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DOI:
10.1016/j.entcs.2016.09.039

## Document Version

Peer reviewed version
Citation for published version (Harvard):
Jakl, T, Jung, A \& Pultr, A 2016, Bitopology and four-valued logic. in L Birkedal \& M Mislove (eds), 32nd Conference on Mathematical Foundations of Programming Semantics: Proceedings. vol. 325, Electronic Notes in Theoretical Computer Science, Elsevier, pp. 201-219, 32nd Conference on Mathematical Foundations of Programming Semantics, Pittsburgh, United States, 23/05/16. https://doi.org/10.1016/j.entcs.2016.09.039

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# Bitopology and four-valued logic 

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#### Abstract

Bilattices and d-frames are two different kinds of structures with a four-valued interpretation. Whereas d-frames were introduced with their topological semantics in mind, the theory of bilattices has a closer connection with logic. We consider a common generalisation of both structures and show that this not only still has a clear bitopological semantics, but that it also preserves most of the original bilattice logic. Moreover, we also obtain a new bitopological interpretation for the connectives of four-valued logic.


Keywords: Bilattices, d-frames, nd-frames, bitopological spaces, four-valued logic.

## 1 Introduction

In 1977, Nuel D. Belnap [5] gave a philosophical justification for distinguishing between two orders when studying information systems: the information order and the logical order. He also suggested that in addition to the classical logical values

[^0]This paper is electronically published in
Electronic Notes in Theoretical Computer Science
URL: www.elsevier.nl/locate/entcs
true and false (denoted $t t$ and $f f$ ) it would also be useful to have the values $T$ and $\perp$ for information-wise maximum and minimum, corresponding to the situation when there is contradicting information and no information, respectively.

Belnap's insights and then Ginsberg's application of those ideas to inference systems [9] motivated Arieli and Avron to develop a formal logical system and to analyse its algebraic models - bilattices [3]. Bilattices proved to be also useful in other areas such as in logic programming [7], algebra [11], and abstract algebraic logic [14].

Later on Jung and Moshier studied bitopological spaces (or bispaces for short) to clarify the interplay of various topologies arising in domain theory. They discovered that bispaces actually also give a very natural semantics to a four-valued logic. The fact that the first topology can represent the observably true predicates and the second the observably false ones gives a good four-valued interpretation/reading and this is even more transparent in the algebraic duals of bitopological spaces, dubbed $d$-frames in [10].

Whereas d-frames were introduced with their topological semantics in mind, the theory of bilattices has a closer connection with logic. Because of this, one might not expect many similarities between both theories but the discovery of the so-called twist-representation of bilattices [6] shows that quite the opposite is the case.

In this paper, we are trying to tackle the obvious question of whether there is a reasonable generalisation of both theories that would give us some better insight into the similarities and differences between bilattices and d-frames. We claim that the answer to this question is yes. As a starting point we take d-frames and we will show that they can be very naturally extended into a new structure which we call $n d$-frames. It seems that this way we get the best from both worlds: We have a clear bitopological semantics while still preserving most of the original logic of bilattices. Moreover, in the nd-frame context the negation of four-valued logic has a new and clear bitopological realisation via interior operators.

This paper contributes to both the study of bilattices and the study of d-frames. For the former, it shows how to generalise bilattices to get four-valued structures where the components are not isomorphic. Contributions to d-frame theory are by giving an explanation of proof-theoretic negation and, moreover, extending this negation to the whole d-frame. By this we also show another connection between geometry (interior operators) and proof theory (cut rules). Moreover, nd-frames allow a finer distinction of bispaces as the class of their spectra is broader than the class of spectra of d-frames. Having a generalisation of both structures allows us to compare partial implication in d-frames with the implication of bilattices and to show that the former is much stronger than latter.

## 2 Preliminaries

Below we give very brief presentations of the two types of structures that this paper aims to combine, bilattices and $d$-frames. As will soon become clear, much of the underlying structure is symmetric with respect to a positive and a negative part; we will respect this when stating a definition or a proposition but will generally restrict proofs to one variant without further comment.

### 2.1 Bilattices

Bilattices are the algebraic manifestation of Belnap's "useful four-valued logic", [5]. One key feature of this logic is paraconsistency, [13], which means that it is not possible to derive an arbitrary proposition from a contradiction. Secondly, the logic is truth-functional in that every connective is characterised by its behaviour on the set of (four) truth values.

Traditionally, bilattices are presented as structures of the type $(A ; \wedge, \vee, \sqcap, \sqcup, f f, t t, \perp, \top, \neg, \supset)$ satisfying a list of axioms. However, a decomposition theorem can be shown for them (see $[4,14,6]$ ) and it is more straightforward to approach the subject from the characterisation that results from it. ${ }^{5}$

Let $H=(H ; \wedge, \vee, 1,0, \rightarrow)$ be a Heyting algebra. On $H \times H$ one defines the bilattice operations by setting, for $\alpha=\left(\alpha_{+}, \alpha_{-}\right), \beta=\left(\beta_{+}, \beta_{-}\right) \in H \times H:{ }^{6}$

$$
\begin{aligned}
\alpha \vee \beta & \stackrel{\text { def }}{\equiv}\left(\alpha_{+} \vee \beta_{+}, \alpha_{-} \wedge \beta_{-}\right), \quad \alpha \wedge \beta \\
\alpha \sqcup \beta & \stackrel{\text { def }}{\equiv}\left(\alpha_{+} \wedge \beta_{+}, \alpha_{-} \vee \beta_{-}\right), \\
& f f \stackrel{\text { def }}{\equiv}\left(\alpha_{+} \vee \beta_{+}, \alpha_{-} \vee \beta_{-}\right), \quad \alpha \sqcap \beta \\
& \stackrel{\text { def }}{\equiv}\left(\alpha_{+} \wedge \beta_{+}, \alpha_{-} \wedge \beta_{-}\right), t t \stackrel{\text { def }}{\equiv}(1,0), \quad \perp \stackrel{\text { def }}{\equiv}(0,0), \quad \top \stackrel{\text { def }}{\equiv}(1,1) .
\end{aligned}
$$

The two final operations deserve to be highlighted: Negation $\neg$ is defined purely by the exchange of components:

$$
\neg \alpha \stackrel{\text { def }}{\equiv}\left(\alpha_{-}, \alpha_{+}\right)
$$

and without reference to the internal logical structure of the component Heyting algebra. Weak implication $\supset$ is the only non-symmetric operation in the signature and it is in fact a remarkable feature of bilattice logic that it can be given in the following way at all:

$$
\alpha \supset \beta \stackrel{\text { def }}{=}\left(\alpha_{+} \rightarrow \beta_{+}, \alpha_{+} \wedge \beta_{-}\right)
$$

The set $H \times H$ together with the four constants and six operations defined above is called the twist-construction over $H$ and denoted by $H^{\bowtie}$. As we said above, the characterisation theorem states that, up to isomorphism, every bilattice arises in this way.

Notice that the "logical" reduct ( $H \times H ; \wedge, \vee, f f, t t)$ and the "informational" reduct ( $H \times H ; \sqcap, \sqcup, \perp, \top$ ) are automatically bounded distributive lattices. The associated orders, $\leq$ and $\sqsubseteq$, however, are not the same; they may be (loosely) said to be "at $90^{\circ}$ to each other", and this helps to explain that negation $\neg$ is antitone w.r.t. the logical order and monotone w.r.t. the informational one.

## $2.2 \quad D$-frames

The motivation for d-frames comes from semantics, in particular, from the observation that "domains" (in the sense of Scott) carry two topologies which are loosely

[^1]connected and in some sense complementary to each other, the Scott-topology and the weak lower topology, [2,8]. Smyth, [15], proposed to interpret open sets as propositions of an "observational logic", and Abramsky fully developed this programme in his celebrated "Domain Theory in Logical Form", [1], but both these works focus entirely on the Scott-topology which begged the question what the logical status of the other topology might be. Taking a step back from domain theory, [10] began the exploration of bitopological spaces under Smyth's interpretation. D-frames are the result of this investigation.

A $d$-frame ${ }^{7}$ is a structure $\mathcal{L}=\left(L_{+} \times L_{-} ;\right.$con, tot $)$where $\left(L_{+} ; \bigvee, \wedge, 0,1\right)$ and ( $L_{-} ; \bigvee, \wedge, 0,1$ ) are frames ${ }^{8}$ and the consistency con $\subseteq L_{+} \times L_{-}$and totality tot $\subseteq$ $L_{+} \times L_{-}$predicates satisfy the following axioms, for all $\alpha, \beta \in L_{+} \times L_{-}$:

```
        (con- \(\downarrow\) ) \(\quad \alpha \in \operatorname{con}\) and \(\beta \sqsubseteq \alpha \Longrightarrow \beta \in \operatorname{con}\),
        (tot-个) \(\alpha \in\) tot and \(\beta \sqsupseteq \alpha \Longrightarrow \beta \in\) tot,
    \((\) con \(-\wedge, \vee) \quad \alpha, \beta \in \operatorname{con} \Longrightarrow \alpha \vee \beta \in \operatorname{con}\) and \(\alpha \wedge \beta \in \operatorname{con}\),
    (tot- \(\wedge, \vee) \quad \alpha, \beta \in\) tot \(\Longrightarrow \alpha \vee \beta \in\) tot and \(\alpha \wedge \beta \in\) tot,
(con,tot-tt,ff) \(\quad t \in\) con and \(t t \in\) tot, \(\quad f f \in \operatorname{con}\) and \(f f \in\) tot,
    (con-tot) \(\quad \alpha \in \operatorname{con}, \beta \in \operatorname{tot}\) and \((\alpha \sqcup \beta=\alpha \wedge \beta\) or \(\alpha \sqcup \beta=\alpha \vee \beta) \Longrightarrow \alpha \sqsubseteq \beta\)
    (con- \(\left.\bigsqcup^{\uparrow}\right) \quad A \subseteq\) con and \(A\) is \(\sqsubseteq\)-directed \(\Longrightarrow \bigsqcup^{\uparrow} A \in\) con.
```

where $\wedge, \vee, \sqcap, \sqcup, f f, t t, \perp, \top$ and the induced logical order $\leq$ and information order $\sqsubseteq$ are defined the same way as in bilattices. In fact, the similarity with bilattices (presented as twist structures) is obvious and it may therefore be helpful to highlight the differences:

- In d-frames, the two component lattices may be different, in bilattices they are identical;
- (consequently) it is not possible to define negation or weak implication on d-frames in the same way as it is done for bilattices;
- frames are complete Heyting algebras (but frame homomorphisms may not preserve Heyting implication);
- the two predicates con and tot are relational, not algebraic structure.

