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Spectral thresholds in the bipartite stochastic block model

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Abstract

¹ We consider a bipartite stochastic block model on vertex sets V_1 and V_2 , with planted partitions in each, and ask at what densities efficient algorithms can recover the partition of the smaller vertex set.

When $|V_2| \gg |V_1|$, multiple thresholds emerge. We first locate a sharp threshold for detection of the partition, in the sense of the results of [Mossel et al. \(2014, 2013\)](#) and [Massoulié \(2014\)](#) for the stochastic block model. We then show that at a higher edge density, the singular vectors of the rectangular biadjacency matrix exhibit a localization / delocalization phase transition, giving recovery above the threshold and no recovery below. Nevertheless, we propose a simple spectral algorithm, Diagonal Deletion SVD, which recovers the partition at a nearly optimal edge density.

The bipartite stochastic block model studied here was used by [Feldman et al. \(2015b\)](#) to give a unified algorithm for recovering planted partitions and assignments in random hypergraphs and random k -SAT formulae respectively. Our results give the best known bounds for the clause density at which solutions can be found efficiently in these models as well as showing a barrier to further improvement via this reduction to the bipartite block model.

Keywords: Stochastic block model, spectral algorithms, random matrices, planted satisfiability

1. Introduction

The stochastic block model is a widely studied model of community detection in random graphs, introduced by [Holland et al. \(1983\)](#). A simple description of the model is as follows: we start with n vertices, divided into two or more communities, then add edges independently at random, with probabilities depending on which communities the endpoints belong to. The algorithmic task is then to infer the communities from the graph structure.

A different class of models of random computational problems with planted solutions is that of planted satisfiability problems: we start with an assignment σ to n boolean variables and then choose clauses independently at random that are satisfied by σ . The task is to recover σ given the random formula. A closely related problem is that of recovering the planted assignment in [Goldreich \(2000\)](#)'s one-way function, see Section 3.1.

A priori, the stochastic block model and planted satisfiability may seem only tangentially related. Nevertheless, two observations reveal a strong connection:

1. Planted satisfiability can be viewed as a k -uniform hypergraph stochastic block model, with the set of $2n$ boolean literals partitioned into two communities of true and false literals under the planted assignment, and clauses represented as hyperedges.

1. Extended abstract. Full version appears as 1506.06737, v2.

2. [Feldman et al. \(2015b\)](#) gave a general algorithm for a unified model of planted satisfiability problems which reduces a random formula with a planted assignment to a bipartite stochastic block model with planted partitions in each of the two parts.

The bipartite stochastic block model in [Feldman et al. \(2015b\)](#) has the distinctive feature that the two sides of the bipartition are extremely unbalanced; in reducing from a planted k -satisfiability problem on n variables, one side is of size $\Theta(n)$ while the other can be as large as $\Theta(n^{k-1})$.

We study this bipartite block model in detail, first locating a sharp threshold for detection and then studying the performance of spectral algorithms.

Our main contributions are the following:

1. When the ratio of the sizes of the two parts diverge, we locate a sharp threshold below which detection is impossible and above which an efficient algorithm succeeds (Theorems 1 and 2). The proof of impossibility follows that of [Mossel et al. \(2014\)](#) in the stochastic block model, with the change that we couple the graph to a broadcast model on a two-type Poisson Galton-Watson tree. The algorithm we propose involves a reduction to the stochastic block model and the algorithms of [Massoulié \(2014\)](#); [Mossel et al. \(2013\)](#).
2. We next consider spectral algorithms and show that computing the singular value decomposition (SVD) of the biadjacency matrix M of the model can succeed in recovering the planted partition even when the norm of the ‘signal’, $\|\mathbb{E}M\|$, is much smaller than the norm of the ‘noise’, $\|M - \mathbb{E}M\|$ (Theorem 3).
3. We show that at a sparser density, the SVD fails due to a localization phenomenon in the singular vectors: almost all of the weight of the top singular vectors is concentrated on a vanishing fraction of coordinates (Theorem 4).
4. We propose a modification of the SVD algorithm, Diagonal Deletion SVD, that succeeds at a sparser density still, far below the failure of the SVD (Theorem 3).
5. We apply the first algorithm to planted hypergraph partition and planted satisfiability problems to find the best known general bounds on the density at which the planted partition or assignment can be recovered efficiently (Theorem 5).

2. The model and main results

The bipartite stochastic block model Fix parameters $\delta \in [0, 2]$, $n_1 \leq n_2$, and $p \in [0, 1/2]$. Then we define the bipartite stochastic block model as follows:

- Take two vertex sets V_1, V_2 , with $|V_1| = n_1$, $|V_2| = n_2$.
- Assign labels ‘+’ and ‘-’ independently with probability $1/2$ to each vertex in V_1 and V_2 . Let $\sigma \in \{\pm 1\}^{n_1}$ denote the labels of the vertices in V_1 and $\tau \in \{\pm 1\}^{n_2}$ denote the labels of V_2 .
- Add edges independently at random between V_1 and V_2 as follows: for $u \in V_1, v \in V_2$ with $\sigma(u) = \tau(v)$, add the edge (u, v) with probability δp ; for $\sigma(u) \neq \tau(v)$, add (u, v) with probability $(2 - \delta)p$.

Algorithmic task: Determine the labels of the vertices given the bipartite graph, and do so with an efficient algorithm at the smallest possible edge density p .

