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# A note on perfect matchings in uniform hypergraphs

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## Abstract

We determine the *exact* minimum  $\ell$ -degree threshold for perfect matchings in  $k$ -uniform hypergraphs when the corresponding threshold for perfect fractional matchings is significantly less than  $\frac{1}{2} \binom{n}{k-\ell}$ . This extends our previous results that determine the minimum  $\ell$ -degree thresholds for perfect matchings in  $k$ -uniform hypergraphs for all  $\ell \geq k/2$  and provides two new (exact) thresholds:  $(k, \ell) = (5, 2)$  and  $(7, 3)$ .

**Keywords:** perfect matchings, hypergraphs, absorbing method

## 1 Introduction

A *perfect matching* in a hypergraph  $H$  is a collection of vertex-disjoint edges of  $H$  which cover the vertex set  $V(H)$  of  $H$ . Given a  $k$ -uniform hypergraph  $H$  with an  $\ell$ -element vertex set  $S$  (where  $0 \leq \ell \leq k - 1$ ) we define  $d_H(S)$  to be the number of edges containing  $S$ . The *minimum  $\ell$ -degree*  $\delta_\ell(H)$  of  $H$  is the minimum of  $d_H(S)$  over all  $\ell$ -element sets of vertices in  $H$ . In recent years the problem of determining the minimum  $\ell$ -degree threshold that ensures a perfect matching in a  $k$ -uniform hypergraph has received much attention (see e.g. [9, 17, 15, 18, 5, 14, 12, 7, 8, 1, 19, 2, 20, 6, 11, 3]). See [16] for a survey on matchings (and Hamilton cycles) in hypergraphs.

Suppose that  $\ell, k, n \in \mathbb{N}$  such that  $k \geq 3$ ,  $\ell \leq k - 1$  and  $k$  divides  $n$ . Let  $m_\ell(k, n)$  denote the smallest integer  $m$  such that every  $k$ -uniform hypergraph  $H$  on  $n$  vertices with  $\delta_\ell(H) \geq m$  contains a perfect matching. The conjectured value of  $m_\ell(k, n)$  comes from two types of constructions. The first type of constructions are referred to as *divisibility*

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*barriers*. Given a set  $V$  of  $n$  vertices with a partition  $A, B$ , let  $E_{\text{odd}}(A, B)$  ( $E_{\text{even}}(A, B)$ ) denote the family of all  $k$ -element subsets of  $V$  that intersect  $A$  in an odd (even) number of vertices. Define  $\mathcal{B}_{n,k}(A, B)$  to be the  $k$ -uniform hypergraph with vertex set  $V$  and edge set  $E_{\text{odd}}(A, B)$ . Note that the complement  $\overline{\mathcal{B}}_{n,k}(A, B)$  of  $\mathcal{B}_{n,k}(A, B)$  has edge set  $E_{\text{even}}(A, B)$ . Define  $\mathcal{H}_{\text{ext}}(n, k)$  to be the collection of the following hypergraphs. First,  $\mathcal{H}_{\text{ext}}(n, k)$  contains all hypergraphs  $\overline{\mathcal{B}}_{n,k}(A, B)$  where  $|A|$  is odd. Second, if  $n/k$  is odd then  $\mathcal{H}_{\text{ext}}(n, k)$  also contains all hypergraphs  $\mathcal{B}_{n,k}(A, B)$  where  $|A|$  is even; if  $n/k$  is even then  $\mathcal{H}_{\text{ext}}(n, k)$  also contains all hypergraphs  $\mathcal{B}_{n,k}(A, B)$  where  $|A|$  is odd. It is easy to see that no hypergraph in  $\mathcal{H}_{\text{ext}}(n, k)$  contains a perfect matching. Define  $\delta(n, k, \ell)$  to be the maximum of the minimum  $\ell$ -degrees among all the hypergraphs in  $\mathcal{H}_{\text{ext}}(n, k)$ . Note that  $\delta(n, k, \ell) = (1/2 + o(1))\binom{n-\ell}{k-\ell}$  but the general formula of  $\delta(n, k, \ell)$  is unknown (see more discussion in [19]).

The other type of extremal constructions are referred to as *space barriers*. Let  $H^*(n, k)$  be the  $k$ -uniform hypergraph on  $n$  vertices whose vertex set is partitioned into two vertex classes  $A$  and  $B$  of sizes  $n/k - 1$  and  $(1 - 1/k)n + 1$  respectively and whose edge set consists precisely of all those edges with at least one endpoint in  $A$ . Then  $H^*(n, k)$  does not have a perfect matching and  $\delta_\ell(H^*(n, k)) = \binom{n-\ell}{k-\ell} - \binom{(1-1/k)n-\ell+1}{k-\ell} \approx \left(1 - \left(\frac{k-1}{k}\right)^{k-\ell}\right) \binom{n-\ell}{k-\ell}$ .

An asymptotic version of the following conjecture appeared in [5, 10] and the minimum vertex degree version was stated in [12].

**Conjecture 1.** *Let  $k, \ell \in \mathbb{N}$  such that  $\ell \leq k - 1$ . Then for sufficiently large  $n \in k\mathbb{N}$ ,*

$$m_\ell(k, n) = \max \left\{ \delta(n, k, \ell), \binom{n-\ell}{k-\ell} - \binom{(1-1/k)n-\ell+1}{k-\ell} \right\} + 1.$$

Note that for all  $1 \leq \ell \leq k - 1$ ,

$$\left(\frac{k-1}{k}\right)^{k-\ell} < \left(\frac{1}{e}\right)^{1-\frac{\ell}{k}} \quad \text{and} \quad 1 - \left(\frac{k-1}{k}\right)^{k \ln 2} \rightarrow \frac{1}{2} \quad \text{as} \quad k \rightarrow \infty,$$

where  $\ln$  denotes the natural logarithm function. Thus, for  $1 \ll k \ll n$ , if  $\ell$  is significantly bigger than  $(1 - \ln 2)k \approx 0.307k$  then  $\delta(n, k, \ell) > \binom{n-\ell}{k-\ell} - \binom{(1-1/k)n-\ell+1}{k-\ell}$ . On the other hand, if  $\ell$  is smaller than  $(1 - \ln 2)k$  then  $\delta(n, k, \ell) < \binom{n-\ell}{k-\ell} - \binom{(1-1/k)n-\ell+1}{k-\ell}$  for sufficiently large  $n$ .

Conjecture 1 has been proven in a number of special cases. Indeed, Rödl, Ruciński and Szemerédi [18] proved the conjecture for  $\ell = k - 1$ . The authors [19, 20] generalized this result by showing  $m_\ell(k, n) = \delta(n, k, \ell) + 1$  for all  $k/2 \leq \ell \leq k - 1$  (independently Czygrinow and Kamat [2] proved this for  $(k, \ell) = (4, 2)$ ). In the case when  $(k, \ell) = (3, 1)$ , Conjecture 1 was confirmed by Kühn, Osthus and Treglown [12] and independently Khan [7]. Khan [8] also resolved the case when  $(k, \ell) = (4, 1)$ . Alon, Frankl, Huang, Rödl, Ruciński and Sudakov [1] determined  $m_\ell(k, n)$  asymptotically in the case when  $(k, \ell) = (5, 1), (5, 2), (6, 2)$ , and  $(7, 3)$ . Other than these results, no other asymptotic or exact results are known (the best known general bounds are due to Kühn, Osthus and Townsend [11]).