These differences are also apparent in the definition of d-frame homomorphism which we take to be a pair of frame homomorphisms $h_{+}: L_{+} \rightarrow M_{+}, h_{-}: L_{-} \rightarrow M_{-}$ such that $h_{+} \times h_{-}\left[\operatorname{con}_{\mathcal{L}}\right] \subseteq \operatorname{con}_{\mathcal{M}}$ and $h_{+} \times h_{-}\left[\operatorname{tot}_{\mathcal{L}}\right] \subseteq \operatorname{tot}_{\mathcal{M}}$. We denote the category of d-frames and d-frame homomorphisms by d-Frm.

As we said before, d-frames arose from consideration of bitopological spaces and indeed, it is straightforward to adapt the open-set functor from spaces to frames to one from the category biTop of bispaces to d-Frm. To this end, we set $\Omega_{d}(X)=$ $\left(\tau_{+}, \tau_{-} ; \operatorname{tot}_{X}, \operatorname{con}_{X}\right)$ for a bispace ( $X ; \tau_{+}, \tau_{-}$) where, for $U \in \tau_{+}$and $V \in \tau_{-}$,

$$
(U, V) \in \operatorname{con}_{X} \stackrel{\text { def }}{\equiv} U \cap V=\emptyset \quad \text { and } \quad(U, V) \in \operatorname{tot}_{X} \stackrel{\text { def }}{\equiv} U \cup V=X .
$$

[^2]Example 2.1 The dual of the one-point bispace $\mathbf{1}=(\{*\} ; \tau, \tau)$ has exactly four elements: $\perp=(\emptyset, \emptyset), f f=(\emptyset,\{*\}), t t=(\{*\}, \emptyset)$, and $T=(\{*\},\{*\})$. These are the truth values of bilattice logic and in that context the structure is usually denoted $\mathcal{F O U R}$. What we are saying here is that $\mathcal{F O U R}$ is also a canonical d-frame with component frames $L_{+}=L_{-}=\mathbf{2}=\{0<1\}$.

## 3 Nd-frames

We saw in the Preliminaries that, in order to define negation and implication for bilattices represented as twist-structures, we heavily use the fact that the carrier is the product of a Heyting algebra with itself. Therefore, we can freely send elements from one component of the product to the other.

Similarly to bilattices, d-frames are also formed of two components but those do not have to be the same and it seems that there are no natural order-preserving mappings between them (as required by the definition of $\neg$ and $\supset$ ). For example, taking pseudocomplements ${ }^{9}$ is antitone and it could be used to define an operation sometimes called conflation, but this is known to be different from negation.

However, looking at the semantic counterparts of d-frames, i.e. bitopological spaces, suggests that we have very natural candidates for maps between both frames of open sets. Let ( $X ; \tau_{+}, \tau_{-}$) be a bispace; then assigning for every $\tau_{+}$-open (or $\tau_{-}$ open) set its interior with respect to the other topology is a monotone map. Moreover, it also distributes over intersections. Let us denote those maps by $m: \tau_{+} \rightarrow \tau_{-}$ and $p: \tau_{-} \rightarrow \tau_{+}$; to wit:

$$
m: U \in \tau_{+} \longmapsto U^{\circ \tau_{-}} \in \tau_{-} \quad \text { and } \quad p: V \in \tau_{-} \longmapsto V^{\circ \tau_{+}} \in \tau_{+}
$$

When translated to the language of d-frames, one can postulate the existence of maps $m: L_{+} \rightarrow L_{-}$and $p: L_{-} \rightarrow L_{+}$satisfying the following axioms:

```
\((\mathrm{pm}-1) \quad m(a \wedge b)=m(a) \wedge m(b), p(a \wedge b)=p(a) \wedge p(b)\),
\((\mathrm{pm}-2) \quad m(1)=1, p(1)=1\),
\((\mathrm{pm}-3) \quad m(0)=0, p(0)=0\),
(pm-4) \(\quad p \circ m \leq \mathrm{id}, m \circ p \leq \mathrm{id}\).
```

Also, the intuition of $p$ and $m$ being the interiors with respect to the other topology justifies the following axioms involving con and tot:

$$
\begin{array}{ll}
\frac{(a \wedge b, c) \in \operatorname{con}}{(a, m(b) \wedge c) \in \operatorname{con}}(\text { con- } m) & \frac{(a, b \wedge c) \in \text { con }}{(a \wedge p(b), c) \in \operatorname{con}}(\text { con- } p) \\
\frac{(a, m(b) \vee c) \in \text { tot }}{(a \vee b, c) \in \text { tot }}(\text { tot- } m) & \frac{(a \vee p(b), c) \in \text { tot }}{(a, b \vee c) \in \text { tot }}(\text { tot- } p)
\end{array}
$$

Definition $3.1\left(L_{+} \times L_{-}\right.$; con, tot; $\left.p, m\right)$ is an $n d-$ frame if $\left(L_{+}, L_{-} ;\right.$con, tot) is a dframe and all axioms for $(p, m)$ mentioned above, i.e. (pm-1), (pm-2), (pm-3), (pm$4)$, (con- $m$ ), (con- $p$ ), (tot- $m$ ) and (tot- $p$ ), are satisfied.

[^3]Let us recall a known fact about continuous maps:
Lemma 3.2 Let $f: X \rightarrow Y$ be a continuous map and let $M \subseteq Y$, then $f^{-1}\left[M^{\circ}\right] \subseteq$ $\left(f^{-1}[M]\right)^{\circ}$.

This motivates the following definition:
Definition 3.3 A d-frame homomorphism $h: \mathcal{L} \rightarrow \mathcal{M}$ between two nd-frames is an nd-frame homomorphism if $h_{+} \circ p \leq p \circ h_{-}$and $h_{-} \circ m \leq m \circ h_{+}$.

Since the component maps are monotone, it follows that nd-frame homomorphisms are closed under composition. We denote the resulting category with nd-Frm.

Example 3.4 We have seen before that the bilatice $\mathcal{F O U \mathcal { R }}$ may alternately be viewed as a d-frame. The axioms (pm-1) - (pm-4) ensure that the identity function is the unique choice for both $p$ and $m$ so that $\mathcal{F O U \mathcal { R }}=\mathbf{2} \times \mathbf{2}$ also qualifies as an nd-frame. Whether we view it as a bilattice, d-frame, or nd-frame, we always denote it with $\mathcal{F O U R}$.

The theory of d-frames works best when the two topologies complement each other, in the sense of the closed sets of one approximating the opens of the other. Examples of this are given by the Scott-topology and the weak lower topology considered in domain theory. If this is not the case, then the two relations con and tot tend to be trivial, by which we mean that $(a, b) \in$ con iff one of two elements equals 0 , and $(a, b) \in$ tot iff one of them equals 1 . For the new structure $m$ and $p$, the situation is exactly reversed. To see this, consider the following two bispaces:
(i) $X_{1}=(\{a, b\}, \tau, \tau)$ where the only non-trivial open of $\tau$ is $\{a\}$.
(ii) $X_{2}=\left(\{a a, a b, b a, b b\}, \tau_{+}, \tau_{-}\right)$where the only non-trivial open of $\tau_{+}$is $\{a a, a b\}$ and that of $\tau_{-}$is $\{a a, b a\}$.

In both cases, con and tot are trivial, which means that $\Omega_{d}\left(X_{1}\right)$ and $\Omega_{d}\left(X_{2}\right)$ are isomorphic. On the other hand, the interior operators on $X_{1}$ are the identity whereas on $X_{2}$ they are trivial. Both bispaces will turn out to be nd-sober, whereas only $X_{2}$ is d-sober.

