The large-time solution of Burgers' equation with time dependent coefficients. I. The coefficients are exponential functions

Leach, John

DOI:
10.1111/sapm.12098

The large-time solution of Burgers' equation with time dependent coefficients. I. The coefficients are exponential functions.

J.A. Leach

School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK

January 22, 2016

Abstract

In this paper, we consider an initial-value problem for Burgers' equation with variable coefficients

$$u_t + \Phi(t)\, uu_x = \Psi(t)\, u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$

where x and t represent dimensionless distance and time respectively and $\Psi(t)$, $\Phi(t)$ are given functions of t. In particular, we consider the case when the initial data has algebraic decay as $|x| \to \infty$, with $u(x,t) \to u_+$ as $x \to \infty$ and $u(x,t) \to u_-$ as $x \to -\infty$. The constant states u_+ and u_- $(\neq u_+)$ are problem parameters. Two specific initial-value problems are considered. In initial-value problem 1 we consider the case when $\Phi(t) = e^t$ and $\Psi(t) = 1$, while in initial-value problem 2 we consider the case when $\Phi(t) = 1$ and $\Psi(t) = e^t$. The method of matched asymptotic coordinate expansions is used to obtain the large-t asymptotic structure of the solution to both initial-value problems over all parameter values.

This paper is to appear in *Stud. Appl. Math.* DOI: 10.1111/sapm.12098.

1 Introduction

In this series of papers we consider the following initial-value problem for Burgers' equation with variable coefficients, namely,

$$u_t + \Phi(t)\, uu_x = \Psi(t)\, u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad -\infty < x < \infty, \tag{1.2}$$

$$u(x,t) \to \begin{cases} u_-, & x \to -\infty, \\ u_+, & x \to \infty, \end{cases} \quad t \geq 0, \tag{1.3}$$

where u_- and $u_+(\neq u_-)$ are parameters and the functions $\Phi(t)$ and $\Psi(t)$ belong to one of the classes outlined below. Further, we assume that the initial data $u_0(x)$ has algebraic decay as $|x| \to \infty$. Specifically,

$$u_0(x) = \begin{cases} u_- + \dfrac{A_L}{(-x)^\gamma} + O\left(e^{-|x|}\right) & \text{as} \quad x \to -\infty, \\ u_+ + \dfrac{A_R}{x^\gamma} + O\left(e^{-|x|}\right) & \text{as} \quad x \to \infty, \end{cases} \tag{1.4}$$

where $A_L\,(\neq 0), A_R\,(\neq 0)$ and $\gamma\,(> 0)$ are parameters. We observe that if $u_+ > u_-$ then $A_L > 0$ and $A_R < 0$, whereas if $u_+ < u_-$ then $A_L < 0$ and $A_R > 0$. We note that equation (1.1) is a canonical equation combining both convection and diffusion and as such arises in a wide range of applications.

Regarding the time dependent functions $\Phi(t)$ and $\Psi(t)$ we will consider the following pairs:

(i) $\Phi(t) = e^t, \quad \Psi(t) = 1.$

(ii) $\Phi(t) = 1, \quad \Psi(t) = e^t.$

(iii) $\Phi(t) = t^\delta \, \big(-1 < \delta \, (\neq 0) \big), \quad \Psi(t) = 1.$

Cases (i) and (ii) where the coefficients are exponential in t will be considered in this current paper while case (iii) when the coefficient of the nonlinear convection term is algebraic in t will be considered in part II of this series of papers [15]. We observe that equation (1.1) with $\Phi(t) = t^{-1}$ and $\Psi(t) = 1$ ($\Phi(t) = 1$ and $\Psi(t) = t^{-1}$) can be transformed to case (ii) (case (i)) respectively, by the change of variable $\tau = \ln t$. The more general situation of equation (1.1) with $\Phi(t) = t^\alpha$ ($\alpha > -1$) and $\Psi(t) = t^\beta$ ($\beta > -1$) where $\alpha \neq \beta$ can be transformed to case (iii) by the change of variables $u = (\beta+1)^{-\delta}\bar{u}$, $(\beta+1)\tau = t^{\beta+1}$, where $\delta = \frac{\alpha-\beta}{\beta+1} \, (\in (-1, \infty))$.

We note that equation (1.1) is related to the generalized Burgers' equation

$$v_\tau + v v_x + f(\tau) v = v_{xx}, \tag{1.5}$$

where $f(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau} \ln\left(\frac{\Psi}{\Phi}\right)$, via the transformation

$$v(x,\tau) = \frac{\Phi(t)}{\Psi(t)} u(x,t), \quad \tau = \int^t \Psi(s)\,\mathrm{d}s. \tag{1.6}$$

When $f(\tau) = 0$ equation (1.5) reduces to the classical Burgers' equation

$$u_\tau + u u_x = u_{xx}.$$

Although this equation is named after J.M. Burgers for his work on the theory of turbulence [3], the equation had already appeared in the works of A.R Forsyth [10] and H. Bateman [2] which preceded Burgers work on turbulence. Burgers' equation is a canonical equation combining diffusion and nonlinear convection and as such arises in the modelling of many physical phenomena involving diffusion-convection processes ([25], [6], [21] and [13]). It can be reduced to the heat equation by the Cole-Hopf transformation ([6] and [13]) and then solved in a straightforward manner.

The generalized Burgers' equation (1.5) when $f(\tau) \neq 0$ has many applications in mathematical physics and was introduced in [18]. For example, when $f(\tau) = -1$ equation (1.5) gives a model of nonlinear interaction of long wave pumping with short wave dissipation [12], while when $f(\tau) = \frac{\lambda}{\tau}$ (where λ is a constant) equation (1.5) models the propagation of finite-amplitude sound waves in variable area ducts [[8], [9], [20] and [22]]. However, it has been established in [19] that there is no Bäcklund transformation for the generalized Burgers' equation (1.5), and hence it is doubtful that a linearizing transformation like the Cole-Hopf transformation exists in this case. Therefore, other methods of solution need to be investigated for this important class of equation.

Our interest in equation (1.1) is motivated by its relationship, via transformation (1.6), to the generalized Burgers' equation (1.5). For example, equation (1.1) with $\Psi(t) = e^t$ and $\Phi(t) = 1$ (case (ii) above) is related to equation (1.5) with $f(\tau) = \frac{1}{\tau}$, whereas equation (1.1) with $\Psi(t) = 1$ and $\Phi(t) = e^t$ (case (i) above) is related to equation (1.5) with $f(\tau) = -1$. However, as we shall see it is more convenient for the analysis presented in this paper to work with equation (1.1) rather than equation (1.5). We note that equation (1.1) when $\Phi(t) = 1$ and where the coefficient of u_{xx} is a function of time was introduced in [14] as a model of viscous effects in sound waves of finite amplitude, and derived in [11] as a model for the propagation of weakly nonlinear acoustic waves under the impact of geometrical spreading. It should be noted that in many applications (see for example, [23] and [5]) the coefficient of u_{xx} is regularly approximated by a constant when in fact it is a function of time. This class of generalized Burgers' equations have also been analyzed by the similarity method in [7] and [4].

In this present paper we use the method of matched asymptotic coordinate expansions (see for example [26], [16] and [17]) to obtain the complete large-time solution of initial-value problem (1.1)-(1.4) in cases (i) (when $\Phi(t) = e^t$ and $\Psi(t) = 1$) and (ii) (when $\Phi(t) = 1$ and $\Psi(t) = e^t$) over all parameter values. Throughout we use the nomenclature of the method of matched asymptotic expansions, as given in [26]. The large-time structure of the solution of the initial-value problem is obtained by careful consideration of the asymptotic structures as $t \to 0$ $(-\infty < x < \infty)$ and as $|x| \to \infty$ $(t \geq O(1))$. The form of the large-time solution of initial-value problem (1.1)-(1.4) when $\Phi(t) = e^t$ and $\Psi(t) = 1$ depends on the problem parameters u_+ and u_- as follows:

- When $u_+ > u_-$ the solution exhibits the formation of the expansion wave.
- When $u_+ < u_-$ the solution exhibits the formation of a localized Taylor shock profile.

In both cases the large-time attractor, the expansion wave or localized Taylor shock profile, connects $u = u_+$ to $u = u_-$. The form of the large-t solution of initial-value (1.1)-(1.4) when $\Phi(t) = 1$ and $\Psi(t) = e^t$ exhibits the formation of an error function profile connecting $u = u_+$ to $u = u_-$. Full details of these results are given in Section 4. In particular, we have shown in Sections 2 and 3, using the method of matched coordinate expansions, how the initial data is connected to the large-time attractor for the given initial-value problem.

2 Initial-value problem 1

In this section we consider the following initial-value problem for Burgers' equation, namely,

$$u_t + e^t\, u u_x = u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{2.1}$$

$$u(x,0) = u_0(x), \quad -\infty < x < \infty, \tag{2.2}$$

$$u(x,t) \to \begin{cases} u_-, & x \to -\infty, \\ u_+, & x \to \infty, \end{cases} \quad t \geq 0, \tag{2.3}$$

where u_- and $u_+ (\neq u_-)$ are parameters. Further, we assume that the initial data $u_0(x)$ has algebraic decay as $|x| \to \infty$. Specifically,

$$u_0(x) = \begin{cases} u_- + \dfrac{A_L}{(-x)^\gamma} + O\left(e^{-|x|}\right) & \text{as} \quad x \to -\infty, \\ u_+ + \dfrac{A_R}{x^\gamma} + O\left(e^{-|x|}\right) & \text{as} \quad x \to \infty, \end{cases} \tag{2.4}$$

where $A_L\,(\neq 0), A_R\,(\neq 0)$ and $\gamma\,(> 0)$ are parameters. In what follows we label initial-value problem (2.1)-(2.4) as **IVP1**.

In this section we develop the asymptotic structure of **IVP1** as $t \to \infty$ when $u_+ > u_-$ and when $u_+ < u_-$. We must first begin by examining the asymptotic structure of the solution of **IVP1** as $t \to 0$.

