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SARASON CONJECTURE ON THE BERGMAN SPACE

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Abstract. We provide a counterexample to the Sarason Conjecture for the Bergman space and present a characterisation of bounded Toeplitz products on the Bergman space in terms of test functions by means of a dyadic model approach. We also present some results about two-weighted estimates for the Bergman projection. Finally, we introduce the class B_∞ and give sharp estimates for the one-weighted Bergman projection.

1. Introduction

Let dA denote Lebesgue area measure on the unit disc $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The Bergman space $A^2(\mathbb{D})$ is the closed subspace of analytic functions in the Hilbert space $L^2(\mathbb{D}, dA)$. Likewise, the Hardy space $H^2(\mathbb{T})$ is the closed subspace of $L^2(\mathbb{T})$ consisting of analytic functions.

The Bergman projection P_B, given by

$$P_B f(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1-\overline{\zeta}z)^2} dA(\zeta),$$

is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$, while the Riesz projection P_R denotes the orthogonal projection from $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$. For each function $f \in L^2(\mathbb{D})$ we have the densely defined Bergman space Toeplitz operator T_f on $A^2(\mathbb{D})$, given by

$$T_f u = P_B f u.$$

In the same way, given $f \in L^2(\mathbb{T})$, the Hardy space Toeplitz operator $\mathcal{T}_f$ on H^2 is given by

$$\mathcal{T}_f v = P_R f v,$$

where u and v are suitable elements in A^2 and H^2, respectively.

For analytic f, it is easy to see that both the Bergman space Toeplitz operator T_f and the Hardy space Toeplitz operator $\mathcal{T}_f$ are bounded, if and only if f is a bounded function on $\mathbb{D}$.

In this paper, we shall study the question as to which pairs of functions $f, g \in A^2(\mathbb{D})$ give rise to a bounded Toeplitz product operator

$$T_f T_g^* : A^2(\mathbb{D}) \to A^2(\mathbb{D}).$$

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This questions has a rich history and interesting connections to Harmonic Analysis, as we outline below.

Sarason [28] conjectured the following:

Conjecture 1.1 (Sarason Conjecture for the Bergman space)**.** *Let $f, g \in A^2(\mathbb{D})$. Then $T_f T_g^*$ is bounded on $A^2(\mathbb{D})$, if and only if*

$$b_{f,g}^2 := \sup_{z\in\mathbb{D}} B(|f|^2)(z)B(|g|^2)(z) < \infty, \tag{1.2}$$

where B denotes the Berezin transform,

$$Bf(z) = \int_{\mathbb{D}} \frac{f(\zeta)(1-|z|^2)^2}{|1-\bar{\zeta}z|^4} dA(\zeta). \tag{1.3}$$

Likewise, he conjectured the following for the case of the Hardy space:

Conjecture 1.4 (Sarason Conjecture for the Hardy space)**.** *Given $f, g \in H^2(\mathbb{T})$, $\mathcal{T}_f \mathcal{T}_g^*$ is bounded in $H^2(\mathbb{T})$ if and only if*

$$\sup_{z\in\mathbb{D}} \mathcal{P}(|f|^2)(z)\mathcal{P}(|g|^2)(z) < \infty, \tag{1.5}$$

where $\mathcal{P}$ denotes the Poisson extension.

Both in the Bergman space and the Hardy space case, these questions are closely connected to very interesting questions in Harmonic Analysis, namely two-weight estimates for the Bergman projection, respectively the Riesz projection.

Cruz-Uribe observed [7] the following commutative diagram in the case of the Hardy space:

$$\begin{array}{ccccc} & H^2(\mathbb{T}) & \overset{\mathcal{T}_f\mathcal{T}_g^*}{\longrightarrow} & H^2(\mathbb{T}) & \\ M_{\bar g} & \downarrow & & \uparrow & M_f \\ & L^2(\frac{1}{|g|^2}, \mathbb{T}) & \overset{P_R}{\longrightarrow} & H^2(|f|^2, \mathbb{T}) & \end{array}$$

Here, $M_{\bar g}$, M_f on the vertical sides denote multiplication with the respective symbols, and these operators are isometric by definition of the weights. A similar argument can be made for the Bergman space,

$$\begin{array}{ccccc} & A^2(\mathbb{D}) & \overset{T_f T_g^*}{\longrightarrow} & A^2(\mathbb{D}) & \\ M_{\bar g} & \downarrow & & \uparrow & M_f \\ & L^2(\frac{1}{|g|^2}, \mathbb{D}) & \overset{P_B}{\longrightarrow} & A^2(|f|^2, \mathbb{D}) & \end{array} \tag{1.6}$$

again with isometric operators on the vertical sides. One can thus see easily that the top row of each diagram is bounded, if and only if the bottom row is bounded.

Hence the question on the boundedness of Toeplitz products can be translated to the problem of boundedness of the two-weighted Bergman projection

$$P_B : L^2(\mathbb{D}, \frac{1}{|g|^2}) \to L^2(\mathbb{D}, |f|^2) \tag{1.7}$$

respectively boundedness of the two-weighted Riesz projection

$$P_R : L^2(\mathbb{T}, \frac{1}{|g|^2}) \to L^2(\mathbb{T}, |f|^2) \tag{1.8}$$

in the case of the Hardy space.

This connection motivated the Sarason conjectures 1.1, 1.4 above. Namely, condition (1.2) is the natural two-weight form of the Békollé-Bonami condition B_2 for a weight function w on $\mathbb{D}$ [4, 5],

$$\sup_{z\in\mathbb{D}} B(w)(z)B(w^{-1})(z) < \infty,$$

which is equivalent to the boundedness of the one-weighted Bergman projection

$$P_B : L^2(\mathbb{D}, w) \to A^2(\mathbb{D}, w), \tag{1.9}$$

and also to the boundedness of the maximal one-weighted Bergman projection

$$P_B^+ : L^2(\mathbb{D}, w) \to L^2(\mathbb{D}, w), \tag{1.10}$$

where

$$P_B^+(f) := \int_{\mathbb{D}} \frac{f(\zeta)}{|1-\bar{\zeta}z|^2} dA(\zeta) \tag{1.11}$$

(see [5]). In the same way, (1.5) is the natural two-weight form of the invariant Muckenhoupt condition A_2 for a weight function v ,

$$\sup_{z\in\mathbb{D}} \mathcal{P}(v)(z)\mathcal{P}(v^{-1})(z) < \infty,$$

which is equivalent to the boundedness of the one-weighted Riesz projection

$$P_R : L^2(\mathbb{D}, v) \to L^2(\mathbb{D}, v), \tag{1.12}$$

or equivalently, the one-weighted Hilbert transform H [12].

The problem of classifying those pairs of weights (ρ, v) for which the two-weighted Riesz projection

$$P_R : L^2(\mathbb{D}, \rho) \to L^2(\mathbb{D}, v), \tag{1.13}$$

or equivalently, the two-weighted Hilbert transform is bounded, is a famous problem in Harmonic Analysis. For a long time, it was conjectured that a version of (1.5) for general weights (ρ, v), the joint invariant A_2 condition

$$\sup_{z\in\mathbb{D}} \mathcal{P}(v)(z)\mathcal{P}(\rho^{-1})(z) < \infty \tag{1.14}$$

characterises (1.13). This would in particular imply Sarason's conjecture on Hardy spaces. However, F. Nazarov disproved both this conjecture and the Sarason conjecture 1.4. in 1997 [22]. The two-weight Hilbert transform problem, the problem of characterising boundedness of (1.13), has been the subject of intense recent research activity, see e.g. [25], [23], [24], [18], [16], [15] and the references therein.

Sarason's Conjecture 1.1 for Toeplitz products on Bergman spaces, in contrast, has remained open till now. The purpose of this paper is to provide a counterexample to this conjecture, depending on a new characterisation of bounded Toeplitz products on Bergman space by means of natural test function conditions. Our main results can be summarised as follows:

Theorem 1.15. *There exist functions $f, g \in A^2(\mathbb{D})$ such that $b_{f,g} < \infty$, but $T_f T_g^*$ is not bounded on $A^2(\mathbb{D})$.*

Theorem 1.16. *Let $P_B^+(\cdot)$ be the maximal Bergman projection on the disc $\mathbb{D}$, and let $f, g \in A^2(\mathbb{D})$. The following are equivalent*

(1) $T_f T_g^* : A^2(\mathbb{D}) \mapsto A^2(\mathbb{D})$ *is bounded;*
(2) $P_B(|g|^2 \cdot) : L^2(\mathbb{D}, |g|^2) \to L^2(\mathbb{D}, |f|^2)$ *bounded;*
(3) $P_B^+(|g|^2 \cdot) : L^2(\mathbb{D}, |g|^2) \to L^2(\mathbb{D}, |f|^2)$ *bounded;*
(4) (a) $\|P_B^+(|g|^2 1_{Q_I})\|_{L^2(\mathbb{D},|f|^2)} \leq C_0 \|1_{Q_I}\|_{L^2(\mathbb{D},|g|^2)}$,
(b) $\|P_B^+(|f|^2 1_{Q_I})\|_{L^2(\mathbb{D},|g|^2)} \leq C_0 \|1_{Q_I}\|_{L^2(\mathbb{D},|f|^2)}$,
for all Carleson boxes Q_I associated to intervals $I \in \mathbb{T}$ and with constant C_0 uniform on I.