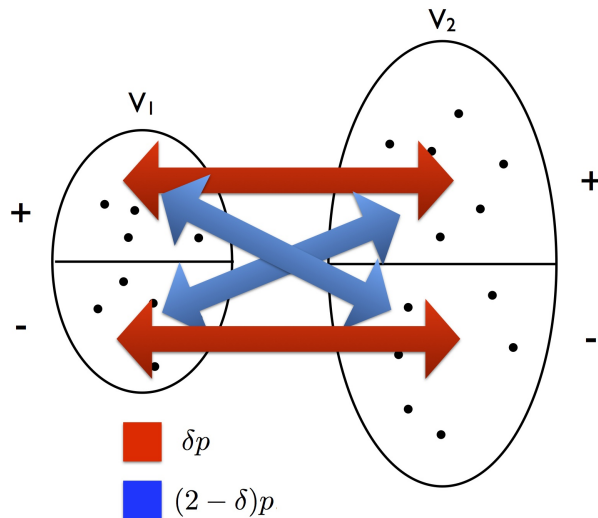


Figure 1: Bipartite stochastic block model on V_1 and V_2 . Red edges are added with probability δp and blue edges are added with probability $(2 - \delta)p$.

Preliminaries and assumptions In the application to planted satisfiability, it suffices to recover σ , the partition of the smaller vertex set, V_1 , and so we focus on that task here; we will accomplish that task even when the number of edges is much smaller than the size of V_2 . For a planted k -SAT problem or k -uniform hypergraph partitioning problem on n variables or vertices, the reduction gives vertex sets of size $n_1 = \Theta(n)$, $n_2 = \Theta(n^{k-1})$, and so the relevant cases are extremely unbalanced.

We will say that an algorithm *detects* the partition if for some fixed $\epsilon > 0$, independent of n_1 , whp it returns an ϵ -correlated partition, i.e. a partition that agrees with σ on a $(1/2 + \epsilon)$ -fraction of vertices in V_1 (again, up to the sign of σ).

We will say an algorithm *recovers* the partition of V_1 if whp the algorithm returns a partition that agrees with σ on $1 - o(1)$ fraction of vertices in V_1 . Note that agreement is up to sign as σ and $-\sigma$ give the same partition.

2.1. Optimal algorithms for detection

On the basis of heuristic analysis of the belief propagation algorithm, [Decelle et al. \(2011\)](#) made the striking conjecture that in the two part stochastic block model, with interior edge probability a/n , crossing edge probability b/n , there is a *sharp threshold* for detection: for $(a - b)^2 > 2(a + b)$ detection can be achieved with an efficient algorithm, while for $(a - b)^2 \leq 2(a + b)$, detection is impossible for any algorithm. This conjecture was proved by [Mossel et al. \(2014, 2013\)](#) and [Massoulié \(2014\)](#).

Our first result is an analogous sharp threshold for detection in the bipartite stochastic block model at $p = (\delta - 1)^{-2}(n_1 n_2)^{-1/2}$, with an algorithm based on a reduction to the SBM, and a lower bound based on a connection with the non-reconstruction of a broadcast process on a tree

associated to a two-type Galton Watson branching process (analogous to the proof for the SBM Mossel et al. (2014) which used a single-type Galton Watson process).

Algorithm: SBM Reduction.

1. Construct a graph G' on the vertex set V_1 by joining u and w if they are both connected to the same vertex $v \in V_2$ and v has degree exactly 2.
2. Randomly sparsify the graph (as detailed in Section 5).
3. Apply an optimal algorithm for detection in the SBM from Massoulié (2014); Mossel et al. (2013); Bordenave et al. (2015) to partition V_1 .

Theorem 1 *Let $\delta \in [0, 2] \setminus \{1\}$ be fixed and $n_2 = \omega(n_1)$. Then there is a polynomial-time algorithm that detects the partition $V_1 = A_1 \cup B_1$ whp if*

$$p > \frac{1 + \epsilon}{(\delta - 1)^2 \sqrt{n_1 n_2}}$$

for any fixed $\epsilon > 0$.

Theorem 2 *On the other hand, if $n_2 \geq n_1$ and*

$$p \leq \frac{1}{(\delta - 1)^2 \sqrt{n_1 n_2}},$$

then no algorithm can detect the partition whp.

Note that for $p \leq \frac{1}{\sqrt{n_1 n_2}}$ it is clear that detection is impossible: whp there is no giant component in the graph. The content of Theorem 2 is finding the sharp dependence on δ .

2.2. Spectral algorithms

One common approach to graph partitioning is spectral: compute eigenvectors or singular vectors of an appropriate matrix and round the vector(s) to partition the vertex set. In our setting, we can take the $n_1 \times n_2$ rectangular biadjacency matrix M , with rows and columns indexed by the vertices of V_1 and V_2 respectively, with a 1 in the entry (u, v) if the edge (u, v) is present, and a 0 otherwise. The matrix M has independent entries that are 1 with probability δp or $(2 - \delta)p$ depending on the label of u and v and 0 otherwise.

A typical analysis of spectral algorithms requires that the second largest eigenvalue or singular value of the expectation matrix $\mathbb{E}M$ is much larger than the spectral norm of the noise matrix, $(M - \mathbb{E}M)$. But here we have $\|M - \mathbb{E}M\| = \tilde{\Theta}(\sqrt{n_2 p})$, which is in fact much larger than $\lambda_2(\mathbb{E}M) = \Theta(p\sqrt{n_1 n_2})$ when $p = o(n_1^{-1})$. Does this doom the spectral approach at lower densities?

Question 1 *For what values of $p = p(n_1, n_2)$ is the singular value decomposition (SVD) of M correlated with the vector σ indicating the partition of V_1 ?*

Algorithm: Singular Value Decomposition.

1. Compute the left singular vector of M corresponding to the second largest singular value.
2. Round the singular vector to a vector $z \in \{\pm 1\}^{n_1}$ by taking the sign of each entry.

In particular, this question was asked by [Feldman et al. \(2015b\)](#). We show that there are two thresholds, both well below $p = n_1^{-1}$: at $p = \tilde{\Omega}(n_1^{-2/3}n_2^{-1/3})$ the second singular vector of M is correlated with the partition of V_1 , but below this density, it is uncorrelated with the partition, and in fact localized. Nevertheless, we give a simple spectral algorithm based on modifications of M that matches the bound $p = \tilde{O}((n_1n_2)^{-1/2})$ achieved with subsampling by [Feldman et al. \(2015b\)](#). In the case of very unbalanced sizes, in particular in the applications noted above, these thresholds can differ by a polynomial factor in n_1 .