## 4 Logic of nd-frames

We introduced $p$ and $m$ to be able to define negation and implication for d-frames similarly to how they are defined for bilattices in their twist-structures representation. Let $\left(L_{+} \times L_{-}\right.$; con, tot; $\left.p, m\right)$ be an nd-frame and define

$$
\neg \varphi \stackrel{\text { def }}{\equiv}\left(p\left(\varphi_{-}\right), m\left(\varphi_{+}\right)\right) \quad \text { and } \quad \varphi \supset \psi \stackrel{\text { def }}{\equiv}\left(\varphi_{+} \rightarrow \psi_{+}, m\left(\varphi_{+}\right) \wedge \psi_{-}\right)
$$

Now any nd-frame $\mathcal{L}$ gives rise to the same signature as bilattices: $\left(L_{+} \times L_{-} ; \wedge, \vee, \sqcap, \sqcup, f f, t t, \perp, \top, \neg, \supset\right)$. Let $\mathbf{L n}$ be the language of bilattices and let $\mathrm{Fm}(\mathbf{L n})$ be the term algebra of $\mathbf{L n}$ (generated by countably many variables). Valuations are the $\mathbf{L n}$-homomorphisms $\operatorname{Fm}(\mathbf{L n}) \rightarrow \mathcal{L}$. We can define (algebraic) semantic validity the same way as it was defined for bilattices [14]; we say $\varphi$ holds in (or, is
valid in) $\mathcal{L}$, and write $\mathcal{L} \models \varphi$ iff

$$
v(\varphi)=v(\varphi \supset \varphi) \text { for all valuations } v: \operatorname{Fm}(\mathbf{L n}) \rightarrow \mathcal{L},
$$

Before we get to the axioms let us first prove the following lemma.
Lemma 4.1 The following holds in all nd-frames $\mathcal{L}$ :
(L1) $\mathcal{L} \models \varphi$ iff $v(\varphi) \sqsupseteq \mathrm{t}$ (or equivalently: $v(\varphi)_{+}=1$ ) for all valuations $v$ into $\mathcal{L}$;
(L2) $\mathcal{L} \models \varphi \supset \psi$ iff $v(\varphi)_{+} \leq v(\psi)_{+}$for all valuations $v$ into $\mathcal{L}$; and
(L3) $\mathcal{L} \models \varphi \equiv \psi$ iff $v(\varphi)_{+}=v(\psi)_{+}$for all valuations $v$ into $\mathcal{L}$, where $\varphi \equiv \psi$ is a shorthand for $(\alpha \supset \beta) \wedge(\beta \supset \alpha)$.

## Proof.

(L1) Right-to-left implication: If $v(\varphi)=(1, a)$ then $v(\varphi \supset \varphi)=v(\varphi) \supset v(\varphi)=$ $(1 \rightarrow 1, m(1) \wedge a)=(1, a)=v(\varphi)$. Reverse direction: $v(\varphi)=v(\varphi \supset \varphi)=$ $v(\varphi) \supset v(\varphi)$ implies that the positive parts are equal and therefore we have $v(\varphi)_{+}=v(\varphi \supset \varphi)_{+}=1$ and $v(\varphi) \sqsupseteq t t$.
(L2) From (L1) we know that $\mathcal{L} \models \varphi \supset \psi$ iff $v(\varphi \supset \psi)_{+}=v(\varphi)_{+} \rightarrow v(\psi)_{+}$is equal to 1 for all valuations $v$, and this is true if and only if $v(\varphi)_{+} \leq v(\psi)_{+}$.
(L3) Follows from (L2) and from the fact that $\mathcal{L} \models v(\varphi \wedge \psi) \sqsupseteq t$ iff $\mathcal{L} \models v(\varphi) \sqsupseteq t$ and $\mathcal{L} \models v(\psi) \sqsupseteq t$. Then, $\mathcal{L} \models v(\varphi \equiv \psi)$ iff $\mathcal{L} \models \varphi \supset \psi$ and $\mathcal{L} \models \varphi \subset \psi$ iff $v(\varphi)_{+} \leq v(\psi)_{+}$and $v(\varphi)_{+} \geq v(\psi)_{+}$for all valuations $v$.

Arieli and Avron, [3], gave a Hilbert-style axiomatisation of a four-valued logic which is sound and complete with respect to bilattices. Here we show that a large part of their logic is still valid in nd-frames.

Theorem 4.2 The following axioms of four-valued logic are valid in any nd-frame: (Weak implication)

$$
\begin{align*}
(\supset 1) & \varphi \supset(\psi \supset \varphi) \\
(\supset \text { 2) } & (\varphi \supset(\psi \supset \gamma)) \supset((\varphi \supset \psi) \supset(\varphi \supset \gamma)) \\
(\neg \neg R) & \neg \neg \varphi \supset \varphi
\end{align*}
$$

(Logical conjunction and disjunction)

$$
\begin{array}{ll}
(\wedge \supset) & (\varphi \wedge \psi) \supset \varphi \text { and }(\varphi \wedge \psi) \supset \psi \\
(\supset \wedge) & \varphi \supset(\psi \supset(\varphi \wedge \psi)) \\
(\supset \mathrm{tt}) & \varphi \supset \mathrm{tt} \\
(\supset \vee) & \varphi \supset(\varphi \vee \psi) \text { and } \psi \supset(\varphi \vee \psi) \\
(\vee \supset) & (\varphi \supset \gamma) \supset((\psi \supset \gamma) \supset((\varphi \vee \psi) \supset \gamma)) \\
(\supset \mathrm{ff}) & \text { ff } \supset \varphi
\end{array}
$$

(Informational conjunction and disjunction)

$$
\begin{array}{ll}
(\sqcap \supset) & (\varphi \sqcap \psi) \supset \varphi \text { and }(\varphi \sqcap \psi) \supset \psi \\
(\supset \sqcap) & \varphi \supset(\psi \supset(\varphi \sqcap \psi))
\end{array}
$$

$$
\begin{align*}
(\supset \top) & \varphi \supset \top \\
(\supset \sqcup) & \varphi \supset(\varphi \sqcup \psi) \text { and } \psi \supset(\varphi \sqcup \psi) \\
(\sqcup \supset) & (\varphi \supset \gamma) \supset((\psi \supset \gamma) \supset((\varphi \sqcup \psi) \supset \gamma)) \\
(\supset \perp) & \perp \supset \varphi \\
\text { (Negation) } & \\
(\neg \wedge L) & \neg(\varphi \wedge \psi) \subset \neg \varphi \vee \neg \psi \\
(\neg \vee) & \neg(\varphi \vee \psi) \equiv \neg \varphi \wedge \neg \psi \\
(\neg \sqcap) & \neg(\varphi \sqcap \psi) \equiv \neg \varphi \sqcap \neg \psi \\
(\neg \sqcup L) & \neg(\varphi \sqcup \psi) \subset \neg \varphi \sqcup \neg \psi \\
(\neg \supset R) & \neg(\varphi \supset \psi) \supset \varphi \wedge \neg \psi
\end{align*}
$$

Furthermore, the rule of Modus Ponens is sound:

$$
\begin{equation*}
\varphi,(\varphi \supset \psi) \vdash \psi \tag{MP}
\end{equation*}
$$

Proof. The axioms ( $\supset 1$ ), ( $\supset$ ), all the logical conjunction and disjunction axioms and the informational conjunction and disjunction axioms hold for the same reason. We know from Lemma 4.1 (L1) that only the first coordinate determines their validity. Moreover, those axioms do not contain negation and so they hold simply because, when projected to the first coordinate, they hold in all Heyting algebras (and therefore also in frames).

From (L2) we know that ( $\neg \wedge \mathrm{L}$ ) is equivalent to $p\left(v(\varphi)_{-} \vee v(\psi)_{-}\right) \geq p\left(v(\varphi)_{-}\right) \vee$ $p\left(v(\psi)_{-}\right)$for all valuations $v$ which is true since $p$ is monotone. The same argument applies for ( $\neg \sqcup \mathrm{L}$ ). From (L3) we know that $(\neg \mathrm{V})$ and $(\neg \square)$ are equivalent to $p\left(v(\varphi)_{-} \wedge v(\psi)_{-}\right)=p\left(v(\varphi)_{-}\right) \wedge p\left(v(\psi)_{-}\right)$for all valuations $v$ and this is true simply because $p$ preserves finite infima.
(L2) implies that $(\neg \supset \mathrm{R})$ is equivalent to $v(\neg(\varphi \supset \psi))_{+} \leq v(\varphi \wedge \neg \psi)_{+}$by expanding the definitions we get $(\neg v(\varphi \supset \psi))_{+}=p\left(v(\varphi \supset \psi)_{-}\right)=p\left(m\left(v(\varphi)_{+}\right) \wedge\right.$ $\left.v(\psi)_{-}\right)=p\left(m\left(v(\varphi)_{+}\right)\right) \wedge p\left(v(\psi)_{-}\right)$which, by $p \circ m \leq \mathrm{id}$, is less or equal to $v(\varphi)_{+} \wedge$ $p\left(v(\psi)_{-}\right)=v(\varphi \wedge \neg \psi)_{+}$as we wanted.

For Modus Ponens, no matter if we interpret comma as $\wedge$ or as $\square$ we get the requirement that $v(\varphi)_{+} \wedge\left(v(\varphi)_{+} \rightarrow v(\psi)_{+}\right)=1$ should imply $v(\psi)_{+}=1$ which is again true for all Heyting algebras.

Remark 4.3 The axioms denoted by $(\star A)$ or $(\star B)$ are the only axioms that differ from the original axioms of bilattices because they are expressed as implications, whereas the original axioms are equivalences. Requiring equivalence instead of implication in the axioms marked by $(\star A)$ is equivalent to requiring that $p \circ m=$ $\mathrm{id}^{10}$ and requiring equivalences for axioms marked by $(\star B)$ is the same as requiring that $p$ preserves finite suprema. Also, in some presentations of bilattices, [3], the following axiom, called Peirce's law, is added
$(\supset 3) \quad((\varphi \supset \psi) \supset \varphi) \supset \varphi$
Assuming this to hold is equivalent to assuming that $L_{+}$is a Boolean frame.

