through that voxel, and the longer it will spend in that voxel, as per  
 193 the ergodic assumption. Hence the raw RTD is proportional to the probability  
 194 density. Finally, to obtain the actual probability density distribution, the raw  
 195 RTD is normalised to the total time that the tracer was tracked for. From  
 196 now on, the RTD will refer to the normalised (and not the raw) residence time  
 197 distribution.



**Figure 5:** The residence time binning procedure: calculating bin boundary crossing times (left) and thus bin residence times (right).

198 The RTD can be interpreted as the most likely instantaneous distribution of

199 material of the class being tracked in the system. It is thus directly useful as a  
 200 measure of the shape of a granular bed, and can be used to assess whether bulk  
 201 behaviours such as segregation or deadzones are arising. It can also be converted  
 202 to a mass or solidicity distribution simply by multiplying it by the total mass or  
 203 volume of the class of particles represented by the tracer respectively (Wildman  
 204 et al., 2000).

205 The procedure for obtaining the RTD also facilitates that for obtaining the  
 206 kinematics of the system. In particular, once the times that the tracer particle  
 207 entered ( $t_i$ ) and exited ( $t_f$ ) a voxel on a transit  $j$  have been obtained, the average  
 208  $s$ -velocity of the tracer on that transit can be written as:

$$\bar{v}_{s,j} = \frac{\int_{t_i}^{t_f} v_s(t) dt}{\int_{t_i}^{t_f} dt} = \frac{s(t_f) - s(t_i)}{\Delta t_j} \quad (4)$$

209 where  $s(t)$  is the relevant piecewise polynomial as defined above,  $v_s(t) = \frac{ds(t)}{dt}$   
 210 and  $\Delta t = t_f - t_i$ . A similar procedure yields the average  $s$ -acceleration during  
 211 transit  $j$ , where  $a_s(t) = \frac{dv_s(t)}{dt} = \frac{d^2s(t)}{dt^2}$ .

212 A single, representative value of each kinematic quantity is assigned to each voxel  
 213 based on a time-weighted average of the values determined for each transit by  
 214 equation 4. This corrects for the variation in transit length due to different  
 215 tracer trajectories through the voxel, and for outlier values. The time-weighted  
 216 average  $s$ -velocity of the  $N$  transits of the tracer through a given voxel is:

$$\bar{v}_s = \frac{\sum_{j=1}^N \Delta t_j \bar{v}_{s,j}}{T} = \frac{\sum_{j=1}^N \{s_j(t_f) - s_j(t_i)\}}{T} \quad (5)$$

217 where  $T = \sum_{j=1}^N \Delta t_j$  is the total time spent by the tracer particle in that voxel  
 218 over all of its transits.

219 This procedure can be performed in each dimension to obtain the  $\hat{x}$ -,  $\hat{y}$ - and  
 220  $\hat{z}$ -velocity flow fields, and a similar procedure can be performed to obtain the  
 221 acceleration flow fields, and thus the full kinematics of particles of the class  
 222 represented by the tracer.

223 Once the grid of voxels has been chosen, and the residence time binning proced-  
 224 ure undertaken, the result is a series of three-dimensional grids containing the  
 225 basic kinematic flow fields of the tracer particle in the Eulerian reference frame  
 226 of the system. These can be combined to produce absolute velocity flow fields,  
 227 or acceleration in the  $xy$ -plane, for instance. They can be sampled to produce  
 228 sections along, or summed or averaged to produce projections onto, planes of  
 229 interest, and– using the positions of the centroids of each voxel relative to an

230 appropriate origin– they can be expressed in a coordinate system more natural  
 231 to the system under study.

232 For this work, the RTD is used directly to investigate the shape of the granular  
 233 bed, and both a cylindrical and Cartesian reference frame are used for various  
 234 velocity distributions. Furthermore, the 3D grids are sectioned to exclude the  
 235 end effects in the rotating drum, and projected onto the transaxial or  $xy$ –planes  
 236 as appropriate.

## 237 The behaviour of a granular bed in a rotating drum

### 238 *The bulk and bed free surfaces*

239 The surface of a dynamic granular bed undergoing cascading motion in a ro-  
 240 tating drum assumes an S-shape (Rajchenbach, 1990; Nakagawa et al., 1993;  
 241 Taberlet et al., 2006). As the rotational speed increases, more and more ma-  
 242 terial is thrown into flight, and the curvature of the S-shaped surface of the  
 243 bed becomes more and more pronounced, until the bed enters the cataracting  
 244 regime.

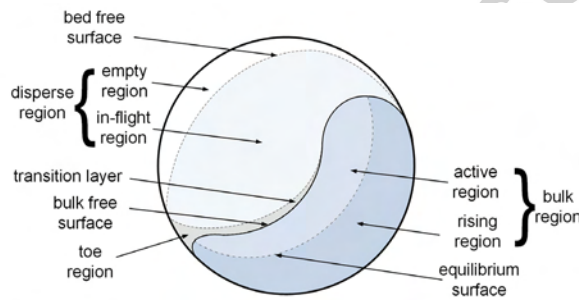


Figure 6: A schematic of the bed regions in a rotating drum.