Algorithm: Diagonal Deletion SVD.

1. Let $B = MM^T - \text{diag}(MM^T)$ (set the diagonal entries of MM^T to 0).
2. Compute the second eigenvector of B .
3. Round the eigenvector to a vector $z \in \{\pm 1\}^{n_1}$ by taking the sign of each entry.

Our results locate two different thresholds for spectral algorithms for the bipartite block model: while the usual SVD is only effective with $p = \tilde{\Omega}(n_1^{-2/3}n_2^{-1/3})$, the modified diagonal deletion algorithm is effective already at $p = \tilde{\Omega}(n_1^{-1/2}n_2^{-1/2})$, which is optimal up to logarithmic factors. In particular, when $n_1 = n, n_2 = n^{k-1}$ for some $k \geq 3$, as in the application above, these thresholds are separated by a polynomial factor in n .

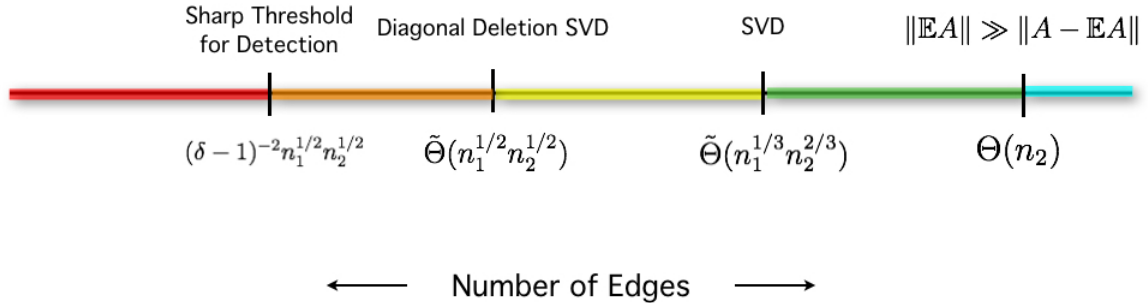


Figure 2: Main theorems illustrated.

First we give positive results for recovery using the two spectral algorithms.

Theorem 3 *Let $n_2 \geq n_1 \log^4 n_1$, with $n_1 \rightarrow \infty$. Let $\delta \in [0, 2] \setminus \{1\}$ be fixed with respect to n_1, n_2 . Then there exists a universal constant $C > 0$ so that*

1. If $p = C(n_1 n_2)^{-1/2} \log n_1$, then whp the diagonal deletion SVD algorithm recovers the partition $V_1 = A_1 \cup B_1$.
2. If $p = C n_1^{-2/3} n_2^{-1/3} \log n_1$, then whp the unmodified SVD algorithm recovers the partition.

Next we show that below the recovery threshold for the SVD, the top left singular vectors are in fact *localized*: they have nearly all of their mass on a vanishingly small fraction of coordinates.

Theorem 4

Let $n_2 \geq n_1 \log^4 n_1$. For any constant $c > 0$, let $p = c n_1^{-2/3} n_2^{-1/3}$, $t \leq n_1^{1/3}$, and $r = n_1 / \log n_1$. Let $\bar{\sigma} = \sigma / \sqrt{n_1}$, and v_1, v_2, \dots, v_t be the top t left unit-norm singular vectors of M .

Then, whp, there exists a set $S \subset \{1, \dots, n_1\}$ of coordinates, $|S| \leq r$, so that for all $1 \leq i \leq t$, there exists a unit vector u_i supported on S so that

$$\|v_i - u_i\| = o(1).$$

That is, each of the first t singular vectors has nearly all of its weight on the coordinates in S . In particular, this implies that for all $1 \leq i \leq t$, v_i is asymptotically uncorrelated with the planted partition:

$$|\bar{\sigma} \cdot v_i| = o(1).$$

One point of interest in Theorem 4 is that in this case of a random biadjacency matrix of unbalanced dimension, the localization and delocalization of the singular vectors can be understood and analyzed in a simple manner, in contrast to the more delicate phenomenon for random square adjacency matrices.

Our techniques use bounds on the norms of random matrices and eigenvector perturbation theorems, applied to carefully chosen decompositions of the matrices of interest. In particular, our proof technique suggested the Diagonal Deletion SVD, which proved much more effective than the usual SVD algorithm on these unbalanced bipartite block models, and has the advantage over more sophisticated approaches of being extremely simple to describe and implement. We believe it may prove effective in many other settings.

Under what conditions might we expect the Diagonal Deletion SVD outperform the usual SVD? The SVD is a central algorithm in statistics, machine learning, and computer science, and so any general improvement would be useful. The bipartite block model addressed here has two distinctive characteristics: the dimensions of the matrix M are extremely unbalanced, and the entries are very sparse Bernoulli random variables, a distribution whose fourth moment is much larger than the square of its second moment. These two facts together lead to the phenomenon of multiple spectral thresholds and the outperformance of the SVD by the Diagonal Deletion SVD. Under both of these conditions we expect dramatic improvement by using diagonal deletion, while under one or the other condition, we expect mild improvement. We expect diagonal deletion will be effective in the more general setting of recovering a low-rank matrix in the presence of random noise, beyond our setting of adjacency matrices of graphs.

3. Planted k -SAT and hypergraph partitioning

Feldman et al. (2015b) reduce three planted problems to solving the bipartite block model: planted hypergraph partitioning, planted random k -SAT, and Goldreich's planted CSP. We describe the reduction here and calculate the density at which our algorithm can detect the planted solution by solving the resulting bipartite block model.

We state the general model in terms of hypergraph partitioning first.

