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The large-time development of the solution to an initial-value problem for the generalized Burgers' equation.

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Abstract

In this paper, we consider an initial-value problem for the generalized Burgers' equation. The normalized Burgers' equation considered is given by

$$u_t + t^\delta u u_x = u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$

where $-\frac{1}{2} \leq \delta \neq 0$, and x and t represent dimensionless distance and time respectively. In particular, we consider the case when the initial data has a discontinuous step, where $u(x,0) = u_+$ for $x \geq 0$ and $u(x,0) = u_-$ for $x < 0$, where u_+ and u_- are problem parameters with $u_+ \neq u_-$. The method of matched asymptotic coordinate expansions is used to obtain the large-t asymptotic structure of the solution to this problem, which exhibits a range of large-t attractors depending on the problem parameters. Specifically, the solution of the initial-value problem exhibits the formation of (i) an expansion wave when $\delta > -\frac{1}{2}$ and $u_+ > u_-$, (ii) a Taylor shock (hyperbolic tangent) profile when $\delta > -\frac{1}{2}$ and $u_+ < u_-$, and (iii) the Rudenko-Soluyan similarity solution when $\delta = -\frac{1}{2}$.

1 Introduction

In this paper we consider the following initial-value problem for the variable coefficient Burgers' equation, namely,

$$u_t + t^\delta u u_x = u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{1.1}$$

$$u(x,0) = \begin{cases} u_-, & x < 0, \\ u_+, & x \geq 0, \end{cases} \tag{1.2}$$

$$u(x,t) \to \begin{cases} u_-, & x \to -\infty, \\ u_+, & x \to \infty, \end{cases} \qquad t \geq 0, \tag{1.3}$$

where u_- , u_+ ($\neq u_-$) and δ are problem parameters, and the initial distribution (1.2) is a discontinuous step. In what follows we label initial-value problem (1.1)-(1.3) as IVP. We will develop the large-time asymptotic solution to IVP for $-\frac{1}{2} \leq \delta \neq 0$ in both the expansive case when $u_+ > u_-$ and the compressive case when $u_+ < u_-$.

The variable coefficient Burgers' equation

$$u_t + \Phi(t)\, u u_x = \Psi(t)\, u_{xx}, \tag{1.4}$$

is related to the generalized Burgers' equation

$$v_\tau + v v_x + f(\tau) v = v_{xx}, \tag{1.5}$$

where $f(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau} \ln\left(\frac{\Psi}{\Phi}\right)$, via the transformation

$$v(x,\tau) = \frac{\Phi(t)}{\Psi(t)} u(x,t), \quad \tau = \int^t \Psi(s)\,\mathrm{d}s. \tag{1.6}$$

When $f(\tau) = 0$ equation (1.5) reduces to the classical Burgers' equation. Although this equation is named after J.M. Burgers for his work on the theory of turbulence [3], the equation had already appeared in the works of A.R Forsyth [11] and H. Bateman [2] which preceded Burgers work on turbulence. Burgers' equation is a canonical equation combining diffusion and nonlinear convection and as such arises in the modelling of many physical phenomena involving diffusion-convection processes ([27], [5], [23] and [13]). It can be reduced to the heat equation by the Cole-Hopf transformation ([5] and [13]) and then solved in a straightforward manner.

The generalized Burgers' equation (1.5) has many applications in mathematical physics and was introduced in [14]. For example, equation (1.5) models the propagation of finite-amplitude sound waves in variable area ducts ([8], [10], [21] and [24]). We note that equation (1.4) where the coefficient of u_{xx} alone is a function of time, given by

$$u_t + uu_x = \Psi(t)\, u_{xx}. \tag{1.7}$$

was introduced in [19] as a model of viscous effects in sound waves of finite amplitude, and derived in [12] as a model for the propagation of weakly nonlinear acoustic waves under the impact of geometrical spreading. J.F. Scott in [24] examined the large-time asymptotics of an initial value for (1.7) having continuous initial data $u(x,0) = u_0(x)$ such that $u_0(x) \to u_\pm$ as $x \to \pm\infty$. He identified three cases:

(i) if $\Psi(t)/t \to \infty$ as $t \to \infty$ (supercylindrical case), then the solution, $u(x,t)$, of the initial value problem tends to an error function as $t \to \infty$.

(ii) if $\Psi(t)/t \to$ constant as $t \to \infty$ (cylindrical case), then the solution, $u(x,t)$, of the initial value problem tends to the Rudenko-Soluyan similarity solution [22] as $t \to \infty$.

(iii) if $\Psi(t)/t \to 0$ as $t \to \infty$ (subcylindrical case), then the solution, $u(x,t)$, of the initial value problem tends to an expansion wave when $u_+ > u_-$ or a Taylor Shock profile when $u_+ < u_-$, as $t \to \infty$

It should be noted that in many applications (see for example, [25] and [7]) the coefficient of u_{xx}, $\Psi(t)$, is regularly approximated by a constant. The basis for this assumption often arises from the fact that diffusivity is only needed in thin shock regions (see [19]). However, it was pointed out by D.G. Crighton in [6] that shocks when they occur may not be thin, and in such cases one needs to consider the full time dependent effective viscosity, $\Psi(t)$. This class of generalized Burgers' equations have also been analyzed using the similarity method (see for example, [9] and [4]). The generalized Burgers' equation (1.5) is therefore central to the study of a number of applications. However, it has been established in [20] that there is no Bäcklund transformation for the generalized Burgers' equation (1.5), and hence it is doubtful that a linearizing transformation like the Cole-Hopf transformation exists in this case. Therefore, other methods of solution need to be investigated for this important class of equation.

Our interest in equation (1.1) is motivated by its relationship, via transformation (1.6), to the generalized Burgers' equation (1.5). However, as we shall see it is more convenient for the analysis presented in this paper to work with equation (1.1) rather than equation (1.5). In a recent paper [15] the large-time solution of an initial value problem for (1.4) with continuous initial data $u(x,0) = u_0(x)$, where $u_0(x) \to u_\pm$ as $x \to \pm\infty$, was examined in the two cases

	$u_+ < u_-$	$u_+ > u_-$
$\delta > -\frac{1}{2}$	Taylor Shock	Expansion Wave
$\delta = -\frac{1}{2}$	Rudenko-Soluyan Similarity Solution	

Table 1: The type of large-time attractor connecting u_+ to u_- in the solution of IVP as $t \to \infty$.

(i) $\Phi(t) = e^t, \quad \Psi(t) = 1,$

(ii) $\Phi(t) = 1, \quad \Psi(t) = e^t.$

We note that case (i) corresponds to equation (1.5) with $f(\tau) = \frac{1}{\tau}$, while case (ii) corresponds to equation (1.5) with $f(\tau) = -1$. As in [15], in this present paper we use the method of matched asymptotic coordinate expansions (see for example [28], [16], [17] and [18]) to obtain the complete large-time solution of IVP over all parameter values. We observe that equation (1.1) corresponds to equation (1.5) with $f(\tau) = \frac{\lambda}{\tau}$ (where λ is a constant), where $\delta = -\frac{\lambda}{\lambda+1}$ $(\in (-1, \infty))$. We note that $\delta = -\frac{1}{2}$ corresponds to Scott's cylindrical case, while $\delta > -\frac{1}{2}$ corresponds to Scott's sub-cylindrical case. Throughout we use the nomenclature of the method of matched asymptotic expansions, as given in [28]. We note that the initial data (1.2) is a discontinuous step and we place a restriction on the parameter δ, in this analysis we consider $\delta \geq -\frac{1}{2}$ and examine the cylindrical and sub-cylindrical cases. The large-time structure of the solution of the initial-value problem is obtained by careful consideration of the asymptotic structures as $t \to 0$ $(-\infty < x < \infty)$ and as $|x| \to \infty$ $(t \geq O(1))$. The form of the large-time attractor for the solution, $u(x,t)$, of IVP as $t \to \infty$ depends on the problem parameters u_+, u_- and δ, and is summarized in Table 1. Complete details of the large-time solution of IVP in each case are given in Section 2 with a summary of the main results being given in Section 3. The results presented are in agreement with Scott's results above for continuous initial data. However, they extend Scott's results by linking the initial data (through the asymptotic structures developed for $t \to 0$ $(-\infty < x < \infty)$ and as $|x| \to \infty$ $(t \geq O(1))$) to the large-time attractor for the solution of IVP as $t \to \infty$. In particular, in the case when $\delta > -\frac{1}{2}$ and $u_+ > u_-$ Scott postulated that the solution to IVP would approach the Talyor shock profile as $t \to \infty$, but was unable to determine the location of the wave-front (see equation (5.27) of [24]). In this paper we establish in this case that the solution to IVP approaches the Taylor shock profile as $t \to \infty$, and that the wave-front of this Taylor shock is located at $x = s(t)$ as $t \to \infty$, where $s(t)$ is given by (3.1).

Finally, to make the results presented in this paper readily accessible to equation (1.4) with $\Phi(t) = t^\kappa$ $(-1 < \kappa \neq 0)$ and $\Psi(t) = t^\omega$ $(-1 < \omega \neq 0)$ we note that equation (1.4) can when $\kappa \neq \omega$ be transformed to equation (1.1) (on dropping the over bars) by

$$u = (\omega+1)^{-\delta}\,\bar{u}, \quad t = \Big((\omega+1)\bar{t}\Big)^{\frac{1}{\omega+1}},$$

where $\delta = \frac{\kappa-\omega}{\omega+1}$ $(\in (-1, \infty))$.

2 Asymptotic solution to IVP as $t \to \infty$

In this section we develop the asymptotic structure of the solution to IVP as $t \to \infty$. To achieve this we must begin by examining the asymptotic structure of the solution to IVP as $t \to 0$.

2.1 Asymptotic solution to IVP as $t \to 0$

Consideration of the initial data (1.2) indicates that the structure of the asymptotic solution to IVP as $t \to 0$ has three asymptotic regions for $x \in (-\infty, \infty)$, namely,

$$\left.\begin{array}{lll} \text{region I:} & x = o(1), & u(x,t) = O(1) \\ \text{region II}^+: & x = O(1)\ (>0), & u(x,t) = u_+ \pm o(1) \\ \text{region II}^-: & x = O(1)\ (<0), & u(x,t) = u_- \pm o(1) \end{array}\right\} \quad \text{as} \quad t \to 0. \tag{2.1}$$

We first consider region I, in which $x = o(1)$ and $u(x,t) = O(1)$ as $t \to 0$. To examine region I, we introduce the scaled coordinate $\eta = xt^{-\gamma} = O(1)$ as $t \to 0$, with $\gamma > 0$, and look for an expansion of the form

$$u(\eta, t) = \bar{u}(\eta) + o(1), \tag{2.2}$$

as $t \to 0$ with $\eta = O(1)$. On substitution of expansion (2.2) into equation (1.1) (when written in terms of η and t) we find that there are two cases to consider, these being $\delta = -\frac{1}{2}$ and $\delta > -\frac{1}{2}$, in both cases the most structured leading order balance (that balance which retains the most terms at leading order in the resulting differential equation) requires $\gamma = \frac{1}{2}$. At leading order we then obtain

$$\bar{u}_{\eta\eta} + \frac{\eta}{2}\bar{u}_\eta = 0 \quad \text{when} \quad \delta > -\frac{1}{2}, \tag{2.3}$$

and

$$\bar{u}_{\eta\eta} - \bar{u}\bar{u}_\eta + \frac{\eta}{2}\bar{u}_\eta = 0 \quad \text{when} \quad \delta = -\frac{1}{2}, \tag{2.4}$$

where $\eta = O(1)$. Equations (2.3) when $\delta > -\frac{1}{2}$ and (2.4) when $\delta = -\frac{1}{2}$ have to be solved subject to matching with regions II$^+$ (as $\eta \to \infty$) and II$^-$ (as $\eta \to -\infty$) and initial condition (1.2). This requires in both cases that

$$\bar{u}(\eta) \to \begin{cases} u_+ & \text{as} \quad \eta \to \infty, \\ u_- & \text{as} \quad \eta \to -\infty. \end{cases} \tag{2.5}$$

We now consider the cases $\delta > -\frac{1}{2}$ and $\delta = -\frac{1}{2}$ separately.