Here, the first equivalence is Cruz-Uribe's observation, the second equivalence is proved in Section 3, and the last equivalence is consequence of the two weight characterization for dyadic positive operators by Lacey, Sawyer and Uriarte-Tuero [17], details are provided in Section 2. We will prove Theorem 1.15 in Section 4. Section 5 is devoted to an application to the proof of sharp estimates for one-weighted Bergman projection.

Sufficient conditions close to Sarason's condition 1.2 for the boundedness of Toeplitz products in the style of the so-called bump conditions can be found in [33] and in [21].

In spite of the formal similarities of the Sarason conjectures in the Hardy space and in the Bergman space settings, the problem is quite different in both settings.

Some aspects of the Bergman space setting are easier, because cancellation plays much less of a role in this setting, as already apparent from the equivalence of (1.9) and (1.10). To characterise boundedness of Toeplitz products, our strategy is thus to replace P_B by P_B^+, and to use established two-weight techniques for dyadic positive operators, via a suitable dyadic model operator introduced in Section 2. Somewhat surprisingly, it turns out that this is possible for the weights $\frac{1}{|g|^2}$, $|f|^2$ in (1.6). This is the equivalence of (2) and (3) in Theorem 1.16, which will be proved in Section 3, and allows us to finally characterise the boundedness of Toeplitz products in Bergman space in terms of test function.

On the other hand, the special rôle played by weights coming from analytic functions, which we exploit in Section 3 and which is in contrast to the situation on the Hardy space, makes it much more difficult to find a counterexample of the Sarason Conjecture on Bergman space (1.2). We prove a counterexample to the Sarason conjecture 1.1 in Section 4. For non-analytic symbols, or even one non-analytic symbol, such examples are much easier to find. In this case, the function g in Lemma 4.3, the construction of which forms the main part of the counterexample, can just be replace by $1-|z|$.

2. A dyadic model for the maximal Bergman projection

In this section we aim to find a dyadic operator that models the behaviour of the maximal Bergman projection. To be precise, we find a dyadic averaging operator that is pointwise comparable to the maximal Bergman projection.

The use of translations of a dyadic system to extend results from a dyadic setting to a continuous one is a well known tool. These ideas go back to the work of Garnett and Jones [10], Christ [6] and also Tao Mei [20]. In our case, we will use two of these dyadic systems to recover the maximal Bergman kernel from dyadic operators.

For $\beta \in \{0, 1/3\}$, we define

$$\mathcal{D}^\beta := \left\{[2^{-j}2\pi m + 2\pi\beta,\, 2^{-j}2\pi(m+1) + 2\pi\beta) : m \in \mathbb{N},\, j \in \mathbb{N},\, j \geq 0,\, 0 \leq m \leq 2^j\right\}.$$

The key fact is that any interval in the torus is contained in one interval belonging to these two families of dyadic grids, moreover the measure of the two intervals is essentially the same. We formulate the result below. Its proof is a well-known exercise that the reader can find in many places, e.g. [20].

Lemma 2.1. *Let I be any interval in $\mathbb{T}$. Then there exists an interval $K \in \mathcal{D}^\beta$ for some $\beta \in \{0, 1/3\}$ such that $I \subset K$ and $|K| \leq 6|I|$.*

We define the family of dyadic operators that will control the maximal Bergman projection (1.11) as the following.

Definition 2.2. Let $\mathcal{D}^\beta$ be one of the dyadic grids in $\mathbb{T}$ described above. For all $z, \xi \in \mathbb{D}$, we define the positive dyadic kernel

$$K^\beta(z,\xi) := \sum_{I \in \mathcal{D}^\beta} \frac{1_{Q_I}(z) 1_{Q_I}(\xi)}{|I|^2}, \tag{2.3}$$

where Q_I is the Carleson box associated to I, namely

$$Q_I := \{r\mathrm{e}^{i\theta} :\ 1 - |I| \leq r < 1 \text{ and } \mathrm{e}^{i\theta} \in I\}, \tag{2.4}$$

and $|I|$ stands for the normalized length of the interval. Associated to this kernel we define the following dyadic operator

$$P^\beta f(z) := \sum_{I \in \mathcal{D}^\beta} \langle f, \frac{1_{Q_I}}{|I|^2} \rangle 1_{Q_I}(z), \tag{2.5}$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(\mathbb{D})$.

The following proposition proves the relation between the kernels (2.8) and the dyadic kernels described in (2.3).

Proposition 2.6. *There exist constants C and $\tilde{C}$ such that for every $\beta_0 \in \{0, 1/3\}$, every $f \in L^1_{loc}$ and $z \in \mathbb{D}$,*

$$\tilde{C} P^{\beta_0} f(z) \leq P^+_B f(z) \leq C \sum_{\beta \in \{0,1/3\}} P^\beta f(z), \tag{2.7}$$

where P^+_B is the maximal Bergman projection as defined in (1.11) and P^β the dyadic operator described in (2.5).

Proof of Proposition (2.6). Let $K(z, \xi)$ denote the kernel associated to the maximal Bergman projection, i.e.,

$$K(z, \xi) = \frac{1}{|1 - z\bar{\xi}|^2}. \tag{2.8}$$

Then it is enough to prove that there exist constants C and $\tilde{C}$ such that for every β_0 and every z, ξ in $\mathbb{D}$ we have the following estimates on the kernel,

$$\tilde{C} K^{\beta_0}(z, \xi) \leq K(z, \xi) \leq C \sum_{\beta \in \{0,1/3\}} K^\beta(z, \xi) \tag{2.9}$$

Let us first prove the left hand side of (2.9). We consider $z = r_0 e^{i\theta_0}$ and $\xi = s_0 e^{i\varphi_0}$. Without loss, we can assume that $r_0 \leq s_0$. We choose $I_0 \in \mathcal{D}^{\beta_0}$ to be the minimal interval such that $|I_0| \geq 1 - r_0$ and $e^{i\theta_0}$, $e^{i\varphi_0} \in I_0$. Then, it is easy to see that $z, \xi \in Q_{I_0}$. It could be that such an interval doesn't exist, in that case the inequality is trivially true. From $z, \xi \in Q_{I_0}$ we can deduce

$$\sum_{I \in \mathcal{D}^\beta} \frac{1_{Q_I(z)} 1_{Q_I(\xi)}}{|I|^2} = \sum_{I, I_0 \subset I} \frac{1}{|I|^2} \leq C \frac{1}{|I_0|^2}. \tag{2.10}$$

To conclude the proof of the left hand side, we need to show

$$|1 - z\bar{\xi}|^2 \leq C|I_0|^2, \tag{2.11}$$

for some uniform constant C. We can write $|1 - z\bar{\xi}|^2$ as

$$|1 - z\bar{\xi}|^2 = (1 - r_0 s_0)^2 + 4 r_0 s_0 \sin^2\left(\frac{\theta_0 - \varphi_0}{2}\right). \tag{2.12}$$

We distinguish two cases, when $(1 - r_0 s_0)^2$ is the majorant term, and when $4 r_0 s_0 \sin^2(\frac{\theta_0 - \varphi_0}{2})$ is the majorant.

(1) Case 1. If $(1 - r_0 s_0)^2 > 4 r_0 s_0 \sin^2(\frac{\theta_0 - \varphi_0}{2})$, then

$$|1 - z\bar{\xi}|^2 \leq 2(1 - r_0 s_0)^2 \leq 8|I_0|^2,$$

as desired.

(2) Case 2. Suppose on the contrary that $(1-r_0s_0)^2 \leq 4r_0s_0\sin^2(\frac{\theta_0-\varphi_0}{2})$. Since $\mathrm{e}^{i\theta_0}$, $\mathrm{e}^{i\varphi_0} \in I_0$, we know that $|I_0| \geq |\theta_0 - \varphi_0|$. Then

$$|1-z\bar{\xi}|^2 \leq 8r_0s_0\sin^2(\frac{\theta_0-\varphi_0}{2}) \leq 2|\theta_0-\varphi_0|^2 \leq 2|I_0|^2,$$

as desired.

Therefore we have concluded the proof of the left hand side of (2.9). We now turn to the right hand inequality in (2.9). Once again let us fix $z, \xi \in \mathbb{D}$, and write them as before as $z = r_0\mathrm{e}^{i\theta_0}$ and $\xi = s_0\mathrm{e}^{i\varphi_0}$.

