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DOI:
10.1016/j.jpaa.2013.10.005

$F_4(2)$ AND ITS AUTOMORPHISM GROUP

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Abstract. We present an identification theorem for the groups $F_4(2)$ and $Aut(F_4(2))$ based on the structure of the centralizer of an element of order 3.

1. Introduction

In the classification of the finite simple groups a fundamental role was played by Timmesfeld's work on groups which contain a large extraspecial 2-subgroup [23]. Timmesfeld determined the structure of the normalizer of such a subgroup and following this achievement several authors contributed to the classification of all the simple groups which contain a large extraspecial 2-subgroup.

The notion of a large extraspecial 2-subgroup of a group is generalized in the work of Meierfrankenfeld, Stellmacher and the second author [13] to the concept of a large p-subgroup where p is an arbitrary prime. The definition of a large p- subgroup is as follows: given a finite group G, a p-subgroup Q of G is *large* if and only if

(L1) $Q = F^*(N_G(Q))$; and
(L2) for all non-trivial subgroups U of $Z(Q)$, $N_G(U) \leq N_G(Q)$.

Recall that condition (L1) is equivalent to $Q = O_p(N_G(Q))$ and $C_G(Q) \leq Q$. If Q is extraspecial and $p = 2$ this definition coincides with Timmesfeld's definition of a large extraspecial 2-group. The classification of groups with a large p-subgroup is sometimes called the MSS-project. The first step of this project is [13], where in contrast to the work of Timmesfeld, it is not the normalizer of Q which is determined but rather structural information about the maximal p-local subgroups of G which are not contained in $N_G(Q)$ is provided.

Suppose now that Q is a large subgroup of a group G and let S be a Sylow p-subgroup of G containing Q. It is an elementary exercise to show that $F^*(N_G(U)) = O_p(N_G(U))$ for all non-trivial normal subgroups U of S ([18, Lemma 2.1]). Groups which satisfy this property are said to be of *parabolic characteristic* p. If $F^*(N_G(U)) = O_p(N_G(U))$

Date: September 13, 2013.

for all $1 \neq U \leq S$, then G is of *local characteristic* p (also called characteristic p-type). In [13] it is assumed that G has local characteristic p. However, there is work in progress which aims to remove this assumption, and so all the successor articles to [13] will be produced under the weaker hypothesis that the group under investigation has a large p-subgroup. One reason for this is that, as mentioned above, a group with a large p-subgroup is of parabolic characteristic p, while demonstrating that a group has local characteristic p may well be hard to verify in applications.

Nevertheless [13] provides us with some p-local structure of the group G and this is all that we require for the next step of the programme in which we aim to recognize G up to isomorphism. For this recognition we typically build a geometry upon which a subgroup of G acts. This means that we take some of the p-local subgroups of G which contain S and consider the subgroup H of G generated by them. The p-local subgroups are selected so that $O_p(H) = 1$. As the generic simple groups with a large p-subgroup are Lie type groups in characteristic p, in many cases we will be able to show that the coset geometry determined by the p-local subgroups in H is a building. The recognition of H is then achieved with help of the classification of buildings of spherical type [24, 25]. At this stage, as a third step of the programme, we would like to show that $G = H$. There is a general approach to achieve this goal. Since H contains S, it also contains Q and so we are able to identify Q as a subgroup of H. Typically $Q = F^*(N_H(R))$ for some root group R in H. We can then determine the structure of $N_G(Q)$. The aim is to show that $N_G(Q) = N_H(Q)$ and from this further show that $N_G(U) = N_H(U)$ for all $1 \neq U \trianglelefteq S$. The final step is to show that, if H is a proper subgroup of G, then H is strongly p-embedded in G and this contradicts the main results in [3] and [21].

However there are situations where it cannot be shown that $N_G(Q) = N_H(Q)$. This happens most frequently when $p = 2$ or 3 and $N_H(Q)$ is soluble. For the final stage of the project one has to analyze exactly these more troublesome configurations; that is determine all the groups G where $F^*(H)$ is a group of Lie type in characteristic p containing a Sylow p-subgroup S of G, $N_H(Q)$ is soluble and $N_H(Q) \neq N_G(Q)$. There are several configurations where this phenomenon arises. For example when $p = 3$ we have $H \cong \mathrm{P}\Omega_6^-(3)$ contained in $G \cong \mathrm{U}_6(2)$. Similarly, there are containments $\mathrm{P}\Omega_6^+(3)$ in $\mathrm{F}_4(2)$, $\mathrm{P}\Omega_7(3)$ in ${}^2\mathrm{E}_6(2)$ and $\mathrm{M}(22)$, and $\mathrm{P}\Omega_8^+(3)$ in $\mathrm{M}(23)$ and F_2. In all these cases Q is an extraspecial 3-group and $N_H(Q)$ is soluble. In a series of papers [17, 19, 20], the larger groups in this list are determined from the approximate structure of the centralizer of an element of order 3, or equivalently from the structure of $N_G(Q)$. In this paper we identify $\mathrm{F}_4(2)$ from the approximate structure of the centralizer of a 3-element. We are motivated by the embedding of $\mathrm{P}\Omega_6^+(3)$ in $\mathrm{F}_4(2)$, but we do not assume that G contains this group as we hope that our work can find broader application. We therefore just assume certain important structural information about the normalizer of Q and, as a consequence, this present article is independent of the results in [13].

This contribution should also be viewed as a companion to the authors' earlier work [17] in which the groups G with $\mathrm{PSU}_6(2) \le G \le \mathrm{Aut}(\mathrm{PSU}_6(2))$ are characterised by such information and this is a second reason why we make no additional assumption on the embedding of $\mathrm{P}\Omega_6^+(3)$ in the present article. Indeed, in such groups, the centralizer of a 3-element has a similar structure to that in $\mathrm{F}_4(2)$ or $\mathrm{Aut}(\mathrm{F}_4(2))$ but in these groups $Z(Q)$ is weakly closed in Q, while in $\mathrm{F}_4(2)$ and its automorphism group it is not. (Recall, for subgroups $X \le Y \le L$, we say X is *weakly closed* in Y with respect to L provided that if $g \in L$ and $X^g \le Y$, then $X^g = X$.) Unfortunately the arguments in these two situations are quite different. The theorems proved in [17] and in this article are employed in [18] to identify the corresponding groups.

We now make precise what we mean by the approximate structure of the centralizer of an element of order 3 in $\mathrm{PSU}_6(2)$ or $\mathrm{F}_4(2)$.

Definition 1.1. *We say that X is similar to a 3-centralizer in a group of type $\mathrm{PSU}_6(2)$ or $\mathrm{F}_4(2)$ provided the following conditions hold.*

(i) *$Q = F^*(X)$ is extraspecial of order 3^5 and $Z(F^*(X)) = Z(X)$; and*
(ii) *X/Q contains a normal subgroup isomorphic to $\mathrm{Q}_8 \times \mathrm{Q}_8$.*

Our main theorem is as follows.

Theorem 1.2. *Suppose that G is a group, $Z \le G$ has order 3. If $C_G(Z)$ is similar to a 3-centralizer in a group of type $\mathrm{PSU}_6(2)$ or $\mathrm{F}_4(2)$ and Z is not weakly closed in $F^*(C_G(Z))$, then $G \cong \mathrm{F}_4(2)$ or $\mathrm{Aut}(\mathrm{F}_4(2))$.*

Combining Theorem 1.2 and the main theorem from [17] we obtain the following statement.

Theorem 1.3. *Suppose that G is a group, $Z \le G$ has order 3. If $C_G(Z)$ is similar to a 3-centralizer in a group of type $\mathrm{PSU}_6(2)$ or $\mathrm{F}_4(2)$ and Z is not weakly closed in a Sylow 3-subgroup of $C_G(Z)$ with respect to G, then either $F^*(G) \cong \mathrm{F}_4(2)$ or $F^*(G) \cong \mathrm{PSU}_6(2)$.*

For groups G with $C_G(Z)$ of type $\mathrm{PSU}_6(2)$ or $\mathrm{F}_4(2)$, the different G-fusion of Z in $C_G(Z)$ manifests itself in the subgroup structure of G very quickly. Indeed, if we let S be a Sylow 3-subgroup of $C_G(Z)$

and $Q = F^*(C_G(Z))$, then we easily determine that $S \in \mathrm{Syl}_3(G)$ and the Thompson subgroup J of S has order 3^4 or 3^5 when Z is weakly closed in Q, whereas, it has order 3^4 if Z is not weakly closed in Q. More strikingly, setting $L = N_G(J)$, we have $F^*(L/Q) \cong \Omega_4^-(3)$ in the first case and in the second case $L/Q \cong \Omega_4^+(3)$.

The paper is set out as follows. In Section 2 we gather pertinent information about that natural and spin modules for $\mathrm{Sp}_6(2)$ and the natural and orthogonal $\mathrm{SU}_4(2)$-module as well as collect together further identification theorems and results which we shall require for the proof of Theorem 1.2. In Section 3 we present Theorem 3.3 which will be used to identify a subgroup P of our target group which is isomorphic to $\mathrm{F}_4(2)$. The proof of Theorem 3.3 involves the construction of a building of type $\mathrm{F}_4(2)$ on which P acts faithfully. The proof of the main theorem commences in Section 4. Thus we assume that G satisfies the hypothesis of Theorem 1.2 and set $M = N_G(Z)$. We remark here that the information that is developed as the proof of Theorem 1.2 unfolds becomes information about the groups $\mathrm{F}_4(2)$ and $\mathrm{Aut}(\mathrm{F}_4(2))$ once the theorem is proved. The initial objective of Section 4 is to determine more information about the structure of M. This is achieved by exploiting the fact that Z is not weakly closed in $Q = O_3(M)$. The first significant result is presented in Lemma 4.8 where it is shown that

$$M/Q \approx (\mathrm{Q}_8 \times \mathrm{Q}_8).\mathrm{Sym}(3) \text{ or } (\mathrm{Q}_8 \times \mathrm{Q}_8).(2 \times \mathrm{Sym}(3)).$$

In Section 4, we then move on, in Lemma 5.3, to the determination of L as described in the previous paragraph. At this stage we have shown that $L \approx 3^4 : \mathrm{GO}_4^+(3)$ or $3^4 : \mathrm{CO}_4^+(3)$. Thus J supports a quadratic form and G-fusion of elements in J is controlled by L. This allows us to parameterize the non-trivial cyclic subgroups of J as singular, plus and minus (the latter two types are fused when $L \approx 3^4 : \mathrm{CO}_4^+(3)$) and also the five types of subgroups of order 9 which we label Type S, Type D+, Type D-, Type N+ and Type N- (the notation is chosen to indicate that the groups are singular, degenerate with three plus groups, degenerate with three minus groups, non-degenerate of plus-type and non-degenerate of minus-type).

We let ρ_1 and ρ_2 be elements of $Q \cap J$ each centralized by a Q_8 (the quaternion group of order 8) subgroup of M and one generating a plus type and the other a minus type cyclic subgroup of J. In Section 6, we show that $C_G(\rho_1) \cong C_G(\rho_2) \cong 3 \times \mathrm{SU}_4(2)$ or $3 \times \mathrm{Sp}_6(2)$. (See Lemmas 6.3 and 6.4.) It is the latter possibility that actually arises in our target groups. There is related work in [6] that we might refer to at this stage but they assume that G is of characteristic 2-type.

We let r_1 and r_2 be central involutions in the subgroup of $C_G(Z)$ isomorphic to $\mathrm{Q}_8 \times \mathrm{Q}_8$ which do not invert Q/Z and, for $i = 1, 2$, we set $K_i = C_G(r_i)$. Again when $L \approx \mathrm{CO}_4^+(3)$ these groups are conjugate. At this stage we know that r_i centralizes the (simple) component of $C_G(\rho_i)$. The heart of the proof of Theorem 1.2 is contained in Sections 7, 8, 9 and 10 where we determine the structure of K_i. Thus the aim is to show that K_1 and K_2 have shape $2^{1+6+8}.\mathrm{Sp}_6(2)$ where $O_2(K_1)$ and $O_2(K_2)$ are commuting products of an extraspecial group of order 2^9 and an elementary abelian group of order 2^7.

We begin our construction of K_i by determining a large 2-group Σ_i which is normalized by $I_i = C_J(r_i)$. It turns out that Σ_i is the extraspecial 2-group of order 2^9 and plus type we are seeking. In the case that $C_G(\rho_i) \cong 3 \times \mathrm{SU}_4(2)$, we are able to show that in fact $K_i = N_G(\Sigma_i)$ and $N_G(\Sigma_i)/\Sigma_i \cong \mathrm{Aut}(\mathrm{SU}_4(2))$ or $\mathrm{Sp}_6(2)$ and this leads to a contradiction as explained in Lemma 8.2. Thus we enter Section 9 knowing that $C_G(\rho_1) \cong C_G(\rho_2) \cong 3 \times \mathrm{Sp}_6(2)$. On the other hand Σ_i is far from being a maximal signalizer for I_i. Thus is Section 9 we construct an even larger signalizer which in the end is a product $\Gamma_i = \Sigma_i \Upsilon_i$ where Υ_i is an elementary abelian group of order 2^7. Thus Γ_i has order 2^{15} and in fact $\Upsilon_i = Z(\Gamma_i)$ and this is proved in Lemma 9.3. We show that $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Sp}_6(2)$ in Lemma 9.6. The final hurdle requires that we show that $K_i = N_G(\Gamma_i)$. This is proved in Lemma 10.8 and requires a sequence of lemmas which begins by showing that Υ_i is strongly closed in Γ_i with respect to K_i and culminates in the statement that Υ_i is strongly closed in a Sylow 2-subgroup of K_i with respect to K_i. At this stage we apply Lemma 2.19 which is essentially Goldschmidt's Strongly Closed Abelian 2-subgroup Theorem [5] to conclude that $K_i = N_G(K_i) \approx 2^{1+6+8}.\mathrm{Sp}_6(2)$. Our final section exploits Theorem 3.3 to produce a subgroup P of G with $P \cong \mathrm{F}_4(2)$. We show that a group closely related to P is strongly 3-embedded in G and finally apply Holt's Theorem [10] in the form presented in Lemma 2.20 to conclude the proof of the Theorem 1.2.

Throughout this article we follow the now standard Atlas [4] notation for group extensions. Thus $X{\cdot}Y$ denotes a non-split extension of X by Y, $X{:}Y$ is a split extension of X by Y and we reserve the notation $X.Y$ to denote an extension of undesignated type (so it is either unknown, or we don't care). Our notation follows that in [1], [7] and [8]. We use the definition of signalizers as given in [8, Definition 23.1]. For odd primes p, the extraspecial groups of exponent p and order p^{2n+1} are denoted by p_+^{1+2n}. The extraspecial 2-groups of order 2^{2n+1} are denoted by 2_+^{1+2n} if the maximal elementary abelian subgroups have order 2^{1+n} and otherwise we write 2_-^{1+2n}. We expect our notation for specific groups is self-explanatory. For a subset X of a group G, X^G denotes the set of G-conjugates of X. If $x, y \in H \le G$, we write $x \sim_H y$ to indicate that x and y are conjugate in H. Often we shall give suggestive descriptions of groups which indicate the isomorphism type of certain composition factors. We refer to such descriptions as the *shape* of a group. Groups of the same shape have normal series with isomorphic sections. We use the symbol $\approx$ to indicate the shape of a group.

Acknowledgement. The first author is grateful to the DFG for their support and thanks the mathematics department in Halle for their generous hospitality from January to August 2011.
Both authors would like to thank the referee for carefully reading our manuscript and for the suggestions which led to improvements included in the final article.

2. Preliminaries

In this section we lay out certain facts about the groups $\mathrm{Sp}_6(2)$ and $\mathrm{Aut}(\mathrm{U}_4(2))$ which play a pivotal role in the proof of our main theorem. We also present other background results that are of key importance to our investigations.

Lemma 2.1. *Suppose that* $X \cong \mathrm{Sp}_6(2)$ *or* $\mathrm{Aut}(\mathrm{SU}_4(2))$. *Then there is a unique irreducible* $\mathrm{GF}(2)X$*-module of dimension* 6 *and a unique irreducible* $\mathrm{GF}(2)X$*-module of dimension 8. All the other non-trivial irreducible* $\mathrm{GF}(2)X$*-modules have dimension at least* 9.

Proof. This is well known. See [12]. □

In this section U will denote the $\mathrm{Aut}(\mathrm{SU}_4(2))$ natural module and the $\mathrm{Sp}_6(2)$ spin module of dimension 8 and V will be the $\mathrm{Aut}(\mathrm{SU}_4(2))$ orthogonal module and the $\mathrm{Sp}_6(2)$ natural module of dimension 6.

For $X \cong \mathrm{Sp}_6(2)$, let X_1, X_2 and X_3 be the minimal parabolic subgroups of X containing a fixed Sylow 2-subgroup S. Set $X_{ij} = \langle X_i, X_j \rangle$ where $1 \le i < j \le 3$ and fix notation so that

$$X_{12}/O_2(X_{12}) \cong \mathrm{SL}_3(2),$$

$$X_{23}/O_2(X_{23}) \cong \mathrm{Sp}_4(2) \text{ and}$$

$$X_{13}/O_2(X_{13}) \cong \mathrm{SL}_2(2) \times \mathrm{SL}_2(2).$$

There are three conjugacy classes of elements of order 3 in X. Let τ_1, τ_2 and τ_3 be representatives of these classes and choose so that on the natural $\mathrm{Sp}_6(2)$-module V, for $1 \le i \le 3$, $\dim[V, \tau_i] = 2i$.

		Centralizer in $\mathrm{Aut}(\mathrm{SU}_4(2))$	Centralizer in $\mathrm{Sp}_6(2)$	$\dim C_U(u_j)$	$\dim C_V(u_j)$
a_2	u_1	$2_+^{1+4}.(\mathrm{SL}_2(2)\times \mathrm{SL}_2(2))$	$2^{1+2+4}.(\mathrm{SL}_2(2)\times \mathrm{SL}_2(2))$	6	4
b_3	u_2	$2\times(\mathrm{Sym}(4)\times 2)$	$2^7.3$	4	3
b_1	u_3	$2\times \mathrm{Sp}_4(2)$	$2^5.\mathrm{Sp}_4(2)$	4	5
c_2	u_4	$2^6.3$	$2^8.\mathrm{SL}_2(2)$	4	4

TABLE 1. Involutions in $\mathrm{Sp}_6(2)$ and $\mathrm{Aut}(\mathrm{SU}_4(2))$. The involutions in the first row are the *unitary transvections.* The involutions labeled with "b" those which are in $\mathrm{Aut}(\mathrm{SU}_4(2))\setminus \mathrm{SU}_4(2)$.

Lemma 2.2. *Suppose that $Y\cong \mathrm{Aut}(\mathrm{SU}_4(2))$ and that $X\cong \mathrm{Sp}_6(2)$ with $Y\le X$. Assume that V and U are the faithful $\mathrm{GF}(2)X$-modules of dimension 6 and 8 respectively.*

- (i) *X and Y each have four conjugacy classes of involutions and for each involution $u\in X$ we have $u^X\cap Y$ is a conjugacy class in Y. In column one of Table 1 we provide the Suzuki names (see* [2, page 16]*) for each class of involutions.*
- (ii) *The shape of the centralizers of involutions in X and Y is given in Table 1.*
- (iii) *For each involution in $u\in X$, $\dim C_V(u)$ and $\dim C_U(u)$ is given in Table 1.*
- (iv) *X does not contain any subgroup of order 2^4 in which all the involutions are conjugate.*
- (v) *X does not contain an extraspecial subgroup of order 2^7.*
- (vi) *If x is an involution of type b_1, then a Sylow 3-subgroup of $C_Y(u)$ contains two conjugates of $\langle\tau_1\rangle$ and two conjugates of $\langle\tau_2\rangle$.*
- (vii) *$E=\langle\tau_1,\tau_2,\tau_3\rangle$ is the Thompson subgroup of a Sylow 3-subgroup of G and every element of order 3 is X-conjugate (Y-conjugate) to an element of E.*

Proof. Parts (i)-(iii) follow from [17, Proposition 2.12, and Table 1].

Suppose that $A\le X$ has order 2^4 and that all the non-trivial elements are conjugate in X. We use the character table of X given in [4, page 47]. Let χ be an irreducible character of X. Then, as $(\chi|_A,1_A)\ge 0$, we have

$$(\chi|_A,1_A)=\frac{1}{|A|}\sum_{a\in A}\chi(a)\ge 0.$$

Taking χ to be the degree 7 character we see that all the non-trivial elements in A are in Suzuki class c_2 (Atlas [4] $2C$). Now considering the character of degree 35 denoted χ_7 in [4] we obtain a contradiction.

Let E be extraspecial of order 2^7. Since X has a faithful 7-dimensional representation in characteristic 0 and the smallest such representation of E is 8-dimensional, E is not isomorphic to a subgroup of X.

Part (vi) follows from the action of $\mathrm{Sp}_4(2)$ on the natural module for $\mathrm{Sp}_6(2)$ as $\mathrm{Sp}_4(2)$ contains no conjugates of τ_3.

Part (vii) is also elementary to verify.