2.1.1 $\delta > -\frac{1}{2}$

The solution to (2.3), (2.5) is readily obtained as

$$\bar{u}(\eta) = u_+ + \frac{(u_- - u_+)}{2}\operatorname{erfc}\left(\frac{\eta}{2}\right), \quad -\infty < \eta < \infty. \tag{2.6}$$

In summary, when $\delta > -\frac{1}{2}$, we have in region I that $x = O(t^{\frac{1}{2}})$ as $t \to 0$, and that

$$u(\eta, t) = \left(u_+ + \frac{(u_- - u_+)}{2}\operatorname{erfc}\left(\frac{\eta}{2}\right)\right) + o(1), \tag{2.7}$$

as $t \to 0$ with $\eta = xt^{-\frac{1}{2}} = O(1)$, and where erfc(.) is the standard complementary error function (see [1]). From (2.7) we observe that

$$u(\eta, t) \sim \begin{cases} u_+ + \frac{(u_- - u_+)}{\eta\sqrt{\pi}} e^{-\frac{\eta^2}{4}} + \ldots & \text{as} \quad \eta \to \infty, \\ u_- - \frac{(u_- - u_+)}{(-\eta)\sqrt{\pi}} e^{-\frac{\eta^2}{4}} + \ldots & \text{as} \quad \eta \to -\infty. \end{cases} \tag{2.8}$$

As $\eta \to \infty$ we move into region II$^+$, where $x = O(1)\,(>0)$ as $t \to 0$. The form of expansion (2.7) for $\eta \gg 1$ given by $(2.8)_1$ suggests that in region II$^+$ we write

$$u(x, t) = u_+ + \mathcal{S}e^{\phi(x,t)} \tag{2.9}$$

with

$$\phi(x,t) = \begin{cases} \phi_0(x)t^{-1} + \phi_1(x)\ln t + \phi_2(x) + o(1) & \text{when} \quad \delta > 0, \\ \phi_0(x)t^{-1} + \phi_1(x)t^{\delta} + \phi_2(x)\ln t + \phi_3(x) + o(1) & \text{when} \quad -\frac{1}{2} < \delta < 0, \end{cases} \tag{2.10}$$

where $x = O(1)\,(> 0)$ as $t \to 0$, $\phi_0(x) < 0$ for $x > 0$ and $\mathcal{S} = \text{sign}(u_- - u_+)$. On substituting (2.9) (with $(2.10)_1$ when $\delta > 0$ or $(2.10)_2$ when $-\frac{1}{2} < \delta < 0$) into equation (1.1) and solving at each order in turn, we find (after matching with (2.7) as $x \to 0^+$) that

$$u(x,t) = u_+ + \begin{cases} \mathcal{S}\exp\left(-\frac{x^2}{4t} + \frac{1}{2}\ln t - \ln x + \ln\left(\frac{|u_- - u_+|}{\sqrt{\pi}}\right) + O\left(t^{\delta}\right)\right) \\ \qquad\qquad \text{when} \quad \delta > 0, \\ \mathcal{S}\exp\left(-\frac{x^2}{4t} + \frac{u_+}{2(\delta+1)}xt^{\delta} + \frac{1}{2}\ln t - \ln x + \ln\left(\frac{|u_- - u_+|}{\sqrt{\pi}}\right) + O\left(t^{2\delta+1}\right)\right) \\ \qquad\qquad \text{when} \quad -\frac{1}{2} < \delta < 0, \end{cases} \tag{2.11}$$

as $t \to 0$ with $x = O(1)(> 0)$, and where $\mathcal{S} = \text{sign}(u_- - u_+)$. Finally, we consider region II^-, where $x = O(1)\,(< 0)$ as $t \to 0$. The details regarding region II^- follow, after minor modification, those given for region II^+ above and are not repeated here. In summary, we have in region II^- that

$$u(x,t) = u_- - \begin{cases} \mathcal{S}\exp\left(-\frac{x^2}{4t} + \frac{1}{2}\ln t - \ln(-x) + \ln\left(\frac{|u_- - u_+|}{\sqrt{\pi}}\right) + O\left(t^{\delta}\right)\right) \\ \qquad\qquad \text{when} \quad \delta > 0, \\ \mathcal{S}\exp\left(-\frac{x^2}{4t} + \frac{u_+}{2(\delta+1)}xt^{\delta} + \frac{1}{2}\ln t - \ln(-x) + \ln\left(\frac{|u_- - u_+|}{\sqrt{\pi}}\right) + O\left(t^{2\delta+1}\right)\right) \\ \qquad\qquad \text{when} \quad -\frac{1}{2} < \delta < 0, \end{cases} \tag{2.12}$$

as $t \to 0$ with $x = O(1)(< 0)$, and where $\mathcal{S} = \text{sign}(u_- - u_+)$. The asymptotic structure as $t \to 0$ is now complete in this case, with the expansions in regions I, II^+ and II^- providing a uniform approximation to the solution of IVP as $t \to 0$.

2.1.2 $\delta = -\frac{1}{2}$

On making the substitution

$$\bar{u} = \frac{1}{\sqrt{2}}\Omega, \quad \eta = \sqrt{2}\,\bar{z}, \tag{2.13}$$

equation (2.4) becomes

$$\Omega_{\bar{z}\bar{z}} - \Omega\Omega_{\bar{z}} + \bar{z}\,\Omega_{\bar{z}} = 0, \quad -\infty < \bar{z} < \infty. \tag{2.14}$$

which was analyzed by Rudenko and Soluyan in [22]. It is straightforward to establish (see for example [24] and [22]) that equation (2.14) admits the family of implicit solutions

$$\left.\begin{aligned} \Omega(\bar{z}) &= \bar{z} - 2F(\xi) \\ \bar{z} &= \gamma + \int_{\xi_0}^{\xi} \frac{\mathrm{d}\xi}{F(\xi)} \end{aligned}\right\} \quad \xi \geq \xi_0, \tag{2.15}$$

where,

$$F(\xi) = \pm\left(\xi - \xi_0 e^{-2(\xi-\xi_0)}\right)^{\frac{1}{2}}, \tag{2.16}$$

and $\gamma, \xi_0\,(> -\frac{1}{2})$ are constants. The wave profile $\Omega(\bar{z})$ is bounded with constant boundary conditions as $|\bar{z}| \to \infty$, and consists of two parts (the plus sign in (2.16) is taken to give $\Omega(\bar{z})$ for $\bar{z} \geq \gamma$, while the negative sign is taken in (2.16) to give $\Omega(\bar{z})$ for $\bar{z} \leq \gamma$) which smoothly join at $\bar{z} = \gamma$. Further, the wave profile $\Omega(\bar{z})$ is monotonically increasing (decreasing) when ξ_0 is negative (positive) respectively. The limiting behaviour of (2.15) is given by

$$\Omega(\bar{z}) \to \gamma \pm \Delta(\xi_0) \quad \text{as} \quad \bar{z} \to \pm\infty$$

with the height of the wave being $2\Delta(\xi_0)$, where $\Delta(\xi_0)$ can be determined numerically. It was established in [22] that in the limits $\xi_0 \to 0$, $\xi_0 \to \infty$ and $\xi_0 \to -\frac{1}{2}$ the similarity solution (2.15) approaches an error function, Taylor shock (hyperbolic tangent) profile and expansion wave profile, respectively. In order to satisfy boundary conditions (2.5) we take $\gamma = \frac{\sqrt{2}(u_+ + u_-)}{2}$, and determine ξ_0 such that $\Delta(\xi_0) = \frac{\sqrt{2}(u_+ - u_-)}{2}$. We observe from (2.13) and (2.15) that

$$\bar{u}(\eta) = u_+ + O\left(\eta^{-1}\exp\left(-\frac{\eta^2}{4} + u_+\eta\right)\right) \quad \text{as} \quad \eta \to \infty. \tag{2.17}$$

The corresponding asymptotic form for $\bar{u}(\eta)$ as $\eta \to -\infty$ follows (2.17) with u_+ replaced by u_-.

As $\eta \to \infty$ we move into region II$^+$, where $x = O(1)\,(>0)$ as $t \to 0$. The details of region II$^+$ in this case follow, after minor modification, those given in Section 2.1.1 and are not repeated here for brevity. Therefore, in region II$^+$ we have that

$$u(x,t) = u_+ + \mathcal{S}_0 \exp\left(-\frac{x^2}{4t} + \frac{x\,u_+}{t^{\frac{1}{2}}} + \frac{1}{2}\ln t - \ln x + \ln|\mathcal{C}_R| + o\,(1)\right) \tag{2.18}$$

as $t \to 0$ with $x = O(1)\,(>0)$, where $\mathcal{C}_R$ is a constant whose value depends on ξ_0 and is given by

$$\mathcal{C}_R = 2\xi_0 e^{2\xi_0 - u_+^2},$$

and $\mathcal{S}_0 = \text{sign}(\mathcal{C}_R)$. We note that $\mathcal{C}_R < 0$ ($\mathcal{C}_R > 0$) when $u_+ > u_-$ ($u_+ < u_-$) respectively. Finally, we consider region II$^-$, where $x = O(1)\,(<0)$ as $t \to 0$. The details regarding region II$^-$ follow, after minor modification, those given in Section 2.1.1 and are not repeated here. In summary, we have in region II$^-$ that

$$u(x,t) = u_- - \mathcal{S}_0 \exp\left(-\frac{x^2}{4t} + \frac{x\,u_-}{t^{\frac{1}{2}}} + \frac{1}{2}\ln t - \ln(-x) + \ln|\mathcal{C}_L| + o\,(1)\right) \tag{2.19}$$

as $t \to 0$ with $x = O(1)\,(>0)$, and where $\mathcal{C}_L = \mathcal{C}_R \exp(u_+^2 - u_-^2)$ is a constant. We note that $\mathcal{C}_R = \mathcal{C}_L$ when $u_+ = u_-$ (the trivial case) or when $u_+ = -u_-$. The asymptotic structure as $t \to 0$ is now complete in this case, with the expansions in regions I, II$^+$ and II$^-$ providing a uniform approximation to the solution of IVP as $t \to 0$.