Planted hypergraph partitioning Fix a function $Q : \{\pm 1\}^k \rightarrow [0, 1]$ so that $\sum_{x \in \{\pm 1\}^k} Q(x) = 1$. Fix parameters n and $p \in (0, 1)$ so that $\max_x Q(x) 2^k p \leq 1$. Then we define the planted k -uniform hypergraph partitioning model as follows:

- Take a vertex set V of size n .
- Assign labels '+' and '-' independently with probability $1/2$ to each vertex in V . Let $\sigma \in \{\pm 1\}^n$ denote the labels of the vertices.
- Add (ordered) k -uniform hyperedges independently at random according to the distribution

$$\Pr(e) = 2^k p \cdot Q(\sigma(e))$$

where $\sigma(e)$ is the evaluation of σ on the vertices in e .

Algorithmic task: Determine the labels of the vertices given the hypergraph, and do so with an efficient algorithm at the smallest possible edge density p .

Usually Q will be symmetric in the sense that $Q(x)$ depends only on the number of $+1$'s in the vector x , and in this case we can view hyperedges as unordered. We assume that Q is not identically 2^{-k} as this distribution would simply be uniform and the planted partition would not be evident.

Planted k -satisfiability is defined similarly: we fix an assignment σ to n boolean variables which induces a partition of the set of $2n$ literals (boolean variables and their negations) into true and false literals. Then we add k -clauses independently at random, with probability proportional to the evaluation of Q on the k literals of the clause.

Planting distributions for the above problems are classified by their *distribution complexity*, $r = \min_{S \neq \emptyset} \{|S| : \hat{Q}(S) \neq 0\}$, where $\hat{Q}(S)$ is the discrete Fourier coefficient of Q corresponding to the subset $S \subseteq [k]$. This is an integer between 1 and k , where k is the uniformity of the hyperedges or clauses.

A consequence of Theorem 1 is the following:

Theorem 5 *There is an efficient algorithm to detect the planted partition in the random k -uniform hypergraph partitioning problem, with planting function Q , when*

$$p > (1 + \epsilon) \min_{S \subseteq [k]} \frac{1}{2^{2k} \hat{Q}(S) 2^{n^{k-|S|/2}}}$$

for any fixed $\epsilon > 0$. Similarly, in the planted k -satisfiability model with planting function Q , there is an efficient algorithm to detect the planted assignment when

$$p > (1 + \epsilon) \min_{S \subseteq [k]} \frac{1}{2^{2k} \hat{Q}(S)^2 (2n)^{k-|S|/2}}.$$

In both cases, if the distribution complexity of Q is at least 3, we can achieve full recovery at the given density.

Proof Suppose Q has distribution complexity r . Fix a set $S \subseteq [k]$ with $\hat{Q}(S) \neq 0$, and $|S| = r$. The first step of the reduction of [Feldman et al. \(2015b\)](#) transforms each k -uniform hyperedge into an r -uniform hyperedge by selecting the vertices indicated by the set S . Then a bipartite block model is constructed on vertex sets V_1, V_2 , with V_1 the set of all vertices in the hypergraph (or literals in the formula), and V_2 the set of all $(r - 1)$ -tuples of vertices or literals. An edge is added by taking each r -uniform edge and splitting it randomly into sets of size 1 and $r - 1$ and joining the associated vertices in V_1 and V_2 . The parameters in our model are $n_1 = n$ and $n_2 \sim n^{r-1}$ (considering ordered $(r - 1)$ -tuples of vertices or literals).

These edges appear with probabilities that depend on the parity of the number of vertices on one side of the original partition in the joined sets, exactly the bipartite block model addressed in this paper; the parameter δ in the model is given by $\delta = 1 + 2^k \hat{Q}(S)$ (see Lemma 1 of [Feldman et al. \(2015b\)](#)). Theorem 1 then states that detection in the resulting block model exhibits a sharp threshold at edge density p^* , with $p^* = \frac{1}{2^{2k} \hat{Q}(S)^2 n^{k-r/2}}$. The difference in bounds in Theorem 5 is due to the two models having n vertices and $2n$ literals respectively.

To go from an ϵ -correlated partition to full recovery, if $r \geq 3$, we can appeal to Theorem 2 of [Bogdanov and Qiao \(2009\)](#) and achieve full recovery using only a linear number of additional hyperedges or clauses, which is lower order than the $\Theta(n^{r/2})$ used by our algorithm. ■

Note that Theorem 2 says that no further improvement can be gained by analyzing this particular reduction to a bipartite stochastic block model.

There is some evidence that up to constant factors in the clause or hyperedge density, there may be no better efficient algorithms [O’Donnell and Witmer \(2014\)](#); [Feldman et al. \(2015a\)](#), unless the constraints induce a consistent system of linear equations. But in the spirit of [Decelle et al. \(2011\)](#), we can ask if there is in fact a sharp threshold for detection of planted solutions in these models. In one special case, such sharp thresholds have been conjectured: [Krzakala et al. \(2014\)](#) have conjectured threshold densities based on fixed points of belief propagation equations. The planted k -SAT distributions covered, however, are only those with distribution complexity $r = 2$: those that are known to be solvable with a linear number of clauses. We ask if there are sharp thresholds for detection in the general case, and in particular for those distributions with distribution complexity $r \geq 3$ that cannot be solved by Gaussian elimination. In particular, in the case of the parity distribution we conjecture that there is a sharp threshold for detection.

