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A CHARACTERISATION OF ALMOST SIMPLE GROUPS WITH SOCLE ${}^2\mathrm{E}_6(2)$ OR $\mathrm{M}(22)$

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ABSTRACT. We show that the sporadic simple group $\mathrm{M}(22)$, the exceptional group of Lie type ${}^2\mathrm{E}_6(2)$ and their automorphism groups are uniquely determined by the approximate structure of the centralizer of an element of order 3 together with some information about the fusion of this element in the group.

1. Introduction

The aim of this article is to identify the groups with minimal normal subgroup $\mathrm{M}(22)$, one of the sporadic simple groups discovered by Fischer, and the exceptional Lie type group ${}^2\mathrm{E}_6(2)$ from certain information about the centralizer of a certain element of order 3.

The results of this paper and its companions [13, 16, 17, 15] is to provide identification theorems for the work in [18] where the following configuration relevant to the classification of groups with a so-called large p-subgroup is considered. We are given a group G, a prime p and a large p-subgroup Q (the definition of a large p-subgroup is not important for this discussion) and we find ourselves in the following situation. Containing a Sylow p-subgroup S of G there is a group H such that $F^*(H)$ is a simple group of Lie type. In the typical situation when one would expect that this group H is in fact the entire group G. However it can exceptionally happen that in fact the normalizer of the large subgroup is not contained in Q. This happens more frequently than one might expect when $F^*(H)$ is defined over the field of 2 or 3 elements and $N_H(Q)$ is soluble. Indeed in [18], the authors determine all the cases when this phenomena appears. This paper fits into the picture when we consider $F^*(H) \cong \Omega_7(3)$. In H, the large subgroup Q is extraspecial of order 3^7 an $N_{F^*(H)}(Q) \approx 3_+^{1+6}.(SL_2(3) \times \Omega_3(3))$. In [18] we show that if $N_G(Q)$ is not contained in H, then we must have $C_H(Z(Q))$ is a centralizer in a group of type either $\mathrm{M}(22)$ or ${}^2\mathrm{E}_6(2)$ where these centralizers are defined as follows.

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Definition 1.1. *We say that X is similar to a 3-centralizer in a group of type* ${}^2\mathrm{E}_6(2)$ *provided*

(i) $Q = F^*(X)$ *is extraspecial of order* 3^{1+6} *and* $Z(F^*(X)) = Z(X)$*; and*
(ii) $O_2(X/Q) \cong \mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8$.

Definition 1.2. *We say that X is similar to a 3-centralizer in a group of type* $\mathrm{M}(22)$ *provided*

(i) $Q = F^*(X)$ *is extraspecial of order* 3^{1+6} *and* $Z(F^*(X)) = Z(X)$*; and*
(ii) $O_2(X/Q)$ *acts on* Q/Z *as a subgroup of order* 2^7 *of* $\mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8$*, which contains* $Z(\mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8)$.

In this paper we will prove the following two theorems

Theorem 1.3. *Suppose that G is a group, $H \le G$ is similar to a 3-centralizer in a group of type* ${}^2\mathrm{E}_6(2)$*, $Z = Z(F^*(H))$ and $H = C_G(Z)$. If $S \in \mathrm{Syl}_3(G)$ and Z is not weakly closed in S with respect to G, then Z is not weakly closed in $O_3(H)$ and* $G \cong {}^2\mathrm{E}_6(2)$, ${}^2\mathrm{E}_6(2).2$, ${}^2\mathrm{E}_6(2).3$ *or* ${}^2\mathrm{E}_6(2).\mathrm{Sym}(3)$.

Theorem 1.4. *Suppose that G is a group, $H \le G$ is similar to a 3-centralizer in a group of type* $\mathrm{M}(22)$*, $Z = Z(F^*(H))$ and $H = C_G(Z)$. If $S \in \mathrm{Syl}_3(G)$ and Z is not weakly closed in S with respect to G, then Z is not weakly closed in $O_3(H)$ and* $G \cong \mathrm{M}(22)$ *or* $\mathrm{Aut}(\mathrm{M}(22))$.

A minor observation that is useful to us in our forthcoming work on $\mathrm{M}(23)$ and the Baby Monster F_2 is that the interim statements that we prove in this paper become observations about the structure of $\mathrm{M}(22)$ and ${}^2\mathrm{E}_6(2)$ once the main theorems have been proved.

The paper is organised as follows. In Section 2, we gather together facts about the 20-dimensional $\mathrm{GF}(2)\mathrm{U}_6(2)$-module, centralizers of involutions in this group and in the spit extension $2^{20} : \mathrm{U}_6(2)$ as well as a transfer theorem for groups of shape $2^{10}.\mathrm{Aut}(\mathrm{Mat}(22))$. We close Section 2 with a collection of theorems and lemmas which will be applied in the proof of our main theorems.

Section 3 contains a proof of the following theorem which we used to determine the structure of the centralizer of an involution in groups satisfying the hypothesis of Theorem 1.3.

Theorem 1.5. *Suppose that X is a group, $O_{2'}(X) = 1$, $H = N_X(A) = AK$ with $H/A \cong K \cong \mathrm{U}_6(2)$ or $\mathrm{U}_6(2) : 2$, $|A| = 2^{20}$ and A a minimal normal subgroup of H. Then H is not a strongly 3-embedded subgroup of X.*

In Section 3, we set $H = C_G(Z)$ and $Q = O_3(H)$ and start by investigating the possible structure of H. Almost immediately from the hypothesis we know that $H/O_3(H)$ embeds into $\mathrm{Sp}_2(3) \wr \mathrm{Sym}(3)$. Lemma 4.5 shows that Z is not weakly closed in Q and we use this information to build a further 3-local subgroup M. It turns out that M is the normalizer of the Thompson subgroup of a Sylow 3-subgroup of G contained in H and further Lemma 4.18 that $O_3(M)$ elementary abelian of order either 3^5 or 3^6 and $F^*(M/O_3(M)) \cong \Omega_5(3)$.

Section 5 is devoted to the proof of Theorem 1.4. From the information gathered in Section 3 we quickly show that the centralizer of an involution has shape $2{}^{\cdot}\mathrm{U}_6(2)$ or $2{}^{\cdot}\mathrm{U}_6(2).2$. From this we can build a further 2-local subgroup of shape $2^{10} : \mathrm{Mat}(22)$ or $2^{10} : \mathrm{Aut}(\mathrm{Mat}(22))$ and use Lemma 2.11 to show that G has a subgroup of index 2 in the latter case. Finally we apply [1, Theorem 31.1] to finally prove Theorem 1.4.

From Section 7 onwards we may assume that H is a 3-centralizer in a group of type ${}^2\mathrm{E}_6(2)$. In particular, we have that $O_2(H/Q) \cong \mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8$ and we let r_1 be an involution in H such that r_1Q is contained in the first direct factor. By the end of Section 7 we know r_1 is a 2-central involution which contains an extraspecial subgroup of order $E \cong 2_+^{1+20}$ in its centralizer and that $F^*(N_G(E)/E) \cong \mathrm{U}_6(2)$. Our next objective is to control the embedding of $N_G(E)$ in $C_G(r_1)$ so that we can show that $C_G(r_1) = N_G(E)$. To do this we first transfer elements of order 2 and order 3 from G. The transfer of an element of order 2 is carried out in Section 8 and then the element of order 3 easily follows in Section 9. At this stage we know that $N_G(E) \approx 2_+^{1+20}.\mathrm{U}_6(2)$, however we still don't know enough about the centralizers of elements of order 3 in $C_G(r_1)$ to be able to show that $N_G(E)$ is strongly 3-embedded in $C_G(r_1)$. Thus in Section 10, we determine the centralizer of a further element of order 3 with the help of Astill's Theorem [4]. With this we can prove that $N_G(E)$ is indeed strongly 3-embedded in $C_G(r_1)$ and conclude from Theorem 1.5 that $C_G(r_1) = N_G(E)$. At this stage, we could apply Aschbacher's Theorem [2] to identify G, however, partly because some of the background material about the simple connectivity of certain graphs related to geometries to type F_4 has not yet been published and also because we would prefer a uniform building theoretic approach to the classification of the groups such as ${}^2\mathrm{E}_6(2)$, in the penultimate section we identify the ${}^2\mathrm{E}_6(2)$ by showing that the coset geometry constructed from certain 2-local subgroups containing the normalizers of a Sylow 2-subgroup of G is in fact a chamber system of type F_4. The Tit's Local Approach Theorem yields that the group generated by these 2-local subgroups is $\mathrm{F}_4(2)$. Finally we apply Holt's Theorem [10] to see that $G \cong {}^2\mathrm{E}_6(2)$. Combining this with the transfer arguments presented earlier finally proves Theorem 1.3 the details being presented in our brief final section.

Throughout this article we follow the now standard Atlas [5] notation for group extensions. Thus $X{}^{\cdot}Y$ denotes a non-split extension of X by Y, $X{:}Y$ is a split extension of X by Y and we reserve the notation $X.Y$ to denote an extension of undesignated type (so it is either unknown, or we do not care). Our group theoretic notation is mostly standard and follows that in [8] for example. For odd primes p, the extraspecial groups of exponent p and order p^{2n+1} are denoted by p_+^{1+2n}. The extraspecial 2-groups of order 2^{2n+1} are denoted by 2_+^{1+2n} if the maximal elementary abelian subgroups have order 2^{1+n} and otherwise we write 2_-^{1+2n}. The extraspecial group of order 8 is denoted by Q_8. We expect our notation for specific groups is self-explanatory. For a subset X of a group G, X^G denotes that set of G-conjugates of X. If $x, y \in H \le G$, we write $x \sim_H y$ to indicate that x and y are conjugate in H. Often we shall give suggestive descriptions of groups which indicate the isomorphism type of certain composition factors. We refer to such descriptions as the *shape* of a group. Groups of the same shape have normal series with isomorphic sections. We use the symbol $\approx$ to indicate the shape of a group.

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2. PRELIMINARY FACTS

Suppose that $X = \mathrm{U}_6(2){:}2$, $Y = \mathrm{U}_6(2)$, $\overline{X} = \mathrm{SU}_6(2){:}2$, $\overline{Y} = \mathrm{SU}_6(2)$ and W is the natural $\mathrm{GF}(4)\overline{Y}$-module. Let $\{w_1, \dots, w_6\}$ be a unitary basis for W. Note that $\overline{X}$ acts on W with the outer elements acting as semilinear transformations. Let $\overline{M}$ be the monomial subgroup of $\overline{Y}$ of shape $3^5{:}\mathrm{Sym}(6)$ and M be its image in Y. Set $J = O_3(M)$. Then J is elementary abelian of order 3^4 and $\overline{J}$ is elementary abelian of order 3^5. Note that M contains a Sylow 3-subgroup of Y. We let e_1, e_2 and e_3 be the images of the diagonal matrices $\mathrm{diag}(\omega, \omega^{-1}, 1, 1, 1, 1)$, $\mathrm{diag}(\omega, \omega, \omega^{-1}, \omega^{-1}, 1, 1)$ and $\mathrm{diag}(\omega, \omega, \omega, \omega^{-1}, \omega^{-1}, \omega^{-1})$ in Y respectively. Then e_1, e_2 and e_3 are representatives of the three conjugacy classes of elements of order 3 in Y.

Lemma 2.1. *Every element of order 3 in X is X-conjugate to an element of J and the centralizers of elements of order 3 are as follows.*

(i) $C_Y(e_1) \cong 3 \times \mathrm{SU}_4(2)$;
(ii) $C_Y(e_2) \cong 3 \times \mathrm{Sym}(3) \wr 3$ *and has order* $2^3.3^5$; *and*
(iii) $C_Y(e_3) \cong (\mathrm{SU}_3(2) \circ \mathrm{SU}_3(2)).3 \approx 3_+^{1+4}.(\mathrm{Q}_8 \times \mathrm{Q}_8).3$.

Proof. Given the descriptions of e_1, e_2 and e_3 above this is an easy calculation. (See also [1, (23.9)] and correct the typographical error.) □

We also need to know the centralizers of involutions in X.

Lemma 2.2. *X has five conjugacy classes of involutions and their centralizers have shapes as follows.*

$$\begin{aligned} C_X(t_1) &\approx 2_+^{1+8} : \mathrm{SU}_4(2).2; \\ C_X(t_2) &\approx 2^{4+8}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)).2; \\ C_X(t_3) &\approx 2^9.3^2.\mathrm{Q}_8.2 \le 2^9 : \mathrm{L}_3(4).2; \\ C_X(t_4) &\approx 2 \times \mathrm{Sp}_6(2); \text{ and} \\ C_X(t_5) &\approx 2 \times (2^5 : \mathrm{Sp}_4(2)). \end{aligned}$$

The involutions t_1, t_2 and t_3 are contained in Y and their centralizers in Y are obtained by dropping the final 2 in their description in X. Furthermore we may suppose that $t_5 = t_4t_1$ and $C_X(t_5) \le C_X(t_4)$.

Proof. This can be found in [3] for the involution t_1, t_2 and t_3 (see also [1, (23.2)] and the following discussion). For the involutions t_4 and t_5 we refer to [9, Proposition 4.9.2]. □

We note that the involutions t_1, t_2, and t_3 are the images in Y of the involutions $\mathrm{diag}(t, I, I)$, $\mathrm{diag}(t, t, I)$ and $\mathrm{diag}(t, t, t)$ respectively, where $t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and I is the 2×2 identity matrix. The conjugates of t_1 are called *unitary transvections.*

Lemma 2.3. *There are no fours groups in X all of whose non-trivial elements are unitary transvections. In particular, if t is a unitary transvection, then $\langle t \rangle$ is weakly closed in $O_2(C_X(t))$.*

Proof. Suppose that F is a fours group in X and that all the non-trivial elements of F are unitary transvections. Let x_1, x_2 and x_3 be the non-trivial elements of F. Since $C_X(x_1)$ is a maximal subgroup of X and $Z(C_X(x_1)) = \langle x_1 \rangle$, $X = \langle C_X(x_1), C_X(x_2) \rangle$. Therefore, $C_W(x_1) \ne C_W(x_2)$. Let $v \in W \setminus C_W(x_1)$ and $w \in C_X(x_2) \setminus C_W(x_1)$. Then $[v, x_3] =$

$[v, x_2]$ and $[w, x_3] = [w, x_2]$. Hence, as $\dim[W, x_3] = 1$, $[W, x_1] = [W, x_2]$ is normalized by X, which is a contradiction. If $O_2(C_G(t))$ containes a unitary transvection s with $t \neq s$, then conjugation in $O_2(C_G(t))$ reveals that all elements of $\langle s, t\rangle$ are unitary transvections and this is impossible as we have just seen. Thus $\langle t\rangle$ is weakly closed in $O_2(C_X(t))$. □

Let P_1 and P_2 be the connected parabolic subgroups of Y containing a fixed Borel subgroup where notation is chosen so that

$$P_1 \approx 2_+^{1+8}{:}\mathrm{SU}_4(2)$$

and

$$P_2 \approx 2^9{:}\mathrm{L}_3(4).$$

Lemma 2.4. *Suppose that* $Y \cong \mathrm{U}_6(2)$ *and that* V *is an irreducible* 20*-dimensional* $\mathrm{GF}(2)Y$*-module. Then* $V \otimes \mathrm{GF}(4)$ *is the exterior cube of* W*. In particular,* $\dim C_V(O_2(P_2)) = 1$ *and* $\dim C_V(e_3) = 2$*.*

Proof. First consider the restriction of V to $O_3(C_Y(e_3))$. This group has no faithful characteristic 2-representation of dimension less than 9 and as e_3 is inverted by a conjugate t of t_3, we see that any characteristic 2 representation of $O_3(C_Y(e_3))\langle t\rangle$ has dimension at least 18. It follows that $\dim C_V(e_3) = 2$ and that V is absolutely irreducible. By Smith's Theorem [20], we now have, for $i = 1, 2$, $C_V(O_2(P_i))$ are irreducible P_i-modules. Suppose that $\dim C_V(O_2(P_2)) > 1$. Then, as $P_2/O_2(P_2) \cong \mathrm{L}_3(4)$ contains an elementary abelian subgroup of order 9 all of whose subgroups of order 3 are conjugate, we have $\dim C_V(O_2(P_2)) \geq 8$. Since $t_1 \in O_2(P_2)$ and since there exists $x \in P_1$ such that $P_1 = \langle O_2(P_2), O_2(P_2)^x\rangle$, we either have $\dim C_V(t_1) \geq 15$ or $\dim C_V(P_1) \geq 2$. The latter possibility violates Smith's Theorem. Hence $\dim C_V(t_1) \geq 15$. Thus $V/C_V(t_1)$ has dimension at most 5. Since $P_1/O_2(P_1) \cong \mathrm{SU}_4(2)$ has Sylow 3-subgroups of order 3^4, we have $[V, P_1] \leq C_V(t_1)$ and so t_1 is a transvection by Smith's Theorem. Since t_1 inverts e_1, we now have $\dim C_V(e_1) \geq 18$ and taking a suitable product of three conjugates of e_1 we obtain a conjugate of e_3 centralizing a 14-space rather than a 2-space. At which stage we conclude $\dim C_V(O_2(P_1)) = 1$. Finally, using [2, 5.5] we obtain the statement of the lemma. □

We note that the 20-dimensional $\mathrm{GF}(2)Y$-module in Lemma 2.4 extends to an action of X (as can be seen in the group ${}^2\mathrm{E}_6(2).2$). Our next gual is to determine the action of elements of X on V described in Lemma 2.4. We recall that $P_1/O_2(P_1) \cong \mathrm{SU}_4(2)$. We call the 4-dimensional $\mathrm{GF}(4)\mathrm{SU}_4(2)$ viewed as an 8-dimensional $\mathrm{GF}(2)$-module the *unitary module* for $\mathrm{SU}_4(2)$ and the 6-dimensional $\mathrm{GF}(2)\mathrm{SU}_4(2)$-module which can be seen as the exterior square of the unitary module is called the *orthogonal module* for $\mathrm{SU}_4(2)$. We will also meet the *symplectic module* for $C_X(t_4)/\langle t_4\rangle \cong \mathrm{Sp}_6(2)$ as well as the *spin module* which has dimension 8 and this is the unique 8-dimensional irreducible $\mathrm{Sp}_6(2)$-module (see [2, 5.4]). Finally, from Lemma 2.6 we have that $C_Y(t_2)/O_2(C_Y(t_2)) \cong \Omega_4^+(2)$ and so this group has an orthogonal module.