2.1 Asymptotic solution as $t \to 0$

We first consider region I, where $x = O(1)$ as $t \to 0$ and expand the solution to **IVP1** as

$$u(x,t) = u_0(x) + u_1(x)t + O\left(t^2\right) \tag{2.5}$$

as $t \to 0$. On substituting into equation (2.1) and applying (2.2) we readily obtain

$$u(x,t) = u_0(x) + \left[u_0''(x) - u_0(x)u_0'(x)\right] t + O\left(t^2\right) \tag{2.6}$$

as $t \to 0$ with $x = O(1)$. Now for $x \gg 1$, expansion (2.5), with $(2.4)_2$, takes the form

$$u(x,t) \sim \left(u_+ + \frac{A_R}{x^\gamma} + \dots\right) + \left(\frac{\gamma u_+ A_R}{x^{\gamma+1}} + \frac{\gamma(\gamma+1)A_R}{x^{\gamma+2}} + \frac{\gamma A_R^2}{x^{2\gamma+1}} + \dots\right) t + \dots \tag{2.7}$$

as $t \to 0$, and we conclude that (2.6) remains uniform for $x \gg 1$ as $t \to 0$. We observe that the ordering of the terms in (2.7) depends on γ, a point we will return to in the next section. Finally, for $(-x) \gg 1$, expansion (2.5), with $(2.4)_1$, takes the form

$$u(x,t) \sim \left(u_- + \frac{A_L}{(-x)^\gamma} + \dots\right) + \left(\frac{\gamma u_- A_L}{(-x)^{\gamma+1}} + \frac{\gamma(\gamma+1)A_L}{(-x)^{\gamma+2}} + \frac{\gamma A_L^2}{(-x)^{2\gamma+1}} + \dots\right) t + \dots \tag{2.8}$$

as $t \to 0$, and we conclude that (2.6) remains uniform for $(-x) \gg 1$ as $t \to 0$.

This completes the asymptotic structure as $t \to 0$, with expansion (2.6) of region I providing a uniform approximation to the solution of **IVP1** as $t \to 0$ with $x = O(1)$.

2.2 Asymptotic solution as $|x| \to \infty$

We now investigate the structure of the solution to **IVP1** as $|x| \to \infty$ for $t = O(1)$. We begin be developing the structure of the solution to **IVP1** as $x \to \infty$ with $t = O(1)$. The form of (2.7) indicates that in this region, region II^+, we expand as

$$u(x,t) = u_+ + \frac{f_0(t)}{x^\gamma} + \frac{f_1(t)}{x^{\gamma+1}} + \frac{f_2(t)}{x^r} + \frac{f_3(t)}{x^s} + o\left(\frac{1}{x^s}\right) \tag{2.9}$$

as $x \to \infty$ with $t = O(1)$. A balancing of terms requires that

$$r = \begin{cases} 2\gamma+1, & 0 < \gamma < 1, \\ 3, & \gamma = 1, \\ \gamma+2, & \gamma > 1, \end{cases}$$

$$s = \begin{cases} \gamma+2, & 0 < \gamma < 1, \\ 4, & \gamma = 1, \\ 2\gamma+1, & \gamma > 1. \end{cases}$$

On substituting (2.9) into equation (2.1) and solving at each order in turn we find (after matching with (2.7) as $t \to 0$) that

$$f_0(t) = A_R, \quad f_1(t) = \gamma u_+ A_R e^t,$$

$$f_2(t) = \begin{cases} \gamma A_R^2 e^t, & 0 < \gamma < 1, \\ u_+^2 A_R e^{2t} + A_R^2 e^t + 2A_R t + (2 - u_+^2)A_R, & \gamma = 1, \\ \frac{\gamma(\gamma+1)u_+^2 A_R}{2} e^{2t} + \gamma(\gamma+1)A_R t + \frac{\gamma(\gamma+1)A_R}{2}(2 - u_+^2), & \gamma > 1. \end{cases}$$

The function $f_3(t)$ is only required in what follows in the case $\gamma > 1$ and for brevity we do not report $f_3(t)$ for $0 < \gamma < 1$ or $\gamma = 1$. We readily obtain that

$$f_3(t) = \gamma A_R^2 e^t, \quad \gamma > 1.$$

Thus, the structure of (2.9) depends upon whether $0 < \gamma < 1$, $\gamma = 1$ or $\gamma > 1$. However, in each case, we observe that expansion (2.9) remains uniform for $t \gg 1$ provided that $x \gg e^t$, but expansion (2.9) becomes nonuniform when $x = O(e^t)$ for $t \gg 1$. We note that when $x = O(e^t)$ for $t \gg 1$ expansion (2.9) fails to provide a uniform asymptotic approximation to the solution to **IVP1** due to the second and third terms of (2.9) becoming of comparable order.

We conclude be developing the structure of the solution to **IVP1** as $x \to -\infty$ with $t = O(1)$. The form of (2.8) indicates that in this region, region II$^-$, we expand as

$$u(x,t) = u_- + \frac{\hat{f}_0(t)}{(-x)^\gamma} + \frac{\hat{f}_1(t)}{(-x)^{\gamma+1}} + \frac{\hat{f}_2(t)}{(-x)^r} + \frac{\hat{f}_3(t)}{(-x)^s} + o\left(\frac{1}{(-x)^s}\right) \tag{2.10}$$

as $x \to -\infty$ with $t = O(1)$, and where r and s are as given above. On substituting (2.10) into equation (2.1) and solving at each order in turn we find (after matching with (2.8) as $t \to 0$) that

$$\hat{f}_0(t) = A_L, \quad \hat{f}_1(t) = -\gamma u_- A_L e^t,$$

$$\hat{f}_2(t) = \begin{cases} -\gamma A_L^2 e^t, & 0 < \gamma < 1, \\ u_-^2 A_L e^{2t} - A_L^2 e^t + 2A_L t + (2 - u_-^2) A_L, & \gamma = 1, \\ \frac{\gamma(\gamma+1)u_-^2 A_L}{2} e^{2t} + \gamma(\gamma+1) A_L t + \frac{\gamma(\gamma+1)A_L}{2}(2 - u_-^2), & \gamma > 1, \end{cases}$$

and

$$\hat{f}_3(t) = -\gamma A_L^2 e^t, \quad \gamma > 1.$$

Thus, the structure of (2.10) again depends upon whether $0 < \gamma < 1$, $\gamma = 1$ or $\gamma > 1$. However, in each case, we observe that expansion (2.10) remains uniform for $t \gg 1$ provided that $-x \gg e^t$, but expansion (2.9) becomes nonuniform when $(-x) = O(e^t)$ for $t \gg 1$.

At this stage of the analysis we need to consider the cases $u_+ > u_-$ and $u_+ < u_-$ separately. We begin by developing the asymptotic structure of **IVP1** as $t \to \infty$ when $u_+ > u_-$.

2.3 Asymptotic solution as $t \to \infty$ when $u_+ > u_-$

We now investigate the structure of **IVP1** as $t \to \infty$ when $u_+ > u_-$. We recall from Section 2.2 that expansions (2.9) and (2.10) of regions II$^+$ ($x \to \infty, t = O(1)$) and II$^-$ ($x \to -\infty, t = O(1)$) respectively, continue to remain uniform provided $|x| \gg e^t$. However, as already noted a nonuniformity develops when $|x| = O(e^t)$. We begin by considering the asymptotic structure as $t \to \infty$ for $x > 0$. To proceed we introduce a new region, region III$^+$. To examine region III$^+$ we introduce the scaled coordinate

$$y = x\, e^{-t}, \tag{2.11}$$

where $y = O(1)$, and look for an expansion of the form

$$u(y,t) = u_+ + \begin{cases} g_0(y)e^{-\gamma t} + g_1(y)e^{-2\gamma t} + o\left(e^{-2\gamma t}\right), & \text{when } 0 < \gamma < 2, \\ g_0(y)e^{-2t} + g_1(y)te^{-4t} + g_2(y)e^{-4t} + o\left(e^{-4t}\right), & \text{when } \gamma = 2, \\ g_0(y)e^{-\gamma t} + g_1(y)te^{-(\gamma+2)t} + g_2(y)e^{-(\gamma+2)t} + g_3(y)e^{-2\gamma t} + o\left(e^{-2\gamma t}\right), & \text{when } \gamma > 2, \end{cases} \tag{2.12}$$

as $t \to \infty$ with $y = O(1)$. We first consider the case when $0 < \gamma < 2$. After substituting $(2.12)_1$ into equation (2.1) (when written in terms of y and t), the leading order problem for $g_0(y)$ becomes

$$(y-u_+)g_0' + \gamma g_0 = 0, \quad y>0, \tag{2.13}$$

$$g_0(y) \sim \frac{A_R}{y^\gamma} + \frac{\gamma u_+ A_R}{y^{\gamma+1}} + \frac{u_+^2 A_R}{y^{\gamma+2}} \quad \text{as} \quad y\to\infty, \tag{2.14}$$

with condition (2.14) arising from matching expansion $(2.12)_1$ ($y \gg 1$) with the far field expansion (2.9) ($x = O(e^t)$). The solution of (2.13), (2.14) is readily obtained as

$$g_0(y) = A_R\,(y-u_+)^{-\gamma}\,, \quad y > u_+. \tag{2.15}$$

Hence, via (2.15), $g_0(y)$ develops a singularity as $y \to (u_+)^+$, and thus expansion $(2.12)_1$ becomes nonuniform when $y = u_+ + o(1)$ as $t\to\infty$. On returning to next order, we obtain the following problem for $g_1(y)$, namely

$$(y-u_+)g_1' + 2\gamma g_1 = g_0 g_0', \quad y>u_+, \tag{2.16}$$

$$g_1(y) \sim \frac{\gamma A_R^2}{y^{2\gamma+1}} \quad \text{as} \quad y\to\infty, \tag{2.17}$$

with condition (2.17) arising from matching expansion $(2.12)_1$ ($y \gg 1$) with the far field expansion (2.9) ($x = O(e^t)$). The solution of (2.16), (2.17) is readily obtained as

$$g_1(y) = \gamma A_R^2\,(y-u_+)^{-(2\gamma+1)}\,, \quad y>u_+. \tag{2.18}$$

Therefore, we have, via $(2.12)_1$, (2.15) and (2.18), that when $0<\gamma<2$ the expansion in region III$^+$ has the form

$$u(y,t) = u_+ + A_R\,(y-u_+)^{-\gamma}\,e^{-\gamma t} + \gamma A_R^2\,(y-u_+)^{-(2\gamma+1)}\,e^{-2\gamma t} + o\left(e^{-2\gamma t}\right) \tag{2.19}$$

as $t\to\infty$ with $y = O(1) (> u_+)$, and where $A_R < 0$. We note that expansion (2.19) becomes nonuniform when

$$y = u_+ + O\left(\exp\left\{-\frac{\gamma}{\gamma+1}t\right\}\right) \tag{2.20}$$

as $t\to\infty$. This nonuniformity occurs due to the second and third terms in expansion (2.19) becoming of the same order when y is given by (2.19).