A connection between  $m_\ell(k, n)$  and the minimum  $\ell$ -degree that forces a *perfect fractional matching* was discovered in [1]. Let  $H$  be a  $k$ -uniform hypergraph on  $n$  vertices. A

*fractional matching* in  $H$  is a function  $w : E(H) \rightarrow [0, 1]$  such that for each  $v \in V(H)$  we have that  $\sum_{e \ni v} w(e) \leq 1$ . Then  $\sum_{e \in E(H)} w(e)$  is the *size* of  $w$ . If the size of the largest fractional matching  $w$  in  $H$  is  $n/k$  then we say that  $w$  is a *perfect fractional matching*. Given  $k, \ell \in \mathbb{N}$  such that  $\ell \leq k - 1$ , define  $c_{k,\ell}^*$  to be the smallest number  $c$  such that every  $k$ -uniform hypergraph  $H$  on  $n$  vertices with  $\delta_\ell(H) \geq (c + o(1)) \binom{n-\ell}{k-\ell}$  contains a perfect fractional matching. It is easy to see that the hypergraph  $H^*(n, k)$  defined earlier contains no perfect fractional matching. Thus  $c_{k,\ell}^* \geq 1 - \left(\frac{k-1}{k}\right)^{k-\ell}$ . Alon et al. [1, Theorem 1.1] showed that for fixed  $k, \ell$ , as  $n \in k\mathbb{N}$  tends to infinity,

$$m_\ell(k, n) = \left( \max \left\{ \frac{1}{2}, c_{k,\ell}^* \right\} + o(1) \right) \binom{n-\ell}{k-\ell}. \quad (1)$$

Furthermore, in [1] the authors conjectured that  $c_{k,\ell}^* = 1 - \left(\frac{k-1}{k}\right)^{k-\ell}$  and confirmed this for  $\ell \geq k - 4$ . Together with (1), this gives the aforementioned (asymptotic) results on  $m_\ell(k, n)$  for  $(k, \ell) = (5, 1), (5, 2), (6, 2)$  and  $(7, 3)$ .

In this note we prove the following refinement of (1).

**Theorem 2.** *Fix  $k, \ell \in \mathbb{N}$  with  $\ell \leq k - 1$  and let  $n \in k\mathbb{N}$ . Then*

$$m_\ell(k, n) = \max \left\{ \delta(n, k, \ell) + 1, (c_{k,\ell}^* + o(1)) \binom{n-\ell}{k-\ell} \right\}.$$

Although it looks like a small improvement, Theorem 2 enables us to determine  $m_\ell(k, n)$  *exactly* whenever  $c_{k,\ell}^* < 1/2$ . A recent result of Kühn, Osthus and Townsend [11, Theorem 1.9] showed that

$$c_{k,\ell}^* \leq \frac{k-\ell}{k} - \frac{k-\ell-1}{k^{k-\ell}}$$

for all  $\ell \leq k - 2$ . Together with  $c_{k,k-1}^* = 1/k$  (see [17]), this implies that  $c_{k,\ell}^* < 1/2$  for all  $k/2 \leq \ell \leq k - 1$ . Consequently Theorem 2 implies the aforementioned results of [19, 20]:  $m_\ell(k, n) = \delta(n, k, \ell) + 1$  for all  $k/2 \leq \ell \leq k - 1$ .

Furthermore, in [1] it was shown that  $c_{5,2}^* = 61/125 < 1/2$  and  $c_{7,3}^* = 1105/2401 < 1/2$ . Therefore an immediate consequence of Theorem 2 is the following corollary.

**Corollary 3.** *Suppose that  $(k, \ell) = (5, 2)$  or  $(7, 3)$ . Then  $m_\ell(k, n) = \delta(n, k, \ell) + 1$  for sufficiently large  $n$ .*

After this paper was submitted, Han [4] showed that  $c_{k,\ell}^* < 1/2$  in the case when  $0.42k \leq \ell < k/2$  or  $(k, \ell) = (12, 5), (17, 7)$ . Thus, together with Theorem 2 this resolves Conjecture 1 in these cases.

Let us highlight the ideas behind the proof of Theorem 2. It is informative to first recall the proof of (1), which is an application of the absorbing method. The authors of [1] first applied a lemma of Hàn, Person and Schacht [5, Lemma 2.4], which states that every  $k$ -uniform hypergraph  $H$  with  $\delta_\ell(H) \geq (1/2 + o(1)) \binom{n}{k-\ell}$  contains a small matching  $M$  that can absorb any vertex set  $W \subset V(H) \setminus V(M)$  of size much smaller than  $M$  (that is, there is matching in  $H$  on  $V(M) \cup W$  exactly). Next they found an almost perfect matching in

$H[V(H) \setminus V(M)]$  by first finding a fractional matching and then converting it to an integer matching. This immediately provides the desired perfect matching of  $H$  because  $M$  can absorb all uncovered vertices in  $V(H) \setminus V(M)$ . In order to prove Theorem 2, we prove a result stronger than [5, Lemma 2.4], Theorem 5, which implies that every  $k$ -uniform hypergraph  $H$  with  $\delta_\ell(H) \geq (1/2 - o(1))\binom{n}{k-\ell}$  either contains the aforementioned  $M$  or looks like a hypergraph in  $\mathcal{H}_{\text{ext}}(n, k)$ . If  $H$  contains  $M$ , then we proceed as in [1] (except that we apply a lemma from [11] when converting a fractional matching to an integer matching); if  $H$  looks like a hypergraph in  $\mathcal{H}_{\text{ext}}(n, k)$  and  $\delta_\ell(H) \geq \delta(n, k, \ell) + 1$ , then we obtain a perfect matching by applying [19, Theorem 4.1].

Theorem 5 is the main contribution of this note – it is stronger than two absorbing theorems in our previous papers [19, Theorem 3.1] and [20, Theorem 3.1], in which we assume that  $\ell \geq k/2$ . The proof of Theorem 5 is actually shorter than those of the two previous absorbing theorems because 1) we use a different absorbing structure which allows us to apply a lemma from [13]; 2) when proving Lemma 10, we avoid using auxiliary hypergraphs and obtain the structure of  $H$  by considering the neighborhoods of the vertices of  $H$  directly.