Proposition 2.5. *Suppose that $X = \mathrm{U}_6(2):2$ and V is the irreducible $\mathrm{GF}(2)X$-module of dimension 20.*

(i) *The following hold:*
  (a) $\dim C_V(t_1) = 14$, *$[V,t_1]$ is the orthogonal module and $C_V(t_1)/[V,t_1]$ is the unitary module for $C_X(t_1)/O_2(C_X(t_1)) \cong \mathrm{SU}_4(2)$;*
  (b) $\dim C_V(t_2) = 12$, *$C_V(t_2)/[V,t_2]$ is the orthogonal module for $C_X(t_2)/O_2(C_X(t_2)) \cong \Omega_4^+(2)$;*
  (c) $\dim C_V(t_4) = 14$, *$[V,t_4]$ is the symplectic module and $C_V(t_4)/[V,t_4]$ is the spin module for $C_X(t_4)/O_2(C_X(t_4)) \cong \mathrm{Sp}_6(2)$;*
  (d) $\dim C_V(t_3) = \dim C_V(t_5) = 10$;

(ii) *The stabilizers of non-zero vectors in V are as follows:*

$$\begin{aligned}
\mathrm{Stab}_X(v_1) &\approx 2^9 : \mathrm{L}_3(4).2;\\
\mathrm{Stab}_X(v_2) &\approx 2^{1+8}.\mathrm{Sp}_4(2).2;\\
\mathrm{Stab}_X(v_3) &\approx 2^8 : 3^2.\mathrm{Q}_8.2;\\
\mathrm{Stab}_X(v_4) &\approx \mathrm{L}_3(4).2.2;\ \textit{and}\\
\mathrm{Stab}_X(v_5) &\approx 3_+^{1+4}.(\mathrm{Q}_8 \times \mathrm{Q}_8).2.2.
\end{aligned}$$

Here v_1, v_2, v_3 are the singular vectors.

Proof. For the involutions t_i, $i = 1,2,3$, $\dim[V,t_i]$ is given in [2, 7.4 (1)]. In particular (i) (c) holds and the dimension statements in (i)(a) and (i)(b) hold.

The remaining parts of (i)(a) can be deduced from [2, (5.6)].

The involution t_2 centralizes the image in X of $\langle a,b\rangle$ where $a = \mathrm{diag}(\omega,\omega,\omega^{-1},\omega^{-1},\omega,\omega^{-1})$ and $b = \mathrm{diag}(\omega^{-1},\omega^{-1},\omega,\omega,\omega,\omega^{-1})$, Thus the Sylow 3-subgroup T of $C_X(t_2)$ contains two conjugates of $\langle e_3\rangle$, a conjugate of $\langle e_1\rangle$ and a conjugate of $\langle e_2\rangle$. Now $C_V(a) = \langle w_1\wedge w_2\wedge w_5, w_3\wedge w_4\wedge w_6\rangle$ and $C_V(b) = \langle w_1\wedge w_2\wedge w_6, w_3\wedge w_4\wedge w_5\rangle$ and so $C_V(T) = 0$. It follows that $C_V(t_2)/[V,t_2]$ admits $C_X(t_2)$ as described in (i)(b).

There is a conjugate of t_4 which centralizes a subgroup isomorphic to $\mathrm{Sp}_4(2)$ in $C_X(t_1)/O_2(C_X(t_1))$. By part (i)(a) $C_X(t_1)$ acts as $\mathrm{O}_6^-(2)$ on $[V,t_1]$ and $V/C_V(t_1)$ and naturally as $\mathrm{SU}_4(2)$ on $C_V(t_1)/[V,t_1]$. Since

t_4 is not a unitary transvection of $C_X(t_1)/O_2(C_X(t_1))$, we see that $\dim[V, t_4] \geq 6$ and $[C_X(t_4), [V, t_4]] \neq 1$. Furthermore $\mathrm{Sp}_4(2)$ acts fixed point freely on $C_W(t_4)/[W, t_4]$ for all $\mathrm{U}_4(2)$ sections in V. Therefore $\mathrm{Sp}_4(2)$ acts fixed point freely on $C_V(t_4)/[V, t_4]$. In particular $|C_V(t_4)/[V, t_4]| = 2^{4x}$ where x is some positive integer. This shows that this module must be the 8-dimensional $\mathrm{Sp}_6(2)$-module and then we deduce $\dim C_V(t_4) = 14$.

We have $t_5 = t_4 t_1$, and $C_X(t_5) \leq C_X(t_4)$. As seen before we have that there is $U = \mathrm{Sym}(3) \times \mathrm{U}_4(2)$ in X such that as an U-module V is a direct sum of the unitary module V_2 with a tensor product of the 2-dimensional Sym(3)- module with the $\mathrm{O}_6^-(2)$-module. We may assume that $t_1 \in \mathrm{Sym}(3)$ and t_5 and t_4 induce an outer automorphism on $\mathrm{U}_4(2)$. As $C_X(t_5)$ does not contain $\mathrm{Sym}(6) \times \mathrm{Sym}(3)$, we see that t_5 acts faithfully on the normal Sym(3), while t_4 centralizes this group. We have that $C_{V_2}(t_5)$ is of order 16. As t_5 inverts an element of order three in Sym(3), which acts fixed point freely on V_1, we get that $C_{V_1}(t_5)$ is of order 64. Hence we have that $\dim C_V(t_5) = 10$.

For part (ii) we refer to Aschbacher [2, 7.5 (4)] for centralizers of singular vectors in V. This gives the centralizers of v_1, v_2 and v_3.

Let $Z = \langle e_3 \rangle$, $Q = O_3(C_X(Z)) \cong 3_+^{1+4}$ and set $U = C_V(Z)$. Then $\dim U = 2$ and $\dim[V, Z] = 18$. Since $Q \leq C_X(Z)'$, we have that Q centralizes U. As none of the singular vectors have such a subgroup centralizing them, we infer that the non-trivial elements of U are all non-singular. Now U is normalized by $N_X(Z)$ and so we have that $C_X(U)$ has index at most 6 in $N_X(Z)$. By Lemma 2.1, there is a conjugate Y of Z in $C_X(Z)$ which is not contained in Q. If $[Y, U] = 1$, then $U = C_V(Y) = C_V(Z)$ and so Y is conjugate to Z in $N_X(U)$, which is not the case. Hence Y acts transitively on $U^\sharp$. This shows that $C_X(v_5)$ is as stated.

Let $L \cong \mathrm{L}_3(4)$ be the Levi complement of the parabolic subgroup of X which is the image of the stabilizer of an isotropic 3-space I of the unitary space W .Then L also stabilizes an isotropic subspace J with $I \cap J = 0$ and in fact I and J are the only such subspaces normalized by L. Now L centralizes $\langle i_1 \wedge i_2 \wedge i_3, j_1 \wedge j_2 \wedge j_3 \rangle$ where $\{i_1, i_2, i_3\}$ and $\{j_1, j_2, j_3\}$ are bases for I and J respectively.

Thus by 2.4 $\dim C_V(L) = 2$ and this space is normalized by $\mathrm{L}_3(4) : 2$. It follows that this group centralizes at least one non-zero vector and this vector must be non-singular as none of the singular vectors have such a stabilizer. By [5] we have that $\mathrm{L}_3(4) : 2$ is a maximal subgroup in $F^*(X)$. Thus we have at least two orbits of non-singular vectors and summing the lengths of these orbits we see that we have accounted for all the orbits of X on V. □

Lemma 2.6. *Assume that* $X \cong \mathrm{U}_6(2):2$ *and that* V *is a* 20*-dimensional* GF(2)X*-module. Let* Y *be the semidirect product of* V *and* X*. Then for* j *an involution in* $Y \setminus V$ *we have one of the following:*

(i) Vj *is a* 2*-central involution in* Y'/V, $|C_V(j)| = 2^{14}$ *and*
  (a) $C_{Y'}(j) \approx 2^{14}.2_+^{1+8}.\mathrm{U}_4(2)$;
  (b) $C_{Y'}(j) \approx 2^{14}.2_+^{1+8}.2^{1+4}.\mathrm{Sym}(3)$;
  (c) $C_{Y'}(j) \approx 2^{14}.2_+^{1+8}.3_+^{1+2}.\mathrm{Q}_8$;

(ii) Vj *is not* 2*-central in* Y'/V *and* $C_{Y'/V}(Vj) = 2^{4+8}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3))$, $|C_V(j)| = 2^{12}$ *and*
  (a) $C_{Y'}(j) \approx 2^{12}.2^{4+8}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3))$;
  (b) $C_{Y'}(j) \approx 2^{12}.2^{4+8}.\mathrm{Sym}(3)$;
  (c) $C_{Y'}(j) \approx 2^{12}.2^{4+8}.2^2$;

(iii) Vj *is not* 2*-central in* Y'/V, $|C_V(j)| = 2^{10}$ *and* $C_{Y'}(j) \approx 2^{10}.2^9.3^2 : \mathrm{Q}_8$;

(iv) $j \in Y \setminus Y'$, $|C_V(j)| = 2^{14}$ *and*
  (a) $C_Y(j) \approx 2^{14}.(2 \times \mathrm{Sp}_6(2))$;
  (b) $C_Y(j) \approx 2^{14}.(2 \times 2^6.\mathrm{L}_3(2))$;
  (c) $C_Y(j) \approx 2^{14}.(2 \times \mathrm{G}_2(2))$; *and*

(v) $j \in Y \setminus Y'$, $C_Y(j) \approx 2^{10}.(2 \times 2^5.\mathrm{Sym}(6))$.

Proof. If $|C_V(j)| = 2^{10}$, then all involutions in Vj are conjugate. Hence (iii) and (v) hold with Proposition 2.5.

Let j be 2-central. Then $C_V(j)/[V, j]$ is the $\mathrm{U}_4(2)$-module by Proposition 2.5 In particular we have three orbits of lengths 1,135, 120, which gives (i) (a) - (c).

If j is as in (iv), then by Proposition 2.5 $C_X(j)$ induces on $C_V(j)/[V, j]$ the spin module and we have again orbits of lengths 1, 135 and 120, which gives (iv) (a) - (c).

Let finally j be as in (ii). Then $|[V, j]| = 2^8$ and by Proposition 2.5 $C_V(j)/[V, j]$ is the $\mathrm{O}_4^+(2)$-module for $C_X(j)$. Hence we have three orbits of lengths 1,6,9, which gives (ii) (a) - (c). □

Lemma 2.7. *Suppose that* $X \cong \mathrm{U}_6(2)$:2 *and that* V *is an irreducible* 20*-dimensional* GF(2)X*-module. Then* V *is not a failure of factorization module.*

Proof. Suppose that $A \le P_1$ is an elementary abelian 2-subgroup of X, $|V : C_V(A)| \le |A|$ and $[V, A, A] = 0$. Then Lemma 2.3 and Proposition 2.5(i) imply that

$$2^8 \le |V : C_V(A)| \le |A| \le 2^9$$

as the 2-rank of X is 9. In particular, Proposition 2.5 implies that all the non-trivial elements of A are conjugate to either t_1 or t_2. As the 2-rank of P_1/Q_1 is 4, $|A \cap Q_1| \geq 2^4$. Since t_1 is weakly closed in Q_1 by Lemma 2.3, there exist $b \in A \cap Q_1$ conjugate to t_2. Hence $C_V(A) = C_V(b) \geq C_V(Q_1)$. Now $C_X(C_V(Q_1)) = Q_1$ by Proposition 2.5 and so $A \leq Q_1$ which is absurd as Q_1 is extraspecial of order 2^9. □

Lemma 2.8. *Suppose that $X = \mathrm{U}_6(2){:}2$ and that $j \sim_X t_2$. Then every normal subgroup of order 8 in a Sylow 2-subgroup of $C_X(j)$ contains a unitary transvection.*

Proof. By Lemma 2.6 we may assume that P_1 contains a Sylow 2-subgroup T of $C_X(j)$ and $j \in Q_1$. Suppose that A is a normal subgroup of T of order 8 with $j \in A$. If $A \cap C_{Q_1}(j) = \langle j \rangle$, then $[A, C_{Q_1}(j)] \leq \langle j \rangle$ and every non-trivial element of AQ_1/Q_1 acts as a unitary transvection on $Q_1/\langle t_1 \rangle$. From [16, Proposition 2.12 (viii)], we have $|AQ_1/Q_1| \leq 2$ which means that $|A| \leq 4$, a contradiction. Thus $A \cap C_{Q_1}(j) \not\leq \langle j \rangle$. Since $C_{Q_1}(j)$ normalizes A and $|Q_1 : C_{Q_1}(j)| = 2$, we now get $t_1 \in A$ and we are done. □

In the next lemma we present some results about the 10-dimensional Todd module for M_{22}. A description of this module may be found in [1, Section 22]. This module is seen to admit the action of $\mathrm{Aut}(\mathrm{M}_{22})$ and we continue to call this module the Todd module. We note that it is a quotient of the natural 22-dimensional permutation module for $\mathrm{Aut}(\mathrm{M}_{22})$ (see [1, (22.3)]) and that the module is uniquely determined by this property. The Todd module for $H = \mathrm{L}_3(4)$ is obtained as an irreducible 9-dimensional quotient GF(2)-permutation module obtained from the action of H on the 21 points of the projective plane. Once tensored with GF(4), it can also be identified with the tensor product $N \otimes N^\sigma$ where N is the natural $\mathrm{SL}_3(4)$-module and σ is the Frobenius automorphism. In particular, if H_1 and H_2 are the two parabolic subgroups of H containing a fixed Borel subgroup of H, then, without loss of generality, H_1 fixes a 1-space and $O_2(H_2)$ centralizes a 4-space one which $H_2/O_2(H_2)$ acts as an orthogonal module.

Lemma 2.9. *Let $X = \mathrm{Aut}(\mathrm{M}_{22})$, $Y = X'$ and V be the irreducible 10-dimensional Todd module for X over* GF(2).

(i) *If $x \in Y$ is an involution, then $|C_V(x)| = 2^6$.*
(ii) *Assume that $M \leq X$ with $M \approx 2^4.\mathrm{Sym}(5)$ and $L = O_2(M)$, then L is elementary abelian of order 16 and $|C_V(L)| = 4$.*
(iii) *Assume that $M \leq X$ with $M \approx 2^4.\mathrm{Alt}(6)$ and $L = O_2(M)$, then L is elementary abelian of order 16, and $|C_V(L)| = 2^5$.*
(iv) *If $x \in X \setminus Y$ centralizes $M \approx 2^3.\mathrm{L}_3(2)$, then $|C_V(x)| = 2^7$ and involves two nontrivial $\mathrm{L}_3(2)$-modules.*

Proof. From the [9, Table 5.3 c], we have that there is just one class of involutions in $Y = \mathrm{M}_{22}$. Let v be some vector in V such that $|v^X| = 22$. Then v is centralized by a subgroup $H \cong \mathrm{L}_3(4)$ and $V/\langle v\rangle$ is the Todd module [1, (22.2) and (22.3.1)]. Hence, by [1, (22.2.1)], there is a parabolic subgroup $H_1 \le H$ fixing a 1-space in $V/\langle v\rangle$ such that, setting $E = O_2(H_1)$, we have $H_1/E \cong \mathrm{SL}_2(4)$ and E is elementary abelian of order 2^4 admitting H_1/E as $\mathrm{SL}_2(4)$. It follows that $|C_V(E)| = 4$. Choose an involution $x \in H_1 \setminus E$, then x inverts some element ω of order 5 with $|[V,\omega]| = 2^8$. Further $[C_V(\omega), x] = 1$. This shows $|C_V(x)| = 2^6$ and proves (i).

Let $H_2 \le H$ be the companion parabolic subgroup to H_1, then, setting $E_2 = O_2(H_2)$, we have $C_{V/\langle v\rangle}(E_2)$ has dimension 4. and it follows that $C_V(E_2)$ has dimension 5.

In Y there is a subgroup $M \approx 2^4.\mathrm{Alt}(6)$ with $L = O_2(M)$ elementary abelian of order 16. As the orbits of Y on V have length 22, 231 and 770, we see that M has no fixed point on V. Hence E is not normalized by M. Hence $N_X(E) \approx 2^4 : \mathrm{Sym}(5)$ and we have (ii). Furthermore E_1 is normalized by M and so E_1 has to centralize the preimage of $C_{V/\langle v\rangle}(E_1)$ and we have (iii).

Now let $x \in X \setminus Y$ be an involution, which centralizes $U \approx 2^3.\mathrm{L}_3(2)$ in Y. As just elements from the orbit v^Y are centralized by an element ν of order 7, we see that $|C_V(\nu)| = 2$ and so V involves three nontrivial $\mathrm{L}_3(2)$-modules. As U is not a subgroup of $\mathrm{L}_3(4)$, we see that $C_V(U) = 1$. In particular $\mathrm{L}_3(2)$ acts nontrivially on $[V,x]$. This now shows that $|[V,x]| = 8$ or 16. In the second case we have that $|C_V(x)/[V,x]| = 4$ and so is centralized by an element of order 7, a contradiction. This shows (iv). □

Our next lemma of this section requires the following transfer theorem.

Theorem 2.10. *Let M be a subgroup of a finite group G with $G = O^2(G)$, $|G : M|$ odd and $M > O^2(M)M'$. Suppose that E is an elementary abelian subgroup of a Sylow 2-subgroup T of M such that E is weakly closed in T and $N_G(E) \le M$. Let T_1 be a maximal subgroup of T with $|M : O^2(M)T_1| = 2$. Then there exists $g \in G \setminus M$ such that $|E^g : E^g \cap M| \le 2$ and $E^g \cap M \not\le O^2(M)T_1$.*

Proof. This is [21, Theorem 2.11 (i)]. □

Lemma 2.11. *Suppose that G is a group, M is a 2-local subgroup of G with $F^*(M) = O_2(M)$. Assume that $M/O_2(M) \cong \mathrm{Aut}(\mathrm{M}_{22})$, $O_2(M)$ is elementary abelian of order 2^{10} and $O_2(M)$ is the Todd module for $M/O_2(M)$. Then*

(i) *For involutions x in $M \setminus O^2(M)$, the 2-rank of $C_M(x)$ is at most 8; and*
(ii) *G has a subgroup of index 2.*

Proof. Let $E = O_2(M)$, $X = M/E$ and $Y = X'$. From [9, Table 5.3 c] we see that X has exactly two conjugacy classes of involutions not in Y one with centralizer of shape $2 \times 2^3 : \mathrm{L}_3(2)$ and the other with centralizer $2 \times 2^4 : (5 : 4)$. Also by [9, Table 5.3 c], the normalizer of a Sylow 11-subgroup of Y has order 55. Hence one class of involutions in $X \setminus Y$ contains elements which normalize, and consequently invert, a Sylow 11-subgroup. Furthermore, such an involution commutes with an element of order 5.

Aiming for a contradiction, let $x \in N_G(E)$ with $Ex \not\in X$ and $F \le C_M(x)$ with F is elementary abelian of order at least 2^9. Since the 2-rank of X is 5, we have $|C_E(F)| \ge 2^4$.

If Ex inverts an element of order 11 in X, then $|C_E(x)| = 2^5$ and $C_X(Ex) \cong 2 \times (2^4 : (5 : 4))$. Let $L = O_2(C_Y(Ex))$. By Lemma 2.9 (ii), we have that $|C_E(L)| \le 2^2$. Since the involutions which invert an element of order 5 in $C_X(Ex)$ can only centralize 2^3 in $C_E(x)$, we infer that $FE/E \le L$. If F centralizes $C_E(x)$ then the normal closure of FE/E in $C_{M/E}(Ex)$ also is abelian and so we may assume that $FE/E = L$ in this case. On the other hand, if F does not centralize $C_E(x)$, then $|FE/E| \ge 2^5$ and we also have $FE/E = L$. Hence in any case $FE/E = L$. However this implies that $|F| \le 2^7$ as $|C_E(L)| \le 4$ and is a contradiction. Hence F contains no such involutions.