It is enough to prove the existence of an interval I_0 in $\mathbb{T}$ such that $z, \xi \in Q_{I_0}$ and $|I_0|^2 \simeq |1-z\bar{\xi}|^2$. If such an interval exists, by Lemma 2.1, we find $K \in \mathcal{D}^\beta$ for some $\beta \in \{0, 1/3\}$ such that $I_0 \subset K$ and $|K| \leq 6|I_0|$. Now the proof of the proposition follows from the set of inequalities below:

$$\begin{aligned}
\frac{1}{|1-z\bar{\xi}|^2} &\lesssim \frac{1}{|I_0|^2} \\
&\lesssim \frac{1}{36|K|^2} \\
&\leq C \sum_{\substack{I\in\mathcal{D}^\beta \\ K\subset I}} \frac{1_{Q_I}(z)1_{Q_I}(\xi)}{|I|^2} \\
&\leq C \sum_{\beta\in\{0,1/3\}} K^\beta(z,\xi).
\end{aligned}$$

Thus we have reduced the problem to prove the existence I_0 interval in $\mathbb{T}$ such that $z, \xi \in Q_{I_0}$ and $|I_0|^2 \simeq |1-z\bar{\xi}|^2$. Notice than we will always have $|\theta_0 - \varphi_0| \leq \pi$, and since $|\sin x| \simeq |x|$ for $|x| \leq \pi/2$, we have

$$|1-z\bar{\xi}|^2 \simeq (1-r_0s_0)^2 + r_0s_0|\theta_0-\varphi_0|^2 \simeq (1-r_0^2)^2 + |\theta_0-\varphi_0|^2. \tag{2.13}$$

by (2.12). Let us choose I_0 to be a minimal interval such that

$$|I_0|^2 = \max((1-r_0^2)^2,\, |\theta_0-\varphi_0|^2)$$

and $\mathrm{e}^{i\theta_0}, \mathrm{e}^{i\varphi_0} \in I_0$. It is easy to see that $z, \xi \in Q_{I_0}$. We have to prove that $|I_0|^2 \simeq |1-z\bar{\xi}|^2$. But this follows directly from (2.13).

This finishes the proof of the proposition.

□

We now establish two-weight estimates for the maximal Bergman projection. We use a two-weight characterization of boundedness for general dyadic positive operators in terms of testing conditions. This characterization was provided by Lacey, Sawyer and Uriarte-Tuero [17], based on previous work of Eric Sawyer in the continuous case [29,30]. To conclude the desired characterization for the maximal Bergman projection, P_B^+, we will use the the dyadic result in combination with inequalities (2.7).

There are three equivalent formulations for two weighted inequalities that we will use in turn. A weight function will be an nonnegative measurable function on $\mathbb{R}^n$, not necessarily locally integrable. Let w, v be weight functions in $\mathbb{R}^n$, let $1 < p < \infty$ and p' its dual exponent. We define $\sigma := v^{1-p'}$, which is usually called the dual weight of v. Let T be an operator. Then the following are equivalent:

(1) $T : L^p(v) \mapsto L^p(w)$
(2) $T(\sigma \cdot) : L^p(\sigma) \mapsto L^p(w)$
(3) $w^{1/p} T(\sigma^{1/p'} \cdot) : L^p \mapsto L^p$.

In this section we will mostly use (2) above, although for the Sarason problem, (3) is more natural and will frequently appear.

Theorem 2.14. *Let $\mathcal{D}^\beta$ be a fixed dyadic grid in $\mathbb{T}$ and let P^β as defined in (2.5). Then*

$$P^\beta(w\cdot) \, : \, L^p(w) \to L^p(\sigma)$$

is bounded, if and only if

$$\Big\| \sum_{\substack{I \in \mathcal{D}^\beta \\ I \subset I_0}} \langle w 1_{Q_{I_0}}, \frac{1_{Q_I}}{|I|^2} \rangle 1_{Q_I} \Big\|^p_{L^p(\sigma)} \leq C_0 w(Q_{I_0}), \tag{2.15}$$

and

$$\Big\| \sum_{\substack{I \in \mathcal{D}^\beta \\ I \subset I_0}} \langle \sigma 1_{Q_{I_0}}, \frac{1_{Q_I}}{|I|^2} \rangle 1_{Q_I} \Big\|^{p'}_{L^{p'}(w)} \leq C_0^* \sigma(Q_{I_0}), \tag{2.16}$$

for all I dyadic interval in $\mathcal{D}^\beta$, where Q_I represents the Carleson box associated to I and the constants C_0 and C_0^* are independent of the intervals I. Moreover, there exists a constant $c > 0$ independent of the weights, such that

$$\big\| P^\beta(w\cdot) \big\|_{L^p(w)^p \to L^p(\sigma)} \leq c(C_0 + C_0^*).$$

The proof of Theorem 2.14 in the disc $\mathbb{D}$ is essentially the one provided by Lacey, Sawyer and Uriarte-Tuero in [17]. A simplified version is given by S. Treil in [32]. In the case of the disc, our dyadic system will be described by the Carleson cubes associated to the intervals in the dyadic grid $\mathcal{D}^\beta$ in $\mathbb{T}$. The details of the proof can be found in an earlier version of this paper [2], Theorem 3.7, or in the survey paper [27].

We obtain the following corollary, which presents a two weight characterization for the maximal Bergman projection.

Corollary 2.17. *Let P_B^+ be the maximal Bergman projection in the disc $\mathbb{D}$, let $1 < p < \infty$ and p' its dual exponent and let w, σ be two weight functions. Then*

$$M_{w^{1/p}} P_B^+ M_{\sigma^{1/p'}} \, : \, L^p(\mathbb{D}) \to L^p(\mathbb{D})$$

is bounded, if and only if

$$\|M_{w^{1/p}}P_B^+M_{\sigma^{1/p'}}(1_{Q_I}\sigma^{1/p})\|_{L^p(\mathbb{D})} \leq C_0\|1_{Q_I}\sigma^{1/p}\|_{L^p(\mathbb{D})} \tag{2.18}$$

and

$$\|M_{\sigma^{1/p'}}P_B^+M_{w^{1/p}}(1_{Q_I}w^{1/p'})\|_{L^{p'}(\mathbb{D})} \leq C_0^*\|1_{Q_I}w^{1/p'}\|_{L^{p'}(\mathbb{D})}, \tag{2.19}$$

for any interval I in $\mathbb{T}$, where the constants C_0 and C_0^* are independent of the choice of interval.

Moreover, there exists a constant $c > 0$ independent of the weights, such that

$$\left\|M_{w^{1/p}}P_B^+M_{\sigma^{1/p'}}\right\|_{L^p\to L^p} \leq c(C_0 + C_0^*).$$

As in the introduction, the operators M_h stand for the operator of multiplication by the symbol h.

Proof. We only have to prove one direction. By the first inequality in (2.7), the testing condition (2.18) and (2.19) imply the corresponding testing condition for each P^β, and therefore the uniform boundedness of all P_β by Theorem 2.14. The second inequality in (2.7) now implies the boundedness of $M_{w^{1/p}}P_B^+M_{\sigma^{1/p'}}$ with the required norm bounds. □

We note that the positivity of P_B^+ and the left hand-side of (2.7) are crucial here to recover the non-dyadic case from the dyadic one. This advantage is not present in the case of cancellative operators such as the Bergman projection itself.

3. P and P_+ are equivalent

Given $f, g \in L^2(\mathbb{D})$, we denote as before

$$b_{f,g} = \sup_{z\in\mathbb{D}} B^{1/2}(|f|^2)(z)B^{1/2}(|g|^2)(z).$$

Theorem 3.1. *Let $f, g \in A^2(\mathbb{D})$. Then $T_fT_g^*$ is bounded on $A^2(\mathbb{D})$ if and only if the operator $P_{f,g}^+$ defined by*

$$P_{f,g}^+u(z) = |f(z)|\int_{\mathbb{D}} \frac{|g(\zeta)|u(\zeta)}{|1-\bar{\zeta}z|^2}dA(\zeta)$$

is bounded on $L^2(\mathbb{D})$.

Notice that the boundedness of $P_{f,g}^+$ on $L^2(\mathbb{D})$ is equivalent to the two-weight estimate $P_B^+(|g|^2\cdot) : L^2(|g|^2) \to L^2(|f|^2)$ by (2) and (3) in page 8. For the proof of the theorem, we need some preliminary estimates and begin with a completely elementary lemma which will play the key role in our argument.

Lemma 3.2. *For $z, \zeta \in \mathbb{D}$ we have*

$$\begin{aligned}\frac{1}{|1-\bar{\zeta}z|^2} &= -\frac{\bar{\zeta}z}{(1-\bar{\zeta}z)^2} + \frac{1-|z\zeta|^2}{(1-\bar{\zeta}z)|1-\bar{\zeta}z|^2}\\ &= -\mathrm{Re}\frac{\bar{\zeta}z}{(1-\bar{\zeta}z)^2} + \frac{1-|z\zeta|^2}{2|1-\bar{\zeta}z|^2} + \frac{(1-|z\zeta|^2)^2}{2|1-\bar{\zeta}z|^4}\,.\end{aligned}$$

Proof. Let $w = \overline{\zeta} z \in \mathbb{D}$, and note that

$$\begin{aligned}\frac{1}{(1-w)^2} + \frac{1}{|1-w|^2} &= \frac{2}{(1-w)}\mathrm{Re}\frac{1}{(1-w)} \\ &= \frac{1}{(1-w)} + \frac{1}{(1-w)}\mathrm{Re}\frac{1+w}{1-w}\,,\end{aligned}$$

and the first identity follows from $\mathrm{Re}\frac{1+w}{1-w} = \frac{1-|w|^2}{|1-w|^2}$. For the second, we just take the real part on both sides of the first and use $\mathrm{Re}\frac{1}{1-w} = \frac{1}{2} + \frac{1-|w|^2}{2|1-w|^2}$. □

The next two lemmas deal with estimates for integral operators whose kernels are involved in the identities above.

