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# $L^p$-Square Function Estimates on Spaces of Homogeneous Type and on Uniformly Rectifiable Sets


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**Abstract**

We establish square function estimates for integral operators on uniformly rectifiable sets by proving a local $T(b)$ theorem and applying it to show that such estimates are stable under the so-called big pieces functor. More generally, we consider integral operators associated with Ahlfors-David regular sets of arbitrary codimension in ambient quasi-metric spaces. The local $T(b)$ theorem is then used to establish an inductive scheme in which square function estimates on so-called big pieces of an Ahlfors-David regular set are proved to be sufficient for square function estimates to hold on the entire set. Extrapolation results for $L^p$ and Hardy space versions of these estimates are also established. Moreover, we prove square function estimates for integral operators associated with variable coefficient kernels, including the Schwartz kernels of pseudodifferential operators acting between vector bundles on subdomains with uniformly rectifiable boundaries on manifolds.



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# Contents

# 1 Introduction

The purpose of this work is three-fold: first, to develop the so-called "local $T(b)$ theory" for square functions in a very general context, in which we allow the ambient space to be of homogeneous type, and in which the "boundary" of the domain is of arbitrary (positive integer) co-dimension; second, to use a special case of this local $T(b)$ theory to establish boundedness, for a rather general class of square functions, on uniformly rectifiable sets of codimension one in Euclidean space; and third, to establish an extrapolation principle whereby an $L^p$ (or even weak-type $L^p$) estimate for a square function, for *one* fixed $p$, yields a full range of $L^p$ bounds. We shall describe these results in more detail below, but let us first recall some of the history of the development of the theory of square functions.

Referring to the role square functions play in mathematics, E. Stein wrote in 1982 (cf. [71]) that "[*square*] *functions are of fundamental importance in analysis, standing as they do at the crossing of three important roads many of us have traveled by: complex function theory, the Fourier transform* (*or orthogonality in its various guises*), *and real-variable methods.*" In the standard setting of the unit disc $\mathbb{D}$ in the complex plane, the classical square function $Sf$ of some $f : \mathbb{T} \to \mathbb{C}$ (with $\mathbb{T} := \partial\mathbb{D}$) is defined in terms of the Poisson integral $u_f(r, \omega)$ of $f$ in $\mathbb{D}$ (written in polar coordinates) by the formula

$$(Sf)(z) := \Big(\int_{(r,\omega)\in\Gamma(x)} |(\nabla u_f)(r,\omega)|^2 \, r \, dr \, d\omega\Big)^{1/2}, \qquad z \in \mathbb{T}, \tag{1.1}$$

where $\Gamma(z)$ stands for the Stolz domain $\{(r,\omega) : |\arg(z) - \omega| < 1 - r < \frac{1}{2}\}$ in $\mathbb{D}$. Let $v$ denote the (normalized) complex conjugate of $u_f$ in $\mathbb{D}$. Then, if the analytic function $F := u_f + iv$ is one-to-one, the quantity $(Sf)(z)^2$ may be naturally interpreted as the area of the region $F(\Gamma(z)) \subseteq \mathbb{C}$ (recall that $\det(DF) = |\nabla u_f|^2$). The operator (1.1) was first considered by Lusin and the observation just made justifies the original name for (1.1) as Lusin's area function (or Lusin's area integral). A fundamental property of $S$, originally proved by complex methods (cf. [13, Theorem 3, pp. 1092-1093], and [31] for real-variable methods) is that

$$\|Sf\|_{L^p(\mathbb{T})} \approx \|f\|_{H^p(\mathbb{T})} \quad \text{for} \quad p \in (0,\infty), \tag{1.2}$$

which already contains the $H^p$-boundedness of the Hilbert transform. Indeed, if $F = u + iv$ is analytic then the Cauchy-Riemann equations entail $|\nabla u| = |\nabla v|$ and, hence, $S(u|_{\mathbb{T}}) = S(v|_{\mathbb{T}})$. In spite of the technical, seemingly intricate nature of (1.1) and its generalizations to higher dimensions, such as

$$(Sf)(x) := \Big(\int_{|x-y|<t} |(\nabla u_f)(y,t)|^2 \, t^{1-n} dy dt\Big)^{1/2}, \qquad x \in \mathbb{R}^n := \partial\mathbb{R}^{n+1}_+, \tag{1.3}$$

a great deal was known by the 1960's about the information encoded into the size of $Sf$, measured in $L^p$, thanks to the pioneering work of D.L. Burkholder, A.P. Calderón, C. Fefferman, R.F. Gundy, N. Lusin, J. Marcinkiewicz, C. Segovia, M. Silverstein, E.M. Stein, A. Zygmund, and others. See, e.g., [11], [12], [13], [31], [68], [70], [71], [72], and the references therein.

Subsequent work by B. Dahlberg, E. Fabes, D. Jerison, C. Kenig and others, starting in the late 1970's (cf. [21], [22], [29], [56], [66]), has brought to prominence the relevance of square function estimates in the context of partial differential equations in non-smooth settings, whereas work by D. Jerison and C. Kenig [54] in the 1980's as well as G. David and S. Semmes in the 1990's (cf. [26], [27]) has lead to the realization that square function estimates are also intimately connected with the geometry of sets (especially geometric measure theoretic aspects). More recently, square function estimates have played an important role in the solution of the Kato problem in [46], [42], [4].

The operator $S$ defined in (1.1) is obviously non-linear but the estimate

$$\|Sf\|_{L^p} \leq C\|f\|_{H^p} \tag{1.4}$$

may be linearized by introducing a suitable (linear) vector-valued operator. Specifically, set $\Gamma := \{(z,t) \in \mathbb{R}^{n+1}_+ : |z| < t\}$ and consider the Hilbert space

$$\mathscr{H} := \Big\{h : \Gamma \to \mathbb{C}^n : \ h \text{ is measurable and } \|h\|_{\mathscr{H}} := \Big(\int_\Gamma |h(z,t)|^2 t^{1-n}\,dtdz\Big)^{\frac{1}{2}} < \infty\Big\}. \tag{1.5}$$

Also, let $\widetilde{S}f : \mathbb{R}^n \to \mathscr{H}$ be defined by the formula

$$\big((\widetilde{S}f)(x)\big)(z,t) := (\nabla u_f)(x-z,t), \qquad \forall\, x \in \mathbb{R}^n,\ \forall\,(z,t) \in \Gamma, \tag{1.6}$$

i.e., $\widetilde{S}$ is the integral operator (mapping scalar-valued functions defined on $\mathbb{R}^n$ into $\mathscr{H}$-valued functions defined on $\mathbb{R}^n$), whose kernel $k : \mathbb{R}^n \times \mathbb{R}^n \setminus \text{diagonal} \to \mathscr{H}$, which is of convolution type, is given by $(k(x,y))(z,t) := (\nabla P_t)(x-y-z)$, for all $x, y \in \mathbb{R}^n$, $x \neq y$, and $(z,t) \in \Gamma$, where $P_t(x)$ is the Poisson kernel in $\mathbb{R}^{n+1}_+$. Then, if $L^p(\mathbb{R}^n, \mathscr{H})$ stands for the Bôchner space of $\mathscr{H}$-valued, $p$-th power integrable functions on $\mathbb{R}^n$, it follows that

$$\|Sf\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{H^p(\mathbb{R}^n)} \iff \|\widetilde{S}f\|_{L^p(\mathbb{R}^n,\mathscr{H})} \leq C\|f\|_{H^p(\mathbb{R}^n)}. \tag{1.7}$$

The relevance of the linearization procedure described in (1.5)-(1.7) is that it highlights the basic role of the case $p = 2$ in (1.4). This is because the operator $\widetilde{S}$ falls within the scope of theory of Hilbert space-valued singular integral operators of Calderón-Zygmund type for which boundedness on $L^2$ automatically extrapolates to the entire scale $L^p$, for $1 < p < \infty$ (the extension to the case when $p \leq 1$ makes use of other specific features of $\widetilde{S}$).

From the point of view of geometry, what makes the above reduction to the case $p = 2$ work is the fact that the upper-half space has the property that $x + \Gamma \subseteq \mathbb{R}^{n+1}_+$ for every $x \in \partial\mathbb{R}^{n+1}_+$. Such a cone property actually characterizes Lipschitz domains (cf. [48]), in which scenario this is the point of view adopted in [65, Theorem 4.11, p. 73].

Hence, $S$ may be eminently regarded as a singular integral operator with a Hilbert space-valued Calderón-Zygmund kernel and, as such, establishing the $L^2$ bound

$$\|\widetilde{S}f\|_{L^2(\mathbb{R}^n,\mathscr{H})} \leq C\|f\|_{L^2(\mathbb{R}^n)} \tag{1.8}$$

is of basic importance to jump-start the study of the operator $S$. Now, as is well-known (and easy to check; see, e.g., [72, pp. 27-28]), (1.8) follows from Fubini's and Plancherel's theorems.

For the goals we have in mind in the present work, it is worth recalling a quote from C. Fefferman's 1974 ICM address [30] where he writes that "*When neither the Plancherel theorem nor Cotlar's lemma applies, $L^2$-boundedness of singular operators presents very hard problems, each of which must* (*so far*) *be dealt with on its own terms.*" For scalar singular integral operators, this situation began to be remedied in 1984 with the advent of the $T(1)$-Theorem, proved by G. David and J.-L. Journé in [24]. This was initially done in the Euclidean setting, using Fourier analysis methods. It was subsequently generalized and refined in a number of directions, including the extension to spaces of homogeneous type by R. Coifman (unpublished, see the discussion in [14]), and the $T(b)$ Theorems proved by A. McIntosh and Y. Meyer in [59], and by G. David, J.L. Journé and S. Semmes in [25]. The latter reference also contains an extension to the class of singular-integral operators with matrix-valued kernels. The more general case of operator-valued kernels has been treated by Figiel [32] and by T. Hytönen and L. Weis [53], who prove $T(1)$ Theorems in the spirit of the original work in [24] for singular integrals associated with kernels taking values in Banach spaces satisfying the UMD property. Analogous $T(b)$ theorems were obtained by Hÿtonen [50] (in Euclidean space) and by Hÿtonen and Martikainen [51] (in a metric measure space). Yet in a different direction, initially motivated by applications to the theory of analytic capacity, $L^2$-boundedness criteria which are local in nature appeared in the work of M. Christ [15]. Subsequently, Christ's local $T(b)$ theorem has been extended to the setting of non-doubling spaces by F. Nazarov, S. Treil and A. Volberg in [67]. Further extensions of the local $T(b)$ theory for singular integrals appear in [6], [8], [7] and [52].

Much of the theory mentioned in the preceding paragraph has also been developed in the context of square functions, as opposed to singular integrals. In the convolution setting discussed above, (1.8) follows immediately from Plancherel's theorem, but the latter tool fails in the case when $\mathbb{R}^{n+1}_+$ is replaced by a domain whose geometry is rough (so that, e.g., the cone property is violated), and/or one considers a square-function operator whose integral kernel $\theta(x,y)$ is no longer of convolution type (as was the case for $\widetilde{S}$). A case in point is offered by the square-function estimate of the type

$$\int_0^\infty \|\Theta_t f\|^2_{L^2(\mathbb{R}^n)}\frac{dt}{t} \leq C\|f\|^2_{L^2(\mathbb{R}^n)}, \tag{1.9}$$

where

$$\big(\Theta_t f\big)(x) := \int_{\mathbb{R}^n} \theta_t(x,y)f(y)\,dy, \qquad x\in\mathbb{R}^n, \qquad t>0, \tag{1.10}$$

with $\{\theta_t(\cdot,\cdot)\}_{t>0}$ a standard Littlewood-Paley family, i.e., satisfying for some exponent $\alpha>0$,

$$|\theta_t(x,y)| \leq C\frac{t^\alpha}{(t+|x-y|)^{n+\alpha}}, \tag{1.11}$$

$$|\theta_t(x,y)-\theta_t(x,y')| \leq C\frac{|y-y'|^\alpha}{(t+|x-y|)^{n+\alpha}}, \text{ if } |y-y'|<t/2. \tag{1.12}$$

Then, in general, linearizing estimate (1.9) in a manner similar to (1.7) yields an integral operator which is no longer of convolution type. As such, Plancherel's theorem is not directly effective in dealing with (1.9) given that the task at hand is establishing the $L^2$-boundedness of a variable kernel (Hilbert-valued) singular integral operator. However, M. Christ and J.-L. Journé have shown in [16] (under the same size/regularity conditions in (1.11)-(1.12)) that the square function estimate (1.9) is valid if the following Carleson measure condition holds:

$$\sup_{Q\subseteq\mathbb{R}^n}\Big(\int_0^{\ell(Q)} ⨍_Q |(\Theta_t 1)(x)|^2\,\frac{dxdt}{t}\Big) < \infty, \tag{1.13}$$

where the supremum is taken over all cubes $Q$ in $\mathbb{R}^n$. The latter result is also implicit in the work of Coifman and Meyer [18]. Moreover, S. Semmes' has shown in [69] that (1.9) holds if there exists a para-accretive function $b$ such that (1.13) holds with "1" replaced by "$b$".

Refinements of Semmes' global $T(b)$ theorem for square functions, in the spirit of M. Christ's local $T(b)$ theorem for singular integrals [15], have subsequently been established in [3], [40], [41]. The local $T(b)$ theorem for square functions which constitutes the main result in [41] reads as follows. Suppose $\Theta_t$ is as in (1.10) with kernel satisfying (1.11)-(1.12) as well as

$$|\theta_t(x,y) - \theta_t(x',y)| \leq C\frac{|x-x'|^\alpha}{(t+|x-y|)^{n+\alpha}} \quad \text{if } \ |x-x'| < t/2. \tag{1.14}$$

In addition, assume that there exist a constant $C_o \in (0,\infty)$, an exponent $q \in (1,\infty)$ and a collection $\{b_Q\}_Q$ of functions indexed by all dyadic cubes $Q$ in $\mathbb{R}^n$ with the following properties:

$$(i) \int_{\mathbb{R}^n} |b_Q(x)|^q\,dx \leq C_o|Q|, \qquad (ii) \Big|\int_{\mathbb{R}^n} b_Q(x)\,dx\Big| \geq \frac{1}{C_o}|Q|,$$
$$(iii) \int_Q \Big(\int_0^{\ell(Q)} |(\Theta_t b_Q)(x)|^2\,\frac{dt}{t}\Big)^{q/2}\,dx \leq C_o|Q|.$$

Then the square function estimate (1.9) holds. The case $q = 2$ of this theorem does not require (1.14) (just regularity in the second variable, as in (1.12))[1], and was already implicit in the solution of the Kato problem in [46], [42], [4]. It was formulated explicitly in [3], [40]. An extension of the result of [41] to the case that the half-space is replaced by $\mathbb{R}^{n+1}\setminus E$, where $E$ is a closed Ahlfors-David regular set (cf. Definition 2.8 below) of Hausdorff dimension $n$, appears in [34]. The latter extension has been used to prove a result of *free boundary* type, in which higher integrability of the Poisson kernel, in the presence of certain natural background hypotheses, is shown to be equivalent to uniform rectifiability (cf. Definition 5.4 below) of the boundary [44], [45]. Further extensions of the result of [41], to the case in which the kernel $\theta_t$ and pseudo-accretive system $b_Q$ may be matrix-valued (as in the setting of the Kato problem), and in which $\theta_t$ need no longer satisfy the pointwise size and regularity conditions (1.11)-(1.12), will appear in the forthcoming Ph.D. thesis of A. Grau de la Herran [33].

A primary motivation for us in the present work is the connection between square function bounds (or their localized versions in the form of "Carleson measure estimates"), and a quantitative, scale invariant notion of rectifiability. This subject has been developed extensively by David and Semmes [26], [27] (but with some key ideas already present in the work of P. Jones [55]). Following [26], [27], we shall give in the sequel (cf. Definition 5.4), a precise definition of the property that a closed set $E$ is "Uniformly Rectifiable" (UR), but for now let us merely mention that UR sets are the ones on which "nice" singular integral operators are bounded on $L^2$. David and Semmes have shown that these sets may also be characterized via certain square function estimates, or equivalently, via Carleson measure estimates. For example, let $E \subset \mathbb{R}^{n+1}$ be a closed set of codimension one, which is ($n$-dimensional) Ahlfors-David regular (ADR) (cf. Definition 2.8). Then $E$ is UR if and only if we have the Carleson measure estimate

$$\sup_B r^{-n} \int_B \big|\big(\nabla^2 \mathcal{S}1\big)(x)\big|^2\,\operatorname{dist}(x,E)\,dx < \infty, \tag{1.15}$$

where the supremum runs over all Euclidean balls $B := B(z,r) \subseteq \mathbb{R}^{n+1}$, with $r \leq \operatorname{diam}(E)$, and center $z \in E$, and where $\mathcal{S}f$ is the harmonic single layer potential of the function $f$, i.e.,

$$\mathcal{S}f(x) := c_n \int_E |x-y|^{1-n} f(y)\,d\mathcal{H}^n(y), \qquad x \in \mathbb{R}^{n+1}\setminus E. \tag{1.16}$$

[1]In fact, even the case $q \neq 2$ does not require (1.14), if the vertical square function is replaced by a conical one; see [33] for details.

Here $\mathcal{H}^n$ denotes $n$-dimensional Hausdorff measure. For an appropriate normalizing constant $c_n|x|^{1-n}$ is the usual fundamental solution for the Laplacian in $\mathbb{R}^{n+1}$. We refer the reader to [27] for details, but see also Section 4 where we present some related results. We note that by "$T1$" reasoning (cf. Section 3 below), (1.15) is equivalent to the square function bound

$$\int_{\mathbb{R}^{n+1}\setminus E} \big|\big(\nabla^2 \mathcal{S} f\big)(x)\big|^2 \operatorname{dist}(x,E)\,dx \leq C \int_E |f(x)|^2\,d\mathcal{H}^n(x)\,. \tag{1.17}$$

Using an idea of P. Jones [55], one may derive, for UR sets, a quantitative version of the fact that rectifiability may be characterized in terms of existence a.e. of approximate tangent planes. Again, a Carleson measure expresses matters quantitatively. For $x \in E$ and $t > 0$, set

$$\beta_2(x,t) := \inf_P \left(\frac{1}{t^n}\int_{B(x,t)\cap E}\left(\frac{\operatorname{dist}(y,P)}{t}\right)^2 d\mathcal{H}^n(y)\right)^{1/2}, \tag{1.18}$$

where the infimum runs over all $n$-planes $P$. Then a closed, ADR set $E$ of codimension one is UR if and only if the following Carleson measure estimate holds on $E \times \mathbb{R}_+$:

$$\sup_{x_0\in E,\, r>0} r^{-n}\int_0^r \int_{B(x_0,t)\cap E} \beta_2(x,t)^2\,d\mathcal{H}^n(x)\frac{dt}{t} \,<\, \infty. \tag{1.19}$$

See [26] for details, and for a formulation in the case of higher codimension. A related result, also obtained in [26], is that a set $E$ as above is UR if and only if, for every odd $\psi \in C_0^\infty(\mathbb{R}^{n+1})$, the following discrete square function bound holds:

$$\sum_{k=-\infty}^{\infty}\int_E\left|\int_E 2^{-kn}\psi\big(2^{-k}(x-y)\big)f(y)\,d\mathcal{H}^n(y)\right|^2 d\mathcal{H}^n(x) \leq C_\psi \int_E |f(x)|^2\,d\mathcal{H}^n(x)\,. \tag{1.20}$$

Again, there is a Carleson measure analogue, and a version for sets $E$ of higher codimension.

The following theorem collects some of the main results in our present work. It generalizes results described earlier in the introduction, which were valid in the codimension one case, and in which the ambient space $\mathscr{X}$ was Euclidean. To state it, recall that (in a context to be made precise below) a measurable function $b : E \to \mathbb{C}$ is called para-accretive if it is essentially bounded and there exist constants $c, C \in (0,\infty)$ such that the following conditions are satisfied:

$$\forall\, Q \in \mathbb{D}(E) \quad \exists\, \widetilde{Q} \in \mathbb{D}(E) \quad \text{such that} \quad \widetilde{Q} \subseteq Q, \quad \ell(\widetilde{Q}) \geq c\ell(Q), \quad \left|\fint_{\widetilde{Q}} b\,d\sigma\right| \geq C. \tag{1.21}$$

Other relevant definitions will be given in the sequel.

**Theorem 1.1.** *Let $0 < d < m < \infty$. Assume that $(\mathscr{X}, \rho, \mu)$ is an $m$-dimensional* ADR *space. Let $\theta : (\mathscr{X}\times\mathscr{X})\setminus\{(x,x) : x \in \mathscr{X}\} \to \mathbb{R}$ denote a Borel measurable function, with respect to the product topology $\tau_\rho \times \tau_\rho$, for which there exist $C_\theta$, $\alpha$, $\upsilon \in (0,\infty)$ such that for all $x$, $y$, $\widetilde{y} \in \mathscr{X}$ with $x \neq y$, $x \neq \widetilde{y}$ and $\rho(y,\widetilde{y}) \leq \frac{1}{2}\rho(x,y)$, the following properties hold:*

$$|\theta(x,y)| \leq \frac{C_\theta}{\rho(x,y)^{d+\upsilon}}, \tag{1.22}$$

$$|\theta(x,y) - \theta(x,\widetilde{y})| \leq C_\theta \frac{\rho(y,\widetilde{y})^\alpha}{\rho(x,y)^{d+\upsilon+\alpha}}. \tag{1.23}$$

*Assume that $E$ is a closed subset of $(\mathscr{X},\tau_\rho)$ and that $\sigma$ is a Borel semiregular measure on $(E,\tau_{\rho|_E})$ such that $\big(E,\rho\big|_E,\sigma\big)$ is a $d$-dimensional* ADR *space, and define the integral operator $\Theta=\Theta_E$ for all functions $f\in L^p(E,\sigma)$, $1\leq p\leq\infty$, by*

$$(\Theta f)(x):=\int_E\theta(x,y)f(y)\,d\sigma(y),\qquad\forall\,x\in\mathscr{X}\setminus E.\tag{1.24}$$

*Let $\mathbb{D}(E)$ denote a dyadic cube structure on $E$ and, for each $Q\in\mathbb{D}(E)$, let $T_E(Q)$ denote the dyadic Carleson tent over $Q$. Also, let $\rho_\#$ denote the regularized version of the quasi-distance $\rho$, as in Theorem 2.1, and for each $x\in\mathscr{X}$, set $\delta_E(x):=\inf\{\rho_\#(x,y):\,y\in E\}$.*

*Then the following properties are equivalent:*

*(1) There exists $C\in(0,\infty)$ such that for each $f\in L^2(E,\sigma)$ it holds that*

$$\int_{\mathscr{X}\setminus E}|(\Theta f)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\leq C\int_E|f(x)|^2\,d\sigma(x).\tag{1.25}$$

*(2) It holds that*

$$\sup_{Q\in\mathbb{D}(E)}\left(\frac{1}{\sigma(Q)}\int_{T_E(Q)}\big|\big(\Theta 1\big)(x)\big|^2\,\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\right)<\infty.\tag{1.26}$$

*(3) There exists $C\in(0,\infty)$ such that for each $f\in L^\infty(E,\sigma)$ it holds that*

$$\sup_{Q\in\mathbb{D}(E)}\left(\frac{1}{\sigma(Q)}\int_{T_E(Q)}|(\Theta f)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\right)^{1/2}\leq C\|f\|_{L^\infty(E,\sigma)}.\tag{1.27}$$

*(4) It holds that*

$$\sup_{x\in E,\,r>0}\left(\frac{1}{\sigma\big(E\cap B_{\rho_\#}(x,r)\big)}\int_{B_{\rho_\#}(x,r)\setminus E}|\Theta 1|^2\delta_E^{2\upsilon-(m-d)}\,d\mu\right)<\infty.\tag{1.28}$$

*(5) There exists a para-accretive function $b:E\to\mathbb{C}$ such that*

$$\sup_{Q\in\mathbb{D}(E)}\left(\frac{1}{\sigma(Q)}\int_{T_E(Q)}|(\Theta b)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\right)<\infty.\tag{1.29}$$

*(6) There exists $C\in(0,\infty)$ such that for each $f\in L^\infty(E,\sigma)$ it holds that*

$$\sup_{x\in E,\,r>0}\left(\frac{1}{\sigma\big(E\cap B_{\rho_\#}(x,r)\big)}\int_{B_{\rho_\#}(x,r)\setminus E}|\Theta f|^2\delta_E^{2\upsilon-(m-d)}\,d\mu\right)^{1/2}\leq C\|f\|_{L^\infty(E,\sigma)}.\tag{1.30}$$

*(7) There exist $C_0\in[1,\infty)$, $c_0\in(0,1]$ and a collection $\{b_Q\}_{Q\in\mathbb{D}(E)}$ of $\sigma$-measurable functions $b_Q:E\to\mathbb{C}$ such that for each $Q\in\mathbb{D}(E)$ the following properties hold:*

$$\int_E|b_Q|^2\,d\sigma\leq C_0\sigma(Q),\quad\left|\int_{\widetilde{Q}}b_Q\,d\sigma\right|\geq\frac{1}{C_0}\,\sigma(\widetilde{Q})\text{ for some }\widetilde{Q}\subseteq Q\text{ with }\ell(\widetilde{Q})\geq c_0\ell(Q),$$
$$\int_{T_E(Q)}|(\Theta\,b_Q)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\leq C_0\sigma(Q).\tag{1.31}$$

*(8) There exists $C_0 \in [1,\infty)$ and, for each surface ball $\Delta = \Delta(x_o, r) := B_{\rho_\#}(x_o, r) \cap E$, where $x_o \in E$ and $r \in (0, \mathrm{diam}_\rho(E)] \cap (0,\infty)$, there exists a $\sigma$-measurable function $b_\Delta : E \to \mathbb{C}$ supported in $\Delta$, such that the following properties hold:*

$$\int_E |b_\Delta|^2 \, d\sigma \leq C_0 \sigma(\Delta), \qquad \left| \int_\Delta b_\Delta \, d\sigma \right| \geq \frac{1}{C_0} \sigma(\Delta),$$
$$\int_{B_{\rho_\#}(x_o, 2C_\rho r)\setminus E} |(\Theta\, b_\Delta)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)} \, d\mu(x) \leq C_0 \sigma(\Delta). \tag{1.32}$$

*(9) The set $E$ has* BPSFE *relative to $\theta$ (see Definition 4.1).*

*(10) The set $E$ has* $(\mathrm{BP})^k$SFE *relative to $\theta$ for some, or any, $k \in \mathbb{N}$ (see Definition 4.1).*

*(11) There exist $p \in (0,\infty)$ and $C, \kappa \in (0,\infty)$ such that for each $f \in L^p(E,\sigma)$ it holds that*

$$\sup_{\lambda>0} \left[ \lambda \cdot \sigma\Big( \Big\{ x \in E : \int_{\Gamma_\kappa(x)} |(\Theta f)(y)|^2 \frac{d\mu(y)}{\delta_E(y)^{m-2\upsilon}} > \lambda^2 \Big\} \Big)^{1/p} \right] \leq C \|f\|_{L^p(E,\sigma)}, \tag{1.33}$$

*where $\Gamma_\kappa(x)$ denotes the nontangential approach region defined in* (6.1).

*(12) Set $\gamma := \min\{\alpha, \mathrm{ind}(E, \rho|_E)\}$, where the index is defined in* (2.6). *If $p \in (\frac{d}{d+\gamma}, \infty)$ and $\kappa \in (0,\infty)$, then $\Theta$ extends to the Hardy space $H^p(E, \rho|_E, \sigma)$ defined in Section 6.4, and there exists $C \in (0,\infty)$ such that for each $f \in H^p(E, \rho|_E, \sigma)$ it holds that*

$$\left[ \int_E \left( \int_{\Gamma_\kappa(x)} |(\Theta f)(y)|^2 \frac{d\mu(y)}{\delta_E(y)^{m-2\upsilon}} \right)^{p/2} d\sigma(x) \right]^{1/p} \leq C \|f\|_{H^p(E,\rho|_E,\sigma)}. \tag{1.34}$$

*(13) Set $\gamma := \min\{\alpha, \mathrm{ind}(E, \rho|_E)\}$. If $p \in (\frac{d}{d+\gamma}, \infty)$ and $q \in (1,\infty)$, then the operator*

$$\delta_E^{\upsilon - m/q} \Theta : H^p(E, \rho|_E, \sigma) \longrightarrow L^{(p,q)}(\mathscr{X}, E) \tag{1.35}$$

*is well-defined, linear and bounded on the mixed norm space $L^{(p,q)}(\mathscr{X}, E)$ defined in* (6.6).