So we have $C_X(Ex) \cong 2 \times 2^3 : \mathrm{L}_3(2)$. Let $L = O_2(C_Y(Ex))$ and $L_1 \le C_X(Ex)$ be such that $L_1 \cong \mathrm{L}_3(2)$. Let $e \in L_1$ be an involution. Then Le contains representatives of two LL_1-conjugacy classes of involutions. As x is not 2-central in X, we have that $x \sim_X x\ell$ for some $1 \ne \ell \in L$. It follows that all the involutions in Lx are conjugate to x in X. Hence we see that the coset Lex contains an involution which is not conjugate to x in X.

Assume that $(F\cap T)E/E \not\le L$. Let $e \in FE/E\cap L_1L\setminus L$. If $|(FE/E)\cap L| > 2$ then $(FE/E\cap L)ex$ is the set of involutions in Lex. But this coset contains an involution which inverts an element of order 11 and we have already seen that such elements cannot be in F. So $|(FE/E) \cap L| \le 2$ and consequently $|FE/E| \le 16$. By Lemma 2.9 (iv), $|C_E(x)| = 2^7$ and, for $e \in FE/E\setminus L\langle Ex\rangle$, as $C_E(x)$ has two non-trivial 3-dimensional composition factors for L_1, $|C_E(x) : C_{C_E(x)}(e)| \ge 4$. Therefore $|C_E(F)| = 2^5$ and $|FE/E| = 2^4$. In L_1 there are two conjugacy classes of fours groups. One which is contained in an elementary abelian group of order 2^5 in M/E and one which is contained in a conjugate of $O_2(C_{M/E}(x))$. If

FE/E is contained in an elementary abelian group F_1 of order 2^5 in $\mathrm{Aut}(\mathrm{M}_{22})$, then, as $|C_E(F)| = 2^5$, we get that $|C_E(F_1)| \geq 2^3$, which contradicts Lemma 2.9 (ii). Therefore FE/E is uniquely determined and is conjugate to $\langle L, Ex\rangle$ in M/E. In particular $|C_E(\langle L, Ex\rangle)| = 2^5$. But then L_1 cannot induce two non-trivial irreducible modules in $C_E(x)$, which contradicts Lemma 2.9(iv).

Suppose that $w \in Lx$ and let $L_w = O_2(C_Y(w))$. We have that $C_{LL_1}(w)/L$ is a parabolic subgroup of LL_1/L. Therefore LL_w has order 2^5 and consequently $L \cap L_w$ has order 2. Now we have $(F \cap Y)E/E \cap L \cap L_w$ which means that $|FE/E| \leq 2^2$ and $|C_E(F)| \geq 2^7$. Using Lemma 2.9 , for $f \in O^2(M) \setminus E$, we have that $|C_E(f)| = 2^6$. Hence $|FE/E| = 2$ and $|C_E(F)| = 2^8$ contrary to Lemma 2.9 (iv). This proves (i).

We recall that V is not a failure of factorization module for X. Thus, for $S \in \mathrm{Syl}_2(M)$, $E = J(S)$ and hence E is weakly closed in S with respect to G. In particular, as $M = N_G(E)$, $S \in \mathrm{Syl}_2(G)$ and M has odd index in G. Therefore (ii) follows from Theorem 2.10 and part (i). □

Lemma 2.12. *Suppose that G is a group, E is an extraspecial subgroup of G, $H = N_G(E) = N_G(Z(E))$, $C_G(E) = Z(E)$ and $S \in \mathrm{Syl}_p(H) \subseteq \mathrm{Syl}_p(G)$. Assume that if $g \in G$ and $Z^g \leq E$ then every element of Z^gZ is conjugate to an element of Z and assume that no element of $S \setminus E$ centralizes a subgroup of index p in E. Then, for all $d \in E$ with $d^G \cap Z = \emptyset$, $\mathrm{Syl}_p(C_H(d)) \subseteq \mathrm{Syl}_p(C_G(d))$ and $d^G \cap E = d^H$.*

Proof. Assume that $d \in E$ is not G-conjugate to an element of Z. Let $T \in \mathrm{Syl}_p(C_G(d))$. Then $Z(T)$ centralizes $C_E(d)$ which has index p in E. Thus $Z(T) \leq E$ and so $Z(T) = Z(C_E(d)) = \langle d\rangle Z$. In particular, Z is the unique G-conjugate of Z contained in $\langle d\rangle Z$. Therefore $N_G(T) \leq H$ and consequently $T \in \mathrm{Syl}_p(C_G(d))$.

Now assume that $e = d^g \in d^G \cap E$ and let $R \in \mathrm{Syl}_p(C_H(e))$. Then, as $T^g \in \mathrm{Syl}_p(C_G(e))$, there exists $h \in C_G(e)$ such that $T^{gw} = R$. But then $Z\langle s\rangle^{gw} = Z\langle e\rangle$ and as Z is the unique conjugate of Z in $Z\langle e\rangle$ we conclude that $Z^{gw} = Z$. Thus $gw \in H$ and $d^{gw} = e^w = e$. Thus $d^G \cap E = d^H$ as claimed. □

Lemma 2.13. *Suppose that p is a prime, G is a group and $P \in \mathrm{Syl}_p(G)$. Assume that $J = J(P)$ is the Thompson subgroup of P. Assume that J is elementary abelian. Then*

(i) *$N_G(J)$ controls G-fusion in J; and*
(ii) *if $J \not\leq N_G(J)'$, then $J \not\leq G'$.*

Proof. Part (i) is well-known see [1, 37.6]. Part (ii) is proved in [16, Lemma 2.2(iii)]. □

The next lemma is a straightforward consequence of Goldschmidt's Theorem on groups with a strongly closed abelian subgroup [6]. Recall that for subgroups $A \le H \le G$, we say that A is *weakly closed* in H with respect to G provided that for $g \in G$, $A^g \le H$ implies that $A^g = A$. We say that A is *strongly closed* in H with respect to G so long as, for all $g \in G$, $A^g \cap H \le A$.

Lemma 2.14. *Suppose that K is a group, $O_{2'}(K) = 1$, E is an abelian 2-subgroup of K and E is strongly closed in $N_K(E)$. Assume that $F^*(N_K(E)/C_K(E))$ is a non-abelian simple group. Then $K = N_K(E)$.*

Proof. See [17, Lemma 2.15]. □

We will also need the following statement of Holt's Theorem [10].

Lemma 2.15. *Suppose that K is a simple group, P is a proper subgroup of K and r is a 2-central element of K. If $r^K \cap P = r^P$ and $C_K(r) \le P$, then $K \cong \mathrm{PSL}_2(2^a)$ ($a \ge 2$), $\mathrm{PSU}_3(2^a)$ ($a \ge 2$), ${}^2\mathrm{B}_2(2^a)$ ($a \ge 3$ and odd) or $\mathrm{Alt}(n)$ where in the first three cases P is a Borel subgroup of K and in the last case $P \cong \mathrm{Alt}(n-1)$.*

Proof. This is [17, Lemma 2.16]. □

Definition 2.16. *We say that X is similar to a 3-centralizer in a group of type $\mathrm{U}_6(2)$ or $\mathrm{F}_4(2)$ provided the following conditions hold.*

(i) *$Q = F^*(X)$ is extraspecial of order 3^5; and*
(ii) *X/Q contains a normal subgroup isomorphic to $\mathrm{Q}_8 \times \mathrm{Q}_8$.*

The main theorems of [16, 17] combine to give the following result which is also recorded in [17].

Theorem 2.17. *Suppose that G is a group, $Z \le G$ has order 3 and set $M = C_G(Z)$. If M is similar to a 3-centralizer of a group of type $\mathrm{U}_6(2)$ or $\mathrm{F}_4(2)$ and Z is not weakly closed in a Sylow 3-subgroup of G with respect to G, then either $F^*(G) \cong \mathrm{U}_6(2)$ or $F^*(G) \cong \mathrm{F}_4(2)$. Furthermore, if $F^*(G) \cong \mathrm{U}_6(2)$, then Z is weakly closed in $O_3(M)$ with respect to G and if $F^*(G) \cong \mathrm{F}_4(2)$, then Z is not weakly closed in $O_3(M)$ with respect to G.*

Definition 2.18. *We say that X is similar to a 3-centralizer in a group of type $\mathrm{Aut}(\Omega_8^+(2))$ provided the following conditions hold.*

(i) *$Q = F^*(X)$ is extraspecial of order 3^5;*
(ii) *$X/Q \cong \mathrm{SL}_2(3)$ or $\mathrm{SL}_2(3) \times 2$;*
(iii) *$[Q, O_{3,2}(X)]$ has order 27.*

Theorem 2.19 (Astill [4]). *Suppose that G is a group, $Z \leq G$ has order 3 and set $M = C_G(Z)$. If M is similar to a 3-centralizer of a group of type $\mathrm{Aut}(\Omega_8^+(2))$ and Z is not weakly closed in $O_3(C_G(Z))$ with respect to G, then either $G \cong \Omega_8^+(2):3$ or $F^*(G) \cong \mathrm{Aut}(\Omega_8^+(2))$.*

3. Strong closure

The main result of this section will be used in the final determination of the centralizer of an involution in ${}^2\mathrm{E}_6(2)$. Remember that for a prime p and a group X a subgroup Y of order divisible by p is *strongly p-embedded* in X so long as $Y \cap Y^g$ has order coprime to p for all $g \in X \backslash Y$.

Lemma 3.1. *Suppose that p is a prime, X is a group and H is strongly p-embedded in X. If $x \in H$, $y \in x^X \cap H$ and p divides both $|C_H(x)|$ and $|C_H(y)|$, then $y \in x^H$.*

Proof. Since H is strongly p-embedded in X and p divides $|C_H(x)|$, $C_H(x)$ contains a Sylow p-subgroup P of $C_X(x)$. Let $g \in X$ be such that $y^g = x$. Since p divides $|C_H(y)|$ there is an element $d \in C_H(y)$ of order p. Then d^g is a p-element of $C_H(x)$ and hence there exists an element $w \in C_G(x)$ such that $d^{gw} \in P$. Then, as H controls p-fusion in X ([8, Prop. 17.11]), there exists $h \in H$ such that $d = d^{gwh}$. As H is strongly p-embedded in G, we now have $gwh \in C_X(d) \leq H$. Hence $gw \in H$, and

$$y^{gw} = x^w = x$$

as claimed. □

Lemma 3.2. *Suppose that X is a group, $H = N_X(A)$ with $H/A \cong \mathrm{U}_6(2)$ or $\mathrm{U}_6(2):2$, $|A| = 2^{20}$ and A a minimal normal subgroup of H. Then $C_H(x)$ contains a Sylow 2-subgroup of $C_X(x)$ for all $x \in A$.*

Proof. Let $S \in \mathrm{Syl}_2(C_X(x))$ with $S \cap H \in \mathrm{Syl}_2(C_H(x))$. As, by Proposition 2.7 (i), A is not a failure of factorization module for H/A, we have $A = J(S \cap H)$ from [8, Lemma 26.7]. In particular, we have $N_S(S \cap H) \leq N_G(J(S \cap H)) = H$. Hence $S = S \cap H$. □

We can now prove Theorem 1.5 which we restate for the convenience of the reader.

Theorem 3.3. *Suppose that X is a group, $O_{2'}(X) = 1$, $H = N_X(A) = AK$ with $H/A \cong K \cong \mathrm{U}_6(2)$ or $\mathrm{U}_6(2):2$, $|A| = 2^{20}$ and A a minimal normal subgroup of H. Then H is not a strongly 3-embedded subgroup of X.*

Proof. Suppose that H is strongly 3-embedded in X. Let $S \in \mathrm{Syl}_2(H)$. Then Lemma 3.2 yields $S \in \mathrm{Syl}_2(X)$. We now claim that A is strongly closed in H with respect to X. Assume that, on the contrary, there is $u \in A$, $g \in X$ and $v \in H \setminus A$ with $v^g = u$. If 3 divides both $|C_H(u)|$ and $|C_H(v)|$, then u and v are H-conjugate by Lemma 3.1. Since A is normal in H, this is impossible. Therefore, as $H = AK$ is a split extension, Proposition 2.5 and Lemma 2.6 together, imply that there is a unique possibility for the conjugacy class of v in H and $C_S(v)A/A$ has index 2 in S/A. In addition, we have $|C_A(v)| = 2^{12}$.

Since $v \in A^{g^{-1}}$, there exists a Sylow 2-subgroup T of $C_X(v)$ which contains both $C_S(v)$ and a conjugate of A which contains v. Let $A_v = J(T)$. If $C_A(v) \leq A_v$, then, as $[A, v] \leq C_A(v)$, $\langle A, A_v\rangle$ normalizes $\langle v, A \cap A_v\rangle$. Because A is the Thompson subgroup of any 2-group which contains A, A and A_v are conjugate in $\langle A, A_v\rangle$. But A does not centralize $\langle v, A_v \cap A\rangle$ while A_v does, which is a contradiction. Thus $C_A(v) \not\leq A_v$.

We have $(A_v \cap C_S(v))A/A$ is an elementary abelian normal subgroup of $C_S(v)A/A$ and, as $(A_v \cap C_S(v))A/A$ only contains elements which are conjugate to Av, we have $|(A_v \cap C_S(v))A/A| \leq 4$ from Lemma 2.8. Combining this with the fact that $A_v \cap C_S(v) \cap A < C_A(v)$, we deduce that $|A_v \cap C_S(v)| \leq 2^{13}$. In particular we have that $|T : A_vC_S(v)| \leq 4$. Now using Lemma 3.2 and Proposition 2.5 we see that v is $H^{g^{-1}}$-conjugate to an element in A_v in class v_1 or v_2 (using the notation as in Proposition 2.5). Furthermore, v is a singular element. Suppose that v is conjugate to v_2. Then $|T : A_vC_S(v)| = 4$ and so $|A_v \cap C_S(v)| = 2^{13}$. But any subgroup of A_v of order 2^{13} is generated by non-singular vectors, and as we have seen such elements are not conjugate to elements in $H \setminus A$, a contradiction. So we have that v is conjugate to v_1. Now let T be a Sylow 2-subgroup of $C_X(v)$, which contains $A_vC_S(v)$. Then $T \in \mathrm{Syl}_2(X)$ by Lemma 3.2. Once again, as $A_v \cap C_S(v)$ is not generated by non-singular vectors, we get that $|A_v \cap C_S(v)| \leq 2^{12}$ and so $|T : A_vC_S(v)| \leq 2$. Further we have $|C_S(v) \cap A_v| \geq 2^{11}$. Therefore, as there are only 891 conjugates of v in A_v, $|(A_v \cap C_S(v)) \setminus A| \leq 891$. It follows that $|A \cap A_v| \leq 2^9$. Since $|(C_S(v) \cap A_v)A/A| \leq 2^2$, we get $|A \cap A_v| = 2^9$ and $|C_S(v) \cap A_v| = 2^{11}$. But then $891 \geq |(A_v \cap C_S(v)) \setminus A| = 1536$ which is a contradiction. Hence A is strongly closed in H.

Since A is strongly closed in H and $O_{2'}(X) = 1$, we now have that $X = H$ by Lemma 2.14 and this is impossible as H is strongly 3-embedded. This completes the proof of the theorem. □

4. The Structure of H

From here on we assume that G satisfies the hypothesis of Theorem 1.3 or Theorem 1.4. We let $H \leq G$ be a subgroup of G which is similar to the 3-centralizer in a group of type ${}^2\mathrm{E}_6(2)$ or M(22). We let $Z = Z(O_3(F^*(H)))$ and assume that $H = C_G(Z)$.

We will use the following notation $Q = O_3(H)$, $S \in \mathrm{Syl}_3(H)$ and $Z = \langle z\rangle = Z(S)$. We select $R \in \mathrm{Syl}_2(O_{3,2}(H))$ such that $S = N_S(R)Q$. Then R is isomorphic to a subgroup of $\mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8$ containing the centre of this group and of order 2^7 when H has type M(22) and order 2^9 when H has type ${}^2\mathrm{E}_6(2)$. Note that $\Omega_1(R)$ is elementary abelian of order 2^3. For $i = 1, 2, 3$, let $\langle r_i\rangle \le \Omega_1(R)$ be chosen so that $C_Q(r_i)$ is extraspecial of order 3^5. We set, for $i = 1, 2, 3$, $Q_i = [Q, r_i]$ and note that Q_i is extraspecial of order 3^3.

If $|R| = 2^9$, we let R_1, R_2 and R_3 be the three normal subgroups of R which are isomorphic to Q_8 such that $[R_i, Q] = Q_i$. Notice that we have $Z(R_i) = \langle r_i\rangle$ in this case. Further we set $B = C_S(\Omega_1(Z(R)))$.

Lemma 4.1. *We have $Q_1 \cong Q_2 \cong Q_3 \cong 3^{1+2}_+$ and that pairwise these subgroups commute.*

Proof. This follows from the Three Subgroup Lemma and the definitions of r_i and Q_i. □

Since each Q_i has exponent 3, Q has exponent 3 and so $\mathrm{Out}(Q) \cong \mathrm{GSp}_6(3)$. For later calculations, for each $i = 1, 2, 3$, we select $q_i, \widetilde{q_i} \in Q_i$ such that $[q_i, S] \le Z$

$$q_i^{r_i} = q_i^{-1},\ \widetilde{q_i}^{r_i} = \widetilde{q_i}^{-1} \text{ and } [q_i, \widetilde{q_i}] = z.$$

We set $\overline{H} = H/Q$. Then the following lemma follows from the structure of $\mathrm{GSp}_6(3)$ and the definition of the 3-centralizers in groups of type M(22) and ${}^2\mathrm{E}_6(2)$.

Lemma 4.2. *We have $\overline{R}$ is normal in $\overline{H}$ and, in particular, $\overline{H}$ is isomorphic to a subgroup of $\mathrm{Sp}_2(3) \wr \mathrm{Sym}(3)$ preserving the symplectic form.*

Proof. This follows from the definition of H. Note also that $\overline{H}$ preserves the "perpendicular" decomposition of Q as the central product of Q_1, Q_2 and Q_3. □

If the Sylow 3-subgroup S of H equal Q, then, as Z is not weakly closed in S by hypothesis, there exists $g \in G$ such that $Z^g \le S = Q$ and $Z \ne Z^g$. Now $C_S(Z^g) \cong 3 \times 3^{1+4}_+$ and so $C_Q(Z^g)' = Z$. However, $C_G(Z^g)$ is 3-closed with Sylow 3-subgroup Q^g and derived subgroup Z^g. Therefore we have

Lemma 4.3. *$S > Q$.*

We draw further information about the structure of $\overline{S}$ from Lemma 4.2.

Lemma 4.4. *The following hold:*

(i) $\overline{S}$ *is isomorphic to a subgroup of* $3 \wr 3$ *and* $|S : BQ| \leq 3$*;*
(ii) *if* $x \in S \setminus BQ$ *has order* 3*, then* $|C_{Q/Z}(x)| = 9$*,* $|[Q/Z, x]| = 3^4$ *and the preimage of* $C_{Q/Z}(x)$ *is equal to the centre of* $[Q, x]$*;*
(iii) *if* $x \in BQ$*, then* $|C_{Q/Z}(x)| \geq 3^3$*;*
(iv) *if* $\overline{S}$ *contains* $\overline{E}$ *of order* 9 *with* $\overline{S} = \overline{EB}$*, then* $|C_{Q/Z}(\overline{E})| = 3$*; and*
(v) *if* $\overline{F} \leq \overline{S}$ *is elementary abelian of order* 27*, then* $\overline{F} = \overline{B}$*.*

Proof. Lemma 4.2 (i) implies that $\overline{S}$ is isomorphic to a subgroup of the wreath product $3 \wr 3$ and, as by design, $\overline{B}$ is the intersection of $\overline{S}$ with the base group of this group, (i) holds.