□

Lemma 2.3. *Let $X \cong \mathrm{Sp}_6(2)$, S a Sylow 2-subgroup of X and V be the $\mathrm{Sp}_6(2)$ natural module. Then the following hold.*

(i) *X acts transitively on the non-zero vectors in V.*

(ii) *V is uniserial as an S-module.*

(iii) *Suppose that, for $1 \le i \le 3$, V_i is an S-invariant subspace of V of dimension i. Then $X_{23} = N_X(V_1)$ and X_{23} acts naturally as $\mathrm{Sp}_4(2)$ on $V_1^\perp/V_1$, $X_{13} = N_X(V_2)$, $O^2(X_3)$ centralizes V_2 and $V/V_2^\perp$, and $O^2(X_1)$ centralizes $V_2^\perp/V_2$ and $X_{12} = N_X(V_3)$ and acts naturally on both V_3 and V/V_3.*

Proof. These are all well known facts about the action of X on V. See for example [15, Lemma 14.37] for (i) and (ii). □

Lemma 2.4. *Let $X \cong \mathrm{Sp}_6(2)$, S a Sylow 2-subgroup of X and U be the $\mathrm{Sp}_6(2)$ spin module.*

(i) *X has exactly two orbits on the non-zero vectors of U one of length 135 and one of length 120.*

(ii) *$N_X(C_U(S)) = X_{12}$ and $C_U(S) = C_U(O_2(X_{12}))$.*

(iii) *If $U_2 \le U$ is S-invariant of dimension 2, then $N_X(U_2) = X_{13}$ and $O^2(X_1)$ centralizes U_2.*

Proof. See [17, Proposition 2.12]. □

Lemma 2.5. *Suppose that $X \cong \mathrm{Sp}_6(2)$ and V is the natural module for X. Let $P = X_{13}$, $T \in \mathrm{Syl}_3(P)$ and $Q = O_2(P)$.*

(i) *$P/Q \cong \mathrm{SL}_2(2) \times \mathrm{SL}_2(2)$.*

(ii) *The subgroups of order 3 in T are as follows: there are two subgroups Z_1 and Z_2 which are X-conjugate to $\langle \tau_3 \rangle$, one subgroup which is X-conjugate to $\langle \tau_1 \rangle$ (which we suppose is $\langle \tau_1 \rangle$) and one subgroup which is X-conjugate to $\langle \tau_2 \rangle$. The two subgroups of T which are conjugate to $\langle \tau_3 \rangle$ are conjugate in $N_P(T)$.*

(iii) *$C_Q(Z_1) \cong C_Q(Z_2) \cong \mathrm{Q}_8$ and $[C_Q(Z_1), C_Q(Z_2)] = 1$.*

(iv) *$C_T(Z(Q)) = \langle \tau_1 \rangle$ and $C_Q(\tau_1) = Z(Q)$.*

(v) *If* $U \le Q$ *has order* 2^3 *and if* U *is* T*-invariant, then either* $U = C_Q(Z_1)$, $U = C_Q(Z_2)$ *or* $U = Z(Q)$.
(vi) *Let* $Q' = \langle t\rangle$. *Then* $t^X \cap Q \not\subseteq Z(Q)$.

Proof. Let Y be the P-invariant isotropic 2-space in V. Then P preserves $0 < Y < Y^\perp < V$. Let I be a hyperbolic line and $J = I^\perp$ be chosen so $Y \le J$. Then the decomposition $I \perp J$ is preserved by $\mathrm{Sp}_2(2) \times \mathrm{Sp}_4(2)$ and the subgroup K of this group which leaves Y invariant has shape $\mathrm{Sp}_2(2) \times (2 \times 2^2).\mathrm{SL}_2(2) \cong \mathrm{SL}_2(2) \times 2 \times \mathrm{Sym}(4)$. In particular, we now have (i) holds. Furthermore, we may suppose the first factor of K contains $\langle \tau_1\rangle$ while the second factor contains $\langle \tau_2^*\rangle$, an X-conjugate of $\langle\tau_2\rangle$, acting fixed point freely on J. Set $T = \langle \tau_1, \tau_2^*\rangle$. Since τ_1 is inverted in the first factor of K, we see the two diagonal products $\tau_1\tau_2^*$ and $\tau_1^2\tau_2^*$ are conjugate in $N_P(T)$. Furthermore these elements act fixed point freely on V and so are X-conjugate to τ_3. This is (ii).

Now consider Q. We know this group has order 2^7. We further have $Q \cap K = O_2(K)$ centralizes $Y + I = Y^\perp$. Consequently $Q\cap K$ is normal in P and as $[V, Q, Q\cap K] = [V, Q\cap K, Q]$ we additionally have $K \cap Q \le Z(Q)$. Note that $\langle\tau_1\rangle$ centralizes $Q\cap K$. Now $C_P(\tau_2^*)$ is contained in K and so we see $C_Q(\tau_2^*) = Z(K)$ has order 2. Now the centralizer in X of τ_3 supports a $\mathrm{GF}(4)$ structure and is isomorphic to $\mathrm{SU}_3(2)$. It follows that $\tau_1\tau_2^*$ and $\tau_1^2\tau_2^*$ can centralize only quaternion subgroups of order 8 in Q. Since $C_Q(\tau_1\tau_2^*)$ and $C_Q(\tau_1^2\tau_2^*)$ both centralize $Z(K)$ and $|Q| = 2^7$ we have $C_Q(\tau_1\tau_2^*) \cong C_Q(\tau_1^2\tau_2^*) \cong \mathrm{Q}_8$ and $C_Q(\tau_1\tau_2^*)' = Z(K)$. Putting $Q_1 = C_Q(\tau_1\tau_2^*)C_Q(\tau_1^2\tau_2^*)$ we have Q_1 is T-invariant. Now $Q = C_Q(\tau_1\tau_2^*)C_Q(\tau_1^2\tau_2^*)(Q\cap K)$,

$$[Q, \tau_1] = [C_Q(\tau_1\tau_2^*), \tau_1][C_Q(\tau_1^2\tau_2^*), \tau_1] = Q_1$$

is a normal subgroup of Q and $Q_1 \cap (Q\cap K) \le Z(K)$. Thus Q_1 is extraspecial and $Q' = Z(K)$ which has order 2. In addition, $Q = C_Q(\tau_1\tau_2^*)[Q, \tau_1\tau_2^*]$ with $C_Q(\tau_1\tau_2^*) \cap [Q, \tau_1\tau_2^*] = Z(K)$. Since

$$[C_Q(\tau_1\tau_2^*), Q, \tau_1\tau_2^*] \le [Z(K), \tau_1\tau_2^*] = 1$$

and $[C_Q(\tau_1\tau_2^*), \tau_1\tau_2^*, Q] = 1$, we also have $[C_Q(\tau_1\tau_2^*), [Q, \tau_1\tau_2^*]] = 1$ by the Three Subgroup Lemma. In particular, as $[Q, \tau_1\tau_2^*] = C_Q(\tau_1^2\tau_2^*)(Q\cap K)$, we now have (iii) and (iv) hold. If U is of order 2^3 and is T-invariant, then $C_T(U) > 1$ and so (v) also follows from the above discussion. To prove (vi), we start with a transvection $r \in Z(Q)$. By Table 1 we have $E = O_2(C_X(r))$ is elementary abelian of order 2^5. Now $|E\cap Q| \ge 2^3$. If $E \cap Q \le Z(Q)$, then, as $E \le C_{N_X(Q)}(E\cap Q)$, we get $|E\cap Q| \ge 2^4$, a contradiction. Hence $E \cap Q \not\le Z(Q)$. Now as $N_X(E)$ acts transitively on $E/\langle r\rangle$, we have any coset of $\langle r\rangle$ in E contains a conjugate of t. In particular $t^X \cap E \cap Q \not\subseteq Z(Q)$. □

Lemma 2.6. *Let $Y = \mathrm{Aut}(\mathrm{SU}_4(2))$ and V be the natural $\mathrm{O}_6^-(2)$-module. Then there is no elementary abelian subgroup E of order 8 in Y such that $|V : C_V(E)| \le 4$.*

Proof. Suppose false and let E be such a subgroup of order 8. From Table 1 we see E cannot contain elements of type b_3. If $E \not\leq Y'$, then E contains exactly four elements of type b_1. As there are at most three hyperplanes in V containing $C_V(E)$, two of these elements have to centralize the same hyperplane of V. But then their product, which is an involution in $E \cap Y$, also centralizes this hyperplane. As $\Omega_6^-(2)$ does not contain transvections, we have $E \le Y'$. Therefore $|V : C_V(E)| = 4$ and $C_V(E) = C_V(e)$ for all $e \in E^\#$. As $C_V(e) = [V, e]^\perp$ we also have $[V, e] = [V, E]$ for all $e \in E^\#$ which means all the involutions in E are conjugate. Now we use the character table of $\mathrm{SU}_4(2)$ as in the proof of Lemma 2.2(iv) to obtain a contradiction. □

Recall that a faithful $\mathrm{GF}(p)G$-module is an *F-module* provided there exists a non-trivial elementary abelian p-subgroup $A \le G$ such that $|V : C_V(A)| \le |A|$. The subgroups $A \le G$ with $|V : C_V(A)| \le |A|$ are called *offenders*.

Lemma 2.7. *Suppose that $X \cong \mathrm{Sp}_6(2)$ or $\mathrm{Aut}(\mathrm{SU}_4(2))$ and W is a $\mathrm{GF}(2)X$-module of dimension 14 which has exactly two composition factors one of dimension 6 and one of dimension 8. Then W is not an F-module.*

Proof. Suppose that $A \le X$ is an offender on W. Then $|A| \ge |W : C_W(A)|$. From Table 1, for $a \in A$, we read $|A| \ge |W : C_W(a)| \ge 2^4$. Since the 2-rank of X is at most 6, we also have that A does not contain any involutions of type b_3.

Suppose that $|A| = 2^4$. Then all the involutions in A must be of type a_2. This contradicts Lemma 2.2(iv). Hence $|A| \ge 2^5$ and $X \cong \mathrm{Sp}_6(2)$ as the 2-rank of $\mathrm{Aut}(\mathrm{SU}_4(2))$ is 4 (see [17, Proposition 2.12 (x)]). We use the notation for involutions from Table 1. We may as well suppose $A \le C_X(u_3)$. Then as the 2-rank of $\mathrm{Sp}_4(2)$ is 3, we have $A \cap O_2(C_X(u_3)) \ne 1$. Since $|C_U(O_2(C_X(u_3)))| = 2^4$ and $|C_V(O_2(C_X(u_3)))| = 2$ certainly $A \ne O_2(C_X(u_3))$. Now $O_2(C_X(u_3))$ contains 15 elements from u_1^X, 15 elements from u_4^X and one element from u_3^X and multiplication by u_3 maps $u_1^X \cap O_2(C_G(u_3))$ to $u_4^X \cap O_2(C_X(u_3))$. Thus, if A contains a conjugate of u_3, then $A \cap u_i^X \ne \emptyset$ for $i = 1, 3, 4$. As $|A| = 2^5$, A does not consist purely of elements of elements from class u_1^X by Lemma 2.2

(iv) and consequently we must have elements from u_4^X in X. It follows now from Table 1 that $|A| = 2^6$. There is a unique such elementary abelian subgroup in a Sylow 2-subgroup of X and its normalizer is a plane stabiliser in the action of X on V. But then $|W : C_W(A)| \geq 2^{10}$ which is a contradiction. □

Lemma 2.8. *Suppose that $X \cong \mathrm{Sp}_6(2)$, W is a 7-dimensional $\mathrm{GF}(2)X$-module with $W/C_W(X)$ the natural $\mathrm{Sp}_6(2)$-module. If $S \in \mathrm{Syl}_2(X)$, then $C_W(S) > C_W(X)$.*

Proof. Consider the subgroup $K = K_1 \times K_2$ of X which preserves the decomposition of $W/C_W(X)$ in to a perpendicular sum of a non-degenerate 2-space $A/C_W(X)$ and a non-degenerate 4-space $B/C_W(X)$ with $K_1 \cong \mathrm{Sp}_2(2)$ and $K_2 \cong \mathrm{Sp}_4(2)$. Let t be an involution in K_1. Since $\dim A = 3$, we have $\dim[A, t] = 1$. Furthermore $B/C_B(t) \cong [B, t]$ as K_2-modules and so we must have $[B, t] = 0$. Thus $[W, t] = [A, t] + [B, t] = [A, t]$ has dimension 1 and so t is a transvection on W. Let $P = C_X(t)$. Then P contains a Sylow 2-subgroup S of X. Since P centralizes $[W, t]$ and $C_W(X)$, P centralizes $L = [W, t] + C_W(X)$ and so $L \leq C_W(S)$. □

Theorem 2.9 (Prince)**.** *Suppose that Y is isomorphic to the centralizer of a 3-central element of order 3 in $\mathrm{PSp}_4(3)$ and that X is a finite group with a non-trivial element d such that $C_X(d) \cong Y$. Let $P \in \mathrm{Syl}_3(C_X(d))$ and E be the elementary abelian subgroup of P of order 27. If E does not normalize any non-trivial $3'$-subgroup of X and d is X-conjugate to its inverse, then either*

(i) *$|X : C_X(d)| = 2$;*
(ii) *X is isomorphic to $\mathrm{Aut}(\mathrm{SU}_4(2))$; or*
(iii) *X is isomorphic to $\mathrm{Sp}_6(2)$.*

Proof. See [22, Theorem 2]. □

Lemma 2.10. *Suppose that X is a group of shape $3_+^{1+2}.\mathrm{SL}_2(3)$, $O_2(X) = 1$ and a Sylow 3-subgroup of X contains an elementary abelian subgroup of order 3^3. Then X is isomorphic to the centralizer of a non-trivial 3-central element in $\mathrm{PSp}_4(3)$.*

Proof. See [14, Lemma 6]. □

Lemma 2.11. *Suppose that F is a field, V is an n-dimensional vector space over F and $G = \mathrm{GL}(V)$. Assume that q is quadratic form of Witt index at least 1 and with non-degenerate associated bilinear form f, where, for $v, w \in V$, $f(v, w) = q(v + w) - q(v) - q(w)$. Let $\mathcal{S}$ be the set of singular 1-dimensional subspaces of V with respect to q. Then the stabiliser in G of $\mathcal{S}$ preserves q up to similarity.*

Proof. See [16, Lemma 2.10]. □

Lemma 2.12. *Suppose that p is an odd prime, $X = \mathrm{GL}_4(p)$ and V is the natural $\mathrm{GF}(p)G$-module. Let $A = \langle a, b\rangle \le X$ be elementary abelian of order p^2 and assume that $[V, a] = C_V(b)$ and $[V, b] = C_V(a)$ are distinct and of dimension 2. Let $v \in V \setminus [V, A]$. Then A leaves invariant a non-degenerate quadratic form with respect to which v is a singular vector. In particular, X contains exactly two conjugacy classes of subgroups such as A. One is conjugate to a Sylow p-subgroup of $\mathrm{GO}_4^+(p)$ and the other to a Sylow p-subgroup of $\mathrm{GO}_4^-(p)$.*

Proof. See [16, Lemma 2.11]. □

The 4-dimensional orthogonal module of +-type will play a prominent role in the proof of our main theorem. We next introduce some notation which will be used in the proof.

Notation 2.13. *Let V be a 4-dimensional non-degenerate orthogonal space of +-type over $\mathrm{GF}(3)$. Assume that X is a non-zero subspace of V. Then $\mathcal{S}(X)$ is the set of singular 1-dimensional subspaces in X, $\mathcal{P}(X)$ the set of 1-dimensional subspaces of +-type in X and $\mathcal{M}(X)$ the set of 1-dimensional subspaces of −-type in X.*

Lemma 2.14. *Let X be a 3-dimensional subspace in a non-degenerate 4-dimensional orthogonal space of +-type over $\mathrm{GF}(3)$. Then $\mathcal{S}(X) \neq \emptyset$.*

Proof. See [1, 21.3]. □

We now introduce some additional notation:

Notation 2.15. *Let V be a 4-dimensional non-degenerate orthogonal space of +-type over $\mathrm{GF}(3)$ and E be a 2-dimensional subspace of V. The type of E is determined by the number of 1-dimensional subspaces of a given type in E. Thus we have*

Type S: $|\mathcal{S}(E)| = 4$.
Type D+: $|\mathcal{S}(E)| = 1$ *and* $|\mathcal{P}(E)| = 3$.
Type D-: $|\mathcal{S}(E)| = 1$ *and* $|\mathcal{M}(E)| = 3$.
Type N+: $|\mathcal{S}(E)| = 2$ *and* $|\mathcal{M}(E)| = |\mathcal{P}(E)| = 1$.
Type N-: $|\mathcal{P}(E)| = |\mathcal{M}(E)| = 2$.

Lemma 2.16. *Let V be a 4-dimensional non-degenerate orthogonal space over $\mathrm{GF}(3)$ of +-type and E be a 2-dimensional subspace of V. Then E is of one of the types in Notation 2.15.*

Proof. The subspaces of V of dimension 2 are either totally singular (S), degenerate with three elements of $\mathcal{P}(V)$ (D+), degenerate with three elements from $\mathcal{M}(V)$ (D-) , non-degenerate of plus type (N+), or non-degenerate of minus type (N-). □

Theorem 2.17. *Suppose that G is a finite group, Q is a subgroup of G and $H = N_G(Q)$. Assume that the following hold*

(i) $H/Q \cong \mathrm{Aut}(\mathrm{SU}_4(2))$ *or* $\mathrm{Sp}_6(2)$*;*
(ii) $Q = C_G(Q)$ *is a minimal normal subgroup of H and is elementary abelian of order* 2^8*;*
(iii) H *controls G-fusion of elements of H of order* 3*; and*
(iv) *if $g \in G \setminus H$ and $d \in H \cap H^g$ has order* 3*, then* $C_Q(d) = 1$*.*

Then $G = HO_{2'}(G)$*.*

Proof. This is [16, Theorem 3.1]. □

Lemma 2.18. *Suppose that G is a group, E is an extraspecial* 2*-group which is normal in G and $x \in G \setminus C_G(E)$ is an involution. If x is not E-conjugate to xe where $e \in Z(E)^{\#}$, then $C_E(x) \geq [E, x]$ and $[E, x]$ is elementary abelian.*

Proof. Certainly $C_{E/Z(E)}(x) \geq [E/Z(E), x]$. Therefore, if $C_E(x) \not\geq [E, x]$, then $[f, x, x] = e$ for some $f \in E$. Setting $w = [f, x]$ we then have $x^w = xe$ which contradicts our hypothesis on x. Hence $C_E(x) \geq [E, x]$.

We now show that every element of $[E, x]$ has order 2. Let $f \in [E, x]$. Then fe has the same order as f. Thus we may suppose that $f = [h, x]$ for some $h \in E$. As $[E, x] \leq C_E(x)$, $x[h, x] = [h, x]x$ and so

$$\begin{aligned} f^2 &= [h, x][h, x] = h^{-1}xhx[h, x] = h^{-1}xh[h, x]x \\ &= h^{-1}xhh^{-1}xhxx = 1 \end{aligned}$$

as required. This proves the lemma. □

For a group X with subgroups $A \leq Y \leq X$, we say that A is *strongly closed in Y with respect to X* provided $A^x \cap Y \leq A$ for all $x \in X$.

Lemma 2.19. *Suppose that K is a group, $O_{2'}(K) = 1$, A is an abelian* 2*-subgroup of K and A is strongly closed in $N_K(A)$ with respect to K. Assume that $F^*(N_K(A)/C_K(A))$ is a non-abelian simple group. Then $K = N_K(A)$.*

Proof. Set $L = \langle A^K \rangle$. Since $O_{2'}(K) = 1$, we have $O_{2'}(L) = 1$. By Goldschmidt [5, Theorem A], $L = O_2(L)E(L)$ and $A = O_2(L)\Omega_1(T)$ where $T \in \mathrm{Syl}_2(L)$ contains A. If $E(L) = 1$, then A is normal in K and we are done. Thus $E(L) \neq 1$. Goldschmidt additionally states that $E(L)$ is a direct product of simple groups of type $\mathrm{PSL}_2(q)$, $q \equiv 3, 5 \pmod 8$, ${}^2\mathrm{G}_2(3^a)$, $\mathrm{SL}_2(2^a)$, $\mathrm{PSU}_3(2^a)$, ${}^2\mathrm{B}_2(2^a)$ for some natural number a, or the sporadic simple group J_1. It follows from the structure of these groups that $N_L(A)$ is a soluble group which is not a 2-group. On the other hand, $N_L(A) = L \cap N_K(A)$ is a normal subgroup of $N_K(A)$. Since $F^*(N_K(A)/C_K(A))$ is a non-abelian simple group and $N_L(A)$ is soluble we now have $N_L(A) \leq C_K(A)$ and this contradicts the structure of $E(L)$. Thus A is normal in K as claimed. □

We will also need the following statement of Holt's Theorem [10].