We next consider $\gamma = 2$. On substituting $(2.12)_2$ into equation (2.1) and solving at each order in turn, we find in region III$^+$ (after matching with (2.9)) that

$$u(y,t) = u_+ + \frac{A_R e^{-2t}}{(y-u_+)^2} + \frac{6A_R t e^{-4t}}{(y-u_+)^4} + \left(\frac{2A_R^2}{(y-u_+)^5} + \frac{3A_R(2-u_+^2)}{(y-u_+)^4}\right)e^{-4t} + o\left(e^{-4t}\right) \tag{2.21}$$

as $t\to\infty$ with $y = O(1)\,(> u_+)$. Expansion (2.21) becomes nonuniform when $y = u_+ + O(t^{-1})$ as $t\to\infty$. To investigate this region which we call region P1 we introduce the scaled variable $\eta = (y-u_+)t = O(1)\,(>0)$ and look for an expansion of the form

$$u(\eta,t) = u_+ + \hat{g}_0(\eta)t^2e^{-2t} + \hat{g}_1(\eta)t^5e^{-4t} + o\left(t^5e^{-4t}\right) \tag{2.22}$$

as $t\to\infty$ with $\eta = O(1)\,(>0)$. On substituting (2.22) into equation (2.1) and solving at each order in turn, we find (after matching with (2.21) as $\eta\to\infty$) that

$$u(\eta,t) = u_+ + \frac{A_R}{\eta^2}t^2e^{-2t} + \left(\frac{6A_R}{\eta^4} + \frac{2A_R^2}{\eta^5}\right)t^5e^{-4t}o\left(e^{-4t}\right) \tag{2.23}$$

as $t\to\infty$ with $\eta = O(1)\,(>0)$, and where $A_R<0$. Expansion (2.23) becomes nonuniform when $\eta = O\left(te^{-2/3t}\right)$ as $t\to\infty$ [that is, when $y = u_+ + O\left(e^{-2/3t}\right)$, which is (2.20) with $\gamma=2$]. On rewriting expansion (2.23) in terms of y we obtain

$$u(y,t) = u_+ + A_R\,(y-u_+)^{-2}\,e^{-2t} + 2A_R^2\,(y-u_+)^{-5}\,e^{-4t} + o\left(e^{-4t}\right) \tag{2.24}$$

as $t \to \infty$ with $(y - u_+) \ll 1$. We observe that expansion (2.24) is just expansion (2.19) with $\gamma = 2$ and therefore region P1 is simply a passive region allowing for the rearrangement of terms in expansion $(2.12)_2$.

Finally, we consider the case when $\gamma > 2$. On substituting $(2.12)_3$ into equation (2.1) and solving at each order in turn, we find in region III$^+$ (after matching with (2.9)) that

$$u(y,t) = u_+ + \frac{A_R}{(y-u_+)^\gamma} e^{-\gamma t} + \frac{\gamma(\gamma+1)A_R}{(y-u_+)^{\gamma+2}} t e^{-(\gamma+2)t} + \frac{\gamma(\gamma+1)A_R(2-u_+^2)}{2(y-u_+)^{\gamma+2}} e^{-(\gamma+2)t} + \frac{\gamma A_R^2}{(y-u_+)^{2\gamma+1}} e^{-2\gamma t} + o\left(e^{-2\gamma t}\right) \tag{2.25}$$

as $t \to \infty$ with $y = O(1)\,(> u_+)$, and where $A_R < 0$. Expansion (2.25) becomes nonuniform when

$$y = u_+ + O\left(\exp\left\{-\frac{(\gamma-2)}{(\gamma-1)}t\right\}\right)$$

as $t \to \infty$. To investigate this region which we call region P2 we introduce the scaled variable $\eta = (y - u_+)\exp\left\{\frac{(\gamma-2)}{(\gamma-1)}t\right\} = O(1)\,(> 0)$ and look for an expansion of the form

$$u(\eta,t) = u_+ + \tilde{g}_0(\eta)e^{-\frac{\gamma}{\gamma-1}t} + \tilde{g}_1(\eta)te^{-\frac{(\gamma+2)}{(\gamma+1)}t} + \tilde{g}_2(\eta)e^{-\frac{(\gamma+2)}{(\gamma+1)}t} + o\left(e^{-\frac{(\gamma+2)}{(\gamma+1)}t}\right) \tag{2.26}$$

as $t \to \infty$ with $\eta = O(1)\,(> 0)$. On substituting (2.26) into equation (2.1) and solving at each order in turn, we find (after matching with (2.25) as $\eta \to \infty$) that

$$\begin{aligned} u(\eta,t) = u_+ &+ \frac{A_R}{\eta^\gamma}e^{-\frac{\gamma}{\gamma-1}t} + \frac{\gamma(\gamma+1)A_R}{\eta^{\gamma+2}}te^{-\frac{(\gamma+2)}{(\gamma+1)}t} \\ &+ \left(\frac{\gamma(\gamma+1)A_R(2-u_+^2)}{2\eta^{\gamma+2}} + \frac{\gamma A_R^2}{\eta^{2\gamma+1}}\right)e^{-\frac{(\gamma+2)}{(\gamma+1)}t} + o\left(e^{-\frac{(\gamma+2)}{(\gamma+1)}t}\right) \end{aligned} \tag{2.27}$$

as $t \to \infty$ with $\eta = O(1)\,(> 0)$. Expansion (2.27) becomes nonuniform when $\eta = O\left(t^{-\frac{1}{\gamma-1}}\right)$ as $t \to \infty$. To investigate this region which we call region P3 we introduce the scaled variable $\xi = \eta t^{\frac{1}{\gamma-1}} = O(1)\,(> 0)$ and look for an expansion of the form

$$u(\xi,t) = u_+ + \bar{g}_0(\xi)t^{\frac{\gamma}{\gamma-1}}e^{-\frac{\gamma}{\gamma-1}t} + \bar{g}_1(\eta)t^{\frac{2\gamma+1}{\gamma-1}}e^{-\frac{(\gamma+2)}{(\gamma+1)}t} + o\left(t^{\frac{2\gamma+1}{\gamma-1}}e^{-\frac{(\gamma+2)}{(\gamma+1)}t}\right) \tag{2.28}$$

as $t \to \infty$ with $\xi = O(1)\,(> 0)$. On substituting (2.28) into equation (2.1) and solving at each order in turn, we find (after matching with (2.27)) that

$$u(\xi,t) = u_+ + \frac{A_R}{\xi^\gamma}t^{\frac{\gamma}{\gamma-1}}e^{-\frac{\gamma}{\gamma-1}t} + \left(\frac{\gamma(\gamma+1)A_R}{\xi^{\gamma+2}} + \frac{\gamma A_R^2}{\xi^{2\gamma+1}}\right)t^{\frac{2\gamma+1}{\gamma-1}}e^{-\frac{(\gamma+2)}{(\gamma+1)}t} + o\left(t^{\frac{2\gamma+1}{\gamma-1}}e^{-\frac{(\gamma+2)}{(\gamma+1)}t}\right) \tag{2.29}$$

as $t \to \infty$ with $\xi = O(1)\,(> 0)$. Expansion (2.29) becomes nonuniform when $\xi = O\left(t^{\frac{1}{\gamma-1}}e^{-\frac{2}{\gamma^2-1}t}\right)$ as $t \to \infty$ [that is, when $y = u_+ + O\left(\exp\left\{-\frac{\gamma}{\gamma+1}t\right\}\right)$ as $t \to \infty$.]. We conclude that, regions P2 and P3 are simply passive regions which allow for the rearrangement of terms in expansion $(2.12)_3$.

Therefore, we encounter a nonuniformity in (2.19) (when $0 < \gamma < 2$), (2.23) (when $\gamma = 2$) and (2.29) (when $\gamma > 2$) when $y = u_+ + O\left(\exp\left\{-\frac{\gamma}{\gamma+1}t\right\}\right)$ as $t \to \infty$. To investigate this region which we call region IV$^+$ we introduce the scaled variable $\zeta = (y - u_+)\exp\left\{\frac{\gamma}{\gamma+1}t\right\} = O(1)$ and look for an expansion of the form

$$u(\zeta,t) = u_+ + F(\zeta)e^{-\frac{\gamma}{\gamma+1}t} + o\left(e^{-\frac{\gamma}{\gamma+1}t}\right) \tag{2.30}$$

as $t \to \infty$ with $\zeta = O(1)$. On substitution of (2.30) into equation (2.1) (when written in terms of ζ and t) we obtain at leading order

$$FF_\zeta - \frac{\zeta}{1+\gamma}F_\zeta - \frac{\gamma}{1+\gamma}F = 0, \quad -\infty < \zeta < \infty, \tag{2.31}$$

$$F(\zeta) \sim \frac{A_R}{\zeta^\gamma} + \frac{\gamma A_R^2}{\zeta^{2\gamma+1}} \quad \text{as} \quad \zeta \to \infty. \tag{2.32}$$

Condition (2.32) arises from matching expansion (2.30) ($\zeta \gg 1$) with (2.19) when $0 < \gamma < 2$, (2.23) when $\gamma = 2$ or (2.29) when $\gamma > 2$. We observe immediately that equation (2.31) admits the exact solution

$$F(\zeta) = \zeta, \quad -\infty < \zeta < \infty.$$

Further, it is straightforward to establish that the solution to (2.31), (2.32) (we recall that in this case $A_R < 0$) is given by

$$F(\zeta - F)^\gamma = A_R, \quad -\infty < \zeta < \infty, \tag{2.33}$$

where $F(\zeta) < 0$ for $\zeta \in (-\infty, \infty)$, and

$$F(\zeta) \sim \begin{cases} \frac{A_R}{\zeta^\gamma} + \frac{\gamma A_R^2}{\zeta^{2\gamma+1}} & \text{as} \quad \zeta \to \infty, \\ \zeta - (-A_R)^{\frac{1}{\gamma}}(-\zeta)^{-\frac{1}{\gamma}} & \text{as} \quad \zeta \to -\infty. \end{cases} \tag{2.34}$$

As $\zeta \to -\infty$ we move out of region IV$^+$ into region V where $u(y) = O(1)$ as $t \to \infty$. To examine region V we look for an expansion of the form

$$u(y,t) = G(y) + O\left(e^{-t}\right) \tag{2.35}$$

as $t \to \infty$ with $y = O(1)\,(< u_+)$. On substitution of (2.35) into equation (2.1) (when written in terms of y and t) we obtain at leading order that

$$G_y(G - y) = 0, \quad y < u_+, \tag{2.36}$$

$$G(y) \sim y \quad \text{as} \quad y \to (u_+)^-, \tag{2.37}$$

where condition (2.37) is the matching condition with region IV$^+$. The solution of (2.36), (2.37) is readily obtained as

$$G(y) = y, \quad y < u_+. \tag{2.38}$$

Therefore, in region V we have that

$$u(y,t) = y + O\left(e^{-t}\right) \tag{2.39}$$

as $t \to \infty$ with $y = O(1)\,(< u_+)$.