245 Figure 6 shows a schematic of a typical granular bed undergoing cascading  
 246 motion in a rotating drum. Under these conditions, the bed can be divided  
 247 into two regions– the *bulk region*, and the *disperse region*. The bulk region  
 248 is defined as that part of the granular bed consisting of particles in continuous  
 249 contact with other particles in the bulk. The disperse region is so dilute that  
 250 its constituent particles hardly interact with each other, except through collisions.  
 251 Between the bulk and disperse regions is a *transition layer* in which particles  
 252 are likely to collide with each other and with the bulk, but are still too dilute  
 253 to be considered part of the bulk. This is most pronounced in the *toe region*,  
 254 which is the most chaotic part of the bed, and the site of reincorporation of  
 255 material from the disperse into the bulk region.

256 The disperse region also consists of two, easily-identifiable regions– the *empty*  
 257 *region* and the *in-flight region*. The empty region is the area of the drum into

258 which no material is ever thrown free of the bulk of the bed. The remainder  
 259 is the in-flight region. The free surfaces that divide the bulk from the in-flight  
 260 region, and the in-flight from the empty region, are called the *bulk free surface*  
 261 and the *bed free surface* respectively. They can be identified from end-window  
 262 filming, for example by Rogovin and Herbst (1989); from X-ray tracking, for  
 263 example by Powell and Nurick (1996a); from computational discrete element  
 264 models, for example by Powell and McBride (2004); or from PEPT data.

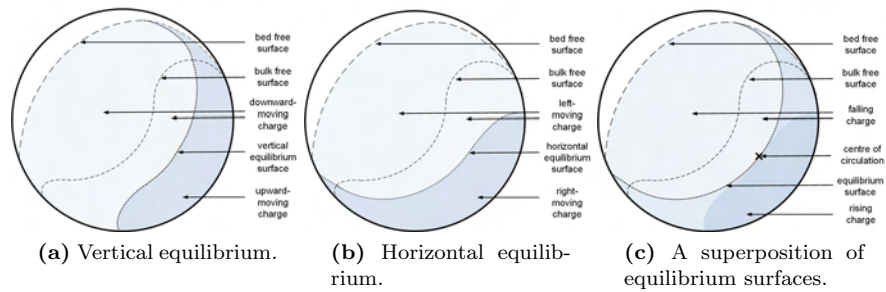
265 For this work, free surfaces were extracted from PEPT data by applying a  
 266 closed-loop *Canny* edge detection routine (Canny, 1986) to transverse RTDs  
 267 of the bed in motion. To deal with internal gaps, particularly in the in-flight  
 268 region, a *Laplacian of a Gaussian* smoothing filter was first applied to the  
 269 RTDs. The ambiguity of the transition layer was removed by setting as the  
 270 edge-detection threshold the solidicity at which spherical particles lose contact  
 271 with each other. The result is a robust numerical routine for extracting the free  
 272 surfaces of a dynamic granular bed from its transverse RTD.

### 273 *The equilibrium surface and the centre of circulation*

274 The bulk region of the bed can be further divided into two distinct regions- the  
 275 *rising region* and the *active region*. The rising region of the bed is that part that  
 276 moves upwards and displays quasi-rigid body motion about the centre of the  
 277 drum (Morrell, 1992). The active region consists of material flowing downwards  
 278 over the rising material under the influence of gravity, and thus has previously  
 279 been called the *descending region* (Powell and McBride, 2004). However, in  
 280 a bed undergoing cascading or cataracting motion, it- together with the toe  
 281 region- is the site of important processes such as mixing and grinding, and so  
 282 is here called the active region.

283 The surface that separates the active from the rising region, shown schematically  
 284 in figure 6, is called the *equilibrium surface*. It is so-called because it divides  
 285 material moving in opposite directions and so, by continuity considerations,  
 286 must consist of a layer of material that has zero net velocity in the appropriate  
 287 direction (Powell and Nurick, 1996a; Powell and McBride, 2004). Figure 7 shows  
 288 three schematics of the Eulerian velocity fields in a typical rotating drum, from  
 289 which can be extracted the equilibrium surface.

290 The shape of the bulk of the bed is such that in its upper part, the rising mater-  
 291 ial is moving essentially upwards and the active material is moving essentially  
 292 downwards. Figure 7a shows the *vertical equilibrium surface*, which divides  
 293 the rising and descending material. However, in the lower part of the bulk,  
 294 the rising material is actually moving rightwards, and the active (and disperse)  
 295 material is moving leftwards. Figure 7b shows the *horizontal equilibrium sur-*  
 296 *face*, which divides the material moving in opposite horizontal directions. The  
 297 combined equilibrium surface, which divides the rising and active regions, is a  
 298 superposition of the two, and is shown in figure 7c.

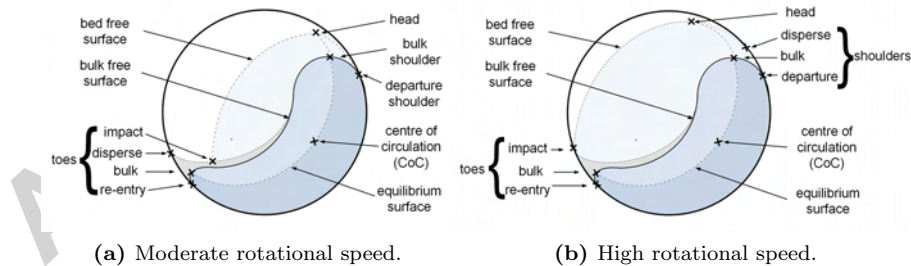


**Figure 7:** Schematics of the dynamic bed in a rotating drum showing the upward- and downward-moving (left), and leftward- and rightward-moving (middle) material separated by equilibrium surfaces, and the combined (right) equilibrium surface.