**Notation:** Given a set  $X$  and  $r \in \mathbb{N}$ , we write  $\binom{X}{r}$  for the set of all  $r$ -element subsets of  $X$ . Let  $H$  be a  $k$ -uniform hypergraph. We write  $V(H)$  for the vertex set and  $E(H)$  for the edge set of  $H$ . Define  $e(H) := |E(H)|$ . Given  $v \in V(H)$ , we write  $N_H(v)$  to denote the *neighborhood of  $v$* , that is, the family of those  $(k-1)$ -subsets of  $V(H)$  which, together with  $v$ , form an edge in  $H$ . Given  $X \subseteq V(H)$ , we write  $H[X]$  for the *subhypergraph of  $H$  induced by  $X$* , namely,  $H[X] := (X, E(H) \cap \binom{X}{k})$ . We denote the *complement of  $H$*  by  $\overline{H}$ . That is,  $\overline{H} := (V(H), \binom{V(H)}{k} \setminus E(H))$ . Suppose that  $n, k \in \mathbb{N}$ . When  $|A| = \lfloor n/2 \rfloor$  and  $|B| = \lceil n/2 \rceil$ , we define  $\mathcal{B}_{n,k} := \mathcal{B}_{n,k}(A, B)$  and  $\overline{\mathcal{B}}_{n,k} := \overline{\mathcal{B}}_{n,k}(A, B)$ .

We will often write  $0 < a_1 \ll a_2 \ll a_3$  to mean that we can choose the constants  $a_1, a_2, a_3$  from right to left. More precisely, there are increasing functions  $f$  and  $g$  such that, given  $a_3$ , whenever we choose some  $a_2 \leq f(a_3)$  and  $a_1 \leq g(a_2)$ , all calculations needed in our proof are valid. Hierarchies with more constants are defined in the obvious way.

## 2 Proof of Theorem 2

The lower bound for  $m_\ell(k, n)$  in Theorem 2 follows from the definitions of  $\delta(n, k, \ell)$  and  $c_{k,\ell}^*$  immediately. The following (more general) result provides the desired upper bound for  $m_\ell(k, n)$ .

**Theorem 4.** *Given any  $\theta > 0$ ,  $k, \ell, \ell' \in \mathbb{N}$  where  $1 \leq \ell, \ell' \leq k-1$  there is an  $n_0 \in \mathbb{N}$  such that the following holds. Let  $n \geq n_0$  where  $k$  divides  $n$ . If  $H$  is a  $k$ -uniform hypergraph on  $n$  vertices with*

$$\delta_\ell(H) > \delta(n, k, \ell) \quad \text{and} \quad \delta_{\ell'}(H) > (c_{k,\ell'}^* + \theta) \binom{n - \ell'}{k - \ell'}, \quad (2)$$

*then  $H$  contains a perfect matching.*

The proof of Theorem 4 splits into *extremal* and *non-extremal* cases, the former case being when  $H$  looks like an element of  $\mathcal{H}_{\text{ext}}(n, k)$ . To make this precise we introduce more notation. Let  $\varepsilon > 0$ . Suppose that  $H$  and  $H'$  are  $k$ -uniform hypergraphs on  $n$  vertices. We say that  $H$  is  $\varepsilon$ -close to  $H'$  if  $H$  becomes a copy of  $H'$  after adding and deleting at most  $\varepsilon n^k$  edges. More precisely,  $H$  is  $\varepsilon$ -close to  $H'$  if there is an isomorphic copy  $\tilde{H}$  of  $H$  such that  $V(\tilde{H}) = V(H')$  and  $|E(\tilde{H}) \Delta E(H')| \leq \varepsilon n^k$ .

Our proof of the non-extremal case uses the absorbing method. Given a  $k$ -uniform hypergraph  $H$ , a set  $S \subseteq V(H)$  is called an *absorbing set* for  $Q \subseteq V(H)$ , if both  $H[S]$  and  $H[S \cup Q]$  contain perfect matchings. In this case, if the matching covering  $S$  is  $M$ , we also say  $M$  *absorbs*  $Q$ .

Our main result, Theorem 5, extends [20, Theorem 3.1]. It ensures that if  $H$  is as in Theorem 4 then  $H$  contains a small absorbing matching or  $H$  is close to one of  $\mathcal{B}_{n,k}$  and  $\overline{\mathcal{B}}_{n,k}$ . We postpone its proof to the next subsection.

**Theorem 5.** *Given any  $\varepsilon > 0$  and integer  $k \geq 2$ , there exist  $0 < \alpha, \xi < \varepsilon$  and  $n_0 \in \mathbb{N}$  such that the following holds. Suppose that  $H$  is a  $k$ -uniform hypergraph on  $n \geq n_0$  vertices. If*

$$\delta_1(H) \geq \left(\frac{1}{2} - \alpha\right) \binom{n-1}{k-1}$$

*then  $H$  is  $\varepsilon$ -close to  $\mathcal{B}_{n,k}$  or  $\overline{\mathcal{B}}_{n,k}$ , or  $H$  contains a matching  $M$  of size  $|M| \leq \xi n/k$  that absorbs any set  $W \subseteq V(H) \setminus V(M)$  such that  $|W| \in k\mathbb{N}$  with  $|W| \leq \xi^2 n$ .*

The next result from [19] ensures a perfect matching when our hypergraph  $H$  is close to one of the extremal hypergraphs  $\mathcal{B}_{n,k}$  and  $\overline{\mathcal{B}}_{n,k}$ .

**Theorem 6.** [19, Theorem 4.1] *Given  $1 \leq \ell \leq k-1$ , there exist  $\varepsilon > 0$  and  $n_0 \in \mathbb{N}$  such that the following holds. Suppose that  $H$  is a  $k$ -uniform hypergraph on  $n \geq n_0$  vertices such that  $n$  is divisible by  $k$ . If  $\delta_\ell(H) > \delta(n, k, \ell)$  and  $H$  is  $\varepsilon$ -close to  $\mathcal{B}_{n,k}$  or  $\overline{\mathcal{B}}_{n,k}$ , then  $H$  contains a perfect matching.*

The final tool required for the proof of Theorem 4 is a weaker version of Lemma 5.6 in [11].

**Lemma 7.** [11] *Let  $k \geq 2$  and  $1 \leq \ell \leq k-1$  be integers, and let  $\varepsilon > 0$ . Suppose that for some  $b, c \in (0, 1)$  and some  $n_0 \in \mathbb{N}$ , every  $k$ -uniform hypergraph  $H$  on  $n \geq n_0$  vertices with  $\delta_\ell(H) \geq cn^{k-\ell}$  has a fractional matching of size  $(b+\varepsilon)n$ . Then there exists an  $n'_0 \in \mathbb{N}$  such that any  $k$ -uniform hypergraph  $H$  on  $n \geq n'_0$  vertices with  $\delta_\ell(H) \geq (c+\varepsilon)n^{k-\ell}$  contains a matching of size at least  $bn$ .*