We next consider the asymptotic structure as $t \to \infty$ for $x < 0$. To proceed we introduce a new region, region III$^-$. The details of this region follow, after minor modification, those given for region III$^+$ and are not repeated here. Therefore, we have in region III$^-$ that

$$u(y,t) = u_- + A_L(u_- - y)^{-\gamma}e^{-\gamma t} + o\left(e^{-\gamma t}\right) \tag{2.40}$$

as $t \to \infty$ with $y = O(1)(\in (-\infty, u_-))$, and where $A_L > 0$. Expansion (2.40) becomes nonuniform when $y = u_- + O\left(\exp\left\{-\frac{\gamma}{\gamma+1}t\right\}\right)$ as $t \to \infty$. To investigate this region which we call region IV$^-$ we introduce the scaled variable $\zeta = (u_- - y)\exp\left\{\frac{\gamma}{\gamma+1}t\right\} = O(1)$ and look for an expansion of the form

$$u(\zeta,t) = u_- + F(\zeta)e^{-\frac{\gamma}{\gamma+1}t} + o\left(e^{-\frac{\gamma}{\gamma+1}t}\right) \tag{2.41}$$

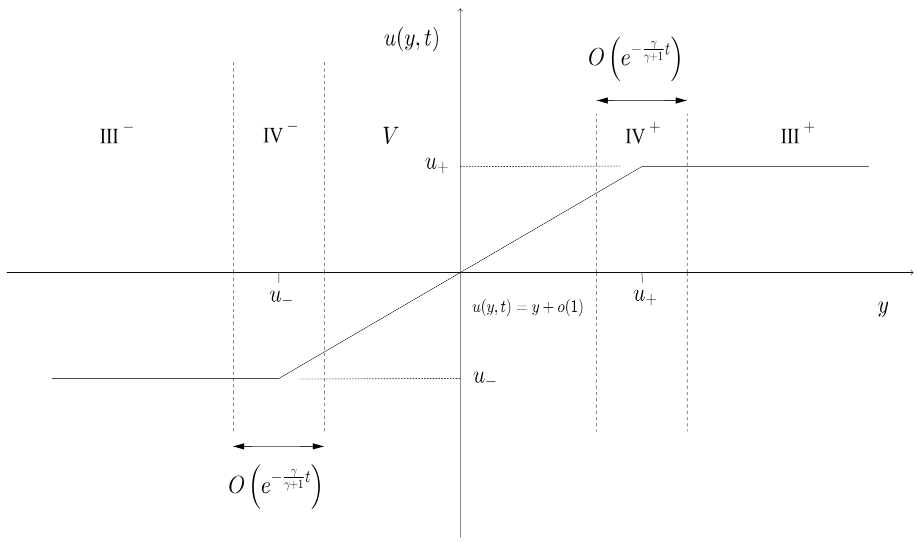

Figure 1: A schematic representation of the asymptotic structure of $u(y,t)$ in the (y,u) plane as $t \to \infty$ for **IVP1** when $u_+ > u_-$.

as $t \to \infty$ with $\zeta = O(1)$. On substitution of (2.41) into equation (2.1) (when written in terms of ζ and t) we obtain at leading order

$$FF_\zeta - \frac{\zeta}{1+\gamma}F_\zeta - \frac{\gamma}{1+\gamma}F = 0, \quad -\infty < \zeta < \infty, \tag{2.42}$$

$$F(\zeta) \sim \frac{A_L}{(-\zeta)^\gamma} - \frac{\gamma A_L^2}{(-\zeta)^{2\gamma+1}} \quad \text{as} \quad \zeta \to -\infty. \tag{2.43}$$

Condition (2.43) arises from matching expansion (2.41) ($-\zeta \gg 1$) with (2.40). We observe immediately that equation (2.42) admits the exact solution $F(\zeta) = \zeta$ for $\zeta \in (-\infty, \infty)$. Further, it is straightforward to establish that the solution to (2.42), (2.43) is given by

$$F(F-\zeta)^\gamma = A_L, \quad -\infty < \zeta < \infty, \tag{2.44}$$

where $F(\zeta) > 0$ for $\zeta \in (-\infty, \infty)$, and

$$F(\zeta) \sim \begin{cases} \frac{A_L}{(-\zeta)^\gamma} - \frac{\gamma A_L^2}{(-\zeta)^{2\gamma+1}} & \text{as} \quad \zeta \to -\infty, \\ \zeta + A_L^{\frac{1}{\gamma}} \zeta^{-\frac{1}{\gamma}} & \text{as} \quad \zeta \to +\infty. \end{cases} \tag{2.45}$$

As $\zeta \to \infty$ we move out of region IV$^-$ into region V, and we conclude by noting that in region V we have that

$$u(y,t) = y + O\left(e^{-t}\right), \quad u_- < y < u_+ \tag{2.46}$$

as $t \to \infty$. Therefore, region V, where $y \in (u_-, u_+)$, is the expansion region with expansion (2.46) being given by the expansion wave at leading order.

The asymptotic structure of the solution of **IVP1** as $t \to \infty$ when $u_+ > u_-$ is now complete. A uniform approximation has been given through regions II$^\pm$, III$^\pm$, IV$^\pm$ and V. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in the case $0 < \gamma < 2$ in Figure 1 (we recall the situation is slightly more complicated for $\gamma \geq 2$ due to the presence of passive regions located at u_+ and u_-). The large-t attractor for the solution of **IVP1** when $u_+ > u_-$ is the expansion wave which allows for the adjustment of the solution form u_+ to u_-.

2.4 Asymptotic solution as $t \to \infty$ when $u_+ < u_-$

We now investigate the structure of **IVP1** as $t \to \infty$ when $u_+ < u_-$. We recall from Section 2.2 that expansions (2.9) and (2.10) of regions II$^+$ ($x \to \infty, t = O(1)$) and II$^-$ ($x \to -\infty, t = O(1)$) respectively, continue to remain uniform provided $|x| \gg e^t$. However, as already noted a nonuniformity develops when $|x| = O(e^t)$. As in Section 2.3, we introduce the scaled coordinate

$$y = x\, e^{-t}, \tag{2.47}$$

where $y = O(1)$, and begin by summarizing the asymptotic structure as $t \to \infty$ in regions III$^+$ and III$-$ (the details of which follow, after minor modification, those given in Section 2.3) and are not repeated here):

Region III$^+$

$$u(y,t) = u_+ + A_R\,(y - u_+)^{-\gamma}\, e^{-\gamma t} + o\left(e^{-\gamma t}\right) \tag{2.48}$$

as $t \to \infty$ with $y = O(1)(\in (u_+, \infty)$, and where $A_R > 0$.

Region III$^-$

$$u(y,t) = u_- + A_L(u_- - y)^{-\gamma} e^{-\gamma t} + o\left(e^{-\gamma t}\right) \tag{2.49}$$

as $t \to \infty$ with $y = O(1)(\in (-\infty, u_-))$, and where $A_L < 0$.

Clearly, in this case when $u_+ < u_-$ expansions (2.48) and (2.49) must become nonuniform as $y \to \alpha$, where $\alpha \in (u_+, u_-)$ and is to be determined. To examine this region which we label as region SS we introduce the scaled coordinate

$$z = (y - \alpha)\psi^{-1} = O(1), \tag{2.50}$$

where $\psi(t) = o(1)$ as $t \to \infty$, is an as yet undetermined gauge function, and expand in the form

$$u(z,t) = U(z) + o(1) \tag{2.51}$$

as $t \to \infty$ with $z = O(1)$. On substituting (2.51) into equation (2.1) (when written in terms of z and t) we find that to obtain the must structured leading order balance that we require

$$\psi(t) = e^{-2t}. \tag{2.52}$$

At leading order we then obtain

$$U_{zz} - UU_z + \alpha U_z = 0, \quad -\infty < z < \infty. \tag{2.53}$$

On integrating (2.53) we obtain

$$U_z = \frac{U^2}{2} - \alpha U + C, \quad -\infty < z < \infty, \tag{2.54}$$

where C is a constant of integration. Equation (2.54) is to be solved subject to the leading order matching conditions

$$U(z) \sim \begin{cases} u_+ & \text{as} \quad z \to \infty, \\ u_- & \text{as} \quad z \to -\infty. \end{cases} \tag{2.55}$$

The solution to (2.54) subject to boundary conditions (2.55) requires that

$$\alpha = \frac{(u_+ + u_-)}{2} \quad \text{and} \quad C = \frac{u_+ u_-}{2},$$

and is given by the Taylor shock profile (see [25])

$$U(z) = \frac{(u_+ + u_-)}{2} - \frac{(u_- - u_+)}{2}\tanh\left(\frac{(u_- - u_+)}{4}z + \phi_c\right), \quad -\infty < z < \infty, \tag{2.56}$$

where ϕ_c is a constant. We note that

$$U(z) \sim \begin{cases} u_+ + (u_- - u_+)\exp\left(-\frac{(u_- - u_+)}{2}z - \phi_c\right) & \text{as} \quad z \to \infty, \\ u_- - (u_- - u_+)\exp\left(\frac{(u_- - u_+)}{2}z + \phi_c\right) & \text{as} \quad z \to -\infty. \end{cases} \tag{2.57}$$

The similarity solution (2.56) represents a wavefront connecting u_+ (as $z \to \infty$) to u_- (as $z \to -\infty$). This wavefront is steepening as $t \to \infty$ and has an accelerating velocity as $t \to \infty$ when $u_+ \neq -u_-$, but is located at $y = 0$ and is stationary if $u_+ = -u_-$. Specifically, we note that when $-u_- < u_+ < u_-$ the wavefront is accelerating in the $+x$ direction, whereas when $u_+ < u_- < -u_+$ the wavefront is accelerating in the $-x$ direction.