[^4]
### 4.1 Implications and a cut rule

One of the nice properties of d-frames is that one can restrict one's attention to the set con of consistent predicates without losing any expressivity, see [10, Proposition 7.4]. We can think of this structure as a semantics for predicates without contradictions. Moreover, there is a binary relation between the elements of con which is in many ways similar to a consequence relation (see Section 7 of [10]). Define, for all $\alpha, \beta \in \mathrm{con}$,

$$
\alpha \prec \beta \quad \stackrel{\text { def }}{\equiv} \quad\left(\beta_{+}, \alpha_{-}\right) \in \text { tot. }
$$

Given the many similarities between d-frames and bilattices one wonders what exactly the relationship between $\prec$ and $\supset$ is. We would like to suggest that the right place to answer this question is within nd-frames as they generalise both notions. Indeed, we will see below that $\prec$ is, from this perspective, much stronger than $\supset$.

Theorem 4.4 Let $\mathcal{L}$ be an nd-frame and let $\alpha, \beta \in \operatorname{con}$ such that $\alpha \prec \beta$. Then the following also hold
(i) $\alpha^{\prime} \supset \beta^{\prime} \sqsupseteq \mathrm{tt}$ for all $\alpha^{\prime}, \beta^{\prime} \in$ con with $\alpha^{\prime} \sqsupseteq \alpha$ and $\beta^{\prime} \sqsupseteq \beta$;
(ii) $\neg \beta^{\prime} \supset \neg \alpha^{\prime} \sqsupseteq \mathrm{tt}$ for all $\alpha^{\prime}, \beta^{\prime} \in \mathrm{con}$ with $\alpha^{\prime} \sqsupseteq \alpha$ and $\beta^{\prime} \sqsupseteq \beta$;

Remark: The logical conjunction of $\alpha \supset \beta$ with $\neg \beta \supset \neg \alpha$ is called strong implication in the bilattice logic literature.

Proof. We know from [10, Proposition 7.1(4)] that $\alpha \prec \beta$ implies $\alpha^{\prime} \prec \beta^{\prime}$ whenever $\alpha^{\prime} \sqsupseteq \alpha$ and $\beta^{\prime} \sqsupseteq \beta$ in con, so it suffices to show the statements for $\alpha$ and $\beta$ : By Lemma $4.1(\mathrm{~L} 2),(\alpha \supset \beta) \sqsupseteq t$ iff $\alpha_{+} \leq \beta_{+}$and this follows, by (con-tot), from $\left(\beta_{+}, \alpha_{-}\right) \in$ tot and $\alpha \in$ con. Similarly, since $(\neg \beta \supset \neg \alpha)=\left(p\left(\beta_{-}\right) \rightarrow\right.$ $\left.p\left(\alpha_{-}\right), m p\left(\beta_{-}\right) \wedge m\left(\alpha_{+}\right)\right)$, then $(\neg \beta \supset \neg \alpha) \sqsupseteq t$ iff $p\left(\beta_{-}\right) \rightarrow p\left(\alpha_{-}\right)=1$ which is equivalent to $p\left(\beta_{-}\right) \leq p\left(\alpha_{-}\right)$. However, we know that $\left(\beta_{+}, \alpha_{-}\right) \in$ tot and $\beta \in \mathrm{con}$, therefore, by (con-tot), $\beta_{-} \leq \alpha_{-}$and, since $p$ is monotone, also $p\left(\beta_{-}\right) \leq p\left(\alpha_{-}\right)$.

The (tot- $m$ ) and (tot- $p$ ) axioms give us immediately the following two rules combining strong implication and negation:

$$
\frac{\alpha \prec \neg \beta \vee \gamma}{\alpha \wedge \beta \prec \gamma} \quad \text { and } \quad \frac{\alpha \wedge \neg \beta \prec \gamma}{\alpha \prec \beta \vee \gamma}
$$

and from these the following cut rule follows by the transitivity of $\prec[10$, Proposition 7.1 (3)]:

$$
\frac{\alpha \prec \neg \neg \beta \vee \gamma \quad \gamma \wedge \neg \neg \alpha^{\prime} \prec \beta^{\prime}}{\alpha \wedge \alpha^{\prime} \prec \beta \vee \beta^{\prime}}
$$

## 5 Stone duality for nd-frames

### 5.1 Spectra of nd-frames

In this section we define a spectrum functor $\Sigma$ : nd-Frm $\rightarrow$ biTop by extending the definition of the spectrum functor for d-frames $\Sigma_{d}$ : d-Frm $\rightarrow$ biTop as presented
in [10]. Let $\mathcal{L}=\left(L_{+} \times L_{-} ;\right.$con, tot; $\left.p, m\right)$ be an nd-frame. Define $\Sigma(\mathcal{L})$ to be the bispace $\left(\Sigma(\mathcal{L}) ; \Phi_{+}\left[L_{+}\right], \Phi_{-}\left[L_{-}\right]\right)$where the underlying set $\Sigma(\mathcal{L})$ is the set of $n d-$ points, that is, the pairs $\left(F_{+}, F_{-}\right)$where $F_{+}$and $F_{-}$are complete prime filters of $L_{+}$and $L_{-}$, respectively, such that, for all $\alpha \in L_{+} \times L_{-}$,

$$
\begin{aligned}
\left(\mathrm{dp}_{\mathrm{con}}\right) & \alpha \in \mathrm{con} \Longrightarrow \alpha_{+} \notin F_{+} \text {or } \alpha_{-} \notin F_{-} ; \\
\left(\mathrm{dp}_{\mathrm{tot}}\right) & \alpha \in \text { tot } \Longrightarrow \alpha_{+} \in F_{+} \text {or } \alpha_{-} \in F_{-} ; \\
\left(\mathrm{dp}_{p}\right) & p\left(\alpha_{-}\right) \in F_{+} \Longrightarrow \alpha_{-} \in F_{-} ; \\
\left(\mathrm{dp}_{m}\right) & m\left(\alpha_{+}\right) \in F_{-} \Longrightarrow \alpha_{+} \in F_{+}
\end{aligned}
$$

Equivalently, we can define the underlying set of $\Sigma(\mathcal{L})$ to be the set of all ndframe homomorphisms from $\mathcal{L}$ to $\mathcal{F O U R}$. The topologies of $\Sigma(\mathcal{L})$ are defined the same way as for spectra of d-frames. That is, $\Phi_{+}\left[L_{+}\right]=\left\{\Phi_{+}(a): a \in L_{+}\right\}$and $\Phi_{-}\left[L_{-}\right]=\left\{\Phi_{-}(b): b \in L_{-}\right\}$where

$$
\Phi_{+}(a)=\left\{\left(F_{+}, F_{-}\right) \mid a \in F_{+}\right\} \quad \text { and } \quad \Phi_{-}(b)=\left\{\left(F_{+}, F_{-}\right) \mid b \in F_{-}\right\} .
$$

Also, similarly to the d-frame spectrum functor, for every nd-frame homomorphism $h: \mathcal{L} \rightarrow \mathcal{M}$, set $\Sigma(h): \Sigma(\mathcal{M}) \rightarrow \Sigma(\mathcal{L})$ to be the map

$$
\Sigma(h): \quad\left(F_{+}, F_{-}\right) \quad \longmapsto \quad\left(h_{+}^{-1}\left[F_{+}\right], h_{-}^{-1}\left[F_{-}\right]\right) .
$$

Proposition 5.1 $\Sigma$ is a contravariant functor from nd-Frm to biTop.
Proof. $\Sigma$ is well defined on objects for the same reason as the corresponding functor for d-frames [10]. When we think of the nd-points of an nd-frame $\mathcal{L}$ as nd-frame homomorphisms $\mathcal{L} \rightarrow \mathcal{F O U \mathcal { R }}$ we see that $\Sigma$ is also well defined on nd-Frm morphisms simply because nd-frame homomorphisms are closed under composition.

### 5.2 Nd-frames from bispaces

Let $X=\left(X ; \tau_{+}, \tau_{-}\right)$be a bispace. Set $\Omega(X)=\left(\tau_{+}, \tau_{-} ; \operatorname{con}_{X}, \operatorname{tot}_{X} ; p_{X}, m_{X}\right)$ where $\operatorname{con}_{X}$ and $\operatorname{tot}_{X}$ are as before and

$$
m_{X}: U_{+} \longmapsto U_{+}^{\circ \tau_{-}} \quad \text { and } \quad p_{X}: U_{-} \longmapsto U_{-}^{\circ \tau_{+}}
$$

Again, $\Omega$ acts on morphisms the same way as the d-frames analogue does, i.e. for a bicontinuous map $f: X \rightarrow Y$ set $\Omega(f): \Omega Y \rightarrow \Omega X$ to be the map

$$
\Omega(f): \quad\left(U_{+}, U_{-}\right) \quad \longmapsto \quad\left(f^{-1}\left[U_{+}\right], f^{-1}\left[U_{-}\right]\right) .
$$

Proposition 5.2 $\Omega$ is a contravariant functor from biTop to nd-Frm.
Proof. $\Omega$ is clearly well defined on objects. From the duality for d-frames, we know that the $\Omega$-image of a bicontinuous map is a d-frame homomorphism. The fact that it is also an nd-frame homomorphism, that is $f(\neg \alpha) \sqsubseteq \neg(f \alpha)$ for all $\alpha$, follows directly from Lemma 3.2.