299 The point (or curve in three dimensions) through which the two equilibrium  
 300 surfaces pass, as shown in figure 7c, is called the *centre of circulation* (CoC).  
 301 Since it is the intersection of the horizontal and vertical velocity surfaces, it  
 302 has zero velocity in both directions; it is the stationary point about which the  
 303 bed revolves. In general, the CoC is not the same as the centre of mass of the  
 304 bed, which is calculable from the RTD by assigned each bin the position of its  
 305 centroid and a mass based on its normalised residence time.

#### 306 *The head, shoulders and toes*

307 Although the free surfaces of the granular bed are a good indicator of the type  
 308 of motion occurring in a rotating drum, it is often more convenient to compare  
 309 different regimes by comparing so-called scalar indicators. Powell and McBride  
 310 (2004) define four of these- the head, shoulder, bulk toe and impact toe of the  
 311 charge. We adapt and extend these to obtain the bed features shown schemat-  
 312 ically in figure 8.



**Figure 8:** Schematics of the bed features in a rotating drum.

313 The *head* of the bed is the highest point attained by the outermost layer of  
 314 in-flight material. It is obtainable from the RTD in one of two ways, depending

315 on whether or not there is material in the in-flight region of bed. That is, if  
 316 there is no material in the in-flight region, as happens at low speeds, the head  
 317 is the intersection of the equilibrium surface and the bed free surface; if there is  
 318 material in the in-flight region, the head is the intersection of the equilibrium  
 319 surface and the bed free surface.

320 Powell and McBride (2004) define the shoulder of the bed as the point at which  
 321 material leaves, and the toes as the highest points at which bulk and in-flight  
 322 material strikes, the drum wall. We replace these with a series of toes and  
 323 shoulders to reflect the greater range of bed behaviours that are possible within  
 324 the cataracting and cascading modes.

325 We define the *departure shoulder* as the point at which material in the bulk of  
 326 the bed leaves the drum wall, and define the *re-entry toe* as the point at which  
 327 material from the active and bulk regions impact the drum wall. They are  
 328 obtainable as the upper and lower intersections of the bulk free surface and the  
 329 drum wall, respectively. Similarly, we define a *disperse shoulder* and *disperse toe*  
 330 as the points at which material in the in-flight region leaves and regains contact  
 331 with the drum wall. These are obtainable as the upper and lower intersections  
 332 of the bed free surface and the drum wall, respectively.

333 At moderate speeds, the bed free surface can be broken down into two segments,  
 334 as shown in figure 8a. The lower part consists of the upper boundary of the  
 335 transition layer. The intersection of this part of the bed free surface and the  
 336 drum wall is the disperse toe. The area between the disperse and re-entry toes  
 337 is called the *toe region*, as shown in figure 6. The upper part of the bed free  
 338 surface consists of the parabolic arc of the outermost material in the in-flight  
 339 region. The *impact toe* is the point at which this parabolic trajectory ends.  
 340 Thus the impact toe is defined as the lower intersection of the upper part of the  
 341 bed free surface and the bulk free surface. However, at high rotational speeds,  
 342 the in-flight material impacts directly on the drum wall, as shown in figure 8b,  
 343 and the impact toe is defined as the lower intersection between the bed free  
 344 surface, which consists then of only one segment, and the drum wall.

345 Finally, we introduce two new features; we define the *bulk shoulder* as the upper  
 346 intersection of the bulk free surface and the equilibrium surface, and the *bulk*  
 347 *toe* as the lower one. These, like the head, are turning points in the bed motion.

348 Not all of these bed features will exist at all drum rotational speeds, as shown  
 349 in figure 8. As discussed before, at low speeds, there will be no material in the  
 350 in-flight region, and so the bed and bulk free surfaces will coincide. In this case,  
 351 the impact toe will not exist, the head will coincide with the bulk shoulder, and  
 352 the disperse toe will coincide with the re-entry toe. Thus, neither the transition,  
 353 nor the toe regions will exist in such a bed.

354 At moderate speeds, as shown in figure 8a, there is material in the in-flight  
 355 region, but it is not thrown so far as to impact directly on the drum wall. In  
 356 this case, the disperse and bulk shoulders may coincide, and the impact toe  
 357 will be at a small angular and radial position than the bulk and disperse toes.

358 In this case, the disperse region will contain not only the in-flight and empty  
359 regions, but also parts of the transition and toe regions.

360 At high speeds, as shown in figure 8b, material in the in-flight region impacts  
361 directly on the drum wall, and so the disperse and impact toes coincide. Fur-  
362 thermore, the transition and toe regions are subsumed in the in-flight region of  
363 the bed.

364 At very high speeds, not shown in figure 8, the outermost layers may never  
365 leave the drum wall, as the motion begins to transition to centrifuging. In this  
366 case, the head of the bed is the highest point in the drum, and neither the  
367 impact toe nor disperse departure shoulder exist, although the bulk departure  
368 shoulder may still. Thus, neither the toe nor the empty regions exist at very  
369 high rotational speeds.