However, we readily observe that matching expansion (2.51) (as $z \to \infty$) to expansion (2.48) (as $y \to \frac{(u_+ + u_-)}{2}^+$) to next order fails and we require a transition region, which we label TR$^+$. To examine region TR$^+$ we introduce the scaled coordinate $\bar{x}$ by

$$y = \frac{u_+ + u_-}{2} + \frac{2\gamma}{u_- - u_+}te^{-2t} + \bar{x}e^{-2t}, \tag{2.58}$$

so that $\bar{x} = O(1)$ as $t \to \infty$ in region TR$^+$ [that is, $z = \frac{2\gamma}{u_- - u_+}t + \bar{x}$]. It is instructive at this point to describe in more detail how (2.58) was obtained. Expansion (2.51) with $(2.57)_1$ (for $z \gg 1$) when written in terms of y is given by

$$u \sim u_+ + (u_- - u_+)\exp\left(-\frac{(u_- - u_+)}{2}\left[y - \frac{(u_- + u_+)}{2}\right]e^{2t} - \phi_c\right) \tag{2.59}$$

as $t \to \infty$. Expansion (2.49) as $y \to \frac{(u_- + u_+)}{2}^+$ (as we move into region TW) gives that

$$u = u_+ + O(e^{-\gamma t}) \tag{2.60}$$

as $t \to \infty$. Comparison of the exponentially small terms in expansions (2.59) and (2.60) then indicates that they are of the same order (that is, of $O(e^{-\gamma t})$ as $t \to \infty$) when

$$y = \frac{(u_- + u_+)}{2} + \frac{2\gamma}{(u_- - u_+)}te^{-2t} + O(e^{-2t})$$

as $t \to \infty$, giving the required scaling for the transition region. The form of expansion (2.56) (for $z \gg 1$) then suggests that in region TR$^+$ we expand as

$$u(\bar{x}, t) = u_+ + F(\bar{x})e^{-\gamma t} + o\left(e^{-\gamma t}\right) \tag{2.61}$$

as $t \to \infty$ with $\bar{x} = O(1)$. On substitution of (2.61) into equation (2.1) (when written in terms of $\bar{x}$ and t) we obtain at leading order that

$$F_{\bar{x}\bar{x}} + \frac{(u_- - u_+)}{2}F_{\bar{x}} = 0, \quad -\infty < \bar{x} < \infty. \tag{2.62}$$

equation (2.62) is to be solved subject to the following matching conditions

$$F(\bar{x}) \sim \begin{cases} \frac{A_R 2^\gamma}{(u_- - u_+)^\gamma} & \text{as} \quad \bar{x} \to \infty, \\ (u_- - u_+)e^{-\phi_c}e^{-\frac{(u_- - u_+)}{2}\bar{x}} & \text{as} \quad \bar{x} \to -\infty. \end{cases} \tag{2.63}$$

The solution to (2.62), (2.63) is readily obtained as

$$F(\bar{x}) = \frac{A_R 2^\gamma}{(u_- - u_+)^\gamma} + (u_- - u_+)e^{-\phi_c}e^{-\frac{(u_- - u_+)}{2}\bar{x}}, \quad -\infty < \bar{x} < \infty. \tag{2.64}$$

Therefore, we have in region TR$^+$ that

$$u(\bar{x}, t) = u_+ + \left(\frac{A_R 2^\gamma}{(u_- - u_+)^\gamma} + (u_- - u_+)e^{-\phi_c}e^{-\frac{(u_- - u_+)}{2}\bar{x}}\right)e^{-\gamma t} + o\left(e^{-\gamma t}\right) \tag{2.65}$$

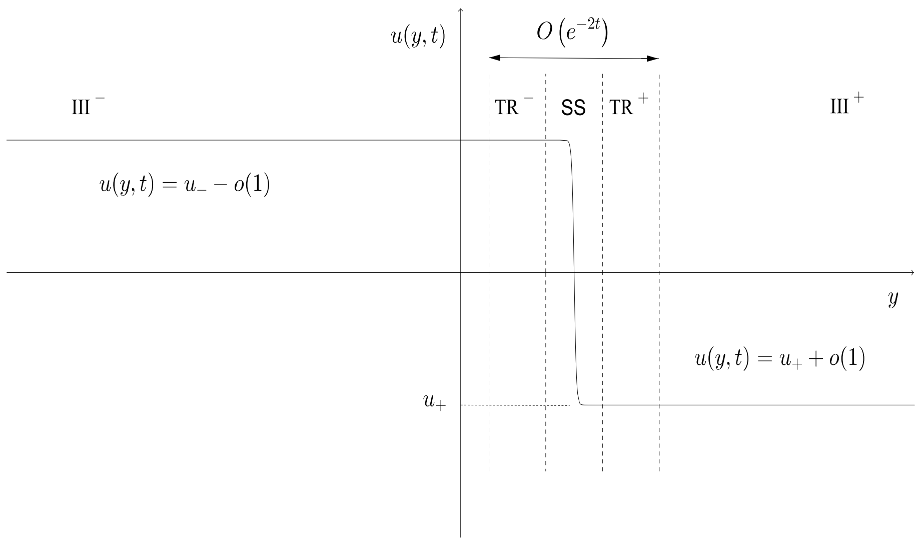

Figure 2: A schematic representation of the asymptotic structure of $u(y,t)$ in the (y,u) plane as $t \to \infty$ for **IVP1** when $u_+ < u_-$. We recall that region SS is located at $y = \frac{u_+ + u_-}{2}$ (with thickness $O\left(e^{-2t}\right)$ as $t \to \infty$), while regions $\mathrm{TR}^{\pm}$ are located at $y = \frac{u_+ + u_-}{2} \pm \frac{2\gamma}{u_- - u_+} t e^{-2t}$ (with thickness $O\left(e^{-2t}\right)$) as $t \to \infty$.

as $t \to \infty$ with $\bar{x} = O(1)$.

Finally, we conclude this case by noting that matching expansion (2.51) (as $z \to -\infty$) to expansion (2.49) (as $y \to \frac{(u_+ + u_-)}{2}^-$) fails and we require a transition region, which we label TR^-. To examine region TR^- we introduce the scaled coordinate $\hat{x}$ by

$$y = \frac{u_+ + u_-}{2} - \frac{2\gamma}{u_- - u_+} t e^{-2t} + \hat{x} e^{-2t}, \tag{2.66}$$

so that $\hat{x} = O(1)$ as $t \to \infty$ in region TR^-. The details of region TR^- follow, after minor modification, those given for region TR^+ are are not repeated here. In summary we have in region TR^- that

$$u(\hat{x},t) = u_- + \left(\frac{A_L 2^{\gamma}}{(u_- - u_+)^{\gamma}} - (u_- - u_+) e^{\phi_c} e^{-\frac{(u_- - u_+)}{2}\hat{x}} \right) e^{-\gamma t} + o\left(e^{-\gamma t}\right) \tag{2.67}$$

as $t \to \infty$ with $\hat{x} = O(1)$.

The asymptotic structure of the solution of **IVP1** as $t \to \infty$ when $u_+ < u_-$ is now complete. A uniform approximation has been given through regions $\mathrm{II}^{\pm}$, $\mathrm{III}^{\pm}$, $\mathrm{TR}^{\pm}$ and SS. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Figure 2. The large-t attractor for the solution of **IVP1** when $u_+ > u_-$ is the Taylor shock profile which allows for the adjustment of the solution form u_+ to u_-.

3 Initial-value problem 2

In this section we consider the following initial-value problem for Burgers' equation, namely,

$$u_t + u u_x = e^t\, u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{3.1}$$

$$u(x,0) = u_0(x), \quad -\infty < x < \infty, \tag{3.2}$$

$$u(x,t) \to \begin{cases} u_-, & x \to -\infty, \\ u_+, & x \to \infty, \end{cases} \quad t \geq 0, \tag{3.3}$$

where u_- and $u_+(\neq u_-)$ are parameters. Further, we assume that the initial data $u_0(x)$ has algebraic decay as $|x| \to \infty$. Specifically,

$$u_0(x) = \begin{cases} u_- + \dfrac{A_L}{(-x)^\gamma} + O\left(e^{-|x|}\right) & \text{as} \quad x \to -\infty, \\ u_+ + \dfrac{A_R}{x^\gamma} + O\left(e^{-|x|}\right) & \text{as} \quad x \to \infty, \end{cases} \tag{3.4}$$

where $A_L\ (\neq 0), A_R\ (\neq 0)$ and $\gamma\ (> 0)$ are parameters. In what follows we label initial-value problem (3.1)-(3.4) as **IVP2**.

In this section we develop the asymptotic structure of the solution of **IVP2** as $t \to \infty$. We first note that the asymptotic structure of **IVP2** as $t \to 0$ follows that given in Section 2.1 and is not repeated here. We begin by developing the asymptotic solution of **IVP2** as $|x| \to \infty$.

3.1 Asymptotic solution as $|x| \to \infty$

We now investigate the structure of the solution to **IVP2** as $|x| \to \infty$ for $t = O(1)$. We begin be developing the structure of the solution to **IVP1** as $x \to \infty$ with $t = O(1)$. The form of (2.7) indicates that in this region, region II$^+$, we expand in the form (2.9) (where r and s are as given in Section 2.2). On substituting (2.9) into equation (3.1) and solving at each order in turn we find (after matching with (2.7) as $t \to 0$) that

$$f_0(t) = A_R, \quad f_1(t) = \gamma u_+ A_R t,$$

$$f_2(t) = \begin{cases} \gamma A_R^2 t, & 0 < \gamma < 1, \\ 2A_R e^t + u_+^2 A_R t^2 + A_R^2 t - 2A_R, & \gamma = 1, \\ \gamma(\gamma+1)A_R e^t + \frac{\gamma(\gamma+1)u_+^2 A_R}{2} t^2 - \gamma(\gamma+1)A_R, & \gamma > 1. \end{cases}$$

The function $f_3(t)$ is only required in what follows in the case $0 < \gamma < 1$ and for brevity we do not report $f_3(t)$ for $\gamma > 1$ or $\gamma = 1$. We readily obtain that

$$f_3(t) = \gamma(\gamma+1)A_R e^t + \frac{\gamma(\gamma+1)u_+^2 A_R}{2} t^2 - \gamma(\gamma+1)A_R, \quad 0 < \gamma < 1.$$

Thus, the structure of (2.9) depends upon whether $0 < \gamma < 1$, $\gamma = 1$ or $\gamma > 1$. When $\gamma \geq 1$ expansion (2.9) remains uniform for $t \gg 1$ provided $x \gg t^{-1}e^t$, becoming nonuniform when $x = O\left(t^{-1}e^t\right)$ for $t \gg 1$. However, when $0 < \gamma < 1$ expansion (2.9) remains uniform for $t \gg 1$ provided $x \gg t^{\frac{1}{\gamma-1}} e^{\frac{t}{1-\gamma}}$, becoming nonuniform when $x = O\left(t^{\frac{1}{\gamma-1}} e^{\frac{t}{1-\gamma}}\right)$ for $t \gg 1$.