**Proof of Theorem 4.** Choose  $\varepsilon > 0$  from Theorem 6. We may additionally assume that  $\varepsilon \ll \theta, 1/k$ . Let  $0 < \alpha, \xi < \varepsilon$  be as in Theorem 5. Let  $n$  be sufficiently large and divisible by  $k$ . Assume that  $H$  is a  $k$ -uniform hypergraph on  $n$  vertices satisfying (2). Since  $\delta_\ell(H) > \delta(n, k, \ell) = (1/2 - o(1)) \binom{n-\ell}{k-\ell}$ , it follows that  $\delta_1(H) \geq (1/2 - \alpha) \binom{n-1}{k-1}$ . By Theorem 5,  $H$  is  $\varepsilon$ -close to  $\mathcal{B}_{n,k}$  or  $\overline{\mathcal{B}}_{n,k}$ , or  $H$  contains a matching  $M$  of size  $|M| \leq \xi n/k$

that absorbs any set  $W \subseteq V(H) \setminus V(M)$  satisfying  $|W| \in k\mathbb{N}$  with  $|W| \leq \xi^2 n$ . In the former case, since  $\delta_\ell(H) > \delta(n, k, \ell)$ , Theorem 6 provides a perfect matching in  $H$ . In the latter case, set  $c^* := c_{k, \ell}^*$ ,  $H' := H[V(H) \setminus V(M)]$  and  $n_1 := |V(H')|$ . Since  $|V(M)| \leq \xi n$ ,

$$\delta_{\ell'}(H') \geq \delta_{\ell'}(H) - |V(M)| \binom{n - \ell' - 1}{k - \ell' - 1} \geq (c^* + \theta - \xi k) \binom{n - \ell'}{k - \ell'} > (c^*/(k - \ell')! + 2\xi^2)n_1^{k - \ell'},$$

where the last inequality follows since  $\xi \ll \theta, 1/k$ . Let  $c := c^*/(k - \ell')! + \xi^2$ . By the definition of  $c^*$ , for sufficiently large  $\tilde{n}$ , every  $k$ -uniform hypergraph  $F$  on  $\tilde{n}$  vertices with  $\delta_{\ell'}(F) \geq c\tilde{n}^{k - \ell'}$  contains a perfect fractional matching. Applying Lemma 7 with  $\xi^2/k$  and  $(1 - \xi^2)/k$  playing the roles of  $\varepsilon$  and  $b$  respectively, we conclude that  $H'$  contains a matching  $M'$  of size at least  $(1 - \xi^2)n_1/k$ . Let  $W$  be the uncovered vertices of  $H'$ . Then  $|W| \leq \xi^2 n$ . We finally absorb  $W$  using the absorbing property of  $M$ .  $\square$

## 2.1 Proof of Theorem 5

The proof of Theorem 5 follows from the following three lemmas. Lemma 8 is a special case of [13, Lemma 1.1] and gives a sufficient condition for a hypergraph  $H$  to contain a small matching that absorbs *any* much smaller set of vertices from  $H$ .

**Lemma 8.** [13] *Let  $k \in \mathbb{N}$  and  $\gamma' > 0$ . Then there exists an  $n_0 \in \mathbb{N}$  such that the following holds. Suppose that  $H$  is a  $k$ -uniform hypergraph on  $n \geq n_0$  vertices so that, for any  $x, y \in V(H)$ , there are at least  $\gamma'n^{2k-1}$   $(2k - 1)$ -sets  $X \subseteq V(H)$  such that both  $H[X \cup \{x\}]$  and  $H[X \cup \{y\}]$  contain perfect matchings. Then  $H$  contains a matching  $M$  so that*

- $|M| \leq (\gamma'/2)^k n / (8k^2(k - 1))$ ;
- $M$  absorbs any  $W \subseteq V(G) \setminus M$  such that  $|W| \in k\mathbb{N}$  and  $|W| \leq (\gamma'/2)^{2k} n / (128k(k - 1)^2)$ .

**Lemma 9.** *Let  $k \in \mathbb{N}$  and  $0 < \gamma' \ll \gamma \ll 1/k$ . There exists an  $n_0 \in \mathbb{N}$  such that the following holds. Let  $H = (V, E)$  be a  $k$ -uniform hypergraph on  $n \geq n_0$  vertices. Suppose that for every  $x, y \in V$  at least one of the following conditions holds.*

- (i)  $|N_H(x) \cap N_H(y)| \geq \gamma n^{k-1}$ ;
- (ii) *There exists at least  $\gamma n$  vertices  $z \in V$  such that  $|N_H(x) \cap N_H(z)| \geq \gamma n^{k-1}$  and  $|N_H(y) \cap N_H(z)| \geq \gamma n^{k-1}$ .*

*Then there are at least  $\gamma'n^{2k-1}$   $(2k - 1)$ -sets  $X \subseteq V$  such that both  $H[X \cup \{x\}]$  and  $H[X \cup \{y\}]$  contain perfect matchings.*

**Lemma 10.** *Let  $k \in \mathbb{N}$  and  $0 < \alpha \ll \gamma \ll \varepsilon, 1/k$ . Then there exists an  $n_0 \in \mathbb{N}$  such that the following holds. Let  $H = (V, E)$  be a  $k$ -uniform hypergraph on  $n \geq n_0$  vertices such that  $\delta_1(H) \geq (1/2 - \alpha) \binom{n-1}{k-1}$ . Suppose that there exists  $x_0, y_0 \in V$  such that*

(i)  $|N_H(x_0) \cap N_H(y_0)| < \gamma n^{k-1}$ ;

(ii) at most  $\gamma n$  vertices  $z \in V$  satisfy  $|N_H(z) \cap N_H(x_0)| \geq \gamma n^{k-1}$  and  $|N_H(z) \cap N_H(y_0)| \geq \gamma n^{k-1}$ .

Then  $H$  is  $\varepsilon$ -close to  $\mathcal{B}_{n,k}$  or  $\overline{\mathcal{B}}_{n,k}$ .

We postpone the proof of Lemmas 9 and 10 and prove Theorem 5 first.

**Proof of Theorem 5.** Given  $\varepsilon > 0$  and  $k \geq 2$ , choose constants  $\alpha, \gamma', \gamma$  so that  $0 < \alpha \ll \gamma' \ll \gamma \ll \varepsilon, 1/k$ . Set  $\xi := (\gamma'/2)^k / \sqrt{128k(k-1)^2}$ . Let  $n$  be sufficiently large and  $H$  be a  $k$ -uniform hypergraph as in the statement of the theorem.

By Lemmas 9 and 10,  $H$  is  $\varepsilon$ -close to  $\mathcal{B}_{n,k}$  or  $\overline{\mathcal{B}}_{n,k}$  or for every  $x, y \in V(H)$  there are at least  $\gamma' n^{2k-1} (2k-1)$ -sets  $X \subseteq V(H)$  such that both  $H[X \cup \{x\}]$  and  $H[X \cup \{y\}]$  contain perfect matchings. In the former case we are done. In the latter case, Lemma 8 implies that  $H$  contains a matching  $M$  so that

- $|M| \leq (\gamma'/2)^k n / (8k^2(k-1)) \leq \xi n / k$ ;
- $M$  absorbs any  $W \subseteq V(H) \setminus M$  such that  $|W| \in k\mathbb{N}$  and  $|W| \leq (\gamma'/2)^{2k} n / (128k(k-1)^2) = \xi^2 n$ ,

as required. □

**Proof of Lemma 9.** Note that (i) and (ii) imply that  $\delta_1(H) \geq \gamma n^{k-1}$  and so  $e(H) \geq \gamma n^k / k$ . Consider any  $x, y \in V$ . First assume that (i) holds. Fix any  $X' \subseteq V$  where  $|X'| = k-1$  and  $X' \cup \{x\}, X' \cup \{y\} \in E$ . By (i) there are at least  $\gamma n^{k-1}$  choices for  $X'$ . Next choose some  $X'' \subseteq V \setminus (X' \cup \{x, y\})$  such that  $|X''| = k$  and  $X'' \in E$ . There are at least  $\gamma n^k / k - (k+1) \binom{n}{k-1} \geq \gamma n^k / (2k)$  choices for  $X''$ . Set  $X := X' \cup X''$ . Note that both  $H[X \cup \{x\}]$  and  $H[X \cup \{y\}]$  contain perfect matchings. Further, since there are at least  $\gamma n^{k-1}$  choices for  $X'$ , at least  $\gamma n^k / (2k)$  choices for  $X''$  and each  $(2k-1)$ -set may be counted at most  $\binom{2k-1}{k-1}$  times, there are at least

$$\gamma n^{k-1} \times \frac{\gamma n^k}{2k} \times \frac{1}{\binom{2k-1}{k-1}} > \gamma' n^{2k-1}$$

choices for  $X$  (as  $\gamma' \ll \gamma \ll 1/k$ ), as desired.