Assume that $x \in S \setminus BQ$. Since $x \notin BQ$, x permutes the set $\{Q_1, Q_2, Q_3\}$ transitively and therefore Q/Z is a sum of two regular representations of $\langle x \rangle$. It follows that $[Q/Z, x]$ has order 81, $|C_{Q/Z}(x)|$ has order 9 and $C_{Q/Z}(x) = [Q/Z, x, x]$. Let J be the preimage of $C_{Q/Z}(x)$. Then $[J, x, Q] = 1$ and $[J, Q, x] = 1$. Hence the Three Subgroup Lemma implies that $J \leq Z([Q, x])$ and as Q is extraspecial, equality follows.

Part(iii) follows from the fact that BQ normalizes each Q_i, $1 \leq i \leq 3$.

For part (iv), we have $\overline{E}$ contains an element which acts nontrivially on each of Q_i, $i = 1, 2, 3$, and a further element which permutes the Q_i transitively. So the result follows.

Finally (v) follows from (i) as $3 \wr 3$ contains a unique elementary abelian subgroup of order 27.

□

The next lemma shows that Z is not weakly closed in Q. As we will see this is not an immediate observation.

Lemma 4.5. *Z is not weakly closed in Q with respect to G.*

Proof. Assume that Z is weakly closed in Q. By hypothesis we have that Z is not weakly closed in S with respect to G. Hence there exists $g \in G$ such that $Y = Z^g \leq S$ and $Y \not\leq Q$.

(4.5.1) We have $Y \leq BQ$.

Suppose that $Y \not\leq BQ$. Then, by Lemma 4.2, $\overline{Y}$ permutes the set $\{Q_1, Q_2, Q_3\}$ transitively and $\overline{Y}$ centralizes $\overline{f} = \overline{r_1 r_2 r_3}$ which has order 2. Furthermore by Lemma 4.4 (ii), $[Q/Z, Y]/C_{Q/Z}(Y)$ and $C_{Q/Z}(Y)$ have order 9. In particular, every element of order 3 in Qz^g is conjugate to an element of Zz^g. Therefore, as Z normalizes R, we may assume that Y normalizes R and so we can further assume that $f = r_1r_2r_3 \in C_R(Y)$.

Let J be the preimage of $C_{Q/Z}(Y)$ and set $E = [J, f]$. Then, as J is abelian by Lemma 4.4 (ii), E has order 9 and is centralized by Y. Hence $J = C_Q(Y) = ZE$. Furthermore, Lemma 4.4 (ii) shows that $[Q, Y] = C_Q(E)$. Since $[Y, f] = 1$ and $[C_Q(E), f] = [Q, Y, f] = [Q, Y]$, the Three Subgroup Lemma (to get the second equality) implies

$$[Q, Y, Y] = [C_Q(E), f, Y] = [C_Q(E), Y, f] = [Q, Y, Y, f] = E.$$

In particular, if $y = z^g$, then every element of the coset Ey is conjugate to z. Hence $Ey \cap Q^g \subseteq \{y, y^{-1}\}$ as $y^G \cap Q^g \subseteq \{y, y^{-1}\}$. Thus $E \cap Q^g = 1$. As f inverts $Q \cap Q^g$ we have that $Q \cap Q^g \leq E$ and so $Q \cap Q^g = 1$. Since $ZE \leq C_G(Y)$, we now have ZEQ^g/Q^g is elementary abelian of order 3^3. It follows from Lemma 4.4 (v) that Z centralizes $\overline{\Omega_1(R^g)Q^g/Q^g}$. Hence $|C_{Q^g/Y}(Z)| \geq 3^3$ by Lemma 4.4(iii). Now we have that $|\overline{C_{Q^g}(Z)}| \geq 3^3$. Since $\overline{Y}$ centralizes $\overline{C_{Q^g}(Z)}$ this is impossible. Hence (4.5.1) holds. ■

Reiterating the statement of (4.5.1), we have $z^G \cap H \subseteq BRQ$.

(4.5.2) We have that $C_Q(Y)$ does not contain a subgroup F isomorphic to $3^2 \times 3^{1+2}_+$.

Suppose false and assume that F is such a subgroup. As $Z \not\leq Q^g$, we have that FQ^g/Q^g is isomorphic to 3^{1+2}_+. Since F centralizes $F \cap Q^g$ which has order 9, we have a contradiction to the fact that $|C_{Q^g/Y}(F)| = 3$, see Lemma 4.4 (iv). ■

(4.5.3) For $\{i, j\} \subset \{1, 2, 3\}$ with $i \neq j$, $[Y, Q_iQ_j] \not\leq Z$.

Assume that $[Y, Q_iQ_j] \leq Z$. Then $C_{Q/Z}(Y)$ has order 3^5 and, letting E_1 be its preimage, we have $E_1 \cong 3 \times 3^{1+4}_+$. If E_1 is centralized by Y, then E_1Q^g/Q^g must be elementary abelian and we have $Z \leq Q^g$ which is a contradiction. So suppose that $[Y, E_1] = Z$. Then $E_2 = C_{E_1}(Y) \cong 3^2 \times 3^{1+2}_+$. But this contradicts (4.5.2). ■

(4.5.4) If $E \leq C_Q(Y)$ with $|E| = 27$, then the non-trivial cyclic subgroups contained in EY but not in E are not all conjugate to Z.

Suppose that every non-trivial cyclic subgroup EY not contained in E is conjugate to Z. Then $E \cap Q^g = 1$ for otherwise $(E \cap Q^g)Y \leq Q^g$ contains a conjugate of Z. Thus (4.5.1) implies that $EY \leq B^{gh}Q^g$ for some appropriate $h \in H^g$. But then there is a subgroup $U \leq EY$,

$U \neq Y$ such that U is G-conjugate to Z and such that U centralizes $(Q_1Q_2)^{gh}/Y$. This violates (4.5.3). ■

(4.5.5) There are non-trivial cyclic subgroups of YZ which are not conjugate to Z. In particular, $C_Q(Y)/Z = C_{Q/Z}(Y)$.

Suppose that statement is false. Let the subgroups of order 3 in YZ be Y_1, Y_2, Y and Z. Then by assumption all these groups are G-conjugate to Z. Let $E = [Q,Y]Z$. Then the cyclic subgroups of EY not contained in E are $Y_1^Q \cup Y_2^Q \cup Y^Q$. Since $|E| \geq 27$ by (4.5.1) and (4.5.3) we have a contradiction to (4.5.4). Let C be the preimage of $C_{Q/Z}(Y)$. Then, as Y and Z are the only G-conjugates of Z in YZ, C centralizes Y and have $C = C_Q(Y)$. ■

(4.5.6) $C_Q(Y)$ is elementary abelian of order 81. In particular, for $i = 1,2,3$, $[Q_i,Y] \not\leq Z$.

Otherwise Y centralizes Q_1/Z say and then $C_Q(Y) \cong 3^2 \times 3_+^{1+2}$ by (4.5.5). Now (4.5.2) gives a contradiction. ■

Since $[Q,Y] = C_Q(Y)$, every subgroup of $[Q,Y]Y$ order 9 containing Z is Q-conjugate to YZ. As $[Q,Y]Y = C_{QY}(C_Q(Y))$ is normalized by $\Omega_1(R)$, we may suppose that $[\Omega_1(R), ZY] = 1$. From (4.5.6) we have $|C_{Q^g}(Z)/Y| = 3^3$ and so Thompson's $A \times B$ Lemma [8, Lemma 11.7] implies that $\Omega_1(R)$ is isomorphic to a subgroup of $\mathrm{GL}_3(3)$. Since all elementary abelian subgroups of order 2^3 in $\mathrm{GL}_3(3)$ contain the centre of $\mathrm{GL}_3(3)$, there exists $x \in \Omega_1(R)$ such that $C_{Q^g}(Z)/Y$ is inverted by x. Hence $C_{Q^g}(Z) = Y[C_{Q^g}(Z), x]$. Because $\overline{C_{Q^g}(Z)}$ normalizes, and is normalized by, $\Omega_1(R)$, we have

$$Q \geq [C_{Q^g}(Z), \Omega_1(R)] = [C_{Q^g}(Z), x].$$

Therefore $C_{Q^g}(Z)Q = YQ$ and $|C_{Q^g}(Z) \cap Q| = |Q \cap Q^g| = 3^3$.

Set $D = Q \cap Q^g$ and $U = ZDY$. Then U is elementary abelian of order 3^5. Let $P = \langle Q, Q^g \rangle$ and note that P normalizes U. Since Z is the only G-conjugate of Z in DZ and P does not normalize Z, we see that there are P-conjugates of Z which are not contained in DZ. Now conjugating by Q, we see that there are 28, 55 or 82 P-conjugates of Z in U. Since 7 and 41 do not divide $|\mathrm{GL}_5(3)|$, we have that there are exactly 55 P-conjugates of Z in U. Similarly, there are 55 P-conjugates of Y and so we infer that Z and Y are P-conjugate. Since DZ and DY each only have one G-conjugate of Z, we have that $U \setminus (DZ \cup DY)$ contains at most two elements which are not conjugate into Z. Since

Q does not normalize Y and does normalize DZ, there is a $u \in P$ with $(ZD)^u \not\subseteq DZ \cup DY$. Set $D_1 = D \cap (DZ)^u$. Then $|D_1| \geq 9$. Choose $x \in (DZ)^u \setminus (ZD \cup DY)$. Then in $\langle D_1, x\rangle$ there are nine subgroups of order three not in $ZD \cup DY$, in particular at least eight of them are conjugate to Z, which is not possible as Z^u is the only conjugate of Z in $(ZD)^u$. This contradiction finally proves that Z is not weakly closed in Q with respect to G.

□

Because of Lemma 4.5 we may and do assume that for some $g \in G$ we have $Y = Z^g \leq Q$ with $Y \neq Z$. Set $V = ZY$ and assume that Y is chosen so that $C_{Q^g}(Z) \leq S$. Set $P = \langle Q, Q^g\rangle$ and $W = C_Q(Y)C_{Q^g}(Z)$.

Lemma 4.6. *The following hold:*

(i) $V \leq Q \cap Q^g$;

(ii) $Q \cap Q^g$ *is normal in* P *and is elementary abelian;*

(iii) $[Q \cap Q^g, P] = V$;

(iv) $P/C_P(V) \cong \mathrm{SL}_2(3)$ *and there are exactly* 4 *conjugates of* Z *in* V*; and*

(v) $|N_G(Z) : H| = 2$.

Proof. We have $C_Q(Y) \cong 3 \times 3^{1+4}_+$ and so, as $C_Q(Y) \leq H^g$, the structure of $\overline{S}$ given in Lemma 4.4 (i) implies that $Z = C_Q(Y)' \leq Q^g$. Hence (i) holds. Since $[Q \cap Q^g, Q] = Z \leq V$ and $[Q \cap Q^g, Q^g] = Y \leq V$, the first part of (ii) and (iii) hold. Of course $\Phi(Q \cap Q^g) \leq Z \cap Y = 1$. Hence the second part of (ii) holds as well. Since $|V| = 3^2$, $[V, Q] = Z$ and $[V, Q^g] = Y$, we get (iv). Finally there is an element in P which inverts V, and so we have $|N_G(Z)/H| = 2$. □

Lemma 4.7. (i) W *is a normal subgroup of* P, $P/W \cong \mathrm{SL}_2(3)$ *and* $W = C_P(V)$;

(ii) $Q \cap Q^g$ *is a maximal abelian subgroup of* Q*, and* $W/(Q \cap Q^g)$ *is elementary abelian of order* 3^4 *which, as a* $P/C_P(V)$*-module, is a direct sum of two natural* $\mathrm{SL}_2(3)$*-modules;*

(iii) $WQ \not\leq BQ$, $\overline{W}$ *has order* 9 *and does not act quadratically on* Q/Z;

(iv) V *is the second centre of* S;

(v) $S = WQ$ *or* $\overline{S}$ *is extraspecial. Furthermore, if* $|R| = 2^7$*, then* $S = WQ$*; and*

(vi) $\overline{W}$ *is inverted by an involution* $t \in N_P(Z) \cap N_G(S)$ *which inverts* Z.

Proof. Since $C_Q(Y)$ normalizes $C_{Q^g}(Z)$, W is a subgroup of G. We have that $[Q, Y, C_{Q^g}(Z)] = [Z, C_{Q^g}(Z)] = 1$ and $[Y, C_{Q^g}(Z), Q] =$

1 and so $[Q, C_{Q^g}(Z), Y] = 1$ by the Three Subgroup Lemma. Thus $[Q, C_{Q^g}(Z)] \le C_Q(Y) \le W$. Hence $[W, Q] \le W$ and similarly $[W, Q^g] \le W$. So W is a normal subgroup of P. Furthermore, $[C_P(V), Q] \le C_Q(Y) \le W$ and $[C_P(V), Q^g] \le C_{Q^g}(Z) \le W$ and so P/W is a central extension of $P/C_P(V)$. Let T be a Sylow 2-subgroup of $O^3(P)$. Then as $O^3(P)/W$ is nilpotent, Q normalizes and does not centralize T. It follows that $P = WTQ$ and then the action of Q on T and the fact that $T/C_T(V) \cong \mathrm{Q}_8$ implies that $T \cong \mathrm{Q}_8$ and that $P/W \cong \mathrm{SL}_2(3)$, as by [11, Satz V.25.3] the Schur multiplier of a quaternion group is trivial. This proves (i).

Since $WQ = C_{Q^g}(Y)Q$ and $Y \le Q$, we have $\overline{W}$ is elementary abelian. Furthermore, as Q is extraspecial and as $Q \cap Q^g$ is elementary abelian by Lemma 4.6 (iii), $Q \cap Q^g$ has index at least 3^3 in Q^g. Because $C_{Q^g}(Y)$ has index 3 in Q^g, there is an integer a such that

$$3^2 \le |\overline{W}| = |WQ^g/Q^g| = 3^a \le 3^3.$$

Furthermore, we have that $W/(Q \cap Q^g) = C_Q(Y)C_{Q^g}(Z)/(Q \cap Q^g)$ has order 3^{2a} and is elementary abelian. If $C_{W/(Q\cap Q^g)}(Q) > C_Q(Y)/(Q \cap Q^g)$, then $C_{W/(Q\cap Q^g)}(Q) \cap C_{Q^g}(Z)/(Q \cap Q^g) > 1$ and is centralized by P. As P acts transitively on the subgroups of V of order 3, we get

$$C_{W/(Q\cap Q^g)}(Q) \cap C_{Q^g}(Z)/(Q \cap Q^g) \le Q/(Q \cap Q^g)$$

which is absurd. Hence $C_{W/(Q\cap Q^g)}(Q) = C_Q(Y)/(Q \cap Q^g)$. In particular, $C_{W/(Q\cap Q^g)}(P) = 1$ and $[W, Q](Q \cap Q^g)/(Q \cap Q^g)$ has order 3^a. Since Q acts quadratically on $W/(Q \cap Q^g)$, as a P/W-module, we have that $W/(Q \cap Q^g)$ is a direct sum of a natural $\mathrm{SL}_2(3)$-modules.

Assume that $|\overline{W}| = 3^3$. Then $WQ = BQ$ and so $|[Q/Z, W]| = |[Q/Z, B]| \le 3^3$. Since $[W, Q \cap Q^g] \ge [Q \cap Q^g, C_{Q^g}(Y)] = Y$ and $|[W/(Q \cap Q^g), Q]| = 3^a = 27$, we infer that $3^3 = |[Q/Z, W]| \ge 3^4$ which is a contradiction. This proves (ii).

Suppose that $WQ \le BQ$ (which is equivalent to W acting quadratically on Q/Z). Then $[Q, W]V/V \le Z(W/V)$ and as $(Q \cap Q^g)/V \le Z(W/V)$, we infer that $C_Q(Y)/V \le Z(W/V)$ and this means that W/V is abelian. Since W is generated by elements of order 3, we then have that W/V is elementary abelian. Letting t be an involution in P, we now have that $W_1 = [W, t]$ has order 3^6, is abelian and is normal in P. Now by (ii) W_1/V is a direct sum of two natural P/W-modules and so there are exactly four normal subgroups of P in W_1/V of order 3^2. Let U be such a subgroup. Then $[U, Q \cap Q^g] \le V$. By (ii) we have $C_Q(Q \cap Q^g) = Q \cap Q^g$ and so $[U, Q \cap Q^g] \ne 1$. As $[U, Q \cap Q^g]$ is normal in P we get $[U, Q \cap Q^g] = V$. Therefore $|[U, Q]/Z| = 3^2$. Now, as $WQ \le BQ$, WQ normalizes Q_1, Q_2 and Q_3, so , as $|[U, Q]/Z| = 3^2$, UQ centralizes exactly one of Q_1/Z, Q_2/Z and Q_3/Z. This is true for all four possibilities for U. Hence there exists two candidates for U centralizing Q_1/Z (say). Thus $\overline{W}$ centralizes Q_1/Z and we get $[Q/Z, W] = [Q_2Q_3/Z, W]$ has order 3^2. Since $|[Q/Z, W]| = 3^3$, this is a contradiction. Hence $W \not\leq BQ$ and W does not act quadratically on Q/Z. This proves (iii).

Since $W \not\leq BQ$ and $|\overline{W} \cap \overline{B}| \neq 1$, we see that $C_{Q/Z}(W) = V/Z$ by using Lemma 4.4 (iv). This then gives (iv).

Note that, by (iv), $S = C_S(Y)Q$ and so WQ is normalized by S. Since, by Lemma 4.4 (i), $\overline{S}$ is isomorphic to a subgroup of $3 \wr 3$ with $\overline{B}$ being the subgroup of $\overline{S}$ meeting the base group of the wreath product, the possibilities for $\overline{S}$ now follow as $\overline{W}$ is normalized by $\overline{S}$. In the case when $|R| = 2^7$, we have that $|R/Z(R)| = 2^4$ and so does not admit an extraspecial group of order 27. Hence in this case we get $\overline{S} = \overline{W}$ has order 9. This proves (v).