Conjecture 1 *Partition a set of n vertices at random into sets A, B . Add k -uniform hyperedges independently at random with probability δp if the number of vertices in the edge from A is even and $(2 - \delta)p$ if the number of vertices from A is odd. Then for any $\delta \in (0, 2)$ there is a constant c_δ so that $p = c_\delta n^{-k/2}$ is a sharp threshold for detection of the planted partition by an efficient algorithm. That is, if $p > (1 + \epsilon)c_\delta n^{-k/2}$, then there is a polynomial-time algorithm that detects the partition whp, and if $p \leq c_\delta n^{-k/2}$ then no polynomial-time algorithm can detect the partition whp.*

This is a generalization to hypergraphs of the SBM conjecture of [Decelle et al. \(2011\)](#); the $k = 2$ parity distribution is that of the stochastic block model. We do not venture a guess as to the precise constant c_δ , but even a heuristic as to what the constant might be would be very interesting.

3.1. Relation to Goldreich’s generator

[Goldreich \(2000\)](#)’s pseudorandom generator or one-way function can be viewed as a variant of planted satisfiability. Fix an assignment σ to n boolean variables, and fix a predicate $P : \{\pm 1\}^k \rightarrow$

$\{0, 1\}$. Now choose m k -tuples of variables *uniformly* at random, and label the k -tuple with the evaluation of P on the tuple with the boolean values given by σ . In essence this generates a uniformly random k -uniform hypergraph with labels that depend on the planted assignment and the fixed predicate P . The task is to recover σ given this labeled hypergraph. The algorithm we describe above will work in this setting by simply discarding all hyperedges labeled 0 and working with the remaining hypergraph.

4. Related work

The stochastic block model has been a source of considerable recent interest. There are many algorithmic approaches to the problem, including algorithms based on maximum-likelihood methods [Snijders and Nowicki \(1997\)](#), belief propagation [Decelle et al. \(2011\)](#), spectral methods [McSherry \(2001\)](#), modularity maximization [Bickel and Chen \(2009\)](#), and combinatorial methods [Bui et al. \(1987\)](#), [Dyer and Frieze \(1989\)](#), [Jerrum and Sorkin \(1998\)](#), [Condon and Karp \(2001\)](#). [Coja-Oghlan \(2010\)](#) gave the first algorithm to detect partitions in the sparse, constant average degree regime. [Decelle et al. \(2011\)](#) conjectured the precise achievable constant and subsequent algorithms [Mas-soulié \(2014\)](#); [Mossel et al. \(2013\)](#); [Bordenave et al. \(2015\)](#); [Abbe and Sandon \(2015\)](#) achieved this bound. Sharp thresholds for full recovery (as opposed to detection) have been found by [Mossel et al. \(2015\)](#); [Abbe et al. \(2016\)](#); [Hajek et al. \(2015a\)](#).

[Bogdanov and Qiao \(2009\)](#) used ideas for reconstructing assignments to random 3-SAT formulas in the planted 3-SAT model to show that Goldreich’s construction of a one-way function in [Goldreich \(2000\)](#) is not secure when the predicate correlates with either one or two of its inputs. For more on Goldreich’s PRG from a cryptographic perspective see the survey of [Applebaum \(2013\)](#).

[Feldman et al. \(2015b\)](#) gave an algorithm to recover the partition of V_1 in the bipartite stochastic block model to solve instances of planted random k -SAT and planted hypergraph partitioning using subsampled power iteration.

A key part of our analysis relies on looking at an auxiliary graph on V_1 with edges between vertices which share a common neighbor; this is known as the one-mode projection of a bipartite graph: [Zhou et al. \(2007\)](#) give an approach to recommendation systems using a weighted version of the one-mode projection. One-mode projections are implicitly used in studying collaboration networks, for example in [Newman \(2001\)](#)’s analysis of scientific collaboration networks. [Larremore et al. \(2014\)](#) defined a general model of bipartite block models, and propose a community detection algorithm that does not use one-mode projection.

The behavior of the singular vectors of a low rank rectangular matrix plus a noise matrix was studied by [Benaych-Georges and Nadakuditi \(2012\)](#). The setting there is different: the ratio between n_1 and n_2 converges, and the entries of the noise matrix are mean 0 variance 1.

[Butucea et al. \(2015\)](#) and [Hajek et al. \(2015b\)](#) both consider the case of recovering a planted submatrix with elevated mean in a random rectangular Gaussian matrix.

Notation All asymptotics are as $n_1 \rightarrow \infty$, so for example, ‘ E occurs whp’ means $\lim_{n_1 \rightarrow \infty} \Pr(E) = 1$.

We write $f(n_1) = \tilde{O}(g(n_1))$ and $f(n_1) = \tilde{\Omega}(g(n_1))$ if there exist constants C, c so that $f(n_1) \leq C \log^c(n_1) \cdot g(n_1)$ and $f(n_1) \geq g(n_1)/(C \log^c(n_1))$ respectively. For a vector, $\|v\|$ denotes the l_2 norm. For a matrix, $\|A\|$ denotes the spectral norm, i.e. the largest singular value (or largest eigenvalue in absolute value for a square matrix). For ease of reading, C will always denote an absolute constant, but the value may change during the course of the proofs.

5. Theorem 1: detection

In this section we sketch the proof of Theorem 1, giving an optimal algorithm for detection in the bipartite stochastic block model when $n_2 = \omega(n_1)$. The main idea of the proof is that almost all of the information in the bipartite block model is in the subgraph induced by V_1 and the vertices of degree two in V_2 . From this induced subgraph of the bipartite graph we form a graph G' on V_1 by replacing each path of length two from V_1 to V_2 back to V_1 with a single edge between the two endpoints in V_1 . We then apply an algorithm from [Massoulié \(2014\)](#); [Mossel et al. \(2013\)](#), or [Bordenave et al. \(2015\)](#) to detect the partition.