We conclude be developing the structure of the solution to **IVP1** as $x \to -\infty$ with $t = O(1)$. The form of (2.7) indicates that in this region, region II$^-$, we expand as in Section 2.2 and look for an expansion of the form (2.10). On substituting (2.10) into equation (3.1) and solving at each order in turn we find (after matching with (2.7) as $t \to 0$) that

$$\hat{f}_0(t) = A_L, \quad \hat{f}_1(t) = -\gamma u_- A_L t,$$

$$f_2(t) = \begin{cases} -\gamma A_L^2 t, & 0 < \gamma < 1, \\ 2A_L e^t + u_-^2 A_L t^2 - A_L^2 t - 2A_L, & \gamma = 1, \\ \gamma(\gamma+1)A_L e^t + \frac{\gamma(\gamma+1)u_-^2 A_L}{2} t^2 - \gamma(\gamma+1)A_L, & \gamma > 1. \end{cases}$$

The function $f_3(t)$ is only required in what follows in the case $0 < \gamma < 1$ and for brevity we do not report $f_3(t)$ for $\gamma > 1$ or $\gamma = 1$. We readily obtain that

$$f_3(t) = \gamma(\gamma+1)A_L e^t + \frac{\gamma(\gamma+1)u_-^2 A_L}{2} t^2 - \gamma(\gamma+1)A_L, \quad 0 < \gamma < 1.$$

Thus, the structure of (2.10) depends upon whether $0 < \gamma < 1$, $\gamma = 1$ or $\gamma > 1$. When $\gamma \geq 1$ expansion (2.10) remains uniform for $t \gg 1$ provided $-x \gg t^{-1}e^{t}$, becoming nonuniform when $-x = O\left(t^{-1}e^{t}\right)$ for $t \gg 1$. However, when $0 < \gamma < 1$ expansion (2.9) remains uniform for $t \gg 1$ provided $-x \gg t^{\frac{1}{\gamma-1}} e^{\frac{t}{1-\gamma}}$, becoming nonuniform when $-x = O\left(t^{\frac{1}{\gamma-1}} e^{\frac{t}{1-\gamma}}\right)$ for $t \gg 1$.

3.2 Asymptotic solution as $t \to \infty$

We now investigate the structure of **IVP2** as $t \to \infty$. We recall from Section 3.1 that we need to consider the cases $\gamma \geq 1$ and $0 < \gamma < 1$ separately. We begin by developing the large-t structure of the solution of **IVP2** when $\gamma \geq 1$.

3.3 $\gamma \geq 1$

In this case expansions (2.9) and (2.10) of regions II$^+$ ($x \to \infty, t = O(1)$) and II$^-$ ($x \to -\infty, t = O(1)$) respectively, continue to remain uniform provided $|x| \gg t^{-1}e^{t}$. However, as already noted a nonuniformity develops when $|x| = O(t^{-1}e^{t})$. We begin by considering the asymptotic structure as $t \to \infty$ for $x > 0$. To proceed we introduce a new region, region III$^+$. To examine region III$^+$ we introduce the scaled coordinate

$$y = x\, t e^{-t}, \tag{3.5}$$

where $y = O(1)$, and look for an expansion of the form

$$u(y,t) = u_{+} + g_0(y) t^{\gamma} e^{-\gamma t} + g_1(y) t^{\gamma+2} e^{-(\gamma+1)t} + o\left(t^{\gamma+2} e^{-(\gamma+1)t}\right) \tag{3.6}$$

as $t \to \infty$ with $y = O(1)$. On substituting (3.6) into equation (3.1) (when written in terms of y and t), the leading order problem for $g_0(y)$ becomes

$$y g_0' + \gamma g_0 = 0, \quad y > 0, \tag{3.7}$$

$$g_0(y) \sim \frac{A_R}{y^{\gamma}} \quad \text{as} \quad y \to \infty, \tag{3.8}$$

with condition (3.8) arising from matching expansion (3.6) ($y \gg 1$) with the far field expansion (2.9) ($x = O(t^{-1}e^{t})$). The solution of (3.7), (3.8) is readily obtained as

$$g_0(y) = A_R y^{-\gamma}, \quad y > 0. \tag{3.9}$$

Hence, via (3.9), $g_0(y)$ develops a singularity as $y \to 0^+$, and thus expansion (3.6) becomes nonuniform when $y = o(1)$ as $t \to \infty$. On returning to next order, we obtain the following problem for $g_1(y)$, namely

$$y g_1' + (\gamma+1) g_1 = -g_0'', \quad y > 0, \tag{3.10}$$

$$g_1(y) \sim \frac{\gamma u_{+} A_R}{y^{\gamma+1}} + \frac{\gamma(\gamma+1) A_R}{y^{\gamma+2}} \quad \text{as} \quad y \to \infty, \tag{3.11}$$

with condition (3.11) arising from matching expansion (3.6) ($y \gg 1$) with the far field expansion (2.9) ($x = O(t^{-1}e^{t})$). The solution of (3.10), (3.11) is readily obtained as

$$g_1(y) = \frac{\gamma u_{+} A_R}{y^{\gamma+1}} + \frac{\gamma(\gamma+1) A_R}{y^{\gamma+2}}, \quad y > 0. \tag{3.12}$$

Therefore, we have, via (3.6), (3.9) and (3.12), that when $\gamma \geq 1$ the expansion in region III$^+$ has the form

$$u(y,t) = u_{+} + \frac{A_R}{y^{\gamma}} t^{\gamma} e^{-\gamma t} + \left(\frac{\gamma u_{+} A_R}{y^{\gamma+1}} + \frac{\gamma(\gamma+1) A_R}{y^{\gamma+2}}\right) t^{\gamma+2} e^{-(\gamma+1)t} + o\left(t^{\gamma+2} e^{-(\gamma+1)t}\right) \tag{3.13}$$

as $t \to \infty$ with $y = O(1)(> 0)$. We note that expansion (3.13) becomes nonuniform when

$$y = O\left(te^{-\frac{t}{2}}\right) \tag{3.14}$$

as $t \to \infty$ [that is, when $x = O\left(e^{\frac{t}{2}}\right)$ as $t \to \infty$]. To examine this region which we label as region SS we introduce the scaled coordinate z by

$$xte^{-t} = y = zte^{-\frac{t}{2}}, \tag{3.15}$$

where $z = O(1)$ as $t \to \infty$, and expand in the form

$$u(z,t) = U(z) + o(1) \tag{3.16}$$

as $t \to \infty$ with $z = O(1)$. On substituting (3.16) into equation (3.1) (when written in terms of z and t) we find at leading order that

$$U_{zz} + \frac{z}{2}U_z = 0, \quad -\infty < z < \infty. \tag{3.17}$$

Equation (3.17) is to be solved subject to the leading order matching conditions

$$U(z) \sim \begin{cases} u_+ & \text{as} \quad z \to \infty, \\ u_- & \text{as} \quad z \to -\infty. \end{cases} \tag{3.18}$$

The solution to (3.17), (3.18) is then obtained as

$$U(z) = \frac{(u_+ + u_-)}{2} - \frac{(u_- - u_+)}{2}\,\text{erf}\left(\frac{z}{2}\right), \quad -\infty < z < \infty, \tag{3.19}$$

where erf[.] is the standard error function (see [1]). We note that

$$U(z) \sim \begin{cases} u_+ - \frac{(u_+ - u_-)}{\sqrt{\pi}} z^{-1} e^{-\frac{z^2}{4}} & \text{as} \quad z \to \infty, \\ u_- + \frac{(u_+ - u_-)}{\sqrt{\pi}} (-z)^{-1} e^{-\frac{z^2}{4}} & \text{as} \quad z \to -\infty. \end{cases} \tag{3.20}$$

The similarity solution (3.19) connects u_+ (as $z \to \infty$) to u_- (as $z \to -\infty$). We observe that this similarity solution is in a stretching frame of reference.