Lemma 3.3. *For $f, g \in A^2(\mathbb{D})$, $u \in L^2(\mathbb{D})$ and $z \in \mathbb{D}$ let*

$$\begin{aligned}C^1_{f,g}u(z) &= |f(z)| \int_{\mathbb{D}} |g|u(\zeta)\frac{1-|z|^2}{|1-\overline{\zeta}z|^2}dA(\zeta)\,,\\ C^2_{f,g}u(z) &= |f(z)| \int_{\mathbb{D}} |g|u(\zeta)\frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\overline{\zeta}z|^4}dA(\zeta)\,,\\ C^3_{f,g}u(z) &= |f(z)| \int_{\mathbb{D}} |g|u(\zeta)\frac{1-|\zeta|^2}{(1-\overline{\zeta}z)|1-\overline{\zeta}z|^2}dA(\zeta)\,,\\ C^4_{f,g}u(z) &= |f(z)| \int_{\mathbb{D}} |g|u(\zeta)\frac{(1-|\zeta|^2)^2}{|1-\overline{\zeta}z|^4}dA(\zeta)\,.\end{aligned}$$

Then for $j = 1, 2$

$$\|C^j_{f,g}u\|_2 \lesssim b_{f,g}\|u\|_2\,.$$

Moreover, for any measurable set $E \subset \mathbb{D}$

$$\|C^3_{f,g}|g|1_E\|_2 \lesssim \|P^+_{f,g}|g|1_E\|_2^{1/2} b_{f,g}^{1/2}\|g1_E\|_2^{1/2}\,,$$

and

$$\|C^4_{f,g}|g|1_E\|_2 \lesssim b_{f,g}\|g1_E\|_2\,.$$

Proof. By the Cauchy-Schwartz inequality we have

$$|C^1_{f,g}u(z)| \le |f(z)|B^{1/2}(|g|^2)(z)\|u\|_2 \le b_{f,g}\|u\|_2\,.$$

Similarly,

$$|C^2_{f,g}u(z)| \le |f(z)|B^{1/2}(|g|^2)(z)\left(\int_{\mathbb{D}}\frac{|u(\zeta)|^2(1-|\zeta|^2)^2}{|1-\overline{\zeta}z|^4}dA(\zeta)\right)^{1/2}\,,$$

so that

$$\|C^2_{f,g}u\|_2^2 \le b^2_{f,g}\int_{\mathbb{D}}\left(\int_{\mathbb{D}}\frac{|u(\zeta)|^2(1-|\zeta|^2)^2}{|1-\overline{\zeta}z|^4}dA(\zeta)\right)dA(z) \lesssim b^2_{f,g}\|u\|_2^2$$

by a standard estimate for integrals (see for example page 10 in [11]). Another application of the Cauchy-Schwartz inequality shows that it will suffice to prove the estimate for $C^4_{f,g}$ since for nonnegative measurable functions u on $\mathbb{D}$ we have

$$|C^3_{f,g}u(z)| \le (P^+_{f,g}u(z))^{1/2}C_4^{1/2}u(z)\,.$$

This follows essentially the argument in [1] (proof of Lemma 3.1). Use the inequality $|1-\overline{\lambda}w| \le |1-\overline{z}w| + |1-\overline{\lambda}z|$ to obtain

$$\begin{aligned}
&\|C^4_{f,g}u\|_2^2 \le \\
&\int_{\mathbb{D}}\int_{\mathbb{D}}\int_{\mathbb{D}} |f(z)|^2 \frac{|gu(\lambda)||gu(w)|(1-|\lambda|^2)^2(1-|w|^2)^2}{|1-\overline{z}w|^4|1-\overline{z}\lambda|^4} dA(\lambda)dA(w)dA(z)\\
&\lesssim \int_{\mathbb{D}}\int_{\mathbb{D}}\int_{\mathbb{D}} |f(z)|^2 \frac{|gu(\lambda)||gu(w)|(1-|\lambda|^2)^2(1-|w|^2)^2}{|1-\overline{\lambda}w|^4|1-\overline{z}\lambda|^4} dA(\lambda)dA(w)dA(z)\\
&+ \int_{\mathbb{D}}\int_{\mathbb{D}}\int_{\mathbb{D}} |f(z)|^2 \frac{|gu(\lambda)||gu(w)|(1-|\lambda|^2)^2(1-|w|^2)^2}{|1-\overline{\lambda}w|^4|1-\overline{z}w|^4} dA(\lambda)dA(w)dA(z)\\
&= 2\int_{\mathbb{D}}\int_{\mathbb{D}}\int_{\mathbb{D}} |f(z)|^2 \frac{|gu(\lambda)||gu(w)|(1-|\lambda|^2)^2(1-|w|^2)^2}{|1-\overline{\lambda}w|^4|1-\overline{z}w|^4} dA(\lambda)dA(w)dA(z)\\
&= 2\int_{\mathbb{D}}\int_{\mathbb{D}} B(|f|^2)(w)|gu(w)| \frac{|gu(\lambda)|(1-|\lambda|^2)^2}{|1-\overline{\lambda}w|^4} dA(\lambda)dA(w)\,.
\end{aligned}$$

When $u = |g|v,\ v \ge 0$, the inequality $|g|^2B(|f|^2) \le b^2_{f,g}$ together with the standard estimate for integrals mentioned above yield

$$\|C^4_{f,g}gv\|^2 \lesssim b^2_{f,g}\|v\|_\infty\|g^2v\|_1\,,$$

and choosing $v = 1_E$ the result follows. □

In what follows we shall use the well known complex differential operators $\partial = \frac{\partial}{\partial z}$, $\overline{\partial} = \frac{\partial}{\partial \overline{z}}$. Let us also note that if $f, g \in A^2(\mathbb{D})$ and $T_fT_g^*$ is bounded on $A^2(\mathbb{D})$ then it is bounded on $L^2(\mathbb{D})$ with the same norm, since for $u \in L^2(\mathbb{D})$ we have

$$T_fT_g^*u = T_fT_g^*P_Bu\,,$$

where P_B is the Bergman projection.

Lemma 3.4. *For $f, g \in A^2(\mathbb{D})$ with $f(0) = 0$, $u \in L^2(\mathbb{D})$ and $z \in \mathbb{D}$ let*

$$Ru(z) = \int_{\mathbb{D}} \frac{|gu(\zeta)|}{|1-\overline{\zeta}z|^2}dA(\zeta)\,,$$

and let

$$Tu(z) = \frac{1}{z}Ru(z) - (1-|z|^2)\partial Ru(z)\,,\quad Su(z) = Ru(z) - \frac{(1-|z|^2)^2}{2}\partial\overline{\partial}Ru(z)\,.$$

Then

$$\|fTg1_E\|_2 \lesssim (\|T_fT_g^*\| + b_{f,g})\|g1_E\|_2 + \|P^+_{f,g}|g|1_E\|_2^{1/2}b_{f,g}^{1/2}\|g1_E\|_2^{1/2}\,,$$

and

$$\|fSg1_E\|_2 \lesssim (\|T_fT_g^*\| + b_{f,g})\|g1_E\|_2\,,$$

for all measurable sets $E \subset \mathbb{D}$.