### 370 **The effect of rotation speed on bed features**

371 In this section we apply the characterisation routines described above to a mono-  
372 disperse bed of 5 mm glass beads in a 300 mm-diameter drum rotated at 50%,  
373 60% and 75% of its critical speed. The drum has an internal length of 270 mm,  
374 and was filled to a load fraction of 31.25%.

#### 375 *The shape of the bed and the free surfaces*

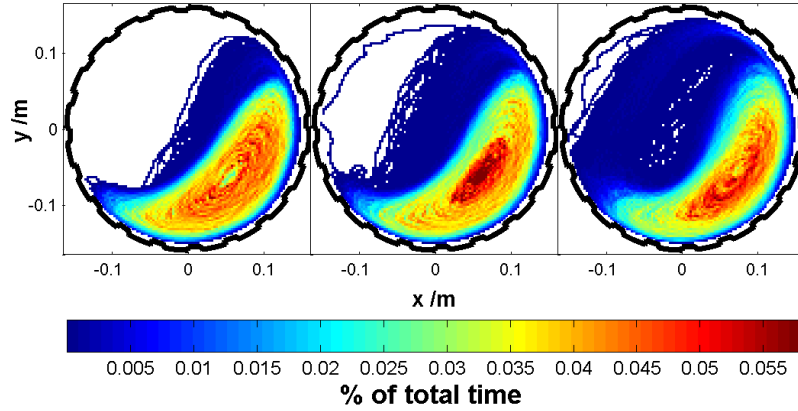
376 Figure 9 shows the transverse RTDs of the dynamic bed under these conditions.  
377 Below them is the scale used for all subsequent RTDs in this paper, with warmer  
378 colours signifying longer residence times, and cooler colours shorter ones.

379 Applying the edge detection routines described above to the dynamic beds  
380 shown in figure 9 allows them to be divided into their bulk, in-flight and empty  
381 regions. In particular, the dashed white lines in figure 10a show their bulk  
382 free surfaces, which become more and more S-shaped with increasing rotational  
383 speed.

384 Fitting a general logistic to these bulk free surfaces, as shown in figure 10b,  
385 simplifies the subsequent procedure of dividing the bed into its bulk and disperse  
386 regions for further analysis. For instance, the centres of mass (CoMs) of the  
387 entire bed and of only the bulk of the bed, at different rotational speeds, are  
388 given in table 1. Here, and for subsequent angular measures, angular position is  
389 measured in degrees from the right-hand half of the horizontal line through the  
390 centre of the drum. The radial position is measured, as usual, from the centre  
391 of the drum.

392 From table 1a, it can be seen that the CoM of the full bed moves upwards  
393 and inwards with increasing rotational speed— that is, towards the centre of  
394 the drum— as more material is thrown into the in-flight region. At the same  
395 time, the CoM of the bulk region moves upwards as the bulk elongates, but





**Figure 9:** The residence time distributions of a 5 mm tracer particle in a 300 mm-diameter drum with a load fraction of 31.25%, rotated at 50% (left), 60% (middle) and 75% (right) of its critical speed.

Rot. speed / $\%v_{crit}$	Full bed		Bulk		Rot. speed / $\%v_{crit}$	In-flight mass fraction	Bed volume fraction
	$\theta / ^\circ$	r / m	$\theta / ^\circ$	r / m			
50	-30.2	0.080	-46.7	0.084	50	0.040	0.445
60	-28.7	0.079	-43.9	0.085	60	0.052	0.554
75	-27.1	0.074	-42.2	0.085	75	0.078	0.697

(a) Centres of mass

(b) Measures of mass distribution

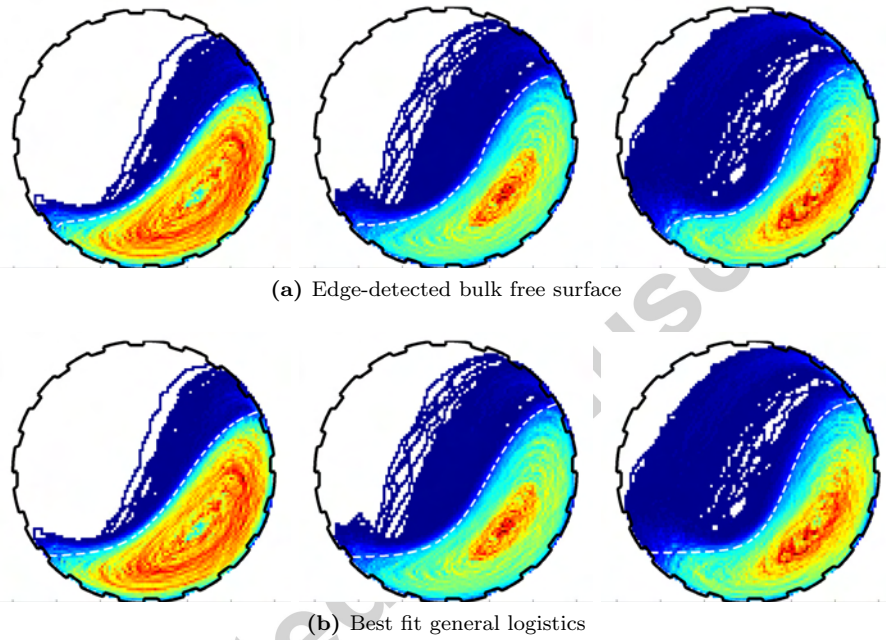
**Table 1:** Angular and radial positions of the centre of mass of the bulk and full bed, the fraction of material in-flight, and the fraction of the drum volume occupied by charge for various rotational speeds.