2.2 Asymptotic solution to IVP as $|x| \to \infty$

We now investigate the asymptotic structure of the solution to IVP as $|x| \to \infty$ with $t = O(1)$. We first determine the structure of the solution to IVP as $x \to \infty$ with $t = O(1)$. The form of expansions (2.11) (when $\delta > -\frac{1}{2}$) and (2.18) (when $\delta = -\frac{1}{2}$) of region II$^+$ for $x \gg 1$ as $t \to 0$ suggests that in this region, which we label as region III$^+$, we write

$$u(x,t) = u_+ + \mathcal{S}e^{-\Theta(x,t)} \tag{2.20}$$

where

$$\Theta(x,t) = \theta_0(t)x^2 + \theta_1(t)x + \theta_2(t)\ln x + \theta_3(t) + o(1) \tag{2.21}$$

as $x \to \infty$ with $t = O(1)$. On substituting (2.20) and (2.21) into equation (1.1) and solving at each order in turn, we find (after matching as $t \to 0^+$ with (2.11) when $\delta > -\frac{1}{2}$ or (2.18) when $\delta = -\frac{1}{2}$) that

$$u(x,t) = u_+ + \begin{cases} \mathcal{S}\exp\Big(-\frac{x^2}{4t} + \frac{u_+}{2(\delta+1)}xt^{\delta} - \ln x + \frac{1}{2}\ln t - \frac{u_+^2}{4(\delta+1)^2}t^{2\delta+1} \\ \qquad\qquad + \ln\left(\frac{|u_- - u_+|}{\sqrt{\pi}}\right) + o\,(1)\Big) \qquad \text{when} \quad \delta > -\frac{1}{2}, \\ \mathcal{S}_0\exp\left(-\frac{x^2}{4t} + (u_+)\,xt^{-\frac{1}{2}} - \ln x + \frac{1}{2}\ln t + \ln|\mathcal{C}_R| + o\,(1)\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{when} \quad \delta = -\frac{1}{2}, \end{cases} \tag{2.22}$$

as $x \to \infty$ with $t = O(1)$. Expansion $(2.22)_1$ remains uniform for $t \gg 1$ provided that $x \gg \lambda(t)$, but becomes nonuniform when $x = O(\lambda(t))$ as $t \to \infty$, where

$$\lambda(t) = \begin{cases} t^{\delta+1}, & \delta > 0, \\ t, & -\frac{1}{2} < \delta < 0, \end{cases} \tag{2.23}$$

whereas, expansion $(2.22)_2$ remains uniform for $t \gg 1$ provided that $x \gg t$, but becomes nonuniform when $x = O(t)$ as $t \to \infty$. We next investigate the structure of the solution structure to IVP as $x \to -\infty$ with $t = O(1)$, which we label as region III$^-$. The details in this case follow, after minor modification, those given above and we obtain in region III$^-$ that

$$u(x,t) = u_- - \begin{cases} \mathcal{S} \exp\Big(-\frac{x^2}{4t} + \frac{u_-}{2(\delta+1)} x t^{\delta} - \ln(-x) + \frac{1}{2}\ln t - \frac{u_-^2}{4(\delta+1)^2} t^{2\delta+1} \\ \qquad\qquad + \ln\left(\frac{|u_- - u_+|}{\sqrt{\pi}}\right) + o(1) \Big) \qquad \text{when} \quad \delta > -\frac{1}{2}, \\ \mathcal{S}_0 \exp\Big(-\frac{x^2}{4t} + (u_-) x t^{-\frac{1}{2}} - \ln(-x) + \frac{1}{2}\ln t + \ln|\mathcal{C}_L| + o(1) \Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{when} \quad \delta = -\frac{1}{2}, \end{cases} \tag{2.24}$$

as $x \to \infty$ with $t = O(1)$. Expansion $(2.24)_1$ remains uniform for $t \gg 1$ provided that $x \gg \lambda(t)$, but becomes nonuniform when $x = O(\lambda(t))$ as $t \to \infty$, where $\lambda(t)$ is given by (2.23). Expansion $(2.24)_2$ remains uniform for $t \gg 1$ provided that $x \gg t$, but becomes nonuniform when $x = O(t)$ as $t \to \infty$.

2.3 Asymptotic solution to IVP as $t \to \infty$

2.3.1 $\delta > 0$ and $u_+ > u_-$

As $t \to \infty$, the asymptotic expansions $(2.22)_1$ and $(2.24)_1$ of regions III$^+$ ($x \to \infty$, $t = O(1)$) and III$^-$ ($x \to -\infty$, $t = O(1)$), respectively, continue to remain uniform provided $|x| \gg t^{\delta+1}$. However, as already noted, a nonuniformity develops when $|x| = O\left(t^{\delta+1}\right)$. We further note that in this case $\mathcal{S} = -1$. We begin by considering the asymptotic structure as $t \to \infty$ for $x > 0$. To proceed we introduce a new region, region IV$^+$ when $x = O\left(t^{\delta+1}\right)$ as $t \to \infty$. To examine region IV$^+$ we introduce the scaled coordinate $y = xt^{-\delta-1}$, where $y = O(1)$ as $t \to \infty$, and write (as suggested by $(2.22))_1$

$$u(y,t) = u_+ - e^{-F(y,t)} \tag{2.25}$$

where

$$F(y,t) = F_0(y)t^{2\delta+1} + F_1(y)\ln t + F_2(y) + o(1) \tag{2.26}$$

as $t \to \infty$ with $y = O(1)$, and where $F_0(y) > 0$. On substituting (2.25) and (2.26) into equation (1.1) (when written in terms of y and t) and solving at each order in turn, we find (after matching with $(2.22)_1$ as $y \to \infty$) that

$$u(y,t) = u_+ - \exp\left(-\frac{1}{4}\left(y - \frac{u_+}{\delta+1}\right)^2 t^{2\delta+1} - \left(\delta + \frac{1}{2}\right)\ln t \right. \\ \left. - \ln\left(y - \frac{u_+}{\delta+1}\right) + \ln\frac{|u_- - u_+|}{\sqrt{\pi}} + o(1) \right) \tag{2.27}$$

as $t \to \infty$ with $y = O(1)$ $\left(\in \left(\frac{u_+}{\delta+1}, \infty\right)\right)$. Expansion (2.27) becomes nonuniform when $y = \frac{u_+}{\delta+1} + O\left(t^{-\delta}\right)$ as $t \to \infty$. To proceed we introduce a localized region, region V$^+$. To investigate region V$^+$ we introduce the scaled coordinate η via

$$\eta = \left(y - \frac{u_+}{\delta+1}\right) t^{\delta}, \tag{2.28}$$

as $t \to \infty$ with $\eta = O(1)$, and look for an expansion of the form (as suggested by (2.27))

$$u(\eta, t) = u_+ - e^{-\hat{F}(\eta,t)} \tag{2.29}$$

where

$$\hat{F}(\eta, t) = \hat{F}_0(\eta)t + \hat{F}_1(\eta)\ln t + \hat{F}_2(\eta) + o(1) \tag{2.30}$$

as $t \to \infty$ with $\eta = O(1)$, and where $\hat{F}_0(\eta) > 0$. On substituting (2.29) and (2.30) into equation (1.1) (when written in terms of η and t) and solving at each order in turn, we find (after matching with (2.27) as $\eta \to \infty$) that

$$u(\eta, t) = u_+ - \exp\left(-\frac{\eta^2}{4}t - \frac{1}{2}\ln t + \hat{H}^+(\eta) + o(1)\right) \tag{2.31}$$

as $t \to \infty$ with $\eta = O(1)$, and where the function $\hat{H}^+ : (0, \infty) \to \mathbb{R}$ is undetermined being a remnant of the global evolution when $t = O(1)$, but having

$$\hat{H}^+(\eta) \sim -\ln\eta + \ln\frac{|u_- - u_+|}{\sqrt{\pi}} \quad \text{as} \quad \eta \to \infty.$$

Expansion (2.31) becomes nonuniform when $\eta = O(t^{-\frac{1}{2}})$ as $t \to \infty$ [that is, when $y = \frac{u_+}{\delta+1} + O\left(t^{-\delta-\frac{1}{2}}\right)$ as $t \to \infty$]. Therefore, we must now introduce a second localized region VI$^+$ in which $y = \frac{u_+}{\delta+1} + O\left(t^{-\delta-\frac{1}{2}}\right)$ as $t \to \infty$. Thus we write

$$y = \frac{u_+}{\delta + 1} + \xi\, t^{-\delta-\frac{1}{2}} \tag{2.32}$$

in region VI$^+$, with $\xi = O(1)$ as $t \to \infty$. It follows from (2.32), (2.28) and expansion (2.31) in region V$^+$, that we should expand as

$$u(\xi, t) = u_+ + \phi(t)G(\xi) + o\left(\phi(t)\right) \quad \text{as} \quad t \to \infty \tag{2.33}$$

with $\xi = O(1)$, and the gauge function $\phi(t) = o(1)$ as $t \to \infty$ is to be determined. On substituting (2.33) into equation (1.1) (when written in terms of ξ and t) we obtain

$$\phi'(t)G - \frac{\xi}{2}G_\xi\frac{\phi(t)}{t} + \phi^2(t)t^{\delta-\frac{1}{2}}G\,G_\xi = \frac{\phi(t)}{t}G_{\xi\xi}. \tag{2.34}$$

A nontrivial balance requires

$$\phi^2(t)t^{\delta-\frac{1}{2}} \sim \frac{\phi(t)}{t} \quad \text{as} \quad t \to \infty,$$

and so, without loss of generality, we take,

$$\phi(t) = t^{-\delta-\frac{1}{2}}. \tag{2.35}$$

We observe that all terms in (2.34) are retained at leading order as $t \to \infty$ and (2.34) becomes

$$G_{\xi\xi} - G\,G_\xi + \frac{\xi}{2}G_\xi + \left(\delta + \frac{1}{2}\right)G = 0, \quad -\infty < \xi < \infty. \tag{2.36}$$

We note that equation (2.36) admits the solution $G(\xi) = (\delta + 1)\xi$. We further note that when $\delta = 0$ equation (2.36) can be integrated to give a solvable Riccati equation and the general solution of (2.36) then obtained. However, for $-\frac{1}{2} < \delta(\neq 0)$ we have been unable to obtain the general solution of (2.36) in closed form. For completeness we observe that when $\delta = -\frac{1}{2}$ equation (2.36) reduces to the equation analyzed by Rudenko and Soluyan in [22], and which we will return to in Section 2.3.4. Therefore, in this case, when $\delta > 0$, we will in what follows have to examine equation (2.36) numerically. Now, matching expansion (2.31) (as $\eta \to 0^+$) with expansion (2.33) (as $\xi \to \infty$) requires first that

$$\hat{H}^+(\eta) \sim 2\delta \ln \eta + \ln \beta \quad \text{as} \quad \eta \to 0^+, \tag{2.37}$$

where $\beta(> 0)$ is a constant to be determined, after which we require that

$$G(\xi) \sim -\beta \xi^{2\delta} e^{-\frac{\xi^2}{4}} \quad \text{as} \quad \xi \to \infty. \tag{2.38}$$

Finally for u to remain bounded as $t \to \infty$ when $y = \frac{u_+}{\delta+1} - O(1)$ then we require,

$$\xi^{-1} G(\xi) \quad \text{bounded as} \quad \xi \to -\infty. \tag{2.39}$$

The leading order problem is now complete, and is given by (2.36), (2.38) and (2.39). The boundary value problem (2.36)-(2.39) is both nonlinear and nonautonomous and there are two parameters δ (fixed) and $\beta(> 0)$. A numerical study of initial-value problem (2.36) and (2.38) using a shooting method reveals that there exists a value $\beta = \beta^*$ such that boundary condition (2.39) is satisfied for each $\beta \in (0, \beta^*]$, whilst for each $\beta \in (\beta^*, \infty)$ the solution blows up at finite-ξ, say $\xi = \xi_0(\beta)$, with

$$G(\xi) \sim -\frac{2}{(\xi - \xi_0(\beta))} \quad \text{as} \quad \xi \to \xi_0(\beta)^+. \tag{2.40}$$

Hence, boundary value problem (2.36), (2.38) and (2.39) has a unique solution for each $\beta \in (0, \beta^*]$ but no solution for $\beta \in (\beta^*, \infty)$. In particular, we have

(i) $0 < \beta < \beta^*$: A unique solution exists for each $\beta \in (0, \beta^*)$, say $G(\xi) = G_N(\xi)$, $-\infty < \xi < \infty$. Moreover, $G_N(\xi) \to 0^+$ as $\xi \to -\infty$.