Proof. Rewrite the first identity in Lemma 3.2 as

$$\begin{aligned}\frac{1}{z|1-\overline{\zeta}z|^2} &= -\frac{\overline{\zeta}}{(1-\overline{\zeta}z)^2} + \frac{1-|z\zeta|^2}{z|1-\overline{\zeta}z|^2} + \frac{(1-|z\zeta|^2)\overline{\zeta}}{(1-\overline{\zeta}z)|1-\overline{\zeta}z|^2} \\ &= -\frac{\overline{\zeta}}{(1-\overline{\zeta}z)^2} + \frac{1-|z|^2}{z|1-\overline{\zeta}z|^2} + \frac{(1-|\zeta|^2)\overline{z}}{|1-\overline{\zeta}z|^2} \\ &+ \frac{(1-|\zeta|^2)\overline{\zeta}|z|^2}{(1-\overline{\zeta}z)|1-\overline{\zeta}z|^2} + \frac{(1-|z|^2)\overline{\zeta}}{(1-\overline{\zeta}z)|1-\overline{\zeta}z|^2}\end{aligned}$$

Let M be the operator of multiplication by the independent variable on $L^2(\mathbb{D})$, $Mv(z) = zv(z)$. It is obvious that M is a bounded operator on $L^2(\mathbb{D})$. As it turns out it also satisfies a bound from below in some cases, namely,

$$\|v\|_{L^2(r\mathbb{D})} \lesssim \|Mv\|_{L^2(r\mathbb{D})}\,, \tag{3.5}$$

and

$$\|v\|_{L^2_{(1-|z|^2)^2}(r\mathbb{D})} \lesssim \|Mv\|_{L^2_{(1-|z|^2)^2}(r\mathbb{D})}\,, \tag{3.6}$$

valid for all subharmonic functions v in $\mathbb{D}$ and all $0 < r \le 1$. These estimates can be easily deduced from the subharmonicity of v. For a measurable function h on $\mathbb{D}$ let $\phi_h(z) = \overline{h(z)}/|h(z)|$, when $h(z) \neq 0$, and $\phi_h(z) = 1$ otherwise, and denote by U_h the unitary operator of multiplication by ϕ_h on $L^2(\mathbb{D})$. Multiply both sides by $|gu(\zeta)|$, integrate on $\mathbb{D}$ w.r.t. $dA(\zeta)$, and note that

$$\partial Ru(z) = \int_{\mathbb{D}} \frac{|gu(\zeta)|\overline{\zeta}}{(1-\overline{\zeta}z)|1-\overline{\zeta}z|^2}dA(\xi)\,.$$

Using the above notations we obtain

$$\begin{aligned}|f|Tu(z) &= \frac{1}{z}|f|Ru(z) - (1-|z|^2)|f|\partial Ru(z) \\ &= -(U_fT_fT_g^*U_g^*M^*|u|)(z) + \frac{1}{z}(C^1_{f,g}|u|(z)) + M^*(C^1_{g,f})^*|u|(z) \\ &+ (M^*MC^3_{f,g}M^*|u|)(z)\,.\end{aligned}$$

If we let $u = g1_E$ then the first estimate in the statement follows directly by Lemma 3.3 together with the fact that $b_{f,g} = b_{g,f}$ and (3.6). The proof of the second estimate is similar. We rewrite the second identity in Lemma 3.2 as

$$\frac{1}{|1-\overline{\zeta}z|^2} = -\mathrm{Re}\frac{\overline{\zeta}z}{(1-\overline{\zeta}z)^2} + \frac{1-|z|^2}{2|1-\overline{\zeta}z|^2} + \frac{|z|^2(1-|\zeta|^2)}{2|1-\overline{\zeta}z|^2} + \frac{(1-|\zeta|^2)^2}{2|1-\overline{\zeta}z|^4}$$

$$+\frac{(1-|z|^2)(1-|\zeta|^2)(|\zeta|^2+|\zeta|^2)}{2|1-\overline{\zeta}z|^4}+\frac{(1-|z|^2)^2|\zeta|^4}{2|1-\overline{\zeta}z|^4},$$

multiply both sides by $|gu(\zeta)|$, integrate on $\mathbb{D}$ w.r.t. $dA(\zeta)$, and note that

$$(1-|z|^2)^2\partial\overline{\partial}Ru(z)=(1-|z|^2)^2\int_{\mathbb{D}}\frac{|gu(\zeta)||\zeta|^2}{|1-\overline{\zeta}z|^4}dA(\zeta)\,.$$

Thus with the notations above we have

$$\begin{aligned}|f|Su(z)&=-\mathrm{Re}(MU_fT_fT_g^*U_g^*M^*|u|)(z)+\frac{1}{2}C^1_{f,g}|u|(z)+\frac{1}{2}M^*M(C^1_{f,g})^*|u|(z)\\&+\frac{1}{2}C^4_{f,g}|u|(z)+\frac{1}{2}(C^2_{f,g}(2M^*M)|u|)(z)+\frac{1}{2}((I-M^*M)C^2_{f,g}M^*M|u|)(z)\,.\end{aligned}$$

If we let $u=g1_E$ then the result follows by another application of Lemma 3.3. $\square$

With the lemmas in hand we can now proceed to the proof of our theorem.

Proof of Theorem 3.1. Of course, the interesting part is to prove the boundedness of $P^+_{f,g}$ under the assumption that $T_fT_g^*$ is bounded. By Corollary 2.17 it suffices to show that

$$\|P^+_{f,g}|g|1_{Q_I}\|_2\lesssim\||g|1_{Q_I}\|_2\,,\quad\|P^+_{g,f}|f|1_{Q_I}\|_2\lesssim\||f|1_{Q_I}\|_2\,,\tag{3.7}$$

for all Carleson boxes Q_I with I an interval in $\mathbb{T}$. To this end, let us assume first that $f(0)=0$, and that $u\in L^2(\mathbb{D})$ is compactly supported. We shall focus our attention on the function

$$\mathbb{E}(z)=(1-|z|^2)^2\partial\overline{\partial}(P^+_{f,g}|u|)^2(z)-\mathrm{Re}\left(\frac{(1-|z|^2)}{z}\overline{\partial}(P^+_{f,g}|u|)^2(z)\right)\tag{3.8}$$

The standard growth estimate for Bergman space functions (see page 54 in [11]) shows that under our assumptions we can apply Stokes' formula and one of the Green's identities to conclude that

$$\begin{aligned}\int_{\mathbb{D}}\mathbb{E}(z)dA(z)&=\int_{\mathbb{D}}(P^+_{f,g}|u|)^2(z)(\partial\overline{\partial}(1-|z|^2)^2)+\mathrm{Re}\left(\frac{1}{z}\overline{\partial}(1-|z|^2)\right)dA(z)\\&=\int_{\mathbb{D}}(P^+_{f,g}|u|)^2(z)(4|z|^2-3)dA(z)\,,\end{aligned}$$

so that

$$\int_{\mathbb{D}}\mathbb{E}(z)dA(z)\le\int_{\mathbb{D}}(P^+_{f,g}|u|)^2(z)dA(z)\,.\tag{3.9}$$

With the notation in Lemma 3.4 we have $(P^+_{f,g}|u|)^2=|f|^2R^2u$, and a direct computation gives

$$\overline{\partial}(P^+_{f,g}|u|)^2=\overline{f'}fR^2u+2|f|^2Ru\overline{\partial}Ru\,.$$

In the formulas below we will commit a convenient abuse of notation and write z also for the identity function on $\mathbb{D}$. Use Lemma 3.4 to obtain

$$\frac{1}{z}\overline{\partial}(P^+_{f,g}|u|)^2(z)=\frac{1}{z}\overline{f'}(z)f(z)R^2u(z)+2(1-|z|^2)|f|^2(z)\partial Ru(z)\overline{\partial}Ru(z)+2|f|^2(z)Tu(z)\overline{\partial}Ru(z)\,.$$

Obviously, $(1-|z|^2)|\overline{\partial}Ru|\le 2Ru$, and $\partial Ru\overline{\partial}Ru\ge 0$, hence

$$\begin{aligned}\text{(3.10)}\quad \mathrm{Re}\left(\frac{(1-|z|^2)}{z}\overline{\partial}(P^+_{f,g}|u|)^2(z)\right)&\le(1-|z|^2)\mathrm{Re}\frac{1}{z}\overline{f'}(z)f(z)R^2u(z)\\&+2(1-|z|^2)^2|f|^2\partial Ru\overline{\partial}Ru+4|f|^2Ru|Tu|\,.\end{aligned}$$

Similarly, we compute

$$\partial\overline{\partial}(P^+_{f,g}|u|)^2=|f'|^2R^2u+4\mathrm{Re}\overline{f'}fRu\partial Ru+2|f|^2\partial Ru\overline{\partial}Ru+2|f|^2Ru\partial\overline{\partial}Ru\,.$$

and apply Lemma 3.4 to obtain

$$\begin{aligned}\text{(3.11)}\quad (1-|z|^2)^2\partial\overline{\partial}(P^+_{f,g}|u|)^2(z)&=(1-|z|^2)^2|f'|^2(z)R^2u(z)\\&+4(1-|z|^2)\mathrm{Re}\left(\frac{1}{z}\overline{f'}(z)f(z)R^2u(z)\right)+2(1-|z|^2)^2|f|^2(z)\partial Ru(z)\overline{\partial}Ru(z)\\&+4|f|^2(z)R^2u(z)-4(1-|z|^2)\mathrm{Re}\left(\overline{f'}(z)f(z)Ru(z)Tu(z)\right)-4|f|^2(z)Ru(z)Su(z)\,.\end{aligned}$$

¿From (3.10) and (3.11) we have

$$\begin{aligned}\text{(3.12)}\quad \mathbb{E}&\ge(1-|z|^2)^2|f'|^2(z)R^2u(z)+3(1-|z|^2)\mathrm{Re}\left(\frac{1}{z}\overline{f'}(z)f(z)R^2u(z)\right)\\&+4|f|^2(z)R^2u(z)-4(1-|z|^2)\mathrm{Re}\left(\overline{f'}(z)f(z)Ru(z)Tu(z)\right)\\&-4|f|^2(z)Ru(z)Su(z)-4|f|^2(z)Ru(z)|Tu(z)|\,.\end{aligned}$$