Now suppose that (ii) holds. Fix any  $z \in V$  such that  $|N_H(x) \cap N_H(z)| \geq \gamma n^{k-1}$  and  $|N_H(y) \cap N_H(z)| \geq \gamma n^{k-1}$ . There are at least  $\gamma n$  choices for  $z$ . Next fix some  $X' \in N_H(x) \cap N_H(z)$  that is disjoint from  $y$ . There are at least  $\gamma n^{k-1} - \binom{n}{k-2} \geq \gamma n^{k-1} / 2$  choices for  $X'$ . Finally, fix some  $X'' \in N_H(y) \cap N_H(z)$  so that  $X''$  is disjoint from  $X' \cup \{x\}$ . There are at least  $\gamma n^{k-1} - k \binom{n}{k-2} \geq \gamma n^{k-1} / 2$  choices for  $X''$ . Set  $X := X' \cup X'' \cup \{z\}$ . So  $|X| = 2k-1$  and both  $H[X \cup \{x\}]$  and  $H[X \cup \{y\}]$  contain perfect matchings. Further, there are at least

$$\gamma n \times \frac{\gamma n^{k-1}}{2} \times \frac{\gamma n^{k-1}}{2} \times \frac{1}{(2k-1) \binom{2k-2}{k-1}} > \gamma' n^{2k-1}$$

choices for  $X$  (as  $\gamma' \ll \gamma \ll 1/k$ ), as desired. □



The rest of this section is devoted to the proof of Lemma 10. We draw on ideas used in the proof of Lemma 5.4 in [20]. We need two results from [20]. The first one implies that if any two vertices in a hypergraph have roughly the same neighborhood, then the hypergraph is near complete or empty.

**Lemma 11.** [20, Lemma 2.2] *Given any  $k \in \mathbb{N}$  and  $\rho > 0$  there exists an  $n_0 \in \mathbb{N}$  such that the following holds. Let  $F = (V, E)$  be a  $k$ -uniform hypergraph on  $n \geq n_0$  vertices with edge density  $|E|/\binom{n}{k} \in [\rho, 1 - \rho]$ . Then there exist two vertices  $v, v' \in V$  such that  $|N_F(v) \Delta N_F(v')| \geq \rho(1 - \rho)n^{k-1}/(k + 1)!$ .*

**Proposition 12.** [20, Proposition 2.3] *For  $r \in \mathbb{N}$ ,  $0 \leq c \leq 1$  and  $n \rightarrow \infty$ ,*

$$\sum_{0 \leq i \leq r, i \text{ even}} \binom{cn}{r-i} \binom{(1-c)n}{i} = \frac{n^r}{2r!} (1 + (2c-1)^r) - O(n^{r-1}),$$

$$\sum_{0 \leq i \leq r, i \text{ odd}} \binom{cn}{r-i} \binom{(1-c)n}{i} = \frac{n^r}{2r!} (1 - (2c-1)^r) - O(n^{r-1}).$$

**Proof of Lemma 10.** Define

$$X := \{v \in V : |N_H(y_0) \cap N_H(v)| < \gamma n^{k-1}\} \quad \text{and} \quad Y := \{v \in V : |N_H(x_0) \cap N_H(v)| < \gamma n^{k-1}\}.$$

Then by Lemma 10 (i),  $x_0 \in X$  and  $y_0 \in Y$ . Let  $V_0 := V \setminus (X \cup Y)$ . We have  $|V_0| \leq \gamma n$  by Lemma 10 (ii). Roughly speaking, our goal is to show that  $|X| \approx |Y| \approx n/2$  and  $H \approx \mathcal{B}_{n,k}(X, Y)$  or  $H \approx \overline{\mathcal{B}}_{n,k}(X, Y)$ .

We first provide several properties of  $X$  and  $Y$ , for example,  $X \cap Y = \emptyset$ , and  $N_H(v) \approx N_H(v')$  whenever  $v, v' \in X$  or  $v, v' \in Y$ .

**Claim 13.** *The following conditions hold.*

- (i) *For all  $v \in X \cup Y$ , we have  $d_H(v) \leq (1/2 + \alpha) \binom{n-1}{k-1} + \gamma n^{k-1}$ .*
- (ii)  $X \cap Y = \emptyset$ .
- (iii) *For any two vertices  $x, x' \in X$ , we have  $|N_H(x) \Delta N_H(x')| < 5\gamma n^{k-1}$ . The same holds for all  $y, y' \in Y$ .*
- (iv) *For any  $x \in X$  and  $y \in Y$ , we have  $|N_H(x) \cap N_H(y)| \leq 4\gamma n^{k-1}$  and  $|\overline{N}_H(x) \cap \overline{N}_H(y)| \leq 4\gamma n^{k-1}$ , where  $\overline{N}_H(x) := \binom{V \setminus \{x\}}{k-1} \setminus N_H(x)$  consists of non-neighbors of  $x$ .*

**Proof.** To see (i), suppose  $x \in X$  and  $y \in Y$ . Since  $d_H(x_0), d_H(y_0) \geq (1/2 - \alpha) \binom{n-1}{k-1}$ , by the definition of  $X$  and  $Y$ ,

$$|N_H(y_0) \setminus N_H(x)|, |N_H(x_0) \setminus N_H(y)| \geq \left(\frac{1}{2} - \alpha\right) \binom{n-1}{k-1} - \gamma n^{k-1}. \quad (3)$$

Consequently, we have  $d_H(x), d_H(y) \leq (1/2 + \alpha) \binom{n-1}{k-1} + \gamma n^{k-1}$ .

To see (ii), suppose that there exists  $v \in X \cap Y$ . Then by (3),

$$\begin{aligned} |(N_H(x_0) \cup N_H(y_0)) \setminus N_H(v)| &\geq |N_H(x_0) \setminus N_H(v)| + |N_H(y_0) \setminus N_H(v)| - |N_H(x_0) \cap N_H(y_0)| \\ &\geq 2 \left( \frac{1}{2} - \alpha \right) \binom{n-1}{k-1} - 3\gamma n^{k-1} \geq \binom{n-1}{k-1} - 4\gamma n^{k-1}, \end{aligned}$$

which implies that  $|N_H(v)| \leq 4\gamma n^{k-1}$ , contradicting the minimum degree condition of  $H$ .