(ii) $\beta = \beta^*$: A unique solution exits, say $G(\xi) = G^*(\xi)$, $-\infty < \xi < \infty$. Moreover, $G^*(\xi)$ is monotone increasing with $-\infty < \xi < \infty$, so that $G^*(\xi) < 0$ for all $\xi \in (-\infty, \infty)$ and

$$G^*(\xi) = (\delta + 1)\xi + O\left((-\xi)^{-\frac{1}{2\delta+1}}\right) \quad \text{as} \quad \xi \to -\infty. \tag{2.41}$$

A graph of $G(\xi)$ against ξ illustrating solutions to (2.36) and (2.38) for representative values of the parameter β in the ranges $\beta^* < \beta < \infty$, $0 < \beta < \beta^*$ and $\beta = \beta^*$ are shown in Figure 1. Thus we have two distinct cases to consider, namely, when (i) $\beta \in (0, \beta^*)$ or when (ii) $\beta = \beta^*$. However, we can reject case (i) as this choice leads to an asymptotic structure which fails to match to the far field $(2.24)_1$ as $y \to -\infty$, the details of which are omitted for brevity. We conclude that we require

$$\beta = \beta^*,$$

and that $G(\xi) = G^*(\xi)$ for $\xi \in (-\infty, \infty)$.

As $\xi \to -\infty$ we move out of region VI$^+$ into region EW where $y = O(1)\,(\in (-\infty, \frac{u_+}{\delta+1}))$. We have from (2.33), (2.35) and (2.41) that

$$u(\xi, t) = u_+ + (\delta + 1)\xi t^{-\delta - \frac{1}{2}} + O\left(t^{-\delta - \frac{1}{2}} (-\xi)^{-\frac{1}{2\delta+1}}\right) \tag{2.42}$$

as $t \to \infty$ with $-\xi \gg 1$. On writing in terms of y we obtain that

$$u(y, t) \sim (\delta + 1) y \tag{2.43}$$

suggesting that in region EW that we look for an expansion of the form

$$u(y, t) = P(y) + o(1) \tag{2.44}$$

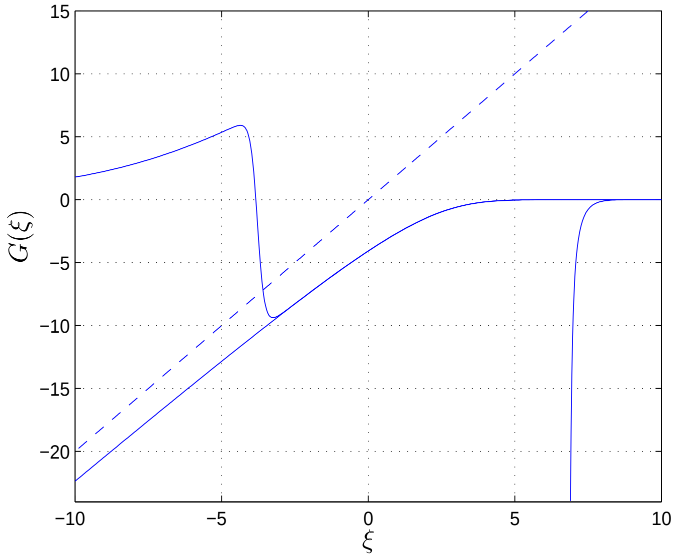

Figure 1: A graph of $G(\xi)$ against ξ illustrating solutions to (2.36) for $\delta = 1$. We note that the dashed line represents the solution $G(\xi) = (\delta + 1)\xi$.

as $t \to \infty$ with $y = O(1)\,(\in (-\infty, \frac{u_+}{\delta+1}))$. On substitution of expansion (2.44) into equation (1.1) (when written in terms of y and t) we obtain at leading order that

$$P_y\left(P - (\delta+1)y\right) = 0, \quad -\infty < y < \frac{u_+}{\delta+1}. \tag{2.45}$$

Equation (2.45) is to be solved subject to the matching condition with region VI$^+$, that is

$$P(y) \sim (\delta+1)y \quad \text{as} \quad y \to \left(\frac{u_+}{\delta+1}\right)^-. \tag{2.46}$$

The solution of (2.45), (2.46) is readily obtained as

$$P(y) = (\delta+1)y, \quad -\infty < y < \frac{u_+}{\delta+1}. \tag{2.47}$$

Therefore, in region EW we have that

$$u(y,t) = (\delta+1)y + o(1) \tag{2.48}$$

as $t \to \infty$ with $y = O(1)\,(\in (-\infty, \frac{u_+}{\delta+1}))$.

We next consider the asymptotic structure of IVP as $t \to \infty$ for $x < 0$. To proceed we introduce a new region, region IV$^-$. The details of this region follow those given for region IV$^+$ and are not repeated here. In region IV$^-$ we have that

$$\begin{aligned} u(y,t) = u_- + \exp\Bigg(&-\frac{1}{4}\left(y - \frac{u_-}{\delta+1}\right)^2 t^{2\delta+1} - \left(\delta + \frac{1}{2}\right)\ln t \\ &- \ln\left(\frac{u_-}{\delta+1} - y\right) + \ln\frac{|u_- - u_+|}{\sqrt{\pi}} + o(1)\Bigg) \end{aligned} \tag{2.49}$$

as $t \to \infty$ with $y = O(1)$ $\left(\in \left(-\infty, \frac{u_-}{\delta+1}\right)\right)$. Expansion (2.49) becomes nonuniform when $y = \frac{u_-}{\delta+1} + O\left(t^{-\delta}\right)$ as $t \to \infty$, and to continue the asymptotic structure in this case we introduce the localized region, region V$^-$. The details of region V$^-$ follow, after minor modification, those given for region V$^+$ and are summarized here for brevity. In region V$^-$ we have that

$$u(\hat{\eta}, t) = u_- + \exp\left(-\frac{\hat{\eta}^2}{4}t - \frac{1}{2}\ln t + \hat{H}^-(\hat{\eta}) + o(1)\right) \tag{2.50}$$

as $t \to \infty$ with $\hat{\eta} = \left(y - \frac{u_-}{\delta+1}\right) t^{\delta} = O(1)$, and where the function $\hat{H}^- : (-\infty, 0) \to \mathbb{R}$ is undetermined (being a remnant of the global evolution when $t = O(1)$), but having

$$\hat{H}^-(\hat{\eta}) \sim -\ln(-\hat{\eta}) + \ln \frac{|u_- - u_+|}{\sqrt{\pi}} \quad \text{as} \quad \hat{\eta} \to -\infty.$$

Expansion (2.31) becomes nonuniform when $\hat{\eta} = O(t^{-\frac{1}{2}})$ as $t \to \infty$ [that is, when $y = \frac{u_-}{\delta+1} + O\left(t^{-\delta-\frac{1}{2}}\right)$ as $t \to \infty$], and to complete the asymptotic structure in this case we introduce a final localized region, region VI$^-$. The details of region VI$^-$ follow those given for region VI$^+$ and are summarized here for brevity. In region VI$^+$ we look for an expansion of the form (as suggested by (2.50))

$$u(\xi, t) = u_- + G(\xi) t^{-\delta-\frac{1}{2}} + o\left(t^{-\delta-\frac{1}{2}}\right) \tag{2.51}$$

as $t \to \infty$ with $\xi = \left(y - \frac{u_-}{\delta+1}\right) t^{\delta+\frac{1}{2}} = O(1)$. On substitution of expansion (2.51) into equation (1.1) (when written in terms of ξ and t) we obtain at leading order

$$G_{\xi\xi} - G\,G_{\xi} + \frac{\xi}{2}G_{\xi} + \left(\delta + \frac{1}{2}\right)G = 0, \quad -\infty < \xi < \infty. \tag{2.52}$$

We first observe that differential equation (2.52) is invariant under the transformation

$$G(\xi) = -G(-\xi).$$

Further, equation (2.52) is to be solved subject to the matching condition with region EW, that is

$$G(\xi) \sim (\delta+1)\xi \quad \text{as} \quad \xi \to \infty. \tag{2.53}$$

Therefore, we select the unique solution $G(\xi) = -G^*(-\xi)$ (see region VI$^+$ above for details of $G^*(\xi)$) when $\beta = \beta^*$, with $G^*(\xi) < 0$ for all $\xi \in (-\infty, \infty)$ [that is, $G(\xi) > 0$ for all $\xi \in (-\infty, \infty)$]. Specifically, we have that

$$G(\xi) \sim \begin{cases} (\delta+1)\xi & \text{as} \quad \xi \to \infty, \\ \beta^*(-\xi)^{2\delta} e^{-\frac{\xi^2}{4}} & \text{as} \quad \xi \to -\infty. \end{cases} \tag{2.54}$$

Finally, matching expansion (2.50) (as $\hat{\eta} \to 0^-$) with expansion (2.51) (as $\xi \to -\infty$) requires that

$$\hat{H}_-(\hat{\eta}) \sim 2\delta \ln(-\hat{\eta}) + \ln \beta^* \quad \text{as} \quad \hat{\eta} \to 0^-,$$

and completes region VI$^+$. On returning to region EW we established that

$$u(y, t) = (\delta+1)y + o(1) \tag{2.55}$$

as $t \to \infty$ with $y = O(1)\ (\in (\frac{u_-}{\delta+1}, \frac{u_+}{\delta+1}))$.

The asymptotic structure of IVP as $t \to \infty$ when $\delta > 0$ and $u_+ > u_-$ is now complete. A uniform approximation has been given through regions III$^\pm$, IV$^\pm$, V$^\pm$, VI$^\pm$ and EW. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given for in Figure 2. The large-t attractor for the solution of IVP in this case is the expansion wave which allows for the adjustment of the solution from u_+ to u_-.