A few comments pertaining to the nature and scope of Theorem 1.1 are in order:

- Theorem 1.1 makes the case that estimating the square function in $L^p$, along with other related issues considered above, may be regarded as "zeroth order calculus", since only integrability and quasi-metric geometry are involved, without recourse to differentiability (or vector space structures). In particular, our approach is devoid of any PDE results and techniques. Compared with works in the upper-half space $\mathbb{R}^n \times (0,\infty)$, or so-called generalized upper-half spaces $E \times (0,\infty)$ (cf., e.g., [35] and the references therein), here we work in an ambient $\mathscr{X}$ with no distinguished "vertical" direction. Moreover, the set $E$ is allowed to have arbitrary ADR co-dimension in the ambient $\mathscr{X}$. In this regard we also wish to point out that Theorem 1.1 permits the consideration of fractal subsets of the Euclidean space (such as the case when $E$ is the von Koch's snowflake in $\mathbb{R}^2$, in which scenario $d = \frac{\ln 4}{\ln 3}$).

- Passing from $L^2$ estimates to $L^p$ estimates is no longer done via a linearization procedure, since the environment no longer permits it, so instead we use tent space theory and exploit the connection between the Lusin and the Carleson operators on spaces of homogeneous type (thus generalizing work from [17] in the Euclidean setting). This reinforces the philosophy that the square-function is a singular integral operator at least in spirit (if not in the letter).

- The various quantitative aspects of the claims in items *(1)-(11)* of Theorem 1.1 are naturally related to one another. The reader is also alerted to the fact that similar results to those contained in Theorem 1.1 are proved in the body of the manuscript for a larger class of kernels (satisfying less stringent conditions) than in the theorem above. The specific way in which Theorem 1.1 follows from these more general results is discussed in Section 7.

- A key feature of items *(12)* and *(13)* of Theorem 1.1 is that the lower bound on the interval of allowable $p \in (\frac{d}{d+\gamma}, \infty)$ required for Hardy space $H^p(E, \rho|_E, \sigma)$ estimates depends on the index $\mathrm{ind}(E, \rho|_E)$ defined in (2.6). In particular, the value of this index is sensitive to the "optimal" quasi-distance equivalent to the given quasi-distance $\rho|_E$ on $E$. This is significant even when $\rho$ is a genuine distance and $(E, \rho|_E)$ is a metric space. For example, if $n \in \mathbb{N}$, then $\mathbb{R}^n$ equipped with the Euclidean distance has index $\mathrm{ind}(\mathbb{R}^n, |\cdot - \cdot|) = 1$, whereas the four-corner planar Cantor set $\mathscr{C}$ equipped with the restriction of the Euclidean distance has index $\mathrm{ind}(\mathscr{C}, |\cdot - \cdot||_{\mathscr{C}}) = \infty$ by [61, Proposition 4.79]. Moreover, the techniques we use to establish this dependence when $\rho$ is a quasi-distance are also required when $\rho$ is a genuine distance. In a nutshell, some quasi-distances are better behaved than others and so one has to make provisions for detecting the optimal quasi-distance compatible with a given quasi-metric space structure.

- Here is an example of a nonstandard geometric setting within the range of applicability of Theorem 1.1. Define $\mathscr{X} := \mathscr{C} \times [0,1] \subset \mathbb{R}^3$ where, as before, $\mathscr{C}$ denotes the four-corner planar Cantor set in $\mathbb{R}^2$, and consider $\rho := |\cdot - \cdot|\big|_{\mathscr{X}}$, and $\mu := \mathcal{H}^2 \lfloor \mathscr{X}$. Then $(\mathscr{X}, \rho, \mu)$ is a 2-dimensional ADR space. Moreover, if we take $E := \mathscr{C} \times \{0\}$, then $E$ is a closed subset of $(\mathscr{X}, \tau_\rho)$, and if $\sigma := \mathcal{H}^1 \lfloor E$, then $\big(E, \rho\big|_E, \sigma\big)$ is a 1-dimensional ADR space. In this scenario, we have $\mathrm{ind}\big(E, \rho\big|_E\big) = \infty$ and hence, in the formulation of items *(12)-(13)* of Theorem 1.1, it holds that $\gamma = \alpha$ and the range of allowable $p$ is $\big(\frac{d}{d+\alpha}, \infty\big)$.

We now describe several consequences of Theorem 1.1 for subsets $E$ of the Euclidean space. First we record the following square function estimate, which extends work from [26].

**Theorem 1.2.** *Suppose that $E$ is a closed subset of $\mathbb{R}^{n+1}$ that is $d$-dimensional* ADR *for some $d \in \{1, \dots, n\}$ and let $\sigma$ denote the surface measure induced by the $d$-dimensional Hausdorff measure on $E$. Assume that $E$ has big pieces of Lipschitz images of subsets of $\mathbb{R}^d$, i.e., there exist $\varepsilon, M \in (0, \infty)$ so that for every $x \in E$ and every $R \in (0, \infty)$, there is a Lipschitz mapping $\varphi$ with Lipschitz norm at most equal to $M$ from the ball $B^d(0, R)$ in $\mathbb{R}^d$ into $\mathbb{R}^{n+1}$ such that*

$$\sigma\big(E \cap B(x, R) \cap \varphi(B^d(0, R))\big) \geq \varepsilon R^d. \tag{1.36}$$

*Let $\psi : \mathbb{R}^{n+1} \to \mathbb{R}$ denote a smooth, compactly supported, odd function and for each $k \in \mathbb{Z}$ set $\psi_k(x) := 2^{-kd}\psi(2^{-k}x)$ for $x \in \mathbb{R}^{n+1}$. If $p \in \big(\frac{d}{d+1}, \infty\big)$, then there exists $C \in (0, \infty)$ such that*

$$\int_E \Big( \sum_{k \in \mathbb{Z}} ⨍_{\Delta(x, 2^k)} \Big| \int_E \psi_k(z - y) f(z)\, d\sigma(z) \Big|^2 d\sigma(y) \Big)^{p/2} d\sigma(x) \leq C \|f\|^p_{H^p(E,\sigma)} \tag{1.37}$$

*for every $f \in H^p(E, \sigma)$, where $\Delta(x, 2^k) := \{y \in E : |y - x| < 2^k\}$ for each $x \in E$ and $k \in \mathbb{Z}$.*

The case when $p = 2$ of Theorem 1.2, in which scenario (1.37) takes the form

$$\sum_{k \in \mathbb{Z}} \int_E \Big| \int_E \psi_k(x - y) f(y)\, d\sigma(y) \Big|^2 d\sigma(x) \leq C \int_E |f|^2\, d\sigma, \tag{1.38}$$

has been treated in [26, Section 3, p. 21]. The main point of Theorem 1.2 is that (1.38) continues to hold, when formulated as in (1.37) for every $p \in \big(\frac{d}{d+1}, \infty\big)$. The proof of this result, presented in the last part of Section 7, relies on Theorem 1.1 and uses the fact that no regularity condition on the kernel $\theta(x, y)$ is assumed in the variable $x$ (compare with (1.22)-(1.23)).

Next, we discuss another consequence of Theorem 1.1 in the Euclidean setting which gives an extension of results due to G. David and S. Semmes.

**Theorem 1.3.** *Suppose that $K$ is a real-valued function satisfying*

$$\begin{gathered} K \in C^2(\mathbb{R}^{n+1} \setminus \{0\}), \quad K \textit{ is odd, and} \\ K(\lambda x) = \lambda^{-n} K(x) \textit{ for all } \lambda > 0,\, x \in \mathbb{R}^{n+1} \setminus \{0\}. \end{gathered} \tag{1.39}$$

*Let $E$ denote a closed subset of $\mathbb{R}^{n+1}$ that is $n$-dimensional* ADR*, let $\sigma$ denote the surface measure induced by the $n$-dimensional Hausdorff measure on $E$, and define the integral operator $\mathcal{T}$ acting on functions $f \in L^p(E, \sigma)$, $1 \le p \le \infty$, by*

$$\mathcal{T} f(x) := \int_E K(x-y) f(y)\, d\sigma(y), \qquad \forall\, x \in \mathbb{R}^{n+1} \setminus E. \tag{1.40}$$

*Let $\mathbb{D}(E)$ denote a dyadic cube structure on $E$ and, for each $Q \in \mathbb{D}(E)$, let $T_E(Q)$ denote the dyadic Carleson tent over $Q$. If the set $E$ is uniformly rectifiable* (UR)*, in the sense of Definition 5.4, then the following properties hold:*

*(1) There exists $C \in (0, \infty)$ such that for each $f \in L^2(E, \sigma)$ it holds that*

$$\int_{\mathbb{R}^{n+1}\setminus E} |(\nabla \mathcal{T} f)(x)|^2 \operatorname{dist}(x, E)\, dx \le C \int_E |f(x)|^2\, d\sigma(x). \tag{1.41}$$

*(2) There exists $C \in (0, \infty)$ such that for each $f \in L^\infty(E, \sigma)$ it holds that*

$$\sup_{Q \in \mathbb{D}(E)} \left( \frac{1}{\sigma(Q)} \int_{T_E(Q)} |(\nabla \mathcal{T} f)(x)|^2 \operatorname{dist}(x, E)\, dx \right)^{1/2} \le C \|f\|_{L^\infty(E,\sigma)}. \tag{1.42}$$

*(3) There exists $C \in (0, \infty)$ such that for each $f \in L^\infty(E, \sigma)$ it holds that*

$$\sup_{x \in E,\, r>0} \left( \frac{1}{\sigma\big(E \cap B(x, r)\big)} \int_{B(x,r)\setminus E} |(\nabla \mathcal{T} f)(y)|^2 \operatorname{dist}(y, E)\, dy \right)^{1/2} \le C \|f\|_{L^\infty(E,\sigma)}. \tag{1.43}$$

*(4) Let $H^p(E, \sigma)$ denote the Lebesgue space $L^p(E, \sigma)$ when $p \in (1, \infty)$, and the Coifman-Weiss Hardy space when $p \in \big(\frac{n}{n+1}, 1\big]$. If $p \in \big(\frac{n}{n+1}, \infty\big)$ and $\kappa \in (0, \infty)$, then $\mathcal{T}$ extends to $H^p(E, \sigma)$ and there exists $C \in (0, \infty)$ such that for all $f \in H^p(E, \sigma)$ it holds that*

$$\left[ \int_E \left( \int_{\Gamma_\kappa(x)} |(\nabla \mathcal{T} f)(y)|^2 \frac{dy}{\operatorname{dist}(y, E)^{n-1}} \right)^{p/2} d\sigma(x) \right]^{1/p} \le C \|f\|_{H^p(E,\sigma)}, \tag{1.44}$$

*where $\Gamma_\kappa(x) := \big\{ y \in \mathbb{R}^{n+1} \setminus E : |x - y| < (1+\kappa) \operatorname{dist}(y, E) \big\}$ for each $x \in E$.*

*(5) If $p \in \left(\frac{n}{n+1}, \infty\right)$ and $q \in (1, \infty)$, then the operator*

$$\operatorname{dist}(\cdot, E)\, \nabla \mathcal{T} : H^p(E, \sigma) \longrightarrow L^{(p,q)}(\mathbb{R}^{n+1}, E) \tag{1.45}$$

*is well-defined, linear and bounded on the mixed norm space $L^{(p,q)}(\mathbb{R}^{n+1}, E)$ (see (6.6)).*

Theorem 1.1 particularized to the setting of Theorem 1.3 gives that conditions *(1)-(5)* above are equivalent. The fact that *(1)* holds in the special case when $\mathcal{T}$ is associated as in (1.40) with each of the kernels $K_j(x) := x_j/|x|^{n+1}$, $1 \leq j \leq n+1$, is due to David and Semmes [27]. The new result here is that *(1)* (hence also all of *(1)-(5)*) holds more generally for the entire class of kernels described in (1.39). We shall prove the latter fact in Corollary 5.7 below. Compared with [26], the class of kernels (1.39) is not tied up to any particular partial differential operator (in the manner that the kernels $K_j(x) := x_j/|x|^{n+1}$, $1 \leq j \leq n+1$, are related to the Laplacian). Moreover, in Section 5.3 we establish a version of Theorem 1.3 for variable coefficient kernels, which ultimately applies to integral operators on domains on manifolds associated with the Schwartz kernels of certain classes of pseudodifferential operators acting between vector bundles.

The condition that the set $E$ is UR in the context of Theorem 1.3 is optimal, as seen from the converse statement stated below. This result is closely interfaced with the characterization of uniform rectifiability, due David and Semmes, in terms of (1.15)-(1.16). In keeping with these conditions, the formulation of our result involves the Riesz-transform operator $\mathcal{R} := \nabla \mathcal{S}$.

**Theorem 1.4.** *Let $E$ denote a closed subset of $\mathbb{R}^{n+1}$ that is $n$-dimensional* ADR*, let $\sigma$ denote the surface measure induced by the $n$-dimensional Hausdorff measure on $E$, and define the vector-valued integral operator $\mathcal{R}$ acting on functions $f \in L^p(E, \sigma)$, $1 \leq p \leq \infty$, by*

$$\mathcal{R}f(x) := \int_E \frac{x-y}{|x-y|^{n+1}} f(y)\, d\sigma(y), \qquad \forall\, x \in \mathbb{R}^{n+1} \setminus E. \tag{1.46}$$

*Let $\mathbb{D}(E)$ denote a dyadic cube structure on $E$ and, for each $Q \in \mathbb{D}(E)$, denote by $T_E(Q)$ the dyadic Carleson tent over $Q$. If any of properties (1)-(9) below hold, then $E$ is a* UR *set:*

*(1) There exists $C \in (0, \infty)$ such that for each $f \in L^2(E, \sigma)$ it holds that*

$$\int_{\mathbb{R}^{n+1}\setminus E} |(\nabla \mathcal{R} f)(x)|^2 \operatorname{dist}(x, E)\, dx \leq C \int_E |f(x)|^2\, d\sigma(x). \tag{1.47}$$

*(2) It holds that*

$$\sup_{Q \in \mathbb{D}(E)} \left( \frac{1}{\sigma(Q)} \int_{T_E(Q)} |(\nabla \mathcal{R} 1)(x)|^2 \operatorname{dist}(x, E)\, dx \right) < \infty. \tag{1.48}$$

*(3) There exists $C \in (0, \infty)$ such that for each $f \in L^\infty(E, \sigma)$ it holds that*

$$\sup_{Q \in \mathbb{D}(E)} \left( \frac{1}{\sigma(Q)} \int_{T_E(Q)} |(\nabla \mathcal{R} f)(x)|^2 \operatorname{dist}(x, E)\, dx \right)^{1/2} \leq C \|f\|_{L^\infty(E,\sigma)}. \tag{1.49}$$

*(4) It holds that*

$$\sup_{x\in E,\, r>0}\left(\frac{1}{\sigma\big(E\cap B(x,r)\big)}\int_{B(x,r)\setminus E}|(\nabla\mathcal{R}1)(y)|^2\,\mathrm{dist}\,(y,E)\,dy\right)<\infty. \tag{1.50}$$

*(5) There exists a para-accretive function $b:E\to\mathbb{C}$ such that*

$$\sup_{Q\in\mathbb{D}(E)}\left(\frac{1}{\sigma(Q)}\int_{T_E(Q)}|(\nabla\mathcal{R}b)(x)|^2\,\mathrm{dist}\,(x,E)\,dx\right)<\infty. \tag{1.51}$$

*(6) There exists $C\in(0,\infty)$ such that for each $f\in L^\infty(E,\sigma)$ it holds that*

$$\sup_{x\in E,\, r>0}\left(\frac{1}{\sigma\big(E\cap B(x,r)\big)}\int_{B(x,r)\setminus E}|(\nabla\mathcal{R}f)(y)|^2\,\mathrm{dist}\,(y,E)\,dy\right)^{1/2}\leq C\|f\|_{L^\infty(E,\sigma)}. \tag{1.52}$$

*(7) There exist $C_0\in[1,\infty)$, $c_0\in(0,1]$ and a collection $\{b_Q\}_{Q\in\mathbb{D}(E)}$ of $\sigma$-measurable functions $b_Q:E\to\mathbb{C}$ such that for each $Q\in\mathbb{D}(E)$ the following properties hold:*

$$\int_E|b_Q|^2\,d\sigma\leq C_0\sigma(Q),\qquad\left|\int_{\widetilde{Q}}b_Q\,d\sigma\right|\geq\frac{1}{C_0}\,\sigma(\widetilde{Q})\ \textit{for some}\ \widetilde{Q}\subseteq Q\ \textit{with}\ \ell(\widetilde{Q})\geq c_0\ell(Q),$$

$$\int_{T_E(Q)}|(\nabla\mathcal{R}b_Q)(x)|^2\,\mathrm{dist}\,(x,E)\,dx\leq C_0\sigma(Q). \tag{1.53}$$

*(8) There exists $C_0\in[1,\infty)$ and, for each surface ball $\Delta=\Delta(x_o,r):=B(x_o,r)\cap E$, where $x_o\in E$ and $r\in(0,\mathrm{diam}_\rho(E)]\cap(0,\infty)$, there exists a $\sigma$-measurable function $b_\Delta:E\to\mathbb{C}$ supported in $\Delta$, such that the following properties hold:*

$$\int_E|b_\Delta|^2\,d\sigma\leq C_0\sigma(\Delta),\qquad\left|\int_\Delta b_\Delta\,d\sigma\right|\geq\frac{1}{C_0}\,\sigma(\Delta),$$

$$\int_{B(x_o,4r)\setminus E}|(\nabla\mathcal{R}\,b_\Delta)(x)|^2\,\mathrm{dist}\,(x,E)\,dx\leq C_0\sigma(\Delta). \tag{1.54}$$

*(9) There exist $p\in(0,\infty)$ and $C,\kappa\in(0,\infty)$ such that for each $f\in L^p(E,\sigma)$ it holds that*

$$\sup_{\lambda>0}\left[\lambda\cdot\sigma\Big(\Big\{x\in E:\int_{\Gamma_\kappa(x)}\frac{|(\nabla\mathcal{R}f)(y)|^2}{\mathrm{dist}\,(y,E)^{n-1}}\,dy>\lambda^2\Big\}\Big)^{1/p}\right]\leq C\|f\|_{L^p(E,\sigma)}, \tag{1.55}$$

*where $\Gamma_\kappa(x):=\big\{y\in\mathbb{R}^{n+1}\setminus E:\,|x-y|<(1+\kappa)\,\mathrm{dist}(y,E)\big\}$ for each $x\in E$.*

The fact that condition *(1)* above implies that $E$ is a UR set has been proved by David and Semmes (see [27, pp. 252-267]). Based on this result, that *(2)*-*(3)* also imply that $E$ is a UR set then follows with the help of Theorem 1.1 upon observing that the components of $\mathcal{R}$ are operators $\mathcal{T}$ as in (1.40) associated with the kernels $K_j(x):=x_j/|x|^{n+1}$, $j\in\{1,...,n+1\}$, which satisfy (1.22)-(1.23). Compared to David and Semmes' result mentioned above (to the effect that the $L^2$ square function for the operators associated with the kernels $K_j$, $1\leq j\leq n+1$, implies that the set $E$ is UR), a remarkable corollary of Theorem 1.4 is that a mere weak-$L^2$ square function estimate for the operators associated with the kernels $K_j(x):=x_j/|x|^{n+1}$, $j\in\{1,...,n+1\}$, as in (1.40), implies that $E$ is a UR set.

Throughout the manuscript, we adopt the following conventions. The symbol $\mathbf{1}_A$ denotes the characteristic function of a set $A$. The letter $C$ represents a finite positive constant that may change from one line to the next. The infinity symbol $\infty$ denotes $+\infty$. The set of positive integers is denoted by $\mathbb{N}$ whilst $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$.

## 2 Analysis and Geometry on Quasi-Metric Spaces

This section contains preliminary material, organized into four subsections dealing, respectively, with: a metrization result for arbitrary quasi-metric spaces, geometrically doubling quasi-metric spaces, approximations to the identity, and a discussion of the nature of Carleson tents in quasi-metric spaces.

### 2.1 A metrization result for general quasi-metric spaces

Here we record some aspects of the sharp quantitative metrization result from [61, Section 3.2], and properties of the Hausdorff outer-measure (cf. Proposition 2.3), on quasi-metric spaces.

We begin by assuming that $\mathscr{X}$ is a set of cardinality at least two and introduce the following notation. A function $\rho : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ is called a `quasi-distance` on $\mathscr{X}$ provided there exist two constants $C_\rho, C_{\tilde{\rho}} \in [1, \infty)$ with the property that for every $x, y, z \in X$, it holds that

$$\rho(x,y) = 0 \Leftrightarrow x = y, \quad \rho(y,x) \leq C_{\tilde{\rho}}\rho(x,y), \quad \rho(x,y) \leq C_\rho \max\{\rho(x,z), \rho(z,y)\}. \tag{2.1}$$

We assume henceforth that $C_\rho$ and $C_{\tilde{\rho}}$ are the smallest such constants. A pair $(\mathscr{X}, \rho)$ is called a `quasi-metric space`. Given a set $E \subseteq \mathscr{X}$ of cardinality at lest two, denote by $\rho\big|_E$ the quasi-distance on $E$ given by the restriction of the function $\rho$ to $E \times E$. The $\rho$-`ball` (or simply `ball` if the quasi-distance $\rho$ is clear from the context) centered at $x \in \mathscr{X}$ with radius $r \in (0, \infty)$ is defined to be $B_\rho(x, r) := \{y \in \mathscr{X} : \rho(x, y) < r\}$. Also, call $E \subseteq \mathscr{X}$ $\rho$-`bounded` if $E$ is contained in a $\rho$-ball, and define its $\rho$-`diameter` (or simply `diameter`) as $\operatorname{diam}_\rho(E) := \sup\big\{\rho(x, y) : x, y \in E\big\}$. The $\rho$-`distance` (or simply `distance`) between two arbitrary, nonempty sets $E, F \subseteq \mathscr{X}$ is naturally defined as $\operatorname{dist}_\rho(E, F) := \inf\{\rho(x, y) : x \in E, \ y \in F\}$. If $E = \{x\}$ for some $x \in \mathscr{X}$ and $F \subseteq \mathscr{X}$, abbreviate $\operatorname{dist}_\rho(x, F) := \operatorname{dist}_\rho(\{x\}, F)$. We define $\tau_\rho$, the `topology canonically induced by` $\rho$ `on` $\mathscr{X}$, to be the largest topology on $\mathscr{X}$ with the property that for each point $x \in \mathscr{X}$ the family $\{B_\rho(x, r)\}_{r>0}$ is a fundamental system of neighborhoods of $x$. Finally, call two functions $\rho_1, \rho_2 : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ `equivalent`, and write $\rho_1 \approx \rho_2$, if there exist $C', C'' \in (0, \infty)$ with the property that $C'\rho_1 \leq \rho_2 \leq C''\rho_1$ on $\mathscr{X} \times \mathscr{X}$.

A few comments are in order. Suppose that $(\mathscr{X}, \rho)$ is a quasi-metric space. It is then clear that if $\rho' : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ is such that $\rho' \approx \rho$ then $\rho'$ is a quasi-distance on $\mathscr{X}$ and $\tau_{\rho'} = \tau_\rho$. Also, it may be checked that

$$\mathcal{O} \in \tau_\rho \iff \mathcal{O} \subseteq \mathscr{X} \text{ and } \forall\, x \in \mathcal{O} \ \exists\, r > 0 \text{ such that } B_\rho(x, r) \subseteq \mathcal{O}. \tag{2.2}$$

As is well-known, the topology induced by the given quasi-distance on a quasi-metric space is metrizable. We now record some aspects of the sharp quantitative metrization theorem from [61, Section 3.2], which is an optimal quantitative version of this fact.

**Theorem 2.1.** *Let $(\mathscr{X}, \rho)$ be a quasi-metric space with $C_\rho, \widetilde{C}_\rho \in [1, \infty)$ as in (2.1) and set $\alpha_\rho := 1/\log_2 C_\rho \in (0, \infty]$. There exists a function $\rho_\# : \mathscr{X} \times \mathscr{X} \to [0, \infty)$, called the* `regularization` *of $\rho$, such that the following properties hold:*

*(1) The function $\rho_\#$ is a symmetric quasi-distance on $\mathscr{X}$ and $\rho_\# \approx \rho$. More specifically, $\rho_\#(x, y) = \rho_\#(y, x)$ and $(C_\rho)^{-2}\rho(x, y) \leq \rho_\#(x, y) \leq \widetilde{C}_\rho\, \rho(x, y)$ for all $x, y \in \mathscr{X}$. Also, $\tau_{\rho_\#} = \tau_\rho$, $C_{\rho_\#} \leq C_\rho$ and $(\rho|_E)_\# \approx \rho|_E \approx (\rho_\#)\big|_E$ for any nonempty set $E \subseteq \mathscr{X}$.*

*(2) If $\beta \in (0, \alpha_\rho]$ is finite, then $d_{\rho,\beta} : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ defined by $d_{\rho,\beta}(x, y) := \rho_\#(x, y)^\beta$, for all $x, y \in \mathscr{X}$, is a distance on $\mathscr{X}$ and $(d_{\rho,\beta})^{1/\beta} \approx \rho$. In particular, $d_{\rho,\beta}$ induces the same topology on $\mathscr{X}$ as $\rho$, hence $\tau_\rho$ is metrizable.*

*(3) If $\beta \in (0, \alpha_\rho]$ is finite, then $\rho_\#$ is Hölder regular of order $\beta$ in the sense that*

$$|\rho_\#(x,y) - \rho_\#(z,w)| \leq \tfrac{1}{\beta} \max\{\rho_\#(x,y)^{1-\beta}, \rho_\#(z,w)^{1-\beta}\}(\rho_\#(x,z)^\beta + \rho_\#(y,w)^\beta) \quad (2.3)$$

*for all $x, y, z, w \in \mathscr{X}$ with, in the case when $\beta \geq 1$ only, $x \neq y$ and $z \neq w$. In particular, $\rho_\# : (\mathscr{X} \times \mathscr{X}, \tau_\rho \times \tau_\rho) \longrightarrow [0, \infty)$ is continuous.*

*(4) If $E$ is a nonempty subset of $(\mathscr{X}, \tau_\rho)$, then the regularized distance function*

$$\delta_E := \mathrm{dist}_{\rho_\#}(\cdot, E) : \mathscr{X} \longrightarrow [0, \infty) \quad (2.4)$$

*is equivalent to $\mathrm{dist}_\rho(\cdot, E)$, and if $\beta \in (0, \min\{1, \alpha_\rho\}]$, then $\delta_E$ is locally Hölder regular of order $\beta$ in the sense that there exists $C \in (0, \infty)$, depending on $C_\rho, \widetilde{C}_\rho, \beta$, such that*

$$\frac{|\delta_E(x) - \delta_E(y)|}{\rho(x,y)^\beta} \leq C \left(\rho(x,y) + \max\{\mathrm{dist}_\rho(x, E)\,,\, \mathrm{dist}_\rho(y, E)\}\right)^{1-\beta} \quad (2.5)$$

*for all $x, y \in \mathscr{X}$ with $x \neq y$. In particular, $\delta_E : (\mathscr{X}, \tau_\rho) \longrightarrow [0, \infty)$ is continuous.*

In view of Theorem 2.1, the "best" quasi-distance equivalent to a given $\rho$ on $\mathscr{X}$ is quantified by the following `index`

$$\mathrm{ind}(\mathscr{X}, \rho) := \sup_{\rho' \approx \rho} \left(\log_2 C_{\rho'}\right)^{-1} = \sup_{\rho' \approx \rho} \left( \log_2 \left[ \sup_{\substack{x,y,z \in \mathscr{X} \\ \text{not all equal}}} \frac{\rho'(x,y)}{\max\{\rho'(x,z), \rho'(z,y)\}} \right] \right)^{-1}, \quad (2.6)$$

which was introduced and studied in [61, 2]. For example, a key feature of Theorem 2.1 is the fact that if $(\mathscr{X}, \rho)$ is any quasi-metric space then $\rho^\beta$ is equivalent to a genuine distance on $\mathscr{X}$ for any finite number $\beta \in (0, \mathrm{ind}\,(\mathscr{X}, \rho))$. This result is sharp and improves upon an earlier version due to R.A. Macías and C. Segovia [57], in which these authors have identified a smaller, non-optimal upper-bound for the exponent $\beta$.