Lemma 2.20. *Suppose that K is a simple group, P is a proper subgroup of K and r is a 2-central element of K. If $r^K \cap P = r^P$ and $C_K(r) \leq P$, then $K \cong \mathrm{PSL}_2(2^a)$ $(a \geq 2)$, $\mathrm{PSU}_3(2^a)$ $(a \geq 2)$, ${}^2\mathrm{B}_2(2^a)$ $(a \geq 3$ and odd$)$ or $\mathrm{Alt}(n)$ $(n \geq 5)$ where in the first three cases P is a Borel subgroup of K and in the last case $P \cong \mathrm{Alt}(n-1)$.*

Proof. Set $\Omega = K/P$ and assume that $P < K$. The conditions $C_K(r) \leq P$ and $r^K \cap P = r^P$ together imply that r fixes a unique point of Ω. Let J be the set of involutions of K which fix exactly one point of Ω. Since r is a 2-central element of K, any 2-group which fixes at least 3 points when it acts on Ω commutes with an element of J. Hence Holt's criterion (*) from [10] is satisfied. In addition, the simplicity of K yields $K = \langle r^K \rangle = \langle J \rangle$. Thus [10, Theorem 1] implies that K is isomorphic to one of the following groups $\mathrm{PSL}_2(2^n)$, $\mathrm{PSU}_3(2^n)$, ${}^2\mathrm{B}_2(2^n)$ ($n \geq 3$ and odd) or $\mathrm{Alt}(\Omega)$ where in the first three classes of groups the stabiliser P is a Borel subgroup and in the latter case it is $\mathrm{Alt}(\Omega \setminus \{P\})$. □

For the final steps in the identification of $\mathrm{F}_4(2)$ we need information about its involutions and their centralizers.

Lemma 2.21. *The group $X = \mathrm{F}_4(2)$ has four conjugacy classes of involutions x_1, x_2, x_3 and x_4 three of which are 2-central. Furthermore we may assume that notation is chosen so that*

(i) $C_X(x_1) \cong C_X(x_2) \approx 2^{1+6+8}.\mathrm{Sp}_6(2)$;
(ii) $C_X(x_3) \approx 2^{1+1+4+1+4+4+1+4}.\mathrm{Sp}_4(2)$; *and*
(iii) $C_X(x_4) \approx 2^{[9]}.(\mathrm{SL}_2(2) \times \mathrm{SL}_2(2))$.

Proof. These facts can be found in Guterman [9, Section 3] (see also [2, Page 45]) . □

3. Identifying $\mathrm{F}_4(2)$

The final step in the proof of Theorem 1.2 demands that we can identify $\mathrm{F}_4(2)$ or $\mathrm{Aut}(\mathrm{F}_4(2))$ from the structure of the centralizer of a certain 2-central involution. In this section we give such an identification. The centralizers of interest are the centralizers of the involutions x_1, x_2 in $\mathrm{F}_4(2)$ as given in Lemma 2.21 (i). Of course, we do not want to specify the isomorphism type of such a centralizer, but only the approximate shape of the group.

Definition 3.1. *We say the group U is similar to a 2-centralizer in a group of type* $\mathrm{F}_4(2)$ *if U has the following properties.*

(i) $U/O_2(U) \cong \mathrm{Sp}_6(2)$;

(ii) $O_2(U)$ *is an product of* $Z(O_2(U))$ *by an extraspecial group of order* 2^9, $Z(O_2(U))$ *is elementary abelian of order* 2^7; *and*

(iii) $U/O_2(U)$ *induces the natural module on* $Z(O_2(U))/O_2(U)'$ *and the spin module on* $O_2(U)/Z(O_2(U))$.

Definition 3.2. *Suppose that G is a group and assume that the following hold:*

(i) *For $i = 1, 2$, there are involutions x_i in G such that* $U_i = C_G(x_i)$ *is similar to a 2-centralizer in a group of type* $\mathrm{F}_4(2)$.

(ii) *There is a Sylow 2-subgroup T of U_1 such that* $Z(T) = \langle x_1, x_2\rangle$.

Then we say that U_1, U_2, T is an F_4 set-up in G.

Our identification theorem in this section is as follows:

Theorem 3.3. *If U_1, U_2, T is an F_4 set-up in G, then* $\langle U_1, U_2\rangle \cong \mathrm{F}_4(2)$.

For the remainder of this section we assume that U_1, U_2 and T is an F_4 set-up in G. Notice that because of Definition 3.1 (ii), for $i = 1, 2$, $O_2(U_i)' = \langle x_i\rangle$ has order 2. The first lemma details the relationship of U_1 with U_2.

Lemma 3.4. *The following hold:*

(i) $U_1 \cap U_2$ *contains* T;

(ii) $(U_1 \cap U_2)/O_2(U_1 \cap U_2) \cong \mathrm{Sp}_4(2)$;

(iii) $O_2(U_1 \cap U_2) = O_2(U_1)O_2(U_2)$; *and*

(iv) $Z(T) = Z(O_2(U_1)) \cap Z(O_2(U_2))$.

Proof. From part (ii) of the definition of an F_4 set-up in G, we have $T \le U_1 \cap U_2$. This proves (i).

Since $Z(U_i)/\langle x_i\rangle$ is a natural $U_i/O_2(U_i)$-module and $|Z(T)| = 4$, Lemma 2.8 implies $Z(T) \le Z(U_1) \cap Z(U_2)$. Therefore, by Lemma 2.3 (iii),

$$\begin{aligned}(U_1 \cap U_2)/O_2(U_1 \cap U_2) &= C_{U_1}(Z(T))/O_2(C_{U_1}(Z(T)) \\ &= C_{U_2}(Z(T))/O_2(C_{U_1}(Z(T)) \cong \mathrm{Sp}_4(2).\end{aligned}$$

Hence (ii) holds.

Since

$$(O_2(U_1) \cap O_2(U_2))' \le O_2(U_1)' \cap O_2(U_2)' = \langle x_1\rangle \cap \langle x_2\rangle = 1,$$

$O_2(U_1) \cap O_2(U_2)$ is abelian. Therefore, as $O_2(U_1)$ contains an extraspecial subgroup of order 2^9, we have

$$|O_2(U_1) : O_2(U_1) \cap O_2(U_2)| \ge 2^4.$$

Furthermore, as $O_2(U_1)O_2(U_2)/O_2(U_1)$ is normal in $(U_1 \cap U_2)/O_2(U_1)$, $O_2(U_1 \cap U_2) = O_2(U_1)O_2(U_2)$ follows from Lemma 2.3 (iii). This is (iii).

Finally, since $O_2(U_1 \cap U_2)$ centralizes $Z(O_2(U_1)) \cap Z(O_2(U_2))$, we deduce $Z(T) = Z(O_2(U_1)) \cap Z(O_2(U_2))$ and this proves (iv). □

Our method to prove Theorem 3.3 is to use the F_4 set-up U_1, U_2, T in G to construct a chamber system of type $\mathrm{F}_4(2)$ using the subgroup $P = \langle U_1, U_2 \rangle$ of G. To accomplish this we first define P_1, P_2, P_3 to be subgroups of U_1 containing T such that $P_j/O_2(U_1)$, $j = 1, 2, 3$, are the minimal parabolic subgroups of $U_1/O_2(U_1)$ containing $T/O_2(U_1)$. We additionally let P_4 be such that $U_2 \geq P_4 \geq T$, $P_4 \not\leq U_1$ and $P_4/O_2(U_2)$ is a minimal parabolic subgroup of $U_2/O_2(U_2)$. For $\emptyset \neq \sigma \subseteq \{1, 2, 3, 4\}$ we set $P_\sigma = \langle P_j \mid j \in \sigma \rangle$.

We may assume that notation has been chosen so that

$$\begin{aligned} P_{12}/O_2(P_{12}) &\cong \mathrm{SL}_3(2); \\ P_{13}/O_2(P_{13}) &\cong \mathrm{SL}_2(2) \times \mathrm{SL}_2(2); \text{ and} \\ P_{23}/O_2(P_{23}) &\cong \mathrm{Sp}_4(2). \end{aligned}$$

Note also that $P_j/O_2(P_j) \cong \mathrm{SL}_2(2)$ for $1 \leq j \leq 4$. By Lemma 3.4 (ii), $P_{23} = U_1 \cap U_2$ and $P = \langle P_1, P_2, P_3, P_4 \rangle$.

Set $\mathcal{I} = \{1, 2, 3, 4\}$, and let

$$\mathcal{C} = (P/T, (P/P_k), k \in \mathcal{I})$$

be the corresponding chamber system. Thus $\mathcal{C}$ is an edge coloured graph with colours from $\mathcal{I} = \{1, 2, 3, 4\}$ and vertex set the right cosets P/T. Furthermore, two cosets Tg_1 and Tg_2 form a k-coloured edge if and only if $Tg_2g_1^{-1} \subseteq P_k$. Obviously P acts on $\mathcal{C}$ by multiplication of cosets on the right and this action preserves the coloured edges. For $\mathcal{J} \subseteq \mathcal{I}$, set $P_\mathcal{J} = \langle P_k \mid k \in \mathcal{J} \rangle$ and $\mathcal{C}_\mathcal{J} = (P_\mathcal{J}/T, (P_\mathcal{J}/P_k), k \in \mathcal{J})$. Then $\mathcal{C}_\mathcal{J}$ is the $\mathcal{J}$-connected component of $\mathcal{C}$ containing the vertex T.

We will show $\mathcal{C}$ locally resembles the corresponding chamber system in $\mathrm{F}_4(2)$. This means that for $\sigma \subset \mathcal{I}$ with $|\sigma| = 2$ we will show $P_\sigma/O_2(P_\sigma)$ is isomorphic to the corresponding group in $\mathrm{F}_4(2)$. Since $U_1/O_2(U_1) \cong \mathrm{Sp}_6(2)$ this is true if $\sigma \subseteq \{1, 2, 3\}$. Hence we may assume that $4 \in \sigma$. There are two possibilities for the relationship between P_2 and P_4 (they are both contained in U_2), but we may have $P_{24}/O_2(P_{24}) \cong \mathrm{SL}_3(2)$ or $P_{24} = P_2P_4$. We shall show that the latter is in fact the case. We will also prove $P_{14} = P_1P_4$. This is the purpose of the next lemma.

Lemma 3.5. *The subgroup $Z_2(T)$ is normalized by P_{14}, $P_{14} = P_1P_4$ and $P_{24} = P_2P_4$.*

Proof. Let $V = Z_2(T)$. Then, by Lemma 3.4 (iv), $V \cap Z(O_2(U_2)) \not\leq Z(O_2(U_1))$.

As $C_{O_2(U_1)/Z(O_2(U_1))}(T)$ has order 2 by Lemma 2.4 and $|V \cap Z(O_2(U_2))| = 2^3$ by Lemma 2.3, we deduce $V = (V \cap Z(O_2(U_1)))(V \cap Z(O_2(U_2)))$ has order 2^4 as $Z(T) = Z(O_2(U_1)) \cap Z(O_2(U_2))$.

Using Lemmas 2.3 and 2.4, $V \cap Z(O_2(U_1))$ and $VZ(O_2(U_1))$ are both normalized by P_1. Set

$$W = \langle V^{P_1} \rangle.$$

Then, as the set V^{P_1} has size at most 3, $W/(V \cap Z(O_2(U_1)))$ has order at most 2^3 and $W = V(W \cap Z(O_2(U_1)))$. Since $(W \cap Z(O_2(U_1)))/(V \cap Z(O_2(U_1)))$ has order at most 2^2, Lemma 2.3 implies $(W \cap Z(O_2(U_1)))/(V \cap Z(O_2(U_1)))$ is centralized by $O^2(P_1)$. But then $W/(V \cap Z(O_2(U_1)))$ is centralized by $O^2(P_1)$. Thus $W = V$. We may apply the same argument to U_2 to see that P_4 also normalizes V and so deduce that P_{14} acts on V which has order 2^4.

We have $[V, O_2(P_1)] \leq Z(O_2(U_1)) \cap Z(O_2(U_2)) = Z(T)$. Hence, as $[V, O_2(P_1)]$ is normalized by P_1, $[V, O_2(P_1)] = \langle x_1 \rangle$. Similarly $[V, O_2(P_4)] = \langle x_2 \rangle$. Therefore $O_2(P_1) \cap O_2(P_4)$ centralizes V and has index 4 in T. Thus $C_T(V) = O_2(P_1) \cap O_2(P_4)$. In particular, $O_2(P_1)$ acts as a transvection on V. Hence $C_V(O_2(P_1))$ has order 2^3 and so $C_V(O_2(P_1)) = V \cap Z(U_1)$ and $C_V(O_2(P_4)) = V \cap Z(O_2(U_2))$. Because $C_G(V) \leq U_1$, we have also shown $C_G(V) = O_2(P_1) \cap O_2(P_4)$.

Set

$$D = \langle O_2(P_1)^{N_G(V)}, O_2(P_4)^{N_G(V)} \rangle C_G(V)/C_G(V).$$

Then $D \cap U_1 = P_1$ and, as x_1 has at most 15 conjugates under the action of D, $|D| \leq 12 \cdot 15$. The structure of $\mathrm{Alt}(8) \cong \mathrm{GL}_4(2)$ therefore shows $D \cong \mathrm{SL}_2(2) \times \mathrm{SL}_2(2)$, or $\mathrm{O}_4^-(2) \cong \mathrm{Sym}(5)$.

Let $Q_{12} = O_2(P_{12})$, W_1 be the preimage of $C_{Z(O_2(U_1))/\langle x_1 \rangle}(Q_{12})$ and define $W = W_1V$. Then W is elementary abelian of order 2^5. Since $V = (V \cap Z(O_2(U_1)))(V \cap Z(O_2(U_2)))$,

$$\begin{aligned}
[W, Q_{12}] &= [W_1(V \cap Z(O_2(U_1)))(V \cap Z(O_2(U_2))), Q_{12}] \\
&\leq \langle r_1 \rangle [(V \cap Z(O_2(U_1)))(V \cap Z(O_2(U_2))), Q_{12}] \\
&= \langle x_1 \rangle [(V \cap Z(O_2(U_2))), Q_{12}] \\
&\leq \langle x_1 \rangle [(V \cap Z(O_2(U_2))), T] \\
&= \langle r_1 \rangle [(V \cap Z(O_2(U_2))), O_2(U_1)O_2(P_4)] \\
&= \langle x_1 \rangle [(V \cap Z(O_2(U_2))), O_2(U_1)] = \langle x_1 \rangle.
\end{aligned}$$

As $O_2(U_1)/Z(O_2(U_1))$ is a spin module for $\mathrm{Sp}_6(2)$,

$$C_{O_2(U_1))/Z(O_2(U_1))}(Q_{12}) = WZ(O_2(U_1))/Z(O_2(U_1))$$

by Lemma 2.4. We deduce that W is the preimage of $C_{O_2(U_1)/\langle x_1\rangle}(Q_{12})$ and thus W is normalized by P_{12}. Since $Z(O_2(U_1)) \cap Z(O_2(U_2)) = Z(T)$, we have $WZ(O_2(U_2))/Z(O_2(U_2))$ has order 2^2. It follows from Lemma 2.4 that $O^2(P_4)$ centralizes $WZ(O_2(U_2))/Z(O_2(U_2))$. Let $W_2 = \langle W^{P_4}\rangle$. Then $W_2 = W(W_2 \cap Z(O_2(U_2)))$. Since W/V has order 2, we infer that W_2/V has order at most 2^3. Thus $(W_2 \cap Z(O_2(U_2)))/(V \cap Z(O_2(U_2)))$ has order at most 2^2. It follows from Lemma 2.3 that $(W_2 \cap Z(O_2(U_2)))/(V \cap Z(O_2(U_2)))$ is centralized by $O^2(P_4)$. Therefore W/V is normalized by $TO^2(P_4) = P_4$. This shows that W is normalized by P_{124}. Notice that along the way we have shown that $P_{24} = P_2P_4$.

Suppose that $P_{14}/O_2(P_{14}) \cong \mathrm{O}_4^-(2)$. Then P_{14} acts irreducibly on V and so, as P_{12} does not normalize V, W is an irreducible P_{124}-module. As P_{14} has orbits of length 10 and 5 on V and $Z(T) \le V$, we have that P_{14} does not centralize any element in $W \setminus V$ and so P_{14} acts transitively on the 16 elements of $W \setminus V$. This means the orbits of P_{14} on the involutions of W have lengths 5, 10 and 16. Since 5 divides the order of D, we get that the number of conjugates of x_1 under P_{124} is divisible by 5 and, as $|x_1^{P_{12}}| = 10$, we conclude $|x_1^{P_{124}}| = 10$ or 15. But then $V = \langle x_1^{P_{124}}\rangle$, contradicting the fact that P_{124} acts irreducibly on W. Hence $P_{14}/O_2(P_{14}) \cong \mathrm{SL}_2(2) \times \mathrm{SL}_2(2)$ with $P_{14} = P_1P_4$ and this concludes the proof of the lemma. □

Proof of Theorem 3.3. Using Lemma 3.5 and the observations before the lemma yields that the chamber systems $\mathcal{C}_{1,2}$, $\mathcal{C}_{3,4}$ are projective planes, $\mathcal{C}_{2,3}$ is a generalized quadrangle and in both cases the parameters are $3,3$ and the remaining $\mathcal{C}_J$ with $|J| = 2$ are all complete bipartite graphs again with parameters $3,3$. Thus $\mathcal{C}$ is a chamber system of type F_4 (see [25]) in which all panels have 3 chambers. Since $U_1/O_2(U_1) \cong \mathrm{Sp}_6(2) \cong U_2/O_2(U_2)$, we have $\mathcal{C}_{1,2,3}$ and $\mathcal{C}_{2,3,4}$ are the $\mathrm{Sp}_6(2)$-building. Hence, as each connected rank 3 residue of $\mathcal{C}$ is a building of type C_3 and all the rank 2 residues of $\mathcal{C}$ are Moufang polygons, applying [25, Corollary 3] yields that the universal covering $\pi : \mathcal{C}' \longrightarrow \mathcal{C}$ has $\mathcal{C}'$ a building of type F_4 which also has three chambers on each panel. By [24, Proof of Theorem 10.2 on page 214] this building is uniquely determined by the two residues of rank three with connected diagram. Thus $\mathcal{C}'$ is isomorphic to the $\mathrm{F}_4(2)$ building and the type preserving automorphism group F of $\mathcal{C}'$ is isomorphic to $\mathrm{F}_4(2)$. Since $\mathcal{C}'$ is a 2-cover of $\mathcal{C}$, there is a subgroup U of F such that U contains U_1 and $U/D \cong P$ for a suitable normal subgroup D of U. As U_1 is isomorphic to a maximal parabolic subgroup of F, we deduce that $U = F$ and $D = 1$. Thus $P \cong F$. □

4. The structure of M

From now on we suppose that G is a group which satisfies the assumptions of Theorem 1.2. We set $M = N_G(Z)$. So $C_G(Z)$ has index at most 2 in M. Let $S \in \mathrm{Syl}_3(M)$ and $Q = F^*(M) = O_3(M)$.

Lemma 4.1. *We have $Z = Z(S) = Z(Q)$, $N_G(S) \leq M$ and $S \in \mathrm{Syl}_3(G)$.*

Proof. Since $C_M(Q) \leq F^*(Q) = Q$, we have that $Z = Z(Q) = Z(S)$. Therefore $N_G(S) \leq N_G(Z) = M$ and, in particular, $S \in \mathrm{Syl}_3(N_G(S)) \subseteq \mathrm{Syl}_3(G)$. □

Let R^* be a normal subgroup of $C_G(Z)$ such that $R^*/Q \cong \mathrm{Q}_8 \times \mathrm{Q}_8$ and let $R \in \mathrm{Syl}_2(R^*)$. We have that M/Q embeds into $\mathrm{Out}(Q)$ and $\mathrm{Out}(Q)$ is isomorphic to $\mathrm{GSp}_4(3)$ by [11, III(13.7)]. We now locate M/Q in $\mathrm{Out}(Q)$. We will show that M/QR is isomorphic to $\mathrm{Sym}(3)$ or $2 \times \mathrm{Sym}(3)$. More precise information will be presented in Lemma 4.8. The next lemma provides our initial restriction on the structure of M.

Lemma 4.2. *We have that M/Q normalizes R^*/Q and is isomorphic to a subgroup of the subgroup $\mathbf{M}$ of $\mathrm{GSp}_4(3)$ which preserves a decomposition of the natural 4-dimensional symplectic space over $\mathrm{GF}(3)$ into a perpendicular sum of two non-degenerate 2-spaces. Furthermore, R/Q maps to $O_2(\mathbf{M})$.*

Proof. See [17, Lemma 3.1]. □

We next introduce a substantial amount of notation. We will use this for the remainder of the paper. We note now that the subgroups Q_1 and Q_2 defined below will be shown to have order 3^3 in Lemma 4.4.

Notation 4.3. (i) *Define R_1 and R_2 to be the two subgroups of R isomorphic to Q_8 which map to normal subgroups of $C_{\mathbf{M}}(Z(R)Q/Q)$.*

(ii) *For $i = 1, 2$, let $r_i \in Z(R_i)^\#$ and $K_i = C_G(r_i)$.*

(iii) *For $i = 1, 2$, define*

$$Q_i = [Q, R_i].$$

(iv) *For $i = 1, 2$, let $A_i \leq Q_i$ be a fixed S-invariant subgroup of Q_i of order 3^2 and set $A = A_1A_2$.*

(v) *For $i = 1, 2$, we let*

$$\langle \rho_i \rangle \leq A_i$$

be such that $\langle \rho_i \rangle$ is inverted by r_i.

(vi) *Set $J = C_S(A)$ and $L = N_G(J)$.*

Most of this paper is devoted to the determination of K_1 and K_2. We will show that K_i is similar to a 2-centralizer in a group of type $\mathrm{F}_4(2)$ as defined in Definition 3.1 and, for $T \in \mathrm{Syl}_2(K_1)$, show that K_1, K_2 and T is an F_4 set-up. We then use Theorem 3.3 to obtain a subgroup $P \cong \mathrm{F}_4(2)$ of G. Our interim goal to achieve this objective is to show that $C_G(\rho_i)$ is isomorphic to the corresponding centralizer in $\mathrm{F}_4(2)$ or $\mathrm{Aut}(\mathrm{F}_4(2))$. We eventually do this in Lemma 8.2. However we begin more modestly by determining the precise structure of M.