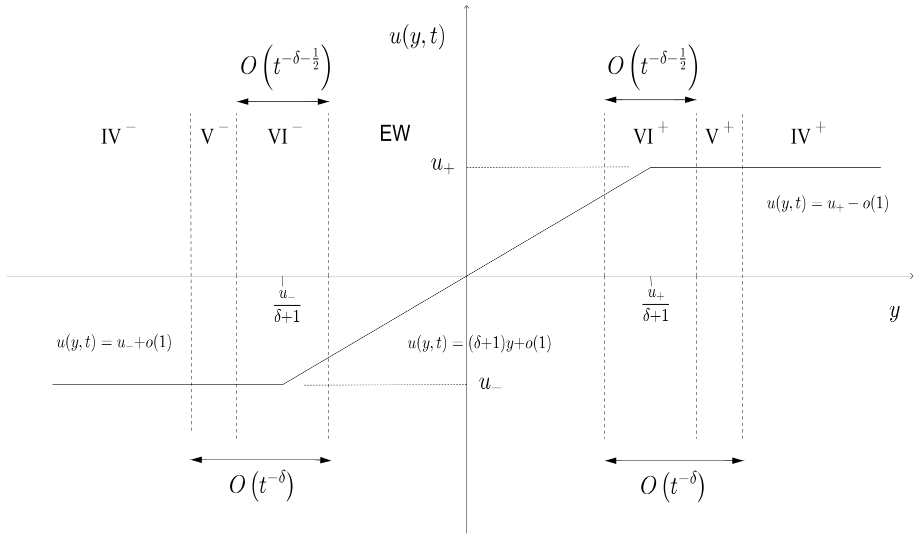

Figure 2: A schematic representation of the asymptotic structure of $u(y,t)$ in the (y,u) plane as $t \to \infty$ for IVP when $\delta > 0$ and $u_+ > u_-$. We recall that in this case $y = xt^{-\delta-1}$.

2.3.2 $\delta > 0$ and $u_+ < u_-$

We now investigate the asymptotic structure of IVP as $t \to \infty$ when $u_+ < u_-$. As $t \to \infty$, the asymptotic expansions $(2.22)_1$ and $(2.24)_1$ of regions III$^+$ ($x \to \infty$, $t = O(1)$) and III$^-$ ($x \to -\infty$, $t = O(1)$), respectively, continue to remain uniform provided $|x| \gg t^{\delta+1}$. However, as already noted, a nonuniformity develops when $|x| = O\left(t^{\delta+1}\right)$. As in Section 2.3.1 we introduce the scaled coordinate $y = xt^{-\delta-1} = O(1)$ as $t \to \infty$ and note in this case that $\mathcal{S} = +1$. We begin by summarizing the asymptotic structure as $t \to \infty$ in regions IV$^+$ and IV$^-$ (the details follow those given in Section 2.3.1 and are not repeated here):

Region IV$^+$

$$u(y,t) = u_+ + \exp\left(-\frac{1}{4}\left(y - \frac{u_+}{\delta+1}\right)^2 t^{2\delta+1} - \left(\delta + \frac{1}{2}\right)\ln t \right.$$
$$\left. - \ln\left(y - \frac{u_+}{\delta+1}\right) + \ln\frac{|u_- - u_+|}{\sqrt{\pi}} + o(1)\right) \tag{2.56}$$

as $t \to \infty$ with $y = O(1)$ $\left(\in \left(\frac{u_+}{\delta+1}, \infty\right)\right)$.

Region IV$^-$

$$u(y,t) = u_- - \exp\left(-\frac{1}{4}\left(y - \frac{u_-}{\delta+1}\right)^2 t^{2\delta+1} - \left(\delta + \frac{1}{2}\right)\ln t \right.$$
$$\left. - \ln\left(\frac{u_-}{\delta+1} - y\right) + \ln\frac{|u_- - u_+|}{\sqrt{\pi}} + o(1)\right) \tag{2.57}$$

as $t \to \infty$ with $y = O(1)$ $\left(\in \left(-\infty, \frac{u_-}{\delta+1}\right)\right)$.

Since $u_+ < u_-$ expansions (2.56) and (2.57) must become nonuniform as $y \to \alpha$, where $\alpha \in (u_+, u_-)$ and is to be determined. To examine this region which we label as region SS we introduce the scaled coordinate

$$z = (y - \alpha)\psi^{-1} = O(1), \tag{2.58}$$

where $\psi(t) = o(1)$ as $t \to \infty$, is an as yet undetermined gauge function, and expand in the form (as suggested by (2.56) and (2.57))

$$u(z, t) = U(z) + o(1) \tag{2.59}$$

as $t \to \infty$ with $z = O(1)$. On substituting (2.59) into equation (1.1) (when written in terms of z and t) we find that to obtain the most structured leading order balance that we require

$$\psi(t) = t^{-2\delta-1}. \tag{2.60}$$

At leading order we then obtain

$$U_{zz} - UU_z + \alpha(\delta + 1)U_z = 0, \quad -\infty < z < \infty. \tag{2.61}$$

On integrating (2.61) we obtain

$$U_z = \frac{U^2}{2} - \alpha(\delta + 1)U + C, \quad -\infty < z < \infty, \tag{2.62}$$

where C is a constant of integration. Equation (2.62) is to be solved subject to the leading order matching conditions

$$U(z) \sim \begin{cases} u_+ & \text{as} \quad z \to \infty, \\ u_- & \text{as} \quad z \to -\infty. \end{cases} \tag{2.63}$$

The solution to (2.62) subject to boundary conditions (2.63) requires that

$$\alpha = \frac{(u_+ + u_-)}{2(\delta + 1)} \quad \text{and} \quad C = \frac{u_+ u_-}{2},$$

and is given by the Taylor shock profile (see [26] and [27])

$$U(z) = \frac{(u_+ + u_-)}{2} - \frac{(u_- - u_+)}{2}\tanh\left(\frac{(u_- - u_+)}{4}z + \phi_c\right), \quad -\infty < z < \infty, \tag{2.64}$$

where ϕ_c is a constant. G.I. Taylor obtained (2.64) in [26] for the structure of a weak thermoviscous shock. We note that

$$U(z) \sim \begin{cases} u_+ + (u_- - u_+)\exp\left(-\frac{(u_- - u_+)}{2}z - \phi_c\right) & \text{as} \quad z \to \infty, \\ u_- - (u_- - u_+)\exp\left(\frac{(u_- - u_+)}{2}z + \phi_c\right) & \text{as} \quad z \to -\infty. \end{cases} \tag{2.65}$$

The similarity solution (2.64) represents a monotone decreasing wavefront connecting u_- ($> u_+$) to u_+. The wavefront is located at $x = \frac{(u_+ + u_-)}{2(\delta+1)}t^{\delta+1}$ and is contained within a region of thickness $x = O\left(t^{-\delta}\right)$ as $t \to \infty$. The wavefront is steepening as $t \to \infty$ and has an accelerating velocity as $t \to \infty$ when $u_+ \neq -u_-$, but is located at $x = 0$ and is stationary when $u_+ = -u_-$. In particular, when $u_+ > -u_-$ the wavefront is accelerating in the $+x$ direction, whereas when $u_+ < -u_-$ the wavefront is accelerating in the $-x$ direction.

Although expansion (2.59) (with$(2.65)_1$) as $z \to \infty$ matches with expansion (2.56) as $y \to \alpha^+$ at leading order matching fails at next order and we require a transition region, which we label TR$^+$. This failure in matching is indicative of the fact that expansion (2.56) will become nonuniform as $y \to (\alpha + \omega)^+$ (where $\omega > 0$ is a constant to be determined), and region TR$^+$ will be required to allow the solution in region

IV$^+$ (where $u = u_+ + o(1)$) to adjust to the solution in region SS (where $u = O(1)$). To examine region TR$^+$ we introduce the scaled coordinate η by

$$z = c(t) + \eta, \tag{2.66}$$

where

$$c(t) = \frac{(u_- - u_+)}{4(\delta+1)(2\delta+1)} t^{2\delta+1} + \frac{2(\delta+1)}{(u_- - u_+)} \ln t \tag{2.67}$$

as $t \to \infty$ with $\eta = O(1)$, and look for an expansion of the form

$$\begin{aligned} u(\eta, t) = u_+ + F(\eta) t^{-\delta-1} \exp\left(-\frac{(u_- - u_+)^2}{8(\delta+1)(2\delta+1)} t^{2\delta+1}\right) \\ + o\left(t^{-\delta-1} \exp\left(-\frac{(u_- - u_+)^2}{8(\delta+1)(2\delta+1)} t^{2\delta+1}\right)\right) \end{aligned} \tag{2.68}$$

as $t \to \infty$ with $\eta = O(1)$. On substituting expansion (2.68) into equation (1.1) (when written in terms of η and t) we obtain at leading order that

$$F_{\eta\eta} + \frac{(u_- - u_+)(2\delta+3)}{4(\delta+1)} F_\eta + \frac{(u_- - u_+)^2}{8(\delta+1)} F = 0, \quad -\infty < \eta < \infty. \tag{2.69}$$

Equation (2.69) has to be solved subject to the matching conditions with regions IV$^+$ and SS, that is

$$F(\eta) \sim \begin{cases} \frac{2(\delta+1)}{\sqrt{\pi}} \exp\left(-\frac{(u_- - u_+)}{4(\delta+1)}\eta\right) & \text{as} \quad \eta \to \infty, \\ (u_- - u_+) e^{-\phi_c} \exp\left(-\frac{(u_- - u_+)}{2}\eta\right) & \text{as} \quad \eta \to -\infty. \end{cases} \tag{2.70}$$

The solution of (2.69) subject to (2.70) is readily obtained as

$$F(\eta) = \frac{2(\delta+1)}{\sqrt{\pi}} \exp\left(-\frac{(u_- - u_+)}{4(\delta+1)}\eta\right) + (u_- - u_+) e^{-\phi_c} \exp\left(-\frac{(u_- - u_+)}{2}\eta\right) \tag{2.71}$$

where $-\infty < \eta < \infty$. Therefore, the expansion in region TR$^+$ is given by (2.68) with (2.71).