To see (iii), consider  $x \in X$ . By the definition of  $X$  and the minimum degree condition of  $H$ , we have  $|N_H(x) \cup N_H(y_0)| \geq 2 \left( \frac{1}{2} - \alpha \right) \binom{n-1}{k-1} - \gamma n^{k-1}$ . Let  $x' \in X \setminus \{x\}$ . Then

$$\begin{aligned} |N_H(x') \setminus N_H(x)| &= |(N_H(x') \setminus N_H(x)) \setminus N_H(y_0)| + |(N_H(x') \setminus N_H(x)) \cap N_H(y_0)| \\ &\leq \left| \binom{V \setminus \{x'\}}{k-1} \setminus (N_H(x) \cup N_H(y_0)) \right| + |N_H(x') \cap N_H(y_0)| \\ &\leq \binom{n-1}{k-1} - 2 \left( \frac{1}{2} - \alpha \right) \binom{n-1}{k-1} + \gamma n^{k-1} + \gamma n^{k-1} \\ &= 2\alpha \binom{n-1}{k-1} + 2\gamma n^{k-1}. \end{aligned} \tag{4}$$

The same bound holds for  $|N_H(x) \setminus N_H(x')|$ . Hence

$$\begin{aligned} |N_H(x) \triangle N_H(x')| &= |N_H(x') \setminus N_H(x)| + |N_H(x) \setminus N_H(x')| \\ &\leq 2 \left( 2\alpha \binom{n-1}{k-1} + 2\gamma n^{k-1} \right) < 5\gamma n^{k-1}. \end{aligned}$$

Analogously we can derive that  $|N_H(y) \triangle N_H(y')| < 5\gamma n^{k-1}$  for any  $y, y' \in Y$ .

To see (iv), consider  $x \in X$  and  $y \in Y$ . By (4), we have  $|N_H(x) \setminus N_H(x_0)| \leq 2\alpha \binom{n-1}{k-1} + 2\gamma n^{k-1}$ . By the definition of  $Y$ , we have  $|N_H(y) \cap N_H(x_0)| < \gamma n^{k-1}$ . Thus

$$\begin{aligned} |N_H(x) \cap N_H(y)| &\leq |N_H(y) \cap N_H(x_0)| + |N_H(x) \setminus N_H(x_0)| \\ &\leq 2\alpha \binom{n-1}{k-1} + 3\gamma n^{k-1} \leq 4\gamma n^{k-1}, \end{aligned} \tag{5}$$

which proves the first assertion of (iv). By the minimum degree condition and (5), we have

$$|N_H(x) \cup N_H(y)| \geq 2 \left( \frac{1}{2} - \alpha \right) \binom{n-1}{k-1} - 3\gamma n^{k-1} - 2\alpha \binom{n-1}{k-1}.$$

It follows that

$$\begin{aligned} |\overline{N}_H(x) \cap \overline{N}_H(y)| &\leq \binom{|V \setminus \{x, y\}|}{k-1} - |N_H(x) \cup N_H(y)| \\ &\leq \binom{n-2}{k-1} - (1 - 2\alpha) \binom{n-1}{k-1} + 3\gamma n^{k-1} + 2\alpha \binom{n-1}{k-1} \leq 4\gamma n^{k-1}, \end{aligned}$$

which proves the second assertion of (iv).  $\square$

Since  $|V_0| \leq \gamma n$  and  $X \cap Y = \emptyset$  we have  $|X| \geq (1-\gamma)n/2$  or  $|Y| \geq (1-\gamma)n/2$ . Without loss of generality we may assume that  $|X| \geq (1-\gamma)n/2 \geq n/3$ . Let  $0 < \gamma_0 < 1/2$  such that  $\frac{\gamma_0(1-\gamma_0)}{(k+1)!} = 5\gamma \cdot 3^{k-1}$ . We apply Lemma 11 to  $F = H[X]$  with  $\rho = \gamma_0$ . Since

$$|N_F(v) \Delta N_F(v')| \leq |N_H(v) \Delta N_H(v')| < 5\gamma n^{k-1} \leq \frac{\gamma_0(1-\gamma_0)}{(k+1)!} |X|^{k-1}$$

for any  $v, v' \in X$  (Claim 13 (iii)), there are two possible cases:

**Case 1**  $e(H[X]) \leq \gamma_0 \binom{|X|}{k}$ ,

**Case 2**  $e(H[X]) \geq (1-\gamma_0) \binom{|X|}{k}$ .

In the rest of the proof we assume that one of the two cases holds. Once we have obtained more information we will prove that  $|X|$  and  $|Y|$  are close to  $n/2$ . At present we require the following weaker lower bounds on  $|X|$  and  $|Y|$ .

**Claim 14.**  $|X|, |Y| \geq (1 - (\frac{1}{2} + 2\gamma_0)^{\frac{1}{k-1}} - \gamma)n$ .

**Proof.** The bound on  $|X|$  follows since  $|X| \geq (1-\gamma)n/2$ . Since  $|X| + |Y| + |V_0| = n$  and  $|V_0| \leq \gamma n$ , to prove the bound on  $|Y|$ , it suffices to show that  $|X| \leq (\frac{1}{2} + 2\gamma_0)^{\frac{1}{k-1}} n$ . In Case 1, there exists a vertex  $x \in X$  such that  $d_{H[X]}(x) \leq \gamma_0 \binom{|X|-1}{k-1}$  and consequently,  $|\bar{N}_{H[X]}(x)| \geq (1-\gamma_0) \binom{|X|-1}{k-1}$ . Together with the minimum degree condition, this gives

$$\left(\frac{1}{2} - \alpha\right) \binom{n-1}{k-1} \leq d_H(x) \leq \binom{n-1}{k-1} - (1-\gamma_0) \binom{|X|-1}{k-1}.$$

In Case 2, there exists a vertex  $x \in X$  such that  $d_{H[X]}(x) \geq (1-\gamma_0) \binom{|X|-1}{k-1}$ . By Claim 13 (i),

$$(1-\gamma_0) \binom{|X|-1}{k-1} \leq d_{H[X]}(x) \leq \left(\frac{1}{2} + \alpha\right) \binom{n-1}{k-1} + \gamma n^{k-1}.$$

In either case we have  $(1-\gamma_0) \binom{|X|-1}{k-1} \leq (\frac{1}{2} + \alpha) \binom{n-1}{k-1} + \gamma n^{k-1}$ , which implies that  $\binom{|X|-1}{k-1} \leq (1+2\gamma_0) \left( (\frac{1}{2} + \alpha) \binom{n-1}{k-1} + \gamma n^{k-1} \right)$ . Letting  $|X| = cn$ , it follows that

$$c^{k-1} \binom{n-1}{k-1} - O(n^{k-2}) \leq \left(\frac{1}{2} + \gamma_0 + 2\alpha\right) \binom{n-1}{k-1} + 2\gamma n^{k-1}$$

Since  $\gamma n^{k-1} \leq \frac{\gamma_0}{5} \binom{n-1}{k-1}$ , we conclude that  $c \leq (\frac{1}{2} + 2\gamma_0)^{\frac{1}{k-1}}$ . □

Given two disjoint subsets  $A, B \subseteq V$  and two integers  $i, j \geq 0$ , we call an  $(i+j)$ -set  $S \subseteq V$  an  $A^i B^j$ -set if  $|S \cap A| = i$  and  $|S \cap B| = j$ , and let  $A^i B^j$  denote the family of all  $A^i B^j$ -sets. Let  $c_0 := 1 - (\frac{1}{2} + 2\gamma_0)^{\frac{1}{k-1}} - \gamma$ , and  $\gamma_i := \gamma_{i-1} + 5\gamma k! / c_0^{k-1}$  for  $i = 1, \dots, k$ .