Fix $\epsilon > 0$. Given an instance G of the bipartite block model with

$$p = (1 + \epsilon)(\delta - 1)^{-2}(n_1 n_2)^{-1/2},$$

we reduce to a graph G' on V_1 as follows:

- Sort V_2 according to degrees and remove any vertices (with their accompanying edges) which are not of degree 2.
- We now have a union of 2-edge paths from vertices in V_1 to vertices in V_2 and back to vertices in V_1 . Create a multi-set of edges \mathcal{E} on V_1 by replacing each 2-path $u - v - w$ by the edge (u, w) .
- Choose N from the distribution $\text{Poisson}((1 + \epsilon)(\delta - 1)^{-4}n_1/2)$.
- If $N > |\mathcal{E}|$, then stop and output ‘failure’. Otherwise, select N edges uniformly at random from \mathcal{E} to form the graph G' on V_1 , replacing any edge of multiplicity greater than one with a single edge.
- Apply an SBM algorithm to G' to partition V_1 .

From the construction above, conditioned on σ the distribution of G' is that of the stochastic block model on V_1 with partition σ : each edge interior to the partition is present with probability a/n_1 , each crossing edge with probability b/n_1 , and all edges are independent.

For σ such that $\beta_1 = o(n^{-1/3})$, we have

$$a = \frac{(1 + \epsilon)(2 - 2\delta + \delta^2)}{(\delta - 1)^4}(1 + o(1))$$

$$b = \frac{(1 + \epsilon)(2\delta - \delta^2)}{(\delta - 1)^4}(1 + o(1))$$

For these values of a and b the condition for detection in the SBM, $(a - b)^2 \geq (1 + \epsilon)2(a + b)$ is satisfied and so whp the algorithms from [Massoulié \(2014\)](#); [Mossel et al. \(2013\)](#); [Bordenave et al. \(2015\)](#) will find a partition that agrees with σ on $1/2 + \epsilon'$ fraction of vertices.

6. Theorem 2: impossibility

The proof of impossibility below the threshold $(a - b)^2 = 2(a + b)$ in [Mossel et al. \(2014\)](#) proceeds by showing that the $\log n$ depth neighborhood of a vertex ρ , along with the accompanying labels, can be coupled to a binary symmetric broadcast model on a Poisson Galton-Watson tree. In this

model, it was shown by [Evans et al. \(2000\)](#) that reconstruction, recovering the label of the root given the labels at depth R of the tree, is impossible as $R \rightarrow \infty$, for the corresponding parameter values (the critical case was shown by [Pemantle and Peres \(2010\)](#)).

In the binary symmetric broadcast model, the root of a tree is labeled with a uniformly random label $+1$ or -1 , and then each child takes its parent's label with probability $1 - \eta$ and the opposite label with probability η , independently over all of the parent's children. The process continues in each successive generation of the tree.

The criteria for non-reconstruction can be stated as $(1 - 2\eta)^2 B \leq 1$, where B is the branching number of the tree T . The branching number is $B = p_c(T)^{-1}$, where p_c is the critical probability for bond percolation on T (see [Lyons \(1990\)](#) for more on the branching number).

Assume first that $n_2 \sim cn_1$ for some constant c , and that $p = d/n_1$. Then there is a natural multitype Poisson branching process that we can associate to the bipartite block model: nodes of type 1, corresponding to vertices in V_1 , have a Poisson(cd) number of children of type 2; nodes of type 2, corresponding to vertices in V_2 , have a Poisson(d) number of children of type 1. The branching number of this distribution on trees is $\sqrt{c} \cdot d$, an easy calculation by reducing to a one-type Galton Watson process by combining two generations into one. Transferring the block model labeling to the branching process gives $\eta = \delta/2$, and so the threshold for reconstruction is given by

$$(\delta - 1)^2 \sqrt{cd} \leq 1$$

or in other words,

$$p \leq \frac{1}{(\delta - 1)^2 \sqrt{n_1 n_2}}$$

exactly the threshold in [Theorem 2](#). In fact, in this case the proof from [Mossel et al. \(2014\)](#) can be carried out in essentially the exact same way in our setting.

Now take $n_2 = \omega(n_1)$. A complication arises: the distribution of the number of neighbors of a node of type 1 does not converge (its mean is $n_2 p \rightarrow \infty$), and the distribution of the number of neighbors of a node of type 2 converges to a delta mass at 0. But this can be fixed by ignoring the vertices in V_2 of degree 0 and 1. Now we explore from a vertex $\rho \in V_1$, but discard any vertices from V_2 that do not have a second neighbor. We denote by \hat{G} the subgraph of G induced by V_1 and the vertices of V_2 of degree at least 2. Let T be the branching process associated to this modified graph: nodes of type 1 have Poisson(d^2) neighbors of type 2, and nodes of type 2 have exactly 1 neighbor of type 1, where here $p = d/\sqrt{n_1 n_2}$. The branching number of this process is d , and the reconstruction threshold is $(\delta - 1)^2 d \leq 1$, again giving the threshold $p \leq \frac{1}{(\delta - 1)^2 \sqrt{n_1 n_2}}$, as required.

As in [Mossel et al. \(2014\)](#), the proof of impossibility in the full version of this papers shows the stronger statement that conditioned on the label of a fixed vertex $w \in V_1$ and the graph G , the variance of the label of another fixed vertex ρ tends to 1 as $n_1 \rightarrow \infty$. The proof of this fact has two main ingredients: showing that the depth R neighborhood of a vertex ρ in the bipartite block model (with vertices of degree 0 and 1 in V_2 removed) can be coupled with the branching process described above, and showing that conditioned on the labels on the boundary of the neighborhood, the label of ρ is asymptotically independent of the rest of the graph and the labels outside of the neighborhood.

7. Theorem 3: Recovery

We follow a similar framework in proving both parts of Theorem 3. Recalling M to be the adjacency matrix, let $B = MM^T - \text{diag}(MM^T)$ and $D_V = \text{diag}(MM^T)$.