Finally, we conclude this case by noting that matching expansion (2.59) (as $z \to -\infty$) to expansion (2.57) (as $y \to \alpha^-$) fails and we require a final transition region, which we label TR$^-$. To examine region TR$^-$ we introduce the scaled coordinate $\hat{\eta}$ by

$$y = -c(t) + \hat{\eta} \tag{2.72}$$

as $t \to \infty$ with $\hat{\eta} = O(1)$, where $c(t)$ is given by (2.67), and look for an expansion of the form

$$\begin{aligned} u(\hat{\eta}, t) = u_- - F(\hat{\eta}) t^{-\delta-1} \exp\left(-\frac{(u_- - u_+)^2}{8(\delta+1)(2\delta+1)} t^{2\delta+1}\right) \\ + o\left(t^{-\delta-1} \exp\left(-\frac{(u_- - u_+)^2}{8(\delta+1)(2\delta+1)} t^{2\delta+1}\right)\right) \end{aligned} \tag{2.73}$$

as $t \to \infty$ with $\hat{\eta} = O(1)$. On substituting expansion (2.73) into equation (1.1) (when written in terms of $\hat{\eta}$ and t) we obtain at leading order that

$$F_{\hat{\eta}\hat{\eta}} - \frac{(u_- - u_+)(2\delta+3)}{4(\delta+1)} F_{\hat{\eta}} + \frac{(u_- - u_+)^2}{8(\delta+1)} F = 0, \quad -\infty < \hat{\eta} < \infty. \tag{2.74}$$

Equation (2.74) has to be solved subject to the matching conditions with regions IV$^-$ and SS, that is

$$F(\hat{\eta}) \sim \begin{cases} \frac{2(\delta+1)}{\sqrt{\pi}} \exp\left(\frac{(u_- - u_+)}{4(\delta+1)}\hat{\eta}\right) & \text{as} \quad \hat{\eta} \to -\infty, \\ (u_- - u_+) e^{\phi_c} \exp\left(\frac{(u_- - u_+)}{2}\hat{\eta}\right) & \text{as} \quad \hat{\eta} \to \infty. \end{cases} \tag{2.75}$$

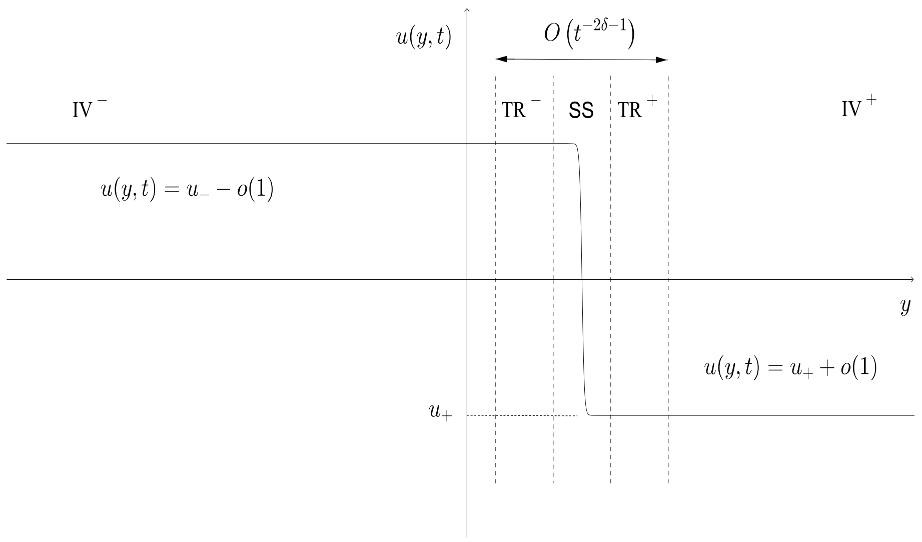

Figure 3: A schematic representation of the asymptotic structure of $u(y,t)$ in the (y,u) plane as $t \to \infty$ for IVP when $u_+ < u_-$. We recall that region SS is located at $y = \frac{u_+ + u_-}{2(\delta+1)}$ (with thickness $O\left(t^{-2\delta-1}\right)$ as $t \to \infty$), while regions TR$^\pm$ are located at $y = \frac{(u_- + u_+)}{2(\delta+1)} \pm \frac{(u_- - u_+)}{4(\delta+1)(2\delta+1)} \pm \frac{2(\delta+1)}{(u_- - u_+)} \frac{\ln t}{t^{2\delta+1}}$ (with thickness $O\left(t^{-2\delta-1}\right)$) as $t \to \infty$.

The solution of (2.74) subject to (2.75) is readily obtained as

$$F(\hat{\eta}) = \frac{2(\delta+1)}{\sqrt{\pi}} \exp\left(\frac{(u_- - u_+)}{4(\delta+1)}\hat{\eta}\right) + (u_- - u_+)e^{\phi_c} \exp\left(\frac{(u_- - u_+)}{2}\hat{\eta}\right) \tag{2.76}$$

where $-\infty < \hat{\eta} < \infty$. Therefore, the expansion in region TR$^-$ is given by (2.73) with (2.76).

The asymptotic structure of the solution of IVP as $t \to \infty$ when $u_+ < u_-$ is now complete. A uniform approximation has been given through regions III$^\pm$, IV$^\pm$, TR$^\pm$ and SS. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Figure 3. The large-t attractor for the solution of IVP when $u_+ > u_-$ is the Taylor shock profile which allows for the adjustment of the solution form u_+ to u_-.

2.3.3 $-\frac{1}{2} < \delta < 0$

As $t \to \infty$, the asymptotic expansion $(2.22)_1$ and $(2.24)_1$ of regions III$^+$ ($x \to \infty$, $t = O(1)$) and III$^-$ ($x \to -\infty$, $t = O(1)$), respectively, continue to remain uniform provided $|x| \gg t$. However, as already noted, a nonuniformity develops when $|x| = O(t)$, We begin by considering the asymptotic structure as $t \to \infty$ for $x > 0$. To proceed we introduce a new region, region IV$^+$ when $x = O(t)$ as $t \to \infty$. To examine region IV$^+$ we introduce the scaled coordinate $\varphi = xt^{-1}$, where $\varphi = O(1)$ as $t \to \infty$, and write (as suggested by $(2.22))_1$

$$u(\varphi, t) = u_+ + \mathcal{S}e^{-G(\varphi,t)} \tag{2.77}$$

where

$$G(\varphi, t) = G_0(\varphi)t + G_1(\varphi)t^{\delta+1} + G_2(\varphi)t^{2\delta+1} + G_3(\varphi)\ln t + G_4(\varphi) + o(1) \tag{2.78}$$

as $t \to \infty$ with $\varphi = O(1)$, and where $G_0(\varphi) > 0$ and $\mathcal{S} = \text{sign}(u_- - u_+)$. On substituting (2.77) and (2.78) into equation (1.1) (when written in terms of φ and t)

and solving at each order in turn, we find (after matching with $(2.22)_1$ as $\varphi \to \infty$) that

$$u(\varphi,t) = u_+ + \mathcal{S}\exp\Bigg(-\frac{1}{4}\varphi^2 t + \frac{u_+}{2(\delta+1)}\varphi t^{\delta+1} - \frac{u_+^2}{4(\delta+1)^2}t^{2\delta+1} - \frac{1}{2}\ln t + H^+(\varphi) + o(1)\Bigg) \tag{2.79}$$

as $t \to \infty$ with $\varphi = O(1)$, and where the function $H^+ : (0,\infty) \to \mathbb{R}$ is undetermined (being a remnant of the global evolution when $t = O(1)$), but having

$$H^+(\varphi) \sim -\ln\varphi + \ln\frac{|u_- - u_+|}{\sqrt{\pi}} \quad \text{as} \quad \varphi \to \infty.$$

Expansion (2.79) becomes nonuniform when $\varphi = O\left(t^{\delta}\right)$ as $t \to \infty$ [that is, when $x = O\left(t^{\delta+1}\right)$ as $t \to \infty$]. To proceed we relabel region IV$^+$ as region IV(a)$^+$, and introduce a new region, region IV(b)$^+$ when $\varphi = O\left(t^{\delta}\right)$ as $t \to \infty$. To investigate region IV(b)$^+$ we introduce the scaled coordinate y via

$$y = \varphi t^{-\delta} = x t^{-\delta-1},$$

where $y = O(1)$ as $t \to \infty$, and expand in the form

$$u(y,t) = u_+ + \mathcal{S}e^{-F(y,t)} \tag{2.80}$$

where

$$F(y,t) = F_0(y)t^{2\delta+1} + F_1(y)\ln t + F_2(y) + o(1) \tag{2.81}$$

as $t \to \infty$ with $y = O(1)$, and where $F_0(y) > 0$ and $\mathcal{S} = \text{sign}(u_- - u_+)$. On substituting (2.80) and (2.81) into equation (1.1) (when written in terms of y and t) and solving at each order in turn, we find (after matching with (2.79) as $y \to \infty$) that

$$u(y,t) = u_+ + \mathcal{S}\exp\left(-\frac{1}{4}\left(y - \frac{u_+}{\delta+1}\right)^2 t^{2\delta+1} - \frac{1}{2}\ln t + \lambda + o(1)\right) \tag{2.82}$$

as $t \to \infty$ with $y = O(1)$ $\left(\in \left(\frac{u_+}{\delta+1}, \infty\right)\right)$, and where λ is a constant. We note that matching expansion (2.80) ($y \to \infty$) with expansion (2.79) ($\varphi \to 0^+$) requires that

$$H^+(\varphi) \sim \lambda \quad \text{as} \quad \varphi \to 0^+.$$

We next consider the asymptotic structure of IVP as $t \to \infty$ for $x < 0$. To proceed we introduce a new region, region IV(a)$^-$. The details of this region follow those given for region IV(a)$^+$ and are not repeated here. In region IV(a)$^-$ we have that

$$u(\hat{\varphi},t) = u_- - \mathcal{S}\exp\Bigg(-\frac{1}{4}\hat{\varphi}^2 t + \frac{u_-}{2(\delta+1)}\hat{\varphi} t^{\delta+1} - \frac{u_-^2}{4(\delta+1)^2}t^{2\delta+1} - \frac{1}{2}\ln t + H^-(\hat{\varphi}) + o(1)\Bigg) \tag{2.83}$$

as $t \to \infty$ with $\hat{\varphi} = \frac{x}{t} = O(1)$, and where the function $H^- : (-\infty,0) \to \mathbb{R}$ is undetermined (being a remnant of the global evolution when $t = O(1)$), but having

$$H^-(\hat{\varphi}) \sim -\ln(-\hat{\varphi}) + \ln\frac{|u_- - u_+|}{\sqrt{\pi}} \quad \text{as} \quad \hat{\varphi} \to -\infty.$$

Expansion (2.83) becomes nonuniform when $\hat{\varphi} = O\left(t^{\delta}\right)$ as $t \to \infty$ [that is, when $-x = O\left(t^{\delta+1}\right)$ as $t \to \infty$]. To proceed we introduce a new region, region IV(b)$^-$ when $\hat{\varphi} = O\left(t^{\delta}\right)$ as $t \to \infty$. The details of region IV(b)$^-$ follow those given for region IV(b)$^+$ above. Therefore, in region IV(b)$^-$ we have that

$$u(y,t) = u_- - \mathcal{S}\exp\left(-\frac{1}{4}\left(y - \frac{u_-}{\delta+1}\right)^2 t^{2\delta+1} - \frac{1}{2}\ln t + \hat{\lambda} + o(1)\right) \tag{2.84}$$

as $t \to \infty$ with $y = \hat{\varphi}t^{-\delta} = O(1)$ $\left(\in \left(-\infty, \frac{u_-}{\delta+1}\right)\right)$, and where $\hat{\lambda}$ is a constant. We note that matching expansion (2.84) ($y \to -\infty$) with expansion (2.83) ($\hat{\varphi} \to 0^-$) requires that

$$H^-(\hat{\varphi}) \sim \hat{\lambda} \quad \text{as} \quad \hat{\varphi} \to 0^-.$$

We now need to consider the cases when $u_+ > u_-$ and when $u_+ < u_-$ separately.