**Claim 15.** (i) *In Case 1, for  $1 \leq i \leq k$ , for any  $y \in Y$ , at least  $(1-\gamma_i) \binom{|X|}{k-i} \binom{|Y|}{i-1} X^{k-i} Y^{i-1}$ -sets are neighbors (respectively, non-neighbors) of  $y$  if  $i$  is odd (respectively, even).*

(ii) In Case 2, for  $1 \leq i \leq k$ , for any  $y \in Y$ , at least  $(1 - \gamma_i) \binom{|X|}{k-i} \binom{|Y|}{i-1}$   $X^{k-i}Y^{i-1}$ -sets are neighbors (respectively, non-neighbors) of  $y$  if  $i$  is even (respectively, odd).

**Proof.** We prove both cases by induction on  $i$ . In Case 1 there exists a vertex  $x_1 \in X$  such that  $d_{H[X]}(x_1) \leq \gamma_0 \binom{|X|-1}{k-1}$  and consequently  $|\overline{N}_H(x_1) \cap \binom{X}{k-1}| \geq (1 - \gamma_0) \binom{|X|-1}{k-1}$ . Fix a vertex  $y \in Y$ . By Claim 13 (iv),  $|\overline{N}_H(x_1) \cap \overline{N}_H(y)| \leq 4\gamma n^{k-1}$ . Thus at least  $(1 - \gamma_0) \binom{|X|-1}{k-1} - 4\gamma n^{k-1}$   $X^{k-1}$ -sets are neighbors of  $y$ . By Claim 14,  $|X|, |Y| \geq c_0 n$ . Then for any  $0 \leq i \leq k - 1$ ,

$$\binom{|X|}{k-i-1} \binom{|Y|}{i} \geq \frac{(c_0 n)^{k-1}}{(k-1)!} - O(n^{k-2}) \geq \frac{c_0^{k-1} n^{k-1}}{k!}. \quad (6)$$

Together with the definition of  $\gamma_1$ , we conclude that at least

$$(1 - \gamma_0) \binom{|X| - 1}{k - 1} - 4\gamma n^{k-1} \geq (1 - \gamma_0) \binom{|X|}{k - 1} - O(n^{k-2}) - \frac{4\gamma k!}{c_0^{k-1}} \binom{|X|}{k - 1} \geq (1 - \gamma_1) \binom{|X|}{k - 1}$$

$X^{k-1}$ -sets are neighbors of  $y$ . This confirms (i) for  $i = 1$ . In Case 2, by averaging, there exists a vertex  $x_1 \in X$  such that  $d_{H[X]}(x_1) \geq (1 - \gamma_0) \binom{|X|-1}{k-1}$ . Fix a vertex  $y \in Y$ . By Claim 13 (iv),  $|N_H(x_1) \cap N_H(y)| \leq 4\gamma n^{k-1}$ . Thus at least  $(1 - \gamma_0) \binom{|X|-1}{k-1} - 4\gamma n^{k-1} \geq (1 - \gamma_1) \binom{|X|}{k-1}$   $X^{k-1}$ -sets are non-neighbors of  $y$ . This confirms (ii) for  $i = 1$ .

For the induction step, we first assume that for some  $1 \leq i \leq k$ , every  $y \in Y$  has at least  $(1 - \gamma_i) \binom{|X|}{k-i} \binom{|Y|}{i-1}$   $X^{k-i}Y^{i-1}$ -sets in its neighborhood. Consequently at least  $(1 - \gamma_i) \binom{|X|}{k-i} \binom{|Y|}{i}$   $X^{k-i}Y^i$ -sets are edges of  $H$ . By averaging, there exists  $x_i \in X$  whose neighborhood contains at least  $(1 - \gamma_i) \binom{|X|-1}{k-i-1} \binom{|Y|}{i}$   $X^{k-i-1}Y^i$ -sets. Fix  $y \in Y$ . By Claim 13 (iv),

$$|(N_H(x_i) \cap X^{k-i-1}Y^i) \setminus N_H(y)| \geq (1 - \gamma_i) \binom{|X| - 1}{k - i - 1} \binom{|Y|}{i} - 4\gamma n^{k-1}.$$

Since  $|\overline{N}_H(y) \cap X^{k-i-1}Y^i| \geq |(N_H(x_i) \cap X^{k-i-1}Y^i) \setminus N_H(y)| - O(n^{k-2})$ , we conclude that at least

$$(1 - \gamma_i) \binom{|X| - 1}{k - i - 1} \binom{|Y|}{i} - 4\gamma n^{k-1} - O(n^{k-2}) \geq (1 - \gamma_{i+1}) \binom{|X|}{k - i - 1} \binom{|Y|}{i}$$

$X^{k-i-1}Y^i$ -sets are non-neighbors of  $y$ , where we use (6) and the definition of  $\gamma_{i+1}$ . Analogously we can show that if for some  $1 \leq i \leq k$ , every  $y \in Y$  has at least  $(1 - \gamma_i) \binom{|X|}{k-i} \binom{|Y|}{i-1}$   $X^{k-i}Y^{i-1}$ -sets as non-neighbors, then at least  $(1 - \gamma_{i+1}) \binom{|X|}{k-i-1} \binom{|Y|}{i}$   $X^{k-i-1}Y^i$ -sets are neighbors of  $y$ . This completes our induction proof.  $\square$

**Claim 16.**  $|X|, |Y| \geq (1 - \eta)n/2$ , where  $\eta := (2\gamma_k)^{1/(k-1)} + \gamma$ .

**Proof.** Suppose that Claim 15 (i) holds (the proof when Claim 15 (ii) holds is analogous).

Let  $\tilde{n} := |X \cup Y|$ . Note that  $\tilde{n} \geq (1 - \gamma)n$  because  $|V_0| \leq \gamma n$ . Let  $c := |X|/\tilde{n}$ . It suffices to show that  $(1 - (2\gamma_k)^{1/(k-1)})/2 \leq c \leq (1 + (2\gamma_k)^{1/(k-1)})/2$  because this implies that

$$|X| = c\tilde{n} \geq \frac{1}{2}(1 - (2\gamma_k)^{1/(k-1)})(1 - \gamma)n > (1 - \eta)\frac{n}{2}$$

and  $|Y| = (1 - c)\tilde{n} \geq \frac{1}{2}(1 - (2\gamma_k)^{1/(k-1)})(1 - \gamma)n > (1 - \eta)n/2$ .