In anticipation of briefly reviewing the notion of Hausdorff outer-measure on a quasi-metric space, we recall a couple of definitions from measure theory. Given an outer-measure $\mu^*$ on an arbitrary set $\mathscr{X}$, consider the collection of all $\mu^*$-measurable sets defined as

$$\mathfrak{M}_{\mu^*} := \{A \subseteq \mathscr{X} : \mu^*(Y) = \mu^*(Y \cap A) + \mu^*(Y \setminus A),\ \forall\, Y \subseteq \mathscr{X}\}. \quad (2.7)$$

Carathéodory's classical theorem allows one to pass from a given outer-measure $\mu^*$ on $\mathscr{X}$ to a genuine measure by observing that

$$\mathfrak{M}_{\mu^*} \text{ is a sigma-algebra, and } \mu^*\big|_{\mathfrak{M}_{\mu^*}} \text{ is a complete measure.} \quad (2.8)$$

The restriction of an outer-measure $\mu^*$ on $\mathscr{X}$ to a subset $E$ of $\mathscr{X}$, denoted by $\mu^*\lfloor E$, is defined naturally by restricting the function $\mu^*$ to the collection of all subsets of $E$. We shall use the same symbol, $\lfloor$ , in denoting the restriction of a measure to a measurable set. In this regard, it is useful to know when the measure associated with the restriction of an outer-measure to a set coincides with the restriction to that set of the measure associated with the given outer-measure. Specifically, it may be checked that if $\mu^*$ is an outer-measure on $\mathscr{X}$, then

$$\left(\mu^*\lfloor E\right)\Big|_{\mathfrak{M}_{(\mu^*\lfloor E)}} = \left(\mu^*\big|_{\mathfrak{M}_{\mu^*}}\right)\lfloor E, \qquad \forall\, E \in \mathfrak{M}_{\mu^*}. \quad (2.9)$$

Next, if $(\mathscr{X}, \tau)$ is a topological space and $\mu^*$ is an outer-measure on $\mathscr{X}$ such that $\mathfrak{M}_{\mu^*}$ contains the Borel sets in $(\mathscr{X}, \tau)$, then call $\mu^*$ a `Borel outer-measure` on $\mathscr{X}$. Furthermore, call such a Borel outer-measure $\mu^*$ a `Borel regular outer-measure` if

$$\forall\, A \subseteq \mathscr{X} \;\; \exists \text{ a Borel set } B \text{ in } (\mathscr{X}, \tau) \text{ such that } A \subseteq B \text{ and } \mu^*(A) = \mu^*(B). \tag{2.10}$$

After this digression, we now proceed to introduce the concept of $d$-dimensional Hausdorff outer-measure for a subset of a quasi-metric space.

**Definition 2.2.** *Let $(\mathscr{X}, \rho)$ be a quasi-metric space. For $d \geq 0$, $A \subseteq \mathscr{X}$ and $\varepsilon > 0$, define*

$$\mathcal{H}^d_{\mathscr{X},\rho,\varepsilon}(A) := \inf\Big\{\sum\nolimits_{j\in\mathbb{N}} (\mathrm{diam}_\rho(A_j))^d :\, A \subseteq \bigcup\nolimits_{j\in\mathbb{N}} A_j \text{ and } \mathrm{diam}_\rho(A_j) \leq \varepsilon \text{ for every } j \in \mathbb{N}\Big\}, \tag{2.11}$$

*where* $\inf \emptyset := +\infty$*, then define the $d$-*`dimensional Hausdorff outer-measure` $\mathcal{H}^d_{\mathscr{X},\rho}(A)$ *by*

$$\mathcal{H}^d_{\mathscr{X},\rho}(A) := \lim\nolimits_{\varepsilon\to 0^+} \mathcal{H}^d_{\mathscr{X},\rho,\varepsilon}(A) = \sup\nolimits_{\varepsilon>0} \mathcal{H}^d_{\mathscr{X},\rho,\varepsilon}(A) \in [0,\infty]. \tag{2.12}$$

*This is abbreviated as $\mathcal{H}^d_{\mathscr{X}}(A)$ when the choice of $\rho$ is irrelevant or clear from the context.*

It is readily verified that $\mathcal{H}^0_{\mathscr{X},\rho}$ is equivalent to the counting measure. Other basic properties of the Hausdorff outer-measure are collected in the proposition below, proved in [62]. To state it, recall that a measure $\mu$ on a quasi-metric space $(\mathscr{X}, \rho)$ is called `Borel` provided the sigma-algebra on which it is defined contains all Borel sets relative to the topological space $(\mathscr{X}, \tau_\rho)$. Also, call a measure $\mu$ on a quasi-metric space $(\mathscr{X}, \rho)$ `Borel regular` provided it is Borel and

$$\forall\, \mu\text{-measurable } A \subseteq \mathscr{X} \;\exists \text{ a Borel set } B \text{ in } (\mathscr{X}, \tau_\rho) \text{ with } A \subseteq B \text{ and } \mu(A) = \mu(B). \tag{2.13}$$

In addition, we require the related notion whereby a measure $\mu$ on a quasi-metric space $(\mathscr{X}, \rho)$ is called `Borel semiregular` provided it is Borel and

$$\begin{gathered}\text{for each } \mu\text{-measurable } A \subseteq \mathscr{X} \text{ with } \mu(A) < \infty \text{ there exists}\\ \text{a Borel set } B \text{ in } (\mathscr{X}, \tau_\rho) \text{ with } \mu((A \setminus B) \cup (B \setminus A)) = 0.\end{gathered} \tag{2.14}$$

Borel regular measures are automatically Borel semiregular. Next, we make the convention for a quasi-metric space $(\mathscr{X}, \rho)$ and $d \geq 0$ whereby

$$\mathscr{H}^d_{\mathscr{X},\rho} \text{ denotes the } \textit{measure} \text{ associated with the outer-measure } \mathcal{H}^d_{\mathscr{X},\rho} \text{ as in (2.8).} \tag{2.15}$$

**Proposition 2.3.** *Let $(\mathscr{X}, \rho)$ be a quasi-metric space and $d \geq 0$. The following hold:*

*(1) $\mathcal{H}^d_{\mathscr{X},\rho}$ is a Borel outer-measure on $(\mathscr{X}, \tau_\rho)$, and $\mathscr{H}^d_{\mathscr{X},\rho}$ is a Borel measure on $(\mathscr{X}, \tau_\rho)$.*

*(2) If $\rho_\#$ is as in Theorem 2.1, then $\mathcal{H}^d_{\mathscr{X},\rho_\#}$ is a Borel regular outer-measure on $(\mathscr{X}, \tau_\rho)$, and $\mathscr{H}^d_{\mathscr{X},\rho_\#}$ is a Borel regular measure on $(\mathscr{X}, \tau_\rho)$.*

*(3) It holds that $\mathcal{H}^d_{\mathscr{X},\rho'} \approx \mathcal{H}^d_{\mathscr{X},\rho}$ whenever $\rho' \approx \rho$, in the sense that there exist $C_1, C_2 \in (0,\infty)$, depending only on $\rho$ and $\rho'$, such that*

$$C_1\, \mathcal{H}^d_{\mathscr{X},\rho}(A) \leq \mathcal{H}^d_{\mathscr{X},\rho'}(A) \leq C_2\, \mathcal{H}^d_{\mathscr{X},\rho}(A) \quad \text{for all } A \subseteq \mathscr{X}. \tag{2.16}$$

*(4) If $E \subseteq \mathscr{X}$, then the d-dimensional Hausdorff outer-measure in the quasi-metric space $(E, \rho|_E)$ is equivalent, in the sense of* (2.16)*, to the restriction to E of the d-dimensional Hausdorff outer-measure in $\mathscr{X}$, that is, $\mathcal{H}^d_{E,\rho|_E} \approx \mathcal{H}^d_{\mathscr{X},\rho}\lfloor E$.*

*(5) If $E \subseteq \mathscr{X}$, then $\mathcal{H}^d_{\mathscr{X},\rho_\#}\lfloor E$ is a Borel regular outer-measure on $(E, \tau_{\rho|_E})$ and the measure associated with it, as in* (2.8)*, is a Borel regular measure on $(E, \tau_{\rho|_E})$. Furthermore, if $E$ is $\mathcal{H}^d_{\mathscr{X},\rho_\#}$-measurable in the sense of* (2.7) *(in particular, if $E$ is a Borel subset of $(\mathscr{X}, \tau_\rho)$), then $\mathscr{H}^d_{\mathscr{X},\rho_\#}\lfloor E$ is a Borel regular measure on $(E, \tau_{\rho|_E})$ and it coincides with the measure associated with the outer-measure $\mathcal{H}^d_{\mathscr{X},\rho_\#}\lfloor E$.*

*(6) If $m \in (d, \infty)$, $E \subseteq \mathscr{X}$ and $\mathcal{H}^d_{\mathscr{X},\rho}(E) < \infty$, then $\mathcal{H}^m_{\mathscr{X},\rho}(E) = 0$.*

## 2.2 Geometrically doubling quasi-metric spaces

In this subsection we shall work in a more specialized setting than that of general quasi-metric spaces considered so far, by considering geometrically doubling quasi-metric spaces, as described in the definition below.

**Definition 2.4.** *A quasi-metric space $(\mathscr{X}, \rho)$ is called* `geometrically doubling` *if there exists a number $N \in \mathbb{N}$, called the geometric doubling constant of $(\mathscr{X}, \rho)$, with the property that any $\rho$-ball of radius $r$ in $\mathscr{X}$ may be covered by at most $N$ $\rho$-balls in $\mathscr{X}$ of radii $r/2$.*

To put this matter into a larger perspective, recall that a subset $E$ of a quasi-metric space $(\mathscr{X}, \rho)$ is said to be `totally bounded` provided that for any $r \in (0, \infty)$ there exists a finite covering of $E$ with $\rho$-balls of radii $r$. Then for a quasi-metric space $(\mathscr{X}, \rho)$ the quality of being geometrically doubling may be regarded as a scale-invariant version of the demand that all $\rho$-balls in $\mathscr{X}$ are totally bounded. In fact it may be readily verified that if $(\mathscr{X}, \rho)$ is a geometrically doubling quasi-metric space, then

$$\begin{array}{l}\exists\, N \in \mathbb{N} \text{ such that } \forall\, \vartheta \in (0,1) \text{ any } \rho\text{-ball of radius } r \text{ in } \mathscr{X} \\ \text{may be covered by at most } N^{-[\log_2 \vartheta]} \ \rho\text{-balls in } \mathscr{X} \text{ of radii } \vartheta r,\end{array} \tag{2.17}$$

where $[\log_2 \vartheta]$ is the smallest integer greater than or equal to $\log_2 \vartheta$. En route, let us also point out that *the property of being geometrically doubling is hereditary*, in the sense that if $(\mathscr{X}, \rho)$ is a geometrically doubling quasi-metric space with geometric doubling constant $N$, and if $E$ is an arbitrary subset of $\mathscr{X}$, then $(E, \rho\big|_E)$ is a geometrically doubling quasi-metric space with geometric doubling constant at most equal to $N^{\log_2 C_\rho} N$.

The relevance of the property (of a quasi-metric space) of being geometrically doubling is apparent from the fact that in such a context a number of useful geometrical results hold, which are akin to those available in the Euclidean setting. A case in point, is the Whitney decomposition theorem discussed in Proposition 2.5 below. A version of the classical Whitney decomposition theorem in the Euclidean setting (as presented in, e.g., [70, Theorem 1.1, p. 167]) has been worked out in [19, Theorem 3.1, p. 71] and [20, Theorem 3.2, p. 623] in the context of bounded open sets in spaces of homogeneous type. Recently, the scope of this work has been further refined in [61] by allowing arbitrary open sets in geometrically doubling quasi-metric spaces, as presented in the following proposition.

**Proposition 2.5.** *Let $(\mathscr{X}, \rho)$ be a geometrically doubling quasi-metric space. If $\lambda \in (1, \infty)$, then there exist $\Lambda \in (\lambda, \infty)$ and $N \in \mathbb{N}$, depending only on $C_\rho, \widetilde{C}_\rho, \lambda$ and the geometric doubling* *constant of $(\mathscr{X},\rho)$, such that if $\mathcal{O}$ is an open, nonempty, proper subset of $(\mathscr{X},\tau_\rho)$, then there exist countable collections $\{x_j\}_{j\in J}$ in $\mathcal{O}$ and $\{r_j\}_{j\in J}$ in $(0,\infty)$ with the following properties:*

(1) $\mathcal{O}=\bigcup_{j\in J} B_\rho(x_j,r_j)$.

(2) *There exists $\varepsilon\in(0,1)$, depending only on $C_\rho,\lambda$ and the geometric doubling constant of $(\mathscr{X},\rho)$, such that $\sup_{x\in\mathcal{O}}\#\big\{j\in J:\, B_\rho\big(x,\varepsilon\,\mathrm{dist}_\rho(x,\mathscr{X}\setminus\mathcal{O})\big)\cap B_\rho(x_j,\lambda r_j)\neq\emptyset\big\}\leq N$. It also holds that $\sum_{j\in J}\mathbf{1}_{B_\rho(x_j,\lambda r_j)}(x)\leq N$ for all $x\in\mathcal{O}$.*

(3) $B_\rho(x_j,\lambda r_j)\subseteq\mathcal{O}$ *and* $B_\rho(x_j,\Lambda r_j)\cap(\mathscr{X}\setminus\mathcal{O})\neq\emptyset$ *for all* $j\in J$.

(4) *$r_i\approx r_j$ uniformly on $\{i,j\in J: B_\rho(x_i,\lambda r_i)\cap B_\rho(x_j,\lambda r_j)\neq\emptyset\}$ and there exists $C\in(0,\infty)$ such that $r_j\leq C\,\mathrm{diam}_\rho(\mathcal{O})$ for all $j\in J$.*

Regarding terminology, we shall frequently employ the following convention:

**Convention 2.6.** *Given a geometrically doubling quasi-metric space $(\mathscr{X},\rho)$, an open, nonempty, proper subset $\mathcal{O}$ of $(\mathscr{X},\tau_\rho)$, and a parameter $\lambda\in(1,\infty)$, we will refer to the balls $B_{\rho_\#}(x_j,r_j)$ obtained by treating $(\mathscr{X},\rho_\#)$ in Proposition 2.5 as* `Whitney cubes`*, denote the collection of these cubes by $\mathbb{W}_\lambda(\mathcal{O})$, and for each $I\in\mathbb{W}_\lambda(\mathcal{O})$, write $\ell(I)$ for the* `radius of` *$I$. Furthermore, if $I\in\mathbb{W}_\lambda(\mathcal{O})$ and $c\in(0,\infty)$, we shall denote by $cI$ the* `dilate of the cube` *$I$* `by factor` *$c$, i.e., the ball having the same center as $I$ and radius $c\ell(I)$.*

Spaces of homogeneous type, reviewed next, are an important subclass of the class of geometrically doubling quasi-metric spaces.

**Definition 2.7.** *A* `space of homogeneous type` *is a triplet $(\mathscr{X},\rho,\mu)$, where $(\mathscr{X},\rho)$ is a quasi-metric space and $\mu$ is a Borel measure on $(\mathscr{X},\tau_\rho)$ with the property that all $\rho$-balls are $\mu$-measurable, and which satisfies, for some finite constant $C\geq 1$, the doubling condition*

$$0<\mu\big(B_\rho(x,2r)\big)\leq C\mu\big(B_\rho(x,r)\big)<\infty,\quad \forall\,x\in\mathscr{X},\ \forall\,r>0. \tag{2.18}$$

*The smallest such constant is denoted $C_\mu$ and called the* `doubling constant` *of $\mu$.*

Iterating (2.18) gives

$$\begin{gathered}\frac{\mu(B_1)}{\mu(B_2)}\leq C_{\mu,\rho}\left(\frac{\text{radius of }B_1}{\text{radius of }B_2}\right)^{D_\mu},\ \text{ for all }\rho\text{-balls } B_2\subseteq B_1,\\ \text{where } D_\mu:=\log_2 C_\mu\geq 0 \text{ and } C_{\mu,\rho}:=C_\mu\big(C_\rho\widetilde{C}_\rho\big)^{D_\mu}\geq 1.\end{gathered} \tag{2.19}$$

The exponent $D_\mu$ is referred to as the `doubling order` of $\mu$. For further reference, let us also record here the well-known fact that

$$\begin{gathered}\text{given a space of homogeneous type } (\mathscr{X},\rho,\mu),\text{ it holds that}\\ \mathrm{diam}_\rho(\mathscr{X})<\infty \text{ if and only if } \mu(\mathscr{X})<\infty.\end{gathered} \tag{2.20}$$

Going further, a distinguished subclass of the class of spaces of homogeneous type, which will play a basic role in this work, is the category of Ahlfors-David regular spaces defined next.

**Definition 2.8.** *Suppose that $d>0$. A $d$*-`dimensional Ahlfors-David regular` (*or simply $d$-dimensional* ADR*, or $d$*-ADR) `space` *is a triplet $(\mathscr{X},\rho,\mu)$, where $(\mathscr{X},\rho)$ is a quasi-metric space and $\mu$ is a Borel measure on $(\mathscr{X},\tau_\rho)$ with the property that all $\rho$-balls are $\mu$-measurable, and for which there exists a constant $C\in[1,\infty)$ such that*

$$C^{-1}\,r^d\leq\mu\big(B_\rho(x,r)\big)\leq C\,r^d,\quad\forall\,x\in\mathscr{X},\quad\textit{for every finite } r\in(0,\operatorname{diam}_\rho(\mathscr{X})]. \tag{2.21}$$

*The constant $C$ in* (2.21) *will be referred to as the* ADR *constant of $\mathscr{X}$.*

As alluded to earlier, if $(\mathscr{X},\rho,\mu)$ is a $d$-dimensional ADR space then, trivially, $(\mathscr{X},\rho,\mu)$ is also a space of homogeneous type. For further reference we note here that (cf., e.g., [62])

$$(\mathscr{X},\rho,\mu)\text{ is }d\text{-ADR}\implies\big(\mathscr{X},\rho_\#,\mathscr{H}^d_{\mathscr{X},\rho_\#}\big)\text{ is }d\text{-ADR}. \tag{2.22}$$

In particular, it follows from (2.22), *(1)* in Theorem 2.1, and *(3)-(5)* in Proposition 2.3 that

$$\left.\begin{array}{r}(\mathscr{X},\rho)\text{ quasi-metric space,}\\ E\text{ Borel subset of }(\mathscr{X},\tau_\rho)\\ \sigma\text{ Borel measure on }(E,\tau_{\rho|_E})\\ \text{such that }(E,\rho|_E,\sigma)\text{ is }d\text{-ADR}\end{array}\right\}\implies\big(E,\rho_\#\big|_E,\mathscr{H}^d_{\mathscr{X},\rho_\#}\lfloor E\big)\text{ is }d\text{-ADR}. \tag{2.23}$$

Also, if $(\mathscr{X},\rho,\mu)$ is $d$-ADR, then there exists a finite constant $C>0$ such that

$$\mathcal{H}^d_{\mathscr{X},\rho_\#}(A)\leq C\inf\big\{\mu(\mathcal{O}):\mathcal{O}\text{ is open and }A\subseteq\mathcal{O}\big\}\text{ for every }A\subseteq\mathscr{X}, \tag{2.24}$$

$$\mu(A)\leq C\mathscr{H}^d_{\mathscr{X},\rho_\#}(A)\text{ for every Borel subset }A\text{ of }(\mathscr{X},\tau_\rho). \tag{2.25}$$

In addition, if $\mu$ is actually a Borel regular measure, then

$$\mu(A)\approx\mathscr{H}^d_{\mathscr{X},\rho_\#}(A),\qquad\text{uniformly for Borel subsets }A\text{ of }(\mathscr{X},\tau_\rho). \tag{2.26}$$

We now discuss a couple of technical lemmas which are going to be useful for us later on.

**Lemma 2.9.** *Let $0<d<m<\infty$. Assume that $(\mathscr{X},\rho,\mu)$ is an $m$-dimensional* ADR *space. If $E$ is a Borel subset of $(\mathscr{X},\tau_\rho)$ and there exists a Borel measure $\sigma$ on $(E,\tau_{\rho|_E})$ such that $(E,\rho\big|_E,\sigma)$ is a $d$-dimensional* ADR *space, then $\mu(E)=0$.*

*Proof.* Fix $x\in E$. Using (2.25), (2.23) and item *(6)* in Proposition 2.3, we obtain

$$\mu(E)\leq C\mathscr{H}^m_{\mathscr{X},\rho_\#}(E)=C\lim_{n\to\infty}\mathscr{H}^m_{\mathscr{X},\rho_\#}(E\cap B_{\rho_\#}(x,n))=0, \tag{2.27}$$

since $\mathscr{H}^d_{\mathscr{X},\rho_\#}(E\cap B_{\rho_\#}(x,n))\leq Cn^d<\infty$ for all $n\in\mathbb{N}$. □

**Lemma 2.10.** *Let $(\mathscr{X},\rho)$ be a quasi-metric space. Suppose that $E$ is a Borel subset of $(\mathscr{X},\tau_\rho)$ and that there exists a Borel measure $\sigma$ on $(E,\tau_{\rho|_E})$ such that $(E,\rho\big|_E,\sigma)$ is a $d$-dimensional* ADR *space for some $d\in(0,\infty)$. Then there exists $c\in(0,\infty)$ such that if $x\in\mathscr{X}$ and $r\in(0,\operatorname{diam}_{\rho_\#}(E)]$ with $B_{\rho_\#}(x,r)\cap E\neq\emptyset$, then $\mathscr{H}^d_{\mathscr{X},\rho_\#}\big(B_{\rho_\#}(x,C_\rho r)\cap E\big)\geq c\,r^d$.*

*Proof.* Let $x\in\mathscr{X}$ with $B_{\rho_\#}(x,r)\cap E\neq\emptyset$. If $y\in B_{\rho_\#}(x,r)\cap E$, then $B_{\rho_\#}(y,r)\subseteq B_{\rho_\#}(x,C_\rho r)$, so by (2.23) and letting $C$ denote the ADR constant of $\big(E,\rho_\#\big|_E,\mathscr{H}^d_{\mathscr{X},\rho_\#}\lfloor E\big)$, we have

$$\mathscr{H}^d_{\mathscr{X},\rho_\#}\big(B_{\rho_\#}(x,C_\rho r)\cap E\big)\geq\mathscr{H}^d_{\mathscr{X},\rho_\#}\big(B_{\rho_\#}(y,r)\cap E\big)\geq C^{-1}r^d, \tag{2.28}$$

as required. □

For further reference, given an ambient quasi-metric space $(\mathscr{X}, \rho)$ and a set $E$ for which there exists a Borel measure $\sigma$ on $(E, \tau_{\rho|_E})$ such that $(E, \rho_\#|_E, \sigma)$ is a space of homogeneous type, we let $M_E$ denote the `Hardy-Littlewood maximal function` defined by

$$(M_E f)(x) := \sup_{r>0} \frac{1}{\sigma\big(B_{\rho_\#}(x,r)\big)} \int_{B_{\rho_\#}(x,r)} |f(y)|\, d\sigma(y), \qquad x \in E. \tag{2.29}$$

Following work in [15] and [23], we now discuss the existence of a `dyadic grid structure` on geometrically doubling quasi-metric spaces. The result below is essentially due to M. Christ [15] but with two refinements. First, Christ's result is established in the presence of a background doubling, Borel regular measure, which is more restrictive than assuming that the ambient quasi-metric space is geometrically doubling. Second, Christ's dyadic grid result involves a scale $\delta \in (0,1)$, which we show may be taken to be $1/2$, as in the Euclidean setting.

**Proposition 2.11.** *Assume that $(E, \rho)$ is a geometrically doubling quasi-metric space and select $\kappa_E \in \mathbb{Z} \cup \{-\infty\}$ such that $2^{-\kappa_E - 1} < \operatorname{diam}_\rho(E) \leq 2^{-\kappa_E}$. For each $k \in \mathbb{Z}$ with $k \geq \kappa_E$, there exist a collection $\mathbb{D}_k(E) := \{Q_\alpha^k\}_{\alpha \in I_k}$ of subsets of $E$ indexed by a nonempty, at most countable set of indices $I_k$, and a collection $\{x_\alpha^k\}_{\alpha \in I_k}$ of points in $E$, such that the collection $\mathbb{D}(E) := \bigcup_{k \in \mathbb{Z},\, k \geq \kappa_E} \mathbb{D}_k(E)$ has the following properties:*

*(1) If $k \in \mathbb{Z}$ with $k \geq \kappa_E$ and $\alpha \in I_k$, then $Q_\alpha^k$ is open in $\tau_\rho$.*

*(2) If $k \in \mathbb{Z}$ with $k \geq \kappa_E$ and $\alpha, \beta \in I_k$ with $\alpha \neq \beta$, then $Q_\alpha^k \cap Q_\beta^k = \emptyset$.*

*(3) If $k, \ell \in \mathbb{Z}$ with $\ell > k \geq \kappa_E$ and $\alpha \in I_k$, $\beta \in I_\ell$, then either $Q_\beta^\ell \subseteq Q_\alpha^k$ or $Q_\alpha^k \cap Q_\beta^\ell = \emptyset$.*

*(4) If $k, \ell \in \mathbb{Z}$ with $k > \ell \geq \kappa_E$ and $\alpha \in I_k$, then there is a unique $\beta \in I_\ell$ such that $Q_\alpha^k \subseteq Q_\beta^\ell$.*

*(5) There exist $0 < a_0 \leq a_1 < \infty$ such that if $k \in \mathbb{Z}$ with $k \geq \kappa_E$ and $\alpha \in I_k$, then*

$$B_\rho(x_\alpha^k, a_0 2^{-k}) \subseteq Q_\alpha^k \subseteq B_\rho(x_\alpha^k, a_1 2^{-k}). \tag{2.30}$$

*In particular, given a measure $\sigma$ on $E$ for which $(E, \rho, \sigma)$ is a space of homogeneous type, there exists $c > 0$ such that if $Q_\beta^{k+1} \subseteq Q_\alpha^k$, then $\sigma(Q_\beta^{k+1}) \geq c\sigma(Q_\alpha^k)$.*

*(6) There exists $N \in \mathbb{N}$ such that if $k \in \mathbb{Z}$ with $k \geq \kappa_E$ and $\alpha \in I_k$, then*

$$\#\big\{\beta \in I_{k+1} : Q_\beta^{k+1} \subseteq Q_\alpha^k\big\} \leq N. \tag{2.31}$$

*Furthermore, if $x \in E$ and $r \in (0, 2^{-k})$, then $\#\big\{Q \in \mathbb{D}_k(E) : Q \cap B_\rho(x,r) \neq \emptyset\big\} \leq N$.*

*(7) If $k \in \mathbb{Z}$ with $k \geq \kappa_E$, then $\bigcup_{\alpha \in I_k} Q_\alpha^k$ is dense in $(E, \tau_\rho)$ and*

$$E = \bigcup\nolimits_{\alpha \in I_k} \big\{x \in E :\, \operatorname{dist}_\rho(x, Q_\alpha^k) \leq \varepsilon 2^{-k}\big\}, \qquad \forall\, \varepsilon > 0. \tag{2.32}$$

*Moreover, if $\alpha \in I_k$, then $\bigcup_{\beta \in I_{k+1},\, Q_\beta^{k+1} \subseteq Q_\alpha^k} Q_\beta^{k+1}$ is dense in $Q_\alpha^k$ and*

$$Q_\alpha^k \subseteq \bigcup\nolimits_{\beta \in I_{k+1},\, Q_\beta^{k+1} \subseteq Q_\alpha^k} \big\{x \in E :\, \operatorname{dist}_\rho(x, Q_\beta^{k+1}) \leq \varepsilon 2^{-k-1}\big\}, \qquad \forall\, \varepsilon > 0. \tag{2.33}$$

*Also, there exist $b_0, b_1 \in (0, \infty)$, depending only on the geometric doubling constant of $E$, such that if $x \in E$ and $r \in (0, \operatorname{diam}_\rho(E)]$, then there exist $k \in \mathbb{Z}$, $k \geq \kappa_E$ and $\alpha \in I_k$ with*

$$Q_\alpha^k \subseteq B_\rho(x,r) \text{ and } b_0 r \leq 2^{-k} \leq b_1 r. \tag{2.34}$$

*(8) If $\sigma$ is a measure on $E$ for which $(E,\rho,\sigma)$ is a space of homogeneous type, then a collection $\mathbb{D}(E)$ can be constructed so that (1)-(7) hold and, in addition, there exist $\vartheta\in(0,1)$ and $c\in(0,\infty)$ such that if $k\in\mathbb{Z}$ with $k\geq\kappa_E$ and $\alpha\in I_k$, then $\big(Q^k_\alpha,\rho|_{Q^k_\alpha},\sigma\lfloor Q^k_\alpha\big)$ is a space of homogeneous type, with doubling constant independent of $k$ and $\alpha$, and*

$$\sigma\big(\big\{x\in Q^k_\alpha:\ \mathrm{dist}_{\rho_\#}(x,E\setminus Q^k_\alpha)\leq t\,2^{-k}\big\}\big)\leq c\,t^{\vartheta}\sigma(Q^k_\alpha),\qquad\forall\,t>0.\tag{2.35}$$

*(9) If $\sigma$ is a measure on $E$ for which $(E,\rho,\sigma)$ is a space of homogeneous type, then a collection $\mathbb{D}(E)$ can be constructed as in (8) such that if $k\in\mathbb{Z}$ with $k\geq\kappa_E$ and $\alpha\in I_k$, then*

$$\sigma\Big(E\setminus\bigcup\nolimits_{\alpha\in I_k}Q^k_\alpha\Big)=\sigma\Big(Q^k_\alpha\setminus\bigcup\nolimits_{\beta\in I_{k+1},\,Q^{k+1}_\beta\subseteq Q^k_\alpha}Q^{k+1}_\beta\Big)=0.\tag{2.36}$$

We now clarify some terminology before discussing the proof of this result. The sets $Q$ in $\mathbb{D}(E)$ will be referred to as `dyadic cubes` on $E$. Also, following a well-established custom, when $Q^{k+1}_\alpha\subseteq Q^k_\beta$, we say that $Q^{k+1}_\alpha$ is a `child` of $Q^k_\beta$, and that $Q^k_\beta$ is a `parent` of $Q^{k+1}_\alpha$. For a given dyadic cube, an `ancestor` is then a parent, or a parent of a parent, or so on. Moreover, for each $k\in\mathbb{Z}$ with $k\geq\kappa_E$, we call $\mathbb{D}_k(E)$ the `dyadic cubes of generation` $k$ and, for each $Q\in\mathbb{D}_k(E)$, define the `side-length` of $Q$ to be $\ell(Q):=2^{-k}$, and the `center` of $Q$ to be the point $x^k_\alpha\in E$ such that $Q=Q^k_\alpha$.