However, we readily observe that matching expansion (3.16) (as $z \to \infty$) to expansion (3.13) (as $y \to 0^+$) at next order fails and we need a transition region, which we label TR$^+$. To examine region TR$^+$ we introduce the scaled coordinate η by

$$z = \sqrt{2\gamma}\, t^{\frac{1}{2}} + \frac{(\gamma - 1)}{\sqrt{2\gamma}} \frac{\ln t}{t^{\frac{1}{2}}} + \eta\, t^{-\frac{1}{2}}, \tag{3.21}$$

as $t \to \infty$. It is instructive at this point to describe in more detail how (3.21) was obtained. Expansion (3.16) with (3.20$_1$) gives

$$u(z,t) \sim u_+ - \frac{(u_+ - u_-)}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4} - \ln z\right) \tag{3.22}$$

as $t \to \infty$ with $z \gg 1$. Expansion (3.13) when written in terms of z gives

$$u(z,t) \sim u_+ + A_R \exp\left(-\frac{\gamma t}{2} - \gamma \ln z\right) \tag{3.23}$$

as $t \to \infty$. Comparison of the exponentially small terms in expansions (3.22) and (3.23) requires that

$$z = \sqrt{2\gamma}\, t^{\frac{1}{2}} + \frac{(\gamma - 1)}{\sqrt{2\gamma}} \frac{\ln t}{t^{\frac{1}{2}}} + O\left(t^{-\frac{1}{2}}\right),$$

as $t \to \infty$, giving the required scaling for the transition region. In region TR$^+$ we then look for an expansion of the form

$$u = u_+ + K(\eta)\, t^{-\frac{\gamma}{2}} e^{-\frac{\gamma t}{2}} + o\left(t^{-\frac{\gamma}{2}} e^{-\frac{\gamma t}{2}}\right) \tag{3.24}$$

as $t \to \infty$ with $\eta = O(1)$. On substitution of expansion (3.24) into equation (3.1) (when written in terms of η and t) we obtain at leading order

$$K_{\eta\eta} + \frac{\sqrt{2\gamma}}{2} K_\eta = 0, \quad -\infty < \eta < \infty. \tag{3.25}$$

Equation (3.25) has to be solved subject to the matching conditions

$$K(\eta) \sim \begin{cases} \dfrac{A_R}{(2\gamma)^{\frac{\gamma}{2}}} & \text{as} \quad \eta \to \infty, \\ -\dfrac{(u_+ - u_-)}{\sqrt{2\gamma\pi}} e^{-\frac{\sqrt{2\gamma}}{2}\eta} & \text{as} \quad \eta \to -\infty. \end{cases} \tag{3.26}$$

The solution of (3.25), (3.26) is readily obtained as

$$K(\eta) = \left(\frac{A_R}{(2\gamma)^{\frac{\gamma}{2}}} - \frac{(u_+ - u_-)}{\sqrt{2\gamma\pi}} e^{-\frac{\sqrt{2\gamma}}{2}\eta} \right), \quad -\infty < \eta < \infty. \tag{3.27}$$

We note that if $u_+ > u_-$ then $A_R < 0$ and $K(\eta) < 0$ for $\eta \in (-\infty, \infty)$, while if $u_+ < u_-$ then $A_R > 0$ and $K(\eta) > 0$ for $\eta \in (-\infty, \infty)$. Therefore, in region TR$^+$ we have that

$$u = u_+ + \left(\frac{A_R}{(2\gamma)^{\frac{\gamma}{2}}} - \frac{(u_+ - u_-)}{\sqrt{2\gamma\pi}} e^{-\frac{\sqrt{2\gamma}}{2}\eta} \right) t^{-\frac{\gamma}{2}} e^{-\frac{\gamma t}{2}} + o\left(t^{-\frac{\gamma}{2}} e^{-\frac{\gamma t}{2}}\right) \tag{3.28}$$

as $t \to \infty$ with $\eta = O(1)$.

We next consider the asymptotic structure as $t \to \infty$ for $x < 0$. To proceed we introduce a new region, region III$^-$. The details of this region follow, after minor modification, those given for region III$^+$ and are not repeated here. Therefore, we have in region III$^-$ that

$$\begin{aligned} u(y,t) = u_- + \frac{A_L}{(-y)^\gamma} t^\gamma e^{-\gamma t} + \left(-\frac{\gamma u_- A_L}{(-y)^{\gamma+1}} + \frac{\gamma(\gamma+1) A_L}{(-y)^{\gamma+2}} \right) t^{\gamma+2} e^{-(\gamma+1)t} \\ + o\left(t^{\gamma+2} e^{-(\gamma+1)t}\right) \end{aligned} \tag{3.29}$$

as $t \to \infty$ with $y = O(1)(< 0)$.

Finally, we conclude this case by noting that matching expansion (3.16) with (3.19) (as $z \to -\infty$) to expansion (3.29) (as $y \to 0^-$) fails and we require a transition region, which we label TR$^-$. To examine region TR$^-$ we introduce the scaled coordinate $\hat{\eta}$ by

$$z = -\sqrt{2\gamma}\, t^{\frac{1}{2}} - \frac{(\gamma - 1)}{\sqrt{2\gamma}} \frac{\ln t}{t^{\frac{1}{2}}} + \hat{\eta}\, t^{-\frac{1}{2}} \tag{3.30}$$

so that $\hat{\eta} = O(1)$ as $t \to \infty$ in region TR$^-$. The details of region TR$^-$ follow, after minor modification, those given for region TR$^+$ are are not repeated here. In summary we have in region TR$^-$ that

$$u = u_- + \left(\frac{A_L}{(2\gamma)^{\frac{\gamma}{2}}} + \frac{(u_+ - u_-)}{\sqrt{2\gamma\pi}} e^{\frac{\sqrt{2\gamma}}{2}\hat{\eta}} \right) t^{-\frac{\gamma}{2}} e^{-\frac{\gamma t}{2}} + o\left(t^{-\frac{\gamma}{2}} e^{-\frac{\gamma t}{2}}\right) \tag{3.31}$$

as $t \to \infty$ with $\hat{\eta} = O(1)$.

The asymptotic structure of the solution of **IVP2** as $t \to \infty$ when $\gamma \geq 1$ is now complete. A uniform approximation has been given through regions II$^\pm$, III$^\pm$, TR$^\pm$ and SS. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Figure 3. The large-t attractor for the solution of **IVP2** when $\gamma \geq 1$ is the error function which allows for the adjustment of the solution form u_+ to u_-.

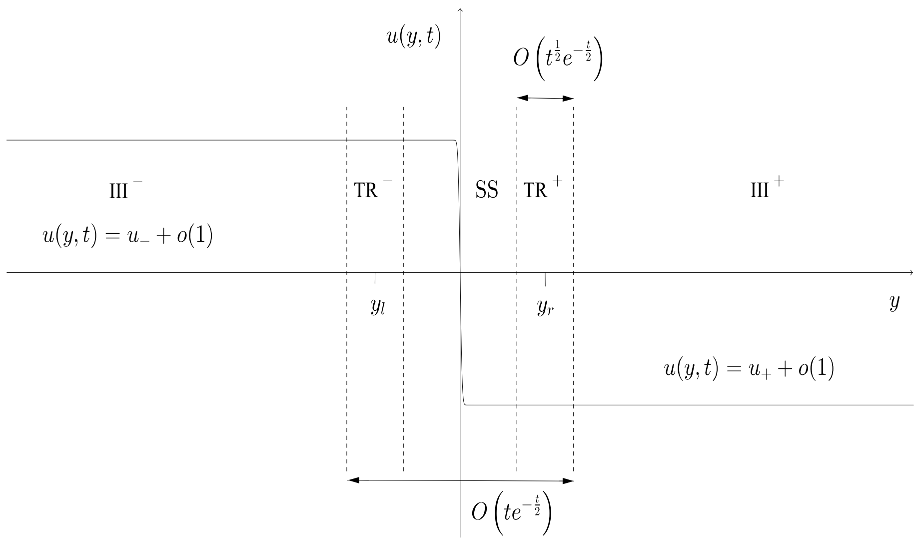

Figure 3: A schematic representation of the asymptotic structure of $u(y,t)$ in the (y,u) plane as $t \to \infty$ for **IVP2** when $\gamma \geq 1$. Here we illustrate the case when $u_+ < u_-$. We recall that $y_l = -y_r = -\sqrt{2\gamma}t^{\frac{3}{2}}e^{-\frac{t}{2}} + (\gamma-1)t^{\frac{1}{2}}e^{-\frac{t}{2}}\ln t$.

3.4 $0 < \gamma < 1$

In this case expansions (2.9) and (2.10) of regions II$^+$ ($x \to \infty, t = O(1)$) and II$^-$ ($x \to -\infty, t = O(1)$) respectively, continue to remain uniform provided $|x| \gg t^{\frac{1}{\gamma-1}}e^{\frac{t}{1-\gamma}}$. However, as already noted a nonuniformity develops when $|x| = O\left(t^{\frac{1}{\gamma-1}}e^{\frac{t}{1-\gamma}}\right)$. We begin by considering the asymptotic structure as $t \to \infty$ for $x > 0$. To proceed we introduce a new region, region E$^+$. To examine region E$^+$ we introduce the scaled coordinate

$$\xi = x\,t^{\frac{-1}{\gamma-1}}e^{-\frac{t}{1-\gamma}}, \tag{3.32}$$

where $\xi = O(1)$, and look for an expansion of the form

$$\begin{aligned} u(y,t) = u_+ &+ h_0(\xi)t^{\frac{\gamma}{1-\gamma}}e^{-\frac{\gamma}{1-\gamma}t} + h_1(\xi)t^{\frac{2}{1-\gamma}}e^{-\frac{(\gamma+1)}{1-\gamma}t} \\ &+ h_2(\xi)t^{\frac{\gamma+2}{1-\gamma}}e^{-\frac{(2\gamma+1)}{1-\gamma}t} + o\left(t^{\frac{\gamma+2}{1-\gamma}}e^{-\frac{(2\gamma+1)}{1-\gamma}t}\right) \end{aligned} \tag{3.33}$$

as $t \to \infty$ with $y = O(1)$. On substituting (3.33) into equation (3.1) (when written in terms of ξ and t) and solving at each order in turn, we find (after matching with (2.9)) that

$$\begin{aligned} u(y,t) = u_+ &+ \frac{A_R}{\xi^\gamma}t^{\frac{\gamma}{1-\gamma}}e^{-\frac{\gamma}{1-\gamma}t} + \frac{\gamma u_+ A_R}{\xi^{\gamma+1}}t^{\frac{2}{1-\gamma}}e^{-\frac{(\gamma+1)}{1-\gamma}t} \\ &+ \left(\frac{\gamma A_R^2}{\xi^{2\gamma+1}} + \frac{\gamma(\gamma+1)A_R}{\xi^{\gamma+2}}\right)t^{\frac{\gamma+2}{1-\gamma}}e^{-\frac{(2\gamma+1)}{1-\gamma}t} + o\left(t^{\frac{\gamma+2}{1-\gamma}}e^{-\frac{(2\gamma+1)}{1-\gamma}t}\right) \end{aligned} \tag{3.34}$$

as $t \to \infty$ with $\xi = O(1)$ (> 0). Expansion (3.34) becomes nonuniform when

$$\xi = O\left(t^{\frac{\gamma}{1-\gamma}}e^{-\frac{\gamma}{1-\gamma}}\right)$$

as $t \to \infty$ [that is, when $x = O\left(t^{-1}e^t\right)$ as $t \to \infty$].