A simple computation shows that the second eigenvector of $\mathbb{E}B$ is the vector σ that we wish to recover; we will consider the different perturbations of $\mathbb{E}B$ that arise with the three spectral algorithms and show that at the respective thresholds, the second eigenvector of the resulting matrix is close to σ . To analyze the diagonal deletion SVD, we must show that the second eigenvector of B is highly correlated with σ (the addition of a constant multiple of the identity matrix does not change the eigenvectors). The main step is to bound the spectral norm $\|B - \mathbb{E}B\|$. Since the entries of B are not independent, we will decompose B into a sequence of matrices based on subgraphs induced by vertices of a given degree in V_2 . This (Lemma 1) is the most technical part of the work.

To analyze the unmodified SVD, we write $MM^T = \mathbb{E}B + (B - \mathbb{E}B) + \mathbb{E}D_V + (D_V - \mathbb{E}D_V)$. The left singular vectors of M are the eigenvectors of MM^T . $\mathbb{E}B$ has σ as its second eigenvector and $\mathbb{E}D_V$ is a multiples of the identity matrix and so adding it does not change the eigenvectors. As above we bound $\|B - \mathbb{E}B\|$ and what remains is showing that the difference of the matrix D_V with its expectation has small spectral norms at the respective thresholds; this involves simple bounds on the fluctuations of independent random variables.

We will assume that σ and τ assign $+1$ and -1 labels to an equal number of vertices; this allows for a clearer presentation, but is not necessary to the argument. We will treat σ and τ as unknown but fixed, and so expectations and probabilities will all be conditioned on the labelings.

The main technical lemma, whose proof is in the full version of the paper, is the following:

Lemma 1 *Define B, D_V as above. Assume n_1, n_2 , and p are as in Theorem 3. Then there exists an absolute constant C so that*

1. $\mathbb{E}B = \lambda_1 J/n_1 + \lambda_2 \sigma \sigma^T/n_1$, with $\lambda_1 = n_1 n_2 p^2$ and $\lambda_2 = (\delta - 1)^2 n_1 n_2 p^2$, where J is the all ones $n_1 \times n_1$ matrix.
2. For $p \geq n_1^{-1/2} n_2^{-1/2} \log n_1$, $\|B - \mathbb{E}B\| \leq C n_1^{1/2} n_2^{1/2} p$ whp.
3. $\mathbb{E}D_V$ is a multiple of the identity matrix.
4. For $p \geq n_1^{-2/3} n_2^{-1/3} \log n_1$, $\|D_V - \mathbb{E}D_V\| \leq C \sqrt{n_2 p \log n_1}$ whp.

We also will use the following lemma from [Lelarge et al. \(2013\)](#) to round a unit vector with high correlation with σ to a ± 1 vector that denotes a partition:

Lemma 2 (Lelarge et al. (2013)) *For any $x \in \{-1, +1\}^n$ and $y \in \mathbb{R}^n$ with $\|y\| = 1$ we have*

$$d(x, \text{sign}(y)) \leq n \left\| \frac{x}{\sqrt{n}} - y \right\|^2,$$

where d represents the Hamming distance.

The next lemma is a classic eigenvector perturbation theorem. Denote by $P_A(S)$ the orthogonal projection onto the subspace spanned by the eigenvectors of A corresponding to those of its eigenvalues that lie in S .

Lemma 3 (Davis and Kahan (1970))

Let A be an $n \times n$ symmetric matrix with $|\lambda_1| \geq |\lambda_2| \geq \dots$, with $|\lambda_k| - |\lambda_{k+1}| \geq 2\delta$. Let B be a symmetric matrix with $\|B\| < \delta$. Let A_k and $(A + B)_k$ be the spaces spanned by the top k eigenvectors of the respective matrices. Then

$$\sin(A_k, (A + B)_k) = \|P_{A_k} - P_{(A+B)_k}\| \leq \frac{\|B\|}{\delta}$$

In particular, If $|\lambda_1| - |\lambda_2| \geq 2\delta$, $|\lambda_2| - |\lambda_3| \geq 2\delta$, $\|B\| < \delta$, and $e_2(A)$, $e_2(A + B)$ are the second (unit) eigenvectors of A and $A + B$, respectively, satisfying $e_2(A) \cdot e_2(A + B) \geq 0$, then $\|e_2(A) - e_2(A + B)\| \leq \frac{4\|B\|}{\delta}$.

Now using Lemmas 1, 2, and 3 we prove parts 1 and 2 of Theorem 3.

Diagonal deletion SVD Let $p \geq n_1^{-1/2} n_2^{-1/2} \log n_1$. Part 1 of Lemma 1 shows that if we had access to the second eigenvector of $\mathbb{E}B$, we would recover σ exactly. (The addition of a multiple of the identity matrix does not change the eigenvectors). Instead we have access to $B = \mathbb{E}B + (B - \mathbb{E}B)$, a noisy version of the matrix we want. We use a matrix perturbation inequality to show that the top eigenvectors of the noisy version are not too far from the original eigenvectors.

Let y_1 and y_2 be the top two eigenvectors of B , and \hat{B} be the space spanned by y_1 and y_2 , and $(\mathbb{E}B)_2$ the space spanned by the top two eigenvectors of $\mathbb{E}B$. Then Lemma 3 gives

$$\sin((\mathbb{E}B)_2, \hat{B}) \leq \frac{C\|B - \mathbb{E}B\|}{\lambda_2} \leq C \frac{n_1^{1/2} n_2^{1/2} p}{(\delta - 1)^2 n_1 n_2 p^2} = O\left(\frac{1}{\log n_1}\right)$$

where the inequality holds whp by Lemma 1. Assuming $\delta \in (0, 2)$, we use the particular case of Lemma 3 to show that $\|y_2 - \sigma/\sqrt{n_1}\| = O(\log^{-1} n_1)$. We round y_2 by signs to get z , and then apply Lemma 2 to show that whp the algorithm recovers $1 - o(1)$ fraction of the coordinates of σ . (If $\delta = 0$ or 2 , then instead of taking the second eigenvector, we take the component of \hat{B} perpendicular to the all ones vector and get the same result).