Henceforth, we make the convention whereby saying that $\mathbb{D}(E)$ is a `dyadic cube structure` (or `dyadic grid`) `on` $E$ indicates that $\mathbb{D}(E)$ is associated with $E$ as in Proposition 2.11. This presupposes that $E$ is the ambient set for a geometrically doubling quasi-metric space, in which case $\mathbb{D}(E)$ satisfies properties *(1)-(7)* above and that, in the presence of a background Borel doubling measure $\sigma$, properties *(8)* and *(9)* also hold.

We are now ready to proceed with the proof of Proposition 2.11.

*Proof of Proposition 2.11.* This is a slight extension and clarification of a result proved by M. Christ in [15], which generalized earlier work by G. David in [23], and we will limit ourselves to discussing only the novel aspects of the present formulation. For the sake of reference, we begin by recalling the main steps in the construction in [15]. For a fixed real number $\delta\in(0,1)$ and for any integer $k\in\mathbb{Z}$, Christ considers a maximal collection of points $z^k_\alpha\in E$ such that

$$\rho_\#(z^k_\alpha,z^k_\beta)\geq\delta^k,\qquad\forall\,\alpha\neq\beta.\tag{2.37}$$

Hence, for each $k\in\mathbb{Z}$, the set $\{z^k_\alpha\}_\alpha$ is $\delta^k$-dense in $E$ in the sense that for each $x\in E$ there exists $\alpha$ such that $\rho_\#(x,z^k_\alpha)<\delta^k$. Then (cf. [15, Lemma 13, p. 8]) there exists a partial order relation $\preceq$ on the set $\{(k,\alpha):\ k\in\mathbb{Z},\alpha\in I_k\}$ with the following properties:

*(a)* If $(k,\alpha)\preceq(l,\beta)$, then $k\geq l$.

*(b)* For each $(k,\alpha)$ and $l\leq k$, there exists a unique $\beta$ such that $(k,\alpha)\preceq(l,\beta)$.

*(c)* If $(k,\alpha)\preceq(k-1,\beta)$, then $\rho_\#(z^k_\alpha,z^{k-1}_\beta)<\delta^{k-1}$.

*(d)* If $\rho_\#(z^l_\beta,z^k_\alpha)\leq 2C_\rho\delta^k$, then $(l,\beta)\preceq(k,\alpha)$.

Having established this, Christ then chooses a number $c\in(0,1/(2C_\rho))$ and defines

$$Q^k_\alpha:=\bigcup\nolimits_{(l,\beta)\preceq(k,\alpha)}B_{\rho_\#}(z^l_\beta,c\delta^{\,l}).\tag{2.38}$$

The novel aspects of the present formulation are as follows.

First, the dyadic cubes in [15, Theorem 11, p. 7] are labeled over all $k \in \mathbb{Z}$, but (2.30) shows that when $E$ is bounded, the index set $I_k$ becomes a singleton, and in particular $\mathbb{D}_k(E) = \{E\}$, when $2^{-k}$ is sufficiently large. While this is not an issue in and of itself, we find it useful to eliminate this redundancy for later considerations and restrict to indices $k \geq \kappa_E$.

Second, the result in [15, Theorem 11, p. 7] is stated with some $\delta \in (0,1)$, and in particular, with $\delta^k$ instead of $2^{-k}$ in (2.30)-(2.35). We will justify this change at the end of the proof.

Third, the result in [15, Theorem 11, p. 7] is formulated in the setting of spaces of homogeneous type (equipped with a symmetric quasi-distance). An inspection of the proof, however, reveals that the arguments in [15, pp. 7-10], applied with the regularization $\rho_\#$ from Theorem 2.1, also prove properties *(1)-(6)* under the weaker assumption that $(E, \rho)$ is a geometrically doubling quasi-metric space.

Fourth, property *(7)* follows from a careful inspection of the proof of [15, Theorem 11, p. 7], which reveals that for each $k \in \mathbb{Z}$ with $k \geq \kappa_E$, and any $j \in \mathbb{N}$ sufficiently large, compared to $k$, the set $\bigcup_{\alpha \in I_k} Q^k_\alpha$ contains a $2^{-j}$-dense subset of $E$ that is maximal with respect to inclusion. Of course, this shows that the union in question is dense in $(E, \tau_\rho)$, and so (2.32) follows.

Fifth, property *(8)* is identical to condition (3.6) in [15, Theorem 11, p. 7] except that the regularization $\rho_\#$ from Theorem 2.1 is used instead of the regularization from [57].

Sixth, property *(9)* corresponds to (3.1) in [15, Theorem 11, p. 7] and the proof therein uses the Lebesgue Differentiation Theorem, which requires that continuous functions vanishing outside bounded subsets of $E$ are dense in $L^1(E, \sigma)$. This density result has been established under the assumption that the measure $\sigma$ is Borel regular in [61, Theorem 7.10], and further refined to the case that $\sigma$ is Borel semiregular in [2]. We avoid having to impose any regularity assumption on the measure by relying on the following special case of the Lebesgue Differentiation Theorem, which we claim is valid for arbitrary Borel measures: For every open set $\mathcal{O} \subseteq E$ in $\tau_\rho$ and every $\{r_j\}_{j \in \mathbb{N}} \subseteq (0, \infty)$ with $\lim_{j \to \infty} r_j = 0$, it holds that

$$\limsup_{j \to \infty} \fint_{B_{\rho_\#}(x, r_j)} \big|\mathbf{1}_{\mathcal{O}}(y) - \mathbf{1}_{\mathcal{O}}(x)\big| \, d\sigma(y) = 0, \qquad \text{for } \sigma\text{-a.e. } x \in E. \tag{2.39}$$

Assuming (2.39), we now prove (2.36). Let $k \in \mathbb{Z}$ with $k \geq \kappa_E$ and set $\mathcal{O}_k := \bigcup_{\alpha \in I_k} Q^k_\alpha$. The second displayed formula on p. 10 of [15] shows that there exists $c \in (0, \infty)$ such that

$$\limsup_{r \to 0^+} \frac{\sigma\big(\mathcal{O}_k \cap B_{\rho_\#}(x, r)\big)}{\sigma\big(B_{\rho_\#}(x, r)\big)} \geq c, \qquad \text{for each } x \in E, \tag{2.40}$$

and since $\mathcal{O}_k$ is open in $\tau_\rho$ by *(1)*, it follows from (2.39) that $\sigma(E \setminus \mathcal{O}_k) = 0$, so (2.36) holds.

To prove (2.39), by working with truncated versions of $\mathbf{1}_{\mathcal{O}}$ via characteristic functions of $\rho_\#$-balls exhausting $E$, we may now assume without loss of generality that $\sigma(\mathcal{O}) < \infty$. For each $j \in \mathbb{N}$, the reasoning in [2] shows that the function $F_j : E \to [0, \infty]$ defined by $F_j(x) := \fint_{B_{\rho_\#}(x, r_j)} \big|\mathbf{1}_{\mathcal{O}}(y) - \mathbf{1}_{\mathcal{O}}(x)\big| \, d\sigma(y)$, for all $x \in E$, is Borel measurable, hence

$$S_\theta := \left\{ x \in E : \limsup_{j \to \infty} \fint_{B_{\rho_\#}(x, r_j)} \big|\mathbf{1}_{\mathcal{O}}(y) - \mathbf{1}_{\mathcal{O}}(x)\big| \, d\sigma(y) > \theta \right\} \tag{2.41}$$

is a Borel set in $(E, \tau_\rho)$ for each $\theta \in [0, \infty)$. Thus, it suffices to prove that $\sigma(S_0) = 0$, and this will follow by proving that $\sigma(S_\theta) = 0$ for each $\theta \in (0, \infty)$. To this end, let $\theta$, $\varepsilon \in (0, \infty)$. Fix

$0 < \beta \leq 1/\log_2 C_\rho$, so by [61, Lemma 4.14, p. 166], there exists a sequence $\{h_\ell\}_{\ell\in\mathbb{N}}$ of $\rho$-Hölder functions of order $\beta$ on $E$ such that $0 \leq h_\ell \leq 1$ and $h_\ell \nearrow \mathbf{1}_{\mathcal{O}}$ pointwise as $\ell \to \infty$. Then, since $\sigma(\mathcal{O}) < \infty$, Lebesgue's Monotone Convergence Theorem implies that $h_\ell \to \mathbf{1}_{\mathcal{O}}$ in $L^1(E,\sigma)$ as $\ell \to \infty$, thus there exists $\ell_o \in \mathbb{N}$ such that $\|\mathbf{1}_{\mathcal{O}} - h_{\ell_o}\|_{L^1(E,\sigma)} < \varepsilon$. If $x \in E$ and $j \in \mathbb{N}$, then

$$\begin{aligned}\fint_{B_{\rho_\#}(x,r_j)} |\mathbf{1}_{\mathcal{O}}(y) - \mathbf{1}_{\mathcal{O}}(x)|\, d\sigma(y) &\leq \fint_{B_{\rho_\#}(x,r_j)} |(\mathbf{1}_{\mathcal{O}} - h_{\ell_o})(y)|\, d\sigma(y) + |(\mathbf{1}_{\mathcal{O}} - h_{\ell_o})(x)| \\ &\quad + \fint_{B_{\rho_\#}(x,r_j)} |h_{\ell_o}(y) - h_{\ell_o}(x)|\, d\sigma(y), \quad (2.42)\end{aligned}$$

so the monotonicity of the limit superior implies that $S_\theta \subseteq A_1 \cup A_2 \cup A_3$, where

$$A_1 := \Big\{x \in E : \limsup_{j\to\infty} \fint_{B_{\rho_\#}(x,r_j)} |(\mathbf{1}_{\mathcal{O}} - h_{\ell_o})(y)|\, d\sigma(y) > \theta/3\Big\}, \quad (2.43)$$

$$A_2 := \Big\{x \in E : |(\mathbf{1}_{\mathcal{O}} - h_{\ell_o})(x)| > \theta/3\Big\}, \quad (2.44)$$

$$A_3 := \Big\{x \in E : \limsup_{j\to\infty} \fint_{B_{\rho_\#}(x,r_j)} |h_{\ell_o}(y) - h_{\ell_o}(x)|\, d\sigma(y) > \theta/3\Big\}. \quad (2.45)$$

It follows by reasoning as in (2.41) that $A_1, A_2, A_3$ are Borel sets in $(E, \tau_\rho)$. We have $A_3 = \emptyset$, since $h_{\ell_o}$ is $\rho$-Hölder, while Tschebyshev's inequality implies that

$$\sigma(A_2) \leq \frac{3}{\theta}\|\mathbf{1}_{\mathcal{O}} - h_{\ell_o}\|_{L^1(E,\sigma)} \leq \frac{3\varepsilon}{\theta}. \quad (2.46)$$

Also, since the Hardy-Littlewood maximal operator is weak-type $(1,1)$ (cf., e.g., [2]), we have

$$\sigma(A_1) \leq \sigma\Big(\big\{x \in E : M_E\big(\mathbf{1}_{\mathcal{O}} - h_{\ell_o}\big)(x) > \theta/3\big\}\Big) \leq \frac{C}{\theta}\|\mathbf{1}_{\mathcal{O}} - h_{\ell_o}\|_{L^1(E,\sigma)} \leq \frac{C\varepsilon}{\theta}, \quad (2.47)$$

for some $C \in (0,\infty)$. Altogether, this shows that $\sigma(S_\theta) \leq C\varepsilon/\theta$ for all $\varepsilon \in (0,\infty)$, hence $\sigma(S_\theta) = 0$. This concludes the proof that $\sigma(S_0) = 0$ and completes the justification of (2.39).

We now return to prove that it is always possible to set $\delta = 1/2$ in the construction of Christ in [15]. To do this, we adopt Christ's convention of labeling the dyadic cubes over all $k \in \mathbb{Z}$, since eliminating the inherent redundancy when $E$ is bounded may be done afterwords. Let $\mathfrak{D}(E) := \bigcup_{k\in\mathbb{Z}} \mathfrak{D}_k(E)$ denote a collection of dyadic cubes satisfying properties *(1)-(9)* but with $\delta^k$ replacing $2^{-k}$ in (2.30)-(2.35). We now construct another collection of dyadic cubes $\mathbb{D}(E) := \bigcup_{k\in\mathbb{Z}} \mathbb{D}_k(E)$ with the same properties but with $\delta = 1/2$. We consider two cases.

*Case I:* $1/2 < \delta < 1$. Set $m_0 := 0$ and, for each integer $k > 0$, let $m_k$ be the largest positive integer such that $\delta^{m_k} \geq 2^{-k}$. Thus,

$$\delta^{m_k+1} < 2^{-k} \leq \delta^{m_k}. \quad (2.48)$$

Similarly, for each $k < 0$, let $m_k$ denote the least integer such that $\delta^{m_k+1} < 2^{-k}$. Thus, again we have (2.48). Of course, we shall have $m_k < 0$ when $k < 0$. The sequence $\{m_k\}_{k\in\mathbb{Z}}$ is strictly increasing. Indeed, for every $k \in \mathbb{Z}$, we have

$$m_k + 1 \leq m_{k+1}. \quad (2.49)$$

To see this in the case that $k \geq 0$, observe that

$$2^{-k-1} = \tfrac{1}{2}2^{-k} \leq \tfrac{1}{2}\delta^{m_k} < \delta^{m_k+1}, \tag{2.50}$$

where in the first inequality we have used (2.48) and in the second that $1/2 < \delta$. Thus, (2.49) holds, since by definition $m_{k+1}$ is the greatest integer for which $2^{-(k+1)} \leq \delta^{m_{k+1}}$. In the case $k \leq 0$, since $1 < 2\delta$, we have

$$\delta^{(m_{k+1}-1)+1} < 2\delta^{m_{k+1}+1} < 2 \cdot 2^{-(k+1)} = 2^{-k}, \tag{2.51}$$

where in the second inequality we have used (2.48). Since $m_k$ is the smallest integer for which $\delta^{m_k+1} < 2^{-k}$, we again obtain (2.49). For each $k \in \mathbb{Z}$, we then define

$$\mathbb{D}_k(E) := \mathfrak{D}_{m_k}(E). \tag{2.52}$$

It is routine to verify that the collection $\mathbb{D}(E) := \bigcup_{k\in\mathbb{Z}} \mathbb{D}_k(E)$ satisfies the desired properties, with some of the constants possibly depending on $\delta$.

*Case II:* $0 < \delta < 1/2$. In this case, we reverse the roles of $1/2$ and $\delta$ in the construction above, to construct a strictly increasing sequence of integers $\{m_k\}_{k\in\mathbb{Z}}$, with $m_0 := 0$, for which

$$2^{-m_k} \leq \delta^k < 2^{-m_k+1}, \qquad \forall\, k \in \mathbb{Z}. \tag{2.53}$$

It then follows that there is a fixed positive integer $q_0 \approx \log_2(1/\delta)$ such that for each $k \in \mathbb{Z}$,

$$m_{k+1} - q_0 \leq m_k < m_{k+1}. \tag{2.54}$$

Indeed, we have

$$2^{-m_k} \leq \delta^k = \frac{1}{\delta}\delta^{k+1} < \frac{1}{\delta}2^{-m_{k+1}+1} = \frac{2}{\delta}2^{-m_{k+1}}, \tag{2.55}$$

where in the two inequalities we have used (2.53). We then obtain (2.54) by taking logarithms. For each $j \in \mathbb{Z}$, there exists a unique $k \in \mathbb{Z}$ such that $m_k \leq j < m_{k+1}$, and we then define

$$\mathbb{D}_j(E) := \mathfrak{D}_k(E), \qquad \text{whenever } m_k \leq j < m_{k+1}. \tag{2.56}$$

It is routine to verify that the collection $\mathbb{D}(E) := \bigcup_{k\in\mathbb{Z}} \mathbb{D}_k(E)$ satisfies the desired properties, with some of the constants possibly depending on $\delta$. In verifying the various properties, it is helpful to observe that by (2.54), it holds that

$$2^{-j} \approx 2^{-m_k} \approx \delta^k, \qquad \text{whenever } m_k \leq j < m_{k+1}. \tag{2.57}$$

This finishes the proof of the proposition. □

## 2.3 Approximations to the identity on quasi-metric spaces

This subsection is devoted to reviewing the definition and properties of approximations to the identity on ADR spaces. To set the stage, we make the following definition.

**Definition 2.12.** *Assume that* $(E, \rho, \sigma)$ *is a d-dimensional* ADR *space for some* $d > 0$ *and recall* $\kappa_E \in \mathbb{Z} \cup \{-\infty\}$ *from Proposition 2.11. A collection* $\{\mathcal{S}_l\}_{l \in \mathbb{Z},\, l \geq \kappa_E}$ *of integral operators*

$$\mathcal{S}_l f(x) := \int_E S_l(x, y) f(y)\, d\sigma(y), \qquad x \in E, \tag{2.58}$$

*with kernels* $S_l : E \times E \to \mathbb{R}$*, is an* `approximation to the identity of order` $\gamma$ *on* $E$ *if there exists* $C \in (0, \infty)$ *such that the following hold for every integer* $l \geq \kappa_E$ *and every* $x, x', y, y' \in E$*:*

*(1)* $0 \leq S_l(x, y) \leq C2^{ld}$*, and if* $\rho(x, y) \geq C2^{-l}$*, then* $S_l(x, y) = 0$.

*(2)* $|S_l(x, y) - S_l(x', y)| \leq C2^{l(d+\gamma)} \rho(x, x')^\gamma$.

*(3)* $\big|[S_l(x, y) - S_l(x', y)] - [S_l(x, y') - S_l(x', y')]\big| \leq C2^{l(d+2\gamma)} \rho(x, x')^\gamma \rho(y, y')^\gamma$.

*(4)* $S_l(x, y) = S_l(y, x)$ *and* $\int_E S_l(x, y)\, d\sigma(y) = 1$.

Starting with the work of Coifman (cf. the discussion in [25, pp. 16-17 and p. 40]), the existence of approximations to the identity of some order $\gamma > 0$ on ADR spaces has been established in [25, p. 40], [36, pp. 10-11], [28, p. 16] (at least when $d = 1$), and [61], for various values of $\gamma$. Quite recently, a version valid for the value of the order parameter $\gamma$ which is optimal in relation to the quasi-metric space structure has been obtained in [2], from which we quote the following result (recall the index from (2.6)):

**Proposition 2.13.** *Assume that* $(E, \rho, \sigma)$ *is a d-dimensional* ADR *space for some* $d > 0$*. If* $0 < \gamma < \mathrm{ind}(E, \rho)$*, then there exists an approximation to the identity* $\{\mathcal{S}_l\}_{l \in \mathbb{Z},\, l \geq \kappa_E}$ *of order* $\gamma$ *on* $E$*. Furthermore, if* $p \in (1, \infty)$*, then* $\sup_{l \in \mathbb{Z},\, l \geq \kappa_E} \|\mathcal{S}_l\|_{L^p(E,\sigma) \to L^p(E,\sigma)} < +\infty$ *and the following properties hold for all* $f \in L^p(E, \sigma)$*:*

*(1) If the measure* $\sigma$ *is Borel semiregular on* $(E, \tau_\rho)$*, then* $\lim_{l \to +\infty} \mathcal{S}_l f = f$ *in* $L^p(E, \sigma)$.

*(2) If* $\mathrm{diam}_\rho(E) = +\infty$*, then* $\lim_{l \to -\infty} \mathcal{S}_l f = 0$ *in* $L^p(E, \sigma)$.

Later on we shall need a Calderón-type reproducing formula involving the conditional expectation operators associated with an approximation to the identity, as discussed above. While this is a topic treated at some length in [25], [28], [36], we prove below a version of this result which best suits the purposes we have in mind.

To state the result, we first record the following preliminaries. A series $\sum_{j \in \mathbb{N}} x_j$ of vectors in a Banach space $\mathscr{B}$ is said to be `unconditionally convergent` if the series $\sum_{j=1}^{\infty} x_{\sigma(j)}$ converges in $\mathscr{B}$ for all permutations $\sigma$ of $\mathbb{N}$, in which case the sum of the series in $\mathscr{B}$ is defined unambiguously as $\sum_{j \in \mathbb{N}} x_j := \sum_{j=1}^{\infty} x_{\sigma(j)}$ for some (hence any) permutation $\sigma$ of $\mathbb{N}$ (cf., e.g., [37, Corollary 3.11, p. 99]). The following useful characterizations of unconditional convergence (in a Banach space setting) will be needed (cf., e.g., [37, Theorem 3.10, p. 94]):

$$\begin{aligned} \sum_{j \in \mathbb{N}} x_j \text{ unconditionally convergent} &\Longleftrightarrow \sum_{j=1}^{\infty} \varepsilon_j x_j \text{ converges } \forall\, \varepsilon_j = \pm 1 \\ &\Longleftrightarrow \begin{cases} \forall\, \varepsilon > 0\ \exists\, N_\varepsilon \in \mathbb{N} \text{ such that } \Big\| \sum_{j \in \mathcal{I}} x_j \Big\| < \varepsilon \\ \forall \text{ finite subsets } \mathcal{I} \subset \mathbb{N} \text{ with } \min \mathcal{I} \geq N_\varepsilon. \end{cases} \end{aligned} \tag{2.59}$$

Moreover, for any countable set $\mathbb{I}$, a series $\sum_{j \in \mathbb{I}} x_j$ of vectors in a Banach space $\mathscr{B}$ is said to be `unconditionally convergent` if there exists a bijection $\varphi : \mathbb{N} \to \mathbb{I}$ such that $\sum_{j \in \mathbb{N}} x_{\varphi(j)}$ is unconditionally convergent, in the sense just defined, in which case the sum of the series in $\mathscr{B}$ is defined as $\sum_{j\in\mathbb{I}} x_j := \sum_{j=1}^{\infty} x_{\varphi(j)}$. This property is independent of the bijection $\varphi$ used and it follows from (2.59) that the following property provides an equivalent characterization:

$$\begin{gathered}\forall\,\{S_i\}_{i\in\mathbb{N}} \text{ such that } S_i \text{ finite and } S_i\subseteq S_{i+1}\subseteq\mathbb{I} \text{ for each } i\in\mathbb{N},\\ \text{the sequence } \Big\{\textstyle\sum_{j\in S_i} x_j\Big\}_{i\in\mathbb{N}} \text{ converges in } \mathscr{B}.\end{gathered} \tag{2.60}$$

We now state the aforementioned Calderón-type reproducing formula.

**Proposition 2.14.** *Assume that $(E,\rho,\sigma)$ is a $d$-dimensional* ADR *space for some $d>0$ and that the measure $\sigma$ is Borel semiregular on $(E,\tau_\rho)$. Fix $0<\gamma<\operatorname{ind}(E,\rho)$, let $\{\mathcal{S}_l\}_{l\in\mathbb{Z},\,l\geq\kappa_E}$ denote an approximation to the identity of order $\gamma$ on $E$ and define the integral operators $D_l:=\mathcal{S}_{l+1}-\mathcal{S}_l$ for each integer $l\geq\kappa_E$. Then there exist a bounded linear operator $R$ on $L^2(E,\sigma)$ and a collection $\{\widetilde{D}_l\}_{l\in\mathbb{Z},\,l\geq\kappa_E}$ of linear operators on $L^2(E,\sigma)$ such that*

$$\sum_{l\in\mathbb{Z},\,l\geq\kappa_E}\|\widetilde{D}_l f\|^2_{L^2(E,\sigma)}\leq C\|f\|^2_{L^2(E,\sigma)},\qquad\forall\, f\in L^2(E,\sigma), \tag{2.61}$$

*and, with $I$ denoting the identity operator on $L^2(E,\sigma)$,*

$$I+\mathcal{S}_{\kappa_E}R=\sum_{l\in\mathbb{Z},\,l\geq\kappa_E} D_l\widetilde{D}_l,\qquad \textit{pointwise unconditionally in } L^2(E,\sigma), \tag{2.62}$$

*with the convention that $\mathcal{S}_{-\infty}:=0$ when $\operatorname{diam}_\rho(E)=+\infty$.*

As a preamble to the proof of the above proposition we momentarily digress and record a version of the Cotlar-Knapp-Stein Lemma which suits our purposes. The result is proved by combining (2.60) and Lemma 2.16 with the well-known version, which is stated as below but with $J$ finite and/or without including property *(3)* (cf., e.g., [72, Theorem 1, p. 280]).

**Lemma 2.15.** *Assume that $\mathscr{H}_0$, $\mathscr{H}_1$ are two Hilbert spaces and consider a family of operators $\{T_j\}_{j\in\mathbb{I}}$, indexed by a countable set $\mathbb{I}$, with $T_j:\mathscr{H}_0\to\mathscr{H}_1$ linear and bounded for every $j\in\mathbb{I}$. If the $T_j$'s are almost orthogonal in the sense that*

$$C_0:=\sup_{j\in\mathbb{I}}\Big(\sum_{k\in\mathbb{I}}\sqrt{\|T_j^*T_k\|_{\mathscr{H}_0\to\mathscr{H}_0}}\Big)<\infty,\quad C_1:=\sup_{k\in\mathbb{I}}\Big(\sum_{j\in\mathbb{I}}\sqrt{\|T_jT_k^*\|_{\mathscr{H}_1\to\mathscr{H}_1}}\Big)<\infty, \tag{2.63}$$

*then for any subset $J\subseteq\mathbb{I}$, the following properties hold:*

*(1) If $x\in\mathscr{H}_0$, then $\sum_{j\in J}T_jx$ converges unconditionally in $\mathscr{H}_1$.*

*(2) If $\Big(\sum_{j\in J}T_j\Big)x:=\sum_{j\in J}T_jx$, for all $x\in\mathscr{H}_0$, then $\Big\|\sum_{j\in J}T_j\Big\|_{\mathscr{H}_0\to\mathscr{H}_1}\leq\sqrt{C_0C_1}$.*

*(3) If $x\in\mathscr{H}_0$, then $\Big(\sum_{j\in\mathbb{I}}\|T_jx\|^2_{\mathscr{H}_1}\Big)^{1/2}\leq 2\sqrt{C_0C_1}\|x\|_{\mathscr{H}_0}$.*

**Lemma 2.16.** *Let $\mathscr{H}$ be a Hilbert space with norm $\|\cdot\|_{\mathscr{H}}$. If $\{x_j\}_{j\in\mathbb{I}}$ is a sequence in $\mathscr{H}$ over a countable set $\mathbb{I}$ and $\mathcal{C}:=\sup\big\{\big\|\sum_{j\in J_o}x_j\big\|_{\mathscr{H}}:J_o\subseteq\mathbb{I}$ is finite$\big\}$, then $\sum_{j\in\mathbb{I}}\|x_j\|^2_{\mathscr{H}}\leq 4\mathcal{C}^2$ and*

$$\sum_{j\in\mathbb{I}}x_j \textit{ is unconditionally convergent}\iff\mathcal{C}<\infty\Longrightarrow\Big\|\sum_{j\in\mathbb{I}}x_j\Big\|_{\mathscr{H}}\leq\mathcal{C}. \tag{2.64}$$

*Proof.* It suffices to assume that $\mathbb{I} = \mathbb{N}$. Let $\{x_j\}_{j\in\mathbb{N}} \subseteq \mathscr{H}$ and assume that $\mathcal{C} < \infty$. Let $\{r_j\}_{j\in\mathbb{N}}$ denote a Rademacher system of functions on $[0,1]$. If $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ stands for the inner product in $\mathscr{H}$, then for any finite set $J_o \subseteq \mathbb{N}$, by orthonormality we have

$$\int_0^1 \Big\|\sum\nolimits_{j\in J_o} r_j(t)x_j\Big\|_{\mathscr{H}}^2 dt = \sum\nolimits_{j\in J_o} \|x_j\|_{\mathscr{H}}^2. \tag{2.65}$$

On the other hand, for each $t \in [0,1]$ we may estimate

$$\Big\|\sum\nolimits_{j\in J_o} r_j(t)x_j\Big\|_{\mathscr{H}} = \Big\|\Big(\sum\nolimits_{j\in J_o,\, r_j(t)=+1} x_j\Big) - \Big(\sum\nolimits_{j\in J_o,\, r_j(t)=-1} x_j\Big)\Big\|_{\mathscr{H}} \leq 2\mathcal{C}. \tag{2.66}$$

By combining (2.65) and (2.66), we obtain $\sum_{j\in J_o} \|x_j\|_{\mathscr{H}}^2 \leq 4\mathcal{C}^2$ for every finite subset $J_o$ of $\mathbb{N}$, from which the required norm estimate readily follows.

