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# A note on coherence of dcpos <sup>☆</sup>

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## Abstract

In this note, we prove that a well-filtered dcpo  $L$  is coherent in its Scott topology if and only if for every  $x, y \in L$ ,  $\uparrow x \cap \uparrow y$  is compact in the Scott topology. We use this result to prove that a well-filtered dcpo  $L$  is Lawson-compact if and only if it is patch-compact if and only if  $L$  is finitely generated and  $\uparrow x \cap \uparrow y$  is compact in the Scott topology for every  $x, y \in L$ .

*Keywords:* coherence, well-filtered dcpo, Lawson compactness, patch topology

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## 1. Introduction

In this paper, we investigate the *coherence* with respect to the Scott topology on directed-complete partial ordered sets (*dcpo*'s for short). Coherence, which states that the intersection of any two compact saturated sets is again compact, is an important property in domain theory [1, 3]. For instance, coherence is equivalent to Lawson compactness on pointed continuous domains [5]. This equivalence enabled the second author to characterise the Lawson compactness of continuous domains by the so-called “property M”, and use this element-level characterization to classify the category of continuous domains with respect to the cartesian closedness [5, 6].

In [9, 8], the equivalence between coherence and Lawson compactness was generalised to quasicontinuous domains. In Chapter 3 of [3], one even sees that on finitely generated quasicontinuous domains the compactness of  $\uparrow x \cap \uparrow y$  for any  $x, y \in L$ , which seems much weaker than what coherence requires, already implies the Lawson compactness of  $L$ . In this note, we greatly generalize this result to *well-filtered* dcpos. Indeed, since every quasicontinuous domain is locally finitary compact and sober (see for example, [4]), our proof drops the locally finitary compact property and only uses well-filteredness, which is even strictly weaker than sobriety [7].

## 2. Preliminaries

We refer to [1, 3] for the standard definitions and notations of order theory and domain theory, and to [4] for topology.

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A topological space is called *well-filtered* if, whenever an open set  $U$  contains a filtered intersection  $\bigcap_{i \in I} Q_i$  of compact saturated subsets, then  $U$  contains  $Q_i$  for some  $i \in I$ . Any sober space is well-filtered (see [3, Theorem II-1.21]). We take *coherence* of a topological space to mean that the intersection of any two compact saturated subsets is compact. A *stably compact* space is a topological space which is compact, locally compact, well-filtered and coherent. We call a dcpo  $L$  well-filtered (respectively, compact, sober, coherent, locally compact, stably compact) if  $L$  with its Scott topology  $\sigma(L)$  is a well-filtered (respectively, compact, sober, coherent, locally compact, stably compact) space. Without further reference, we always equip  $L$  with the Scott topology  $\sigma(L)$ . Finally, a dcpo  $L$  is said to be *core-compact* if its Scott topology  $\sigma(L)$  is a continuous lattice in the inclusion order.

For a topological space  $X$ , we denote the set of all compact saturated sets of  $X$  by  $\mathcal{Q}(X)$ . We consider the *upper Vietoris topology*  $v$  on  $\mathcal{Q}(X)$ , generated by the sets

$$\Box U = \{K \in \mathcal{Q}(X) \mid K \subseteq U\},$$

where  $U$  ranges over the open subsets of  $X$ . We use  $\mathcal{Q}_v(X)$  to denote the resulting topological space. For a dcpo  $L$ , we use  $\mathcal{Q}_v(L)$  to denote  $\mathcal{Q}_v((L, \sigma(L)))$ .

### 3. Main results

**Lemma 3.1.** *Let  $L$  be a well-filtered dcpo. Then  $L$  is coherent if and only if  $\uparrow x \cap \uparrow y$  is compact for all  $x, y \in L$ .*

*Proof.* If  $L$  is coherent, it is obvious that  $\uparrow x \cap \uparrow y$  is compact for all  $x, y \in L$ , since  $\uparrow x, \uparrow y$  are compact saturated.

For the reverse, suppose  $\uparrow x \cap \uparrow y$  is compact for all  $x, y \in L$ . We proceed to prove that for any compact saturated sets  $A, B \subseteq L$ ,  $A \cap B$  is compact in  $L$ . To this end, fix some element  $a \in L$ ; we define a function  $f$  from  $L$  to  $\mathcal{Q}_v(L)$  by sending an element  $x$  to the compact saturated set  $\uparrow x \cap \uparrow a$ . We claim that  $f$  is continuous. Indeed, for every Scott open subset  $U \subseteq L$ ,  $f^{-1}(\Box U) = \{x \mid \uparrow x \cap \uparrow a \subseteq U\}$  is obviously an upper set. Let  $D \subseteq L$  be a directed subset with  $\sup D \in f^{-1}(\Box U)$ , then one has  $\uparrow(\sup D) \cap \uparrow a \subseteq U$ , that is,  $\bigcap_{d \in D} (\uparrow d \cap \uparrow a) \subseteq U$ . Note that  $L$  is well-filtered and  $\{\uparrow d \cap \uparrow a \mid d \in D\}$  is a filtered family of compact saturated sets by assumption, so we have some  $d \in D$  such that  $\uparrow d \cap \uparrow a \subseteq U$ , i. e.,  $d \in f^{-1}(\Box U)$ . Hence  $f$  is continuous.