Finally we note that the involution t in a Sylow 2-subgroup of P inverts Z, normalizes S and also inverts $\overline{W}$. So (vi) holds. □

Lemma 4.8. *One of the following holds:*

(i) $|R| = 2^9$, $S = WQ$ *and either* $|H| = 2^9 \cdot 3^9$, $H = WRQ$ *and*

$$\overline{H} \approx (\mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8).3^2$$

or $|H| = 2^{10} \cdot 3^9$, $H/BRQ \cong \mathrm{Sym}(3)$ *and*

$$\overline{H} \approx (\mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8).3.\mathrm{Sym}(3);$$

(ii) $|R| = 2^9$, $\overline{S}$ *is extraspecial and either* $|H| = 2^9 \cdot 3^{10}$, $H = SR$

$$\overline{H} \approx (\mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8).3^{1+2}_+$$

or $|H| = 2^{10} \cdot 3^{10}$, $H/BRQ \cong \mathrm{Sym}(3)$ *and*

$$\overline{H} \approx (\mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8).3^{1+2}_+.2;$$

or

(iii) $|R| = 2^7$, $S = WQ$ *and either* $|H| = 2^7 \cdot 3^9$, $H = QRW$ *and*

$$\overline{H} \approx 2^7.3^2$$

or $|H| = 2^8 \cdot 3^9$, $H/BRQ \cong \mathrm{Sym}(3)$ *and*

$$\overline{H} \approx 2^7.3.\mathrm{Sym}(3).$$

Proof. This is a summary of things we have learnt in Lemma 4.7 combined with the fact that $\overline{H}$ embeds into $\mathrm{Sp}_2(3) \wr \mathrm{Sym}(3)$. □

We may now fill in the details of the structure of $N_G(Z)$ and while doing so establish some further notation which will be used throughout the remainder of the paper.

By Lemma 4.7 (i), $\overline{W}$ does not act quadratically on Q/Z. Thus $W \not\leq QB$. It follows that $N_S(R)$ contains an element w which permutes $\{Q_1, Q_2, Q_3\}$ transitively (w is a wreathing element). Furthermore, as $\overline{W}$ is abelian, $\overline{W} \cap \overline{B}$ contains an a cyclic subgroup which is centralized by wQ. We let x_{123} be the corresponding element in $N_S(R)$ (here the notation should remind the readers (and the authors) that x_{123} acts non-trivially on Q_1/Z, Q_2/Z and Q_3/Z and on $R_1/\langle r_1\rangle$, $R_2/\langle r_2\rangle$, $R_3/\langle r_3\rangle$. Since x_{123} centralizes $r_1r_2r_3$, it normalizes $\langle q_1, q_2, q_3\rangle$ and consequently

$$[x_{123}, \langle q_1, q_2, q_3\rangle] \leq \langle q_1, q_2, q_3\rangle \cap Z = 1.$$

Hence $x_{123} \in C_S(\langle q_1, q_2, q_3\rangle)$.

If $S > QW$, then $|\overline{B}|$ has order 9 and is normalized by w. Thus $N_S(R)$ contains an element $x_2x_3^{-1}$, which as with x_{123} centralizes $\langle q_1, q_2, q_3, Z\rangle$. Note that at this stage it may be that x_{123} and $x_2x_3^{-1}$ do not commute. We continue our investigations under the assumption that if $S = WQ$, then $x_2x_3^{-1}$ is the identity element and $J = J_0$.

Set $A = [Q, B] = \langle Z, q_1, q_2, q_3\rangle$,

$$J = C_{QW}(A) = \langle A, x_{123}\rangle$$

and

$$J_0 = C_S(A) = \langle A, x_{123}, x_2x_3^{-1}\rangle.$$

Lemma 4.9. (i) *$J = J(W)$ is the Thompson subgroup of W, $(Q\cap Q^g)J/(Q\cap Q^g)$ is a non-central P-chief factor and $A \neq Q\cap Q^g$;*

(ii) *if $S > QW$ then J_0 is elementary abelian and $\overline{J_0} = \overline{B}$;*

(iii) *x_{123} has order 3 and, if $S > QW$, $x_2x_3^{-1}$ also has order 3 and commutes with x_{123};*

(iv) *if $S = WQ$, then $J = J(S)$ and, if $S > WQ$, then $J_0 = J(S)$; and*

(v) *if $S > QW$, then $|J_0| = 3^6$ and $S = QWJ_0$.*

Proof. Because A has index 3 in J, J is abelian. As J centralizes V and $J \leq QW$, $J \leq C_{QW}(V) = W$. As, by Lemma 4.7 (ii), $W/(Q \cap Q^g)$ is a direct sum of two natural $\mathrm{SL}_2(3)$-modules, there is a normal subgroup W_0 of P such that $(Q \cap Q^g) \leq W_0 \leq W$ and

$$\overline{W}_0 = \overline{\langle x_{123}\rangle} \leq \overline{B}.$$

We have $|W_0 \cap Q : Q \cap Q^g| = 3$. Thus, as $Q \cap Q^g$ is a maximal abelian subgroup of Q by Lemma 4.7 (ii), $Z(W_0 \cap Q)$ has index 3 in $Q \cap Q^g$ and contains V. Hence $Z(W_0 \cap Q)$ is normal in P by Lemma 4.6 (iii) and this means that $Z(W_0) = Z(W_0) \cap Q$. From the definition of A and of W_0, we have $[A, W_0] \leq Z$. On the other hand, $Z(W_0) \leq C_Q(W_0) \leq A$. Thus W_0 centralizes a subgroup of A of index 3. It follows that W_0 induces a group of order 3 on A. Hence $C_{W_0}(A) = J$ and $W_0 = (Q \cap Q^g)J$. As $[W_0, Q \cap Q^g] = V$, W_0 is not abelian and hence J is a maximal abelian subgroup of W_0.

If $J^* \leq W_0$ is abelian with $|J^*| = |J|$ and $J \neq J^*$, then $W_0 = JJ^*$ and $Z(W_0) \geq J \cap J^*$. Since $Q \cap Q^g \not\leq Z(W_0)$ and $W_0/(Q \cap Q^g)$ is a P-chief factor, we get $W_0 = Z(W_0)(Q \cap Q^g)$ which means that W_0 is abelian and is a contradiction. Hence $J = J(W_0)$ is normal in P and, as $J = [J, Q][J, Q^g]$ is generated by elements of order 3, J is elementary abelian.

Since J contains a P-chief factor, we have $C_P(J) = C_W(J) = J$. Assume that $\widetilde{A}$ is an abelian subgroup of QW with $|\widetilde{A}| \geq |J| = 3^5$. If $\widetilde{A}Q \not\leq BQ$, then $|C_{Q/Z}(\widetilde{A})| \leq 3^2$ by Lemma 4.4 (ii). Hence $|\widetilde{A} \cap Q| \leq 3^3$ which means that $\widetilde{A}Q = WQ$ and so we have $|C_{Q/Z}(\widetilde{A})| = 3$ by Lemma 4.4 (iv). But then $\overline{W}$ has order greater than 9, a contradiction. So $\widetilde{A} \leq W_0Q$ and $|\widetilde{A} \cap Q| = 3^4$, it follows that $\widetilde{A} \cap Q = A$ and $\widetilde{A} \leq J$. Thus $J = J(WQ)$ and if $S = QW$ we even have $J = J(S)$. This completes the proof of (i) and shows that x_{123} has order 3. Since J does not centralize $Q \cap Q^g$, $A \neq Q \cap Q^g$.

Now we consider J_0 and suppose that $S > QW$. Then $S = J_0QW$. Because A is normalized by S, J_0 is a normal subgroup of S and $x_2x_3^{-1} \in J_0 \setminus J$. Set $A_1 = A \cap Q \cap Q^g$. Then, as $W_0 \cap Q = A(Q \cap Q^g)$, we have A_1 has order 3^3 and is centralized by W_0J_0. It follows that $W_0J_0 = C_S(A_1)$. Since A_1 is normalized by P by Lemma 4.6(iii) and $C_{PS}(A_1) \leq O_3(PS)$, we have J_0W_0 is normalized by PS and that J_0W_0/J is centralized by $O^3(P)$. As J_0 is normalized by S, we have that J_0 is a normal subgroup of PS. Employing the fact that $A \leq Z(J_0)$, yields $J = \langle A^P \rangle \leq Z(J_0)$. Hence J_0 is abelian. As J is elementary abelian, $\Phi(J_0)$ has order at most 3 and as P does not normalize Z we have J_0 is elementary abelian. This then implies that $x_2x_3^{-1}$ has order 3 and $[x_{123}, x_2x_3^{-1}] = 1$. Since $|J_0| = 3^6$, we also have that $J_0 = J(S)$ in this case. □

The next lemma just reiterates what we have discovered in Lemma 4.9 (iii).

Lemma 4.10. *$B = \langle x_{123}, x_2x_3^{-1}, z \rangle$ is elementary abelian.* □

Lemma 4.11. *The subgroup $N_G(J(S))$ controls G-fusion of elements in $J(S)$.*

Proof. This follows from lemma 2.13 (i) as $J(S)$ is elementary abelian. □

Lemma 4.12. *$N_G(Z)$ controls G-fusion of elements of order 3 in Q which are not conjugate to z. In particular, q_1, q_1q_2 and z represent distinct G-conjugacy classes of elements of Q.*

Proof. From Lemma 4.7 (iii) and (v) no element of $\overline{S}$ centralizes a subgroup of index 3 in Q. Furthermore, if $Z^g \le Q$, then all the elements of ZZ^g are G-conjugate to elements of Z by Lemma 4.6 (iv). Hence $N_G(Z)$ controls G-fusion of elements of order 3 in Q which are not conjugate to elements of z by Lemma 2.12.

By Lemma 4.7(iv) any conjugate of z in Q is in the second centre of some Sylow 3-subgroup of $N_G(Z)$ and so q_1 and q_1q_2 both are not conjugate to z in G. □

Lemma 4.13. *We have $N_H(J) = \Omega_1(Z(R))N_H(S)$.*

Proof. We know by direct calculation that $N_{\overline{H}}(\overline{J}) = \overline{\Omega_1(Z(R))}N_{\overline{H}}(\overline{S})$ and so the result follows.

□

Recall that, for $i = 1, 2, 3$, $Q_i = \langle q_i, \widetilde{q}_i\rangle$ where $[q_i, \widetilde{q}_i] = z$ are specifically defined. In the next lemma we give precise descriptions, some of which we have already seen, of a number of the key subgroups of Q.

Lemma 4.14. *The following hold:*

(i) $V = \langle z, q_1q_2q_3\rangle$;
(ii) $C_Q(V) = \langle A, \widetilde{q}_1\widetilde{q}_2^{-1}, \widetilde{q}_1\widetilde{q}_2\widetilde{q}_3\rangle$;
(iii) $A = \langle z, q_1, q_2, q_3\rangle$;
(iv) $A \cap Q^g = \langle V, q_1q_2^{-1}\rangle = \langle V, q_2q_3^{-1}\rangle$; *and*
(v) $Q \cap Q^g = \langle A \cap Q^g, \widetilde{q}_1\widetilde{q}_2\widetilde{q}_3\rangle$.

Proof. We have that V is centralized by W and $\overline{W} = \langle wQ, x_{123}Q\rangle$, hence (i) holds and (ii) follows from that. Part (iii) is the definition of A. Since $A \le C_Q(V) \le W$, $[A, W] = [A, w] \le Q^g$ and this gives (iv). Finally, since $[Q \cap Q^g, W] = V$ and so we get (v). □

Lemma 4.15. *A contains exactly 13 conjugates of Z and $A \cap Q^g$ contains exactly 4 G-conjugates of Z.*

Proof. Since the images of G-conjugates of Z contained in Q are 3-central in $N_G(Z)/Z$ by Lemma 4.7 (iv), the conjugates of Z in Q are $N_G(Z)$-conjugate to $\langle q_1q_2q_3\rangle$ by Lemma 4.12. Therefore, in $A = \langle z, q_1, q_2, q_3\rangle$ we have thirteen candidates for such subgroups and they are in the four groups

$$\langle Z, q_1q_2q_3\rangle\ , \langle Z, q_1q_2^{-1}q_3\rangle\ , \langle Z, q_1q_2q_3^{-1}\rangle \text{ and } \langle Z, q_1q_2^{-1}q_3^{-1}\rangle.$$

As all these groups are conjugate in $\Omega_1(R)Q$, we see that A contains exactly thirteen conjugates of Z. Now $A \cap Q^g = \langle z, q_1q_2^{-1}, q_2q_3^{-1}\rangle$ contains four conjugates of Z all of which are contained in V. □

Lemma 4.16. *J_0 contains exactly 40 subgroup which are G-conjugate to Z and they are all contained in J. In particular, $N_G(J) \geq N_G(J_0)$ and $|N_G(J)/J_0| = 2^{7+i}\cdot 3^4\cdot 5$ where i is such that $2^{i+2} = |N_G(S)/S| \leq 8$.*

Proof. By Lemma 4.15, we have that $A = J \cap Q$ contains exactly thirteen conjugates of Z and $J \cap Q^g \cap Q = A \cap Q \cap Q^g = \langle z, q_1q_2^{-1}, q_2q_3^{-1}\rangle$ contains exactly four conjugates of Z. We have that both J and $J \cap Q \cap Q^g$ are normal in P. As $J/(J \cap Q \cap Q^g)$ is a natural P-module by Lemma 4.7(ii), we see that $J = \cup_{x\in P}(J \cap Q)^x$ is a union of four conjugates of $J \cap Q$ pairwise meeting in $J \cap Q \cap Q^g$. This gives, using the inclusion exclusion principle and Lemma 4.12, that there are exactly $4\cdot 13 - 3\cdot 4 = 40$ conjugates of Z in J. In particular, $J_0 = \langle Z^g \mid Z^g \leq J_0\rangle$.

Suppose that $J_0 > J$. Then $|R| = 2^9$ and $\overline{S} \cong 3_+^{1+2}$. If $N_G(J_0)$ normalizes J then Lemma 4.11 delivers the result. So we may assume that $N_G(J)$ does not normalize J_0. Suppose that X is a subgroup of J of order 3 and that $X \not\leq J_0$. Then $\overline{X} \leq \overline{B}$ and $\overline{X} \neq \overline{J}_0$ is conjugate to $\overline{\langle x_2x_3^{-1}\rangle}$ and so we have that $C_Q(X)$ is conjugate to Q_1A which has order 3^5. Thus XA is normalized by Q, $|X^Q| = 3^2$ and, as $|(XQ)^S| = 3$, $|X^S| = 27$.

Hence, taking X to be a conjugate of Z, yields that there are $40+27i$ conjugates of Z contained in J_0 where $1 \leq i \leq 9$. If there is some non-trivial element of A which has all its G-conjugates contained in some proper subgroup of J, then we have that this subgroup is normal in $N_G(J_0) \geq S$ and so contains Z. But then Z is trapped in this subgroup, a contradiction. By Lemma 4.12 there are at least two G-conjugacy classes of cyclic subgroups different from Z in A and so there are at least 54 cyclic subgroups of J_0 not in J, which are not G-conjugate to Z. It follows that $i \leq 7$. Now the only non-zero i which has $40 + 27i$ dividing $|\mathrm{GL}_6(3)|$ is $i = 3$. This means that there are 121 conjugates of Z in J_0 and that $N_G(J_0)$ contains a cyclic group D of order 121. Let $J_1 \leq J$ have order 3^5 be normalized by D. Then D acts transitively on the cyclic subgroups of J_1 and consequently $J_1 \cap Q = J_1 \cap A$ which has order 27 has only one G-class of cyclic subgroups. As $Z \not\leq J_1 \cap A$, we get that $(J_1 \cap A)Z = A$. Now all elements of A not in Z are conjugate, which contradicts Lemma 4.15. Now we have that all the G-conjugates of Z in J_0 are contained in J. Thus $N_G(J_0) \leq N_G(J)$.

□

Lemma 4.17. *There are* 36 *conjugates of* $\langle q_1\rangle$ *in* J. *In particular,* $\langle q_1\rangle$ *is centralized by an element of order* 5 *in* $N_G(J_0)$

Proof. In $J \cap Q$, there are nine $N_H(J)$-conjugates of $\langle q_1\rangle$ (which are already conjugate in QW) and in $Q \cap Q^g \cap J$ there are none by Lemmas 4.12 and 4.14 (iv). Again as J is the union of the four P-conjugates of $J \cap Q$, we have $4 \cdot 9$ conjugates of $\langle q_1\rangle$ in J. Since, by Lemma 4.16, $|N_G(J_0)|$ is divisible by 5, we have that some element of order 5 in $N_G(J_0)$ centralizes $\langle q_1\rangle$. □

Lemma 4.18. $N_G(J)/J_0 \cong \Omega_5(3).2$ *or* $\Omega_5(3).2 \times 2$. *In particular,* r_1 *centralizes an element of order* 5 *in* $N_G(J)$.

Proof. Let $M = N_G(J)$, $\mathcal{P} = Z^M$ and $\mathcal{L} = V^M$. We call the elements of $\mathcal{P}$ points and those in $\mathcal{L}$ lines. For $X \in \mathcal{P}$ and $Y \in \mathcal{L}$, declare X and Y to be incident if and only if $X \le Y$. We claim the this makes $(\mathcal{P}, \mathcal{L})$ into a generalized quadrangle with parameters $(3, 3)$.

For $X = Z^m \in \mathcal{P}$, $m \in M$, we set $Q_x = O_3(C_G(x)) = Q^m$.

By Lemma 4.6 (iv), we have 4 points on each line. Suppose that $Z \le V^m \in \mathcal{L}$. Then either $Z^m = Z$ or $Z^m \ne Z$ and $Z \le Q^m$. In the first case $m \in H \cap M$ and $V^m \le J \cap Q_Z$ and, in the second case, we have $Z^m \le Q$ by Lemma 4.6 (i) and so $V^m \le Q_Z$ again. Thus, if $X \in \mathcal{P}$ is incident to a line $L \in \mathcal{L}$, then $L \le J \cap Q_X$.

By Lemma 4.15 there are twelve M-conjugates of Z in $(J \cap Q) \setminus Z$ and each of them forms a line with Z. Thus Z is contained in exactly 4 lines and, furthermore, any two lines containing Z meet in exactly Z and any two points determine exactly one line.

Now suppose that $L \in \mathcal{L}$ is a line which is not incident to $X \in \mathcal{P}$. Then, as $|J : J \cap Q_X| = 3$, we have $L \cap (J \cap Q_X)$ is a point and this is the unique point of L which is collinear to X. It follows that $(\mathcal{P}, \mathcal{L})$ is a generalized quadrangle with parameters $(3, 3)$. By [19] there is up to duality a unique such quadrangle. Hence we have that $N_G(J)/J_0$ induces a subgroup of $\Omega_5(3).2$ on the quadrangle. Using Lemma 4.16, we see that the full group is induced. As there might be some element which inverts J and so acts trivially on $(\mathcal{P}, \mathcal{L})$, we get the two possibilities as stated.

Finally, as r_1 acts as a reflection on J, we see that r_1 centralizes an element of order 5. □

Lemma 4.19. *We have* $F^*(C_{N_G(J)}(q_1)/J_0) \cong \mathrm{Alt}(6) \cong \Omega_4^-(3)$.

Proof. Because q_1 is inverted by r_1 and r_1 acts on J as a reflection, we have that $F^*(C_{N_G(J)}(q_1)/J_0)$ is an orthogonal group in dimension 4. Since, by Lemma 4.17, q_1 commutes with an element of order 5, we have $F^*(C_{N_G(J)}(q_1)/J_0) \cong \Omega_4^-(3) \cong \mathrm{Alt}(6)$. □

5. The Fischer group M(22) and its automorphism group

In this section we will assume that $|R| = 2^7$ and determine the isomorphism type of G. Set $r = r_1$ and $K = C_G(r)$. Recall that R is a subgroup of $R_1 \times R_2 \times R_3 \cong \mathrm{Q}_8 \times Q_8 \times \mathrm{Q}_8$ and $R \geq \langle r_1, r_2, r_3 \rangle = \Omega_1(Z(R))$.