Moving on, assume that $\mathcal{C} < \infty$ and that $\sum_{j\in\mathbb{N}} x_j$ does not converge unconditionally to seek a contradiction. Then (cf. the first equivalence in (2.59)), there exists a choice of signs $\varepsilon_j \in \{\pm 1\}$, $j \in \mathbb{N}$, with the property that the sequence of partial sums of the series $\sum_{j\in\mathbb{N}} \varepsilon_j x_j$ is not Cauchy in $\mathscr{H}$. In turn, this implies that there exist $\vartheta > 0$ along with two sequences $\{a_i\}_{i\in\mathbb{N}}$, $\{b_i\}_{i\in\mathbb{N}}$ of numbers in $\mathbb{N}$, such that

$$a_i \leq b_i < a_{i+1} \quad \text{and} \quad \Big\|\sum\nolimits_{a_i\leq j\leq b_i} \varepsilon_j x_j\Big\|_{\mathscr{H}} \geq \vartheta, \quad \text{for every } i \in \mathbb{N}. \tag{2.67}$$

Next, by (2.67), the sequence $\{y_i\}_{i\in\mathbb{N}}$ in $\mathscr{H}$ defined by $y_i := \sum_{a_i\leq j\leq b_i} \varepsilon_j x_j$ satisfies

$$\|y_i\|_{\mathscr{H}} \geq \vartheta, \quad \text{for every } i \in \mathbb{N}. \tag{2.68}$$

Now fix an arbitrary finite subset $I_o$ of $\mathbb{N}$ and set $J_o := \big\{j \in \mathbb{N} : \exists\, i \in I_o \text{ such that } a_i \leq j \leq b_i\big\}$. Thus, $J_o$ is a finite subset of $\mathbb{N}$ and we have

$$\begin{aligned}\Big\|\sum\nolimits_{i\in I_o} y_i\Big\|_{\mathscr{H}} &= \Big\|\sum\nolimits_{i\in I_o}\Big(\sum\nolimits_{a_i\leq j\leq b_i} \varepsilon_j x_j\Big)\Big\|_{\mathscr{H}} \\ &= \Big\|\Big(\sum\nolimits_{j\in J_o,\, \varepsilon_j=+1} x_j\Big) - \Big(\sum\nolimits_{j\in J_o,\, \varepsilon_j=-1} x_j\Big)\Big\|_{\mathscr{H}} \leq 2\mathcal{C},\end{aligned} \tag{2.69}$$

where the second equality relies on the fact from (2.67) that $a_i \leq b_i < a_{i+1}$. Hence,

$$\sup\Big\{\Big\|\sum\nolimits_{i\in I_o} y_i\Big\|_{\mathscr{H}} : I_o \subseteq \mathbb{N} \text{ is finite}\Big\} \leq 2\mathcal{C}. \tag{2.70}$$

It follows that $\sum_{i\in\mathbb{N}} \|y_i\|_{\mathscr{H}}^2 \leq 16\mathcal{C}^2 < \infty$, which forces $\lim\limits_{i\to\infty} \|y_i\|_{\mathscr{H}} = 0$, and contradicts (2.68).

This shows that if $\mathcal{C} < \infty$, then $\sum_{j\in\mathbb{N}} x_j$ is unconditionally convergent. Once the (norm) convergence is established, then $\mathcal{C} < \infty$ implies $\|\sum_{j\in\mathbb{N}} x_j\|_{\mathscr{H}} \leq \limsup_{N\to\infty} \|\sum_{j=1}^N x_j\|_{\mathscr{H}} \leq \mathcal{C}$, which is the second implication in (2.64). Therefore, it remains to prove that $\mathcal{C} < \infty$ when the series $\sum_{j\in\mathbb{N}} x_j$ is unconditionally convergent. Let $N_1 \in \mathbb{N}$ denote $N_\varepsilon$ with $\varepsilon := 1$ from (2.59) and set $M := \sup\big\{\|\sum_{j\in I_o} x_j\|_{\mathscr{H}} : I_o \subseteq \{1,...,N_1\}\big\} < \infty$. If $J_o \subset \mathbb{N}$ is finite, then

$$\Big\|\sum\nolimits_{j\in J_o} x_j\Big\|_{\mathscr{H}} \leq \Big\|\sum\nolimits_{j\in J_o\cap\{1,\ldots,N_1\}} x_j\Big\|_{\mathscr{H}} + \Big\|\sum\nolimits_{j\in J_o\setminus\{1,\ldots,N_1\}} x_j\Big\|_{\mathscr{H}} \leq M+1, \tag{2.71}$$

from which it follows that $\mathcal{C} < \infty$. □

We next present the proof of Proposition 2.14.

*Proof of Proposition 2.14.* For each $l \in \mathbb{Z}$ with $l \geq \kappa_E$, denote by $h_l(\cdot,\cdot)$ the integral kernel of the operator $D_l$. Thus, $h_l(\cdot,\cdot) = S_{l+1}(\cdot,\cdot) - S_l(\cdot,\cdot)$ and, as a consequence of properties *(1)-(4)* in Definition 2.12, we see that $h_l(\cdot,\cdot)$ is a symmetric function on $E \times E$, and there exists $C \in (0,\infty)$ such that for each $l \in \mathbb{Z}$ with $l \geq \kappa_E$ and all $x, x', y \in E$, we have

$$|h_l(x,y)| \leq C2^{ld}\mathbf{1}_{\{\rho(x,y)\leq C2^{-l}\}}, \tag{2.72}$$

$$|h_l(x,y) - h_l(x',y)| \leq C2^{l(d+\gamma)}\rho(x,x')^\gamma, \tag{2.73}$$

$$\int_E h_l(x,y)\,d\sigma(x) = 0. \tag{2.74}$$

Of course, due to the symmetry of $h$, smoothness and cancellation conditions in the second variable, similar to (2.73) and (2.74), respectively, also hold.

Furthermore, for each $j,k \in \mathbb{Z}$ with $j,k \geq \kappa_E$, using first (2.74), then (2.72) and (2.73), and then the fact that $(E,\rho,\sigma)$ is $d$-ADR, we may write

$$\begin{aligned}
\left|\int_E h_j(x,z)h_k(z,y)\,d\sigma(z)\right| &= \left|\int_E [h_j(x,z) - h_j(x,y)]h_k(z,y)\,d\sigma(z)\right| \\
&\leq C2^{j(d+\gamma)}\int_E \rho_\#(y,z)^\gamma 2^{kd}\mathbf{1}_{\{\rho_\#(y,\cdot)\leq C2^{-k}\}}(z)\,d\sigma(z) \\
&\leq C2^{j(d+\gamma)}2^{-k\gamma}.
\end{aligned} \tag{2.75}$$

Combining (2.75), and the analogous estimate obtained by interchanging the roles of $j$ and $k$, with the support condition (2.72), it follows that for each $j,k \in \mathbb{Z}$ with $j,k \geq \kappa_E$, it holds that (compare with [25, p. 15] and [28, (1.14), p. 16])

$$\left|\int_E h_j(x,z)h_k(z,y)\,d\sigma(z)\right| \leq C2^{-|j-k|\gamma}\,2^{d\cdot\min(j,k)}\mathbf{1}_{\{\rho(x,y)\leq C2^{-\min(j,k)}\}}, \quad \forall\, x,y \in E. \tag{2.76}$$

Note that for each $j,k \in \mathbb{Z}$ with $j,k \geq \kappa_E$ we have that $D_jD_k : L^2(E,\sigma) \to L^2(E,\sigma)$ is a linear and bounded integral operator whose integral kernel is given by $\int_E h_j(x,z)h_k(z,y)\,d\sigma(z)$, for $x,y \in E$. Based on this and (2.76) we may then conclude that for each $j,k \in \mathbb{Z}$ with $j,k \geq \kappa_E$,

$$\begin{aligned}
\big|(D_jD_kf)(x)\big| &\leq C2^{-|j-k|\gamma}\fint_{B_{\rho_\#}(x,C2^{-\min(j,k)})}|f(y)|\,d\sigma(y) \\
&\leq C2^{-|j-k|\gamma}M_E(f)(x), \qquad \forall\, x \in E,
\end{aligned} \tag{2.77}$$

for every $f \in L^1_{loc}(E,\sigma)$. In turn, the boundedness of $M_E$ and (2.77) yield

$$\|D_jD_k\|_{L^2(E,\sigma)\to L^2(E,\sigma)} \leq C2^{-|j-k|\gamma}, \qquad \forall\, j,k \in \mathbb{Z},\ j,k \geq \kappa_E. \tag{2.78}$$

Having established (2.78), it follows that the family of linear operators $\{D_l\}_{l\in\mathbb{Z},\, l\geq\kappa_E}$, from $L^2(E,\sigma)$ into itself, is almost orthogonal. As such, Lemma 2.15 implies that

$$\sup\Big\{\Big\|\sum_{l\in J,\, l\geq\kappa_E} D_l\Big\|_{L^2(E,\sigma)\to L^2(E,\sigma)} : J \subset \mathbb{Z} \text{ is finite}\Big\} \leq C < \infty, \tag{2.79}$$

that the Littlewood-Paley estimate

$$\Big(\sum_{l\in\mathbb{Z},\, l\geq\kappa_E}\|D_l f\|^2_{L^2(E,\sigma)}\Big)^{1/2}\leq C\|f\|_{L^2(E,\sigma)},\quad \forall\, f\in L^2(E,\sigma), \tag{2.80}$$

holds and, in combination with Proposition 2.13, that

$$\begin{gathered}\big(I-\mathcal{S}_{\kappa_E}\big)f=\sum_{l\in\mathbb{Z},\, l\geq\kappa_E}D_l f\quad\text{ for each } f\in L^2(E,\sigma),\\ \text{where the series converges unconditionally in } L^2(E,\sigma).\end{gathered} \tag{2.81}$$

To proceed, fix a number $N\in\mathbb{N}$. Based on (2.79), we may square (2.81) and obtain, pointwise in $L^2(E,\sigma)$,

$$\begin{aligned}\big(I-\mathcal{S}_{\kappa_E}\big)^2&=\lim_{M\to\infty}\Big[\Big(\sum_{j\in\mathbb{Z},\, j\geq\kappa_E,\, |j|\leq M}D_j\Big)\Big(\sum_{k\in\mathbb{Z},\, k\geq\kappa_E,\, |k|\leq M}D_k\Big)\Big]\\ &=\lim_{M\to\infty}\Big(\sum_{\substack{|j-k|\leq N\\ j,k\geq\kappa_E,\, |j|,|k|\leq M}}D_jD_k+\sum_{\substack{|j-k|>N\\ j,k\geq\kappa_E,\, |j|,|k|\leq M}}D_jD_k\Big).\end{aligned} \tag{2.82}$$

Going further, fix $i\in\mathbb{Z}$ and consider the family $\{T_l\}_{l\in J_i}$ of operators on $L^2(E,\sigma)$, where

$$T_l:=D_{l+i}D_l\quad\text{ for every }\quad l\in J_i:=\big\{l\in\mathbb{Z}:\, l\geq\max\{\kappa_E,\kappa_E-i\}\big\}. \tag{2.83}$$

Then, with $\|\cdot\|$ temporarily abbreviating $\|\cdot\|_{L^2(E,\sigma)\to L^2(E,\sigma)}$, for each $j,k\in J_i$ we may estimate

$$\begin{aligned}\|T_j^*T_k\|&\leq\min\Big\{\|D_j\|\|D_{j+i}D_{k+i}\|\|D_k\|\,,\,\|D_jD_{j+i}\|\|D_{k+i}\|\|D_k\|\Big\}\\ &\leq C\,\min\Big\{2^{-|k-j|\gamma}\,,\,2^{-|i|\gamma}\Big\},\end{aligned} \tag{2.84}$$

thanks to (2.79) and (2.78). This readily implies that $\sup\limits_{j\in J_i}\Big(\sum\limits_{k\in J_i}\sqrt{\|T_j^*T_k\|}\Big)\leq C(1+|i|)2^{-|i|\gamma/2}$ and $\sup\limits_{k\in J_i}\Big(\sum\limits_{j\in J_i}\sqrt{\|T_j\,T_k^*\|}\Big)\leq C(1+|i|)2^{-|i|\gamma/2}$ for some $C\in(0,\infty)$ independent of $i$. Hence, for each $i\in\mathbb{Z}$, the family $\big\{D_{l+i}D_l\big\}_{l\in\mathbb{Z},\, l\geq\max\{\kappa_E,\kappa_E-i\}}$ is almost orthogonal, and by Lemma 2.15 there exists some constant $C\in(0,\infty)$ independent of $i$ such that for every set $J\subseteq J_i$ we have that $\sum\limits_{l\in J}D_{l+i}D_l$ converges pointwise unconditionally in $L^2(E,\sigma)$ and

$$\Big\|\sum_{l\in J}D_{l+i}D_l\Big\|_{L^2(E,\sigma)\to L^2(E,\sigma)}\leq C(1+|i|)2^{-|i|\gamma/2}. \tag{2.85}$$

Next, fix $N\in\mathbb{N}$ and let $\mathscr{I}$ be an arbitrary finite subset of $\{(l,m)\in\mathbb{Z}\times\mathbb{Z}:\, l,\,m\geq\kappa_E\}$. Then for each function $f\in L^2(E,\sigma)$ with $\|f\|_{L^2(E,\sigma)}=1$, using (2.85) we may estimate

$$\begin{aligned}&\Big\|\sum_{(j,k)\in\mathscr{I},\, |j-k|>N}D_jD_kf\Big\|_{L^2(E,\sigma)}=\Big\|\sum_{i\in\mathbb{Z},\, |i|>N}\Big(\sum_{l\in\mathbb{Z},\, (l+i,l)\in\mathscr{I}}D_{l+i}D_lf\Big)\Big\|_{L^2(E,\sigma)}\\ &\leq\sum_{i\in\mathbb{Z},\, |i|>N}\Big\|\sum_{l\in\mathbb{Z},\, (l+i,l)\in\mathscr{I}}D_{l+i}D_lf\Big\|_{L^2(E,\sigma)}\leq\sum_{i\in\mathbb{Z},\, |i|>N}C(1+|i|)2^{-|i|\gamma/2}\leq C_\gamma N2^{-N\gamma/2},\end{aligned} \tag{2.86}$$

for some finite constant $C_\gamma > 0$ independent of $N$. It then follows from (2.64) and (2.86) that

$$R_N := \sum\nolimits_{j,k\geq\kappa_E,\,|j-k|>N} D_j D_k \text{ converges pointwise unconditionally in } L^2(E,\sigma) \tag{2.87}$$

$$\text{and there exists } C_\gamma \in (0,\infty) \text{ such that } \|R_N\|_{L^2(E,\sigma)\to L^2(E,\sigma)} \leq C_\gamma N 2^{-N\gamma/2}. \tag{2.88}$$

In a similar fashion to (2.85)-(2.87), we may also deduce that

$$T_N := \sum\nolimits_{j,k\geq\kappa_E,\,|j-k|\leq N} D_j D_k \text{ converges pointwise unconditionally in } L^2(E,\sigma). \tag{2.89}$$

We set $D_l^N := \sum_{i\geq\kappa_E-l,\,|i|\leq N} D_{l+i}$ for each $l\in\mathbb{Z}$, so $T_N = \sum_{l\in\mathbb{Z},\,l\geq\kappa_E} D_l D_l^N$, where the sum converges pointwise unconditionally in $L^2(E,\sigma)$. Then (2.82), (2.87) and (2.89) imply that

$$\big(I-\mathcal{S}_{\kappa_E}\big)^2 = R_N + T_N \quad \text{on} \quad L^2(E,\sigma), \tag{2.90}$$

which is convenient to further re-write as

$$I = R_N + \widetilde{T}_N \quad \text{on} \quad L^2(E,\sigma), \qquad \text{where} \quad \widetilde{T}_N := T_N + \mathcal{S}_{\kappa_E}\big(2I - \mathcal{S}_{\kappa_E}\big). \tag{2.91}$$

Thanks to the estimate in (2.87), it follows from (2.91) that

$$\widetilde{T}_N : L^2(E,\sigma) \to L^2(E,\sigma) \text{ is boundedly invertible for } N\in\mathbb{N} \text{ sufficiently large.} \tag{2.92}$$

Hence, for $N$ sufficiently large and fixed, based on (2.92) we may write that $I = \widetilde{T}_N(\widetilde{T}_N)^{-1}$, and keeping in mind (2.91), we arrive at the following Calderón-type reproducing formula

$$I = \Big(\sum_{l\in\mathbb{Z},\,l\geq\kappa_E} D_l\widetilde{D}_l\Big) + \mathcal{S}_{\kappa_E}\big(2I-\mathcal{S}_{\kappa_E}\big)(\widetilde{T}_N)^{-1}, \tag{2.93}$$

where the sum converges pointwise unconditionally in $L^2(E,\sigma)$, and

$$\widetilde{D}_l := D_l^N(\widetilde{T}_N)^{-1}, \qquad \forall\, l\in\mathbb{Z} \ \text{ with } \ l\geq\kappa_E. \tag{2.94}$$

From this (2.62) follows with $R := \big(\mathcal{S}_{\kappa_E} - 2I\big)(\widetilde{T}_N)^{-1}$. Finally, (2.61) is a consequence of (2.94), the fact that the sum defining $D_l^N$ has a finite number of terms, (2.92) and (2.80). $\square$

### 2.4 Dyadic Carleson tents

Suppose that $(\mathscr{X},\rho)$ is a geometrically doubling quasi-metric space and that $E$ is a nonempty, closed, proper subset of $(\mathscr{X},\tau_\rho)$. It follows from the discussion below Definition 2.4 that $(E,\rho\big|_E)$ is also a geometrically doubling quasi-metric space. We now introduce dyadic Carleson tents in this setting. These are sets in $\mathscr{X}\setminus E$ that are adapted to $E$ in the same way that classical Carleson boxes or tents in the upper-half space $\mathbb{R}^{n+1}_+$ are adapted to $\mathbb{R}^n$. We require a number of preliminaries before we introduce these sets in (2.97) below. First, fix a collection $\mathbb{D}(E)$ of dyadic cubes contained in $E$ as in Proposition 2.11. Second, choose $\lambda\in[2C_\rho,\infty)$ and fix a Whitney covering $\mathbb{W}_\lambda(\mathscr{X}\setminus E)$ of balls contained in $\mathscr{X}\setminus E$ as in Proposition 2.5. Following Convention 2.6, we refer to these $\rho_\#$-balls as Whitney cubes, and for each $I\in\mathbb{W}_\lambda(\mathscr{X}\setminus E)$, we use the notation $\ell(I)$ for the radius of $I$. Third, choose $C_*\in[1,\infty)$, and for each $Q\in\mathbb{D}(E)$, define the following collection of Whitney cubes:

$$W_Q := \{I\in\mathbb{W}_\lambda(\mathscr{X}\setminus E) : C_*^{-1}\ell(I)\leq\ell(Q)\leq C_*\ell(I) \text{ and } \operatorname{dist}_\rho(I,Q)\leq\ell(Q)\}. \tag{2.95}$$

Fourth, for each $Q \in \mathbb{D}(E)$, define the following subset of $(\mathscr{X}, \tau_\rho)$:

$$\mathcal{U}_Q := \bigcup_{I \in W_Q} I. \tag{2.96}$$

Since from Theorem 2.1 we know that the regularized quasi-distance $\rho_\#$ is continuous, it follows that the $\rho_\#$-balls are open. As such, that each $I$ in $W_Q$, hence $\mathcal{U}_Q$ itself, is open. Finally, for each $Q \in \mathbb{D}(E)$, the `dyadic Carleson tent` $T_E(Q)$ `over` $Q$ is defined as follows:

$$T_E(Q) := \bigcup_{Q' \in \mathbb{D}(E),\, Q' \subseteq Q} \mathcal{U}_{Q'}. \tag{2.97}$$

For most of the subsequent work we will assume that the Whitney covering $\mathbb{W}_\lambda(\mathscr{X} \setminus E)$ and the constant $C_*$ are chosen as in the following lemma. The reader should be aware that even when (2.98) holds, there may exist $Q \in \mathbb{D}(E)$ for which $\mathcal{U}_Q$ is empty.

**Lemma 2.17.** *Let $(\mathscr{X}, \rho)$ be a geometrically doubling quasi-metric space and suppose that $E$ is a nonempty, closed, proper subset of $(\mathscr{X}, \tau_\rho)$. Fix a collection $\mathbb{D}(E)$ of dyadic cubes in $E$ as in Proposition 2.11. Next, choose $\lambda \in [2C_\rho, \infty)$, fix a Whitney covering $\mathbb{W}_\lambda(\mathscr{X} \setminus E)$ of $\mathscr{X} \setminus E$, and let $\Lambda$ denote the constant associated with $\lambda$ as in Proposition 2.5.*

*If $C_* \in [4C_\rho^4\,\Lambda, \infty)$, then there exists $\epsilon \in (0,1)$, depending only on $\lambda$ and geometry, such that the collection $\{\mathcal{U}_Q\}_{Q \in \mathbb{D}(E)}$ associated with $\mathbb{W}_\lambda(\mathscr{X} \setminus E)$ and $C_*$ as in* (2.95)-(2.96)*, satisfies*

$$\big\{x \in \mathscr{X} \setminus E : \, \delta_E(x) < \epsilon \, \mathrm{diam}_\rho(E)\big\} \subseteq \bigcup_{Q \in \mathbb{D}(E)} \mathcal{U}_Q. \tag{2.98}$$

*Proof.* If $\mathrm{diam}_\rho(E) = \infty$, then both sides of (2.98) are equal to $\mathscr{X} \setminus E$ for all $\epsilon \in (0,1)$, since the Whitney cubes cover $\mathscr{X} \setminus E$, so the result is immediate. Now assume that $\mathrm{diam}_\rho(E) < \infty$. Fix some integer $N \in \mathbb{N}$, to be specified later, and consider an arbitrary point $x \in \mathscr{X} \setminus E$ with $\delta_E(x) < 2^{-N}\mathrm{diam}_\rho(E)$. Then by (2.2) and the definition of $\kappa_E$, we have $0 < \delta_E(x) < 2^{-N-\kappa_E}$, hence there exists $k \in \mathbb{Z}$ with $k \geq \kappa_E$ such that $2^{-N-k-1} \leq \delta_E(x) < 2^{-N-k}$. Now, select a ball $I = B_{\rho_\#}(x_I, \ell(I)) \in \mathbb{W}_\lambda(\mathscr{X} \setminus E)$ such that $x \in I$. Then, by *(3)* in Proposition 2.5, there exists $z \in E$ such that $\rho_\#(x_I, z) < \Lambda \ell(I)$. Consequently,

$$\delta_E(x) \leq \rho_\#(x,z) \leq C_\rho \max\{\rho_\#(x, x_I), \rho_\#(x_I, z)\} < C_\rho \Lambda \ell(I). \tag{2.99}$$

In addition, *(3)* in Proposition 2.5 implies that $B_{\rho_\#}(x_I, \lambda\ell(I)) \subseteq \mathscr{X} \setminus E$, hence, for every $y \in E$

$$2C_\rho \ell(I) \leq \lambda \ell(I) \leq \rho_\#(x_I, y) \leq C_\rho\, \ell(I) + C_\rho\, \rho_\#(x,y). \tag{2.100}$$

After canceling like-terms in (2.100) and taking the infimum over all $y \in E$, we arrive at

$$\ell(I) \leq \delta_E(x). \tag{2.101}$$

Next, since $\delta_E(x) < 2^{-N-k}$, there exists $x_0 \in E$ such that $\rho_\#(x, x_0) < 2^{-N-k}$. Furthermore, by invoking *(7)* in Proposition 2.11 we may choose $Q \in \mathbb{D}_k(E)$ with the property that $B_{\rho_\#}(x_0, 2^{-N-k}) \cap Q$ contains at least one point $x_1$. Thus, by *(1)* in Theorem 2.1 we have

$$\begin{aligned}\mathrm{dist}_\rho(I, Q) &\leq \mathrm{dist}_\rho(x, Q) \leq \rho(x, x_1) \leq C_\rho^2 \rho_\#(x, x_1) \\ &\leq C_\rho^2 C_{\rho_\#} \max\{\rho_\#(x, x_0), \rho_\#(x_0, x_1)\} < C_\rho^3 2^{-N-k} = C_\rho^3 2^{-N} \ell(Q).\end{aligned} \tag{2.102}$$

Starting with (2.101) and keeping in mind that $\delta_E(x) < 2^{-N-k}$, we obtain

$$\ell(I) < 2^{-N-k} = 2^{-N}\ell(Q) \leq \ell(Q). \tag{2.103}$$

Next, by (2.99) we write $2^{-N-1}\ell(Q) = 2^{-N-k-1} \leq \delta_E(x) \leq C_\rho \Lambda \ell(I)$, which further entails

$$C_* \geq 2^{N+1} C_\rho \Lambda \implies \ell(I) \geq C_*^{-1}\ell(Q). \tag{2.104}$$

Finally, if $C_* \geq 4C_\rho^4\,\Lambda$, then by (2.102)-(2.104), upon choosing $N \in \mathbb{N}$ such that

$$N - 1 \leq \log_2\big(C_\rho^3\big) < N, \tag{2.105}$$

it follows that $I \in W_Q$, hence $x \in I \subseteq \mathcal{U}_Q$, and so (2.98) holds with $\epsilon := 2^{-N}$. □

We now return to the context introduced in the first paragraph of this subsection, where $\lambda \in [2C_\rho, \infty)$ and $C_* \in [1, \infty)$. Then there exists $C_o \in [1, \infty)$ such that

$$C_o^{-1}\ell(Q) \leq \delta_E(x) \leq C_o\ell(Q), \qquad \forall\, Q \in \mathbb{D}(E) \text{ and } \forall\, x \in \mathcal{U}_Q. \tag{2.106}$$

Indeed, an inspection of (2.99), (2.101), (2.95) and (2.96) shows that (2.106) holds when

$$C_o := C_* C_\rho \Lambda, \tag{2.107}$$

where $\Lambda$ is the constant associated with $\lambda$ as in Proposition 2.5. We will need the containments below for the dyadic Carleson tents $\{T_E(Q)\}_{Q\in\mathbb{D}(E)}$ from (2.97).

**Lemma 2.18.** *Assume all of the hypotheses contained in the first paragraph of Lemma 2.17. If $C_* \in [1, \infty)$, then there exists $C \in (0, \infty)$, depending only on $C_*$ and $\rho$, such that*

$$T_E(Q) \subseteq B_\rho\big(x, C\ell(Q)\big) \setminus E, \quad \forall\, Q \in \mathbb{D}(E),\ \forall\, x \in Q. \tag{2.108}$$

*If $C_* \in [4C_\rho^4\,\Lambda, \infty)$, then there exists $\varepsilon \in (0,1)$, depending only on $\lambda$ and geometry, such that*

$$B_{\rho_\#}\big(x_Q, \varepsilon\ell(Q)\big) \setminus E \subseteq T_E(Q), \qquad \forall\, Q \in \mathbb{D}(E). \tag{2.109}$$

*Proof.* The containment in (2.108) follows from (2.97). To prove (2.109), let $C_* \in [4C_\rho^4\,\Lambda, \infty)$. Fix $\varepsilon \in (0,1)$ to be specified later and choose $N$ as in (2.105). Let $Q \in \mathbb{D}(E)$ and fix $x \in B_{\rho_\#}\big(x_Q, \varepsilon\ell(Q)\big) \setminus E$. Then $\rho_\#(x_Q, x) < \varepsilon\ell(Q)$. We now restrict $\varepsilon < 2^{-N-1}$ to obtain

$$\delta_E(x) \leq \rho_\#(x_Q, x) < \varepsilon\ell(Q) \leq 2\varepsilon\,\mathrm{diam}_\rho(E) < 2^{-N}\mathrm{diam}_\rho(E). \tag{2.110}$$

Thus, as in the first part of the proof of Lemma 2.17, we have $\delta_E(x) < \min\{2^{-N-k}, \varepsilon\ell(Q)\}$ for some $k \in \mathbb{Z}$, $k \geq \kappa_E$. Hence, there exists $x_0 \in E$ such that $\rho_\#(x, x_0) < \min\{2^{-N-k}, \varepsilon\ell(Q)\}$. By property *(7)* in Proposition 2.11, we choose $Q' \in \mathbb{D}_k(E)$ such that $B_{\rho_\#}\big(x_0, \varepsilon\ell(Q)\big) \cap Q' \neq \emptyset$.