396 stays roughly at the same radial distance from the centre of the drum. This  
 397 is because the large increase in *bed volume*— the volume of the drum with a  
 398 non-zero residence time— from  $\sim 45\%$  to  $\sim 70\%$  is due only to a very small mass  
 399 fraction—  $\sim 4\%$  of the total mass of the bed, as shown in table 1b.

#### 400 *Scalar measures of the bed shape*

401 Once the equilibrium surfaces have been recovered from the RTDs by edge-  
 402 detection, the shoulders and toes of the bed can be obtained from their inter-  
 403 section with the drum. Table 2 shows these intersections for the RTDs shown  
 404 in figure 9.

405 Here, the angular compression of bulk of the bed is captured by the combination  
 406 of a relatively small increase in the angular position of its departure shoulder  
 407 with increasing speed, and a relatively large increase in the angular position of  
 408 its re-entry toe. Some of this is due to the development of a dense central region



**Figure 10:** Edge-detected equilibrium surfaces (top), and their general logistic approximations (bottom), for the bed in a drum rotated at 50% (left), 60% (middle) and 75% (right) of its critical speed.

Speed ( $\%v_{crit}$ )	Shoulder ( $\theta / ^\circ$ )		Toe ( $\theta / ^\circ$ )		
	Departure	Disperse	Re-entry	Disperse	Impact
50	23.6	47.3	-153.4	-154.1	-121.5
60	27.3	50.8	-144.1	-157.9	-130.6
75	28.7	69.7	-135.0	-167.1	-167.1

**Table 2:** Angular positions of the departure and disperse shoulders; and re-entry, disperse and impact toes of a mono-disperse bed of 5 mm glass beads in a 300 mm-diameter drum with a load fraction of 31.25%, rotated at various speeds.

409 of the bed, but it also reflects the increasing amount of material in the in-flight,  
410 transition and toe regions.

411 As the rotational speed of the drum increases, more and more material enters  
412 the in-flight region as it is carried out of the bulk along the surface of the drum,  
413 leading to the increasing angular position of the disperse shoulder, and to the  
414 increasing angular separation of the disperse and departure shoulders.

415 As more charge enters the disperse region, a chaotic transition and toe region  
416 develops. The development of the toe region is evidenced by the increasing  
417 angular separation between the re-entry and disperse toes with increasing speed.  
418 At very low speeds, there is essentially no toe region, and at very high speeds,  
419 it is significant in size.

420 Finally, from table 2, it is possible to draw some inferences about the nature  
421 of the toe region from the angular position of the impact toe. At 50% and  
422 60% of critical speed, the in-flight material lands on the surface of the bulk, as  
423 evidenced by having impact toes below their re-entry toes. In the 50% case,  
424 the small amount of material in the in-flight region lands so high up on the  
425 bulk, that it is effectively flowing with the active region by the time it reaches  
426 the re-entry toe, resulting in essentially no toe region. In the 60% case, there  
427 is more material landing lower down the bulk, leading to a small but chaotic  
428 toe region. In the 75% case, the highest material in the in-flight region impacts  
429 directly on the drum wall, and so there is no difference between the disperse  
430 and impact toes. The result is a large and chaotic toe region.

Rot. speed $/ \%v_{crit}$	Head	
	$\theta / ^\circ$	h / % max
50	62.7	90.4
60	72.9	94.2
75	81.9	97.1

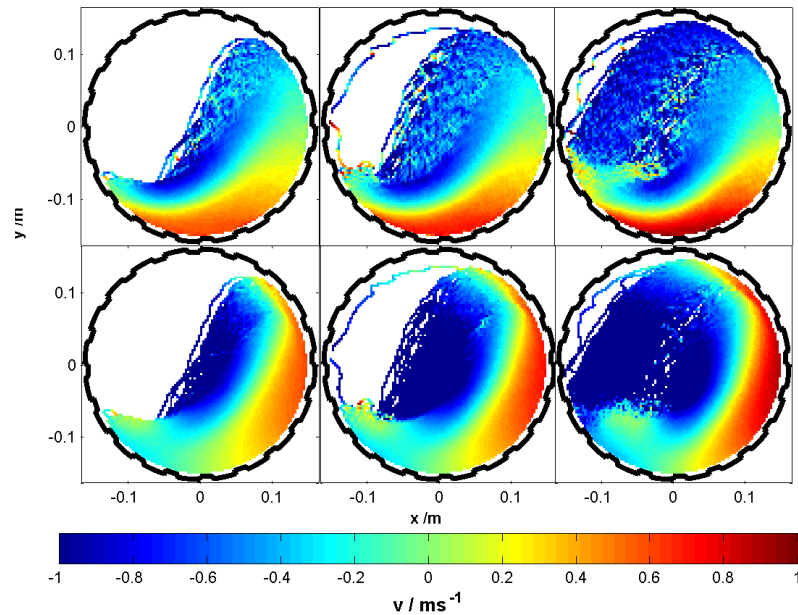
**Table 3:** The angular and normalised vertical positions of the highest point reached by a mono-disperse bed of 5 mm glass beads in a 300 mm-diameter drum with a load fraction of 31.25%, rotated at various speeds.