### 5.3 Sobriety, spatiality and the adjunction

We say that a bispace $X$ is $n d$-sober if there exists an nd-frame $\mathcal{L}$ such that $X \cong$ $\Sigma(\mathcal{L})$. We also have the usual embedding into the sobrification of a space (the unit of adjunction) $\eta_{X}: X \rightarrow \Sigma \Omega(X)$ defined as $x \mapsto\left(\mathcal{U}_{+}(x), \mathcal{U}_{-}(x)\right)$ where $\mathcal{U}_{+}(x)$ and $\mathcal{U}_{-}(x)$ are the neighbourhood filters in $\tau_{+}$and $\tau_{-}$, respectively. For the same reason as in the case of d-frames, $\eta_{X}$ is natural in $X$.

Theorem 5.3 For a bitopological space $X$, the following are equivalent:
(i) $X$ is nd-sober;
(ii) $X$ is bihomeomorphic to $\Sigma \Omega(X)$;
(iii) The unit map $\eta_{X}$ is a bihomeomorphism;
(iv) The unit map $\eta_{X}$ is a bijection.

The reader may now check that the two examples we gave earlier (at the end of Section 3) are indeed nd-sober.

We say that an nd-frame $\mathcal{L}$ is spatial if there exists a bitopological space $X$ such that $\mathcal{L} \cong \Omega(X)$. Again, similarly to d-frame theory, we have the (co-unit) map $\epsilon_{\mathcal{L}}: \mathcal{L} \rightarrow \Omega \Sigma(L)$ defined as $(a, b) \mapsto\left(\Phi_{+}(a), \Phi_{-}(b)\right)$. This, again, is natural in $\mathcal{L}$.

Theorem 5.4 For an nd-frame $\mathcal{L}$, the following are equivalent:
(i) $\mathcal{L}$ is spatial.
(ii) $\mathcal{L} \cong \Omega \Sigma(\mathcal{L})$.
(iii) The co-unit $\epsilon_{\mathcal{L}}$ is an isomorphism.
(iv) The co-unit $\epsilon_{\mathcal{L}}$ is injective, reflects con and tot, and $\epsilon_{\mathcal{L}}^{-1}(\neg \alpha) \sqsubseteq \neg \epsilon_{\mathcal{L}}^{-1}(\alpha)$ for all $\alpha \in \Omega \Sigma(\mathcal{L})$.
(v) $\mathcal{L}$ satisfies the following conditions:

$$
\begin{array}{cl}
\left(s_{+}\right) & \forall x \not \leq x^{\prime} \in L_{+} \exists\left(F_{+}, F_{-}\right) \in \Sigma(\mathcal{L}) \cdot x \in F_{+}, x^{\prime} \notin F_{+} ; \\
\left(s_{-}\right) & \forall y \not \leq y^{\prime} \in L_{-} \exists\left(F_{+}, F_{-}\right) \in \Sigma(\mathcal{L}) \cdot y \in F_{-}, y^{\prime} \notin F_{-} ; \\
\left(s_{\text {con }}\right) & \forall \alpha \notin \operatorname{con} \exists\left(F_{+}, F_{-}\right) \in \Sigma(\mathcal{L}) \cdot \alpha_{+} \in F_{+}, \alpha_{-} \in F_{-} ; \\
\left(s_{\text {totot }}\right) & \forall \alpha \notin \text { tot } \exists\left(F_{+}, F_{-}\right) \in \Sigma(\mathcal{L}) \cdot \alpha_{+} \notin F_{+}, \alpha_{-} \notin F_{-} ; \\
\left(s_{p}\right) & \forall a \not 又 p(x) \in L_{+} \exists\left(F_{+}, F_{-}\right) \in \Sigma(\mathcal{L}) \cdot a \in F_{+}, x \notin F_{-} ; \\
\left(s_{m}\right) & \forall b \not 又 m(y) \in L_{-} \exists\left(F_{+}, F_{-}\right) \in \Sigma(\mathcal{L}) \cdot y \notin F_{+}, b \in F_{-} .
\end{array}
$$

Corollary 5.5 $\Omega$ and $\Sigma$ form a (dual) adjunction with $\eta$ and $\epsilon$ being the unit and co-unit, respectively. Moreover, the restriction of $\Sigma$ and $\Omega$ to spatial nd-frames and nd-sober bispaces, respectively, forms a duality of categories.

It is a good sign that extending Stone duality for d-frames to nd-frames is quite straightforward as it points towards the robustness of the theory. All the additional assumptions make sense topologically given that $p$ and $m$ should correspond to interior operators. The only difference with d-frame duality is that we added the conditions $\left(\mathrm{dp}_{p}\right)$ and $\left(\mathrm{dp}_{m}\right)$. Similarly, in the characterisation of spatial nd-frames we needed to assume $\left(s_{p}\right)$ and $\left(s_{m}\right)$ in addition to the original conditions for spatial d-frames.

On the other hand, the language of nd-frames is definitely more expressive in
the sense that more bispaces are nd-sober than d-sober. We gave an example of this at the end of Section 3.

## 6 Canonical ( $p, m$ )

Every d-frame can be turned into an nd-frame in a trivial way, just augment ( $L_{+} \times L_{-}$; con, tot) with $p^{\text {triv }}$ and $m^{\text {triv }}$ where $p^{\text {triv }}$ and $m^{\text {triv }}$ are trivial in the sense that they send 1 to 1 and everything else to 0 . It is easy to see that this construction provides a left adjoint to the forgetful functor from nd-Frm to d-Frm that erases $p$ and $m$.

The purpose of this section is to demonstrate that under mild conditions on a d-frame, a more interesting choice for $p$ and $m$ is available, one which interacts well with Stone duality. For motivation we begin by reviewing the notion of regularity for d-frames.

For a d-frame $\mathcal{L}=\left(L_{+} \times L_{-} ;\right.$con, tot $)$and for $c, a \in L_{+}$we say that $c$ is wellinside $a$ (and write $c \triangleleft_{+} a$ ) if there exists a $d \in L_{-}$such that $(c, d) \in$ con and $(a, d) \in$ tot. We define $c \triangleleft_{-} a$ for $c, a \in L_{-}$dually, that is, with the roles of $L_{+}$and $L_{-}$switched. We say that $\mathcal{L}$ is $d$-regular if

$$
a=\bigvee\left\{c \in L_{+} \mid c \triangleleft_{+} a\right\} \quad \text { and } \quad b=\bigvee\left\{c \in L_{-} \mid c \triangleleft_{-} b\right\}
$$

for all $a \in L_{+}$and $b \in L_{-}$. Finally, we say that a bitopological space $X$ is $d$-regular if $\Omega_{d}(X)$ is. Note that the well-inside relation has a clear bitopological reading. For $U, V \in \tau_{+}, U \triangleleft_{+} V$ just means that $\tau_{-}$-closure of $U$ is a subset of $V$.

We can express the interior operations of d-regular bispaces explicitly in the language of d-frames. Indeed, let $X=\left(X ; \tau_{+}, \tau_{-}\right)$be d-regular. Then, for a $U \in \tau_{+}$,

$$
U^{\circ \tau_{-}}=\bigcup\left\{V \in \tau_{-} \mid \exists V^{\prime} \in \tau_{+} . V \cap V^{\prime}=\emptyset \text { and } V^{\prime} \cup U=X\right\}
$$

Let $\mathcal{L}=\Omega_{d}(X)$, then the term above becomes, for an $a \in L_{+}$,

$$
m^{r}(a)=\bigvee\left\{x_{-} \in L_{-} \mid \exists x_{+} \in L_{+} .\left(x_{+}, x_{-}\right) \in \text { con and } x_{+} \vee a=1\right\}
$$

Notice the similarity of the relationship between $x_{-}$and $a$ in this definition, and the well-inside relation defined above, except that here it is between elements from the two different components of $\mathcal{L}$. Also note that the definition of $m^{r}(a)$ does not presuppose regularity of the underlying d-frame.

To simplify the definition of $m^{r}$ a bit further, recall for any $x \in L_{-}$the pseudocomplement $x^{*}$ of $x$ is defined as $\bigvee\left\{c \in L_{+} \mid(c, x) \in \operatorname{con}\right\}$. This allows us to define our candidate interior operators as follows

$$
m^{r}(a)=\bigvee\left\{x \in L_{-} \mid x^{*} \vee a=1\right\} \quad \text { and } \quad p^{r}(b)=\bigvee\left\{x \in L_{+} \mid x^{*} \vee b=1\right\}
$$

and to prove some of the required properties. To begin we see that $m^{r}(1)=$ $\bigvee\left\{x \mid x^{*} \vee 1=1\right\} \geq \bigvee\{1\}=1$. For the preservation of 0 recall that $x^{*}=1$ implies $x=0$ because $\left(x^{*}, x\right)=(1, x) \in$ con and $(1,0) \in$ tot gives, by (con-tot), $x \leq 0$. Therefore $m^{r}(0)=\bigvee\left\{x \mid x^{*} \vee 0=1\right\}=\bigvee\{0\}=0$.