Now, considering the asymptotic structure as $t \to \infty$ when $x < 0$ leads us to introduce region E$^-$ (the details of which follow, after minor modification those given for region E$^+$). In region E$^-$ we have that

$$u(y,t) = u_- + \frac{A_L}{(-\xi)^\gamma} t^{\frac{\gamma}{1-\gamma}} e^{-\frac{\gamma}{1-\gamma}t} - \frac{\gamma u_- A_L}{(-\xi)^{\gamma+1}} t^{\frac{2}{1-\gamma}} e^{-\frac{(\gamma+1)}{1-\gamma}t}$$
$$+ \left(-\frac{\gamma A_L^2}{(-\xi)^{2\gamma+1}} + \frac{\gamma(\gamma+1)A_L}{(-\xi)^{\gamma+2}} \right) t^{\frac{\gamma+2}{1-\gamma}} e^{-\frac{(2\gamma+1)}{1-\gamma}t} + o\left(t^{\frac{\gamma+2}{1-\gamma}} e^{-\frac{(2\gamma+1)}{1-\gamma}t} \right) \tag{3.35}$$

as $t \to \infty$ with $\xi = O(1)$ (< 0). Expansion (3.34) becomes nonuniform when

$$\xi = O\left(t^{\frac{\gamma}{1-\gamma}} e^{-\frac{\gamma}{1-\gamma}} \right)$$

as $t \to \infty$ [that is, when $x = O\left(t^{-1}e^t\right)$ as $t \to \infty$].

To examine the asymptotic structure of the solution of **IVP2** when $\xi = o(1)$ we proceed by introducing the scaled coordinate y by

$$y = xte^{-t},$$

as $t \to \infty$ with $y = O(1)$. We notice that this is the scaling (3.5) that we encountered in Section 3.3, with regions E$^\pm$ being passive allowing for the rearrangement of terms. Therefore, the remaining asymptotic structure of the solution of **IVP2** as $t \to \infty$ when $0 < \gamma < 1$, given by regions III$^\pm$, TR$^\pm$ and SS, follows that given in Section 3.3 and is not repeated here. A uniform approximation has been given through regions II$^\pm$,E$^\pm$, III$^\pm$, TR$^\pm$ and SS. The large-t attractor for the solution of **IVP2** when $0 < \gamma < 1$ is the error function which allows for the adjustment of the solution form u_+ to u_-.

4 Summary

In this paper we have obtained, via the method of matched asymptotic coordinate expansions, the uniform asymptotic structure of the large-t solution to the initial-value problems **IVP1** and **IVP2**.

The form of the large-t solution of initial-value problem **IVP1** depends on the problem parameters u_+ and u_- as follows:

(i) When $u_+ > u_-$ the solution exhibits the formation of the expansion wave, where

$$u(x,t) \sim \begin{cases} u_+, & x > u_+e^t, \\ xe^{-t}, & u_-e^t < x < u_+e^t, \\ u_-, & x < u_-e^t, \end{cases} \tag{4.1}$$

as $t \to \infty$.

(ii) When $u_+ < u_-$ the solution exhibits the formation of a localized Taylor shock profile, where

$$u\left(s(t) + ze^{-t}, t\right) = \left[\frac{(u_+ + u_-)}{2} - \frac{(u_- - u_+)}{2} \tanh\left(\frac{(u_- - u_+)}{4} z + \phi_c \right) \right] + o(1) \tag{4.2}$$

as $t \to \infty$ with $z = O(1)$, where $z = (x - s(t))e^t$, ϕ_c is a constant and

$$s(t) = \frac{(u_+ + u_-)}{2} e^t$$

as $t \to \infty$. In particular, we observe that:

(a) When $-u_- < u_+ < u_-$ the Taylor shock profile is located at $x = s(t)$ and is accelerating in the $+x$ direction.

(b) When $u_+ < u_- < -u_+$ the Taylor shock profile is located at $x = s(t)$ and is accelerating in the $-x$ direction.

(c) When $u_+ = -u_-$ the Taylor shock profile is stationary and located at $x = 0$.

In each case the thickness of the localized region is of $O(e^{-t})$ as $t \to \infty$.

The large-time asymptotic behaviour of the solution of **IVP1** has been considered in [12] for the case $u_+ = -u_-$. It is established in [12] that when $u_+ = -1$ the solution of **IVP1** tends to the stationary solution, while when $u_+ = +1$ the solution of **IVP1** tends to the expansion (or rarefaction) wave. These results being in agreement with the cases (i) and (ii)(c) above. A full discussion of expansion waves and Taylor shocks can be found in [25].

The form of the large-t solution of initial-value **IVP2** exhibits the formation of an error function profile, where

$$u\left(ze^{\frac{t}{2}}, t\right) = \left[\frac{(u_+ + u_-)}{2} - \frac{(u_- - u_+)}{2}\,\mathrm{erf}\left(\frac{z}{2}\right)\right] + o(1) \tag{4.3}$$

as $t \to \infty$ with $z = xe^{-\frac{t}{2}} = O(1)$. We observe that error function profile is in a stretching frame of reference of thickness $O(e^{\frac{t}{2}})$ as $t \to \infty$. We conclude by noting that equation (3.1) is of supercylindrical type and has been investigated by a number of authors (see for example [22] and [24]). In particular, in [22] the global stability of this solution is established with it being shown that

$$\left| u(x,t) - \left[\frac{(u_+ + u_-)}{2} - \frac{(u_- - u_+)}{2}\,\mathrm{erf}\left(\frac{xe^{-\frac{t}{2}}}{2}\right)\right]\right| \to 0 \quad \text{as} \quad t \to \infty,$$

uniformly in x. It was noted in [5] and [22] that the leading order term in expansion (4.3) (when written in terms of x and t) is the error function solution of the old-age equation

$$u_t = e^t u_{xx}.$$

It is interesting to note that although the parameter γ plays an important role in the development and structure of the large-t solution of the initial-value problems considered in this paper it does not appear at leading order in expansions (4.1), (4.2) and (4.3) (which are dependent only on the constant states u_+ and u_-). The correction terms to these leading order profiles will however depend on the parameter γ.

Finally, the asymptotic predictions for the large-time attractors for the problems **IVP1** and **IVP2** considered are in agreement with the known results for the application areas that motivate this work. In particular, in the important area of acoustic waves (see for example, [5], [8], [11], [9] and [18]) the results presented are in excellent agreement with the existing results. However, the analysis presented here links for the first time these large-time attractors to the given initial data through a number of asymptotic regions giving the complete large-time solution for each initial-value problem considered over all values of the problem parameters.

References

[1] M. Abramowitz and I. Stegun *Handbook of Mathematical Functions.* Dover (1965).

[2] H. Batemann. Some recent researches on the motion of fluids. *Monthly Weather Review* 43, 163-170 (1915).

[3] J.M. Burgers. A mathematical model illustrating the theory of turbulence. *Adv. Appl. Mech.* 1, 171-199 (1948).

[4] Q. Changzheng. Allowed transformations and symmetry classes of variable coefficient Burgers equation. IMA J. Appl. Math. 54, 203-225 (1995).

[5] D.G. Crighton. Basic nonlinear acoustics. In: Sette D, editor. Frontiers in physical acoustics. Amsterdam: North-Holland; (1986).

[6] J.D. Cole. On a quasilinear parabolic equation occuring in aerodyamics. *Quart. Appl. Math.* 9, 225-236 (1951).

[7] J. Doyle and M.J. Englefield. Similarity solutions of a generalized Burgers equation. *IMA J. Appl. Math.* 50, 142-152 (1990).

[8] D.G. Crighton and J.F. Scott. Asymptotic solutions of model equations in nonlinear acoustics. *Phil. Trans. R. Soc. Lond.* A292, 101-134 (1979).

[9] B.O. Enflo and O.V. Rudenko. To the theory of generalized Burgers' equations. *Acta Acust* 88, 1-8 (2002)

[10] A.R. Forsyth. *Theory of Differential Equations* Part 4 (Vol. 5-6). Reprinted: Dover, New York (1959).

[11] P.W. Hammerton and D.G. Crighton. Approximate solution methods for nonlinear acoustic propagation over long ranges. *Proc. R. Soc. Lond.* A 426:125152 (1989).

[12] N. Hayashi and P.I. Naumkin. Asymptotics for the Burgers equation with pumping. *Comm. Math. Phys.* 239, 287-307 (2003).

[13] E. Hopf. The partial differential equation $u_t + uu_x = \mu u_{xx}$. *Comm. Pure Appl. Math.* 3, 201-230 (1950).

[14] M.J. Lighthill. Viscosity effects in sound waves of finite amplitude. In: *Surveys in Mechanics*. Cambridge University Press, 250-351 (1956).

[15] J.A. Leach. The large-time solution of Burgers' equation with time dependent coefficients. II. The coefficients are algebraic functions. (In preparation)

[16] J.A. Leach and D.J. Needham Matched Asymptotic Expansions in Reaction-Diffusion Theory. Springer Monographs in Mathematics (2003).

[17] J.A. Leach and D.J. Needham The large-time development of the solution to an initial-value problem for the Korteweg-de Vries equation: I. Initial data has a discontinuous expansive step. *Nonlinearity* 21, 2391-2408 (2008).

[18] S. Leibovich and A. R. Seebass (Eds.) Nonlinear waves. Cornell University Press, Ithaca, N.Y.London, 1974.

[19] J.J.C. Nimmo and D.G. Crighton. Bäcklund transformations for nonlinear parabolic equations: The general results. *Proc. Roy. Soc. Lond.* A384, 381-401 (1982).

[20] J.J.C. Nimmo and D.G. Crighton. Geometrical and diffusive effects in nonlinear acoustic propagation over long ranges. *Proc. Roy. Soc. Lond.* A320, 1-35 (1986).

[21] P.L. Sachdev. Nonlinear diffusive waves. Cambridge University Press, (2009).

[22] J.F. Scott. The long time asymptotics of the solutions to the generalized Burgers equation. *Proc. Roy. Soc. Lond.* A373, 443-456 (1981).

[23] P.N. Sionoid and A.T. Cates. The generalized Burgers and ZabolotskayaKhokhlov equations: transformations, exact solutions and qualitative properties. *Proc. Roy. Soc. Lond.* A447,253270 (1994).

[24] E. Soewono and L. Debnath. Asymptotic stability of solutions of the generalized Burgers equation. *J. Math. Anal. appl.* 153, 179-189 (1990).

[25] G.B. Whitham. Linear and Nonlinear Waves. Wiley, New York (1974).

[26] M. Van Dyke Perturbation Methods in Fluid Dynamics. Parabolic Press, Stanford, CA, (1975).