431 Also from the bed free surface, which separates the in-flight from the empty

432 regions, it is possible to obtain the head of the bed. Table 3 shows the angular  
 433 position of the head with increasing rotational speed, as well as its vertical  
 434 height above the lowest point of the drum, normalised to the drum diameter.  
 435 This is of interest because it determines the maximum potential energy available  
 436 to a particle in the bed. As expected, the position of the head approaches the  
 437 highest point in the drum as more and more material is thrown into the in-flight  
 438 region. At lower speeds than shown here, the head of the bed would coincide  
 439 with its bulk shoulder.

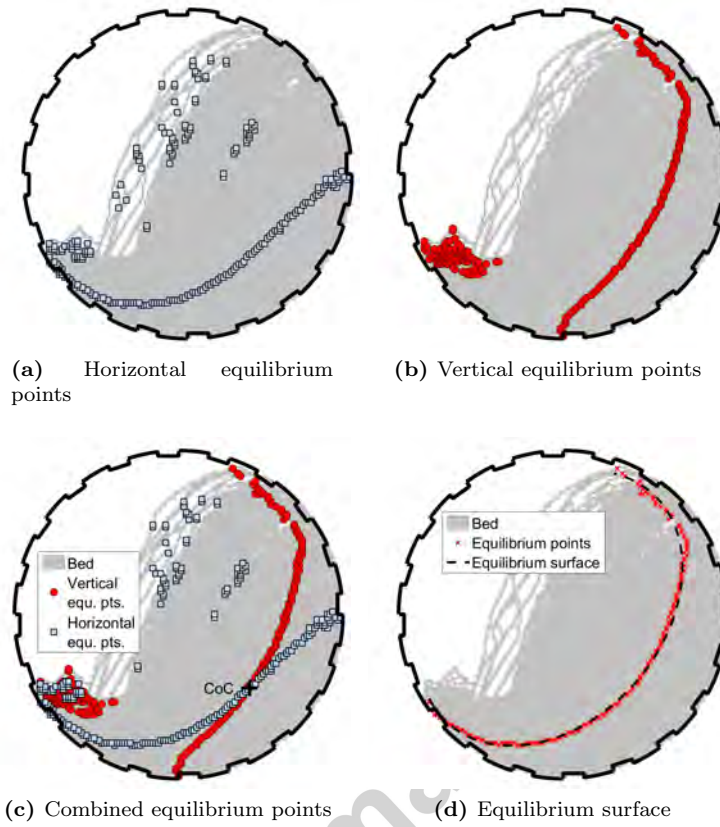
440 *The motion of the bed and the equilibrium surface*

441 Figure 11 shows the  $\hat{x}$ - and  $\hat{y}$ -velocity distributions corresponding to the RTDs  
 442 shown in figure 9. Below them is the scale used for all subsequent velocity  
 443 distributions in this paper, with warmer colours signifying large positive speeds,  
 444 and cooler colours large negative ones.



**Figure 11:** Time-weighted  $\hat{x}$ - (top) and  $\hat{y}$ -velocity (bottom) distributions for the bed in a drum rotated at 50% (left), 60% (middle) and 75% (right) of its critical speed.

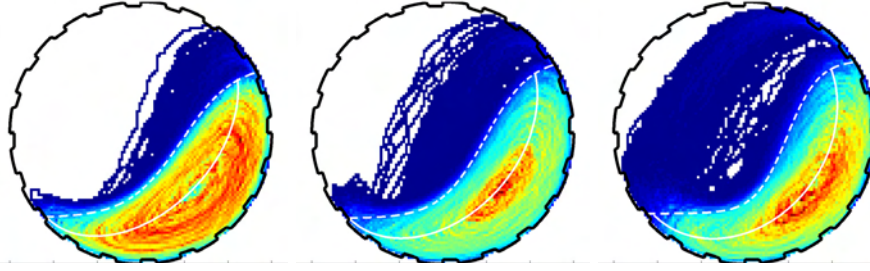
445 The velocity distributions in figure 11 clearly demonstrate the separation of  
 446 leftward- and rightward-moving material, and upward- and downward-moving  
 447 material by surfaces of zero velocity, as shown schematically in figure 7. These  
 448 zero-velocity surfaces are obtained quantitatively from the zero-velocity *equilib-*  
 449 *rium points* in each direction, as follows.



**Figure 12:** The horizontal and vertical equilibrium points of the velocity distributions of a granular bed in a drum rotated at 60% of its critical speed, separately (above), combined (below left), and with the equilibrium surface superimposed (below right).

450 The top row of figure 12 shows the equilibrium points in the transverse  $\hat{x}$ - and  $\hat{y}$ -  
 451 velocity distributions of the bed in a drum rotated at 60% of its critical speed,  
 452 that is, of the distributions in the central column of figure 11. Figure 12c shows  
 453 the two equilibrium surfaces defined by these equilibrium points intersecting at  
 454 the centre of circulation (CoC) of the bed, and figure 12d shows the equilibrium  
 455 points that make up the final equilibrium surface with an elliptical arc fitted to  
 456 them.