Lemma 5.1. *We have that* $\Omega_1(Z(R)) \leq \Phi(R)$.

Proof. Assume that $\Omega_1(Z(R)) \not\leq \Phi(R)$. As w acts transitively on the set $\{r_1, r_2, r_3\}$, we may assume that $r_i \notin \Phi(R)$ for $1 \leq i \leq 3$. Let U be a hyperplane in $\Omega_1(Z(R))$ which contains $\Phi(R)$. Then, as w normalizes R, we may assume that $\{r_1, r_2, r_3\} \cap U = \emptyset$. An easy inspection of the maximal subgroups of $\Omega_1(Z(R))$ yields $U = \langle r_1r_2, r_2r_3 \rangle$. Therefore $(R_1 \times R_2 \times R_3)/U$ is an extraspecial group of order 2^7. We have that R/U is of order 2^5, hence R/U is not abelian. However $\Phi(R) \not\leq U$, which is a contradiction. □

Recall from Lemma 4.8 (iii), either $H = QRW$ or $H/BRQ \cong \mathrm{Sym}(3)$ and in either case $S = WQ$. If $H/BRQ \cong \mathrm{Sym}(3)$, then there is an element iRQ of order 2 which permutes Q_2 and Q_3 and centralizes r. We let $i \in H$ be such an element where for convenience we understand that $i = 1$ if $H = QRW$. Thus in any case $H = QRW\langle i \rangle$. By Lemma 4.7 (vi), $|N_G(Z) : H| = 2$ and $\overline{W}$ is inverted by an involution j in $N_G(Z) \cap N_G(S)$. Again, we can choose j to centralize $rQ \in HQ$ and consequently it can be further chosen to centralize r. Thus we have $N_K(Z) = Q_2Q_3RC_S(r)\langle i, j \rangle$ and this group has order $3^6 \cdot 2^9$.

Lemma 5.2. *Suppose that* $|R| = 2^7$. *Then* $K \cong 2{}^{\cdot}\mathrm{U}_6(2)$ *or* $2{}^{\cdot}\mathrm{U}_6(2).2$.

Proof. We have $N_K(Z) = Q_2Q_3RC_S(r)\langle i, j \rangle$. Since $Z(C_S(r)R/\langle r \rangle)$ acts faithfully on Q_2Q_3 and centralizes the fours group $\Omega_1(R)/\langle r \rangle$, we see that $N_K(Z)/\langle r \rangle$ when embedded into $\mathrm{GSp}_4(3)$ preserves the decomposition of the associated symplectic space into a perpendicular sum of two non-degenerate spaces and has $R/\langle r \rangle \cong \mathrm{Q}(8) \times \mathrm{Q}(8)$ as a normal subgroup. Therefore, as $Q_1Q_2 \cong F^*(N_K(Z)/\langle r \rangle)$ is extraspecial of order 3^5, we have $N_K(Z)/\langle r \rangle$ is similar to a normalizer in a group of $\mathrm{U}_6(2)$-type. By Lemma 4.12, no conjugate of Z is G-conjugate to an element of $Q_1Q_2 \setminus Z$ and so Z is weakly closed in Q_1Q_2 with respect to K. Since, by Lemma 4.18, $C_{N_G(J)}(r)$ has an element f of order 5, we have $Z^f \leq C_J(r)$ and, of course, $Z^f \neq Z$. It follows that $Z\langle r \rangle/\langle r \rangle$ is not weakly closed in $C_S(r)\langle r \rangle/\langle r \rangle$ with respect to $C_G(r)/\langle r \rangle$. Therefore, as $C_S(r)Q_2Q_3/Q_2Q_3$ has order 3, Theorem 2.17 implies that $C_G(r)/\langle r \rangle \cong \mathrm{U}_6(2)$ or $\mathrm{U}_6(2).2$. Since $R \leq C_G(r)$ and

$r \in R'$ by Lemma 5.1, $F^*(C_G(r))$ does not split over $\langle r\rangle$. It follows that $F^*(C_G(r)) \cong 2{\cdot}\mathrm{U}_6(2)$ or $2{\cdot}\mathrm{U}_6(2).2$ as claimed. □

Let $K_1 = F^*(K) \cong 2{\cdot}\mathrm{U}_6(2)$ and fix some Sylow 2-subgroup T of K_1. In $T/\langle r\rangle$ there is a unique elementary abelian group of order 2^9 with normalizer of shape $2^9 : \mathrm{PSL}_3(4)$ (the stabilizer of a totally isotropic subspace of dimension 3). Let E be the preimage of this subgroup. Then $\mathrm{PSL}_3(4)$ acts irreducibly on $E/\langle r\rangle$ and $|E| = 2^{10}$, we get that E is elementary abelian of order 2^{10} with $N_{K_1}(E)/E \cong \mathrm{PSL}_3(4)$ and $C_G(E) = C_K(E) = E$. By [1, (23.5 .5)], E is an indecomposable module for $N_K(E)/E$.

Lemma 5.3. *We have that $N_G(E)/E \cong \mathrm{M}_{22}$ or $\mathrm{Aut}(\mathrm{M}_{22})$.*

Proof. As $r^H \cap R' \neq \{r\}$ we have that $r^G \cap K_1 \neq \{r\}$. As all involutions of $\mathrm{U}_6(2)$ are conjugate into E (see [1, (23.3)]), we have that $r^{N_G(E)} \neq \{r\}$. Recall that $E/\langle r\rangle$ is just the Todd module for $\mathrm{L}_3(4)$ and so $N_K(E)$ has orbits of length 1, 21 ,21, 210, 210, 280 and 280 on E (where some of these lengths may double as E is indecomposable) by [1, (22.2)].

Then, as $Z(T) \le E$ has order 4 by [9, Table 5.3t], $N_K(Z(T))$ has shape $2.2^{1+8}_+.\mathrm{SU}_4(2)$. In particular, we can choose $t \in Z(T)$ such that t is a square in K_1 and $Z(T) = \langle r, t\rangle$. Since r is not a square in K_1 by [1, (23.5.3)], we have t is not $N_G(E)$-conjugate to r. Now taking in account that $|N_G(E)/E|$ has to divide $|\mathrm{GL}_{10}(2)|$, we see that $|r^{N_G(E)}| = 2 \cdot 11$, 2^9 or 561. If $|r^{N_G(E)}| = 561$, then $|N_G(E)/E| = 2^a \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 17$, where $a = 6$ or 7. As the normalizer of a Sylow 17-subgroup in $\mathrm{GL}_{10}(2)$ has order $2^4 \cdot 3^2 \cdot 5 \cdot 17$, Sylow's Theorem implies that there must be $2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ Sylow 17-subgroups in $N_G(E)/E$. In particular the Sylow 3-subgroup D of the normalizer of the subgroup of order 17 has order 9 and is elementary abelian. Two of the cyclic subgroups of D are fixed point free on E, one has centralizer of order 4 and the final one centralizes a subgroup of order 2^8. As the Sylow 3-subgroups of $N_G(E)$ have order 3^3, at least one of these subgroups is conjugate in to $N_K(E)$ and there we see that such groups all have centralizer of order 2^4 in E. This shows that this configuration cannot arise.

So assume that $|r^{N_G(E)}| = 2^9$. Then $|N_G(E)/E| = 2^a \cdot 3^2 \cdot 5 \cdot 7$, $a = 15$ or 16. Since some orbit on E is of odd length, we must have an orbit of length 21, 231 or 301 or 511. As we know $|N_G(E)|$, we get an orbit of length 21. From the action of $\mathrm{L}_3(4)$ on this set, we see that no element of odd order fixes more than 3 points. Let $T \in \mathrm{Syl}_2(N_G(E)/E)$. Now $\mathrm{Sym}(21)$ has Sylow 2-subgroups of order 2^{18} and $\mathrm{Sym}(8)$ has Sylow 2-subgroups of order 2^6. Hence, as $|T| \ge 2^{15}$, there is an involution $j \in T$ which fixes at least 13 points and the product of two such involutions fixes at least 5 points. It follows that $\langle j, j^x\rangle$ is a 2-group for all $x \in N_G(E)/E$. Hence $O_2(N_G(E)) > E$ by the Baer-Suzuki Theorem and this contradicts the fact that $N_G(E)$ acts irreducibly on E and $C_G(E) = E$.

So we have that $|r^{N_G(E)}| = 22$. In particular we have that $N_G(E)/E$ acts triply transitive on 22 points with point stabilizer $\mathrm{L}_3(4)$ or $\mathrm{L}_3(4):2$. Using, for example [12], get that $N_G(E)/E$ is isomorphic to M_{22} or $\mathrm{Aut}(\mathrm{M}_{22})$, the assertion. □

Proof of Theorerm 1.4. If $K = K_1$, then, as r is not weakly closed in a Sylow 2-subgroup of G (its conjugate to r_2 for example) we have $G \cong \mathrm{M}(22)$ by [1, Theorem 31.1]. If $K > K_1$, then also $N_G(E)/E \cong \mathrm{Aut}(\mathrm{M}_{22})$ and Lemma 2.11 (ii) implies that G has a subgroup G_1 of index 2. We have $K_1 = K \cap G_1$ and $G_1 \cong \mathrm{M}(22)$ by [1, Theorem 31.1]. □

6. Some notation

From here on we may suppose that $|R| = 2^9$. In this brief section we are going to reinforce some of our earlier notation in preparation for determining the centralizers of various elements in the coming sections.

We begin by recalling our basic notation which has already been established. We have R_1, R_2, R_3 are the normal quaternion groups of R and $Q_i = [Q, R_i]$ extraspecial of order 27. We have defined $Z(R_i) = \langle r_i\rangle$ so that $Z(R) = \Omega_1(R) = \langle r_1, r_2, r_3\rangle$. We have for $B = C_S(Z(R))$ and that $B = \langle Z, x_{123}, x_2x_3^{-1}\rangle$, where the last element is non-trivial just when $WQ < S$. By Lemma 4.10 B is elementary abelian. Further we have some $w \in N_H(R)$ with $Q_1^w = Q_2$, $Q_2^w = Q_3$ and $Q_3^w = Q_1$.

From Lemma 4.8 (ii) and (iii) we have $|H| = 2^{9+a} \cdot 3^{10}$ or $2^{9+a} \cdot 3^9$ where $a = 0, 1$. When $a = 1$, just as in the case when $|R| = 2^7$, there exists a further involution $i \in N_H(S)$. This involution can be chosen to centralize Z and normalize R. Since, by Lemma 4.8, $\overline{H}$ is isomorphic to a subgroup of $\mathrm{Sp}_2(3) \wr \mathrm{Sym}(3)$, we see that i can be selected so that Q_1 is centralized by i, and so that $Q_2^i = Q_3$.

We take the involution $t \in N_P(Z) \cap N_G(S)$ from Lemma 4.7 (vi). Since t normalizes QR and $Q \leq P$, we may assume that t normalizes R. Since t inverts $\overline{W}$, t inverts wQ and so t permutes R_1, R_2 and R_3 as a 2-cycle. Thus we may suppose that t normalizes R_1 and exchanges R_2 and R_3. In particular, t centralizes r_1 and acts on Q_1 inverting Z. Since $W/(Q \cap Q^g)$ is inverted by t, we see, using Lemma 4.14 (iv), that $q_1(Q \cap Q^g)$ is inverted by t. Similarly $\widetilde{q_1}W$ is centralized by t. It follows that $[Q_1, t] = Z\langle q_1\rangle$ and that t inverts q_1.

Lemma 6.1. *With the notation just established, we have* $N_{N_G(Z)}(R) = R\langle z, x_{123}, x_2x_3^{-1}, w\rangle\langle i, t\rangle$. *Furthermore,*

(i) $q_1^t = q_1^{-1}$.
(ii) t *inverts* $\langle z, x_{123}, w\rangle$ *which is abelian and* t *centralizes* $x_2x_3^{-1}$.
(iii) $w^i = w^{-1}$ *and* $(x_2x_3^{-1})^i = (x_2x_3^{-1})^{-1}$.

Proof. We have already discussed (i). By Lemma 4.7(iv), t inverts $\overline{W} = \overline{\langle x_{123}, w\rangle}$ and t inverts Z. Thus, we may choose notation so that that t inverts $\langle z, x_{123}, w\rangle$ (i) holds. Furthermore, we may suppose that t centralizes $x_2x_3^{-1}$. Now $C_X(i) = \langle Z, x_{123}\rangle$ and $[X, i]$ has order 9. In particular, $[X, i] \cap [X, t]$ has order 3. We choose w such that $[X, i] \cap [X, t] = \langle w\rangle$. Finally we may suppose that $x_2x_3^{-1}$ is chosen so that it is inverted by i. □

7. A SIGNALIZER

Recall from Lemma 4.7 (vii) that there is an involution $t \in P$ which inverts both Z and $\overline{W}$ and that further properties of t are listed in Section 6. We set

$$H_0 = QWR\langle t\rangle$$

and note that, as t inverts $\overline{W}$, H_0 is a normal subgroup of $N_G(Z)$.

Lemma 7.1. *The following hold.*

(i) $F^*(C_G(q_1)) \cong 3 \times \mathrm{U}_6(2)$;
(ii) $|N_G(\langle q_1\rangle) : C_G(q_1)| = 2$; *and*
(iii) $C_G(q_1)/F^*(C_G(q_1)) \cong N_G(Z)/H_0$ *and is isomorphic to a subgroup of* $\mathrm{Sym}(3)$.

Furthermore $[r_1, E(C_G(q_1))] = 1$.

Proof. We have $O^2(C_H(q_1)) = C_Q(q_1)(R_2R_3)B$ which has shape $(3 \times 3_+^{1+4}).(\mathrm{Q}_8 \times \mathrm{Q}_8).3^k$ where $3^k = |\overline{B}|$ with $k = 1, 2$. From Lemma 6.1 (i), we have that t inverts q_1 and, by definition t inverts Z, since r_1 inverts q_1 and centralizes Z, we have that $r_1t \in N_{C_H(q_1)}(Z)$. Thus

$$C_{N_G(Z)}(q_1) = \langle q_1\rangle Q_2Q_3R_2R_3J_0\langle i, r_1t\rangle.$$

Now we see that $O_3(C_{N_G(Z)}(q_1)/\langle q_1\rangle) = Q_2Q_3\langle q_1\rangle/\langle q_1\rangle$ is extraspecial of order 3^5 and that

$$O_2(C_{N_G(Z)}(q_1)/Q_2Q_3\langle q_1\rangle) = R_2R_3Q_2Q_3\langle q_1\rangle/Q_2Q_3\langle q_1\rangle/\langle q_1\rangle \cong \mathrm{Q}_8 \times \mathrm{Q}_8.$$

Thus $C_G(q_1)/\langle q_1\rangle$ is similar to a 3-centralizer in either $\mathrm{U}_6(2)$ or $\mathrm{F}_4(2)$ (see Definition 2.16). By Lemma 4.17, q_1 is centralized by an element f of order 5 in $N_G(J)$. Furthermore, f does not normalize Z as 5 does not divide the order of H. Since $Z^f \le J$ and $f \in C_G(q_1)$, we see that

Z is not weakly closed in $C_S(q_1)$ and so it follows from Theorem 2.17 that $F^*(C_H(q_1))/\langle q_1\rangle \cong \mathrm{U}_6(2)$ or $\mathrm{F}_4(2)$ and that $C_H(q_1)/F^*(C_H(q_1)) \cong H/H_0$. Finally, as $N_{C_G(q_1)}(J)$ involves $\mathrm{Alt}(6)$ by Lemma 4.19, the subgroup structure of $\mathrm{F}_4(2)$ implies that

$$F^*(C_H(q_1)/\langle q_1\rangle) \cong \mathrm{U}_6(2).$$

Now $\langle q_1\rangle$ is normalized by the involution r_1 and r_1 centralizes $C_H(q_1)/\langle q_1\rangle$. Hence, by Proposition 2.2, r_1 centralizes $C_G(q_1)/\langle q_1\rangle$. Since $C_H(q_1)$ splits over q_1, we now have $F^*(C_G(q_1)) \cong 3 \times \mathrm{U}_6(2)$. This proves (i). Part (ii) follows as r_1 (and t) invert q_1.

We also easily have $C_G(q_1)/F^*(C_G(q_1)) \cong N_G(Z)/H_0$. □

Let $K = E(C_G(q_1))$. Then $K \cong \mathrm{U}_6(2)$ by Lemma 7.1. Since $R_2 \le C_G(q_1)$, we have $r_2 \in K$. As r_2 centralizes $Q_3 \cong 3^{1+2}_+$ in K, Proposition 2.2 yields

$$C_K(r_2) \cong 2^{1+8}_+ : \mathrm{U}_4(2).$$

Notice that r_3 is also in K and therefore q_2 and $q_3 \in K$. From the structure of $C_S(q_1)$ we also have that $z \in K$.

Furthermore, we have $|J_0 \cap K|$ is elementary abelian of order 3^4 and that $A \cap K = \langle Z, q_2, q_3\rangle = C_A(r_1)$. Using [16, Theorem 4.8], we get that

$$F = N_K(J \cap K) \cong 3^4 : \mathrm{Sym}(6).$$

Furthermore [16, Lemma 4.2] indicates that Z has exactly 10 conjugates under the action of F. As $A \cap K = J \cap O_3(C_K(Z))$ we see that $(A \cap K)^F$ has order 10 and F acts 2-transitively on this set.

We also have that F commutes with $\langle q_1, r_1\rangle \le C_G(K)$ and $A \cap K = C_A(r_1)$. Let $f \in F$ be such that $C = (A \cap K) \cap (A \cap K)^f = \langle q_2, q_3\rangle$. Then, as q_1 and q_2 are G-conjugate, we obtain

$$L = C_G(C)^\infty \le C_{C_G(q_2)}(q_3)^\infty \cong \mathrm{U}_4(2)$$

from Lemma 7.1. In addition, C commutes with $R_1R_1^fN_J(R_1)N_J(R_2)$ and therefore $R_1R_1^f \le L \cong \mathrm{U}_4(2)$. If $R_1 = R_1^f$, then R_1 centralizes $J \cap K$. However, $C_J(R_1) \le Q$ and $J \cap K \not\le Q$. Therefore $R_1 \ne R_1^f$ and this means that r_1 is a 2-central involution of L. Hence $R_1R_1^f \cong 2^{1+4}_+$ and we deduce that R_1 and R_1^f commute as $R_1R_1^f$ contains exactly two subgroups isomorphic to Q_8. As F acts 2-transitively on the set $(A \cap K)^F$, we deduce that any two F-conjugates of R_1 commute and so

$$E = \langle R_1^F\rangle \cong 2^{1+20}_+$$

and this is a 2-signalizer for F.

Lemma 7.2. *The following hold.*

(i) *E is extraspecial of order 2^{21} and plus type;*
(ii) *$C_E(Z) = R_1$;*
(iii) *E is the unique maximal 2-signalizer for Q_2Q_3 in $C_G(r_1)$; and*
(iv) *$C_G(\langle r_1, q_1\rangle)$ normalizes E.*

In particular, K normalizes E.