If, in addition, $\varepsilon < a_0 C_\rho^{-4}$, then

$$B_{\rho_\#}\big(x_0, \varepsilon\ell(Q)\big) \cap E \subseteq Q \quad \text{ and } \quad Q' \cap Q \neq \emptyset, \tag{2.111}$$

since if $y \in B_{\rho_\#}\big(x_0, \varepsilon\ell(Q)\big) \cap E$, then

$$\rho_\#(x_Q, y) \leq C_\rho \max\{C_\rho \max\{\rho_\#(x_0, x), \rho_\#(x, x_Q)\}, \rho_\#(x_0, y)\} < \varepsilon C_\rho^2 \ell(Q) \tag{2.112}$$

shows that $y \in Q$ (recall that $B_{\rho_\#}\big(x_Q, a_0 C_\rho^{-2}\ell(Q)\big) \cap E \subseteq Q$ by (2.30)) and so $Q' \cap Q \neq \emptyset$. Using the reasoning in the proof of Lemma 2.17 that yielded (2.102)-(2.105) we obtain

$$x \in I \subseteq \mathcal{U}_{Q'}. \tag{2.113}$$

Thus, using also (2.106), we have

$$\ell(Q') \leq C_o \delta_E(x) \leq C_o \rho_\#(x_Q, x) < C_o \varepsilon \ell(Q). \tag{2.114}$$

Hence, under the additional restriction $\varepsilon < C_o^{-1}$, we have $\ell(Q') < \ell(Q)$, which when combined with (2.111) and *(3)* in Proposition 2.11, forces $Q' \subseteq Q$. In concert with (2.113) and (2.97), this shows that $x \in T_E(Q)$ when $0 < \varepsilon < \min\{2^{-N-1}, a_0 C_\rho^{-4}, C_o^{-1}\}$, as required. □

We conclude this section with a finite overlap property for the sets $\{\mathcal{U}_Q\}_{Q\in\mathbb{D}(E)}$ from (2.96).

**Lemma 2.19.** *Let $(\mathscr{X}, \rho)$ be a geometrically doubling quasi-metric space and suppose that $E$ is a nonempty, closed, proper subset of $(\mathscr{X}, \tau_\rho)$. Fix $a \in [1,\infty)$, a collection $\mathbb{D}(E)$ of dyadic cubes in $E$ as in Proposition 2.11, and $C_* \in [1,\infty)$.*

*If $\lambda \in [a,\infty)$, and we fix a Whitney covering $\mathbb{W}_\lambda(\mathscr{X}\setminus E)$ of $\mathscr{X}\setminus E$ as in Proposition 2.5, with Whitney cubes $\{W_Q\}_{Q\in\mathbb{D}(E)}$ as in* (2.95)*, then there exists $N \in \mathbb{N}$, depending only on $\lambda$, $C_*$ and geometry, such that*

$$\sum_{Q\in\mathbb{D}(E)} \mathbf{1}_{\mathcal{U}_Q^*} \leq N, \tag{2.115}$$

*where $\mathcal{U}_Q^* := \bigcup_{I\in W_Q} aI$ (compare with* (2.96)*).*

*Proof.* Fix $a \in [1,\infty)$, $\lambda \in [a,\infty)$ and a Whitney covering $\mathbb{W}_\lambda(\mathscr{X}\setminus E)$, so by Proposition 2.5 we have $\sum_{I\in\mathbb{W}_\lambda(\mathscr{X}\setminus E)} \mathbf{1}_{\lambda I} \leq N_1$ for some $N_1 \in \mathbb{N}$. Now define $\mathcal{I} := \bigcup_{Q\in\mathbb{D}(E)} W_Q \subseteq \mathbb{W}_\lambda(\mathscr{X}\setminus E)$ and, for each $I \in \mathcal{I}$, set $q_I := \{Q \in \mathbb{D}(E) : I \in W_Q\}$. We claim that $\#\, q_I \leq N_2$ for all $I \in \mathbb{W}_\lambda(\mathscr{X}\setminus E)$ and some $N_2 \in \mathbb{N}$. To see this, consider $I \in \mathbb{W}_\lambda(\mathscr{X}\setminus E)$ and $Q \in \mathbb{D}(E)$ such that $I \in W_Q$. Then, from (2.95) we deduce that

$$C_*^{-1}\ell(I) \leq \ell(Q) \leq C_*\ell(I) \quad \text{and} \quad \operatorname{dist}_\rho(I,Q) \leq C_*\ell(I), \tag{2.116}$$

and the claim follows from the fact that $\big(E, \rho\big|_E\big)$ is geometrically doubling. We then have

$$\sum_{Q\in\mathbb{D}(E)} \mathbf{1}_{\mathcal{U}_Q^*} \leq \sum_{Q\in\mathbb{D}(E)} \sum_{I\in W_Q} \mathbf{1}_{\lambda I} = \sum_{I\in\mathcal{I}} (\#\, q_I)\cdot \mathbf{1}_{\lambda I} \leq N_1 N_2, \tag{2.117}$$

so (2.115) holds with $N := N_1 N_2$. □

# 3 $T(1)$ and local $T(b)$ Theorems for Square Functions

This section consists of two parts, dealing with a $T(1)$ Theorem and a local $T(b)$ Theorem for square functions on sets of arbitrary co-dimension, relative to an ambient quasi-metric space (the notion of dimension refers to the degree of Ahlfors-David regularity). The $T(1)$ Theorem generalizes the Euclidean co-dimension one result proved by M. Christ and J.-L. Journé in [16] (cf. also [14, Theorem 20]). The local $T(b)$ Theorem generalizes the Euclidean co-dimension one result implicit in the solution of the Kato problem in [46, 42, 4] and explicit in [3, 40, 47].

We consider the following context. Fix two real numbers $d, m$ such that $0 < d < m$, an $m$-dimensional ADR space $(\mathscr{X}, \rho, \mu)$, a closed subset $E$ of $(\mathscr{X}, \tau_\rho)$, and a Borel measure $\sigma$ on $(E, \tau_{\rho|_E})$ with the property that $(E, \rho\big|_E, \sigma)$ is a $d$-dimensional ADR space. Suppose that

$$\begin{gathered}\theta : (\mathscr{X} \setminus E) \times E \longrightarrow \mathbb{R} \quad \text{is Borel measurable with respect to} \\ \text{the relative topology induced by the product topology } \tau_\rho \times \tau_\rho \text{ on } (\mathscr{X} \setminus E) \times E,\end{gathered} \tag{3.1}$$

and has the property that there exist $C_\theta$, $\alpha$, $\upsilon \in (0, \infty)$ and $a \in [0, \upsilon)$ such that for all $x \in \mathscr{X} \setminus E$, $y \in E$ and $\widetilde{y} \in E$ with $\rho(y, \widetilde{y}) \leq \frac{1}{2}\rho(x, y)$, the following hold:

$$|\theta(x, y)| \leq \frac{C_\theta}{\rho(x, y)^{d+\upsilon}} \Big(\frac{\operatorname{dist}_\rho(x, E)}{\rho(x, y)}\Big)^{-a}, \tag{3.2}$$

$$|\theta(x, y) - \theta(x, \widetilde{y})| \leq C_\theta \frac{\rho(y, \widetilde{y})^\alpha}{\rho(x, y)^{d+\upsilon+\alpha}} \Big(\frac{\operatorname{dist}_\rho(x, E)}{\rho(x, y)}\Big)^{-a-\alpha}. \tag{3.3}$$

Then define the integral operator $\Theta$ for all functions $f \in L^p(E, \sigma)$, $1 \leq p \leq \infty$, by

$$(\Theta f)(x) := \int_E \theta(x, y) f(y)\, d\sigma(y), \qquad \forall\, x \in \mathscr{X} \setminus E. \tag{3.4}$$

We note that Lemma 3.4 below guarantees that this integral is absolutely convergent. Also, the terms in parentheses in (3.2)-(3.3) are greater than or equal to 1, since $\rho(x, y) \geq \operatorname{dist}_\rho(x, E) > 0$, and so the inclusion of these terms weakens the hypotheses.

We proceed to prove square function versions of the $T(1)$ Theorem and the local $T(b)$ Theorem for $\Theta$. As usual, we prove the local $T(b)$ Theorem by applying the $T(1)$ Theorem.

## 3.1 An arbitrary codimension $T(1)$ theorem for square functions

The main result in this subsection is the following $T(1)$ theorem for square functions.

**Theorem 3.1.** *Let $0 < d < m < \infty$. Assume that $(\mathscr{X}, \rho, \mu)$ is an $m$-dimensional* ADR *space, $E$ is a closed subset of $(\mathscr{X}, \tau_\rho)$, and $\sigma$ is a Borel semiregular measure on $(E, \tau_{\rho|_E})$ with the property that $(E, \rho\big|_E, \sigma)$ is a $d$-dimensional* ADR *space.*

*Suppose that $\Theta$ is an integral operator with kernel $\theta$ satisfying* (3.1)-(3.4)*. Let $\mathbb{D}(E)$ denote a dyadic cube structure on $E$, consider a Whitney covering $\mathbb{W}_\lambda(\mathscr{X} \setminus E)$ of $\mathscr{X} \setminus E$ and a constant $C_*$ as in Lemma 2.17 with the corresponding dyadic Carleson tents from* (2.97)*. If*

$$\sup_{Q \in \mathbb{D}(E)} \left( \frac{1}{\sigma(Q)} \int_{T_E(Q)} |\Theta 1(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \right) < \infty, \tag{3.5}$$

*then there exists $C \in (0, \infty)$, depending only on $C_\theta$, the* ADR *constants of $E$ and $\mathscr{X}$, and the value of the supremum in* (3.5)*, such that*

$$\int_{\mathscr{X} \setminus E} |(\Theta f)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \leq C \int_E |f(x)|^2\, d\sigma(x), \quad \forall\, f \in L^2(E, \sigma). \tag{3.6}$$

*Conversely, under the original background assumptions, excluding* (3.3)*, if*

$$\int_{x \in \mathscr{X},\, 0 < \delta_E(x) < \eta \operatorname{diam}_\rho(E)} |(\Theta f)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \leq C \int_E |f(x)|^2\, d\sigma(x),\ \forall\, f \in L^2(E, \sigma), \tag{3.7}$$

*for some $C, \eta \in (0, \infty)$, then* (3.5) *holds.*

We record some preliminaries. The following discrete Carleson Estimate is well-known.

**Lemma 3.2.** *Let $(E,\rho,\sigma)$ be a space of homogeneous type and denote by $\mathbb{D}(E)$ a dyadic cube structure on $E$. If a sequence $\{B_Q\}_{Q\in\mathbb{D}(E)}\subseteq[0,\infty]$ satisfies the discrete Carleson Condition*

$$C:=\sup_{R\in\mathbb{D}(E)}\Big[\frac{1}{\sigma(R)}\sum_{Q\in\mathbb{D}(E),\,Q\subseteq R}B_Q\Big]<\infty, \tag{3.8}$$

*then for every sequence $\{A_Q\}_{Q\in\mathbb{D}(E)}\subseteq\mathbb{R}$ it holds that*

$$\sum_{Q\in\mathbb{D}(E)}A_QB_Q\leq C\int_E A^*\,d\sigma, \tag{3.9}$$

*where $A^*(x):=\sup_{Q\in\mathbb{D}(E),\,x\in Q}|A_Q|$ if $x\in\bigcup_{Q\in\mathbb{D}(E)}Q$ and $A^*(x):=0$ otherwise.*

The following quantitative version of the classical Urysohn Lemma is in [61] (cf. also [1]).

**Lemma 3.3.** *Let $(E,\rho)$ be a quasi-metric space and suppose that $0<\beta\leq[\log_2 C_\rho]^{-1}$. If $F_0,F_1\subseteq E$ are nonempty and $\mathrm{dist}_\rho(F_0,F_1)>0$, then there exists $\eta:E\to\mathbb{R}$ such that*

$$0\leq\eta\leq 1\quad\text{on}\quad E,\qquad \eta\equiv 0\quad\text{on}\quad F_0,\qquad \eta\equiv 1\quad\text{on}\quad F_1, \tag{3.10}$$

*and for which there exists a finite constant $C>0$, depending only on $\rho$, such that*

$$\sup_{x,y\in E,\,x\neq y}\frac{|\eta(x)-\eta(y)|}{\rho(x,y)^\beta}\leq C\big(\mathrm{dist}_\rho(F_0,F_1)\big)^{-\beta}. \tag{3.11}$$

The next two preliminary lemmas are from geometric measure theory.

**Lemma 3.4.** *Let $(\mathscr{X},\rho)$ be a quasi-metric space. Suppose that $E\subseteq\mathscr{X}$ is nonempty and $\sigma$ is a measure on $E$ such that $(E,\rho\big|_E,\sigma)$ is a $d$-dimensional* ADR *space for some $d>0$. If $m>d$, then there exists $C\in(0,\infty)$, depending only on $m$, $\rho$, and the* ADR *constant of $E$, such that*

$$\int_E\frac{1}{\rho_\#(x,y)^m}\,d\sigma(y)\leq C\delta_E(x)^{d-m},\qquad\forall\,x\in\mathscr{X}\setminus E. \tag{3.12}$$

*Moreover, if $\varepsilon>0$ and $c>0$, then there exists $C\in(0,\infty)$, depending only on $\varepsilon$, $c$, $\rho$, and the* ADR *constant of $E$, such that*

$$\int_{y\in E,\,\rho_\#(y,x)>cr}\frac{r^\varepsilon}{\rho_\#(y,x)^{d+\varepsilon}}f(y)\,d\sigma(y)\leq C\,M_E(f)(x),\qquad\forall\,x\in E,\quad\forall\,r>0, \tag{3.13}$$

*for every $\sigma$-measurable function $f:E\to[0,\infty]$, where $M_E$ is as in (2.29).*

*Proof.* If $m>d$, then since $(E,\rho_\#|_E,\sigma)$ is a $d$-dimensional ADR space, we have

$$\begin{aligned}\int_E\frac{1}{\rho_\#(y,x)^m}\,d\sigma(y)&\leq\sum_{j=0}^\infty\int_E\mathbf{1}_{\{z:\,\rho_\#(z,x)\in[2^j\delta_E(x),2^{j+1}\delta_E(x))\}}(y)\frac{1}{\rho_\#(y,x)^m}\,d\sigma(y)\\&\leq C\sum_{j=0}^\infty\frac{1}{[2^j\delta_E(x)]^m}\sigma\left(B_{\rho_\#}(x,2^{j+1}\delta_E(x))\cap E\right)\\&\leq C\delta_E(x)^{d-m}\end{aligned} \tag{3.14}$$

for all $x\in\mathscr{X}\setminus E$, which proves (3.12). The estimate in (3.13) is proved similarly by decomposing the domain of integration in dyadic annuli centered at $x$ and at scale $r$. □

The proof of the following result can be found in [62].

**Lemma 3.5.** *Let $0 < d < m < \infty$. Assume that $(\mathscr{X}, \rho, \mu)$ is an $m$-dimensional* ADR *space, $E$ is a closed nonempty subset of $(\mathscr{X}, \tau_\rho)$, and there exists a measure $\sigma$ on $E$ such that $(E, \rho\big|_E, \sigma)$ is a $d$-dimensional* ADR *space. If $\gamma < m - d$, then there exists $C_0 \in (0, \infty)$, depending only on $\gamma$ and the* ADR *constants of $E$ and $\mathscr{X}$, such that*

$$\int_{x \in B_\rho(x_0, R),\, \delta_E(x) < r} \delta_E(x)^{-\gamma}\, d\mu(x) \leq C_0\, r^{m-d-\gamma} R^d, \tag{3.15}$$

*for every $x_0 \in E$ and every $r, R > 0$.*

At this stage, we are ready to present the proof of Theorem 3.1.

*Proof of Theorem 3.1.* Set $A_Q f := \fint_Q f\, d\sigma$ for all $Q \in \mathbb{D}(E)$ and measurable $f : E \to \mathbb{C}$.

`Step I.` *We claim that for each $r \in (1, \infty)$, there exist constants $C, \beta \in (0, \infty)$ such that*

$$\sup_{x \in \mathcal{U}_Q} \big|\delta_E(x)^\upsilon \big(\Theta(D_l g)(x) - (\Theta 1)(x) A_Q(D_l g)\big)\big| \leq C 2^{-|k-l|\beta} \inf_{w \in Q} \Big[M_E^2(|g|^r)(w)\Big]^{\frac{1}{r}}, \tag{3.16}$$

*for all $l, k \in \mathbb{Z}$ with $l, k \geq \kappa_E$, all $Q \in \mathbb{D}_k(E)$ and all locally integrable functions $g : E \to \mathbb{R}$, where $D_l := \mathcal{S}_{l+1} - \mathcal{S}_l$ is the integral operator defined in Proposition 2.14 with kernel $h_l(\cdot, \cdot)$.* We prove (3.16) by distinguishing two cases below. Fix $k_0 \in \mathbb{N}_0$ to be specified later.

*Case I: $k + k_0 \geq l$.* In this case, since $|k - l| \leq k - l + 2k_0$, we have

$$2^{-(k+k_0-l)} \approx 2^{-|k-l|}, \tag{3.17}$$

where the comparability constants depend only on $k_0$. For all $x \in \mathcal{U}_Q$, we write

$$\delta_E(x)^\upsilon \big(\Theta(D_l g)(x) - (\Theta 1)(x) A_Q(D_l g)\big) = \int_E \left[\int_E \Phi(x, y) h_l(y, z)\, d\sigma(y)\right] g(z)\, d\sigma(z), \tag{3.18}$$

where

$$\Phi(x, y) := \delta_E(x)^\upsilon \Big[\theta(x, y) - \tfrac{1}{\sigma(Q)}(\Theta 1)(x) \mathbf{1}_Q(y)\Big], \quad \forall\, x \in \mathscr{X} \setminus E, \quad \forall\, y \in E. \tag{3.19}$$

Note that, by design,

$$\int_E \Phi(x, y)\, d\sigma(y) = 0, \qquad \forall\, x \in \mathscr{X} \setminus E, \tag{3.20}$$

and we claim that

$$|\Phi(x, y)| \leq \frac{C}{\sigma(Q)}, \qquad \forall\, x \in \mathcal{U}_Q, \quad \forall\, y \in E. \tag{3.21}$$

Indeed, if $x \in \mathcal{U}_Q$, then $\delta_E(x) \approx \ell(Q)$, so (3.2) and *(4)* in Theorem 2.1 imply that

$$\delta_E(x)^\upsilon |\theta(x, y)| \leq \frac{C \delta_E(x)^{\upsilon - a}}{\rho(x, y)^{d+\upsilon-a}} \leq \frac{C \delta_E(x)^{\upsilon - a}}{\delta_E(x)^{d+\upsilon-a}} \leq \frac{C}{\ell(Q)^d} \leq \frac{C}{\sigma(Q)}, \qquad \forall\, y \in E, \tag{3.22}$$

whilst (3.12) implies that

$$\delta_E(x)^\upsilon|(\Theta 1)(x)| \leq C\delta_E(x)^{\upsilon-a}\int_E \frac{d\sigma(y)}{\rho_\#(x,y)^{d+\upsilon-a}} \leq C, \qquad \forall\, x \in \mathcal{U}_Q. \tag{3.23}$$

Fix $\varepsilon \in (0,1)$ and $C_0 > 0$, to be specified later. If $w \in Q$ and $z \in E$, then (3.20) gives

$$\left|\int_E \Phi(x,y)h_l(y,z)\,d\sigma(y)\right| \leq \int_E |\Phi(x,y)||h_l(y,z) - h_l(w,z)|\,d\sigma(y) =: I_1 + I_2, \tag{3.24}$$

where $I_1$ is the integral restricted to the set $\{y \in E,\ \rho_\#(y,x_Q) \leq C_0 2^{(k+k_0-l)\varepsilon}\ell(Q)\}$ and $x_Q$ is the center of $Q$. To estimate $I_1$, we claim that, for $\gamma$ from (2.72), if $\widetilde{C}$ is sufficiently large, then

$$\begin{aligned} &|h_l(y,z) - h_l(w,z)| \leq C\,2^{\,l(d+\gamma)}2^{(k+k_0-l)\varepsilon\gamma}\ell(Q)^\gamma \mathbf{1}_{\{\rho_\#(w,\cdot)\leq \widetilde{C}2^{-l}\}}(z) \\ &\text{whenever } z \in E,\ y \in E,\ w \in Q \text{ and } \rho_\#(y,x_Q) \leq C_0 2^{(k+k_0-l)\varepsilon}\ell(Q). \end{aligned} \tag{3.25}$$

To prove this claim, first note that if $C_0$ is large, then since $k + k_0 - l \geq 0$, we have

$$y \in E,\ w \in Q \text{ and } \rho_\#(y,x_Q) \leq C_0 2^{(k+k_0-l)\varepsilon}\ell(Q) \ \Rightarrow\ \rho_\#(y,w) \leq CC_0 2^{(k+k_0-l)\varepsilon}\ell(Q). \tag{3.26}$$

From now on, assume that $C_0$ is large enough to ensure the validity of (3.26). Second, if $y, w$ are as in (3.26) and $z \in E$ is such that $\rho_\#(z,w) \geq \widetilde{C}2^{-l}$, then

$$\widetilde{C}2^{-l} \leq \rho_\#(z,w) \leq C\rho_\#(z,y) + 2^{k_0}C_0 C 2^{-l}, \tag{3.27}$$

for some geometric constant $C > 0$. Setting $\widetilde{C} := 2^{k_0+1}C_0 C$ and $C_1 := 2^{k_0}C_0 C$, we now have

$$\left.\begin{array}{r} y \in E,\ w \in Q,\ \rho_\#(y,x_Q) \leq C_0 2^{(k+k_0-l)\varepsilon}\ell(Q) \\ \text{and } \ \rho_\#(z,w) \geq \widetilde{C}2^{-l} \end{array}\right\} \Longrightarrow \rho_\#(z,y) > C_1 2^{-l}. \tag{3.28}$$

Moreover, we can further increase $C_0$ and, in turn, $\widetilde{C}$ to insure that the constant $C_1$ in the last inequality in (3.28) is larger that the constant $C$ in (2.72). Henceforth, assume that such a choice has been made. Then a combination of (3.26), (3.28) and (2.73) yields (3.25).

Next, we choose $0 < \varepsilon < \frac{\gamma}{d+\gamma} < 1$, which ensures that $\beta_1 := \gamma - \varepsilon(d+\gamma) > 0$, and use (3.21), (3.25) and the $d$-ADR property to estimate

$$I_1 \leq C\,2^{\,dl}\,2^{-(k+k_0-l)\beta_1}\,\mathbf{1}_{\{\rho_\#(w,\cdot)\leq \widetilde{C}2^{-l}\}}(z), \qquad \forall\, z \in E. \tag{3.29}$$

To estimate the contribution of $I_1$ in (3.18), based on (3.29) and (3.17), we obtain

$$\int_E I_1|g(z)|\,d\sigma(z) \leq C2^{-(k+k_0-l)\beta_1} \fint_{z\in E,\,\rho_\#(z,w)\leq C2^{-l}} |g(z)|\,d\sigma(z) \leq C2^{-|k-l|\beta_1}M_E g(w), \tag{3.30}$$

uniformly for all $w \in Q$.

Next, we turn our attention to $I_2$ from (3.24). Note that since we are currently assuming that $k + k_0 \geq l$, the condition $\rho_\#(y,x_Q) \geq C_0 2^{(k+k_0-l)\varepsilon}\ell(Q)$ forces $y \notin c_1 Q$ for some finite positive constant $c_1$, which may be further increased as desired by suitably increasing the value of $C_0$. Thus, assuming that $C_0$ is sufficiently large to guarantee $c_1 > 1$, we obtain $\mathbf{1}_Q(y) = 0$

if $y \in E$ and $\rho_\#(y, x_Q) \geq C_0 2^{(k+k_0-l)\varepsilon}\ell(Q)$. In turn, this implies that $\Phi(x,y) = \delta_E(x)^\upsilon \theta(x,y)$ on the domain of integration in $I_2$. Thus, for each $z \in E$, we have

$$\begin{aligned} I_2 &\leq C2^{-k\upsilon} \int_{y\in E,\, \rho_\#(y,x_Q)>C_0 2^{(k+k_0-l)\varepsilon}\ell(Q)} |\theta(x,y)|\,|h_l(y,z)|\,d\sigma(y) \\ &\quad + C2^{-k\upsilon}|h_l(w,z)| \int_{y\in E,\, \rho_\#(y,x_Q)>C_0 2^{(k+k_0-l)\varepsilon}\ell(Q)} |\theta(x,y)|\,d\sigma(y) =: I_3 + I_4. \end{aligned} \tag{3.31}$$

We also remark that the design of $\mathcal{U}_Q$ and the fact that $k + k_0 - l \geq 0$ ensure that

$$y \in E,\ \ \rho_\#(y,x_Q) > C_0 2^{(k+k_0-l)\varepsilon}\ell(Q) \implies \begin{cases} \rho_\#(x,y) \approx \rho_\#(w,y) \approx \rho_\#(x_Q,y), \\ \text{uniformly for } x \in \mathcal{U}_Q \text{ and } w \in Q. \end{cases} \tag{3.32}$$

Using (3.2), (3.32), the comparability $\delta_E(x) \approx \ell(Q)$ for $x \in \mathcal{U}_Q$, and (2.72), we have

$$\begin{aligned} I_3 &\leq C\,2^{-k\upsilon} \int_{y\in E,\, \rho_\#(y,w)>C2^{(k+k_0-l)\varepsilon}\ell(Q)} \frac{\delta_E(x)^{-a}}{\rho_\#(y,w)^{d+\upsilon-a}}\,|h_l(y,z)|\,d\sigma(y) \\ &\leq C\,2^{-(k+k_0-l)\varepsilon(\upsilon-a)} 2^{dl} \int_{y\in E,\, \rho_\#(y,w)\geq Cr} \frac{r^{\upsilon-a}}{\rho_\#(y,w)^{d+\upsilon-a}}\,\mathbf{1}_{\{\rho_\#(y,\cdot)\leq C2^{-l}\}}(z)\,d\sigma(y), \end{aligned} \tag{3.33}$$

for all $z \in E$, where $r := 2^{(k+k_0-l)\varepsilon-k}$. Then, by (3.13) and (3.17), we obtain

$$\begin{aligned} \int_E I_3 |g(z)|\,d\sigma(z) &\leq C\,2^{-(k+k_0-l)\varepsilon(\upsilon-a)} \int_{y\in E,\, \rho_\#(y,w)\geq Cr} \frac{r^{\upsilon-a}}{\rho_\#(y,w)^{d+\upsilon-a}}\,(M_E g)(y)\,d\sigma(y) \\ &\leq C\,2^{-|k-l|\varepsilon(\upsilon-a)}(M_E^2 g)(w), \quad \text{uniformly for all } w \in Q. \end{aligned} \tag{3.34}$$

Similarly, we have

$$I_4 \leq C2^{-(k+k_0-l)\varepsilon(\upsilon-a)}\,2^{dl}\mathbf{1}_{\{\rho_\#(w,\cdot)\leq C2^{-l}\}}(z) \int_{y\in E,\, \rho_\#(y,x_Q)>Cr} \frac{r^{\upsilon-a}}{\rho_\#(y,x_Q)^{d+\upsilon-a}}\,d\sigma(y) \tag{3.35}$$

for all $z \in E$. Then, by (3.13) and (3.17), we obtain

$$\int_E I_4|g(z)|\,d\sigma(z) \leq C\,2^{-|k-l|\varepsilon(\upsilon-a)}(M_E g)(w), \quad \text{uniformly for all } w \in Q. \tag{3.36}$$

If $r \in [1, \infty)$, then Lebesgue's Differentiation Theorem, which holds since $\sigma$ is Borel semiregular (see [2]), combined with the monotonicity of $M_E$, and Hölder's inequality, implies that

$$M_E g \leq \big[M_E^2(|g|^r)\big]^{\frac{1}{r}} \quad \text{and} \quad M_E^2 g \leq \big[M_E^2(|g|^r)\big]^{\frac{1}{r}} \qquad \text{pointwise in } E. \tag{3.37}$$

Thus, in *Case I*, (3.30), (3.34), (3.36) prove (3.16) with $\beta$ given by $\beta_2 := \min\{\beta_1, \varepsilon(\upsilon - a)\} > 0$.