Since  $f$  is continuous, for the given compact saturated subset  $A \subseteq L$ ,  $f(A) = \{\uparrow x \cap \uparrow a \mid x \in A\}$  is a compact subset of  $\mathcal{Q}_v(L)$ . We now claim that the union of  $f(A)$ , which is just  $A \cap \uparrow a$ , is compact in  $L$ . Indeed, for any compact subset  $\mathcal{C}$  of  $\mathcal{Q}_v(L)$ , let  $\{U_\alpha\}$  be a directed family of open sets of  $L$  covering  $\bigcup \mathcal{C}$ . By compactness, every element  $K$  of  $\mathcal{C}$  is already covered by one  $U_\alpha$ ; in other words,  $K \in \Box U_\alpha$ . It follows that  $\{\Box U_\alpha\}$  is a directed family covering  $\mathcal{C}$ , and now the compactness of  $\mathcal{C}$  tells us that  $\mathcal{C} \subseteq \Box U_\alpha$  for some  $\alpha$ . Hence  $\bigcup \mathcal{C} \subseteq U_\alpha$  for this  $\alpha$ . (This argument is similar to the one employed by Andrea Schalk in [10, Chapter 7] for showing that  $\bigcup: \mathcal{Q}_v(\mathcal{Q}_v(X)) \rightarrow \mathcal{Q}_v(X)$  is well-defined.)

Now for such  $A$  the above argument enables us to define another function  $g$  from  $L$  to  $\mathcal{Q}_v(L)$  as:  $g(x) = \uparrow x \cap A$  for every  $x \in L$ . A similar deduction shows that  $g$  is continuous. So for the compact saturated subset  $B$  of  $L$ ,  $g(B)$  is compact in  $\mathcal{Q}_v(L)$ , and again the union of  $g(B)$ , which is  $A \cap B$ , is compact in  $L$ . So  $L$  is coherent.  $\square$

**Corollary 3.2.** *Every well-filtered complete lattice  $L$  is coherent.*

*Proof.* For every  $x, y \in L$  the intersection of  $\uparrow x$  and  $\uparrow y$ , which is  $\uparrow(x \vee y)$ , is always compact, so the statement follows from Lemma 3.1.  $\square$

The following fact about core-compact complete lattices is essentially due to G. Gierz and K.H. Hofmann [2]; we collect it here as a corollary to the previous result.

**Corollary 3.3.** *For a complete lattice  $L$ , the following statements are equivalent:*

1.  $L$  is core-compact, i. e.,  $\sigma(L)$  is a continuous lattice;
2.  $(L, \sigma(L))$  is stably compact.

*Proof.* The only interesting part is that 1 implies 2. Suppose  $L$  is a complete lattice and  $\sigma(L)$  is continuous, then  $(L, \sigma(L))$  is a locally compact sober space by [3, Proposition VII-4.1]. Since sober spaces are well-filtered,  $L$  is coherent by Corollary 3.2. Finally,  $L$  is obviously compact in its Scott topology since it has a least element.  $\square$

We now come to a characterization of the compactness of Lawson and patch topologies on  $L$ . Recall that the patch topology on  $L$  arises by taking all Scott closed sets together with all compact saturated sets as a subbasis for the closed sets; whereas the (coarser) Lawson topology is generated by the Scott closed subsets and principal upper sets  $\uparrow x$ . The following theorem is a generalization of [3, Theorem III-5.8] which is stated for quasicontinuous domains.

**Theorem 3.4.** *Let  $L$  be a well-filtered dcpo. Then the following statements are equivalent:*

1.  $L$  is patch-compact, i. e.,  $L$  is compact in the patch topology;
2.  $L$  is Lawson-compact;
3.  $L$  is compact and  $\uparrow x \cap \uparrow y$  is compact for every  $x, y \in L$ ;
4.  $L$  is finitely generated and  $\uparrow x \cap \uparrow y$  is compact for every  $x, y \in L$ ;
5.  $L$  is finitely generated and coherent.

*Proof.* (1 $\Rightarrow$ 2): That 1 implies 2 is true for all dcpos since the patch topology is finer than the Lawson topology.

(2 $\Rightarrow$ 3): It is obvious that  $L$  is compact since the Lawson topology is finer than the Scott topology. For every  $x, y \in L$ ,  $\uparrow x \cap \uparrow y$  is Lawson closed; therefore it is Lawson-compact, thus Scott compact.

(3 $\Rightarrow$ 4): Since  $L$  is compact, by the Hausdorff Maximality Principle, every element is above some minimal element of  $L$ . Denote the set of all minimal elements of  $L$  by  $M$ ;  $M$  must be finite. Otherwise, the family  $\{M \setminus F \mid F \subseteq_{\text{fin}} M\}$  is a filtered set of non-empty Scott closed sets with an empty intersection, which contradicts compactness.

(4 $\Rightarrow$ 5): This is from Lemma 3.1.

(5 $\Rightarrow$ 1): This is a straightforward consequence of [3, Lemma VI-6.5]<sup>1</sup>.  $\square$

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<sup>1</sup>This lemma works for sober dcpos in [3]. However, one can find that its proof only uses well-filteredness.

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