Proof. We have already remarked that (i) is true. Also, we know that $Q_2Q_3 \le F$ and so E is a 2-signalizer for Q_2Q_3. Suppose that D is a 2-signalizer for Q_2Q_3 in $C_G(r_1)$. Then

$$D = \langle C_D(x) \mid x \in \langle z, q_2\rangle^{\#}\rangle$$

and observe that $\langle z, q_2\rangle$ contains three Q_2-conjugates of $\langle q_2\rangle$. Now in $C_K(z)$ the only 2-subgroup which is normalized by Q_2Q_3 is R_1 and this is contained in E. In particular, (ii) holds. So we consider signalizers for $\langle q_2, Q_3\rangle$ in $C_{C_G(r_1)}(q_2)$. First we note that R_1 commutes with q_2 and so we have that $r_1 \in K_2 = C_G(q_2)^\infty \cong \mathrm{U}_6(2)$ and, as $Q_1Q_3 \le O_3(C_{K_2}(Z))$, we have that $Q_3 \le C_{K_2}(r_1)$ and this means that r_1 is a 2-central element of K_2 by Proposition 2.2. As an extraspecial group of order 27 in $\mathrm{U}_4(2)$ does not normalize a non-trivial 2-group, we now have that the maximal signalizer for Q_3 in $C_{C_G(q_2)}(r_1)$ is $O_2(C_{K_2}(r_1)) \cong 2^{1+8}_+$. We have that $\langle Z, q_2\rangle$ acts on E and $C_E(\langle Z, q_2\rangle) = C_E(Z) = R_1$. Since

$$E = \langle C_E(x) \mid x \in \langle z, q_2\rangle^{\#}\rangle,$$

we have $|C_E(q_2)| = 2^9$ and $C_E(q_2) = O_2(C_{K_2}(r_1))$. Therefore $C_D(q_2) \le E$. It now follows that $D \le E$ as claimed in (iii).

From the construction of E, we have that E is normalized by F and (ii) implies that $N_{C_G(\langle q_1,r_1\rangle)}(Q_2Q_3) = N_{C_G(\langle q_1,r_1\rangle)}(Z)$ also normalizes E. Now either using [5] or [16] we have that $C_G(\langle q_1, r_1\rangle)$ normalizes E. This is (iii). Since $K \le C_G(\langle q_1, r_1\rangle)$ by Lemma 7.1, we have $K \le N_G(E)$ as well. □

Lemma 7.3. *$F^*(N_G(E)/E) = KE/E \cong \mathrm{U}_6(2)$.*

Proof. Note that $N_G(E) = N_{C_G(r_1)}(E)$. In $N_{C_G(r_1)}(E)/E$ we have that $N_K(Z)E/E$ is a 3-normalizer of type $\mathrm{U}_6(2)$. Therefore, as Z is not weakly closed in $C_S(r)E/E$ with respect to $N_{C_G(r_1)}(E)/E$, we have that $F^*(N_{C_G(r_1)}(E)/E) = EK/E$ from Theorem 2.17. □

Lemma 7.4. *$N_G(E)/E$ acts irreducibly on $E/\langle r_1\rangle$ and $N_G(E)$ contains a Sylow 2-subgroup of G.*

Proof. We know that $F^*(N_G(E)/E) \cong \mathrm{U}_6(2)$ and that $|E/\langle r_1\rangle| = 2^{20}$. The action of F and E, shows that $E/\langle r_1\rangle$ is irreducible. Thus Lemma 2.7 implies that $E/\langle r_1\rangle$ is not a failure of factorization module for $N_G(E)/E$. In particular, if $T \in \mathrm{Syl}_2(N_G(E))$, we have that

$Z(T) = \langle r_1 \rangle$ and the Thompson Subgroup of $T/\langle r_1 \rangle$ is $E/\langle r_1 \rangle$ by [8, Lemma 26.15]. Thus $N_G(T) \le N_G(E)$ and so $T \in \mathrm{Syl}_2(G)$. □

We close this section with a technical detail that we shall need later.

Lemma 7.5. *We have* $C_K(q_2) \cong 3 \times \mathrm{U}_4(2)$.

Proof. Set $X = \langle q_2, Q_3, (J \cap K)R_3 \rangle \approx 3 \times 3^{1+2}_+.\mathrm{Q}_8.3$. Then $X \le C_K(q_2)$. As $\langle q_2 \rangle = [J \cap K, r_2]$, we have that $N_K(J \cap K)/(J \cap K) \cong \mathrm{O}_4^-(3)$, we get $C_{N_K(J \cap K)}(q_2) \approx 3^4 : \mathrm{Sym}(4)$. Hence $C_K(q_2) \cong 3 \times \mathrm{U}_4(2)$ as is seen in [5]. □

8. The centralizer of an outer involution

In this section we continue our investigation of the situation when $|R| = 2^9$, assume that $H/BRQ \cong \mathrm{Sym}(3)$ and show that G has a subgroup of index 2. Thus, by Lemma 4.8,

$$\overline{H} \approx (\mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8).3.\mathrm{Sym}(3)$$

or

$$\overline{H} \approx (\mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8).3^{1+2}_+.2.$$

Since $H/BRQ \cong \mathrm{Sym}(3)$, Lemma 4.2 implies that the Sylow 2-subgroup of H is isomorphic to the Sylow 2-subgroup of $\mathrm{Sp}_2(3) \wr \mathrm{Sym}(3)$ and hence we may select an the involution d which conjugates Q_2 to Q_3 and centralizes an extraspecial "diagonal" subgroup of Q_2Q_3 and in addition centralizes Q_1 and normalizes S.

Lemma 8.1. *We have* $C_G(d)/\langle d \rangle \cong \mathrm{F}_4(2)$.

Proof. Since d centralizes Z, we have $C_Q(d)$ is extraspecial of order 3^{1+4}. Furthermore, as $\overline{B}$ has order 3 or 3^2 we have $|C_{\overline{B}}(d)| = 3$. Thus $C_S(d)$ has order 3^6. Furthermore, $C_R(d) = R_1 \times C_{R_2R_3}(d)$ is a direct product of two quaternion groups. It follows that $C_{C_G(d)}(Z)$ is a 3-centralizer in a group of type $\mathrm{U}_6(2)$ or $\mathrm{F}_4(2)$. Since d normalizes S, d normalizes $Z_2(S) = V$ and, as $V = Z\langle q_1q_2q_3 \rangle$, d centralizes V (see Lemma 4.6). From the definition of P, we now have that d normalizes P. Since d centralizes V, we have that $C_{P\langle d \rangle}(V) = \langle d \rangle W$. A Frattini Argument now shows that $C_{P\langle d \rangle}(d)W = P\langle d \rangle$. Therefore $C_P(d)$ acts transitively on the non-trivial elements of V. Hence Z is not weakly closed in $C_Q(d)$. Now Theorem 2.17 implies that $C_G(d)/\langle d \rangle \cong \mathrm{F}_4(2)$ or $\mathrm{Aut}(\mathrm{F}_4(2))$. Since $|C_H(d)| = 2^7 \cdot 3^6$ it transpires that $C_G(d)/\langle d \rangle \cong \mathrm{F}_4(2)$ as claimed. □

Theorem 8.2. *If* $H/BRQ \cong \mathrm{Sym}(3)$*, then* G *has a subgroup* G^* *of index* 2 *which satisfies the hypothesis of Theorem 1.3 and in addition has* $|H \cap G^*/BRQ| = 3$.

Proof. Now let $T \in \mathrm{Syl}_2(N_G(E))$ and $T_0 = T \cap EK$. By Lemma 7.4, $T \in \mathrm{Syl}_2(G)$. Assume that G does not have a subgroup of index 2. Then by [8, Proposition 15.15] we have that there is a conjugate d^* of d in T_0 such that $C_T(d^*) \in \mathrm{Syl}_2(C_G(d^*))$. In particular, we must have $|C_{EK\langle d\rangle}(d^*)| = 2^{25}$. Using Lemma 2.5 (ii) we see that $d^* \notin E$. Now note that

$$C_{EK\langle d\rangle}(d^*)EK = EK\langle d\rangle$$

by Lemma 2.6 and so we require $|C_{EK/\langle r_1\rangle}(d^*\langle r_1\rangle)| = 2^{23}$ or 2^{24} where in the latter case, we must have

$$C_{EK/\langle r_1\rangle}(d^*\langle r_1\rangle) > C_{EK}(d^*)\langle r_1\rangle/\langle r_1\rangle.$$

We now apply Lemma 2.6. As $d^* \in Y'$ in the notation of Lemma 2.6, this shows that (iv) and (v) not apply. But then Lemma 2.6 provides no possibility for d^*. □

9. Transferring the element of order 3

Because of Theorem 8.2, from here on we suppose that H/BRQ has order 3. In this section we show that if $S > QW$, then G has a normal subgroup of index 3 which satisfies the hypothesis of Theorem 1.3. So assume that $S > QW$. Then, by Lemma 4.8 (ii), $\overline{S}$ is extraspecial and $|H| = 2^9 \cdot 3^{10}$ with

$$\overline{H} \approx (\mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8).3_+^{1+2}.$$

Lemma 9.1. *Suppose that $S > QW$ and $|H| = 2^9 \cdot 3^{10}$. Then G has a normal subgroup G^* of index of index 3 and $C_G(Z) \cap G^* = QWR\langle t\rangle$ is similar to a 3-centralizer on type ${}^2\mathrm{E}_6(2)$ and Z is not weakly closed in $S \cap G^*$ with respect to G^*.*

Proof. We know that $S = QJ_0W$ and $N_G(Z) = QRWJ_0\langle t\rangle$ by Lemma 4.9(v). From Lemma 4.7(vi), t inverts $\overline{W}$ and so, as $\overline{S}$ is extraspecial, $J_0Q/JQ \cong J_0/J$ is centralized by t. Therefore $J_0 \not\leq N_G(Z)'$ and $S/J = J_0/J \times QW/J$. Since J_0/J is a normal subgroup of $N_G(J_0/J)$ we now have that $J_0 \not\leq N_G(J_0)'$. As J_0 is abelian, we may use Lemma 2.13 (ii)to obtain $J_0 \not\leq G'$. Let G^* be a normal subgroup of G of index 3. Then, as $\overline{W}$ is inverted by t and $Q = [Q, R]$, $S \cap G^* = QW$. It follows that $C_{G^*}(Z) = QWR$ and $M \cap G^* = N_{G^*}(J) \not\leq H$, in particular, Z is not weakly closed in $S \cap G^*$ with respect to G^*. This proves the lemma. □

10. The centralizer of an involution

Because of Lemma 9.1, we may now assume that G satisfies the hypothesis of the Theorem 1.3 with $S = QW$ and $H = QRW$. Thus we now have

$$S = QW = Q\langle x_{123}, w\rangle$$

where x_{123} and w are as introduced just before Lemma 4.9.

Lemma 10.1. *We have*

$$C_G(q_2q_3^{-1})/\langle q_2q_3^{-1}\rangle \cong \Omega_8^+(2) : 3.$$

Proof. Set $x = q_2q_3^{-1}$. Then

$$C_Q(x) = \langle q_1, \widetilde{q}_1, q_2, q_3, \widetilde{q}_2\widetilde{q}_3^{-1}\rangle.$$

Furthermore $[x_{123}, x] = 1$ and $[w, x] \notin Z$. Hence we see that

$$C_S(x) = C_Q(x)\langle x_{123}\rangle.$$

We also have $C_R(x) = R_1$. So we have

$$C_H(x) = \langle q_1, \widetilde{q}_1, q_2, q_3, \widetilde{q}_2\widetilde{q}_3^{-1}, x_{123}, R_1\rangle$$

and $C_H(x)/O_3(C_{C_G(Z)}(x)) \cong \mathrm{SL}_2(3)$. Furthermore, $[C_Q(x), R_1] = Q_1$ has order 27 and $C_Q(x)/\langle x\rangle$ is extraspecial of order 3^5.

By Lemma 4.14 we see that $x \in Q \cap Q^g$ and $[P, x] \leq V = ZZ^g$ by Lemma 4.6(iii). Since all the elements of the coset Vx are conjugate in P, it follows that we may assume that there is $U \leq P$ with $U \cong \mathrm{Q}_8$ with $[U, x] = 1$. Then Z and Z^g are conjugate by an element of U. It follows that Z is not weakly closed in $C_Q(x)$ with respect to $C_G(x)$. Now we have $C_G(x)/\langle x\rangle \cong \mathrm{P}\Omega_8^+(2) : 3$ by Astill's Theorem 2.19. □

Recall the subgroup $E = \langle R_1^F\rangle$ from Lemma 7.2 is normalized by $C_J(r_1) = J \cap K$ and that $F = N_K(J \cap K) \approx 3^4 : \mathrm{O}_4^-(3)$. Since r_1 centralizes $q_2q_3^{-1}$, we have that $q_2q_3^{-1} \in J \cap K$. Furthermore, we note that F has exactly 3-orbits on the subgroups of order 3 in $J \cap K$ representatives being Z, $\langle q_2\rangle$ and $\langle q_2q_3^{-1}\rangle$ and that these subgroups are in different G-conjugacy classes by Lemma 4.12. The next goal is to show that $N_G(E)$ is strongly 3-embedded in $C_G(r_1)$. The next lemma facilitates this aim.

Lemma 10.2. *The following hold:*

(i) $C_E(q_2q_3^{-1}) \cong 2_+^{1+8}$;
(ii) r_1 *is a 2-central involution in* $E(C_G(q_2q_3^{-1}))$;
(iii) $C_G(r_1) \cap C_G(\langle q_2q_3^{-1}\rangle) \leq N_G(E)$;
(iv) $O_2(C_{E(C_G(q_2q_3^{-1}))}(r_1)) = C_E(q_2q_3^{-1})$; *and*
(v) $r_1^{C_G(q_2q_3^{-1})} \cap E \neq \{r_1\}$.

Proof. Let $D = E(C_G(q_2q_3^{-1}))$. Then $D \cong \Omega_8^+(2)$ by Lemma 10.1 as the Schur multiplier of $\Omega_8^+(2)$ is a 2-group.

We have seen that R_1 centralizes $q_2q_3^{-1}$ and so $r_1 \in D$. As $\langle z, q_2q_3^{-1}\rangle \leq J \cap K$ acts on E and $C_E(Z) = R_1 \cong \mathrm{Q}_8$ by Lemma 7.2 (ii), by decomposing E under the action of $\langle z, q_2q_3^{-1}\rangle$ we see that

$$C_E(q_2q_3^{-1}) \cong 2_+^{1+8}.$$

Hence (i) holds. Additionally, we have $S \cap K = C_S(\langle r_1, q_1\rangle) = Q_2Q_3\langle q_1, x_{123}\rangle$ and therefore

$$C_{S\cap EK}(q_2q_3^{-1}) = \langle q_2, q_3, \widetilde{q}_2\widetilde{q}_3^{-1}, x_{123}\rangle$$

has order 3^4. Using this and [5] we infer that r_1 is a 2-central element of $E(C_G(q_2q_3^{-1}))$ which is (ii).

Since r_1 is 2-central in D,

$$C_{C_G(q_2q_3^{-1})}(r_1) \approx ((2_+^{1+8}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3) \times \mathrm{Sym}(3)).3) \times 3$$

with $O_2(C_{C_G(q_2q_3^{-1})}(r_1)) = C_E(q_2q_3{-}1)$ normalized by $C_J(r_1)$. It follows that

$$C_{C_G(\langle q_2q_3^{-1}\rangle)}(r_1) = O_2(C_{C_G(q_2q_3^{-1})}(r_1))N_{C_G(r_1)}(C_J(r_1)) \leq N_G(E).$$

Thus (iii) and (iv) hold.

This proves the main part of the lemma and the remaining part follows as r_1 is not weakly closed in $C_E(r_1)$ in D. □

Lemma 10.3. *If $N_G(E) < C_G(r_1)$, then $N_G(E) = KE$ is strongly 3-embedded in $C_G(r_1)$.*

Proof. Let $d \in N_G(E)$ be a 3-element. Then d is conjugate in $N_G(E)$ to an element of $C_J(r_1)$ by Lemma 2.1. We have $N_{C_G(r_1)}(S \cap KE) = N_{C_G(r_1)}(Z)$ and so to prove the lemma it suffices to show that

$$C_{C_G(r_1)}(\langle d\rangle) \leq N_G(E)$$

for all $d \in C_J(r_1)^\#$ by [8, Proposition 17.11]. By Lemma 10.2 (iii) we have that

$$C_{C_G(r_1)}(\langle q_2q_3^{-1}\rangle) \leq N_G(E).$$

By Lemma 7.2 we have that

$$C_{C_G(r_1)}(Z) \leq N_G(E).$$

Further we have that $C_{N_G(E)}(q_2)E/E = C_K(q_2)E/E \cong 3 \times \mathrm{U}_4(2)$ from Lemma 7.5. Using Lemma 7.1 this shows that also

$$C_{C_G(r_1)}(\langle q_2\rangle) \leq N_G(E).$$

By Lemma 4.12 these subgroups $\langle q_2\rangle$, $\langle q_2q_3^{-1}\rangle$ and Z are in different conjugacy classes of G and as $N_K(J\cap K)$ has three orbits on the non-trivial cyclic subgroups of $J\cap K$ we have accounted for all conjugacy classes of three elements in $N_G(E)$ and consequently $N_K(E)$ is strongly 3-embedded in $C_G(r_1)$. □

Theorem 10.4. $C_G(r) = N_G(E) = KE \approx 2_+^{1+20} : \mathrm{U}_6(2)$.

Proof. This now follows from Lemma 10.3 and Theorem 1.5. □

11. The identification of G

For the section we set $r = r_1$, $L = C_G(r)$ and $K = E(C_G(q_1))$. From Theorem 10.4 we have $L = N_G(E)$ and from Lemma 7.1 and Lemma 7.3 we have $K \cong \mathrm{U}_6(2)$ with $L = KE \approx 2_+^{1+20}.\mathrm{U}_6(2)$. In particular, E is extraspecial of order 2^{21}.

Lemma 11.1. *Suppose that $r^g \in E\setminus\langle r\rangle$ for some $g\in G$. Define $F = \langle C_E(r^g), C_{E^g}(r)\rangle$ and $X = \langle E, E^g\rangle$. Then*

(i) *$E\cap E^g$ is elementary abelian of order 2^{11} and is a maximal elementary abelian subgroup of E.*
(ii) *$C_{E^g}(r) \le L$ and $C_{E^g}(r)E/E$ is elementary abelian of order 2^9.*
(iii) *$C_L(r^g)E/E \cong 2^9.\mathrm{L}_3(4)$ and $O_2(C_L(r^g)E) = (E^g\cap L)E$.*
(iv) *F is normal in X, $X/F\cong \mathrm{Sym}(3)$ and $[X, E\cap E^g] = \langle r, r^g\rangle$.*
(v) *If $h\in G$ and $r^h\in E\setminus\langle r\rangle$, then there is some $k\in EK$ such that $r^{hk} = r^g$.*

Proof. Since E is extraspecial of order 2^{1+20}, $C_E(r^g)$ is a direct product of $\langle r^g\rangle$ with an extraspecial group of order 2^{1+18}. As $|L^g/E^g|$ is not divisible by 2^{19}, there is no such extraspecial group in L^g/E^g and therefore $r\in E^g$.