*Case II: $k + k_0 < l$.* In this case, we write

$$\delta_E(x)^\upsilon \Theta(D_l g)(x) = \int_E \Psi(x,z)\,g(z)\,d\sigma(z), \qquad \forall\, x \in \mathcal{U}_Q, \tag{3.38}$$

where

$$\Psi(x,z) := \delta_E(x)^\upsilon \int_E \theta(x,y) h_l(y,z)\,d\sigma(y), \qquad \forall\, x \in \mathscr{X} \setminus E, \quad \forall\, z \in E. \tag{3.39}$$

To proceed, fix $x \in \mathcal{U}_Q$ arbitrary. Based on (2.74) and (2.72), we have

$$|\Psi(x,z)| \leq \delta_E(x)^\upsilon \int_{y\in E,\, \rho_\#(y,z)\leq C2^{-l}} |\theta(x,y)-\theta(x,z)||h_l(y,z)|\,d\sigma(y), \qquad \forall\, z \in E. \tag{3.40}$$

As a consequence of (3.3), we have

$$|\theta(x,y)-\theta(x,z)| \leq C\,\frac{\rho(y,z)^\alpha \delta_E(x)^{-a-\alpha}}{\rho(x,y)^{d+\upsilon-a}} \qquad \text{if } \; y,z\in E,\; \rho(y,z) < \tfrac{1}{2}\rho(x,y). \tag{3.41}$$

Observe that, since here $k+k_0<l$, if the points $y,z\in E$ are such that $\rho_\#(y,z)\leq C2^{-l}$, then

$$\rho(y,z) \leq C\rho_\#(y,z) \leq C2^{-l} \leq C2^{-k_0}2^{-k} \leq C2^{-k_0}\delta_E(x) \leq C2^{-k_0}\rho(x,y) < \tfrac{1}{2}\rho(x,y), \tag{3.42}$$

where the last inequality follows by choosing $k_0$ large. We henceforth fix such a $k_0\in\mathbb{N}_0$. Then (3.41) holds for all $y$ in (3.40). For $w\in Q$ arbitrary, we claim that

$$\left.\begin{array}{l}\exists\, C'>0 \text{ such that } \forall\, z,y\in E\\ \text{satisfying } \rho(y,z)\leq C2^{-l}\end{array}\right\} \text{ it holds that } \left\{\begin{array}{l}\rho(x,y)\geq C'[2^{-k}+\rho_\#(w,z)],\\ \text{uniformly for } x\in\mathcal{U}_Q,\ w\in Q.\end{array}\right. \tag{3.43}$$

Indeed, if $y,z$ are as in the left side of (3.43), then $\rho(x,y)\geq C\delta_E(x)\approx \ell(Q)\approx 2^{-k}$ and

$$\rho_\#(w,z) \leq C\rho(w,z) \leq C(2^{-k}+\rho(x,y)+2^{-l}) \leq C\rho(x,y). \tag{3.44}$$

This proves (3.43). Combining (3.43), (3.41), (2.72) and (3.40), we obtain

$$\begin{aligned}|\Psi(x,z)| &\leq C2^{-k\upsilon}\int_{y\in E,\,\rho_\#(y,z)\leq C2^{-l}} \frac{2^{k(a+\alpha)}2^{-l\alpha}}{(2^{-k}+\rho_\#(w,z))^{d+\upsilon-a}}\,2^{dl}\,d\sigma(y)\\ &\leq C2^{-|k-l|\alpha}\frac{2^{-k(\upsilon-a)}}{(2^{-k}+\rho_\#(w,z))^{d+\upsilon-a}}, \qquad \forall\, z\in E,\end{aligned} \tag{3.45}$$

where the second inequality uses the $d$-ADR property and that $k<l$. Thus, using (3.13) and (3.37), for each $r\in[1,\infty)$, we obtain

$$\big|\delta_E(x)^\upsilon\Theta(D_l g)(x)\big| \leq C2^{-|k-l|\alpha}\big[M_E^2(|g|^r)(w)\big]^{\frac{1}{r}}, \quad \text{uniformly for all } w\in Q. \tag{3.46}$$

To estimate $|\delta_E(x)^\upsilon(\Theta 1)(x)A_Q(D_l g)|$, first note that (3.23) holds in this case, so we have

$$|\delta_E(x)^\upsilon(\Theta 1)(x)A_Q(D_l g)| \leq C\left|\int_E \frac{1}{\sigma(Q)}\int_Q h_l(y,z)\,d\sigma(y)\; g(z)\,d\sigma(z)\right|. \tag{3.47}$$

To continue, for some fixed $\varepsilon\in(0,1)$, define

$$S_Q := \big\{x\in Q:\ \mathrm{dist}_{\rho_\#}(x,E\setminus Q)\leq C2^{-|k-l|\varepsilon}\ell(Q)\big\} \quad \text{and} \quad F_Q := Q\setminus S_Q. \tag{3.48}$$

Also, consider a function $\eta_Q: E\to\mathbb{R}$ such that $\operatorname{supp}\eta_Q\subseteq Q$, $0\leq\eta_Q\leq 1$, $\eta_Q=1$ on $F_Q$, and

$$|\eta_Q(x)-\eta_Q(y)| \leq C\Big(\frac{\rho(x,y)}{2^{-|k-l|\varepsilon}\ell(Q)}\Big)^\gamma, \qquad \forall\, x\in E,\ \forall\, y\in E, \tag{3.49}$$

for some $\gamma \in (0,1)$. That such a function exists is a consequence of Lemma 3.3. Hence,

$$\left|⨍_Q h_l(y,z)\,d\sigma(y)\right| \le \frac{1}{\sigma(Q)}\left|\int_E \big(\mathbf{1}_Q - \eta_Q(y)\big)h_l(y,z)\,d\sigma(y)\right| + \frac{1}{\sigma(Q)}\left|\int_E \eta_Q(y)h_l(y,z)\,d\sigma(y)\right| =: II_1(z) + II_2(z), \quad \forall\, z \in E. \tag{3.50}$$

Fix $z \in E$. To estimate $II_2(z)$, use (2.74), (2.72), (3.49) and the $d$-ADR property to obtain

$$II_2(z) \le \frac{1}{\sigma(Q)}\int_{y\in E,\,\rho_\#(y,z)\le C2^{-l}} |\eta_Q(y) - \eta_Q(z)||h_l(y,z)|d\sigma(y) \le \frac{1}{\sigma(Q)}\left[\frac{2^{-l}}{2^{-|k-l|\varepsilon}2^{-k}}\right]^\gamma. \tag{3.51}$$

In addition, since whenever $y \in \operatorname{supp}\eta_Q \subseteq Q$ and $\rho_\#(y,z) \le C2^{-l} \le C\ell(Q)$ one necessarily has $\rho_\#(w,z) \le C\ell(Q)$, for all $w \in Q$, it follows that one may strengthen (3.51) to

$$II_2(z) \le C\frac{1}{\sigma(Q)}2^{-|k-l|(1-\varepsilon)\gamma}\mathbf{1}_{\{\rho_\#(w,\cdot)\le C\ell(Q)\}}(z), \qquad \text{for all } w \in Q. \tag{3.52}$$

Hence, by recalling (3.37), for each $r \in [1,\infty)$ and for all $w \in Q$, we further obtain

$$\int_E II_2(z)|g(z)|\,d\sigma(z) \le C2^{-|k-l|(1-\varepsilon)\gamma}(M_E g)(w) \le C2^{-|k-l|(1-\varepsilon)\gamma}\big[M_E^2(|g|^r)(w)\big]^{1/r}. \tag{3.53}$$

To estimate $II_1(z)$, fix $r \in (1,\infty)$ and set $\frac{1}{r} + \frac{1}{r'} = 1$. Property *(8)* of Proposition 2.11 implies that there exists $c > 0$ and $\tau \in (0,1)$ such that $\sigma(S_Q) \le c2^{-|k-l|\varepsilon\tau}\sigma(Q)$. Therefore, since $\operatorname{supp}(\mathbf{1}_Q - \eta_Q) \subseteq S_Q$, by applying (2.72), Hölder's inequality and (3.37), we obtain

$$\begin{aligned}\int_E II_1(z)|g(z)|\,d\sigma(z) &\le C\frac{1}{\sigma(Q)}\int_{S_Q}\int_{z\in E,\,\rho_\#(y,z)\le C2^{-l}} 2^{dl}|g(z)|\,d\sigma(z)\,d\sigma(y)\\ &\le C\frac{1}{\sigma(Q)}\int_Q \mathbf{1}_{S_Q}(y)(M_E g)(y)\,d\sigma(y)\\ &\le C\left[\frac{\sigma(S_Q)}{\sigma(Q)}\right]^{\frac{1}{r'}}\left[⨍_Q (M_E g)^r\,d\sigma\right]^{\frac{1}{r}}\\ &\le C2^{-|k-l|\varepsilon\tau/r'}\big[M_E^2(|g|^r)(w)\big]^{\frac{1}{r}}, \quad \text{for all } w \in Q. \end{aligned} \tag{3.54}$$

Thus, in *Case II*, (3.53) and (3.54) prove (3.16) with $\beta$ given by $\beta_3 := \min\{(1-\varepsilon)\gamma, \frac{\varepsilon\tau}{r'}, \alpha\} > 0$.

This completes Step I, since *Case I* and *Case II* prove (3.16) with $\beta := \min\{\beta_2, \beta_3\} > 0$.

`Step II.` *We claim that there exists $C \in (0,\infty)$ such that, for all $f \in L^2(E,\sigma)$, we have*

$$\sum_{k\in\mathbb{Z},\,k\ge\kappa_E}\ \sum_{Q\in\mathbb{D}_k(E)} \int_{\mathcal{U}_Q} \left|\delta_E(x)^\upsilon\big((\Theta f)(x) - (\Theta 1)(x)A_Q f\big)\right|^2 \frac{d\mu(x)}{\delta_E(x)^{m-d}} \le C\int_E |f|^2\,d\sigma. \tag{3.55}$$

To prove this claim, fix $r \in (1,2)$ and $\beta > 0$ such that (3.16) holds. Then, by (2.62), we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z},\,k\ge\kappa_E}\ \sum_{Q\in\mathbb{D}_k(E)} \int_{\mathcal{U}_Q} \left|\delta_E(x)^\upsilon\big(\Theta - (\Theta 1)(x)A_Q\big)(f)(x)\right|^2 \frac{d\mu(x)}{\delta_E(x)^{m-d}} \qquad (3.56)\\ &\le 2\sum_{k\in\mathbb{Z},\,k\ge\kappa_E}\ \sum_{Q\in\mathbb{D}_k(E)} \int_{\mathcal{U}_Q} \Big|\sum_{l\in\mathbb{Z},\,l\ge\kappa_E} \delta_E(x)^\upsilon\big(\Theta - (\Theta 1)(x)A_Q\big)(D_l\widetilde{D}_l f)(x)\Big|^2 \frac{d\mu(x)}{\delta_E(x)^{m-d}}\\ &\quad + 2\sum_{k\in\mathbb{Z},\,k\ge\kappa_E}\ \sum_{Q\in\mathbb{D}_k(E)} \int_{\mathcal{U}_Q} \left|\delta_E(x)^\upsilon\big(\Theta - (\Theta 1)(x)A_Q\big)(\mathcal{S}_{\kappa_E}(Rf))(x)\right|^2 \frac{d\mu(x)}{\delta_E(x)^{m-d}} =: A_1 + A_2.\end{aligned}$$

Pick now $\varepsilon \in (0,\beta)$ arbitrary and proceed to estimate $A_1$ as follows:

$$
\begin{aligned}
A_1 &= 2\sum_{k\geq\kappa_E}\sum_{Q\in\mathbb{D}_k(E)}\int_{\mathcal{U}_Q}\Big|\sum_{l\geq\kappa_E}2^{-|k-l|\varepsilon}2^{|k-l|\varepsilon}\delta_E(x)^{\upsilon}\big(\Theta-(\Theta 1)(x)A_Q\big)(D_l\widetilde{D}_l f)(x)\Big|^2\frac{d\mu(x)}{\delta_E(x)^{m-d}}\\
&\leq C\sum_{l\geq\kappa_E}\sum_{k\geq\kappa_E}\sum_{Q\in\mathbb{D}_k(E)}2^{2|k-l|\varepsilon}\int_{\mathcal{U}_Q}\big|\delta_E(x)^{\upsilon}\big(\Theta-(\Theta 1)(x)A_Q\big)(D_l\widetilde{D}_l f)(x)\big|^2\frac{d\mu(x)}{\delta_E(x)^{m-d}}\\
&\leq C\sum_{l\geq\kappa_E}\sum_{k\geq\kappa_E}\sum_{Q\in\mathbb{D}_k(E)}2^{2|k-l|\varepsilon}2^{-2|k-l|\beta}\inf_{w\in Q}\Big[M_E^2\big(|\widetilde{D}_l f|^r\big)(w)\Big]^{\frac{2}{r}}\int_{\mathcal{U}_Q}2^{k(m-d)}\,d\mu\\
&\leq C\sum_{l\geq\kappa_E}\sum_{k\geq\kappa_E}2^{-2|k-l|(\beta-\varepsilon)}\int_E\Big[M_E^2\big(|\widetilde{D}_l f|^r\big)\Big]^{\frac{2}{r}}\,d\sigma\leq C\sum_{l\geq\kappa_E}\int_E|\widetilde{D}_l f|^2\,d\sigma\leq C\int_E|f|^2\,d\sigma,
\end{aligned}
\tag{3.57}
$$

where the first inequality uses the Cauchy-Schwarz inequality and $\sup_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}2^{-2|k-l|\varepsilon}<\infty$, the second inequality uses (3.16), and the final line uses $\mu(\mathcal{U}_Q)\leq C2^{-km}$ and $2^{-kd}\leq C\sigma(Q)$ for all $Q\in\mathbb{D}_k(E)$, $\sup_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}2^{-2|k-l|(\beta-\varepsilon)}<\infty$, the $L^{\frac{2}{r}}(E,\sigma)$ boundedness of $M_E$ and(2.61).

To estimate $A_2$, note that if $E$ is unbounded, then $\kappa_E=-\infty$, so $A_2=0$ (recall $\mathcal{S}_{-\infty}:=0$). Now assume that $E$ is bounded, hence $\kappa_E\in\mathbb{Z}$. In *Case I* above, we may replace $D_l$ with $\mathcal{S}_l$ in the proof of (3.16), since only the regularity of the kernel is used. Therefore, if $r\in(1,\infty)$, then there exist $C,\beta\in(0,\infty)$ such that for all integers $k\geq l\geq\kappa_E$ and all $Q\in\mathbb{D}_k(E)$, we have

$$
\sup_{x\in\mathcal{U}_Q}\big|\delta_E(x)^{\upsilon}\big(\Theta(\mathcal{S}_l g)(x)-(\Theta 1)(x)A_Q(\mathcal{S}_l g)\big)\big|\leq C2^{-|k-l|\beta}\inf_{w\in Q}\Big[M_E^2(|g|^r)(w)\Big]^{\frac{1}{r}},\tag{3.58}
$$

for all locally integrable $g:E\to\mathbb{R}$. Applying (3.58) with $l:=\kappa_E$ and $g:=Rf$ then yields

$$
\begin{aligned}
A_2 &\leq C\sum_{k\geq\kappa_E}\sum_{Q\in\mathbb{D}_k(E)}2^{-2|k-\kappa_E|\beta}\inf_{w\in Q}\Big[M_E^2\big(|Rf|^r\big)(w)\Big]^{\frac{2}{r}}\int_{\mathcal{U}_Q}2^{k(m-d)}\,d\mu\\
&\leq C\sum_{k\geq\kappa_E}\sum_{Q\in\mathbb{D}_k(E)}2^{-2|k-\kappa_E|\beta}\int_Q\Big[M_E^2\big(|Rf|^r\big)\Big]^{\frac{2}{r}}\,d\sigma\\
&\leq C\int_E\Big[M_E^2\big(|Rf|^r\big)\Big]^{\frac{2}{r}}\,d\sigma\leq C\int_E|f|^2\,d\sigma,
\end{aligned}
\tag{3.59}
$$

since $R$ is a bounded on $L^2(E,\sigma)$. Now (3.57) and (3.59) imply (3.55), as required.

`Step III.` *The end-game in the proof of the implication "(3.5)⇒(3.6)".* Fix $\epsilon\in(0,1)$ as in Lemma 2.17 (here we need $C_*\in[4C_\rho^4\Lambda,\infty)$). Then (2.98) and (2.115) imply

$$
\begin{aligned}
&\int_{\{x\in\mathscr{X}\setminus E:\,\delta_E(x)<\epsilon\,\mathrm{diam}_\rho(E)\}}|(\Theta f)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\\
&\qquad\leq C\sum_{k\geq\kappa_E}\sum_{Q\in\mathbb{D}_k(E)}\int_{\mathcal{U}_Q}\big|(\Theta f)(x)-(\Theta 1)(x)A_Q f\big|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\\
&\qquad\quad+C\sum_{k\geq\kappa_E}\sum_{Q\in\mathbb{D}_k(E)}\int_{\mathcal{U}_Q}\big|(\Theta 1)(x)A_Q f\big|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x).
\end{aligned}
\tag{3.60}
$$

If we set $B_Q := \int_{\mathcal{U}_Q} |(\Theta 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x)$, then (2.97), (2.115) and (3.5) imply

$$\sum_{Q' \in \mathbb{D}(E),\, Q' \subseteq Q} B_Q \leq C \int_{T_E(Q)} |(\Theta 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \leq C\sigma(Q), \quad \forall\, Q \in \mathbb{D}(E). \tag{3.61}$$

Thus, the sequence $\{B_Q\}_{Q\in\mathbb{D}(E)}$ satisfies (3.8), and so Lemma 3.2 implies

$$\sum_{Q\in\mathbb{D}(E)} \int_{\mathcal{U}_Q} \big|(\Theta 1)(x) A_Q f\big|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \leq C \int_E (M_E f)^2\, d\sigma \leq C \int_E |f|^2\, d\sigma. \tag{3.62}$$

By combining (3.60), (3.55) and (3.62), we obtain

$$\int_{\{x\in\mathscr{X}\setminus E:\, \delta_E(x)<\epsilon\, \mathrm{diam}_\rho(E)\}} |(\Theta f)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \leq C \int_E |f|^2\, d\sigma. \tag{3.63}$$

This proves (3.6) when $\mathrm{diam}_\rho(E) = \infty$. Now assume that $E$ is bounded and fix $R := \mathrm{diam}_\rho(E)$, $\mathcal{O} := \{x \in \mathscr{X} \setminus E : \epsilon R \leq \delta_E(x)\}$ and $x_0 \in E$. For each $x \in \mathcal{O}$, there exists $y \in E$ such that $\rho_\#(x,y) < 2\delta_E(x)$, and so $\rho(x,x_0) \leq C_\rho^2 \rho_\#(x,x_0) \leq C_\rho^3 \max\{2, \frac{1}{\epsilon}\}\delta_E(x)$. This implies that $\rho(x,x_0) \approx \delta_E(x)$ for all $x \in \mathcal{O}$, so by (3.2) we have $|(\Theta f)(x)|^2 \leq CR^d \|f\|^2_{L^2(E,\sigma)} \rho(x,x_0)^{-2(d+\upsilon)}$. Thus, for some sufficiently small $c > 0$ and some $C \in (0,\infty)$, independent of $f$ and $R$, we have

$$\begin{aligned} &\int_{\mathcal{O}} |(\Theta f)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\qquad\qquad \leq CR^d \|f\|^2_{L^2(E,\sigma)} \int_{\mathscr{X}\setminus B_{\rho_\#}(x_0,cR)} \rho_\#(x,x_0)^{-m-d}\, d\mu(x) \leq C\|f\|^2_{L^2(E,\sigma)}. \end{aligned} \tag{3.64}$$

Now (3.6) follows by combining (3.63) and (3.64).

`Step IV.` *The proof of the converse implication "(3.7)⇒(3.5)".* We no longer assume (3.3) and suppose that (3.7) holds for some $\eta \in (0,\infty)$. We may assume, without loss of generality, that $\eta$ is as large as desired. This is trivial in the case $\mathrm{diam}_\rho(E) = \infty$, whilst in the case $\mathrm{diam}_\rho(E) < \infty$, if $0 < \eta < \eta_o < \infty$, then

$$\int_{\{x\in\mathscr{X}\setminus E:\, \eta\, \mathrm{diam}_\rho(E) \leq \delta_E(x) < \eta_o\, \mathrm{diam}_\rho(E)\}} |(\Theta f)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \leq C \int_E |f|^2\, d\sigma, \tag{3.65}$$

since (3.2) shows that $|(\Theta f)(x)|^2 \leq C\|f\|^2_{L^2(E,\sigma)} [\mathrm{diam}_\rho(E)]^{-d-2\upsilon}$ for all $x$ in the integral above.

For all $Q \in \mathbb{D}(E)$ and some large finite positive constant $C_o$, we have

$$\begin{aligned} &\frac{1}{\sigma(Q)} \int_{T_E(Q)} |\Theta 1(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\quad \leq \frac{2}{\sigma(Q)} \int_{T_E(Q)} \big|\big(\Theta \mathbf{1}_{E\cap B_{\rho_\#}(x_Q, C_o\ell(Q))}\big)(x)\big|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\qquad + \frac{2}{\sigma(Q)} \int_{T_E(Q)} \big|\big(\Theta \mathbf{1}_{E\setminus B_{\rho_\#}(x_Q, C_o\ell(Q))}\big)(x)\big|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) =: \mathcal{I}_1 + \mathcal{I}_2. \end{aligned} \tag{3.66}$$

By choosing $\eta$ sufficiently large, (2.108), (3.7) and the doubling property of $\sigma$, we have

$$\begin{aligned} \mathcal{I}_1 &\leq \frac{2}{\sigma(Q)} \int_{\{x\in\mathscr{X}:\, 0<\delta_E(x)<\eta\, \mathrm{diam}_\rho(E)\}} \big|\big(\Theta \mathbf{1}_{E\cap B_{\rho_\#}(x_Q, C_o\ell(Q))}\big)(x)\big|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\leq \frac{C}{\sigma(Q)} \int_E \big|\mathbf{1}_{E\cap B_{\rho_\#}(x_Q, C_o\ell(Q))}(x)\big|^2\, d\sigma(x) \leq C. \end{aligned} \tag{3.67}$$

Next, by (3.2), (3.13) and the choice of $C_o$ (cf. (3.32)), there exists $C \in (0,\infty)$ such that

$$\begin{aligned}\big|\big(\Theta \mathbf{1}_{E\setminus B_{\rho_\#}(x_Q, C_o\ell(Q))}\big)(x)\big| &\leq C \int_{E\setminus B_{\rho_\#}(x_Q, C_o\ell(Q))} \frac{\delta_E(x)^{-a}}{\rho_\#(x,y)^{d+\upsilon-a}}\, d\sigma(y) \\ &\leq C\ell(Q)^{-(\upsilon-a)}\delta_E(x)^{-a}, \qquad \forall\, x \in T_E(Q). \end{aligned} \tag{3.68}$$

Consequently, applying (2.108), (3.15) (recall that $\upsilon - a > 0$) and $\sigma(Q) \approx \ell(Q)^d$, we obtain

$$\mathcal{I}_2 \leq \frac{C}{\sigma(Q)} \ell(Q)^{-2(\upsilon-a)} \int_{B_{\rho_\#}(x_Q, C\ell(Q))} \delta_E(x)^{2(\upsilon-a)-(m-d)}\, d\mu(x) \leq C. \tag{3.69}$$

Estimates (3.67) and (3.69) together prove (3.5), which completes the proof of the theorem. □

### 3.2 An arbitrary codimension local $T(b)$ theorem for square functions

The main result in this subsection is the following local $T(b)$ theorem for square functions.

**Theorem 3.6.** *Let $0 < d < m < \infty$. Assume that $(\mathscr{X}, \rho, \mu)$ is an $m$-dimensional* ADR *space, $E$ is a closed subset of $(\mathscr{X}, \tau_\rho)$, and $\sigma$ is a Borel semiregular measure on $(E, \tau_{\rho|_E})$ with the property that $(E, \rho\big|_E, \sigma)$ is a $d$-dimensional* ADR *space.*

*Suppose that $\Theta$ is an integral operator with kernel $\theta$ satisfying* (3.1)-(3.4)*. Let $\mathbb{D}(E)$ denote a dyadic cube structure on $E$, consider a Whitney covering $\mathbb{W}_\lambda(\mathscr{X}\setminus E)$ of $\mathscr{X}\setminus E$ and a constant $C_*$ as in Lemma 2.17 with the corresponding dyadic Carleson tents from* (2.97)*. If there exist $C_0 \in [1,\infty)$, $c_0 \in (0,1]$ and for each $Q \in \mathbb{D}(E)$ a $\sigma$-measurable function $b_Q : E \to \mathbb{C}$ such that*

*(i)* $\int_E |b_Q|^2\, d\sigma \leq C_0 \sigma(Q)$,

*(ii) there exists* $\widetilde{Q} \in \mathbb{D}(E)$, $\widetilde{Q} \subseteq Q$, $\ell(\widetilde{Q}) \geq c_0 \ell(Q)$, *and* $\left|\int_{\widetilde{Q}} b_Q\, d\sigma\right| \geq \frac{1}{C_0}\sigma(\widetilde{Q})$,

*(iii)* $\int_{T_E(Q)} |(\Theta\, b_Q)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \leq C_0 \sigma(Q)$,

*then there exists $C \in (0,\infty)$, depending only on $C_0$, $C_\theta$, the* ADR *constants of $E$ and $\mathscr{X}$, and (when $E$ is bounded)* $\operatorname{diam}_\rho(E)$*, such that*

$$\int_{\mathscr{X}\setminus E} |(\Theta f)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \leq C \int_E |f(x)|^2\, d\sigma(x), \quad \forall f \in L^2(E,\sigma). \tag{3.70}$$

We will use the following stopping-time construction to prove Theorem 3.6.

**Lemma 3.7.** *Let $(E, \rho, \sigma)$ be a space of homogeneous type. If there exist $C_0 \in [1,\infty)$, $c_0 \in (0,1]$ and, for each $Q \in \mathbb{D}(E)$, a $\sigma$-measurable function $b_Q : E \to \mathbb{C}$ and a cube $\widetilde{Q} \in \mathbb{D}(E)$ such that*

$$\int_E |b_Q|^2\, d\sigma \leq C_0 \sigma(Q), \quad \widetilde{Q} \subseteq Q, \quad \ell(\widetilde{Q}) \geq c_0 \ell(Q), \quad \left|\int_{\widetilde{Q}} b_Q\, d\sigma\right| \geq \frac{1}{C_0}\sigma(\widetilde{Q}), \tag{3.71}$$

*then there exists a number $\eta \in (0,1)$ such that, for each cube $Q \in \mathbb{D}(E)$ and corresponding cube $\widetilde{Q}$ in* (3.71)*, there exists a sequence $\{Q_j\}_{j\in J} \subseteq \mathbb{D}(E)$ of pairwise disjoint cubes such that*

*(1)* $Q_j \subseteq \widetilde{Q}$ *for every* $j \in J$, $\sigma\Big(\widetilde{Q} \setminus \bigcup_{j\in J} Q_j\Big) \geq \eta\, \sigma(\widetilde{Q})$,

*(2)* $\left|\fint_{Q'} b_Q\,d\sigma\right| \geq \frac{1}{2}$ *for every* $Q' \in \big\{Q' \in \mathbb{D}(E) : Q' \subseteq \widetilde{Q}$ *and* $Q' \cap \Big(\bigcup_{j\in J} Q_j\Big) = \emptyset\big\} =: \mathcal{F}_Q$.

*Proof.* It follows from *(3)* and *(9)* in Proposition 2.11 that

$$\sigma\Big(Q \setminus \bigcup_{Q'\subseteq Q,\, Q'\in\mathbb{D}_l}(E)Q'\Big) = 0, \quad \forall Q \in \mathbb{D}_k(E),\ \forall k,l \in \mathbb{Z} \text{ with } l \geq k \geq \kappa_E. \tag{3.72}$$

We use (3.71) to normalize each $b_Q$ so that $\fint_{\widetilde{Q}} b_Q\,d\sigma = 1$, thus $\int_E |b_Q|^2\,d\sigma \leq C_0^3\sigma(Q)$. Also

$$\sigma(Q) \leq C_1\sigma(\widetilde{Q}) \quad \text{for some } C_1 \in [1,\infty) \text{ independent of } Q, \widetilde{Q}. \tag{3.73}$$

Fix $Q$ and $\widetilde{Q}$ as in (3.71). We perform a stopping-time argument for $\widetilde{Q}$ by successively dividing it into dyadic sub-cubes $Q' \subseteq \widetilde{Q}$ and stopping whenever $\mathrm{Re}\fint_{Q'} b_Q\,d\sigma \leq \frac{1}{2}$. This is possible by (3.72) and the normalization of $b_Q$. We obtain a family of cubes $\{Q_j\}_{j\in J} \subseteq \mathbb{D}(E)$ such that:

*(a)* $Q_j \subseteq \widetilde{Q} \subseteq Q$ for each $j \in J$ and $Q_j \cap Q_{j'} = \emptyset$ whenever $j, j' \in J$, $j \neq j'$.

*(b)* $\mathrm{Re}\fint_{Q_j} b_Q\,d\sigma \leq \frac{1}{2}$ for each $j \in J$.

*(c)* the family $\{Q_j\}_{j\in J}$ is maximal with respect to *(a)* and *(b)* above, i.e., if $Q' \in \mathbb{D}(E)$ is such that $Q' \subseteq \widetilde{Q}$, then either there exists $j_0 \in J$ such that $Q' \subseteq Q_{j_0}$, or $\mathrm{Re}\fint_{Q'} b_Q\,d\sigma > \frac{1}{2}$.