It is clear that $m^{r}$ is monotone, so in order to show that it preserves binary meets, it suffices to check that $m^{r}\left(a \wedge a^{\prime}\right) \geq m^{r}(a) \wedge m^{r}\left(a^{\prime}\right)$ for all $a, a^{\prime} \in L_{+}$. Assume $x^{*} \vee a=1$ and $x^{*} \vee a^{\prime}=1$, then because pseudocomplement is antitone, we have for $x^{\prime \prime}=x \wedge x^{\prime}, x^{\prime \prime *} \vee a=1$ and $x^{\prime \prime *} \vee a^{\prime}=1$ from which it follows that $x^{\prime \prime *} \vee\left(a \wedge a^{\prime}\right)=1$ and hence $x^{\prime \prime} \leq m^{r}\left(a \wedge a^{\prime}\right)$. Frame distributivity now allows us to conclude the desired inequality.

We are also able to show $($ con $-m)$. For this let $(a \wedge b, c) \in$ con and let $x \in L_{-}$be such that $x^{*} \vee b=1$. Since, $\left(x^{*}, x\right) \in$ con, by (con- $\vee$ ), we have that $(a \wedge b, c) \vee\left(x^{*}, x\right) \in$ con. Therefore, since con is $\sqsubseteq$-downwards closed,

$$
(a \wedge b, c) \vee\left(x^{*}, x\right)=\left(\left(a \vee x^{*}\right) \wedge\left(b \vee x^{*}\right), c \wedge x\right) \sqsupseteq(a, c \wedge x) \in \mathrm{con}
$$

where the inequality follows from $x^{*} \vee b=1$. Since $(a, c \wedge x) \in$ con for all $x$ such that $x^{*} \vee b=1$, then by (con $-\bigsqcup^{\uparrow}$ ) and frame distributivity we get that also $\left(a, c \wedge m^{r}(b)\right)=\left(a, c \wedge \bigvee\left\{x \mid x^{*} \vee b=1\right\}\right) \in$ con.

However, we can show neither ( $\mathrm{pm}-4$ ) nor ( $\mathrm{tot}-\mathrm{p}$ ) at this level of generality. To make progress, recall the following two infinitary cut rules for d-frames (already discussed in [10], but not part of the definition of d-frames):

$$
\begin{aligned}
& \frac{\left(x, y \vee \bigvee_{i \in I} b_{i}\right) \in \text { tot, }, \forall i \in I .\left(x \vee a_{i}, y\right) \in \text { tot, }\left(a_{i}, b_{i}\right) \in \operatorname{con}}{(x, y) \in \operatorname{tot}}\left(\mathrm{CUT}_{r}\right) \\
& \frac{\left(x \vee \bigvee_{i \in I} a_{i}, y\right) \in \text { tot }, \forall i \in I .\left(x, y \vee b_{i}\right) \in \text { tot, }\left(a_{i}, b_{i}\right) \in \mathrm{con}}{(x, y) \in \text { tot }}\left(\mathrm{CUT}_{l}\right)
\end{aligned}
$$

These two rules are precisely what we need to complete our construction:
Proposition 6.1 Let $\mathcal{L}=\left(L_{+} \times L_{-}\right.$; con, tot) be a d-frame satisfying the infinitary cut rules. Then, $\left(\mathcal{L} ; p^{r}, m^{r}\right)=\left(L_{+} \times L_{-} ;\right.$con, tot $\left.; p^{r}, m^{r}\right)$ is an $n d$-frame.

Proof. Only (pm-4) and (tot $-p$ ) remain to be shown. The former says that $m^{r} p^{r}(b) \leq b$ for all $b \in L_{-}$. Since

$$
m^{r} p^{r}(b)=\bigvee\left\{y \in L_{-} \mid y^{*} \vee p^{r}(b)=1\right\}
$$

it is enough to show that every $y \in L_{-}$, such that $y^{*} \vee p^{r}(b)=1$, is less or equal to $b$. From the definition of $p^{r}$ we have $(1,0)=\left(y^{*} \vee \bigvee\left\{x \mid x^{*} \vee b=1\right\}, 0\right) \in$ tot, therefore, from (tot- $\uparrow$ ), we get

$$
\begin{equation*}
\left(y^{*} \vee \bigvee\left\{x \mid x^{*} \vee b=1\right\}, b\right) \in \operatorname{tot} \tag{1a}
\end{equation*}
$$

(2a) for all $x$ such that $x^{*} \vee b=1:\left(y^{*}, x^{*} \vee b\right) \in$ tot
(3a) $\quad\left(x, x^{*}\right) \in$ con.
By applying $\left(\mathrm{CUT}_{l}\right)$ to (1a), (2a) and (3a) we obtain $\left(y^{*}, b\right) \in$ tot, and from (contot) that $y \leq b$ as we wanted.

To show (tot $-p$ ), let $\left(a \vee p^{r}(b), c\right) \in$ tot. Again, by unwrapping the definitions we get $\left(a \vee \bigvee\left\{x \mid x^{*} \vee b=1\right\}, c\right) \in$ tot and this, by (tot- $\uparrow$ ), gives us
(1b) $\quad\left(a \vee \bigvee\left\{x \mid x^{*} \vee b=1\right\}, b \vee c\right) \in$ tot
(2b) for all $x$ such that $x^{*} \vee b=1:\left(a, x^{*} \vee b \vee c\right) \in$ tot.

$$
(3 \mathrm{~b}) \quad\left(x, x^{*}\right) \in \mathrm{con} .
$$

Therefore, $\left(\mathrm{CUT}_{l}\right)$ applied to $(1 \mathrm{~b}),(2 \mathrm{~b})$ and $(3 \mathrm{~b})$ gives us that $(a, b \vee c) \in$ tot.
Proposition 6.2 The mapping $\mathrm{N}: \mathbf{d - F r m}_{C U T} \rightarrow$ nd-Frm assigning $\mathcal{L} \mapsto$ $\left(\mathcal{L} ; p^{r}, m^{r}\right)$ is functorial, where $\mathbf{d - F r m}{ }_{C U T}$ is the category of d-frames satisfying the infinitary cut rules.

Proof. N is well defined on objects by Proposition 6.1. For morphisms, let $h: \mathcal{L} \rightarrow$ $\mathcal{M}$ be a d-frame homomorphism between two d-frames that satisfy the infinitary cut rules. We need to show that $h(\neg \alpha) \sqsubseteq \neg h(\alpha)$ for all $\alpha \in L_{+} \times L_{-}$. From the definition we see that the corresponding plus coordinates are computed as follows:

$$
(h(\neg \alpha))_{+}=h_{+}\left(p^{r}\left(\alpha_{-}\right)\right)=h_{+}\left(\bigvee\left\{x \mid x^{*} \vee \alpha_{-}=1\right\}\right)=\bigvee\left\{h_{+}(x) \mid x^{*} \vee \alpha_{-}=1\right\}
$$

and

$$
(\neg h(\alpha))_{+}=p^{r}\left(h_{-}\left(\alpha_{-}\right)\right)=\bigvee\left\{w \mid w^{*} \vee h_{-}\left(\alpha_{-}\right)=1\right\}
$$

It is sufficient to show that $x^{*} \vee \alpha_{-}=1$ implies $h_{+}(x)^{*} \vee h_{-}\left(\alpha_{-}\right)=1$. This is true because from $\left(x, x^{*}\right) \in$ con we get that $\left(h_{+}(x), h_{-}\left(x^{*}\right)\right) \in$ con and hence $h_{+}(x)^{*} \geq$ $h_{-}\left(x^{*}\right)$. Therefore, by applying the frame homomorphism $h_{-}$to $x^{*} \vee \alpha_{-}=1$ we obtain $h_{-}\left(x^{*}\right) \vee h_{-}\left(\alpha_{-}\right)=1$ and this implies $h_{+}(x)^{*} \vee h_{-}\left(\alpha_{-}\right)=1$.

Remark 6.3 The d-frame of truth values $\mathcal{F O U \mathcal { O }}$ satisfies the cut rules (as it is spatial, for example) so we can also apply the functor N to equip it with interior operators. However, only the identity maps $\mathbf{2} \rightarrow \mathbf{2}$ are available, so this is what N will produce.

### 6.1 Spectra and comparison with the interior operations

We are now ready to show that the spectrum of a $\left(p^{r}, m^{r}\right)$ enriched d-frame is the same as the spectrum of the original d-frame.

Proposition 6.4 Let $\mathcal{L}$ be a d-frame satisfying the infinitary cut rules. Then, the spectra of $\mathcal{L}$ and $\left(\mathcal{L} ; p^{r}, m^{r}\right)$ are the same; that is

$$
\Sigma_{d}(\mathcal{L})=\Sigma\left(\mathcal{L} ; p^{r}, m^{r}\right)
$$

Proof. This follows from the functoriality of N as proved in Proposition 6.2: Every d-point of $\mathcal{L}$ viewed as a d-frame homomorphism $p: \mathcal{L} \rightarrow \mathcal{F O U \mathcal { R }}$ is also an nd-point $\mathrm{N}(p):\left(\mathcal{L} ; p^{r}, m^{r}\right) \rightarrow \mathcal{F} \mathcal{O} \mathcal{R}$. The converse inclusion is immediate.