Because $\Phi(E\cap E^g) \le \langle r\rangle\cap\langle r^g\rangle = 1$, $E\cap E^g$ is elementary abelian. Hence, as E is extraspecial, we have $|E\cap E^g| \le 2^{11}$. In particular, as $|C_{E^g}(r)| = 2^{20}$, we have that $C_{E^g}(r)E/E$ is an elementary abelian group of order at least 2^9. Since the 2-rank of L/E is 9, we deduce that $|C_{E^g}(r)E/E| = 2^9$ and $|E\cap E^g| = 2^{11}$. Furthermore $(E^g\cap L)E/E$ is uniquely determined. This completes the proof of parts (i) and (ii).

By Lemma 2.7, we have $|C_{E/\langle r\rangle}(C_{E^g}(r))| = 2$ and therefore

$$C_{E/\langle r\rangle}(C_{E^g}(r)) = \langle r, r^g\rangle/\langle r\rangle.$$

Hence we have that $C_{L/\langle r\rangle}(\langle r, r^g\rangle/\langle r\rangle) = N_L(C_{E^g}(r))E$. This proves (iii).

As $C_E(r^g)$ and $C_{E^g}(r)$ normalize each other, F is a 2-group and

$$[E, C_{E^g}(r)] \le C_E(r^g) \text{ and } [E^g, C_E(r^g)] \le C_{E^g}(r)$$

which means that F is normal in X. In addition, $[E, E \cap E^g] \leq \langle r \rangle$ and $[E^g, E \cap E^g] \leq \langle r^g \rangle$. So the group $(E \cap E^g)/\langle r, r^g \rangle$ is centralized by X. Suppose that $f \in C_X(\langle r, r^g \rangle)$ has odd order. Then f is in L and centralizes $E \cap E^g$. As $E \cap E^g$ is a maximal elementary abelian subgroup of E we now have that E is centralized by f and this contradicts Lemma 7.3. Thus $C_X(\langle r, r^g \rangle)$ is a 2-group. Modulo F the group X is generated by two conjugate involutions, X/F is dihedral. This shows that $X/F \cong \mathrm{Sym}(3)$, and proves (iv).

Suppose that $r^h \in E \setminus \langle r \rangle$ for some $h \in G$. Then by (iii) $r^h\langle r \rangle$ is centralized by a maximal parabolic subgroup of L/E of shape $2^9.\mathrm{L}_3(4)$. But this group has a 1-dimensional centralizer in $E/\langle r \rangle$ and so r^h is conjugate to r^g in L which proves (v). □

We now fix some Sylow 2-subgroup T of L. From Lemma 10.2 we have that

$$r^{C_G(q_2 q_3^{-1})} \cap E \neq \{r\}.$$

Thus there $g \in G$ with $s = r^g \neq r$ and $s \in E$. By Lemma 11.1 we may assume that $Z_2(T) = \langle r, s \rangle$. We set $X = \langle E, E^g \rangle$,

$$B = N_L(T)$$

and

$$P_1 = BX.$$

For $2 \leq j \leq 4$, we let $P_j \geq B$ be such that P_j/E is a minimal parabolic subgroups in L/E containing B/E and $L = \langle P_2, P_3, P_4 \rangle$. Set $I = \{1, 2, 3, 4\}$ and for $\mathcal{J} \subseteq I$ define $P_{\mathcal{J}} = \langle P_j \mid j \in \mathcal{J} \rangle$ and $M = P_I$.

We further choose notation such that

$$\begin{aligned} P_{34}/O_2(P_{34}) &\cong \mathrm{L}_3(4) \\ P_{23}/O_2(P_{23}) &\cong \mathrm{U}_4(2) \text{ and} \\ P_{24}/O_2(P_{24}) &\cong \mathrm{SL}_2(2) \times \mathrm{SL}_2(4). \end{aligned}$$

Let $\mathcal{C} = (M/B, (M/P_k), k \in I)$ be the corresponding chamber system. Thus $\mathcal{C}$ is an edge coloured graph with colours from $I = \{1, 2, 3, 4\}$ and vertex set the right cosets M/B. Furthermore, two cosets Bg_1 and Bg_2 form a k-coloured edge if and only if $Bg_2 g_1^{-1} \subseteq P_k$. Obviously M acts on $\mathcal{C}$ by multiplying cosets on the right and this action preserves the colours. For $\mathcal{J} \subseteq I$, set $M_{\mathcal{J}} = \langle P_k \mid k \in \mathcal{J} \rangle$ and

$$\mathcal{C}_{\mathcal{J}} = (M_{\mathcal{J}}/B, (M_{\mathcal{J}}/P_k), k \in \mathcal{J}) \subseteq \mathcal{C}.$$

Then $\mathcal{C}_{\mathcal{J}}$ is the $\mathcal{J}$-coloured connected component of $\mathcal{C}$ containing the vertex B.

Lemma 11.2. *The following hold.*

(i) $|P_1 : B| = 3$.
(ii) $\mathcal{C}_{1,3} =$ *and* $\mathcal{C}_{1,4}$ *are generalized digons.*

Proof. By Lemma 11.1 (iii), P_{34} normalizes $Z_2(T)$. Hence P_{34} acts on the set $\{E^h \mid r^h \in Z_2(T)\}$ and consequently P_{34} normalizes $X = \langle E, E^g\rangle$. In particular, we have $P_1 = BX$ and, as $X/O_2(X) \cong \mathrm{Sym}(3)$, (i) holds. Now note that

$$P_1P_3 = XBP_3 = XP_3 = P_3X = P_3BX = P_3P_1.$$

In particular, the cosets of B in $\mathcal{C}_{1,3}$ correspond to the edges in a generalized digon with one part having valency 3 and the other 5. The same is true for $\mathcal{C}_{1,4}$ and so (ii) holds. □

Because of Lemma 11.2, have that $\mathcal{C}_1$ and $\mathcal{C}_2$ have three chambers and $\mathcal{C}_3$ and $\mathcal{C}_4$ each have 5-chambers. Furthermore, from the choice of notation we also have that $\mathcal{C}_{3,4}$ is the projective plane $\mathrm{PG}(2,4)$ and that $\mathcal{C}_{2,3}$ is the generalised polygon associated with $\mathrm{SU}_4(2)$. Furthermore, we have that $\mathcal{C}_{2,3,4}$ is the $\mathrm{U}_6(2)$ polar space.

Lemma 11.3. *We have* $P_{12}/O_2(P_{12}) \cong \mathrm{SL}_3(2) \times 3$ *and* $P_{124} = P_{12}P_4$. *In particular,* $\mathcal{C}_{12}$ *is the projective plane* $\mathrm{PG}(2,2)$.

Proof. We have that $C_{E/\langle r\rangle}(O_2(P_2))$ is 2-dimensional by Smith's Lemma [20] and additionally $P_2/C_{P_2}(C_{E/\langle r\rangle}(O_2(P_2))) \cong \mathrm{SL}_2(2)$. It follows that

$$C_{E/\langle r\rangle}(O_2(P_2)) = Z_3(T)/\langle r\rangle.$$

Hence P_2 acts on $Z_3(T)$ and $O^3(P_2)$ induces $\mathrm{Sym}(4)$ on $Z_3(T)$ with the normal fours group inducing all transvections to $\langle r\rangle$. As $(E\cap E^g)/Z_2(T)$ is non-trivial and normal in T, we have that $Z_3(T) \le E \cap E^g$. Thus Lemma 11.1(iv) yields that P_1 normalizes and induces $\mathrm{Sym}(4)$ on $Z_3(T)$ where now the normal fours group induces all transvections to $Z_2(T)$. Hence $\langle O^3(P_1), O^3(P_2)\rangle$ induces $\mathrm{SL}_3(2)$ on $Z_3(T)$. Furthermore, we have that $P_{12} = \langle O^3(P_1), O^3(P_2)\rangle C_G(Z_3(T))$.

We now see that

$$X = \langle O^3(P_1), O^3(P_2)\rangle = \langle E^h \mid r^h \in Z_3(T)\rangle.$$

Since, by Lemma 11.2 (ii) and choice of notation, X is normalized by P_4 and $\mathrm{SL}_2(4)$ is not isomorphic to a section of $\mathrm{SL}_3(2)$ we infer that $O^2(P_4) \le C_L(Z_3(T))$ and normalizes $\langle P_1, O^3(P_2)\rangle$. This shows that $C_{\langle P_1, O^3(P_2)\rangle}(Z_3(T)) = O_2(\langle P_1, O^3(P_2)\rangle)$ as well as $P_{124} = P_{14}P_4$. Recall that $P_2 = O^3(P_2)N_G(T)$ and $P_1 = O^3(P_1)N_G(T)$. So $P_{12} = \langle O^3(P_1), O^3(P_2)\rangle N_G(T)$ and this completes the proof. □

Lemma 11.4. *We have that* $P_{123}/O_2(\langle P_{123}\rangle) \cong \Omega_8^-(2)$.

Proof. Let U_{23} be the preimage in E of $C_E(O_2(P_{23}))$. Then, by Lemma 2.5, $U_{23} = [E, Et_1]$ where Et_1 is centralized by P_{23}/E. In particular, we have that $U_{23}/Z(E)$ is an orthogonal module for $P_{23}/O_2(P_3) \cong \mathrm{U}_4(2)$ and, furthermore, $U_{23}/Z(E)$ is totally singular which means that U_{23} is elementary abelian. Since U_{23} is normal in T, $Z_2(T) \le U_{23} \le E \cap L^g$ which is the unique T-invariant subgroup of E of index 2. Now $P_3/O_2(P_3) \cong \mathrm{SL}_2(4) \cong \Omega_4^-(2)$ and

$$(E^g \cap L)O_2(P_{23})/O_2(P_{23}) = O_2(P_3)/O_2(P_{23}).$$

As P_3 normalizes a hyperplane in $U_{23}/Z(E)$, we have $[U_{23}, E^g \cap L]$ has order 2^6 and $[U_{23}, E^g \cap L]$. In particular, $U_{23} \not\le E^g$ and, in fact, $|U_{23}E^g/E^g| = 2$ and is centralized by $O_2(P_1)E^g/E^g \in \mathrm{Syl}_2(L^g/E^g)$. Thus

$$[U_{23}, E^g] = U_{23}^g \text{ and } [U_{23}^g, E] = U_{23}.$$

Set $U_4 = U_{23}U_{23}^g$. Then, as $[U_{23}, U_{23}^g] \le Z(E) \cap Z(E^g) = 1$, we have U_4 is elementary abelian. Furthermore, $[U_4, E^g] = U_{23}^g \le U_4$ and $[U_4, E] \le U_{23} \le U_4$ and consequently U_4 is normalized by X. Since X normalizes P_3 by Lemma 11.3 (i), we now have $\langle X, P_3\rangle = P_1P_3$ normalizes U_4. Note that $U_4E = U_{23}^gE = E\langle t_1\rangle$ and so $C_E(U_4)$ has order 2^{15} by Lemma 2.5. Because U_4 is elementary abelian, we have $U_4 \le C_E(U_4)U_4$ and, as a $P_{23}/O_2(P_{23})$-module, $C_E(U_4)U_4/U_{23}$ has a natural 8-dimensional composition factor and a trivial factor. Since U_4/U_{23} is stabilized by P_3 and the composition factors of P_3 on $C_E(U_4)/U_{23}$ are both non-trivial, we find that U_4 is normalized by P_{123}.

Let

$$\mathcal{P} = \langle r\rangle^{P_{123}} \text{ and } \mathcal{L} = \langle r, s\rangle^{P_{123}}$$

and define incidence between elements $x \in \mathcal{P}$ and $y \in \mathcal{L}$ if and only if $x \le y$. Of course all the points and lines are contained in U_4. We claim that $(\mathcal{P}, \mathcal{L})$ is a polar space. Because of the transitivity of P_{123} on $\mathcal{P}$, we only need to examine the relationship between $\langle r\rangle$ and an arbitrary member of $\mathcal{L}$. So let $l \in \mathcal{L}$. Then every involution of l is G-conjugate to r. Hence if $r^* \in l \cap E$ $(= l \cap U_{23})$, then, by Lemma 11.1 (v), r^* is L-conjugate to r^g. In particular, we have that r^* is a vector of type v_1 in the notation of Lemma 2.5. Since P_{23} has 3-orbits on its 6-dimensional module and since $U_{23}/\langle r\rangle$ contains representatives of the three classes of singular vectors in $E/\langle r\rangle$, we infer that r^* is P_{123}-conjugate to an element of $\langle r, r^g\rangle$ .Thus $\langle r, r^*\rangle \in \mathcal{L}$. Since $|U_4 : U_{23}| = 2$, we have that $\langle r\rangle$ is incident to at least one point of l. Assume that $\langle r\rangle$ is incident to at least two points, p_1, p_2 of l. Then $\langle r, p_1\rangle \le E$ and $\langle r, p_2\rangle \le E$. Hence $l \le E$. But then r is incident to every point on l. Thus we have shown that $(\mathcal{P}, \mathcal{L})$ is a polar space. Since $Z_3(T) \le U_{23}$, we have that

$(\mathcal{P}, \mathcal{L})$ has rank either 3 or 4. As the P_{123} induces $\Omega_6^-(2)$ on the lines through $\langle r \rangle$, we get with [22, Theorem on page 176] that $(\mathcal{P}, \mathcal{L})$ is the polar space associated to $\Omega_8^-(2)$, the assertion. $\square$

Combining Lemmas 11.2 and 11.3 we now have that $\mathcal{C}$ is a chamber system of type F_4 with local parameters in which the panels of type 1 and 2 have three chambers and the panels of type 3 and 4 have five chambers.

Proposition 11.5. *We have* $\mathcal{C}$ *is a building of type* F_4 *with automorphism group* $\mathrm{Aut}({}^2\mathrm{E}_6(2))$*. In particular,* $M \cong {}^2\mathrm{E}_6(2)$*.*

Proof. The chamber systems $\mathcal{C}_{1,2}$, $\mathcal{C}_{3,4}$ are projective planes with parameters $3, 3$ and $5, 5$ and $\mathcal{C}_{2,3}$ is a generalized quadrangle with parameters $3, 5$. The remaining $\mathcal{C}_{\mathcal{J}}$ with $|J| = 2$ are all complete bipartite graph. Thus, using the language of Tits in [23], $\mathcal{C}$ is a chamber system of type F_4. Now suppose that J of $\{1, 2, 3, 4\}$ has cardinality three. Then $\mathcal{C}_{1,2,3}$ is the $\mathrm{O}_8^-(2)$-building by Lemma 11.4 and, as $L/E \cong \mathrm{U}_6(2)$, we have $\mathcal{C}_{2,3,4}$ is a building of type $\mathrm{U}_6(2)$. Finally, Lemma 11.3 implies that $\mathcal{C}_{1,3,4}$ and $\mathcal{C}_{1,2,4}$ are both buildings. Since each rank 3-residue is a building, if $\pi : \mathcal{C}' \longrightarrow \mathcal{C}$ is the universal 2-covering of $\mathcal{C}$, then $\mathcal{C}'$ is a building of type F_4 by [23, Corollary 3]. By [22, Proof of Theorem 10.2 on page 214] this building is uniquely determined by the two residues of rank three with connected diagram (i.e. $\mathrm{U}_6(2)$, $\Omega_8^-(2)$) and so $F^*(\mathrm{Aut}(\mathcal{C}')) \cong {}^2\mathrm{E}_6(2)$. Now we have that there is a subgroup U of $\mathrm{Aut}(\mathcal{C}')$ such that U contains L and $U/D \cong M$ for a suitable normal subgroup D of U. As $L = L'$, we have that $L \leq F^*(\mathrm{Aut}(\mathcal{C}'))$ and so L is a maximal parabolic of $F^*(\mathrm{Aut}(\mathcal{C}'))$. As $U \cap F^*(\mathrm{Aut}(\mathcal{C}')) > L$, we get $F^*(\mathrm{Aut}(\mathcal{C}')) \leq U$. As $F^*(\mathrm{Aut}(\mathcal{C}'))$ is simple this implies that $U = M$ and therefore $M \cong {}^2\mathrm{E}_6(2)$. $\square$

Theorem 11.6. *The group* G *is isomorphic to* ${}^2\mathrm{E}_6(2)$*.*

Proof. By [3] we have that M has exactly three conjugacy classes of involutions. In $E \setminus \langle r \rangle$ we also have three classes $C_M(r)$-classes by Lemma 2.5. Using Lemmas 11.1 (iv) and (v) and the fact that $E/\langle r \rangle$ does not admit transvections from L, we may apply Lemma 2.12 to see that $x^G \cap E = x^L$ for all $x \in E \setminus \{z\}$. In particular, the three conjugacy classes of involutions in M all have representatives in E. Further, if $x \in G$ with $r^x \in M$, then there is $h \in M$ such that $r^{xh} \in E$. But now by Lemma 11.1 we may assume that $r^{xh} = r$. Then $xh \in L \leq M$ and so $x \in M$. Hence M controls fusion of 2-central elements in M.

If Y is a normal subgroup of G, then, as M contains the normalizer of a Sylow 3-subgroup of G and is simple, we either have $M \leq Y$ which means that $Y = G$ or Y is a $3'$-group. Suppose the latter. Since r_1 is in M and is non-central, we have $C_Y(r_1) \neq 1$. But then $C_Y(r_1) \leq M$ a contradiction. Thus $Y = 1$ and G is a simple group. As $C_G(r_1) < M$ and $r_1^G \cap M = r_1^M$ we get with Lemma 2.15 that G is isomorphic to one of the following groups $\mathrm{PSL}_2(2^n)$, $\mathrm{PSU}_3(2^n)$, ${}^2\mathrm{B}_2(2^n)$ ($n \geq 3$ and odd) or $\mathrm{Alt}(\Omega)$. In the first three classes of groups the point stabiliser in question is soluble and in the latter case it is $\mathrm{Alt}(n-1)$. Since M is neither soluble nor isomorphic to $\mathrm{Alt}(\Omega\backslash\{M\})$, we have a contradiction. Hence $M = G$ and the proof of Theorem 11.6 is complete. □

12. The proof of Theorem 1.3

Here we assemble the mosaic which proves Theorem 1.3. Thus here we have $C_G(Z)$ is a centralizer of type ${}^2\mathrm{E}_6(2)$ and so $|R| = 2^9$. Lemma 4.8 (i) and (ii) gives the possibilities for the structure of $\overline{H} = H/Q$. If $|H|_2 = 2^{10}$, then Theorem 8.2 implies that G has a subgroup of index 2 which satisfies the hypothesis of Theorem 1.3 . Thus it suffices to prove the result for groups in which $|H|_2 = 2^9$. This means that $S = QW$ or $S > QW$ and $\equiv S/Q \cong 3_+^{1+2}$. The latter situation is addressed in Lemma 9.1 where is shown that if $S > QW$ then G has a normal subgroup of index 3 which also satisfies the hypothesis of Theorem 1.3. Thus we may assume that $S = QW$. Under this hypothesis in Section 10 we prove Theorem 10.4 which asserts that $C_G(r_1) = N_G(E) = KE \approx 2_+^{1+20} : \mathrm{U}_6(2)$. Finally, in Section 11, we prove Theorem 11.6 which shows that under the hypothesis that $C_G(r_1) = N_G(E) = KE$, $G \cong {}^2\mathrm{E}_6(2)$. Thus we have $F^*(G) \cong {}^2\mathrm{E}_6(2)$ and the theorem is validated.

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