Lemma 4.4. *The following hold.*

(i) $|S/Q| \leq 3^2$.
(ii) $Q_1 = C_Q(r_2)$ *and* $Q_2 = C_Q(r_1)$ *and both are normal in* S*; and*
(iii) $Q_1 \cong Q_2 \cong 3^{1+2}_+$, $[Q_1, Q_2] = 1$ *and* $Q = Q_1Q_2$*;*
(iv) A *is elementary abelian of order* 3^3.

In particular, Q *has exponent* 3.

Proof. Part (i) follows from Lemma 4.2.

That Q_1 and Q_2 are normalized by S follows from the action of M on Q, as R_1Q/Q and R_2Q/Q are normalized by S/Q.

For $i = 1, 2$, we have that $C_Q(r_i)$ and $Q_i = [Q, r_i]$ commute by the Three Subgroup Lemma. Since Q_i has order 3^3 it follows that $Q_i \cong 3^{1+2}_+$. As r_1r_2 inverts Q/Z, r_2 inverts $C_{Q/Z}(r_1)$ and so $C_Q(r_1) = Q_2$ and $C_Q(r_2) = Q_1$. In particular, Q_1 and Q_2 commute and $Q = Q_1Q_2$. This proves (ii) and (iii). Finally (iv) follows from (ii) and (iii). □

Lemma 4.5. *Every element of* Q *is* M*-conjugate to an element of* A.

Proof. It suffices to prove that every element of Q/Z is conjugate to an element of A/Z. Let $w \in Q/Z$. Then $w = x_1x_2$ where $x_i \in Q_i/Z$ by Lemma 4.4 (iii). Since, from the definition of A, for $i = 1, 2$, $(A \cap Q_i)/Z = A_i/Z$ has order 3 and R_i acts transitively on Q_i/Z, there exists $s_i \in R_i$ such that $w^{s_1s_2} = x_1^{s_1}x_2^{s_2} \in A/Z$. This proves the claim. □

Recall that by hypothesis Z is not weakly closed in Q. Hence there is a $g \in G$ such that $Y = Z^g \leq Q$ and $Y \neq Z$. We set

$$\begin{aligned} V &= ZY; \\ H &= \langle Q, Q^g \rangle; \text{ and} \\ W &= C_{Q^g}(Z)C_Q(Y). \end{aligned}$$

Notice that $C_Q(Y)$ normalizes $C_{Q^g}(Z)$ and so W is indeed a subgroup of G. Because of Lemma 4.5 we may and do suppose that $V \leq A$. In particular, V is normalized by S. Before we continue our study of M, we investigate H.

Lemma 4.6. *The following statements hold.*

(i) $S > Q$;

(ii) $Q \cap Q^g$ *is elementary abelian of order* 3^3 *and is a normal subgroup of* S;

(iii) $W = C_Q(Y)C_{Q^g}(Y)$ *is a normal subgroup of* H, $H/W \cong \mathrm{SL}_2(3)$, $WQ \in \mathrm{Syl}_3(H)$ *and* $W/(Q \cap Q^g)$ *is a natural* H/W *module;*

(iv) *for* $i = 1, 2$, $V \cap Q_i = Z$ *and* $A \neq Q \cap Q^g$;

(v) $A = [Q, W] \leq W$, $A/Z = C_{Q/Z}(S) = C_{Q/Z}(W)$ *and* A *is normal in* $N_G(S)$; *and*

(vi) *for* $i = 1, 2$, $[WQ/Q, R_iQ/Q] \neq 1$.

Proof. As Q is extraspecial, $C_Q(Y)$ is non-abelian of order 3^4. By Lemma 4.1, M^g/Q^g has Sylow 3-subgroups of order at most 9 and $C_Q(Y) \leq M^g$ so we have $Z = C_Q(Y)' \leq Q^g$. In particular we now have $S > Q$ for else $C_Q(Y) \leq Q^g$ and then $Z = C_Q(Y)' \leq (Q^g)' = Y$ which is a contradiction. In particular, (i) holds.

Since $\Phi(Q \cap Q^g) \leq Z \cap Y = 1$, $Q \cap Q^g$ is elementary abelian.

Because $V \leq Q \cap Q^g$, we have $[V, Q] = Z$ and $[V, Q^g] = Y$ and so H normalizes and acts non-trivially on V with $H/C_H(V) \cong \mathrm{SL}_2(3)$.

Turning our attention to W, we have

$$[W, Q] = [C_Q(Y)C_{Q^g}(Z), Q] = Z[C_{Q^g}(Z), Q].$$

Since $[[C_{Q^g}(Z), Y], Q] = 1 = [Q, Y, C_{Q^g}(Z)]$, the Three Subgroup Lemma implies that $[C_{Q^g}(Z), Q] \leq C_Q(Y) \leq W$. Therefore

$$[Q, W] \leq C_Q(Y) \leq W$$

and, similarly, $[W, Q^g] \leq C_{Q^g}(Z) \leq W$. Hence H normalizes W and of course $W \leq C_G(V)$.

As $[C_H(V), Q] \leq C_Q(V) = C_Q(Y) \leq W$, H/W is a central extension of $\mathrm{SL}_2(3)$. Since H acts transitively on the four subgroups of order 3 in V, and each such subgroup determines uniquely a subgroup of H we have that Q^H has exactly 4 members. Now $O^3(H)W/W$ is a central extension of a nilpotent group and is thus nilpotent. Let T be a Sylow 2-subgroup of $O^3(H)$. Then as $O^3(H)W/W$ is nilpotent, Q normalizes and does not centralize TW/W. It follows that $H = WTQ$ and then the action of Q on TW/W and the fact that $T/C_T(V) \cong \mathrm{Q}_8$ imply that $T \cong \mathrm{Q}_8$ and that $H/W \cong \mathrm{SL}_2(3)$, as by [11, Satz V.25.3] the Schur multiplier of a quaternion group is trivial.

Using that $O^3(H)$ acts transitively on $V^\#$, we see that $O^3(H)$ does not normalize any non-trivial subgroup of $(W \cap Q)/(Q \cap Q^g)$.

Assume $Q \cap Q^g = V$. Then $|W| = 3^6$. As $W' \leq V$, W is generated by groups of exponent 3 and W is non-abelian, we have $\Phi(W) = V$.

Let $f \in H$ be an involution. Then $fW \in Z(H/W)$ and, by Burnside's Lemma, f does not centralize $W/\Phi(W)$ and neither does it invert $W/\Phi(W)$, for then, as f inverts V, W would be abelian. Therefore, setting $W_0 = C_W(f)V$, we have $W_0 > V$. Then, as the faithful representations of $\mathrm{SL}_2(3)$ in characteristic 3 have even dimension and the minimal faithful representation for $\mathrm{PSL}_2(3)$ is 3, $|W_0/V| = 3^2$ and W_0 is centralized by $O^3(H)$ and normalized by Q; in particular, $Q \cap W_0 \le V$ by the comments at the end of the last paragraph. But then $(W \cap Q)W_0 = W_0(W \cap Q^g) = W$ which means that

$$[W,Q] = [W_0,Q][W \cap Q, Q] \le V.$$

Consequently $O^3(H)$ centralizes W/V which is a contradiction, as we have already remarked that f does not centralize W/V. Therefore $Q \cap Q^g > V$.

Since $Q \cap Q^g$ is abelian and Q is extraspecial of order 3^5, we now have that $|Q \cap Q^g| = 3^3$ and $W/(Q \cap Q^g)$ is a natural $\mathrm{SL}_2(3)$-module. This completes the proof of the first two statements in (ii) and all of (iii).

Since H acts 2-transitively on the non-trivial cyclic subgroups of V, $N_G(V) = (N_M(V) \cap N_{M^g}(V))H$ and therefore $N_G(V)$ normalizes $Q \cap Q^g$. From the choice of $V \le A$, we have $S \le N_G(V)$. This is the last statement in (ii).

Suppose that $V \le Q_i$ for some $i \in \{1,2\}$. Then $C_M(V) \ge R_{3-i}$ and so R_{3-i} acts on $Q \cap Q^g$. Since $|Q \cap Q^g : V| = 3$, we obtain $Q \cap Q^g \le C_Q(r_{3-i}) = Q_i$ contrary to $Q \cap Q^g$ being elementary abelian of order 3^3. Hence V is not contained in Q_i for $i = 1, 2$. If $A = Q \cap Q^g$, then

$$Y = [A, C_{Q^g}(Z)] \le [A, S] = Z,$$

which is impossible. Hence we also know that $A \ne Q \cap Q^g$. Thus (iv) holds.

If $[Q_1, W] \le Z$, then $[Q,W] = [Q_1,W][Q_2,W] \le A_2$. Therefore using (iv),

$$[C_Q(V), W] = [C_Q(V), C_{Q^g}(V)]Z \le Q \cap Q^g \cap A_2 = Z.$$

Since $|Q \cap Q^g| = 3^3$ by (ii), $Y = [Q \cap Q^g, C_{Q^g}(V)] \le [Q,W] = Z$ which is impossible. Thus $[Q_1,W] = A_1$ and similarly $[Q_2,W] = A_2$. Now $[Q,W] = A$ and consequently $[Q,S] = A$. This proves (v).

Finally, suppose that $[WQ, R_1Q] \le Q$. Then $[Q_1,W] \le A_1$ and is R_1-invariant. Hence $[Q_1,W] \le Z$ and this contradicts (v). Thus $[WQ, R_1Q] \not\le Q$ and (vi) holds. □

Now we are in a position to determine M. For this set

$$M_0 = RQ$$

and let f be an involution in H. Then f inverts V and thus $f \in M$. We refine our choice of R so that $R\langle f\rangle$ is a Sylow 2-subgroup of $M_0S\langle f\rangle$.

Lemma 4.7. *We have that Z is the unique G-conjugate of Z in both Q_1 and Q_2.*

Proof. Suppose that $g \in G$, $Z^g \le Q_1$ with $Z^g \ne 1$. Then, using Z^g in place of Y, Lemma 4.6 (iv) applies to give a contradiction. □

Lemma 4.8. *The following hold.*

(i) *$S = WQ$ and $|S/Q| = 3$; and*
(ii) *One of the following holds:*
  (a) *$M = M_0S\langle f\rangle$, $C_M(Z) = M_0S$ and $M/M_0 \cong \mathrm{Sym}(3)$; or*
  (b) *$|M : M_0S\langle f\rangle| = 2$, $C_M(Z) = M_0S\langle t\rangle$ where t is an involution which exchanges R_1 and R_2, centralizes V and inverts SM_0/M_0 and $M/M_0 = \langle t, f\rangle SM_0/M_0 \cong 2 \times \mathrm{Sym}(3)$ with centre $\langle tf\rangle M_0/M_0$.*

Proof. We have seen in Lemma 4.6 (i) and (v) that $|S/Q| \ge 3$ and $A/Z = C_{Q/Z}(S) = C_{Q/Z}(W)$.

Suppose that $|S/Q| = 3^2$ and assume that B is an abelian subgroup of Q which is normal in S of order 3^3 with $B \ne A$. For $i = 1, 2$, let $s_i \in S$ be such that $[s_i, R_{3-i}] \le Q$. Then $[B, s_i] \le B \cap A \cap Q_i \le A_i$. Thus if s_i does not centralizes B/Z, then $A_i \le B$. Since $S = Q\langle s_1, s_2\rangle$ and $B \ne A$, without loss of generality we may suppose that $A_1 \le B$ and $[B, s_2] \le Z$. In particular, $B \le Q_1A$ as $C_{Q/Z}(s_2) = Q_1A/Z$. But then A_1 is centralized by $AB = Q_1A$ and we have a contradiction as $Z(Q_1A) = A_2$. Thus, if $B \le Q$ is a normal abelian subgroup of S of order 3^3, then $B = A$. Taking $B = Q\cap Q^g$, we now have that $Q\cap Q^g = A$ a possibility which is eliminated by Lemma 4.6 (iv). Thus $|S/Q| = 3$. This proves (i).

We know that f inverts $W/(Q \cap Q^g)$ and so WQ/Q is inverted by f. In particular, $M_0S\langle f\rangle/M_0 \cong \mathrm{Sym}(3)$. If $M = M_0S\langle f\rangle$, then (ii)(a) holds. So assume that $M > M_0S\langle f\rangle$. As M inverts Z, we have $M = C_M(Z)\langle f\rangle$. Since, by Lemma 4.2, $C_M(Z)/Q$ is isomorphic to a subgroup of $\mathrm{Sp}_2(3) \wr 2$ and since S/Q has order 3, Lemma 4.6 (vi) implies that $C_M(Z)/M_0 \cong 3 \times 2$ or $\mathrm{Sym}(3)$. Especially, there is a 2-element $t \in C_M(Z) \setminus M_0$ which normalizes $R\langle f\rangle$ and swaps R_1 and R_2. Because $R\langle t\rangle$ is isomorphic to a Sylow 2-subgroup of $\mathrm{Sp}_2(3) \wr 2$, we may as well assume that t is an involution and that t normalizes S.

Since t normalizes S and swaps R_1 and R_2, t also interchanges Q_1 and Q_2 and normalizes A. It follows that t normalizes V. Without loss of generality we may now additionally assume that t normalizes Y. Thus t normalizes $Q \cap Q^g$ as well as A. Since t centralizes Z, $[Q, t]$ is extraspecial of order 3^{1+2}. Hence either t centralizes V and $Q/C_Q(V)$ or t inverts V/Z and $Q/C_Q(V)$. Multiplying t by r_1r_2, we may assume that t centralizes V. If S/Q is centralized by t, we now have $S/C_Q(V)$ is centralized by t. However, as $[Q,S](Q\cap Q^g)=C_Q(V)/(Q\cap Q^g)$, we see that $S/(Q\cap Q^g)$ is extraspecial and since t centralizes $S/C_Q(V)$, Burnside's Lemma implies that t centralizes $S/(Q\cap Q^g)$. Then t also centralizes Q which is a contradiction. Hence t inverts S/Q and therefore $C_M(Z)/M_0$ has the structure described in (ii)(b). □

5. The structure of $L = N_G(J)$

In this section we continue to use the notation introduced in 4.3. We also recall $H=\langle Q,Q^g\rangle$ and f is an involution in $H\cap M$ which inverts Z.

We will show that J is the Thompson subgroup of S and determine $L=N_G(J)$.

Set

$$H_1=H^{r_1}, W_1=W^{r_1} \text{ and } V_1=V^{r_1}.$$

Lemma 5.1. *We have $W\neq W_1$ and $H\neq H_1$.*

Proof. Notice that r_1 inverts A_1/Z and centralizes A_2/Z. Therefore, $V^{r_1}\neq V$. Since

$$W'=[C_Q(V),C_{Q^g}(V)]V\le Q\cap Q^g\cap[Q,W]=Q\cap Q^g\cap A=V,$$

we see $W'=V$ and $W_1'=V_1$. Thus W and W_1 are not equal and so also $H\neq H_1$. □

Lemma 5.2. *For $i=1,2$, we have ρ_i is not G-conjugate to an element of Z. In particular, A contains exactly seven G-conjugates of Z.*

Proof. By definition $\langle\rho_i\rangle\le Q_i$ for $i=1,2$. Hence Lemma 4.7 gives $\langle\rho_i\rangle$ is not a G-conjugate of Z.

Since $V\cup V_1\subset A$, we now see A contains exactly seven G-conjugates of Z, three Q-conjugates of $\langle\rho_1\rangle$, and three Q-conjugates of $\langle\rho_2\rangle$. □

We can now describe the structure of L.

Lemma 5.3. *The following hold.*

(i) *$J=J(S)$ is elementary abelian of order 3^4.*
(ii) *L controls G-fusion of elements of J.*
(iii) *$J=C_G(J)$.*
(iv) *L preserves a quadratic form q of $+$-type on J up to similarity.*
(v) *Set $L_*=\langle H,H_1,r_1,r_2\rangle$. Then $L_*/J\cong \mathrm{GO}_4^+(3)$ and either*
   (a) *if $M=M_0S\langle f\rangle$, then $L=L_*$; or*

(b) *if* $M > M_0S\langle f\rangle$*, then* $L/J \cong \mathrm{CO}_4^+(3)$*. (Here* $\mathrm{CO}_4^+(3)$ *is the group which preserves* q *up to similarity.)*

Proof. By construction A is elementary abelian and so $A \le C_Q(V) \le W$ and $A \le C_Q(V_1) \le W_1$. Since S centralizes A/Z and since in $\mathrm{GL}_3(3)$ such a centralizer has order 18, we infer that $J = C_S(A)$ has order 3^4. Since A has index 3 in J, J is abelian. Suppose that B is an abelian subgroup of S of order at least 3^4. We may assume that $B \ge Z$. Thus by Lemma 4.8, $B \cap Q$ is an abelian subgroup of Q of order at least 3^3 and hence of order exactly 3^3. Using that $(B\cap Q)/Z$ is centralized by $QB = S$, Lemma 4.6 (iii) yields $B \cap Q = A$. But then $B \le C_S(A) = J$ and we have $B = J$. Hence $J = J(S)$ is the Thompson subgroup of S. Since J centralizes V, $J \le S \cap C_G(V) = W$. Thus $J = J(W)$ and similarly $J = J(W_1)$. In particular, $L \ge \langle H, H_1\rangle N_G(S)$. Since J contains A, if J is not elementary abelian, then $\Phi(J) = Z$. But then Z is normalized by H, which is a contradiction as H acts irreducibly on V. Thus J is elementary abelian. This proves (i). Part (ii) follows from [1, 37.6] as J is abelian.

We have that $C_G(J) \le C_G(Z) < M$. Since J acts non-trivially on both R_1Q/Q and R_2Q/Q, and JM_0/M_0 is inverted by t when $M > M_0S\langle f\rangle$ (see Lemma 4.8 (ii)), we have $C_M(J) \le S\langle r_1, r_2\rangle$. Since r_1Q and r_2Q act non-trivially on A/Z, we have $C_G(J) \le S$. Hence $J \le C_G(J) = C_S(J) \le C_S(A) \le J$ and this proves (iii).

Define

$$\mathcal{S}(J) = \{j \in J^\# \mid j^l \in Z \text{ for some } l \in L\}.$$

Consider $S/J = Q_1Q_2J/J$. Then $S/J \in \mathrm{Syl}_3(L_*/J) \subseteq \mathrm{Syl}_3(L/J)$. We have $[J, Q_1] = A_1 = C_J(Q_2)$ and $[J, Q_2] = A_2 = C_J(Q_1)$. In addition, $[J, S] = [J, Q] = [W, Q] = A$ and $C_J(S) = Z$.

Now $\langle Z^{L_*}\rangle \ge \langle Z^H\rangle\langle Z^{H_1}\rangle = VV_1 = A$ and, as $A \not\le Q \cap Q^g$, A is not normalized by H. Hence $\langle Z^{L_*}\rangle = J$ and, in particular, L_* and, consequently, L acts irreducibly on J. Thus there are members of $\mathcal{S}(J)$ in $J\backslash A$. By Lemma 5.2 there are exactly 14 elements of $\mathcal{S}(J)$ in A and in $J\backslash A$ there are a multiple of 18 such elements. Thence $|\mathcal{S}(J)| = 14+n\cdot 18$ for some integer $n \ge 1$. Since $|J| = 3^4$, using the fact that $|\mathcal{S}(J)|$ divides $|\mathrm{GL}_4(3)|$ we infer that $|\mathcal{S}(J)| = 32$.

Using Lemma 2.12 with $\langle a\rangle = Q_1J/J$ and $\langle b\rangle = Q_2J/J$, yields that S preserves a quadratic form with any element of $\mathcal{S}(J)$ as a singular vector. Since S/J contains W_1/J and W_2/J which both act quadratically on J with $[J, W] = [J, J(Q \cap Q^g)] = [J, (Q \cap Q^g)] = V$ and $[J, W] = [J, W]^{r_1} = V_1$ we see that for any such form V and V_1 would consist of singular vectors. It follows that $\mathcal{S}(J)$ is the set of singular vector of a $+$-type quadratic form on J. Since this set is by design invariant under the action of L, we have L/J is isomorphic to a subgroup of $\mathrm{CO}_4^+(3)$ by Lemma 2.11. Thus (iv) is true. Now HH_1 contains $S = WW_1$ which is a Sylow 3-subgroup of G, H acts irreducibly on V and H_1 acts irreducibly on V_1, it follows that $HH_1/J \cong \Omega_4^+(3)$. Conjugation by r_1 exchanges H and H_1, $\langle r_1r_2\rangle H_1/W_1 \cong \mathrm{GL}_2(3)$ and so we infer that $L_*/J \cong \mathrm{GO}_4^+(3)$ and L_* is normal in L. By the Frattini Argument, $L = N_L(S)L_* = N_M(S)L_*$ and so (v) holds. □

Lemma 5.4. *We have ρ_1 is G-conjugate to ρ_2 if and only if $SR\langle f\rangle$ has index 2 in M.*

Proof. This is a consequence of Lemma 5.3(ii) and (v). □

Recall the notation introduced in 2.13 and 2.15.