From the fact that spatial d-frames satisfy the infinitary cut rules (Lemma 5.10 and Corollary 5.13 in [10]), we have:

Corollary 6.5 Let $\mathcal{L}$ be a spatial d-frame. Then $\left(\mathcal{L} ; p^{r}, m^{r}\right)$ is an nd-frame and, moreover, the spectra of $\mathcal{L}$ and $\left(\mathcal{L} ; p^{r}, m^{r}\right)$ are the same.

Note that this still does not mean that, for a spatial d-frame $\mathcal{L}, p^{r}$ and $m^{r}$ are the interior operations of the corresponding bispace, but in the d-regular case everything works out:

Proposition 6.6 Let $\mathcal{L}$ be a spatial d-regular d-frame. Then the nd-frame $\left(\mathcal{L} ; p^{r}, m^{r}\right)$ is spatial.

Proof. We need to prove that the conditions $\left(\mathrm{s}_{p}\right)$ and $\left(\mathrm{s}_{m}\right)$ hold in $\left(\mathcal{L} ; p^{r}, m^{r}\right)$. For ( $\mathrm{s}_{p}$ ), assume $a \not \leq p^{r}(b)$ for some $a \in L_{+}$and $b \in L_{-}$. From d-regularity, there exists $c \in L_{+}$such that $\left(a, c^{*}\right) \in$ tot and $c \not \leq p^{r}(b)$. Since $p^{r}(b)=\bigvee\left\{x \mid x^{*} \vee b=1\right\}$, the latter condition on $c$ implies that $c^{*} \vee b \neq 1$ or, in other words, $\left(0, c^{*} \vee b\right) \notin$ tot. From spatiality of $\mathcal{L}$, there exists a point $\left(F_{+}, F_{-}\right)$such that $c^{*} \vee b \notin F_{-}$and, therefore, also $c^{*} \notin F_{-}$and $b \notin F_{-}$. Finally, we know that $\left(a, c^{*}\right) \in$ tot, therefore it has to be the case that $a \in F_{+}$.

### 6.2 Maximality of $\left(p^{r}, m^{r}\right)$

We saw before that d-regularity is enough to ensure that $p^{r}$ and $m^{r}$ correspond to the interior operations on the corresponding bispace. Here we show (assuming just d-regularity) that ( $p^{r}, m^{r}$ ) is "larger" than any other ( $p, m$ ) pair:

Proposition 6.7 Let $\mathcal{L}$ be a d-regular d-frame and let $(p, m)$ be such that $(\mathcal{L} ; p, m)$ is an nd-frame. Then, $p \leq p^{r}$ and $m \leq m^{r}$ in the pointwise order.

Proof. Let $b \in L_{-}$. Since $\mathcal{L}$ is d-regular, $p(b)=\bigvee\left\{c \mid\left(p(b), c^{*}\right) \in\right.$ tot $\}$. But, any time $\left(p(b), c^{*}\right) \in$ tot, from (tot- $p$ ), we also have that $\left(0, c^{*} \vee b\right) \in$ tot and this is equivalent to $c^{*} \vee b=1$. Therefore, $p \leq p^{r}$.

The fact that $\left(p^{r}, m^{r}\right)$ is maximal says that it frame-theoretically mimics the interior operations as closely as possible. Indeed, if ( $p^{\circ}, m^{\circ}$ ) were the interior operators of a (spatial) d-regular d-frame then, since $\left(p^{\circ}, m^{\circ}\right)$ satisfies the $(p, m)$ axioms, the previous proposition says that $\left(p^{\circ}, m^{\circ}\right)$ is pointwise smaller than $\left(p^{r}, m^{r}\right)$. On the other hand, $p^{r}(b)$ is computed as a join of $\tau_{+}$-open elements well-inside $b$, whereas $p^{\circ}(b)$ is computed as the join of all $\tau_{+}$-open subsets of $b$. Therefore, $\left(p^{r}, m^{r}\right)$ is also pointwise smaller than $\left(p^{\circ}, m^{\circ}\right)$. This means that in the spatial case $\left(p^{r}, m^{r}\right)$ and ( $p^{\circ}, m^{\circ}$ ) coincide.

### 6.3 Proof-theoretic negation

Assuming the Gentzen cut rule in the original paper [10] allowed Jung and Moshier to give a proof-theoretic characterisation of negation. For a $\gamma \in \mathcal{L}$, let $\bar{I} \gamma=\{\varphi \in$ con $\mid \varphi \wedge \gamma \prec f f\}$ and $\bar{F} \gamma=\{\psi \in \operatorname{con} \mid t \prec \gamma \vee \psi\}$. Then, define the proof-theoretic negation of $\gamma$ as

$$
\bar{\gamma} \stackrel{\text { def }}{\equiv}\left(\bigvee_{\varphi \in \bar{I}_{\gamma}}^{\uparrow} \varphi_{+}, \bigvee_{\psi \in \overline{F \gamma}}^{\uparrow} \psi_{-}\right)
$$

We can now observe that this negation is actually exactly the same as the one obtained from the canonical $\left(p^{r}, m^{r}\right)$ :
Theorem 6.8 Let $\mathcal{L}$ be a d-frame. Then, $\bar{\gamma}=\left(p^{r}\left(\gamma_{-}\right), m^{r}\left(\gamma_{+}\right)\right)$for all $\gamma \in$ $L_{+} \times L_{-}$.

Proof. For " $\sqsubseteq$ ", let $\varphi \in \bar{I} \gamma$. Notice that $\varphi \wedge \gamma \prec f f$ is equivalent to $\left(0, \varphi_{-} \vee \gamma_{-}\right) \in$ tot which is the same as $\varphi_{-} \vee \gamma_{-}=1$. Since $\varphi \in \operatorname{con},\left(\varphi_{+}\right)^{*} \geq \varphi_{-}$and so $\left(\varphi_{+}\right)^{*} \vee \gamma_{-}=1$,
therefore $\varphi_{+} \leq p^{r}\left(\gamma_{-}\right)$. Dually also $\psi \in \bar{F} \gamma$ implies $\psi_{-} \leq m^{r}\left(\gamma_{+}\right)$. For " $\supseteq$ ", let $\varphi_{+} \in L_{+}$be such that $\left(\varphi_{+}\right)^{*} \vee \gamma_{-}=1$. Define

$$
\chi \stackrel{\text { def }}{\equiv}\left(\varphi_{+},\left(\varphi_{+}\right)^{*}\right) .
$$

Obviously $\chi \in$ con and $\chi \wedge \gamma \prec f f$ (this is exactly the condition $\chi_{-} \vee \gamma_{-}=1$ ). Therefore, $\chi \in \bar{I} \gamma$ and $\varphi_{+}=\chi_{+} \leq \bar{\gamma}_{+}$. Dually, for every $\varphi_{-} \in L_{-}$such that $\left(\varphi_{-}\right)^{*} \vee \gamma_{+}=1, \varphi_{-} \leq \bar{\gamma}_{-}$holds.

In fact, the proof that $m^{r} p^{r} \leq \mathrm{id}$ and $p^{r} m^{r} \leq \mathrm{id}$ in Proposition 6.1 is a direct translation of the proof that $\overline{\bar{\gamma}} \sqsubseteq \gamma$ in [10]. The only, but very important, difference is that $\bar{\gamma}$ was originally defined only for the consistent predicates whereas $\left(p^{r}, m^{r}\right)$ is defined for the whole d-frame. On top of that, the previous theorem provides the proof-theoretic negation with a bitopological interpretation.

## Acknowledgements

We would like to thank the referees of the MFPS 32 conference for their detailed comments on an earlier version of this paper.

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## A Appendix: Proofs omitted from the main text

## A.1 Proofs related to Remark 4.3

Proposition A. 1 Satisfying any one of the following axioms is equivalent to requiring that $p \circ m=i d$ :

$$
\begin{array}{ll}
(\neg \neg L) & \neg \neg \varphi \subset \varphi \\
(\neg \supset L) & \neg(\varphi \supset \psi) \subset \varphi \wedge \neg \psi
\end{array}
$$

and satisfying any one of the following axioms is equivalent to requiring that $p$ preserves finite suprema:

$$
\begin{array}{ll}
(\neg \wedge R) & \neg(\varphi \wedge \psi) \supset \neg \varphi \vee \neg \psi \\
(\neg \sqcup R) & \neg(\varphi \sqcup \psi) \supset \neg \varphi \sqcup \neg \psi
\end{array}
$$

Proof. $(\neg \neg \mathrm{L})$ is equivalent to $p \circ m=\mathrm{id}$ as, by Lemma 4.1 (L2), $\mathcal{L} \models \varphi \supset \neg \neg \varphi$ is equivalent to $v(\varphi)_{+} \leq v(\neg \neg \varphi)_{+}=p\left(m\left(v(\varphi)_{+}\right)\right)$for all valuations $v$. To show the same for $(\neg \supset \mathrm{L})$, again by Lemma $4.1(\mathrm{~L} 2), \mathcal{L} \models \varphi \wedge \neg \psi \supset \neg(\varphi \supset \psi)$ is equivalent to $v(\varphi)_{+} \wedge p\left(v(\psi)_{-}\right) \leq p\left(m\left(v(\varphi)_{+}\right)\right) \wedge p\left(v(\psi)_{-}\right)$for all valuations $v$. This has to hold for all $\varphi$ and $\psi$. If $\psi$ is such that $\psi_{-}=1$ we get $v(\varphi)_{+} \wedge p(1) \leq p\left(m\left(v(\varphi)_{+}\right)\right) \wedge p(1)$ and, since both $p$ and $m$ preserve 1 , we get $v(\varphi)_{+} \leq p\left(m\left(v(\varphi)_{+}\right)\right)$as we wanted.