Fix $\delta\in(\frac{3}{4},1)$ and use the inequalities

$$\begin{aligned}\delta(1-|z|^2)^2|f'|^2(z)R^2u(z)+3(1-|z|^2)\mathrm{Re}\left(\frac{1}{z}\overline{f'}(z)f(z)R^2u(z)\right)&\ge-\frac{9}{4\delta|z|^2}|f|^2(z)R^2u(z)\\&=-\frac{9}{4\delta}|f|^2(z)R^2u(z)-\frac{9}{4\delta|z|^2}|f|(z)Ru(z)C^1_{f,g}|u|\,,\end{aligned}$$

$$(1-\delta)(1-|z|^2)^2|f'|^2(z)R^2u(z)-4(1-|z|^2)\mathrm{Re}\left(\overline{f'}(z)f(z)Ru(z)Tu(z)\right)\ge-\frac{4}{1-\delta}|f|^2(z)|Tu|^2(z)$$

that come from completing squares to conclude that

$$\begin{aligned}\mathbb{E}(z)&\ge(4-\frac{9}{4\delta})|f|^2(z)Ru^2(z)-\frac{4}{1-\delta}|f|^2(z)|Tu|^2(z)-4|f|^2(z)Ru(z)Su(z)\\&-4|f|^2(z)Ru(z)|Tu|(z)-\frac{9}{4\delta|z|^2}|f|(z)Ru(z)C^1_{f,g}|u|(z)\,.\end{aligned}$$

Now recall that $|f|^2R^2u = (P^+_{f,g}|u|)^2$, and use the previous inequality in (3.9) together with the Cauchy-Schwartz inequality and the estimates (3.5) and (3.6) to obtain

$$\begin{aligned}(3-\frac{9}{4\delta})\|P^+_{f,g}|u|\|_2^2 &\le \frac{4}{1-\delta}\|fTu\|_2^2 + 4\|P^+_{f,g}|u|\|_2\|fSu\|_2\\ &+ 4\|P^+_{f,g}|u|\|_2\|fTu\|_2 + \frac{9k}{4\delta}\|P^+_{f,g}|u|\|_2\|C^1_{f,g}|u|\|_2\,,\end{aligned}$$

where k is the constant in (3.5). Now let $u = g1_E$ for a measurable set E with $\overline{E}\subset\mathbb{D}$. By the last inequality and the lemmas 3.4 and 3.3 we have

$$\begin{aligned}(3-\frac{9}{4\delta})\|P^+_{f,g}|g|1_E\|_2^2 &\lesssim (\|T_fT_g^*\| + b_{f,g})^2\|g1_E\|_2^2\\ &+ \|P^+_{f,g}|g|1_E\|_2(\|T_fT_g^*\|+b_{f,g})\|g1_E\|_2 + \|P^+_{f,g}|g|1_E\|_2^{3/2}b_{f,g}^{1/2}\|g1_E\|_2^{1/2}\,.\end{aligned}\tag{3.13}$$

Since $(3-\frac{9}{4\delta})>0$, this immediately implies that

$$\|P^+_{f,g}|g|1_E\|_2 \lesssim (\|T_fT_g^*\|+b_{f,g})\|g1_E\|_2\,.$$

The assumption that $\overline{E}\subset\mathbb{D}$ is easily removed by an approximation argument and Fatou's lemma, while the assumption $f(0)=0$ can be removed by another use of (3.5). Finally, the remaining estimate in (3.7) is obtained by interchanging f and g, so that the proof is complete. □

4. A counterexample to Sarason's conjecture for Bergman space

Recall that for $f,g\in A^2(\mathbb{D})$, we have denoted by $b_{f,g}$ the supremum of the product of the Berezin transforms of $|f|^2$ and $|g|^2$. In this section we will prove Theorem 1.15. The proof requires several steps. We begin with the following notations. The Dirichlet space D consists of analytic functions u in $\mathbb{D}$ whose derivative belongs to $A^2(\mathbb{D})$, and the norm is defined by

$$\|u\|_D^2 = |u(0)|^2 + \|u'\|_2^2\,.$$

Given $f\in A^2(\mathbb{D})$ we denote by

$$\gamma^2(f) = \sup_I \log\frac{2\pi}{|I|}\int_{Q_I}|f|^2dA\,,$$

where the supremum is taken over all arcs $I\subset\mathbb{T}$, and by

$$\delta^2(f) = \sup_{\|u\|_D\le 1}\int_{\mathbb{D}}|fu|^2dA\,.$$

It is well known and easy to prove that

$$\gamma(f)\lesssim\delta(f)\,.$$

The fact that these quantities are not comparable was discovered by Stegenga [31] and will play an essential role in our argument.

The next lemma relates these numbers to the boundedness of Toeplitz products and products of Berezin transforms.

Lemma 4.1. *Let $f \in A^2(\mathbb{D})$, and let g be a Lipschitz analytic function in $\mathbb{D}$ with*

$$|g(z)| \geq c(1-|z|)\,, \tag{4.2}$$

for some constant $c > 0$ and all $z \in \mathbb{D}$.
(i) If $fg \in H^\infty$ and $\gamma(f) < \infty$ then $b_{f,g} < \infty$.
(ii) If $T_f T_g^*$ is bounded then $\delta(f) < \infty$.

Proof. (i) Since g is Lipschitz we have

$$\begin{aligned} B(|g|^2)(z) &\lesssim |g(z)|^2 + (1-|z|^2)^2 \int_{\mathbb{D}} \frac{|g(\zeta)-g(z)|^2}{|1-\bar{\zeta}z|^4} dA(\zeta) \\ &\lesssim |g(z)|^2 + (1-|z|^2)^2 \int_{\mathbb{D}} \frac{1}{|1-\bar{\zeta}z|^2} dA(\zeta) \\ &\lesssim |g(z)|^2 + (1-|z|^2)^2 \log\frac{2}{1-|z|}\,. \end{aligned}$$

Similarly,

$$\begin{aligned} |g(z)|^2 B(|f|^2)(z) &\lesssim (1-|z|^2)^2 \int_{\mathbb{D}} \frac{|g(\zeta)-g(z)|^2|f(\zeta)|^2}{|1-\bar{\zeta}z|^4} dA(\zeta) + B(|fg|^2)(z) \\ &\lesssim \|f\|_2^2 + \|fg\|_\infty^2\,. \end{aligned}$$

Moreover, let

$$A_k(z) = \mathbb{D} \cap \{2^k(1-|z|) \leq |1-\bar{z}\zeta| \leq 2^{k+1}(1-|z|)\}\,,$$

and note that

$$\begin{aligned} (1-|z|)^2 \log\frac{2}{1-|z|} B(|f|^2)(z) &\sim \log\frac{2}{1-|z|} \sum_{2^k(1-|z|)\leq 2} 2^{-4k} \int_{A_k(z)} |f|^2 dA \\ &\leq \sum_{2^k(1-|z|)\leq 2} \left(\log\frac{2}{2^k(1-|z|)} + k\right) 2^{-4k} \int_{A_k(z)} |f|^2 dA\,, \end{aligned}$$

Each set $A_k(z)$ is contained in a Carleson box of perimeter comparable to $2^k(1-|z|)$. Indeed, if $2^k(1-|z|) > \frac{1}{2}$, we take the box to be the whole unit disc and if $2^k(1-|z|) \leq \frac{1}{2}$, we note that $A_k(z)$ is the intersection of the unit disc with an annulus centered at $\frac{1}{\bar{z}}$, which is contained in a disc centered at $\frac{\bar{z}}{|z|}$, of radius comparable to $2^k(1-|z|)$. Then it is easy to see that such a disc is contained in a Carleson box with a comparable perimeter.

$$\left(\log\frac{2}{2^k(1-|z|)} + k\right) \int_{A_k(z)} |f|^2 dA \lesssim \gamma^2(f) + k\|f\|_2^2$$

which implies

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^2 \log\frac{2}{1-|z|} B(|f|^2)(z) \lesssim \gamma^2(f) + \|f\|_2^2\,.$$

Thus

$$b_{f,g} \lesssim \|f\|_2^2 + \|fg\|_\infty^2 + \gamma^2(f)\,.$$

(ii) Let

$$Ru(z) = \int_{\mathbb{D}} \frac{(1-|\zeta|)u(\zeta)}{(1-\bar{\zeta}z)^2} dA(\zeta)\,, \quad u \in A^2(\mathbb{D})\,, z \in \mathbb{D}\,.$$

It is well known and easy to show that R is a bounded invertible operator from $A^2(\mathbb{D})$ onto the Dirichlet space D. Moreover, if g satisfies (4.2) we have the obvious inequality

$$|f(z)Ru(z)| \le P^+_{f,g}|u|(z)\,.$$

If $T_f T_g^*$ is bounded then by Theorem 3.1 we have

$$\|P^+_{f,g}|u|\|_2 \lesssim \|u\|_2\,,$$

for all $u \in A^2(\mathbb{D})$, hence, by the above argument

$$\|fv\|_2 \lesssim \|v\|_D\,,$$

for all $v \in D$, and the proof is complete. □

We now construct a special Lipschitz function g with the property (4.2). Consider sequences $\alpha = (\alpha_j)$, where all but finitely many terms are zero, and the remaining ones are equal to one. Let

$$\lambda_0 = 1\,, \quad \lambda_j = 2^{-2^j}\,, j = 1, 2 \ldots\,.$$

Given a sequence α as above, let

$$x_\alpha = \sum_{j\ge 1} \alpha_j (1-\lambda_j)\lambda_0 \cdots \lambda_{j-1}\,,$$

and let $E_1 \subset \mathbb{R}$ be the closure of the set of points x_α. Finally, let E be the preimage of E_1 by the conformal map $\phi(z) = i\frac{1+z}{1-z}$, from the unit disc onto the upper half-plane. The following lemma is a direct application of a result in [8] and was suggested to us by Konstantin Dyakonov.