Lemma 5.5. *The sets $\mathcal{P}(J)$ and $\mathcal{M}(J)$ are fused in L if $L > L_*$ and we have $|\mathcal{S}(J)| = 16$, $|\mathcal{P}(J)| = |\mathcal{M}(J)| = 12$.*

Proof. This follows directly from Lemma 5.3. □

Lemma 5.6. *For $i = 1, 2$, $C_L(r_i) = C_{L_*}(r_i)$, $[J, r_i] = \langle \rho_i\rangle$, $|C_J(r_i)| = 3^3$ and $C_L(r_i)/C_J(r_i)\langle r_i\rangle \cong \mathrm{GO}_3(3) \cong 2 \times \mathrm{Sym}(4)$.*

Proof. We have that $|C_S(r_i)| = 3^4$ and r_i inverts Q_iJ/J. Hence $|C_J(r_i)| = 3^3$. It follows that both r_1 and r_2 are reflections on J. If $L > L_*$, then $r_1^t = r_2$ and so $C_L(r_i) = C_{L_*}(r_i)$. Since r_1 and r_2 are reflections and since $L_*/J \cong \mathrm{GO}_4^+(3)$ by Lemma 5.3, we have $C_L(r_i)/C_J(r_i)\langle r_i\rangle \cong \mathrm{GO}_3(3) \cong 2 \times \mathrm{Sym}(4)$. □

From Lemma 5.6 we have $[J, r_1] = \langle \rho_1\rangle$ and $[J, r_2] = \langle \rho_2\rangle$ are non-singular 1-dimensional spaces in J. We fix notation so that $\langle \rho_1\rangle \in \mathcal{P}(J)$ and $\langle \rho_2\rangle \in \mathcal{M}(J)$.

Lemma 5.7. *The following hold:*

(i) *V and V_1 are of Type S;*
(ii) *A_1 is of Type D+;*
(iii) *A_2 is of Type D-;*
(iv) *$\langle \rho_1, \rho_2\rangle$ is of type N+;*
(v) *$|\mathcal{S}(C_J(r_1))| = 4$, $|\mathcal{M}(C_J(r_1))| = 6$ and $|\mathcal{P}(C_J(r_1))| = 3$; and*
(vi) *$|\mathcal{S}(C_J(r_2))| = 4$, $|\mathcal{M}(C_J(r_2))| = 3$ and $|\mathcal{P}(C_J(r_2))| = 6$.*

Proof. Parts (i)–(iv) are obvious. By Lemma 5.6 we have that $|C_J(r_i)| = 3^3$ for $i = 1, 2$. Since J is a quadratic space of plus type, it follows that $C_J(r_1)$ has an orthonormal basis consisting of members of $\mathcal{P}(J)$ and $C_J(r_2)$ has an orthonormal basis consisting of elements of $\mathcal{M}(J)$. Thus (v) and (vi) hold. □

Lemma 5.8. *If $\widetilde{\rho}_i \in C_J(r_i)$ is L_*-conjugate to ρ_i, then $\langle \rho_i, \widetilde{\rho}_i\rangle$ has Type N-. In particular, $|\mathcal{P}(\langle \rho_i, \widetilde{\rho}_i\rangle)| = |\mathcal{M}(\langle \rho_i, \widetilde{\rho}_i\rangle)| = 2$.*

Proof. Suppose that $\widetilde{\rho}_i \in C_J(r_i)$ is L_*-conjugate to $\langle \rho_i\rangle$. Then, as $\langle \rho_i\rangle = [J, r_i]$, ρ_i is perpendicular to $C_J(r_i)$. It follows that $\widetilde{\rho}_i$ is perpendicular to ρ_i and this means that $\langle \rho_i, \widetilde{\rho}_i\rangle$ is of Type N-. □

6. Two 3-centralizers

In this section we determine the structure of $C_G(\rho_1)$ and $C_G(\rho_2)$. We first show that these centralisers do not have non trivial normal $3'$-subgroups. Recall the notation of 4.3 and that $f \in M$ is an involution inverting Z.

Lemma 6.1. *J does not normalize any non-trivial $3'$-subgroups.*

Proof. Suppose that Y is a non-trivial $3'$-subgroup normalized by J. Then, as every subgroup of J of order 27 contains a conjugate of Z by Lemma 2.14, we may assume that $X = C_Y(Z) \neq 1$. As X is normalized by $A = J \cap Q$ and X normalizes Q, $[A, X] \leq Q \cap X = 1$ and hence $X \leq C_M(A) = J$ as A is a maximal abelian subgroup of Q. But then $X = 1$ which is a contradiction. This proves the lemma. □

Lemma 6.2. *For $i = 1, 2$, $C_M(\rho_i) = Q_{3-i}R_{3-i}J\langle fr_i\rangle$ and $C_{C_M(Z)}(\rho_i)/\langle \rho_i\rangle$ is isomorphic to the centralizer of a non-trivial 3-central element in $\mathrm{PSp}_4(3)$ and Z is inverted in $C_M(\rho_i)$.*

Proof. Since $\rho_i \in A_i \leq J$ and since $[Q_1, Q_2] = 1$ and $[Q_i, R_{3-i}] = 1$, we certainly have $C_M(\rho_i) \geq Q_{3-i}R_{3-i}J$. Furthermore, f inverts J and so f inverts ρ_i and as r_i also inverts ρ_i, we have $C_M(\rho_i) \geq Q_{3-i}R_{3-i}J\langle fr_i\rangle$ which has index either 24 or 48 in M dependent upon whether or not $M = RS\langle f\rangle$ respectively. Since Q_i contains twelve Q-conjugates of $\langle \rho_i\rangle$, Lemma 5.4 implies $C_M(\rho_i) \geq Q_{3-i}R_{3-i}J\langle fr_i\rangle$.

Because $r_i f$ inverts Z, we have $C_{C_M(Z)}(\rho_i)/\langle \rho_i\rangle = Q_{3-i}R_{3-i}J/\langle \rho_i\rangle$ with R_{3-i} acting faithfully on Q_{3-i}. Thus the final statement also is valid by Lemma 2.10. □

In the next two lemmas we pin down two possible structures of $C_G(\rho_1)$ and $C_G(\rho_2)$. In fact in $\mathrm{F}_4(2)$ we have that both are isomorphic to $3 \times \mathrm{Sp}_6(2)$. That this is the case in our group will be proved later in Lemma 8.2.

Lemma 6.3. *For $i = 1, 2$ either $C_G(\rho_i) \cong 3 \times \mathrm{Aut}(\mathrm{SU}_4(2))$ or $C_G(\rho_i) \cong 3 \times \mathrm{Sp}_6(2)$. Furthermore, r_i inverts ρ_i and centralizes $C_G(\rho_i)/\langle \rho_i\rangle$.*

Proof. We consider $C_G(\rho_i)/\langle\rho_i\rangle$. By Lemma 6.2, $C_{C_M(Z)}(\rho_i)/\langle\rho_i\rangle$ is isomorphic to a 3-centralizer in $\mathrm{PSp}_4(3)$. Since $J/\langle\rho_i\rangle$ normalizes no nontrivial $3'$-subgroup of $C_G(\rho_i)$ by Lemma 6.1 and Z is inverted by fr_i, we may apply Theorem 2.9 to obtain $C_G(\rho_i)/\langle\rho_i\rangle \cong \mathrm{Aut}(\mathrm{SU}_4(2))$ or $\mathrm{Sp}_6(2)$ or that $C_G(\rho_i) = C_M(\rho_i)$. The latter possibility is dismissed as $C_L(\rho_i)$ has index 2 in $\langle\rho_i\rangle C_{L_*}(r_i)$ and so, by Lemma 5.6,

$$C_L(\rho_i) \cong 3 \times 3^3 : (2 \times \mathrm{Sym}(4))$$

does not normalize Z.

The Sylow 3-subgroup of $C_G(\rho_i)$ is $\langle\rho_i\rangle \times Q_{3-i}C_J(r_i)$ and hence the extension $C_G(\rho_i)/\langle\rho_i\rangle$ splits by Gaschütz Theorem. Finally we have that r_i centralizes $Q_{3-i}J/\langle\rho_i\rangle$ and, as no automorphism of either $\mathrm{Aut}(\mathrm{SU}_4(2))$ or $\mathrm{Sp}_6(2)$ of order 2 centralizes such a subgroup, we infer that r_i centralizes $C_G(\rho_i)/\langle\rho_i\rangle$ and of course we also know that ρ_i is inverted by r_i. □

Lemma 6.4. *We have $C_G(\rho_1) \cong C_G(\rho_2)$.*

Proof. By Lemma 6.3, $C_G(\rho_1)/\langle\rho_1\rangle \cong \mathrm{Sp}_6(2)$ or $\mathrm{Aut}(\mathrm{SU}_4(2))$.

Assume that $C_G(\rho_1)/\langle\rho_1\rangle \cong \mathrm{Sp}_6(2)$. Using Lemma 5.7 (v), we have some $\widetilde{\rho_1} \in \mathcal{P}(C_J(\rho_1))$ and as $|\mathcal{P}(C_J(\rho_1))| = 3$, $C_{E(C_G(\rho_1))}(\widetilde{\rho_1}) \cong 3 \times \mathrm{Sp}_4(2)$ from the structure of $\mathrm{Sp}_6(2)$. Therefore $E(C_G(\langle\rho_1, \widetilde{\rho_1}\rangle)) \cong \mathrm{Sp}_4(2)'$. Lemma 5.8, yields that $\mathrm{Sp}_4(2)'$ is involved in the centralizer of a 3-element in $C_G(\rho_2)$. As there are no such 3-elements in $\mathrm{SU}_4(2)$ [4], Lemma 6.3 implies $E(C_G(\rho_2))/\langle\rho_2\rangle \cong \mathrm{Sp}_6(2)$. Hence Lemma 6.4 holds. □

7. Building a signalizer in the centralizers of r_1 and r_2

In this section we begin the construction $K_i = C_G(r_i)$ for $i = 1, 2$. We give a brief overview of our plans for $i = 1$ to guide the reader through the technicalities involved. Our final aim is to show that K_1 is similar to a 2-centralizer in a group of type $\mathrm{F}_4(2)$ (see Definition 3.1). Hence we aim to show that K_1 is an extension of a 2-group by $\mathrm{Sp}_6(2)$. Further we show this 2-group is a product of an extraspecial group of order 2^9 by an elementary abelian group. Our first aim is to construct the extraspecial group Σ_1, and show that it is normalized by $C_L(r_1)$. Note that $C_J(r_1) \le C_L(r_1)$ and the former group is elementary abelian of order 3^3.

We briefly consider the situation in our target group. In $\mathrm{F}_4(2)$ there are exactly four maximal subgroups of $C_J(r_1)$ with centralizers in Σ_1 which properly contain $\langle r_1\rangle$ and these maximal subgroups centralize a quaternion group of order eight in Σ_1. In our group G, the first problem is to find these quaternion groups. For this we pick a set of four maximal subgroups of $C_J(r_1)$, which are conjugate to A_2. They all contain a conjugate of ρ_2. By Lemma 6.3 there are exactly two possibilities for the structure of $C_G(\rho_2)$. Examining these structures shows $C_{C_G(\rho_2)}(A_2)/\langle\rho_2\rangle \cong 3^{1+2}_+{:}\mathrm{SL}_2(3)$. Hence $C_{C_G(\rho_2)\cap C_G(r_1)}(A_2)/\langle\rho_2\rangle \cong \mathrm{SL}_2(3)$. This shows that $O_2(C_{C_G(\rho_2)\cap C_G(r_1)}(A_2)) \cong \mathrm{Q}_8$, and this is one of the quaternion groups we are looking for. As A_2 has four conjugates under $C_L(r_1)$, we now get a set of four quaternion groups. The problem is now to show these four quaternion groups generate a 2-group Σ_1 which is extraspecial of order 2^9. This will be done in Lemma 7.12. Furthermore, the very construction guarantees that $C_L(r_1)$ acts on Σ_1.

We continue to use the notation from 2.13, 2.15 and 4.3. Additionally we introduce

Notation 7.1. *For $i = 1, 2$, $I_i = C_J(r_i)$ and $F_i = C_L(r_i)$.*

Notice that by Lemma 5.6, F_i acts on I_i and $F_i/I_i\langle r_i\rangle \cong 2 \times \mathrm{Sym}(4)$. As explained above we intend to determine a large signalizer for I_i (a $3'$-group which is normalized by I_i). We begin with two easy observations.

Lemma 7.2. *For $i = 1, 2$, $C_{C_M(Z)}(r_i) = Q_{3-i}R_1R_2I_i$ and $C_S(r_i) = Q_{3-i}I_i \in \mathrm{Syl}_3(C_M(r_i)) \subseteq \mathrm{Syl}_3(K_i)$.*

Proof. Obviously $C_{C_M(Z)}(r_i) \geq Q_{3-i}R_1R_2C_J(r_i)$ and so Lemma 4.8 (ii) yields equality. Therefore, $C_S(r_i) = Q_{3-i}I_i \in \mathrm{Syl}_3(C_M(r_i))$ and $Z(C_S(r_i)) = Z$. Thus $N_{K_i}(C_S(r_i)) \leq N_G(Z) = M$. In particular, $C_S(r_i) \in \mathrm{Syl}_3(K_i)$. □

Lemma 7.3. *We have r_1 is G-conjugate to r_2 if and only if r_1 is M-conjugate to r_2.*

Proof. Obviously if r_1 and r_2 are conjugate in M then they are conjugate in G. Suppose then that $r_1 = r_2^g$ for some $g \in G$. By Lemma 7.2, for $i = 1, 2$, $C_S(r_i) \in \mathrm{Syl}_3(C_G(r_i))$ and $Z = Z(C_S(r_i))$. Since $r_1 = r_2^g$, $C_S(r_2)^g \in \mathrm{Syl}_3(C_G(r_1))$. Thus there is $h \in C_G(r_1)$ such that $C_S(r_2)^{gh} = C_S(r_1)$. But then

$$Z^{gh} = Z(C_S(r_2))^{gh} = Z(C_S(r_1)) = Z$$

which means that $gh \in M$. Hence r_1 and r_2 are M-conjugate. □

Recall, for $i = 1, 2$,

$$I_i = C_J(r_i) = J \cap E(C_G(\rho_i))$$

as, by Lemma 6.3, $E(C_G(\rho_i)) = C_{C_G(\rho_i)}(r_i)$.

Lemma 7.4. *Suppose that $\widetilde{\rho}_1 \in \mathcal{P}(I_1)$ and $\widetilde{\rho}_2 \in \mathcal{M}(I_2)$. Then, for $i = 1, 2$, in $E(C_G(\widetilde{\rho}_i))\langle r_i\rangle$, r_i is an involution which has $\mathrm{Sp}_4(2)'$ as* *a composition factor of its centralizer. Moreover,* $I_i \cap E(C_G(\widetilde{\rho}_i))$ *is of Type N-.*

Proof. For $i = 1, 2$, the definition of I_i, yields $r_i \in C_G(\widetilde{\rho}_i)$. Now r_i normalizes $E(C_G(\widetilde{\rho}_i))$ and centralizes $I_i \cap E(C_G(\widetilde{\rho}_i))$ which has order 9.

On the other hand, in $C_G(\rho_i)$, as there are only three conjugates of $\langle\widetilde{\rho}_i\rangle$ in I_i by Lemma 5.7(v) and (vi), we have that

$$C_{E(C_G(\rho_i))}(\widetilde{\rho}_i) \approx 3 \times 3^2.\mathrm{Dih}(8)$$

if $E(C_G(\rho_i)) \cong \mathrm{SU}_4(2)$ and

$$C_{E(C_G(\rho_i))}(\widetilde{\rho}_i) \approx 3 \times \mathrm{Sp}_4(2)$$

if $E(C_G(\rho_i)) \cong \mathrm{Sp}_6(2)$. As $I_i \le E(C_G(\rho_i))$, it follows that

$$I_i \cap [I_i, C_{E(C_G(\rho_i))}(\widetilde{\rho}_i)]$$

is of Type N-. Now deploying Lemmas 2.2 and 2.5 (ii), $C_{E(C_G(\widetilde{\rho}_i))}(r_i) \cong \mathrm{Sp}_4(2)$ if $E(C_G(\widetilde{\rho}_i)) \cong \mathrm{SU}_4(2)$ and has shape $2^5.\mathrm{Sp}_4(2)$ when $E(C_G(\widetilde{\rho}_i)) \cong \mathrm{Sp}_6(2)$. In particular, the main claim in the lemma is true. We have already observed that $I_i \cap [I_i, C_{E(C_G(\rho_i))}(\widetilde{\rho}_i)]$ has Type N- and as this group is $I_i \cap E(C_G(\widetilde{\rho}_i))$ we have the last part of the lemma. □

We can now locate the four maximal subgroups of I_i, whose centralizers contain the quaternion groups we are looking for. Recall that, for $i = 1, 2$, $A_{3-i} = A \cap Q_{3-i}$ is a hyperplane of I_i which with respect to the quadratic form on J is a degenerate 2-dimensional subspace which contains one conjugate of Z and three conjugates of $\langle\rho_i\rangle$. Therefore A_1 has Type D+ and has A_2 Type D- in the sense of Notation 2.15. Consequently the set $A_{3-i}^{F_i}$ has order 4. We let the four F_i-conjugates of A_{3-i} be $I_i^1 = A_{3-i}$, I_i^2, I_i^3 and I_i^4. Then, for $1 \le j < k \le 4$, we have $I_i^j \cap I_i^k$ is an M-conjugate of $\langle\rho_{3-i}\rangle$. We further select notation so that

$$I_i^1 \cap I_i^2 = \langle\rho_{3-i}\rangle.$$

The next lemma follows immediately from the 2-transitive action of F_i on the set $\{I_i^1, I_i^2, I_i^3, I_i^4\}$.

Lemma 7.5. *For* $1 \le l \le 4$ *and* $1 \le j < k \le 4$ *we have*

(i) I_1^l *has Type D- and* $I_1^j \cap I_1^k \in \mathcal{M}(I_1)$*; and*

(ii) I_2^l *has Type D+ and* $I_2^j \cap I_2^k \in \mathcal{P}(I_2)$*.*

□

With these comments we have the following lemma directly from Lemmas 6.3 and 6.4.

Lemma 7.6. *For $i = 1, 2$ and for $1 \le j < k \le 4$, we have*

$$C_G(I_i^k \cap I_i^j) \cong 3 \times \mathrm{Aut}(\mathrm{SU}_4(2)) \text{ or } 3 \times \mathrm{Sp}_6(2).$$

Furthermore, the isomorphism type of $C_G(I_i^k \cap I_i^j)$ does not depend on i, j or k.

Recall the Type N+ subgroups of order 9 are just the non-degenerate subgroups of J of plus type.

Lemma 7.7. *$I_1 \cap I_2$ is of Type N+.*

Proof. We know that $I_1 \cap I_2 = C_J(\langle r_1, r_2\rangle)$ and is consequently non-degenerate. Since $Z \le I_1 \cap I_2$, it has Type N+. □

The next lemma is an adaptation of Lemma 5.3(ii) to K_i.

Lemma 7.8. *$F_i = N_{K_i}(I_i)$ controls K_i-fusion of elements in I_i.*

Proof. By Lemma 7.2, $C_S(r_i) \in \mathrm{Syl}_3(K_i)$ and thus I_i is the Thompson subgroup of $C_S(r_i)$ and is elementary abelian. It follows from [1, 37.6] that $N_{K_i}(I_i)$ controls fusion in I_i. As $C_G(I_i) \le M$, we calculate that $C_G(I_i) = J\langle r_i\rangle$. Hence $C_{K_i}(I_i) = I_i\langle r_i\rangle$ and $N_{K_i}(I_i) = L \cap K_i = F_i$. □

For $i \in \{1, 2\}$ and $1 \le j < k \le 4$,

$$E_i^{j,k} = E(C_G(I_i^j \cap I_i^k)).$$

So $E_i^{j,k} \cong \mathrm{SU}_4(2)$ or $\mathrm{Sp}_6(2)$ and we note again that the isomorphism type of this group does not depend on i, j or k. At least one potential avenue for confusion is caused by this notation so please note that $E_i^{j,k}$ does not centralize r_i. Rather it centralizes a conjugate of r_{3-i}. Indeed $E_1^{1,2} = E(C_G(\rho_2))$ centralizes r_2 and $E_2^{1,2} = E(C_G(\rho_1))$ centralizes r_1 by Lemma 6.3.

Notice that I_i is centralized by r_i and so r_i is contained in $C_G(I_i^j \cap I_i^k)$ and it centralizes $I_i \cap E_i^{j,k}$ and this contains Z. It follows that $I_i \cap E_i^{j,k}$ is of Type N+ as it must also be non-degenerate. This means that r_i acts as an involution of type a_2 on $E_i^{j,k}$ in the sense of Table 1. Therefore, Lemma 2.2(ii) gives the following result:

Lemma 7.9. *We have*

$$\begin{aligned} C_{K_i}(I_i^j \cap I_i^k) &= C_{C_G(I_i^j \cap I_i^k)}(r_i) \\ &\approx \begin{cases} 3 \times 2_+^{1+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)) & E_i^{j,k} \cong \mathrm{SU}_4(2) \\ 3 \times 2^{1+2+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)) & E_i^{j,k} \cong \mathrm{Sp}_6(2) \end{cases}. \end{aligned}$$

□

The next lemma now is the key. It shows that the groups $O_2(C_{K_i}(I_j^i))$ are quaternion groups of order eight which pairwise commute and so generate an extraspecial group of order 2^9.