For the second part we use Lemma 4.1 (L2) once again to show that $(\neg \wedge \mathrm{R})$ is equivalent to $p\left(\varphi_{-} \vee \psi_{-}\right)=p\left(\varphi_{-}\right) \vee p\left(\psi_{-}\right)$for all $\varphi$ and $\psi$. Notice that, $p(x \vee y) \geq$ $p(x) \vee p(y)$ holds always from monotonicity of $p$, and $\mathcal{L} \models \neg(\varphi \wedge \psi) \supset \neg \varphi \vee \neg \psi$ is equivalent to $p\left(v(\varphi)_{-} \vee v(\psi)_{-}\right) \leq p\left(v(\varphi)_{-}\right) \vee p\left(v(\psi)_{-}\right)$for all valuations $v$. The same argument applies for $(\neg \sqcup \mathrm{R})$.

We can also assume the following axiom

$$
(\supset 3) \quad(\varphi \supset \psi) \supset \varphi) \supset \varphi
$$

which is equivalent to assuming that $L_{+}$is a Boolean algebra as it forces $\left(\left(v(\varphi)_{+} \rightarrow\right.\right.$ $\left.\left.v(\psi)_{+}\right) \rightarrow v(\varphi)_{+}\right) \rightarrow v(\varphi)_{+}=1$ to hold for all valuations $v$.

## A. 2 Proofs of the main theorems in Section 5

Lemma A. 2 Let $\mathcal{L}$ be an nd-frame. Then, $\left(\Phi_{+}(x)\right)^{\circ \tau_{-}} \supseteq \quad \Phi_{-}(m(x))$ and $\left(\Phi_{-}(y)\right)^{\circ \tau_{+}} \supseteq \Phi_{+}(p(y))$ in $\Sigma(\mathcal{L})$, for all $x \in L_{+}$and $y \in L_{-}$.

Proof. The second statement is true because, from $\left(\operatorname{dp}_{p}\right), \Phi_{+}(p(y)) \subseteq \Phi_{-}(y)$.
Proof of Theorem 5.3 (following the proof of Theorem 4.1 in [10]). Clearly, (iii) implies (ii) which implies (i). As in the original paper, $\eta_{X}$ is bicontinuous, biopen onto the image and natural in $X$ we see that (iii) and (iv) are equivalent. To show that (i) implies (iv) assume that $\iota: X \cong \Sigma(\mathcal{L})$. If we prove that $\eta_{\Sigma(\mathcal{L})}$ is a bijection, we get that also $\eta_{X}$ is because the following square commutes (from naturality of $\eta$ ):


Observe that $\eta_{\Sigma(\mathcal{L})}$ is injective for all $\mathcal{L}$. For the surjectivity of $\eta_{\Sigma(\mathcal{L})}$, take a $\left(\mathcal{F}_{+}, \mathcal{F}_{-}\right) \in \Sigma \Omega \Sigma(L)$ and define

$$
F_{+}=\left\{x \in L_{+} \mid \Phi_{+}(x) \in \mathcal{F}_{+}\right\} \quad F_{-}=\left\{y \in L_{-} \mid \Phi_{-}(y) \in \mathcal{F}_{i}\right\}
$$

We will show that $\left(F_{+}, F_{-}\right)$is an nd-point of $L$. As in [10], the pair $\left(F_{+}, F_{-}\right)$is a dpoint. To show that it is actually an nd-point, we need to show that it satisfies $\left(\mathrm{dp}_{p}\right)$ and $\left(\mathrm{dp}_{m}\right)$. For the former, let $p(x) \in F_{+}$. This is equivalent to $\Phi_{+}(p(x)) \in \mathcal{F}_{+}$. From Lemma A. 2 we also know that $\Phi_{+}(p(x)) \subseteq\left(\Phi_{-}(x)\right)^{\circ \tau_{+}} \in \mathcal{F}_{+}$. And, since $\left(\mathcal{F}_{+}, \mathcal{F}_{-}\right)$is an nd-point, we know that $\Phi_{-}(x) \in \mathcal{F}_{-}$. Finally, from the definition of $F_{-}$we get that $x \in F_{-}$as we wanted. The proof of $\left(\mathrm{dp}_{m}\right)$ is the same but dual.

The argument that $\eta_{\Sigma(\mathcal{L})}\left(F_{+}, F_{-}\right)=\left(\mathcal{F}_{+}, \mathcal{F}_{-}\right)$is exactly the same as in [10].
Lemma A. $3 \epsilon_{\mathcal{L}}$ is an onto nd-frame homomorphism. Moreover, $\epsilon$ is natural in $\mathcal{L}$.
Proof. Since $\epsilon$ is defined the same way as for d-frames, we see that $\epsilon_{\mathcal{L}}$ is a d-frame homomorphism and that $\epsilon$ is natural in $\mathcal{L}$. We need to show that it is indeed an ndframe homomorphism. That is that it satisfies $\epsilon_{\mathcal{L}}(\neg \alpha) \sqsubseteq \neg \epsilon_{\mathcal{L}}(\alpha)$ for all $\alpha \in L_{+} \times L_{-}$ but this is exactly the same statement as Lemma A.2.

Proof of Theorem 5.4 (following the proof of Theorem 5.1 in [10]). The implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are immediate, and (iv) $\Longrightarrow$ (iii) follows from the fact that $\epsilon_{L}$ is onto (by Lemma A.3).

To show that (i) implies (v), it is enough to show that $\left(\mathrm{s}_{p}\right)$ and ( $\mathrm{s}_{m}$ ) hold for all images of $\Omega$ (the other conditions were already proved in [10] for d-frames). Clearly, for $U_{+} \in \tau_{+}$and $V_{-} \in \tau_{-}, U_{+} \subseteq V_{-}^{0 \tau_{+}}$iff $U_{+} \subseteq V_{-}$. Therefore, if $U_{+} \nsubseteq V_{-}^{0 \tau_{+}}$, then there exists an $x \in U_{+} \backslash V_{-}$such that $U_{+} \in \mathcal{U}_{+}(x)$ and $V_{-} \notin \mathcal{U}_{-}(x)$. Moreover, $\left(\mathcal{U}_{+}(x), \mathcal{U}_{-}(x)\right)$ is an nd-point.

Finally, (v) implies (iv). As already proved in [10], $\epsilon_{L}$ is injective and reflects con and tot. It remains to prove that $\epsilon_{L}^{-1}\left(\neg\left(\Phi_{+}(x), \Phi_{-}(y)\right)\right) \sqsubseteq \neg \epsilon_{L}^{-1}\left(\Phi_{+}(x), \Phi_{-}(y)\right)$. Let us focus on the plus coordinates, that is to prove that

$$
\left(\epsilon_{L}\right)_{+}^{-1}\left(\Phi_{-}(y)^{\circ \tau_{+}}\right) \leq p\left(\epsilon_{L}\right)_{-}^{-1}\left(\Phi_{-}(y)\right)
$$

Observe that $\Phi_{-}(y)^{\circ} \tau_{+}=\Phi_{+}(x)$ for some $x \in L_{+}$because $\epsilon_{\mathcal{L}}$ is onto. Since $\epsilon_{L}$ is injective, $\left(\epsilon_{L}\right)_{+}^{-1}\left(\Phi_{-}(y)^{\circ \tau_{+}}\right)=\left(\epsilon_{L}\right)_{+}^{-1}\left(\Phi_{+}(x)\right)=x$ and also $p\left(\left(\epsilon_{L}\right)_{-}^{-1}\left(\Phi_{-}(y)\right)\right)=$ $p(y)$. Now, assume for a contradiction that $x \not \leq p(y)$. Then, from ( $\left.\mathrm{s}_{p}\right)$, there exists an nd-point $\left(F_{+}, F_{-}\right)$such that $x \in F_{+}$and $y \notin F_{-}$but this is impossible since $\Phi_{+}(x)=\Phi_{-}(y)^{\circ \tau_{+}} \subseteq \Phi_{-}(y)$.


[^0]:    1 The work was supported by the grant SVV-2016-260332 and by the CE-ITI grant, GAČR P202/12/G061.
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[^1]:    5 The decomposition theorem is surprisingly robust; very little of the structure of bilattices is required. While this is an intriguing aspect of the theory, it has also led to a proliferation of terminology, not all of which is universally accepted. Our choice of "bilattice" for the purposes this paper is really just for brevity and simplicity as we will have no need to consider any variations in the axiomatisation.
    ${ }^{6}$ Note the overloading of $\wedge$ and $\vee$ as operations on both the Heyting algebra and the bilattice. We hope that the context will always make clear what we are referring to.

[^2]:    7 This definition of d-frames agrees with the definition of reasonable d-frames in [10].
    8 Frames are complete lattices satisfying the equation: $b \wedge\left(\bigvee_{i} a_{i}\right)=\bigvee_{i}\left(b \wedge a_{i}\right)$. Frame homomorphisms are monotone maps distributing over all joins and finite meets. For more information see [12].

[^3]:    9 See Section 6 for the definition.

[^4]:    ${ }^{10}$ Notice that assuming $p \circ m=$ id implies that $m$ is a one-one frame homomorphism and $p$ is its right adjoint.