The SVD Let $p \geq n_1^{-2/3} n_2^{-1/3} \log n_1$. Let y_1 and y_2 be the top two left singular vectors of M , and M_2 be the space spanned by y_1 and y_2 . y_1 and y_2 are the top two eigenvectors of $MM^T = B + D_V$. Again Lemma 3 gives that whp,

$$\sin((\mathbb{E}B)_2, M_2) \leq C \frac{\|B - \mathbb{E}B\| + \|D_V - \mathbb{E}D_V\|}{\lambda_2} \leq \frac{C_1 n_1^{1/2} n_2^{1/2} p + C_2 \sqrt{n_2 p \log n_1}}{(\delta - 1)^2 n_1 n_2 p^2} = O\left(\frac{1}{\log n_1}\right).$$

This gives $\|y_2 - \sigma/\sqrt{n_1}\| = O(\log^{-1} n_1)$, and shows that the SVD algorithm recovers σ whp. Note that in this case $\|D_V - \mathbb{E}D_V\| \gg \|B - \mathbb{E}B\|$. It is these fluctuations on the diagonal that explain the poor performance of the SVD and its need for a higher edge density for success.

8. Theorem 4: Failure of the vanilla SVD

Here we again use a matrix perturbation lemma, but in the opposite way: we will show that the ‘noise matrix’ ($D_V - \mathbb{E}D_V$) has a large spectral norm (and an eigenvalue gap), and thus adding the

‘signal matrix’ approximately preserves the space spanned by the top eigenvalues. This shows that the top t eigenvectors of $B + D_V$ have almost all their weight on a small number of coordinates and is enough to conclude that they cannot be close to the planted vector σ .

The perturbation lemma we use is a generalization of the Davis-Kahan theorem found in [Bhatia \(1997\)](#).

Lemma 4 (Bhatia (1997)) *Let A and B be $n \times n$ symmetric matrices with the eigenvalues of A ordered $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Suppose $r > k$, $\lambda_k - \lambda_r > 2\delta$, and $\|B\| \leq \delta$. Let A_r denote the subspace spanned by the first r eigenvectors of A and likewise for $(A + B)_k$. Then*

$$\|P_{A_r^\perp} P_{(A+B)_k}\| \leq \frac{\|B\|}{\delta}.$$

In particular, if v_k is the k^{th} unit eigenvector of $(A + B)$, then there is some unit vector $u \in A_r$ so that

$$\|u - v_k\| \leq \frac{4\|B\|}{\delta}.$$

We also need to analyze the degrees of the vertices in V_1 . The following lemma gives some basic information about the degree sequence:

Lemma 5 *Let d_1, \dots, d_{n_1} be the sequence of degrees of vertices in V_1 . Then there exist constants c_1, c_2, c_3 so that*

1. *The d_i 's are independent and identically distributed, with distribution $d_i \sim \text{Bin}(n_2/2, \delta p) + \text{Bin}(n_2/2, (2 - \delta)p)$.*
2. $\mathbb{E}d_i = n_2 p$.
3. *Whp, $\max_i d_i \leq n_2 p + c_1 \sqrt{n_2 p \log n_1}$.*
4. *Whp, $|\{i : d_i \geq n_2 p + c_2 \sqrt{n_2 p \log n_1}\}| \geq n_1^{1/3}$.*
5. *Whp, $|\{i : d_i \geq n_2 p + c_3 \sqrt{n_2 p \log \log n_1}\}| \leq n_1 / \log n_1$.*

Now we can prove [Theorem 4](#). Let $p = cn_1^{-2/3} n_2^{-1/3}$. The left singular vectors of M are the eigenvectors of $B + D_V$. Recall that D_V is a diagonal matrix with the i th entry the degree of the i th vertex of V_1 . $\mathbb{E}D_V$ is therefore a multiple of the identity matrix, and so subtracting $\mathbb{E}D_V$ from $B + D_V$ does not change its eigenvectors. The standard basis vectors form an orthonormal set of eigenvectors of $D_V - \mathbb{E}D_V$.

For the constants c_2, c_3 in [Lemma 5](#), let $\eta_1 = c_2 \sqrt{n_2 p \log n_1}$ and $\eta_2 = c_3 \sqrt{n_2 p \log \log n_1}$. Order the eigenvalues of $D_V - \mathbb{E}D_V$ as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and let r be the smallest integer such that $\lambda_r < \eta_2$. Then we have $\lambda_i - \lambda_r \geq c \sqrt{n_2 p \log n_1}$ for all $1 \leq i \leq t$. From [Lemma 5](#), $r \leq n_1 / \log n_1$.

We now bound

$$\|B\| \leq \|\mathbb{E}B\| + \|B - \mathbb{E}B\| \leq n_1 n_2 p^2 + C n_1^{1/2} n_2^{1/2} p.$$

Now Lemma 4 says that if v_i is the i th eigenvector of $D_V - \mathbb{E}D_V + B$, then there is a vector u in the span of the first r eigenvectors of $D_V - \mathbb{E}D_V$ so that

$$\|v_i - u\| \leq C \frac{n_1 n_2 p^2 + n_1^{1/2} n_2^{1/2} p}{\sqrt{n_2 p \log n_1}} = O\left(\frac{1}{\sqrt{\log n_1}}\right).$$

The span of the first r eigenvectors of $D_V - \mathbb{E}D_V$ is supported on only r coordinates, so u is far from $\bar{\sigma} = \sigma/\sqrt{n_1}$:

$$\|u - \bar{\sigma}\| \geq \sqrt{2 - 2\sqrt{r/n_1}} = \sqrt{2} - O(1/\sqrt{\log n_1}).$$

By the triangle inequality, v_i must also be far from $\bar{\sigma}$: $|v_i \cdot \bar{\sigma}| = O(1/\sqrt{\log n_1})$. This proves Theorem 4.

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