For any  $y \in Y$ , by Claim 15 (i),

$$d_H(y) \geq \sum_{1 \leq i \leq k, i \text{ odd}} (1 - \gamma_i) \binom{|X|}{k-i} \binom{|Y|}{i-1} \geq (1 - \gamma_k) \sum_{0 \leq j \leq k-1, j \text{ even}} \binom{|X|}{k-1-j} \binom{|Y|}{j}.$$

Hence, by Proposition 12,  $d_H(y) \geq (1 - \gamma_k) \frac{\tilde{n}^{k-1}}{2(k-1)!} (1 + (2c-1)^{k-1}) - O(n^{k-2})$ . If  $(2c-1)^{k-1} \geq 2\gamma_k$ , then

$$d_H(y) \geq (1 - \gamma_k) \frac{(1 - \gamma)^{k-1} n^{k-1}}{2(k-1)!} (1 + 2\gamma_k) - O(n^{k-2}) \geq \left(1 + \frac{\gamma_k}{2}\right) \frac{n^{k-1}}{2(k-1)!},$$

as  $5\gamma(k-1)! < \gamma_k \ll 1$ . This contradicts Claim 13 (i). Thus  $(2c-1)^{k-1} < 2\gamma_k$ . If  $c \geq 1/2$ , then  $c < (1 + (2\gamma_k)^{1/(k-1)})/2$ ; if  $c < 1/2$  and  $k-1$  is even, then  $(1 - 2c)^{k-1} = (2c-1)^{k-1} < 2\gamma_k$  and thus  $c > (1 - (2\gamma_k)^{1/(k-1)})/2$ . In either case we are done. Otherwise assume that  $c < 1/2$  and  $k-1$  is odd. By Claim 15 (i),

$$\bar{d}_H(y) \geq \sum_{1 \leq i \leq k, i \text{ even}} (1 - \gamma_i) \binom{|X|}{k-i} \binom{|Y|}{i-1} \geq (1 - \gamma_k) \sum_{0 \leq j \leq k-1, j \text{ odd}} \binom{|X|}{k-1-j} \binom{|Y|}{j},$$

where  $\bar{d}_H(y) := |\bar{N}_H(y)|$ . By Proposition 12, we have  $\bar{d}_H(y) \geq (1 - \gamma_k) \frac{\tilde{n}^{k-1}}{2(k-1)!} (1 - (2c-1)^{k-1}) - O(n^{k-2})$ . If  $1 - (2c-1)^{k-1} = 1 + (1 - 2c)^{k-1} \geq 1 + 2\gamma_k$ , then we obtain a contradiction as before because  $\bar{d}_H(y) \leq (\frac{1}{2} + \alpha) \binom{n-1}{k-1}$ . Hence,  $(1 - 2c)^{k-1} < 2\gamma_k$  and consequently  $c > (1 - (2\gamma_k)^{1/(k-1)})/2$ , as desired.  $\square$

By Claim 16, there exists a partition  $X', Y'$  of  $V$  such that  $|X'| = \lceil n/2 \rceil$ ,  $|Y'| = \lfloor n/2 \rfloor$  and  $|X \cap X'|, |Y \cap Y'| \geq (1 - \eta)n/2$ . We claim that  $H$  is  $\varepsilon$ -close to  $\mathcal{B}_{n,k}(Y', X')$  in Case 1 and  $\varepsilon$ -close to  $\bar{\mathcal{B}}_{n,k}(Y', X')$  in Case 2. Indeed, set  $\mathcal{B} := \mathcal{B}_{\tilde{n},k}(Y, X)$ , where  $\tilde{n} := |X \cup Y|$ , and  $H' := H[X \cup Y]$ . By definition,  $E(\mathcal{B})$  consists of all  $X^{k-i}Y^i$ -sets for all odd  $0 \leq i \leq k$ . If Claim 15 (i) holds, then  $|E(\mathcal{B}) \cap E(H')| \geq (1 - \gamma_k)|E(\mathcal{B})|$  and  $|E(\bar{\mathcal{B}}) \cap E(\bar{H}')| \geq (1 - \gamma_k)|E(\bar{\mathcal{B}})|$ . Thus

$$|E(\mathcal{B}) \Delta E(H')| = |E(\mathcal{B}) \setminus E(H')| + |E(\bar{\mathcal{B}}) \setminus E(\bar{H}')| \leq \gamma_k |E(\mathcal{B})| + \gamma_k |E(\bar{\mathcal{B}})| \leq \gamma_k \binom{n}{k}.$$

Let  $V' := (X \cap X') \cup (Y \cap Y')$ . Then  $|V'| \geq (1 - \eta)n$  and

$$\left| (E(\mathcal{B}_{n,k}(Y', X')) \Delta E(H)) \setminus \binom{V'}{k} \right| \leq \eta n \binom{n-1}{k-1}.$$

Since  $\gamma \ll \varepsilon, 1/k$ , we have that  $\gamma_k, \eta \ll \varepsilon$ . Therefore,

$$|E(\mathcal{B}_{n,k}(Y', X')) \Delta E(H)| \leq \eta n \binom{n-1}{k-1} + |E(\mathcal{B}) \Delta E(H')| \leq \eta n \binom{n-1}{k-1} + \gamma_k \binom{n}{k} \leq \varepsilon \binom{n}{k}.$$

which implies that  $H$  is  $\varepsilon$ -close to  $\mathcal{B}_{n,k}(Y', X')$ . Analogously we can show that  $H$  is  $\varepsilon$ -close to  $\overline{\mathcal{B}}_{n,k}(Y', X')$  in Case 2. This completes the proof of Lemma 10.  $\square$

### 3 An application of Lemma 9

The following simple application of Lemma 9 implies that the minimum  $\ell$ -degree condition that forces a perfect fractional matching also forces a perfect matching in a  $k$ -uniform hypergraph  $H$ , if we additionally assume that  $H$  has a small number of vertices of large degree.

**Theorem 17.** *Given any  $0 < \varepsilon \leq \delta'$  and  $k, \ell \in \mathbb{N}$  where  $\ell < k$ , there is an  $n_0 \in \mathbb{N}$  such that the following holds. Let  $H$  be a  $k$ -uniform hypergraph on  $n \geq n_0$  vertices where  $k$  divides  $n$ . Suppose that  $\delta_1(H) \geq \delta' \binom{n-1}{k-1}$  and  $\delta_\ell(H) \geq (c_{k,\ell}^* + \varepsilon) \binom{n-\ell}{k-\ell}$ . If there are at least  $\varepsilon n$  vertices  $x \in V(H)$  so that  $d_H(x) \geq (1 - \delta' + \varepsilon) \binom{n-1}{k-1}$  then  $H$  contains a perfect matching.*

**Sketch proof.** It is easy to see that  $H$  satisfies Lemma 9 (ii) (where we choose  $0 < \gamma \ll \varepsilon$ ) and so by Lemma 8,  $H$  contains a small absorbing matching  $M$ . Let  $H' := H \setminus V(M)$ . Then  $\delta_\ell(H') \geq (c_{k,\ell}^* + \varepsilon/2) \binom{n-\ell}{k-\ell}$  and so by Lemma 7,  $H'$  contains a matching covering all but a very small set of vertices. After absorbing the uncovered vertices by  $M$ , we obtain a perfect matching in  $H$ .  $\square$

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