457 Figure 13 shows the RTDs of the granular bed in the drum rotated at the three  
 458 speeds. Superimposed thereon are not only approximations to the bulk free  
 459 surfaces shown in figure 10, but also to the equilibrium surfaces of the bed in  
 460 motion. As with the general logistics fitted to the bulk free surfaces before, the  
 461 elliptical arcs fitted to the equilibrium surfaces do not represent an underlying  
 462 model, but are adopted merely to simplify further analysis.



**Figure 13:** General logistic and elliptic approximations to the bulk free (dashed white lines) and equilibrium (solid white lines) surfaces of a granular bed in a drum rotated at 50% (left), 60% (middle) and 75% (right) of its critical speed.

463 In this case, the equilibrium surface divides the bulk of the bed into its rising  
 464 and active regions, allowing a more quantitative discussion of the behaviour  
 465 of the bulk of the bed. For instance, the total mass and volume fractions  
 466 can be obtained for these two regions, as they were for the bulk and disperse  
 467 regions in table 1b. From these, the density of the bed in the rising and active  
 468 regions, shown in table 4 normalised to that of close-packed glass spheres, can  
 469 be calculated.

Rot. speed / $\%v_{crit}$	Relative density		
	In-flight	Active	Rising
50	0.091	0.953	0.993
60	0.069	0.931	0.989
75	0.066	0.855	0.969

**Table 4:** The density of the bed in the rising, active and in-flight regions, normalised to the density of close-packed glass spheres.

470 The change in densities with increasing rotational speed confirm that— since it  
 471 is able to dilate more or less freely, constrained only by the very dilute disperse  
 472 region— the active region of the bed fluidises more quickly than the rising region,  
 473 which continues to act as a more or less solid body as the rotational speed  
 474 increases. Furthermore, the relatively density of the in-flight material in the  
 475 slowest case reflects the fact that the material in the transition and toe regions  
 476 are effectively just a dilute part of the active region.

#### 477 *Scalar measures of the bed motion*

478 As well as dividing the bulk region of the bed into its rising and active regions,  
 479 the equilibrium surface can be used to obtain the centre of circulation (CoC)  
 480 of the bed, and its bulk toe and shoulder. The CoC is obtained as part of  
 481 the procedure for obtaining the equilibrium surface, as shown in figure 12d,

482 while the bulk toe and shoulder are the upper and lower intersections of the  
483 equilibrium with the bulk free surface.

484 The re-entry toe and departure shoulder, as given in table 2, mark the beginning  
485 and end of a special type of motion: circular motion of the bed with the drum,  
486 and the disperse shoulder and toe mark the beginning and end of the motion  
487 of material in free fall. In contrast, the bulk shoulder and toe do not mark  
488 the beginning or end of any type of motion, but rather turning points in the  
489 motion of the bed. In this respect, they are like the head of the bed, which  
490 marks a turning point in the free fall motion of material in the in-flight region.  
491 In fact, all of the turning points of the bed motion are joined by the equilibrium  
492 surface— the head can be defined as the intersection of the equilibrium and bed  
493 free surfaces— and so can be considered special equilibrium points. The CoC  
494 is also a special equilibrium point— it is the only stationary point in the bed.  
495 Table 5 shows the radial and angular positions of these three features at various  
496 drum rotational speeds.

Rot. speed / $\%v_{crit}$	CoC		Bulk shoulder		Bulk toe	
	$\theta / ^\circ$	$r / m$	$\theta / ^\circ$	$r / m$	$\theta / ^\circ$	$r / m$
50	-52.6	0.091	28.7	0.127	-149.0	0.141
60	-47.7	0.093	32.2	0.126	-151.9	0.142
75	-44.1	0.098	36.3	0.124	-153.4	0.135

**Table 5:** Angular and radial positions of the CoC, and bulk shoulder and toe of the bed in a drum rotated at various speeds.

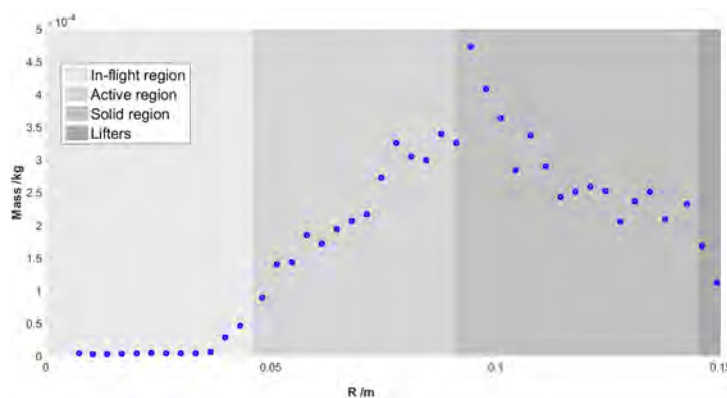
497 The bulk shoulder and toe give an indication of the extent of the bulk region.  
498 Table 5 shows the gradual angular elongation of the bulk with increasing speed,  
499 with slightly more of an effect in the position of the shoulder than the toe. It  
500 also shows the gradual radial thickening of the bulk at the toe and shoulder  
501 with increasing speed, with a significant thickening of the toe region at high  
502 speeds. At the same time, the radial position of the CoC increases, reflecting  
503 the increasing S-shape of the bulk. The CoC also moves higher up in the bulk,  
504 relative to both the bulk toe and shoulder, as the bulk itself moves up in the  
505 drum.