(a) $\boldsymbol{u_+ > u_-}$

The remaining asymptotic structure of the solution to IVP as $t \to \infty$ in this case follows, after minor modification, that given in Section 2.3.1 for regions V$^\pm$, VI$^\pm$ and EW. We note that in this case $\mathcal{S} = -1$. In summary, we have that

Region V$^+$. $y = \frac{u_+}{\delta+1} + O\left(t^{-\delta}\right)$ as $t \to \infty$

$$u(\eta,t) = u_+ - \exp\left(-\frac{\eta^2}{4}t - \frac{1}{2}\ln t + \hat{H}^+(\eta) + o(1)\right) \tag{2.85}$$

as $t \to \infty$ with $\eta = \left(y - \frac{u_+}{\delta+1}\right) = O(1)$, and where the function $\hat{H}^+ : (0,\infty) \to \mathbb{R}$ is undetermined, but having

$$\hat{H}^+(\eta) \sim \begin{cases} \lambda & \text{as} \quad \eta \to \infty, \\ 2\delta\ln\eta + \ln\beta^* & \text{as} \quad \eta \to 0^+. \end{cases} \tag{2.86}$$

Region VI$^+$. $y = \frac{u_+}{\delta+1} + O\left(t^{-\delta-\frac{1}{2}}\right)$ as $t \to \infty$

$$u(\xi,t) = u_+ + G^*(\xi)t^{-\delta-\frac{1}{2}} + o\left(t^{-\delta-\frac{1}{2}}\right) \tag{2.87}$$

as $t \to \infty$ with $\xi = \left(y - \frac{u_+}{\delta+1}\right)t^{\delta+\frac{1}{2}} = O(1)$, and where $G^*(\xi)$ is the solution to boundary value problem (2.36), (2.38) and (2.39) when $\beta = \beta^*$. Also,

$$G^*(\xi) \sim \begin{cases} -\beta^*\xi^{2\delta}e^{-\frac{\xi^2}{4}} & \text{as} \quad \xi \to \infty, \\ (\delta+1)\xi & \text{as} \quad \xi \to -\infty, \end{cases} \tag{2.88}$$

Region EW. $y = O(1)$ $\left(\in \left(\frac{u_-}{\delta+1}, \frac{u_+}{\delta+1}\right)\right)$ as $t \to \infty$

$$u(y,t) = (\delta+1)y + o(1) \tag{2.89}$$

as $t \to \infty$ with $y = O(1)$ $\left(\in \left(\frac{u_-}{\delta+1}, \frac{u_+}{\delta+1}\right)\right)$.

Region VI$^-$. $y = \frac{u_-}{\delta+1} + O\left(t^{-\delta-\frac{1}{2}}\right)$ as $t \to \infty$

$$u(\xi,t) = u_- + G(\xi)t^{-\delta-\frac{1}{2}} + o\left(t^{-\delta-\frac{1}{2}}\right) \tag{2.90}$$

as $t \to \infty$ with $\xi = \left(y - \frac{u_-}{\delta+1}\right) t^{\delta+\frac{1}{2}} = O(1)$, and $G(\xi) = -G^*(-\xi)$ where $G^*(\xi)$ is the solution to boundary value problem (2.36), (2.38) and (2.39) when $\beta = \beta^*$. Also,

$$G^*(\xi) \sim \begin{cases} \beta^*(-\xi)^{2\delta} e^{-\frac{\xi^2}{4}} & \text{as} \quad \xi \to -\infty, \\ (\delta+1)\xi & \text{as} \quad \xi \to \infty. \end{cases} \tag{2.91}$$

Region V$^-$. $\quad y = \frac{u_-}{\delta+1} + O\left(t^{-\delta}\right)$ as $t \to \infty$

$$u(\hat{\eta}, t) = u_- + \exp\left(-\frac{\hat{\eta}^2}{4}t - \frac{1}{2}\ln t + \hat{H}^-(\hat{\eta}) + o(1)\right) \tag{2.92}$$

as $t \to \infty$ with $\hat{\eta} = \left(y - \frac{u_-}{\delta+1}\right) = O(1)$, and where the function $\hat{H}^- : (-\infty, 0) \to \mathbb{R}$ is undetermined, but having

$$\hat{H}^-(\hat{\eta}) \sim \begin{cases} \hat{\lambda} & \text{as} \quad \hat{\eta} \to -\infty, \\ 2\delta \ln(-\hat{\eta}) + \ln \beta^* & \text{as} \quad \hat{\eta} \to 0^-. \end{cases} \tag{2.93}$$

The asymptotic structure of the solution of IVP as $t \to \infty$ when $u_+ > u_-$ and $-\frac{1}{2} < \delta < 0$ is now complete. A uniform approximation has been given through regions III$^\pm$, IV$(a)^\pm$, IV(b)$^\pm$, V$^\pm$, VI$^\pm$ and EW. The large-t attractor for the solution of IVP when $u_+ > u_-$ is the expansion wave which allows for the adjustment of the solution form u_+ to u_-.

(b) $\boldsymbol{u_+ < u_-}$

The remaining asymptotic structure of the solution to IVP as $t \to \infty$ in this case follows, after minor modification, that given in Section 2.3.2 for regions TR$^\pm$ and SS. We note that $\mathcal{S} = 1$. In summary, we have that

Region TR$^+$. $\quad y = \frac{(u_+ + u_-)}{2(\delta+1)} + \hat{c}(t)t^{-2\delta-1} + O\left(t^{-2\delta-1}\right)$ as $t \to \infty$

$$\begin{aligned} u(\eta, t) = u_+ + \Bigg(& e^{\lambda} \exp\left[-\frac{(u_- - u_+)}{4(\delta+1)}\eta\right] + (u_- - u_+)e^{-\phi_c} \exp\left[-\frac{(u_- - u_+)}{2}\eta\right]\Bigg) \\ & \times t^{-\frac{\delta+1}{2\delta+1}} \exp\left(-\frac{(u_- - u_+)^2}{8(\delta+1)(2\delta+1)}t^{2\delta+1}\right) \\ & + o\left(t^{-\frac{\delta+1}{2\delta+1}} \exp\left(-\frac{(u_- - u_+)^2}{8(\delta+1)(2\delta+1)}t^{2\delta+1}\right)\right) \end{aligned} \tag{2.94}$$

as $t \to \infty$ with $\eta = \left(y - \frac{(u_- + u_+)}{2(\delta+1)} - \hat{c}(t)t^{-2\delta-1}\right) t^{2\delta+1} = O(1)$, and where

$$\hat{c}(t) = \frac{(u_- - u_+)}{4(\delta+1)(2\delta+1)}t^{2\delta+1} + \frac{2(\delta+1)}{(u_- - u_+)(2\delta+1)}\ln t \tag{2.95}$$

as $t \to \infty$.

Region SS. $\quad y = \frac{(u_+ + u_-)}{2(\delta+1)} + O\left(t^{-2\delta-1}\right)$ as $t \to \infty$

$$u(z, t) = \left(\frac{(u_+ + u_-)}{2} - \frac{(u_- - u_+)}{2}\tanh\left(\frac{(u_- - u_+)}{4}z + \phi_c\right)\right) + o(1) \tag{2.96}$$

as $t \to \infty$ with $z = \left(y - \frac{(u_+ + u_-)}{2(\delta+1)}\right) t^{2\delta+1} = O(1)$, and where ϕ_c is a constant. The solution (2.96) represents a monotone decreasing wavefront connecting u_- ($> u_+$) to

u_+. The wavefront is located at $x = \frac{(u_+ + u_-)}{2(\delta+1)} t^{\delta+1}$ and is contained within a region of thickness $x = O\left(t^{-\delta}\right)$ (recall that $-\frac{1}{2} < \delta < 0$ in this case) as $t \to \infty$. The wavefront is stretching as $t \to \infty$ and has an decelerating velocity as $t \to \infty$ when $u_+ \neq -u_-$, but is located at $x = 0$ and is stationary when $u_+ = -u_-$. In particular, when $u_+ > -u_-$ the wavefront is decelerating in the $+x$ direction, whereas when $u_+ < -u_-$ the wavefront is decelerating in the $-x$ direction.

Region TR$^-$. $\quad y = \frac{(u_+ + u_-)}{2(\delta+1)} - \hat{c}(t) t^{-2\delta-1} + O\left(t^{-2\delta-1}\right)$ as $t \to \infty$

$$\begin{aligned} u(\hat{\eta}, t) = u_- - &\left(e^{\hat{\lambda}} \exp\left[\frac{(u_- - u_+)}{4(\delta+1)} \hat{\eta} \right] + (u_- - u_+) e^{\phi_c} \exp\left[\frac{(u_- - u_+)}{2} \hat{\eta} \right] \right) \\ &\times t^{-\frac{\delta+1}{2\delta+1}} \exp\left(-\frac{(u_- - u_+)^2}{8(\delta+1)(2\delta+1)} t^{2\delta+1} \right) \\ &+ o\left(t^{-\frac{\delta+1}{2\delta+1}} \exp\left(-\frac{(u_- - u_+)^2}{8(\delta+1)(2\delta+1)} t^{2\delta+1} \right) \right) \end{aligned} \tag{2.97}$$

as $t \to \infty$ with $\hat{\eta} = \left(y - \frac{(u_- + u_+)}{2(\delta+1)} + \hat{c}(t) t^{-2\delta-1} \right) t^{2\delta+1} = O(1)$, and where $\hat{c}(t)$ is given by (2.95).

The asymptotic structure of the solution of IVP as $t \to \infty$ when $u_+ < u_-$ and $-\frac{1}{2} < \delta < 0$ is now complete. A uniform approximation has been given through regions III$^\pm$, IV(a)$^\pm$, IV(b)$^\pm$, TR$^\pm$ and SS. The large-t attractor for the solution of IVP when $u_+ < u_-$ is the Taylor shock profile which allows for the adjustment of the solution form u_+ to u_-.