Lemma 4.3. *There exists a Lipschitz analytic function g in $\mathbb{D}$ which satisfies* (4.2) *and vanishes on $E \cup \{1\}$.*

Proof. We claim that E satisfies the condition (K) in [8], that is

$$|I| \lesssim \sup_{z\in I} dist(z, E)\,,$$

for all arcs $I \subset \mathbb{T}$. If we assume the claim, then by Theorem 4 in [8] there exists an outer function $w_{1/2}$ in $\mathbb{D}$ such that

$$|w_{1/2}(z)| \sim dist(z, E)^{1/2}\,, \quad |w'_{1/2}(z)| \lesssim dist(z, E)^{-1/2}\,, \quad z \in \mathbb{D}\,.$$

If we set $g_1 = w^2_{1/2}$ then clearly,

$$(1-|z|) \le dist(z, E) \sim |g_1(z)|\,, \quad |g'_1(z)| = 2|w'_{1/2}w_{1/2}(z)| \lesssim 1\,, \quad z \in \mathbb{D}\,,$$

i.e. g_1 is Lipschitz, vanishes on E and satisfies (4.2). Since $1 \notin E$ it follows that $g(z) = (1-z)g_1(z)$ has the properties required in the statement.

To verify the claim, note first that since ϕ^{-1} is analytic and one-to-one in a neighborhood of E_1, it will suffice to verify the condition (K) for E_1 and all intervals $I \subset \mathbb{R}$. To this end, we use the obvious inequality

$$\tau = \sup_{j\geq 1} \frac{\sum_{m>j}(1-\lambda_m)\lambda_0 \cdots \lambda_{m-1}}{(1-\lambda_j)\lambda_0 \ldots \lambda_{j-1}} < \frac{1}{2}\,. \tag{4.4}$$

Indeed,

$$\frac{\sum_{m>j}(1-\lambda_m)\lambda_0 \cdots \lambda_{m-1}}{(1-\lambda_j)\lambda_0 \ldots \lambda_{j-1}} < \frac{\lambda_j}{1-\lambda_j}\left(1 + \sum_{m>j+1} 2^{-m}\right) < \frac{1}{3}(1+2^{-2})\,.$$

In particular, (4.4) shows that if $x_\alpha < x_\beta$ then there exists $j \geq 1$ such that $\beta_j - \alpha_j = 1$, and $\alpha_m = \beta_m$ for $m < j$. Moreover, in this case we have that

$$\left|\frac{x_\alpha + x_\beta}{2} - x_{\alpha'}\right| > k(x_\beta - x_\alpha)\,, \tag{4.5}$$

for some $k > 0$ independent of α, α', β. To see this note that the inequality holds with $k = \frac{1}{2}$, when $x_{\alpha'}$ lies outside (x_α, x_β). When $x_{\alpha'}$ lies inside this interval, with j given above we have by (4.4)

$$\begin{aligned}\left|\frac{x_\alpha + x_\beta}{2} - x_{\alpha'}\right| &> \frac{1}{2}(1-\lambda_j)\lambda_0 \cdots \lambda_{j-1} - \sum_{m>j}(1-\lambda_m)\lambda_0 \cdots \lambda_{m-1} \\ &> (\frac{1}{2} - \tau)(1-\lambda_j)\lambda_0 \cdots \lambda_{j-1} \\ &> \frac{\frac{1}{2} - \tau}{1+\tau}(x_\beta - x_\alpha)\,.\end{aligned}$$

Finally, (4.5) immediately implies (K). If (a, b) is any interval and

$$dist(a, E_1), dist(b, E_1) > \frac{1}{3}(b-a)\,,$$

then (K) holds with constant $\frac{1}{3}$. If

$$dist(a, E_1), dist(b, E_1) < \frac{1}{3}(b-a)\,,$$

we can find $x_\alpha, x_\beta \in E_1$ such that

$$|x_\alpha - a|\,, |x_\beta - b| < \frac{1}{3}(b-a),$$

and then $\frac{x_\alpha + x_\beta}{2} \in (a, b)$, and $x_\beta - x_\alpha > \frac{1}{3}(b-a)$, so that the result follows from above. □

Proof of Theorem 1.15. Assume the contrary. Fix a function g as in Lemma 4.3, this function clearly belongs to $A^2(\mathbb{D})$, and consider the space X_g consisting of functions $f \in A^2(\mathbb{D})$ with

$$\|f\| = \|fg\|_\infty + \gamma(f) < \infty\,.$$

It is obviously a Banach space. By Lemma 4.1 (i) we have $b_{f,g} < \infty$ for all $f \in X_g$, by assumption this implies that $T_f T_g^*$ is bounded on $A^2(\mathbb{D})$ whenever $f \in X_g$, and finally, by Lemma 4.1 (ii) we obtain that $\delta(f) < \infty,\ f \in X_g$. Since the space of functions $u \in A^2(\mathbb{D})$ with $\delta(u) < \infty$ and norm given by $u \to \delta(u)$ is at its turn a Banach space, we can apply the closed graph theorem to conclude that there exists $c > 0$ such that

$$\delta(f) \leq c(\|fg\|_\infty + \gamma(f)) , \tag{4.6}$$

for all $f \in X_g$. We will show that this leads to a contradiction.

Recall that $\phi(z) = i\frac{1+z}{1-z}$ is the conformal map from the unit disc onto the upper half-plane. With the notations preceding Lemma 4.3, D. Stegenga ([31], p. 136) has constructed a sequence (f_n) in $A^2(\mathbb{D})$ of the form

$$f_n(z) = 2^{-n/2} p_n \sum_{k=1}^{2^n} \frac{1}{(\phi(z) - z_{nk})^2} \phi'(z) ,$$

where $p_n = \lambda_0 \ldots \lambda_n$, $\text{Im} z_{nk} < 0$, with

$$dist(z_{nk}, E_1) = -\text{Im} z_{nk} \sim p_n \tag{4.7}$$

such that

$$\limsup_{n\to\infty} \gamma(f_n) < \infty , \tag{4.8}$$

$$\lim_{n\to\infty} \delta(f_n) = \infty , \tag{4.9}$$

$$\sup_{\substack{z\in\mathbb{D}\\ n\in\mathbb{N}}} 2^{-n/2} p_n \sum_{k=1}^{2^n} \frac{1}{|\phi(z) - z_{nk}|} < \infty . \tag{4.10}$$

A simple calculation gives

$$\frac{\phi'(z)}{\phi(z) - z_{nk}} = \frac{2i}{(z_{nk} + i)(1 - z)(z - \phi^{-1}(z_{nk}))} ,$$

and by (4.7) there exist points $\zeta_{nk} \in E$ with

$$|\phi^{-1}(z_{nk}) - \zeta_{nk}| \sim |\phi^{-1}(z_{nk})| - 1 = dist(\phi^{-1}(z_{nk}), \mathbb{T}) .$$

Since g has the properties in Lemma 4.3,that is, it is Lipschitz and vanishes at 1, ζ_{nk}, it follows immediately that

$$\left| \frac{g(z)\phi'(z)}{\phi(z) - z_{nk}} \right| \leq C ,$$

for some absolute constant $C > 0$, all $k, n \in \mathbb{N}$ with $1 \leq k \leq 2^n$, and all $z \in \mathbb{D}$. From (4.10) we have that $f_n g$ are uniformly bounded in H^∞. Thus by (4.8) we have that (f_n) is a bounded sequence in X_g, hence (4.9) and (4.6) yield a contradiction which proves the theorem. $\square$

5. The class B_∞ and improved estimates in terms of the Békollé constant

In our last section we include an application of the two weight result for the maximal Bergman projection, namely we obtain sharp Békollé estimates by establishing sharp estimates for the testing conditions (2.15) and (2.16). We provide sharper estimates than the ones discussed by Pott and Reguera in [26].