506 The thinning of the central part of the rising region, suggested by the movement

Rot. speed / $\%v_{crit}$	Active \m	Rising \m	Relative thickness
50	0.046	0.059	0.771
60	0.045	0.057	0.799
75	0.051	0.052	0.989

**Table 6:** Relative thicknesses of the rising and active regions of the bed in a drum rotated at various speeds.

507 of the CoC outwards with increasing mill speed, is more clearly captured in  
 508 the ratio of the thickness of the rising and active regions along a line passing  
 509 through both the centre of the drum, and the CoC of the bed. Table 6 contains  
 510 these ratios, which far better illustrate the significant change in bed behaviour  
 511 between 60% and 75% of critical speed, evident from a visual inspection of figure  
 512 9.



**Figure 14:** The mass profile of the dynamic granular bed in a drum rotated at 60% of its critical speed, sampled along the line passing through the centre of the drum and the centre of circulation of the bed.

513 Any line passing through the CoC of the bed must have the following property:  
 514 the flux of material passing through it must be zero. In addition, any line  
 515 passing through both the centre of the drum, and the CoC of the bed must  
 516 pass through all of the distinct regions of the bed shown in figure 6, except the  
 517 chaotic toe region. This makes such a line an ideal one along which to sample  
 518 Eulerian distributions to produce graphs such as the mass profile shown in figure  
 519 14.

520 Figure 14 shows a mass profile obtained by selecting voxels from the total mass-  
 521 weighted RTD of the bed in a drum rotated at 60% of its critical speed along  
 522 the line defined above. It provides a more detailed idea of the density in various  
 523 regions than does table 4. For ease of interpretation, the mass curve is plotted  
 524 on top of grey blocks representing the empty, in-flight, active and rising regions  
 525 (from left to right in the image). From this, the dense core visible in the central  
 526 graph in figure 9 is evident as a peak in the mass profile. Also, the ambiguity  
 527 of the transition zone is illustrated by the continuous mass curve between the  
 528 active and in-flight regions.

## 529 Conclusions

530 In this paper we have dissected the dynamic bed of granular material in a  
 531 rotating drum, and shown how the shape and behaviour of such a bed changes



532 with increasing rotational speed. In so-doing, we have defined various surfaces  
533 that divide the bed into distinct regions, and obtained various bed features  
534 from the intersections of these surfaces. We have described the procedure for  
535 converting Lagrangian data of the type obtainable from PEPT into Eulerian  
536 flow fields more amenable to investigation and modelling of bed behaviour, and  
537 the procedures for obtaining the surfaces and features described below from  
538 such Eulerian flow fields. Finally, we have illustrated their application with real  
539 data, and gained an insight into how the behaviour of a dynamic granular bed  
540 changes with increasing rotational speed of the drum that it is in.

541 In particular, we have divided the bed into the dense *bulk region* consisting of  
542 particles in continuous contact with each other, and the complementary dilute  
543 *disperse region*, separated by an intermediate-density *transition region* consist-  
544 ing of the most highly-fluidised material at the surface of the bulk region, and  
545 material from the disperse region re-joining the bulk. We have divided the bulk  
546 into the *rising region*, consisting of densely-packed particles rising in solid-body  
547 motion with the drum, and the *active region*, consisting of dilated material flow-  
548 ing downwards over the rising region, and the disperse region into the *in-flight*  
549 *region*, into which material is thrown by the action of the rotating drum, and  
550 the *empty region*, which the material of the bed never reaches. Further, we  
551 have identified the more or less chaotic *toe region* at the base of the bulk; the  
552 confluence of material leaving the in-flight, transition and active regions, and  
553 re-entering the rising region.

554 We have made this division of the bed by defining the *bed free surface*, which  
555 divides the in-flight region from the empty region; the *bulk free surface*, which  
556 divides the bulk from the disperse region of the bed; and the *equilibrium surface*,  
557 which divides the rising, en-masse material from the falling material in the bed  
558 and, in the bulk, separates the rising and active regions.

559 Finally, we have defined various bed features as the intersection of these surfaces  
560 with each other, and with the drum. Due to the changing shape and behaviour  
561 of the bed with increasing drum rotational speed, not all of these are defined at  
562 all speeds, but we attempt to deal with redundancies. For instance, we define  
563 the *head* of the bed as the intersection of the equilibrium and bed free surfaces  
564 unless there is no material in the in-flight region, in which case, the bed and  
565 bulk free surfaces are degenerate, and the head coincides with the *bulk shoulder*.  
566 Similarly, we define the *impact toe* and *disperse shoulder* as the intersection of  
567 the bed free surface and the drum, unless either there is no material in flight,  
568 and the disperse shoulder coincides with the *departure shoulder*, or the in-flight  
569 material strikes the surface of the bulk and not the drum, in which case we  
570 define a *disperse toe* as the upper extent of the *toe region*. And so on.

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