2.3.4 $\delta = -\frac{1}{2}$

As $t \to \infty$, the asymptotic expansion $(2.22)_2$ and $(2.24)_2$ of regions III$^+$ ($x \to \infty$, $t = O(1)$) and III$^-$ ($x \to -\infty$, $t = O(1)$), respectively, continue to remain uniform provided $|x| \gg t$. However, as already noted, a nonuniformity develops when $|x| = O(t)$, We begin by considering the asymptotic structure as $t \to \infty$ for $x > 0$. To proceed we introduce a new region, region IV$^+$ when $x = O(t)$ as $t \to \infty$. To examine region IV$^+$ we introduce the scaled coordinate $y = xt^{-1}$, where $y = O(1)$ as $t \to \infty$, and write (as suggested by $(2.22))_2$

$$u(y, t) = u_+ + \mathcal{S}_0 e^{-F(y,t)} \tag{2.98}$$

where

$$F(y, t) = F_0(y) t + F_1(y) t^{\frac{1}{2}} + F_2(y) \ln t + F_3(y) + o(1) \tag{2.99}$$

as $t \to \infty$ with $y = O(1)$, and where $F_0(y) > 0$ and $\mathcal{S}_0 = \text{sign}(\mathcal{C}_R)$. On substituting (2.98) and (2.99) into equation (1.1) (when written in terms of y and t) and solving at each order in turn, we find (after matching with $(2.22)_2$ as $y \to \infty$) that

$$u(y, t) = u_+ + \mathcal{S}_0 \exp\left(-\frac{1}{4} y^2 t + (u_+) y t^{\frac{1}{2}} - \frac{1}{2} \ln t + H_0(y) + o(1) \right) \tag{2.100}$$

as $t \to \infty$ with $y = O(1)$, and where the function $H_0 : (0, \infty) \to \mathbb{R}$ is undetermined (being a remnant of the global evolution when $t = O(1)$), but having

$$H_0(y) \sim -\ln y + \ln |\mathcal{C}_R| \quad \text{as} \quad y \to \infty.$$

We now make the assumption (which we will verify as consistent) that

$$H_0(y) \sim -\ln y + \beta_R \quad \text{as} \quad y \to 0^+,$$

where β_R is a constant to be determined. Expansion (2.100) becomes nonuniform when $y = O\left(t^{-\frac{1}{2}}\right)$ as $t \to \infty$ [that is, when $x = O\left(t^{\frac{1}{2}}\right)$ as $t \to \infty$]. To proceed we introduce a new region, region SS when $y = O\left(t^{-\frac{1}{2}}\right)$ as $t \to \infty$. To investigate region SS we introduce the scaled coordinate z via

$$z = yt^{\frac{1}{2}} = xt^{-\frac{1}{2}},$$

where $z = O(1)$ as $t \to \infty$ and look for an expansion of the form

$$u(z,t) = U(z) + o(1) \tag{2.101}$$

as $t \to \infty$ with $z = O(1)$. On substitution of (2.101) into equation (1.1) (when written in terms of z and t) we obtain at leading order

$$U_{zz} - UU_z + \frac{z}{2}U_z = 0, \quad -\infty < z < \infty. \tag{2.102}$$

Equation (2.102) has to be solved subject to matching with region IV$^+$, that is

$$U(z) \sim u_+ + \frac{\mathcal{S}_0 e^{\beta_R}}{z} \exp\left(-\frac{z^2}{4} + u_+ z\right) \quad \text{as} \quad z \to \infty. \tag{2.103}$$

On making the substitution

$$U = \frac{1}{\sqrt{2}}\Omega, \quad z = \sqrt{2}\,\bar{z}, \tag{2.104}$$

equation (2.102) becomes equation (2.14) of Section 2.1.2, which admits the similarity solution (2.15). Therefore, in region SS we have at leading order in expansion (2.101) the similarity solution of Rudenko and Soluyan which connects u_+ (as $z \to \infty$) to u_- (as $z \to -\infty$). This similarity solution is in a stretching frame of reference of thickness $x = O(t^{\frac{1}{2}})$ as $t \to \infty$. We note that expansion (2.101) then satisfies the matching condition (2.103) and fixes

$$\beta_R = \ln|\mathcal{C}_R|.$$

We next consider the asymptotic structure as $t \to \infty$ for $x < 0$. To proceed we introduce a new region, region IV$^-$. The details of this region follow, after minor modification, those given for region IV$^+$ above and are not repeated here. Therefore, we have in region IV$^-$ that

$$u(y,t) = u_- - \mathcal{S}_0 \exp\left(-\frac{1}{4}y^2 t + (u_-)yt^{\frac{1}{2}} - \frac{1}{2}\ln t + \hat{H}_0(y) + o(1)\right) \tag{2.105}$$

as $t \to \infty$ with $y = O(1)$, and where the function $\hat{H}_0 : (0,\infty) \to \mathbb{R}$ is undetermined (being a remnant of the global evolution when $t = O(1)$), but having

$$\hat{H}_0(y) \sim -\ln(-y) + \ln|\mathcal{C}_L| \quad \text{as} \quad y \to \infty.$$

Expansion (2.105) becomes nonuniform when $y = O\left(t^{-\frac{1}{2}}\right)$ as $t \to \infty$ [that is, when $x = O(t^{\frac{1}{2}})$ as $t \to \infty$]. As $y \to 0^-$ we move into region SS. Matching expansion (2.105) (as $y \to 0^-$) with expansion (2.101) (as $z \to -\infty$) follows directly, and requires that

$$\hat{H}_0(y) \sim -\ln(-y) + \ln|\mathcal{C}_L| \quad \text{as} \quad y \to 0^-.$$

The asymptotic structure of the solution to IVP as $t \to \infty$ when $\delta = -\frac{1}{2}$ is now complete. A uniform approximation has been given through regions III$^\pm$, IV$^\pm$ and SS. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Figure 4. The large-time attractor for the solution of IVP when $\delta = -\frac{1}{2}$ is the similarity solution of Rudenko and Soluyan which allows for the adjustment of the solution from u_+ to u_-.

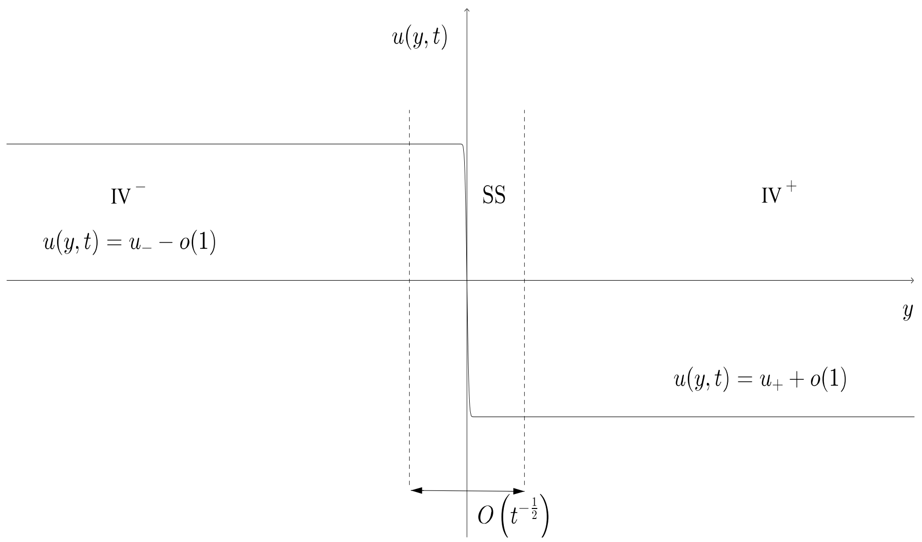

Figure 4: A schematic representation of the asymptotic structure of $u(y,t)$ in the (y,u) plane as $t \to \infty$ for IVP when $\delta = -\frac{1}{2}$. Here we illustrate the case when $u_+ < u_-$, and recall that $y = \frac{x}{t}$.

3 Summary

In this paper we have obtained, via the method of matched asymptotic coordinate expansions, the uniform asymptotic structure of the large-time solution to the initial-value problem IVP over all parameter values. In each case the large-time structure was obtained by careful consideration of the asymptotic structures as $t \to 0$ $(-\infty < x < \infty)$ and as $|x| \to \infty$ $(t \geq O(1))$. The form of the large-time attractor for the solution, $u(x,t)$, of IVP as $t \to \infty$ depends on the problem parameters, and is summarized in Table 1. Although the large-time attractors described in Table 1 (the Taylor shock, expansion wave and the Rudenko-Soluyan similarity solution) are generic coherent structures associated with equation (1.1), the analysis presented in Section 2 has for the first time allowed these structures to be rationally embedded into the large-time solution of initial-value problem IVP.

The form of the large-time attractor for the solution of initial-value problem IVP depends on the problem parameters δ, u_+ and u_- as follows:

(i) When $\delta > -\frac{1}{2}$ and $u_+ > u_-$ the solution $u(x,t)$ to IVP exhibits the formation of an expansion wave profile, where

$$u(x,t) \sim \begin{cases} u_+, & x > \frac{u_+}{\delta+1}t^{\delta+1}, \\ (\delta+1)xt^{-(\delta+1)}, & \frac{u_-}{\delta+1}t^{\delta+1} < x < \frac{u_+}{\delta+1}t^{\delta+1}, \\ u_-, & x < \frac{u_-}{\delta+1}t^{\delta+1}, \end{cases}$$

as $t \to \infty$. We observe that $u(x,t) = (\delta+1)xt^{-(\delta+1)}$ is the degenerate solution of equation (1.1).

(ii) When $\delta > -\frac{1}{2}$ and $u_+ < u_-$ the solution $u(x,t)$ to IVP exhibits the formation of a Taylor shock (hyperbolic tangent) profile, where

$$u\left(\frac{(u_+ + u_-)}{2(\delta+1)}t^{\delta+1} + zt^{-\delta}, t\right) = \left[\frac{(u_+ + u_-)}{2} - \frac{(u_- - u_+)}{2}\tanh\left(\frac{(u_- - u_+)}{4}z + \phi_c\right)\right] + o(1)$$

as $t \to \infty$ with $z = O(1)$, and ϕ_c is a globally determined constant. It follows that the Taylor shock wave-front is at $x = s(t)$, where

$$s(t) = \frac{(u_+ + u_-)}{2(\delta+1)} t^{\delta+1} + ct^{-\delta} + o\left(t^{-\delta}\right) \tag{3.1}$$

as $t \to \infty$, with c being a globally determined constant. The Taylor shock propagation speed is then

$$\dot{s}(t) = \frac{(u_+ + u_-)}{2} t^{\delta} - \delta c t^{-(\delta+1)} + o\left(t^{-(\delta+1)}\right)$$

as $t \to \infty$. Therefore, when $u_+ = -u_-$ the Taylor shock profile is decelerating as $t \to \infty$ with $\dot{s}(t) = O\left(t^{-(\delta+1)}\right)$ as $t \to \infty$, whilst $s(t) \to 0$ ($ct^{|\delta|}$) as $t \to \infty$ for $\delta > 0$ ($-\frac{1}{2} < \delta < 0$) respectively. In each case the solution to IVP approaches a stationary Taylor shock profile located at $x = 0$ and contained in a region of thickness $O(t^{-\delta})$ as $t \to \infty$ (within this region the profile is steepening for $\delta > 0$, whereas the profile is stretching for $-\frac{1}{2} < \delta < 0$). Further, we observe that:

(a) When $\delta > 0$ the Taylor shock profile is located at $x = s(t)$ and contained in a localized region of thickness $O(t^{-\delta})$ as $t \to \infty$ (the profile steepens as $t \to \infty$). The Taylor shock is accelerating in the $+x$ ($-x$) direction as $t \to \infty$ when $u_+ > -u_-$ ($u_+ < -u_-$) respectively.

(b) When $-\frac{1}{2} < \delta < 0$ the Taylor shock profile is located at $x = s(t)$ and contained within a region of thickness $O(t^{-\delta})$ as $t \to \infty$ (the profile becomes stretched $t \to \infty$). The Taylor shock is decelerating in the $+x$ ($-x$) direction as $t \to \infty$ when $u_+ > -u_-$ ($u_+ < -u_-$) respectively.

(iii) When $\delta = -\frac{1}{2}$ the solution $u(x,t)$ to IVP exhibits the formation of the similarity solution found by Rudenko and Soluyan [22], and given by (2.15). We observe from Section 2.1.2 that this structure forms in the small-time solution of IVP when $x = O(t^{\frac{1}{2}})$ as $t \to 0$, and that this similarity profile is in a stretching frame of reference of thickness $O(t^{\frac{1}{2}})$ as $t \to \infty$.

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