Lemma 7.10. *Assume that $i = 1, 2$ and $1 \leq j < k \leq 4$.*

(i) *For $m \in \{j, k\}$, $I_i^m \cap E_i^{j,k}$ is a 3-central element of G and of $E_i^{j,k}$;*

(ii) $C_G(I_i^k) = (I_i^k \cap I_i^j) \times C_{E_i^{j,k}}(I_k \cap E_i^{j,k}) \approx 3 \times 3_+^{1+2}.\mathrm{SL}_2(3)$*;*

(iii) (a) $O_2(C_{K_i}(I_i^j)) \cong O_2(C_{K_i}(I_i^k)) \cong \mathrm{Q}_8$*;*

(b) $O_2(C_{K_i}(I_i^j))O_2(C_{K_i}(I_i^k)) \leq O_2(C_{K_i}(I_i^j \cap I_i^k))$ *with equality if $E_i^{j,k} \cong \mathrm{SU}_4(2)$; and*

(c) $[O_2(C_{K_i}(I_i^j)), O_2(C_{K_i}(I_i^k))] = 1$*; and*

(iv) $C_{I_i}(O_2(C_{K_i}(I_i^j))O_2(C_{K_i}(I_i^k))) = I_i^j \cap I_i^k$.

Proof. It suffices to prove part (i) for I_i^1 as then the result will follow by conjugating by F_i

So consider $I_i^1 \cap I_i^2 = \langle \rho_{3-i} \rangle$. Then, by Lemma 6.2, $C_S(\rho_{3-i}) = Q_i J$ and $C_S(\rho_{3-i})' \cap Z(C_S(\rho_{3-i})) = Z$. Thus $Z \leq I_i^1 \cap E_i^{1,j}$ is 3-central in G and in $E_i^{1,j}$. Part (i) follows as F_i acts 2-transitively on $\{I_i^j \mid 1 \leq j \leq 4\}$.

Part (ii) follows from (i) as the centralizer of a 3-central element in $\mathrm{Sp}_6(2)$ and $\mathrm{SU}_4(2)$ has shape $3_+^{1+2}.\mathrm{SL}_2(3)$.

To deduce part (iii), we first note that

$$O_2(C_{K_i}(I_i^k)) \cong O_2(C_{K_i}(I_i^j)) \cong \mathrm{Q}_8$$

follows from (ii) as r_i is an involution in $C_G(I_i^k)$. We have $l \in \{j, k\}$, $O_2(C_{K_i}(I_i^l)) \leq C_{K_i}(I_i^j \cap I_i^k)$ and is normalized by $I_i^j I_i^k = I_i$. Since

$$\begin{aligned} C_{K_i}(I_i^j \cap I_i^k) &= C_{C_G(I_i^j \cap I_i^k)}(r_i) \\ &\approx \begin{cases} 3 \times 2_+^{1+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)) & E_i^{j,k} \cong \mathrm{SU}_4(2) \\ 3 \times 2^{1+2+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)) & E_i^{j,k} \cong \mathrm{Sp}_6(2) \end{cases} \end{aligned}$$

by Lemma 7.9, it follows that $O_2(C_{K_i}(I_i^l)) \leq O_2(C_{K_i}(I_i^j \cap I_i^k))$. Now we apply Lemma 2.5(iii) to see that $[O_2(C_{K_i}(I_i^k)), O_2(C_{K_i}(I_i^k))] = 1$. (Recall that $O_2(C_{\mathrm{SU}_4(2)}(r_i)) \leq O_2(C_{\mathrm{Sp}_6(2)}(r_i))$.)

Part (iv) follows as $I_i \cap E_i^{j,k}$ acts faithfully on $O_2(C_{K_i}(I_i^j))O_2(C_{K_i}(I_i^k))$. □

We now introduce some further notation

Notation 7.11. *For $i = 1, 2$, $1 \leq k \leq 4$,*

$$\Sigma_i^k = O_2(C_{K_i}(I_i^k)) \cong \mathrm{Q}_8$$

and

$$\Sigma_i = \langle \Sigma_i^k \mid 1 \le k \le 4\rangle = \langle O_2(C_{K_i}(I_i^k)) \mid 1 \le k \le 4\rangle.$$

Note that $\Sigma_1^1 = O_2(C_{K_1}(A_2)) = R_1$ and $\Sigma_2^1 = O_2(C_{K_2}(A_1)) = R_2$.

Lemma 7.12. *We have Σ_i is extraspecial of order 2^9 and plus type, $Z(\Sigma_i) = \langle r_i\rangle$ and $F_i/\langle r_i\rangle$ acts faithfully on Σ_i.*

Proof. The structure of Σ_i follows from Lemma 7.10 (iii) as the generating subgroups commute pairwise. To see the last part is suffices to show that I_i acts faithfully on Σ_i as every normal subgroup of F_i which strictly contains $\langle r_i\rangle$ contains I_i. Using Lemma 7.10 (iv) we see that $C_{I_i}(\Sigma_i) = \bigcap_{j=1}^4 I_i^j = 1$. □

At this stage we have constructed the extraspecial group of order 2^9 on which F_i acts.

Lemma 7.13. *The following hold:*

(i) $C_{\Sigma_1}(Z) = R_1$, $C_{\Sigma_1}(I_1^j \cap I_1^k) = \Sigma_1^j\Sigma_1^k$ *and, if* $\langle x\rangle \in \mathcal{P}(I_1)$, *then* $C_{\Sigma_1}(x) = \langle r_1\rangle$.

(ii) $C_{\Sigma_2}(Z) = R_2$, $C_{\Sigma_2}(I_2^j \cap I_2^k) = \Sigma_2^j\Sigma_2^k$ *and, if* $\langle x\rangle \in \mathcal{M}(I_2)$, *then* $C_{\Sigma_2}(x) = \langle r_2\rangle$.

Proof. We prove (i) the proof of (ii) being the same. Let $1 \le j \le 4$. We know that $\Sigma_1 = \Sigma_1^1\Sigma_1^2\Sigma_1^3\Sigma_1^4$. Since I_1 acts faithfully on Σ_1, we have that $C_{I_1}(\Sigma_1^j) = I_1^j$. Thus the elements of $\mathcal{P}(I_1)$ act non-trivially on each Σ_1^j and so $C_{\Sigma_1}(x) = \langle r_1\rangle$ for $\langle x\rangle \in \mathcal{P}(I_1)$. Since we know that Z centralizes exactly $R_1 = \Sigma_1^1$ on Σ_1 we now have that (i) holds. □

8. The structure of $C_G(\rho_1)$

We continue to use our standard notation. In this section we are going to show that $C_G(\rho_1)$ is isomorphic to the corresponding centralizer in $\mathrm{F}_4(2)$. So our aim is to show that $C_G(\rho_1) \cong 3 \times \mathrm{Sp}_6(2)$. By Lemma 6.3 we have that $C_G(\rho_1)$ either is as in $\mathrm{F}_4(2)$ or is isomorphic to $3 \times \mathrm{Aut}(\mathrm{SU}_4(2))$. We will show the latter case yields a contradiction.

Lemma 8.1. *Suppose that $C_G(\rho_i) \cong 3 \times \mathrm{Aut}(\mathrm{SU}_4(2))$. Then Σ_i is the unique maximal signalizer for I_i^1 in K_i.*

Proof. We simplify our notation by assuming that $i = 1$. The argument for $i = 2$ is the same. Notice that

$$\{I_1^1 \cap I_1^j \mid 2 \le j \le 4\} = \mathcal{M}(I_i^1).$$

The only other proper subgroup of I_1^1 is Z by Lemma 7.5. Hence, as $E_1^{1,j} \cong \mathrm{SU}_4(2)$ by assumption, Lemma 7.10 (iii)(b) implies that

$$\Sigma_1 \ge O_2(C_{K_1}(I_1^k \cap I_1^j)) = O_{3'}(C_{K_1}(I_1^k \cap I_1^j)).$$

Suppose that Θ is a signalizer for I_1^1. Then

$$\Theta = \langle C_\Theta(a) \mid a \in I_1^{1\#} \rangle.$$

However,

$$C_\Theta(Z) \le O_{3'}(M \cap K_1) = R_1 \le \Sigma_1$$

and, for $1 < j \le 4$, by Lemma 7.9,

$$C_\Theta(I_1^1 \cap I_1^j) \le O_{3'}(C_{K_i}(I_1^1 \cap I_1^j)) = \Sigma_1\Sigma_j \le \Sigma_1.$$

Hence $\Theta \le \Sigma_1$. □

The next lemma puts us firmly on the track of $\mathrm{F}_4(2)$ and $\mathrm{Aut}(\mathrm{F}_4(2))$.

Lemma 8.2. *We have $C_G(\rho_1) \cong C_G(\rho_2) \cong 3 \times \mathrm{Sp}_6(2)$.*

Proof. Suppose that the lemma is false. Then by Lemmas 6.3 and 6.4

$$C_G(\rho_1) \cong C_G(\rho_2) \cong 3 \times \mathrm{Aut}(\mathrm{SU}_4(2)).$$

We claim that, for $i = 1, 2$, Σ_i is self-centralizing in K_i. Let $W_i = C_G(\Sigma_i)$. Then $W_i \le K_i$ and, as $C_S(r_i) \in \mathrm{Syl}_3(K_i)$ by Lemma 7.2 and since this group acts faithfully on Σ_i by Lemma 7.12, we have that W_i is a $3'$-group which is normalized by I_i^1. By Lemma 8.1, Σ_i is the unique maximal signalizer for I_i^1 and hence $\Sigma_i \ge W_i$.

Since Σ_i is the unique maximal signalizer for I_i^1 in K_i it is also the unique maximal signalizer of $Q_{3-i} \ge I_i^1$ and $I_i \ge I_i^1$ in K_i. It follows that $N_G(\Sigma_i) \ge \langle F_i, C_M(r_i)\rangle$ as Q_{3-i} is a normal subgroup of $C_M(r_i)$. Now

$$C_M(r_i)\Sigma_i/\Sigma_i = I_iQ_{3-i}R_{3-i}\langle f\rangle\Sigma_i/\Sigma_i$$

as $R_i \le \Sigma_i$. We now deduce $C_{C_M(Z)}(r_i)\Sigma_i/\Sigma_i$ is isomorphic to a 3-centralizer in $\mathrm{PSp}_4(3)$. Furthermore, as Σ_i is the unique maximal signalizer for I_i in K_i, we have that I_i does not normalize any non-trivial $3'$-subgroup of $N_G(\Sigma_i)/\Sigma_i$ and f inverts Z. Therefore, since $F_i \le N_G(\Sigma_i)$, Prince's Theorem 2.9 yields

$$N_G(\Sigma_i)/\Sigma_i \cong \mathrm{Aut}(\mathrm{SU}_4(2)) \text{ or } \mathrm{Sp}_6(2).$$

Observe that $N_G(\Sigma_i) \ge \langle F_i, C_M(r_i)\rangle \ge E(C_G(\rho_i))$.

We claim $N_G(\Sigma_i) = K_i$. To prove this we intend to apply Theorem 2.17 to $K_i/\langle r_i\rangle$. We have already verified hypotheses (i) and (ii) of that theorem.

As $N_G(\Sigma_i)/\Sigma_i \cong \mathrm{Aut}(\mathrm{SU}_4(2))$ or $\mathrm{Sp}_6(2)$, every element of $C_S(r_i)\Sigma_i/\Sigma_i$ is $N_G(\Sigma_i)/\Sigma_i$-conjugate to an element of $I_i\Sigma_i/\Sigma_i = J(C_S(r_i))\Sigma_i/\Sigma_i$ the Thompson subgroup of $C_S(r_i)\Sigma_i/\Sigma_i$. Since F_i controls fusion in I_i by Lemma 7.8, we also have hypothesis (iii) of Theorem 2.17.

Again to simplify notation, assume that $i = 1$. Suppose that d is an element of order 3 with $d \in N_G(\Sigma_1) \cap N_G(\Sigma_1)^h$ for some $h \in K_1$ such that $C_{\Sigma_1}(d) \neq \langle r_1 \rangle$. Then, by Lemma 7.13 (i), we may suppose that $\langle d \rangle = Z$ or $\langle d \rangle = I_1^1 \cap I_1^2 = \langle \rho_2 \rangle$. Then, as $N_{K_1}(Z) = C_M(r_1) \leq N_G(\Sigma_1)$ and $C_{K_1}(\rho_2) = C_{C_G(\rho_2)}(r_1) \leq N_G(\Sigma_1)$, we deduce

$$C_{K_1}(d) \leq N_G(\Sigma_1).$$

On the other hand, $C_{N_G(\Sigma_1)^h}(d)$ contains a K_1-conjugate X of I_1. Since $X \leq C_{K_1}(d) \leq N_G(\Sigma_1)$, we may suppose that $N_G(\Sigma_1) \cap N_G(\Sigma_1)^h \geq I_1$. But then $\Sigma_1 = \Sigma_1^h$ and $N_G(\Sigma_1) = N_G(\Sigma_1)^h$ as Σ_1 is the unique maximal signalizer for I_1 in K_1 by Lemma 8.1. Thus the hypothesis of Theorem 2.17 fulfilled and therefore $K_1 = N_G(\Sigma_1)$.

Suppose that $N_G(\Sigma_1)/\Sigma_1 \cong \mathrm{Aut}(\mathrm{SU}_4(2))$. Let $\widetilde{\rho_1} \in \mathcal{P}(I_1)$. Then, as $|\mathcal{P}(I_1)| = 3$,

$$C_{N_G(\Sigma_1)/\Sigma_1}(\widetilde{\rho_1}\Sigma_1) \cong 3^3.\mathrm{Dih}(8)$$

by Lemma 5.7 (v). On the other hand, by Lemma 7.4 this group is non-soluble which is a contradiction. We conclude that $N_G(\Sigma_1)/\Sigma_1 \cong \mathrm{Sp}_6(2)$. Repeating the arguments for $N_G(\Sigma_2)$ yields $N_G(\Sigma_2)/\Sigma_2 \cong \mathrm{Sp}_6(2)$. Furthermore, the elements from $\mathcal{P}(I_1)$ act fixed point freely on $\Sigma_1/\langle r_1 \rangle$ and the elements of $\mathcal{M}(I_2)$ act fixed point freely on $\Sigma_2/\langle r_2 \rangle$. In both cases, $i = 1, 2$, $\Sigma_i/\langle r_i \rangle$ is the spin module for $N_G(\Sigma_i)/\Sigma_i$.

Since r_2 commutes with $I_1 \cap I_2 \leq N_G(\Sigma_1)$ which has Type N+ by Lemma 7.7, Table 1 indicates that r_2 acts as a unitary transvection on $\Sigma_1/\langle r_1 \rangle$. Therefore $|C_{\Sigma_1/\langle r_1 \rangle}(r_2)| = 2^6$ and

$$2^6 \leq |C_{\Sigma_1}(r_2)| \leq 2^7.$$

Since $\langle r_1, r_2 \rangle$ is centralized by $I_1 \cap I_2$, $C_{\Sigma_1}(r_2)$ is $(I_1 \cap I_2)$-invariant. Because the elements of $\mathcal{P}(I_1 \cap I_2)$ act fixed point freely on $\Sigma_1/\langle r_1 \rangle$ (see Lemma 2.4) we infer that $|C_{\Sigma_1}(r_2)| = 2^7$. Now, as $K_i = N_G(\Sigma_i)$ for $i = 1, 2$, $C_{\Sigma_1}(r_2)$ normalizes $C_{\Sigma_2}(r_1)$ and vice versa, and so

$$[C_{\Sigma_1}(r_2), C_{\Sigma_2}(r_1)] \leq \Sigma_1 \cap \Sigma_2.$$

Since $r_1 \notin \Sigma_2$ and $r_2 \notin \Sigma_1$, $\Sigma_1 \cap \Sigma_2$ is abelian and is centralized by $C_{\Sigma_1}(r_2)C_{\Sigma_2}(r_1)$. In particular, $\Sigma_1 \cap \Sigma_2 \leq Z(C_{\Sigma_1}(r_2))$. Thus, as $|C_{\Sigma_1}(r_2)| = 2^7$ and Σ_1 is extraspecial it follows that $\Sigma_1 \cap \Sigma_2$ has order at most 2^2 as $r_1 \notin \Sigma_2$. We have that $I_1 \cap I_2$ acts on $\Sigma_1 \cap \Sigma_2$. Since $|I_1 \cap I_2| = 3^2$, there is $w \in C_{I_1 \cap I_2}(\Sigma_1 \cap \Sigma_2)^\#$. Now $(\Sigma_1 \cap \Sigma_2)\langle r_1 \rangle$ is elementary abelian. Since, for $a \in \mathcal{S}(I_1 \cap I_2)$, we have $C_{\Sigma_1}(a) \cong \mathrm{Q}_8$ and, for $a \in \mathcal{P}(I_1 \cap I_2)$, we have $C_{\Sigma_1}(a) = \langle r_1 \rangle$, we must have $\langle w \rangle \in \mathcal{M}(I_1 \cap I_2)$. But then $\Sigma_1 \cap \Sigma_2 \leq C_{\Sigma_2}(w) = 1$ by Lemma 7.13. This means that $\Sigma_1 \cap \Sigma_2 = 1$ which then forces $[C_{\Sigma_1}(r_2), C_{\Sigma_2}(r_1)] = 1$ and Lemma 2.2 (iv) provides a contradiction. $\square$

9. Some subgroups in the centralizer of the involutions r_1 and r_2

In this section, we finally construct $O_2(K_i)$ where $K_i = C_G(r_i)$. Recall from Definition 3.1, we expect $O_2(K_i)$ to be a product of an elementary abelian group of order 2^7 by an extraspecial group of order 2^9. We have already located the extraspecial group Σ_i. In this section we uncover the elementary abelian group. We consider the situation for K_1. In the previous section we proved that $C_G(\rho_2) \cong 3 \times \mathrm{Sp}_6(2)$. With this additional information we study $C_{K_1}(\rho_2)$. This group has shape $3 \times 2^{1+2+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3))$. For us it is important that $Z(O_2(C_{K_1}(\rho_2)))$ is elementary abelian of order 8. Furthermore $I_1 = C_J(r_1)$ normalizes this group. This time there are six conjugates of this group under the action $C_L(r_1)$ and we define a group Υ_1 generated by these six conjugates. We show that Υ_1 is elementary abelian of order 2^7 and centralizes Σ_1, the extraspecial group found earlier. Hence the product of both gives a 2-group Γ_1 of order 2^{15}, which is in fact isomorphic to the corresponding group in $\mathrm{F}_4(2)$. Furthermore we show that $N_G(\Gamma_1)/\Gamma_1 \cong \mathrm{Sp}_6(2)$ and so $N_G(\Gamma_1)$ is similar to a 2-centralizer in $\mathrm{F}_4(2)$. In the next section show $K_1 = N_G(\Gamma_1)$.

We use our, by now, standard notation. In particular recall the definition of Σ_i from 7.11 and I_i^j the conjugates of A_{3-i} under $F_i = C_L(r_i)$. Our first goal is to construct a signalizer for I_i^1, $i = 1, 2$, which contains Σ_i properly. So, for $1 \le j < k \le 4$, we define

$$\Theta_i^{j,k} = Z(O_2(C_{K_i}(I_i^j \cap I_i^k)))$$

and put

$$\Upsilon_i = \langle \Theta_i^{j,k} \mid 1 \le j < k \le 4 \rangle.$$

We will shortly show that Υ_i is elementary abelian of order 2^7.

As $C_G(I_i^j \cap I_i^k) \cong 3 \times \mathrm{Sp}_6(2)$, Lemma 7.9 yields

$$C_{K_i}(I_i^j \cap I_i^k) \approx 2^{1+2+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)).$$

Hence, by Lemmas 2.5 (iii) and (iv) and 7.10(iii), $\Theta_i^{j,k}$ is elementary abelian of order 2^3 and

$$O_2(C_{K_i}(I_i^j \cap I_i^k)) = \Sigma_i^j \Sigma_i^k \Theta_i^{j,k}.$$

We record this latter equality.

Lemma 9.1. *For $i = 1, 2$ and $1 \le j < k \le 4$, $O_2(C_{K_i}(I_i^j \cap I_i^k)) = \Sigma_i^j \Sigma_i^k \Theta_i^{j,k}$.* □

Lemma 9.2. *Suppose that $i = 1, 2$ and $\{j, k, l, m\} = \{1, 2, 3, 4\}$. Then*

(i) $\Theta_i^{j,k}$ *is elementary abelian of order* 2^3*, contains* r_i *and a* G*-conjugate* s_{3-i} *of* r_{3-i} *with* $s_{3-i} \neq r_i$*.*
(ii) $\Theta_i^{j,k} = \Theta_i^{l,m}$.
(iii) Υ_i *centralizes* Σ_i*.*
(iv) $\Theta_i^{j,k}\Theta_i^{k,l}$ *is elementary abelian of order* 2^5*.*
(v) Υ_i *is elementary abelian of order* 2^7 *and is normalized by* I_i*.*

Proof. To reduce the notational complexity of our argument we present the proof for $i = 1$ the proof when $i = 2$ is the same but we have to be careful when following the members of $\mathcal{M}(J)$ and $\mathcal{P}(J)$ in the arguments.