5.1. The classes B_p and B_∞. Following Békollé and Bonami [5], we say that a weight, i.e., a measurable positive function w, belongs to the class B_p for $1 < p < \infty$, if and only if

$$B_p(w) := \sup_{\substack{I \text{ interval} \\ I \subset \mathbb{T}}} \left(\frac{1}{|Q_I|} \int_{Q_I} w dA \right) \left(\frac{1}{|Q_I|} \int_{Q_I} w^{1-p'} dA \right)^{p-1} < \infty \tag{5.1}$$

The following result was proved in [26]:

Theorem 5.2. *Let $w \in B_2$ be a Bekollé weight with constant $B_2(w)$ and let P_B^+ be the positive Bergman projection. Then*

$$\|P_B^+ f\|_{L^2(w)} \leq C B_2(w) \|f\|_{L^2(w)}, \tag{5.3}$$

with C independent of the weight w. Moreover, this result is sharp, in the sense that the power of the Bekollé constant $B_2(w)$ cannot be improved.

However, in the following we will improve this result by replacing the B_2 constant partially by an appropriately defined B_∞ constant.

Definition 5.4. We say that a weight w belongs to the class B_∞, if and only if

$$B_\infty(w) := \sup_{\substack{I \text{ interval} \\ I \subset \mathbb{T}}} \frac{1}{w(Q_I)} \int_{Q_I} M(w 1_{Q_I}) < \infty, \tag{5.5}$$

where M stands for the Hardy-Littlewood maximal function over Carleson cubes.

This definition of B_∞ is motivated by the version of the Muckenhoupt A_∞ condition given by Fujii in [9] and Wilson in [34–36]. This A_∞ definition appears in the recent works of Lerner [19], Hytönen and Pérez [14] and Hytönen and Lacey [13] among others, where it is used to find sharp estimates in terms of the Muckenhoupt A_p and A_∞ constants.

In particular, B_∞ contains any of the classes B_p:

Proposition 5.6. *Let w be a weight and $1 < p < \infty$. Then*

$$B_\infty(w) \leq B_p(w).$$

Proof. Let $w \in B_p$ and recall that $B_p(w) = B_{p'}(w')^{\frac{1}{p'-1}}$, where $w' = w^{1-p'}$. Hence for any Carleson cube Q_I,

$$\begin{aligned} \int_{Q_I} M(1_{Q_I} w) &\leq \left(\int_{Q_I} M(1_{Q_I} w)^{p'} w' \right)^{1/p'} \left(\int_{Q_I} w \right)^{\frac{1}{p}} \\ &\leq \|M(w\cdot)\|_{L^{p'}(w) \to L^{p'}(w')} w(Q_I)^{1/p'} w(Q_I)^{1/p} \end{aligned}$$

$$= \|M\|_{L^{p'}(w')\to L^{p'}(w')} w(Q_I) \le B_p(w')^{\frac{1}{p'-1}} w(Q_I),$$

where we have used the estimate (4.7) from [26] for the maximal function in the last line. □

5.2. $B_2 - B_\infty$ **estimates.** The main result in this section is the following:

Theorem 5.7. *Let $w \in B_2$ be a Bekollé weight with constant $B_2(w)$ and let P_B^+ be the positive Bergman projection. Then*

$$\|P_B^+ f\|_{L^2(w)} \le C B_2(w)^{1/2}(B_\infty(w)^{1/2} + B_\infty(w^{-1})^{1/2})\|f\|_{L^2(w)}, \tag{5.8}$$

with C independent of the weight w.

Corollary 5.9. *The same result holds for the Bergman projection P_B.*

The method of proof will be as follows. We will consider the dyadic operators P^β and use Theorem 2.14 to obtain the sharp bound in the Békollé constants, which will be independent of the choice of the grid. An averaging operation will now yield the desired result.

The following lemma is known for Muckenhoupt A_∞ weights in case that the collection of cubes appearing in the sum is sparse, this can be found in [14]. In our case, we consider Carleson cubes associated to a fixed dyadic grid $\mathcal{D}^\beta$. This is a sparse family of cubes on the disc. The lemma reads as follows.

Lemma 5.10. *Let $\sigma \in B_\infty$, then*

$$\sum_{\substack{K:\, K\subset I\\ K\in\mathcal{D}^\beta}} \sigma(Q_K) \le 2B_\infty(\sigma)\sigma(Q_I). \tag{5.11}$$

Proof. Given $K \in \mathcal{D}^\beta$, we denote the top-half of the Carleson cube Q_K by T_K, that is, $T_K := \{re^{i\theta} :\ 1-|K| \le r < 1-|K|/2 \text{ and } e^{i\theta}\in K\}$, and $|K|$ stands for the normalized length of the interval. Notice that, given $I \in \mathcal{D}^\beta$, the top halves T_K where $K \subset I$ and $K \in \mathcal{D}^\beta$ tile the whole Carleson cube Q_I.

$$\begin{aligned}
\sum_{\substack{K:\, K\subset I\\ K\in\mathcal{D}^\beta}} \sigma(Q_K) &= \sum_{\substack{K:\, K\subset I\\ K\in\mathcal{D}^\beta}} \frac{\sigma(Q_K)}{|Q_K|}|Q_K| \\
&\le 2\sum_{\substack{K:\, K\subset I\\ K\in\mathcal{D}^\beta}} \frac{\sigma(Q_K)}{|Q_K|}|T_K| \\
&\le 2\sum_{\substack{K:\, K\subset I\\ K\in\mathcal{D}^\beta}} \int_{T_K} M(\sigma 1_{Q_I})dm \\
&\le 2B_\infty(\sigma)\sigma(Q_I)
\end{aligned}$$

□

We turn to proving the desired bound for the two testing conditions.

Proof of Theorem 5.7. We use Theorem 2.14 for the weights w and $\sigma = w' = w^{p'-1}$. We only have to show the appropriate bounds for the test function conditions, and we will only focus on one of the conditions, as the study of the other is analogous. In what follows, let $I \in \mathcal{D}^\beta$, all other intervals interfering in this proof also belong to $\mathcal{D}^\beta$. We want to prove

$$\Big\| \sum_{K\colon K\subset I} \langle w^{-1} 1_{Q_I}, \frac{1_{Q_K}}{|K|}\rangle \frac{1_{Q_K}}{|K|} \Big\|^2_{L^2(w)} \lesssim B_2(w) B_\infty(w^{-1}) w^{-1}(Q_I), \tag{5.12}$$

where the implicit constant does not depend on the chosen grid D^β or the weight w.

$$\begin{aligned}
&\Big\| \sum_{K\colon K\subset I} \langle w^{-1} 1_{Q_I}, \frac{1_{Q_K}}{|K|}\rangle \frac{1_{Q_K}}{|K|} \Big\|^2_{L^2(w)} = \int_{Q_I} \left| \sum_{K\colon K\subset I} \langle w^{-1} 1_{Q_I}, \frac{1_{Q_K}}{|K|}\rangle \frac{1_{Q_K}}{|K|} \right|^2 w dA \\
=& \int_{Q_I} \sum_{K\colon K\subset I} \langle w^{-1} 1_{Q_I}, \frac{1_{Q_K}}{|K|}\rangle^2 \frac{1_{Q_K}}{|K|^2} w dA \\
&+ 2 \int_{Q_I} \sum_{K'\colon K'\subset I} \sum_{K\colon K\subset K'} \langle w^{-1} 1_{Q_I}, \frac{1_{Q_K}}{|K|}\rangle \langle w^{-1} 1_{Q_I}, \frac{1_{Q_{K'}}}{|K'|}\rangle \frac{1_{Q_K}}{|K||K'|} w dA \\
:=& D + 2OD,
\end{aligned}$$

where the terminology for D and OD comes from the diagonal and the off-diagonal. Let us treat each term in turn.

$$\begin{aligned}
D &= \sum_{K\colon K\subset I} \frac{w^{-1}(Q_K)^2}{|K|^2} \frac{w(Q_K)}{|K|^2} \\
&\leq B_2(w) \sum_{K\colon K\subset I} w^{-1}(Q_K) \\
&\leq B_2(w) B_\infty(w^{-1}),
\end{aligned}$$

where the last inequality follows from Lemma 5.10. The off-diagonal term is equally simple,

$$\begin{aligned}
OD &= \sum_{K'\colon K'\subset I} \sum_{K\colon K\subset K'} \frac{w^{-1}(Q_{K'})}{|K'|} \frac{w^{-1}(Q_K)}{|K|} \frac{w(Q_K)}{|K||K'|} \\
&= \sum_{K'\colon K'\subset I} w^{-1}(Q_{K'}) \frac{1}{|K'|^2} \sum_{K\colon K\subset K'} \frac{w^{-1}(Q_K)}{|K|^2} \frac{w(Q_K)}{|K|^2} |K|^2 \\
&\leq B_2(w) \sum_{K'\colon K'\subset I} w^{-1}(Q_{K'}) \frac{1}{|K'|^2} \sum_{K\colon K\subset K'} |K|^2 \\
&\leq C B_2(w) \sum_{K'\colon K'\subset I} w^{-1}(Q_{K'})
\end{aligned}$$

$$\leq \quad 2CB_2(w)B_\infty(w^{-1}),$$

where in the second line we have multiplied and divided by $|K|^2$ to use the Bekollé constant. $\square$

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