By definition
$$\Theta_1^{j,k} = Z(O_2(C_{K_1}(I_1^j \cap I_1^k))).$$
We know $I_1^j \cap I_1^k \in \mathcal{M}(J)$ from Lemma 7.5 and we know $C_{K_1}(I_1^j \cap I_1^k) \cap E_1^{j,k}$ is a line stabiliser in the natural symplectic representation of $E_1^{j,k} \cong \mathrm{Sp}_6(2)$. Thus $\Theta_1^{j,k}$ is elementary abelian of order 2^3 by Lemma 2.5 and of course $\Theta_1^{j,k}$ contains r_1 and, by Lemma 7.4, r_2 is a 2-central involution in E_1^{jk} and so $\Theta_1^{j,k}$ also contains a conjugate of r_2. This proves (i).

Now $J \cap E_1^{j,k}$ centralizes a conjugate of r_2 and is thus G-conjugate to I_2. It follows from Lemma 5.7 that $|\mathcal{S}(J \cap E_1^{j,k})| = 4$, $|\mathcal{P}(J \cap E_1^{j,k})| = 6$ and $|\mathcal{M}(J \cap E_1^{j,k})| = 3$. Now using Lemma 2.5 (iv), we have
$$X_1^{j,k} = C_{I_1 \cap E_1^{j,k}}(\Theta_1^{j,k}) \in \mathcal{M}(I_1 \cap E_1^{j,k}).$$
Observe $X_1^{j,k} \leq I_1$ and so $X_1^{j,k}$ normalizes Σ_1.

Since $X_1^{j,k} \in \mathcal{M}(I_1)$, $C_{\Sigma_1}(X_1^{j,k})$ has order 2^5 by Lemma 7.13. As $[\Sigma_1^j\Sigma_1^k, X_1^{j,k}] = \Sigma_1^j\Sigma_1^k$ and Σ_1 is extraspecial, we deduce
$$C_{\Sigma_1}(X_1^{j,k}) = \Sigma_1^l\Sigma_1^m = C_{\Sigma_1}(\Sigma_1^j\Sigma_1^k).$$
In particular, we now have $X_1^{j,k} = I_1^l \cap I_1^m$ by Lemma 7.13. This implies $\Theta_1^{j,k} \leq C_G(I_1^l \cap I_1^m)$ and $\Theta_1^{j,k}$ is normalized by I_1; therefore
$$\langle \Theta_1^{j,k}, \Sigma_1^l\Sigma_1^m \rangle = O_2(C_{K_1}(I_1^l \cap I_1^m)).$$
Since $\Theta_1^{j,k}$ is I_1-invariant and elementary abelian, we infer $\Theta_1^{j,k} = \Theta_1^{l,m}$ and that $\Theta_1^{j,k}$ commutes with $\Sigma_1^j\Sigma_1^k$ as well as with $\Sigma_1^l\Sigma_1^m$. Since $\Sigma_1 = \Sigma_1^j\Sigma_1^k\Sigma_1^l\Sigma_1^m$, we have now proved claims (ii) and (iii).

Because $\Theta_1^{j,k} = \Theta_1^{l,m}$ we have that $\Theta_1^{j,k}$ is centralized by $\langle X_1^{j,k}, X_1^{l,m} \rangle = \langle I_1^i \cap I_1^j, I_1^l \cap I_1^m \rangle$ which has Type N- as $\Theta_1^{j,k}$ does not commute with a conjugate of Z. Hence $\langle \Theta_1^{j,k}, \Theta_1^{k,l} \rangle$ is centralized by
$$Y = \langle X_1^{j,k}, X_1^{l,m} \rangle \cap \langle X_1^{k,l}, X_1^{j,m} \rangle \in \mathcal{P}(J).$$

Now $C_G(Y) \cong 3 \times \mathrm{Sp}_6(2)$ and $I_1 \cap E(C_G(Y))$ is of Type N- by Lemma 7.4. Since $\langle \Theta_1^{j,k}, \Theta_1^{k,l} \rangle$ centralizes r_1 and is normalized by I_1 we infer that r_1 is an involution of $E(C_G(Y))$ with centralizer of shape $2^5.\mathrm{Sp}_4(2)$ and that $\langle \Theta_1^{j,k}, \Theta_1^{k,l} \rangle \le O_2(C_{E(C_G(Y))}(r_1))$ which is elementary abelian. But then

$$\langle \Theta_1^{j,k}, \Theta_1^{k,l} \rangle = \Theta_1^{j,k}\Theta_1^{k,l}$$

is elementary abelian of order at most 2^5. It now follows that $\Upsilon_1 = \Theta_1^{1,2}\Theta_1^{2,3}\Theta_1^{2,4}$ has order at most 2^7 and is I_1-invariant. We have seen that $C_{I_1}(\Theta_1^{j,k}\Theta_1^{k,l})$ is $I_1^j \cap I_1^k$. Thus $C_{I_1}(\Upsilon_i) \le I_1^1 \cap I_1^2 \cap I_1^3 \cap I_1^4 = 1$. Hence I_1 acts faithfully on Υ_1 and so $|\Upsilon_1| = 2^7$. This completes the proof of (iv) and (v) and the verification of the statements in the lemma. □

For $i = 1, 2$, we now set

$$\Gamma_i = \Sigma_i \Upsilon_i.$$

Lemma 9.3. *For $i = 1, 2$, we have that Γ_i has order 2^{15} and is normalized by F_i. Furthermore the following hold.*

(i) $Z(\Gamma_i) = \Upsilon_i$; *and*
(ii) $[\Gamma_i, \Gamma_i] = \langle r_i \rangle$.

Proof. By Lemmas 7.12 and 9.2, Σ_i has order 2^9 and is extraspecial and $|\Upsilon_i| = 2^7$ and centralizes Σ_i. This yields $\Upsilon_i \cap \Sigma_i = \langle r_i \rangle$ and Γ_i has order 2^{15}. Furthermore, as Σ_i is extraspecial, Υ_i is elementary abelian and Υ_i commutes with Σ_i we have that $\Upsilon_i = Z(\Gamma_i)$ and $[\Gamma_i, \Gamma_i] = \langle r_i \rangle$. Hence points (i) and (ii) hold.

By the construction of Σ_i and Υ_i, F_i normalizes both groups and consequently also normalizes their product, completing the proof. □

Lemma 9.4. *For $i = 1, 2$, Γ_i is the unique maximal signalizer for I_i^1 in K_i.*

Proof. Assume that W is an I_i^1 signalizer in K_i. Then

$$W = \langle C_W(x) \mid x \in (I_i^1)^\# \rangle.$$

If $\langle x \rangle = Z \in \mathcal{S}(I_i^1)$, then

$$O_{3'}(C_{K_i}(Z)) = R_i = \Sigma_i^1 \le \Sigma_i \le \Gamma_i$$

is the unique maximal I_i^1 signalizer in $C_{K_i}(Z)$. All the other subgroups of order 3 in I_i^1 are conjugate to $\langle \rho_{3-i} \rangle$ by an element of $Q_{3-i} \le F_i$. Hence we only need to consider I_i^1 signalizers in $C_{K_i}(\rho_{3-i})$.

By Lemma 8.2, $C_G(\rho_{3-i}) = C_G(I_i^1 \cap I_i^2) \cong 3 \times \mathrm{Sp}_6(2)$ and we know from Lemma 7.9 that

$$C_{K_i}(\rho_{3-i}) \approx 3 \times 2^{1+2+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)).$$

Set $D = C_{K_i}(\rho_{3-i})$. Then

$$O_2(D) = \Sigma_i^1\Sigma_i^2\Theta_i^{1,2} \le \Gamma_i$$

and, Lemma 2.5(ii), implies $ZO_2(D)/O_2(D)$ is diagonal in $D/O_2(D)$. Since $C_W(\rho_{3-i})$ is normalized by Z we infer that $C_W(\rho_{3-i}) \le \Gamma_i$ as claimed. □

Lemma 9.5. *For $i = 1, 2$, there is a G-conjugate of r_i in $\Gamma_i \setminus \Upsilon_i$.*

Proof. This fusion can already be seen in

$$C_{K_i}(\rho_{3-i}) \approx 3 \times 2^{1+2+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3))$$

as r_i is not weakly closed in $O_2(C_{K_i}(\rho_{3-i}))$ with respect to $C_G(\rho_{3-i})$ by Lemma 2.5 (vi). □

We are now able to determine the structure of $N_G(\Gamma_i)$.

Lemma 9.6. *For $i = 1, 2$, the following hold.*

(i) $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Sp}_6(2)$;
(ii) *as $N_G(\Gamma_i)/\Gamma_i$-modules, Γ_i/Υ_i is a spin module and $\Upsilon_i/\langle r_i\rangle$ is a natural module;*
(iii) $\mathrm{Syl}_2(N_G(\Gamma_i)) \subseteq \mathrm{Syl}_2(K_i)$; *and*
(iv) *if $T \in \mathrm{Syl}_2(N_G(\Gamma_i))$, then $\Gamma_i/\langle r_i\rangle = J(T/\langle r_i\rangle)$, $Z(T) \le \Upsilon_i$ and $Z(T)$ has order 4.*

In particular, $N_G(\Gamma_i)$ is similar to a 2-centralizer in $\mathrm{F}_4(2)$.

Proof. We already know that Γ_i is normalized by F_i and we have that Γ_i is the unique maximal I_i^1-signalizer in K_i by Lemma 9.4. It follows that Γ_i is also the unique maximal signalizer for $Q_{3-i} \ge I_i^1$ in K_i. Therefore $N_{E(C_G(\rho_i))}(Q_{3-i})$ also normalizes Γ_i. It follows from [4, page 46] that

$$X = \langle F_i, N_{E(C_G(\rho_i))}(Q_{3-i})\rangle \cong \mathrm{Aut}(\mathrm{SU}_4(2))$$

and X normalizes Γ_i.

Since $C_{K_i}(Z)\Gamma_i/\Gamma_i$ is a 3-centralizer of type $\mathrm{PSp}_4(3)$, Γ_i is a maximal signalizer for I_i^1 and Z is inverted in $N_G(\Gamma_i)/\Gamma_i$, we deduce $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Sp}_6(2)$ or $\mathrm{Aut}(\mathrm{SU}_4(2))$ by using Theorem 2.9.

We know that I_i acts faithfully on both Γ_i/Υ_i and $\Upsilon_i/\langle r_i\rangle$. In particular, as $|\Upsilon_i/\langle r_i\rangle| = 2^6$, if $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Aut}(\mathrm{SU}_4(2))$ then $\Upsilon_i/\langle r_i\rangle$ is an orthogonal module and if $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Sp}_6(2)$ then $\Upsilon_i/\langle r_i\rangle$ is a natural module. Similarly since $C_{\Sigma_i}(Z) = \Sigma_i^1$ and since this subgroup is not normalized by F_i and $|\Gamma_i/\Upsilon_i| = 2^8$, if $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Aut}(\mathrm{SU}_4(2))$, then Γ_i/Υ_i is an natural module and, if $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Sp}_6(2)$, then Γ_i/Υ_i is a spin module (see Lemma 2.1). So once we have proved part (i), part (ii) will also be proved.

Next we prove (iii) and the first part of (iv). Let $T \in \mathrm{Syl}_2(N_G(\Gamma_i))$. Since, by Lemma 2.7, $\Gamma_i/\langle r_i\rangle$ is not an F-module for $N_G(\Gamma_i)/\Gamma_i$, [8, Lemma 26.15] implies that $\Gamma_i/\langle r_i\rangle$ is the Thompson subgroup of $T/\langle r_i\rangle$. It follows that $N_{K_i}(T) \le N_G(\Gamma_i)$ and, in particular, $T \in \mathrm{Syl}_2(K_i)$ and $N_{K_i}(T) = T$. Notice furthermore that $N_G(\Gamma_i)/\langle r_i\rangle$ controls $K_i/\langle r_i\rangle$-fusion in $\Gamma_i/\langle r_i\rangle$. The last two parts of (iv) follow from the fact that Σ_i is extraspecial and Lemma 2.8.

It remains to prove (i). Assume that $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Aut}(\mathrm{SU}_4(2))$. Using Lemma 9.5, there exists $g \in G$ and $s \in \Gamma_i \setminus \Upsilon_i$ such that $s = r_i^g$. Since $N_G(\Gamma_i^g)$ contains a Sylow 2-subgroup of $C_G(s)$, there is a $h \in C_G(s)$ such that $C_{\Gamma_1}(s)^h \le N_G(\Gamma_i^g)$ and we have $s = r_i^{gh}$ so we may suppose g was chosen so $C_{\Gamma_1}(s) \le N_G(\Gamma_i^g)$. Note that, as $s \in \Gamma_i \setminus \Upsilon_i$, s is conjugate in Γ_i to sr_i and, as $N_G(\Gamma_i)/\langle r_i\rangle$ controls $K_i/\langle r_i\rangle$-fusion in $\Gamma_i/\langle r_i\rangle$, s is not K_i-conjugate to an element of Υ_i.

Since $C_{\Gamma_1}(s)$ contains an extraspecial group of order 2^7 with derived group $\langle r_i\rangle$, and $\mathrm{Aut}(\mathrm{SU}_4(2))$ does not (by Lemma 2.2), we have $r_i \in \Gamma_i^g$. It follows that $C_{\Gamma_i^g}(r_i)$, which has index at most 2 in Γ_i^g, also contains an extraspecial group of order 2^7. As $T \in \mathrm{Syl}_2(K_i)$, there is $f \in K_i$ such that $C_{\Gamma_i^g}(r_i)^f = C_{\Gamma_i^{gf}}(r_i) \le T$. It follows that $s^f \in \Gamma_i \setminus \Upsilon_i$ and we may as well suppose that $s = s^f$ (though we may no longer have $C_{\Gamma_1}(s) \le N_G(\Gamma_i^g)$). With this choice of s, $|\Gamma_i^g : \Gamma_i^g \cap N_G(\Gamma_i)| \le 2$. Now

$$\Phi(\Gamma_i^g \cap \Gamma_i) \le \Phi(\Gamma_i^g) \cap \Phi(\Gamma_i) = \langle s\rangle \cap \langle r_i\rangle = 1$$

which means $\Gamma_i^g \cap \Gamma_i$ is elementary abelian. As Γ_i contains Σ_i which is extraspecial of order 2^9, this yields $|\Gamma_i^g \cap \Gamma_i| \le 2^{11}$ and so

$$|(\Gamma_i^g \cap N_G(\Gamma_i))\Gamma_i/\Gamma_i| \ge 2^3.$$

Further

$$[\Upsilon_i \cap \Gamma_i^g, N_G(\Gamma_i) \cap \Gamma_i^g] \le [\Gamma_i^g, \Gamma_i^g] \cap \Upsilon_i = \langle s\rangle \cap \Upsilon_i = 1.$$

Hence, as $|(\Gamma_i^g \cap N_G(\Gamma_i))\Gamma_i/\Gamma_i| \ge 2^3$, Lemma 2.2(iii) (which says that $\mathrm{Aut}(\mathrm{SU}_4(2))$ contains no fours group of unitary transvections) implies $|\Upsilon_1 \cap \Gamma_i^g| \le 2^5$. Therefore $|\Gamma_i \cap \Gamma_i^g| \le 2^9$. We have now shown $|(\Gamma_i^g \cap N_G(\Gamma_i))\Gamma_i/\Gamma_i| \ge 2^5$ which, as this group is elementary abelian and the 2-rank of $\mathrm{Aut}(\mathrm{SU}_4(2))$ is 4, is a contradiction. Therefore $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Sp}_6(2)$ and this completes the proof of part (i) and thereby also (ii). □

10. The centralisers of r_1 and r_2

In this section we finally determine the structure of $K_i = C_G(r_i)$. We will prove $K_i = N_G(\Gamma_i)$ and hence conclude that K_i is similar to a 2-centralizer in $\mathrm{F}_4(2)$. The plan is to show Υ_i is strongly closed in a Sylow 2-subgroup of K_i with respect to K_i and then to quote Goldschmidt's Theorem in the form of Lemma 2.19 to show that $K_i = N_G(\Gamma_i)$. To achieve this we study K_i-fusion of involutions. As most of the centralizers of involutions in $N_G(\Gamma_i)$ have order divisible by three, this will be reduced to fusion of 3-elements. Hence the first lemma we prove in this section will be that $N_G(\Gamma_i)$ is strongly 3-embedded in K_i, which means that we have control of fusion of elements of order 3 in K_i.

We use all our previous notation and furthermore for this section we set $H_i = N_G(\Gamma_i)$.

Lemma 10.1. *For $i = 1, 2$, H_i is strongly 3-embedded in K_i. In particular, H_i controls fusion of elements of order 3 in H_i.*

Proof. Suppose that $x \in H_i$ has order 3. We will show $C_{K_i}(x)$ normalizes Γ_i. Recall $C_S(r_i) \in \mathrm{Syl}_3(K_i)$ and $C_S(r_i) \leq F_i \leq H_i$ so $C_S(r_i)$ normalizes Γ_i. Since every element of order 3 in $C_S(r_i)$ is H_i-conjugate into I_i, we may suppose $x \in I_i$.

Again to simplify our notation slightly we consider the case when $i = 1$. Thus $|\mathcal{S}(I_1)| = 4$, $|\mathcal{M}(I_1)| = 6$ and $|\mathcal{P}(I_1)| = 3$ by Lemma 5.6. If $\langle x\rangle \in \mathcal{S}(I_1)$, then we may suppose that $\langle x\rangle = Z$. In this case, by Lemma 7.2

$$C_{K_1}(Z) = Q_2R_1R_2I_1 \leq H_1.$$

So suppose that $\langle x\rangle = \langle \rho_2\rangle \in \mathcal{M}(I_1)$. Then, by Lemma 9.1,

$$C_{K_1}(\rho_2) = \Sigma_1^1\Sigma_1^2\Theta_1^{1,2}N_{F_1}(I_1 \cap E_1^{12}) \leq \Gamma_1F_1 \leq H_1.$$

Suppose $\langle x\rangle = \widetilde{\rho_1} \in \mathcal{P}(I_1)$. Then, by Lemma 7.4, $C_{K_1}(\widetilde{\rho_1}) \approx 3 \times 2^5{:}\mathrm{Sp}_4(2)$ and this has the same order as $C_{H_1}(\widetilde{\rho_1})$. Thus $C_{K_1}(\widetilde{\rho_1}) \leq H_1$. Finally, $N_{K_1}(C_S(r_1)) \leq N_{K_1}(Z)$ and so H_1 is strongly 3-embedded in K_1 by [8, Lemma 17.11]. □

We next show $H_i = K_i$ for $i = 1, 2$. The proof is accomplished through a series of lemmas. It suffices to prove this with $i = 1$ as the proof for $i = 2$ is the same. By Lemma 9.6 (ii), $Z(H_1) = \langle r_1\rangle$, $\Upsilon_1/Z(H_1)$ is the natural $\mathrm{Sp}_6(2)$-module and Γ_1/Υ_1 is the spin module for $\mathrm{Sp}_6(2)$. Let T be a Sylow 2-subgroup of H_1. From Lemma 9.6 (iv) we have $T \in \mathrm{Syl}_2(K_1)$.

Lemma 10.2. (i) *If $x \in \Upsilon_1^{\#}$ and $s \in x^{K_1}$, then s and sr_1 are not K_1-conjugate.*
(ii) *Υ_1 is strongly closed in Γ_1 with respect to K_1.*

Proof. (i) Obviously, if $x = r_1$, the result is true. So we may suppose $x \in \Upsilon_1 \setminus \langle r_1\rangle$. Since H_1 acts transitively on $(\Upsilon_1/\langle r_1\rangle)^{\#}$, we may additionally assume $x\langle r_1\rangle \in C_{\Upsilon_1/\langle r_1\rangle}(T)$ which has order 2 by Lemma 2.3. As by Lemma 2.8 the preimage of $C_{\Upsilon_1/\langle r_1\rangle}(T)$ is centralized by T we have $x \in Z(T)$.

Suppose that x is K_1-conjugate to xr_1. Then as x and $xr_1 \in Z(T)$, this conjugation must happen in $N_{K_1}(T)$. Since $T \in \mathrm{Syl}_2(K_1)$, this is impossible and it follows that x is not K_1-conjugate to xr_1. This proves (i)

Now consider $y \in \Gamma_1 \setminus \Upsilon_1$. Then $[y, \Gamma_1] = \langle r_1\rangle$ and so y is conjugate to $r_1 y$ in Γ_1. Therefore (i) implies (ii). □

Lemma 10.3. *Let $x \in \Upsilon_1$, $g \in K_1$ and assume that $s = x^g$ with $s \in T \setminus \Gamma_1$. Then s normalizes an H_1-conjugate of $I_1\Gamma_1$ and Σ_1.*

Proof. Since in $H_1/\Gamma_1 \cong \mathrm{Sp}_6(2)$ every involution is conjugate into $N_{H_1/\Gamma_1}(I_1\Gamma_1/\Gamma_1)$, we may as well suppose that s normalizes $I_1\Gamma_1$. In particular by Lemma 7.12 we may additionally assume $\Sigma_1^s = \Sigma_1$. □

Lemma 10.4. *Let $x \in \Upsilon_1$, $g \in K_1$ and assume that $s = x^g$ with $s \in T \setminus \Gamma_1$. Then the following hold:*

(i) $C_{\Gamma_1/\Upsilon_1}(s) = C_{\Gamma_1}(s)\Upsilon_1/\Upsilon_1$*; and*
(ii) $C_{H_1}(s)$ *is a $3'$-group.*

Proof. By Lemma 10.3 we may assume that s normalizes both $I_1\Gamma_1$ and Σ_1. Let $w\Upsilon_1 \in C_{\Gamma_1/\Upsilon_1}(s)$ and write $w = w_*u$ where $w_* \in \Sigma_1$ and $u \in \Upsilon_1$. Then

$$[w, s] = [w_*u, s] = [w_*, s][u, s] \in \Upsilon_1.$$

As s normalizes Σ_1, this means that $[w_*, s] \in \Sigma_1 \cap \Upsilon_1 = \langle r_1\rangle$. Since x is not K_1-conjugate to sr_1, we deduce that w_* is centralized by s and this proves (i).

Suppose that $W \in \mathrm{Syl}_3(C_{H_1}(s))$ and let $U \in \mathrm{Syl}_3(C_{H_1}(x))$. Then, as $\Upsilon_1/\langle r_1\rangle$ is the natural $\mathrm{Sp}_6(2)$-module, U has order 3^2 by Lemma 2.3. Since by Lemma 10.1 H_1 is strongly 3-embedded in K_1 we know that $U \in \mathrm{Syl}_3(C_{K_1}(x))$ and so $U^g \in \mathrm{Syl}_3(C_{K_1}(s))$. Thus there exists $h \in C_{K_1}(s)$ so that $U^{gh} \geq W$. Consequently $W \leq H_1 \cap H_1^{gh}$. If $W \neq 1$, Lemma 10.1 yields $gh \in H_1$ which contradicts the fact that $s = x^{gh}$, $s \in T \setminus \Sigma_1\Upsilon_1$ and $x \in \Upsilon_1$. Hence $W = 1$, proving (ii). □

Suppose that $s^* \in s\Gamma_1$ is an involution which is conjugate to s in K_1.

Then $ws = s^*$ with $w \in \Gamma_1$. We claim that $w \in C_{\Gamma_1}(s)$. To see this we note that the other possibility is that $w^s = w^{-1} = wr_1$ and then we calculate

$$s^{*s} = (ws)^s = w^s s = w^{-1}s = wr_1 s = s^* r_1$$

which contradicts Lemma 10.2(i).

Let $q \in C_{\Gamma_1}(s)$ and assume that $[w, q] \neq 1$. Then, by Lemma 9.3, $[w, q] = r_1$ and

$$s^{*q} = (ws)^q = w^q s = w[w, q]s = wsr_1 = s^* r_1,$$

which is also impossible. Therefore $w \in Z(C_{\Gamma_1}(s))$. Since s normalizes Σ_1 and Σ_1 is extraspecial, the Three Subgroup Lemma implies $Z(C_{\Sigma_1}(s)) = [\Sigma_1, s]$. Thus Lemma 10.2(i) implies that

Lemma 10.5. *Let $x \in \Upsilon_1$, $g \in K_1$ and assume that $s = x^g$ with $s \in T \setminus \Gamma_1$. If s is H_1-conjugate to $s^* = ws$ where $w \in \Gamma_1$, then $w \in Z(C_{\Gamma_1}(s)) \leq [\Gamma_1, s]\Upsilon_1$. In particular, $s\Upsilon_1$ is Γ_1/Υ_1-conjugate to $s^*\Upsilon_1$ and $C_{H_1/\Gamma_1}(s\Upsilon_1) = C_{H_1/\Upsilon_1}(s)\Gamma_1/\Gamma_1$.*

Now we are going to identify the involution $s\Gamma_1$ in $H_1/\Gamma_1 \cong \mathrm{Sp}_6(2)$.

Lemma 10.6. *Let $x \in \Upsilon_1$, $g \in K_1$ and assume that $s = x^g$ with $s \in T \setminus \Gamma_1$. Then $s\Gamma_1$ is an involution of type c_2 and all K_1-conjugates of x in $H_1 \setminus \Gamma_1$ project to elements of this type.*

Proof. By Lemma 2.2 (i), $s\Gamma_1$ is an involution of type a_2, b_1, b_3 or c_2 in $H_1/\Gamma_1 \cong \mathrm{Sp}_6(2)$. If $s\Gamma_1$ is of type b_3, then Lemma 2.2 implies that $[\Gamma_1/\langle r_1\rangle, s] = C_{\Gamma_1/\langle r_1\rangle}(s)$ and consequently 3 divides $|C_{H_1}(s)|$. Hence $s\Gamma_1$ is not of type b_3 by Lemma 10.4 (ii).

If $s\Gamma_1$ is of type b_1 or a_2, then, by Lemma 10.5, $|C_{H_1/\Upsilon_1}(s)|$ is divisible by 3^2. If $s\Gamma_1$ is of type a_2, then Lemma 2.2 implies

$$|C_{\Upsilon/\langle r_1\rangle}(s)/[\Upsilon/\langle r_1\rangle, s]| = 4$$

and so s is centralized by an element of order 3 contrary to Lemma 10.4 (ii). Thus $s\Gamma_1$ is not of type a_2. If $s\Gamma_1$ is of type b_1, then Lemma 2.2 yields $C_{\Upsilon/\langle r_1\rangle}(s)/[\Upsilon/\langle r_1\rangle, s]$ is the natural $\mathrm{Sp}_4(2)$-module and, as $\mathrm{Sp}_4(2)$ acts transitively on the non-trivial elements of this module, we again see s is centralized by a 3-element, a contradiction. Thus $s\Gamma_1$ must be of type c_2. □

Lemma 10.7. *Υ_1 is strongly 2-closed in T with respect to K_1.*

Proof. Let $x \in \Upsilon_1$, $g \in K_1$ and assume that $s = x^g$ with $s \in T \setminus \Gamma_1$. By Lemma 10.6, s acts as an element of type c_2 on the natural $\mathrm{Sp}_6(2)$-module.

Let $F = C_{\Sigma_1}(s) = [\Sigma_1, s]$. Then F has order 2^5 by Lemma 2.2. Thus the coset Fs consists solely of conjugates of s and of sr_1 and $F \cap \Upsilon_1 = \langle r_1\rangle$.

Recall that we may suppose that $x \in Z(T)$. So s is a 2-central element of K_1. Hence, as F is a 2-group which centralizes s, F is contained in a Sylow 2-subgroup T_0 of K_1 which centralizes s. Let Γ_1^* be the preimage of $J(T_0/\langle r_1\rangle)$, $\Upsilon_1^* = Z(\Gamma_1^*)$ and $H^* = N_G(\Gamma_1^*)$. By Lemma 9.6 we have that Γ_1^* is conjugate to Γ_1 in K_1. Then also H^* is K_1-conjugate to H_1 and $H^*/\Gamma_1^* \cong \mathrm{Sp}_6(2)$.

Assume that $y \in F \setminus \langle r_1 \rangle$. Then ys is conjugate to either s or sr_1. In particular any coset of $\langle r_1 \rangle$ in F contains some y such that ys is conjugate to s in K_1. If $y \in \Gamma_1^*$, then, as $y \in \Gamma_1 \setminus \Upsilon_1$, Lemma 10.2 (ii) yields $y \notin \Upsilon_1^*$ and consequently we also have $ys \in \Gamma_1^* \setminus \Upsilon_1^*$ which contradicts Lemma 10.2. Thus $y \notin \Gamma_1^*$ and the coset $y\Gamma_1^*$ contains ys. We deduce with Lemma 10.6 that $y\Gamma_1^*$ is of type c_2 in $N_{K_1}(\Gamma_1^*)/\Gamma_1^*$ and $F\Gamma_1^*/\Gamma_1^*$ is a subgroup of order 2^4 in which all the non-trivial elements are in class c_2. Since $\mathrm{Sp}_6(2)$ has no such subgroups by Lemma 2.2, we have a contradiction. Therefore Υ_1 is strongly 2-closed in T with respect to K_1. □

Next we can prove the main result of this section:

Lemma 10.8. *For $i = 1, 2$, we have $H_i = K_i$. In particular, K_1 and K_2 are similar to 2-centralizers in* $\mathrm{F}_4(2)$.

Proof. Again it is enough to prove the lemma for $i = 1$. By Lemma 10.7 we have that Υ_1 is strongly 2-closed in T with respect to K_1. Therefore Lemma 2.19 yields $K_1 \le N_G(\Upsilon_1)$. Now $C_{K_1}(\Upsilon_1) \cap C_S(r_1) = 1$ and so $C_{K_1}(\Upsilon_1)$ is a $3'$-group. Since, by Lemma 9.4, Γ_1 is the unique maximal I_1^1-signalizer in K_1, we conclude $\Gamma_1 \ge C_{K_1}(\Upsilon_1)$ and thus $\Gamma_1 = C_{K_1}(\Upsilon_1)$. It follows that $K_1 = N_{K_1}(\Upsilon_1) = N_{K_1}(\Gamma_1)$ as claimed. □

11. Proof of Theorem 1.2

Having determined the shapes of the centralizers of the involutions r_1 and r_2, in this section we accomplish the final identification of G.

Let $T \in \mathrm{Syl}_2(K_1)$, where $K_1 = C_G(r_1)$, and recall that $\Gamma_1 = \Sigma_1\Upsilon_1 = O_2(K_1)$. The conclusion of the work of the previous sections is that K_1 is similar to a 2-centralizer in $\mathrm{F}_4(2)$.

By Lemma 9.2, Υ_1 contains a G-conjugate s_2 of r_2 with $s_2 \ne r_1$. As K_1 acts transitively on the non-trivial elements of $\Upsilon_1/\langle r_1 \rangle$, Lemma 2.8 shows that we may further suppose that $s_2 \in Z(T)$ and $Z(T) = \langle r_1, s_2 \rangle$. Define $U_2 = C_G(s_2)$. We have U_2 is G-conjugate to $K_2 = C_G(r_2)$ and thus, as $|K_1| = |K_2|$, we have $T \in \mathrm{Syl}_2(U_2)$.

We will use the two groups to construct a subgroup $P = \langle K_1, U_2 \rangle \cong F_4(2)$ using Theorem 3.3. Recall Definition 3.2, and note that K_1, U_2, T is an F_4 set-up.

Lemma 11.1. $P = \langle K_1, U_2 \rangle \cong \mathrm{F}_4(2)$.

Proof. This follows directly from Theorem 3.3. □

In fact we have the following corollary:

Corollary 11.2. *If X is any group which satisfies the assumptions of Theorem 1.2, then X contains a subgroup isomorphic to* $\mathrm{F}_4(2)$.

Proof. This follows immediately from Lemma 11.1. $\square$

Our aim is to show that G is isomorphic to either $\mathrm{F}_4(2)$ or $\mathrm{Aut}(\mathrm{F}_4(2))$. For this we will show that P is normal in G. As a first step we show that P is normalized by M and that $P_0 = PM$ is either $\mathrm{F}_4(2)$ or $\mathrm{Aut}(\mathrm{F}_4(2))$. We then produce a normal subgroup G_* of G of index at most two such that $P_0 \cap G_* = P$. Our objective is then to show $G_* = P$. This will be done using Holt's Theorem (Lemma 2.20). Hence we have to gain control of G_*-fusion of involutions in P. For this we show that P_0 is strongly 3-embedded in G_*, which will imply that P controls G_*-fusion in P. We start with the proof that M normalizes P.

We have $C_P(\rho_1) \cong C_P(\rho_2) \cong 3 \times \mathrm{Sp}_6(2)$ and so, by Lemma 8.2, $C_G(\rho_i) = C_P(\rho_i)$, $i = 1, 2$. As $\langle C_M(\rho_1), C_M(\rho_2)\rangle = M \cap P$, we see $\langle C_G(\rho_1), C_G(\rho_2)\rangle$ satisfies the assumptions of Theorem 1.2. By Corollary 11.2 we get that $\langle C_G(\rho_1), C_G(\rho_2)\rangle$ contains a subgroup isomorphic to $\mathrm{F}_4(2)$. As $P \cong \mathrm{F}_4(2)$, we obtain

Lemma 11.3. $\langle C_G(\rho_1), C_G(\rho_2)\rangle = P$. $\square$

Lemma 11.4. *M normalizes P.*

Proof. Since $P \cong \mathrm{F}_4(2)$ and ρ_1 and ρ_2 are not conjugate in P, we have that $M \cap P = RS\langle f\rangle$. If $M \leq P$, we have nothing to do. If $M > M \cap P = RS\langle f\rangle$, then, by Lemma 4.8, there is an involution t of $M \setminus M \cap P$ such that $\rho_1^t = \rho_2$. This element normalizes P by Lemma 11.3. Thus M normalizes P. $\square$

Define $P_0 = PM$.

Lemma 11.5. *P_0 is strongly 3-embedded in G.*

Proof. Since $P \cong \mathrm{F}_4(2)$, there are three conjugacy classes of elements of order 3 in P and they are all witnessed in J. For $\langle x\rangle \in \mathcal{S}(J)$, we have $N_G(\langle x\rangle) = M \leq P_0$ and for $\langle x\rangle \in \mathcal{M}(J) \cup \mathcal{P}(J)$ we have $C_G(x) = C_P(x)$ by Lemma 8.2. Since also $N_G(S) \leq M \leq P_0$ we have P_0 is strongly 3-embedded in G by [8, Lemma 17.11]. $\square$

We can now determine the structure of P_0.

Lemma 11.6. *We have P_0 contains a Sylow 2-subgroup of G and either $P_0 = P$ or $P_0 \cong \mathrm{Aut}(\mathrm{F}_4(2))$.*

Proof. Assume that $T \notin \mathrm{Syl}_2(G)$ and let $T_1 > T$ normalize T. Then T_1 normalizes $Z(T) = \langle r_1, s_2\rangle$. Since $K_1 \leq P$ and $U_2 \leq P$, there exists $x \in T_1$ such that $r_1^x \neq r_1$ and $s_2^x \neq s_2$. Since $Z(T)$ has order 4, we deduce that $r_1^x = s_2$ and thus that $K_1^x = U_2$. Hence x normalizes $P = \langle K_1, U_2\rangle$ and $P_0 = P\langle x\rangle \cong \mathrm{Aut}(\mathrm{F}_4(2))$.

Now let $T_0 \in \mathrm{Syl}_2(P_0)$ ($P_0 = P$ or $P_0 = \mathrm{Aut}(P)$) and assume that $w \in N_G(T_0)$. As $r_1 \in T' \leq T_0' \leq T$, we have $r_1^w \in T \leq P$. Employing Lemma 2.21 we see that all involutions of P commute with elements of order 3. By Lemma 11.5 $C_{P_0}(r_1^w)$ contains a Sylow 3-subgroup of $C_G(r_1^w)$. Hence it follows that $r_1^w \in r_1^{P_0} \cup s_2^{P_0}$. Then there is $x \in P_0$ such that $r_1 = r_1^{wx}$ or $s_2 = r_1^{wx}$. Since $\langle K_1, U_2\rangle = P$, we have $wx \in P$. However this means $w \in P_0$ and we infer $T_0 \in \mathrm{Syl}_2(G)$. □

Now we produce the normal subgroup G_* with $G_* \cap P_0 = P$.

Lemma 11.7. *If $P_0 > P$, then G has a subgroup G_* of index 2 with $P = P_0 \cap G_*$. Furthermore G_* satisfies the hypothesis of Theorem 1.2.*

Proof. We let $T_0 \in \mathrm{Syl}_2(P_0)$ and $T \in \mathrm{Syl}_2(P)$ with $T_0 > T$. Suppose that $t \in T_0$ is an involution and $C_{P_0}(t)$ has a non-trivial Sylow 3-subgroup D. Then as P_0 is strongly 3-embedded by Lemma 11.5 we have that $D \in \mathrm{Syl}_3(C_G(t))$. Now by Lemma 2.21 P has four conjugacy classes of involutions and their centralizers have 3-parts of their orders 3^4, 3^4, 3^2 and 3^2. On the other hand, if we let $x \in T_0 \setminus T$ with $C_{P_0}(x) \cong 2 \times {}^2\mathrm{F}_4(2)$, then $C_P(x)$ has Sylow 3-subgroups which are extraspecial of order 3^3. It follows that x is not conjugate to any element in T and consequently G has a subgroup G_* of index 2 by Thompson's Transfer Lemma [8, Lemma 15.16]. Obviously then $P_0 \cap G_* = P$ and G_* satisfies the hypothesis of Theorem 1.2. □

We finally prove that $G \cong \mathrm{F}_4(2)$ or $\mathrm{Aut}(\mathrm{F}_4(2))$.

Proof of Theorem 1.2. By Lemma 11.7, we may suppose that $P = P_0$. Using Lemma 2.21, P has exactly four conjugacy classes of involutions and each such involution t has $|C_P(t)|_3 \neq 1$. Since P is strongly 3-embedded in G, $C_P(t)$ contains a Sylow 3-subgroup of $C_G(t)$. Thus, as $|C_P(r_1)|_3 = 3^4$, we have $r_1^G \cap P \subseteq r_1^P \cup r_2^P$. Since r_1 and r_2 are not G-conjugate by Lemma 7.3 and 11.7, we get that $r_1^G \cap P = r_1^P$. We note that if N is a non-trivial normal subgroup of G, then, as $C_G(r_1) \leq P$ and $r_1 \notin Z(P)$, $1 \neq C_N(r_1) \leq N \cap P$ which means that $P \leq N$. Because $N_G(S) \leq P$, the Frattini Argument implies $G = N_G(S)N \leq PN = N$. Hence G is a simple group. Now an application of Lemma 2.20 and the observation that P is neither soluble nor an alternating group yields $G = P$ and the proof is complete. □

References

[1] Michael Aschbacher, Finite group theory, Cambridge University Press 1986.

[2] Michael Aschbacher and Garry Seitz, Involutions in Chevalley groups over fields of even order, Nagoya Math. J. 63, (1976), 1–91.
[3] Helmut Bender, Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festläßt, J. Algebra 17 1971 527–554.
[4] John H. Conway, Robert T. Curtis, Simon P. Norton, Richard A. Parker and Robert A. Wilson, Atlas of finite groups, Oxford University Press 1985.
[5] David Goldschmidt, 2-fusion in finite groups, Ann. of Math. 99 (1974), 70–117.
[6] Larry Finkelstein and Daniel Frohardt, Standard 3-components of type $\mathrm{Sp}(6,2)$, Trans. Amer. Math. Soc. 266 (1981), no. 1, 71–92.
[7] Daniel Gorenstein, Finite groups, Harper & Row, Publishers, New York-London 1968.
[8] Daniel Gorenstein, Richard Lyons and Ronald Solomon, The classification of the finite simple groups Number 2, Mathematical Surveys and Monographs, 40.2. American Mathematical Society, Providence, RI, 1996.
[9] Martin M. Guterman, A characterization of the groups $\mathrm{F}_4(2^n)$, J. Algebra 20 (1972), 1–23.
[10] Derek Holt, Transitive permutation groups in which an involution central in a Sylow 2-subgroup fixes a unique point, Proc. LMS 37 (1978), 165 – 192.
[11] Bertram Huppert, Endliche Gruppen I, Springer 1967.
[12] Christoph Jansen, Klaus Lux, Richard Parker and Robert Wilson, An Atlas of Brauer Characters, Oxford Science Publications 1995.
[13] Ulrich Meierfrankenfeld, Bernd Stellmacher and Gernot Stroth, The structure theorem for finite groups with a large p-subgroup, preprint 2011.
[14] Chris Parker, A 3-local characterization of $\mathrm{U}_6(2)$ and Fi_{22}, J. Algebra 300 (2006), no. 2, 707–728.
[15] Chris Parker and Peter Rowley, Symplectic amalgams, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2002.
[16] Chris Parker and Peter Rowley, A 3-local characterization of Co_2, J. Algebra 323 (2010), no. 3, 601–621.
[17] Chris Parker and Gernot Stroth, Groups with socle $\mathrm{PSU}_6(2)$, J. Australian Math. Soc., Journal of the Australian Math. Soc. 93 (2012), 277–310.
[18] C. Parker and G. Stroth, Groups which are almost groups of Lie type in characteristic p, preprint, arXiv:1110.1308.
[19] Chris Parker and Gernot Stroth, An identification theorem for the sporadic simple groups F_2 and M(23), to appear Journal of Group Theory, J. Group Theory 16 (2013), 319–352.
[20] C. Parker, M. R. Salarian and G. Stroth, A characterisation of ${}^2\mathrm{E}_6(2)$, M(22) and Aut(M(22)) from a characteristic 3 perspective, submitted Forum Mathematicum, preprint arXiv:1108.1894.
[21] Chris Parker and Gernot Stroth, Strongly p-embedded subgroups, Pure and Applied Mathematics Quarterly Volume 7, Number 3 (Special Issue: In honor of Jacques Tits) 797–858, 2011.
[22] Alan Prince, characterization of the simple groups $\mathrm{PSp}(4,3)$ and $\mathrm{PSp}(6,2)$, J. Algebra 45 (1977), no. 2, 306–320.
[23] F.G. Timmesfeld, Finite simple groups in which the generalized Fitting group of the centralizer of some involution is extraspecial, Ann. of Math. 107 (1978), no. 2, 297–369.

[24] Jaques Tits, Buildings of spherical type and finite BN-pairs, Lecture Notes in Math. 386. Springer Berlin 1974.

[25] Jaques Tits, A local approach to Buildings. In: The geometric vein (Coxeter Festschrift), 519 - 547, Springer New York 1981.

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