We combine the above results to obtain

$$\begin{aligned}
\sigma(\widetilde{Q}) = \int_{\widetilde{Q}} b_Q\,d\sigma &= \mathrm{Re}\int_{\widetilde{Q}\setminus(\bigcup_{j\in J} Q_j)} b_Q\,d\sigma + \sum_{j\in J}\mathrm{Re}\int_{Q_j} b_Q\,d\sigma \\
&\leq \Big(\int_E |b_Q|^2 d\sigma\Big)^{\frac{1}{2}}\sigma\big(\widetilde{Q}\setminus\cup_{j\in J}Q_j\big)^{\frac{1}{2}} + \tfrac{1}{2}\sum_{j\in J}\sigma(Q_j) \\
&\leq C_1^{\frac{1}{2}}C_0^{\frac{3}{2}}\sigma(\widetilde{Q})^{\frac{1}{2}}\sigma\big(\widetilde{Q}\setminus\cup_{j\in J}Q_j\big)^{\frac{1}{2}} + \tfrac{1}{2}\sigma(\widetilde{Q}).
\end{aligned} \tag{3.74}$$

Hence, $\sigma\big(\widetilde{Q}\setminus\cup_{j\in J}Q_j\big) \geq \eta\sigma(\widetilde{Q})$ with $\eta := \frac{1}{4C_1C_0^3}$, which proves *(1)*, whilst *(c)* implies *(2)*. □

We are now ready to present the proof of Theorem 3.6.

*Proof of Theorem 3.6.* By Theorem 3.1, it suffices to show that $|\Theta 1|^2\delta_E^{2v-(m-d)}\,d\mu$ is a Carleson measure in $\mathscr{X}\setminus E$ relative to $E$, that is, that (3.5) holds. We first show that (3.5) holds for $\Theta$ replaced by some truncated operators. More precisely, for each $i \in \mathbb{N}$ consider the kernel

$$\theta_i(x,y) := \mathbf{1}_{\{1/i<\delta_E<i\}}(x)\theta(x,y), \qquad \forall\, x \in \mathscr{X}\setminus E, \quad \forall\, y \in E, \tag{3.75}$$

and introduce the integral operator

$$(\Theta_i f)(x) := \int_E \theta_i(x,y)f(y)\,d\sigma(y), \qquad \forall\, x \in \mathscr{X}\setminus E. \tag{3.76}$$

Clearly,

$$(\Theta_i f)(x) = \mathbf{1}_{\{1/i<\delta_E<i\}}(x)(\Theta f)(x), \qquad \forall\, x \in \mathscr{X}\setminus E, \quad \forall\, i \in \mathbb{N}, \tag{3.77}$$

and, with $C_\theta$ as in (3.2), for all $x \in \mathscr{X} \setminus E$, $y \in E$ and $\widetilde{y} \in E$ with $\rho(y,\widetilde{y}) \leq \frac{1}{2}\rho(x,y)$, we have

$$|\theta_i(x,y)| \leq C_\theta \frac{\delta_E(x)^{-a}}{\rho(x,y)^{d+\upsilon-a}}, \tag{3.78}$$

$$|\theta_i(x,y) - \theta_i(x,\widetilde{y})| \leq C_\theta \frac{\rho(y,\widetilde{y})^\alpha \delta_E(x)^{-a-\alpha}}{\rho(x,y)^{d+\upsilon-a}}. \tag{3.79}$$

For each $i \in \mathbb{N}$ and $x \in \mathscr{X} \setminus E$, by (3.76), (3.78) and Lemma 3.4 (since $\upsilon - a > 0$), we have

$$\begin{aligned} |(\Theta_i 1)(x)| &\leq C\mathbf{1}_{\{1/i<\delta_E<i\}}(x) \int_E \frac{\delta_E(x)^{-a}}{\rho_\#(x,y)^{d+\upsilon-a}}\, d\sigma(y) \leq C\mathbf{1}_{\{1/i<\delta_E<i\}}(x)[\delta_E(x)]^{-\upsilon} \\ &\leq Ci^\upsilon \mathbf{1}_{\{1/i<\delta_E<i\}}(x). \end{aligned} \tag{3.80}$$

Using (2.108) and Lemma 3.5, for all $Q \in \mathbb{D}(E)$, with $x_Q$ denoting the center of $Q$, we obtain

$$\begin{aligned} &\int_{T_E(Q)} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\qquad \leq Ci^{2\upsilon} \int_{x \in B_{\rho_\#}(x_Q, C\ell(Q)),\, \delta_E(x)<i} \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \leq Ci^{4\upsilon}\sigma(Q), \end{aligned} \tag{3.81}$$

where $C \in (0,\infty)$ does not depend on $Q$ nor on $i$. Hence, if we now define

$$c_i := \sup_{Q \in \mathbb{D}(E)} \frac{1}{\sigma(Q)} \int_{T_E(Q)} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x), \qquad \forall\, i \in \mathbb{N}, \tag{3.82}$$

then $0 \leq c_i \leq Ci^{4\upsilon} < \infty$ for all $i \in \mathbb{N}$. We must now show that $\sup_{i\in\mathbb{N}} c_i < \infty$. Fix $Q \in \mathbb{D}(E)$ and the corresponding cube $\widetilde{Q}$ satisfying *(ii)* in Theorem 3.6 with $c_0 \in (0,1]$. We have

$$\exists\, p, q \in \mathbb{Z} \quad \text{satisfying} \quad 0 \leq p - q \leq \log_2(1/c_0) \quad \text{such that} \quad \widetilde{Q} \in \mathbb{D}_p(E),\, Q \in \mathbb{D}_q(E). \tag{3.83}$$

Next, recall the notation in Lemma 3.7 to define $E_Q^* := \bigcup_{Q' \in \mathcal{F}_Q} \mathcal{U}_{Q'}$. Then by (2.97) we have

$$T_E(Q) \subseteq E_Q^* \cup \Big(\bigcup_{j\in J} T_E(Q_j)\Big) \cup \Big(\bigcup_{\substack{Q'' \in \mathbb{D}_p(E) \\ Q''\subseteq Q,\, Q''\neq\widetilde{Q}}} T_E(Q'')\Big) \cup \Big(\bigcup_{\substack{r\in\mathbb{Z} \\ q\leq r<p}} \bigcup_{\substack{Q''\in\mathbb{D}_r(E) \\ Q''\subseteq Q}} \mathcal{U}''_Q\Big). \tag{3.84}$$

Consequently, for each $i \in \mathbb{N}$ we may write

$$\begin{aligned} &\int_{T_E(Q)} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\quad \leq \int_{E_Q^*} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) + \sum_{j\in J} \int_{T_E(Q_j)} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\quad\quad + \sum\nolimits_{Q''\in\mathbb{D}_p(E),\, Q''\subseteq Q,\, Q''\neq\widetilde{Q}} \int_{T_E(Q'')} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\quad\quad + \sum_{r=q}^{p-1} \sum\nolimits_{Q''\in\mathbb{D}_r(E),\, Q''\subseteq Q} \int_{\mathcal{U}_{Q''}} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x). \end{aligned} \tag{3.85}$$

To estimate the first integral on the right in (3.85), we combine (2.115), (3.71) and *(2)* in Lemma 3.7 with the fact, implied by (3.77), that $|\Theta_i 1| \le |\Theta 1|$ for all $i \in \mathbb{N}$, to obtain

$$\begin{aligned}\int_{E^*_Q} |\Theta_i 1(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) &\le C \sum_{Q'\in\mathcal{F}_Q} \int_{\mathcal{U}_{Q'}} |(\Theta 1)(x) A_{Q'} b_Q|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x)\\ &\le C \int_{T_E(Q)} |(\Theta b_Q)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) + C\int_E |b_Q|^2\, d\sigma \le C\sigma(Q), \quad (3.86)\end{aligned}$$

where the final line uses (2.97), (3.55) with $f := b_Q$, and assumptions *(i)* and *(iii)* of Theorem 3.6. For the first sum in (3.85), we use (3.82) and Lemma 3.7 to obtain

$$\sum_{j\in J} \int_{T_E(Q_j)} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \le c_i\, \sigma\big(\bigcup_{j\in J} Q_j\big) \le c_i\,(1-\eta)\,\sigma(\widetilde{Q}). \qquad (3.87)$$

For the second sum in (3.85), we use (3.82) to obtain

$$\sum\nolimits_{Q''\in\mathbb{D}_p(E),\, Q''\subseteq Q,\, Q''\neq\widetilde{Q}} \int_{T_E(Q'')} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \le c_i \sigma(Q\setminus\widetilde{Q}). \qquad (3.88)$$

For the final two sums in (3.85), we use (3.80), (2.106), (2.108) and (3.83) to obtain

$$\sum_{r=q}^{p-1} \sum\nolimits_{Q''\in\mathbb{D}_r(E),\, Q''\subseteq Q} \int_{\mathcal{U}_{Q''}} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \le C \log_2(1/c_0)\,\sigma(Q). \qquad (3.89)$$

Combining (3.85)-(3.89) and (3.73), there exists $C \in (0,\infty)$ such that for all $i \in \mathbb{N}$, we have

$$\begin{aligned}\int_{T_E(Q)} |\Theta_i 1|^2 \delta_E^{2\upsilon-(m-d)}\, d\mu &\le c_i\,(1-\eta)\,\sigma(\widetilde{Q}) + c_i\sigma(Q\setminus\widetilde{Q}) + C\sigma(Q)\\ &\le c_i\,(1-\eta C_1^{-1})\,\sigma(Q) + C\sigma(Q),\end{aligned} \qquad (3.90)$$

for all $Q \in \mathbb{D}(E)$. It follows that $c_i \le c_i\,(1-\eta C_1^{-1}) + C$, hence $\sup_{i\in\mathbb{N}} c_i \le \eta^{-1} C_1 C < \infty$.

We may now apply (3.77) and Lebesgue's Monotone Convergence Theorem to obtain

$$\begin{aligned}\int_{T_E(Q)} |(\Theta 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) &= \lim_{i\to\infty} \int_{T_E(Q)} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x)\\ &\le (\sup\nolimits_{i\in\mathbb{N}} c_i)\,\sigma(Q) \le C\sigma(Q),\end{aligned} \qquad (3.91)$$

for all $Q \in \mathbb{D}(E)$. This completes the proof of (3.5) and finishes the proof of Theorem 3.6. □

# 4 An Inductive Scheme for Square Function Estimates

We now apply the local $T(b)$ Theorem from the previous section to establish an inductive scheme for square function estimates. We show that an integral operator $\Theta$, associated with an Ahlfors-David regular set $E$ as in (3.4), satisfies square function estimates whenever the set $E$ contains (uniformly, at all scales and locations) so-called big pieces of sets on which square function estimates for $\Theta$ hold. In short, we say that big pieces of square function estimates (BPSFE) imply square function estimates (SFE). We emphasize that this "big pieces functor" is applied to square function estimates for a fixed operator $\Theta$. Thus, the result in this section is not a consequence of the stability of UR sets under the so-called big pieces functor, as our particular square function bounds may *not* be equivalent to the property that $E$ is UR.

We work in the context introduced at the beginning of Section 3, except we must assume in addition that the integral kernel $\theta$ is not adapted to a fixed set $E$. In particular, fix two real numbers $0 < d < m$ and an $m$-dimensional ADR space $(\mathscr{X}, \rho, \mu)$. Suppose that

$$\begin{gathered}\theta : (\mathscr{X} \times \mathscr{X}) \setminus \{(x,x) : x \in \mathscr{X}\} \longrightarrow \mathbb{R} \\ \text{is Borel measurable with respect to the product topology } \tau_\rho \times \tau_\rho,\end{gathered} \tag{4.1}$$

and has the property that there exist $C_\theta$, $\alpha$, $\upsilon \in (0,\infty)$ such that for all $x$, $y$, $\widetilde{y} \in \mathscr{X}$ with $x \neq y$, $x \neq \widetilde{y}$ and $\rho(y,\widetilde{y}) \leq \frac{1}{2}\rho(x,y)$, the following hold:

$$|\theta(x,y)| \leq \frac{C_\theta}{\rho(x,y)^{d+\upsilon}}, \tag{4.2}$$

$$|\theta(x,y) - \theta(x,\widetilde{y})| \leq C_\theta \frac{\rho(y,\widetilde{y})^\alpha}{\rho(x,y)^{d+\upsilon+\alpha}}. \tag{4.3}$$

For a closed subset $E$ of $(\mathscr{X}, \tau_\rho)$ and a Borel semiregular measure $\sigma$ on $(E, \tau_{\rho|_E})$ such that $(E, \rho\big|_E, \sigma)$ is a $d$-dimensional ADR space, define the operator $\Theta_E$ for all $f \in L^p(E,\sigma)$, $1 \leq p \leq \infty$, by

$$(\Theta_E f)(x) := \int_E \theta(x,y) f(y)\, d\sigma(y), \qquad \forall\, x \in \mathscr{X} \setminus E. \tag{4.4}$$

We then say that $E$ has `Square Function Estimates` (`SFE`) relative to $\theta$ if

$$\int_{\mathscr{X}\setminus E} |\Theta_E f(z)|^2 \operatorname{dist}_{\rho_\#}(z,E)^{2\upsilon-(m-d)}\, d\mu(z) \leq C \int_E |f|^2\, d\sigma, \quad \forall f \in L^2(E,\sigma). \tag{4.5}$$

We now define what it means for a set to have big pieces of square function estimates.

**Definition 4.1.** *Let $0 < d < m < \infty$ denote real numbers, suppose that $(\mathscr{X}, \rho, \mu)$ is an $m$-dimensional* ADR *space and assume that $\theta$ satisfies* (4.1)-(4.3)*. A set $E \subseteq \mathscr{X}$ is said to have* `Big Pieces of Square Function Estimate` (`BPSFE`) *relative to $\theta$ if it is closed in $(\mathscr{X}, \tau_\rho)$, there exist a Borel semiregular measure $\sigma$ on $(E, \tau_{\rho|_E})$, such that $\big(E, \rho\big|_E, \sigma\big)$ is a $d$-dimensional* ADR *space, and constants $\eta, C_1, C_2 \in (0,\infty)$, referred to as the* `BPSFE character` *of $E$, with the following property: for each $x \in E$ and each $r \in (0, \operatorname{diam}_{\rho_\#}(E)]$, there exists a closed subset $E_{x,r}$ of $(\mathscr{X}, \tau_\rho)$ such that $\big(E_{x,r}, \rho\big|_{E_{x,r}}, \sigma_{x,r}\big)$, where $\sigma_{x,r} := \mathscr{H}^d_{\mathscr{X},\rho_\#} \lfloor E_{x,r}$ is given by* (2.15)*, is a $d$-dimensional* ADR *space, with* ADR *constant at most equal to $C_1$, satisfying*

$$\sigma\big(E_{x,r} \cap E \cap B_{\rho_\#}(x,r)\big) \geq \eta\, r^d \tag{4.6}$$

*and*

$$\int_{\mathscr{X}\setminus E_{x,r}} |\Theta_{E_{x,r}} f(z)|^2 \operatorname{dist}_{\rho_\#}(z,E_{x,r})^{2\upsilon-(m-d)}\, d\mu(z) \leq C_2 \int_{E_{x,r}} |f|^2\, d\sigma_{x,r}, \tag{4.7}$$

*for all $f \in L^2(E_{x,r}, \sigma_{x,r})$, where $\Theta_{E_{x,r}}$ is the operator associated with $E_{x,r}$ as in* (4.4)*.*

*For each integer $k \geq 2$, a set $E \subseteq \mathscr{X}$ is said to have* `(BP)`$^{k+1}$`SFE` *relative to $\theta$ if the above properties hold but with* (4.7) *replaced by the requirement that*

$$E_{x,r} \text{ has } \texttt{(BP)}^k\texttt{SFE} \text{ relative to } \theta, \text{ with } \texttt{(BP)}^k\texttt{SFE} \text{ character controlled by } C_2, \tag{4.8}$$

*where* `(BP)`$^1$`SFE` *denotes* `BPSFE`*.*

**Remark 4.2.** *This property may be discretized with respect to a dyadic cube structure $\mathbb{D}(E)$. In particular, it follows from (2.30) and (2.34) that BPSFE is equivalent to the property that for each $Q \in \mathbb{D}(E)$, there exists a set $E_Q$ with the properties required of $E_{x,r}$ in Definition 4.1.*

We now state and prove the main result in this section.

**Theorem 4.3.** *Let $0 < d < m < \infty$ denote real numbers, suppose that $(\mathscr{X}, \rho, \mu)$ is an $m$-dimensional* ADR *space, and assume that $\theta$ is as in (4.1)-(4.3). If the set $E \subseteq \mathscr{X}$ has* BPSFE *relative to $\theta$ then there exists a finite constant $C > 0$, depending only on $\rho$, $m$, $d$, $\upsilon$, $C_\theta$, the* BPSFE *character of $E$, and the* ADR *constants of $E$ and $\mathscr{X}$, such that*

$$\int_{\mathscr{X}\setminus E} |\Theta_E f(x)|^2 \, \delta_E(x)^{2\upsilon-(m-d)} \, d\mu(x) \le C \int_E |f|^2 \, d\sigma, \qquad \forall\, f \in L^2(E,\sigma), \tag{4.9}$$

*where $\sigma := \mathscr{H}^d_{\mathscr{X},\rho_\#} \lfloor E$ is given by (2.15).*

*Proof.* Suppose that $E$ has BPSFE relative to $\theta$, so by Remark 4.2, for each $Q \in \mathbb{D}(E)$ there exists $E_Q$ such (4.6)-(4.7) hold with $E_{x,r}$ replaced by $E_Q$, and define $b_Q : E \to \mathbb{R}$ by setting $b_Q(y) := \mathbf{1}_{Q\cap E_Q}(y)$ for all $y \in E$. We will prove (4.9) by applying Theorem 3.6, so we only need to prove that *(i)–(iii)* in Theorem 3.6 hold for $\{b_Q\}_{Q\in\mathbb{D}(E)}$. Condition *(i)* is immediate whilst condition *(ii)*, with $\widetilde{Q} := Q$, is a consequence of (4.6). To verify condition *(iii)*, let $Q \in \mathbb{D}(E)$, fix $C_1 \in (1,\infty)$ to be specified later, and recall the notation in (2.4) to write

$$\begin{aligned}
&\int_{T_E(Q)} |\Theta_E b_Q(x)|^2 \delta_E(x)^{2\upsilon-(m-d)} \, d\mu(x) \\
&\quad = \int_{T_E(Q)} |\Theta_E b_Q(x)|^2 \mathbf{1}_{\{z\in\mathscr{X}:\, \delta_{E_Q}(z) > C_1\delta_E(z)\}}(x) \, \delta_E(x)^{2\upsilon-(m-d)} \, d\mu(x) \\
&\qquad + \int_{T_E(Q)} |\Theta_E b_Q(x)|^2 \mathbf{1}_{\{z\in\mathscr{X}:\, C_1^{-1}\delta_E(z) \le \delta_{E_Q}(z) \le C_1\delta_E(z)\}}(x) \, \delta_E(x)^{2\upsilon-(m-d)} \, d\mu(x) \\
&\qquad + \int_{T_E(Q)} |\Theta_E b_Q(x)|^2 \mathbf{1}_{\{z\in\mathscr{X}:\, \delta_{E_Q}(z) < C_1^{-1}\delta_E(z)\}}(x) \, \delta_E(x)^{2\upsilon-(m-d)} \, d\mu(x) \\
&\quad =: I_1 + I_2 + I_3.
\end{aligned} \tag{4.10}$$

To estimate $I_1$, we obtain a pointwise bound for $\Theta_E b_Q$. To this end, first observe that

$$\mathcal{O} := \big\{ z \in \mathscr{X} : \delta_{E_Q}(z) > C_1 \delta_E(z) \big\} \implies \mathcal{O} \cap E_Q = \emptyset. \tag{4.11}$$

Hence, by (4.2) and (3.12) in Lemma 3.4, we have

$$|\Theta_E b_Q(x)| \le \int_{E_Q} |\theta(x,y)| \, d\sigma(y) \le \frac{C}{\delta_{E_Q}(x)^\upsilon}, \qquad \forall\, x \in \mathcal{O}. \tag{4.12}$$

We fix $x_0 \in Q \cap E_Q$, since $Q \cap E_Q \neq \emptyset$ by (4.6), so by (2.108), there exists $c \in (0,\infty)$ such that

$$I_1 \le C \int_{B_{\rho_\#}(x_0, c\ell(Q))\cap\mathcal{O}} \delta_{E_Q}(x)^{-2\upsilon} \delta_E(x)^{2\upsilon-(m-d)} \, d\mu(x). \tag{4.13}$$

Now fix $M \in (C_\rho^2, \infty]$, choose $C_1 \in (M,\infty)$, and observe that if $x \in \mathcal{O}$ then $\delta_{E_Q}(x) > M\delta_E(x)$, hence $B_{\rho_\#}\big(x, \delta_{E_Q}(x)/M\big) \cap E \neq \emptyset$, so by Lemma 2.10, there exists $C \in (0,\infty)$ such that

$$\frac{\delta_{E_Q}(x)^d}{M^d} \le C \mathscr{H}^d_{\mathscr{X},\rho_\#}\Big( B_{\rho_\#}\Big(x, C_\rho \frac{\delta_{E_Q}(x)}{M}\Big) \cap E \Big), \qquad \forall\, x \in \mathcal{O}. \tag{4.14}$$

Using this in (4.13) we obtain

$$I_1 \le C \int_{B_{\rho_\#}(x_0,c\ell(Q))\cap\mathcal{O}} \int_{B_{\rho_\#}(x,C_\rho\delta_{E_Q}(x)/M)\cap E} 1\, d\mathscr{H}^d_{\mathscr{X},\rho_\#}(z)\, \delta_{E_Q}(x)^{-2\upsilon-d}\delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x). \tag{4.15}$$

We make the claim that for each $\vartheta \in (0,1)$

$$\left.\begin{array}{r}\text{if } x\in\mathscr{X}\setminus E_Q \text{ and } z\in\mathscr{X} \text{ are}\\ \text{such that } \rho_\#(x,z)<\frac{\vartheta}{C_\rho}\delta_{E_Q}(x)\end{array}\right\} \Longrightarrow \frac{1-\vartheta}{C_\rho}\delta_{E_Q}(x)\le\delta_{E_Q}(z)\le C_\rho\delta_{E_Q}(x). \tag{4.16}$$

Indeed, for each $\eta > 1$ close to 1, there exists $y \in E_Q$ satisfying $\rho_\#(y,x) < \eta\delta_{E_Q}(x)$, thus $\delta_{E_Q}(z) \le \rho_\#(y,z) \le C_\rho\eta\delta_{E_Q}(x)$, and $\delta_{E_Q}(z) \le C_\rho\delta_{E_Q}(x)$ follows. Also, if $w \in E_Q$, then $\delta_{E_Q}(x) \le \rho_\#(x,w) \le \vartheta\delta_{E_Q}(x) + C_\rho\rho_\#(z,w)$, which implies $\delta_{E_Q}(x) \le \frac{C_\rho}{1-\vartheta}\delta_{E_Q}(z)$, as required.

Going further, fix $x \in B_{\rho_\#}(x_0, c\ell(Q))\cap\mathcal{O}$ and $z \in B_{\rho_\#}\big(x, C_\rho\delta_{E_Q}(x)/M\big)\cap E$ and make two observations. First, an application of (4.16) with $\vartheta := C_\rho^2/M \in (0,1)$ yields

$$\frac{M-C_\rho^2}{MC_\rho}\delta_{E_Q}(x)\le\delta_{E_Q}(z)\le C_\rho\delta_{E_Q}(x) \quad\text{and}\quad \rho_\#(z,x)<\frac{C_\rho}{M}\delta_{E_Q}(x)\le\frac{C_\rho^2}{M-C_\rho^2}\delta_{E_Q}(z), \tag{4.17}$$

hence $x \in B_{\rho_\#}\big(z, \frac{C_\rho^2}{M-C_\rho^2}\delta_{E_Q}(z)\big)$. Second, since $x_0 \in E_Q$, $M > C_\rho^2$ and $C_\rho \ge 1$, we obtain

$$\begin{aligned}\rho_\#(x_0,z) &\le C_\rho\max\big\{\rho_\#(x_0,x),\rho_\#(x,z)\big\} < C_\rho\max\big\{c\,\ell(Q),\tfrac{C_\rho}{M}\delta_{E_Q}(x)\big\}\\ &\le C_\rho\max\big\{c\,\ell(Q),\tfrac{1}{C_\rho}\delta_{E_Q}(x)\big\}\le C_\rho\max\big\{c\,\ell(Q),\tfrac{1}{C_\rho}\rho_\#(x_0,x)\big\}=C_\rho c\,\ell(Q),\end{aligned} \tag{4.18}$$

thus $z \in B_{\rho_\#}(x_0, C_\rho c\ell(Q))$. Setting $C_M := C_\rho^2/(M-C_\rho^2)$, by (4.17) and (4.11), we now have

$$\begin{aligned}I_1 &\le C\int_{B_{\rho_\#}(x_0,C_\rho c\ell(Q))\cap(E\setminus E_Q)}\delta_{E_Q}(z)^{-2\upsilon-d}\int_{B_{\rho_\#}(z,C_M\delta_{E_Q}(z))\setminus E_Q}\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\,d\mathscr{H}^d_{\mathscr{X},\rho_\#}(z)\\ &\le C\int_{B_{\rho_\#}(x_0,C_\rho c\ell(Q))\cap(E\setminus E_Q)}\delta_{E_Q}(z)^{-2\upsilon-d}\delta_{E_Q}(z)^{2\upsilon+d}\,d\mathscr{H}^d_{\mathscr{X},\rho_\#}(z)\\ &\le C\mathscr{H}^d_{\mathscr{X},\rho_\#}\big(B_{\rho_\#}(x_0,C_\rho c\,\ell(Q))\cap E\big)\le C\ell(Q)^d\le C\sigma(Q),\end{aligned} \tag{4.19}$$

where we used Lemma 3.5 and the fact that $\big(E, \rho\big|_E, \mathscr{H}^d_{\mathscr{X},\rho_\#}\lfloor E\big)$ is $d$-ADR and that $x_0 \in E$.

To estimate $I_3$, we first note that since $T_E(Q)\cap E = \emptyset$, then (4.2) and (3.12) imply that

$$|\Theta_E b_Q(x)| = \left|\int_E \theta(x,y)\,b_Q(y)\,d\sigma(y)\right| \le \int_E |\theta(x,y)|\,d\sigma(y) \le \frac{C}{\delta_E(x)^\upsilon}, \quad \forall\, x\in T_E(Q). \tag{4.20}$$

Also, we have $|\Theta_E b_Q(x)| \le C\delta_{E_Q}(x)^{-\upsilon}$ for each $x \in T_E(Q)\setminus E_Q$ (cf. (4.12)). Next, fix $\alpha, \beta > 0$ such that $\alpha+\beta = \upsilon$. A logarithmic convex combination of these inequalities then yields

$$|\Theta_E b_Q(x)| \le C\delta_{E_Q}(x)^{-\alpha}\delta_E(x)^{-\beta} \qquad \forall\, x\in T_E(Q)\setminus E_Q. \tag{4.21}$$

By Lemma 2.9, we have $\mu(E_Q) = 0$, so using (4.21) in place of (4.12), we obtain (cf. (4.13))

$$I_3 \le C\int_{B_{\rho_\#}(x_0,c\ell(Q))\cap(\widetilde{\mathcal{O}}\setminus E_Q)} \delta_E(x)^{-2\beta+2\upsilon-(m-d)}\delta_{E_Q}(x)^{-2\alpha}\,d\mu(x), \tag{4.22}$$

where we set $\widetilde{\mathcal{O}} := \{z \in \mathscr{X} : \delta_E(z) > C_1 \delta_{E_Q}(z)\}$. The same reasoning leading up to (4.19), applied with $E$ and $E_Q$ interchanged, then gives

$$\begin{aligned} I_3 &\leq C \int_{B_{\rho_\#}(x_0, C_\rho c\ell(Q)) \cap (E_Q \setminus E)} \delta_E(z)^{-2\beta+2\upsilon-m} \int_{B_{\rho_\#}(z, C_M \delta_E(z)) \setminus E} \delta_E(x)^{-2\alpha}\, d\mu(x)\, d\mathscr{H}^d_{\mathscr{X},\,\rho_\#}(z) \\ &\leq C \int_{B_{\rho_\#}(x_0, C_\rho c\ell(Q)) \cap (E_Q \setminus E)} \delta_E(z)^{-2\beta+2\upsilon-m} \delta_E(z)^{-2\alpha+m}\, d\mathscr{H}^d_{\mathscr{X},\,\rho_\#}(z) \\ &\leq C \mathscr{H}^d_{\mathscr{X},\,\rho_\#}\big(B_{\rho_\#}(x_0, C_\rho c\,\ell(Q)) \cap E_Q\big) \leq C\ell(Q)^d \leq C\sigma(Q) \end{aligned} \tag{4.23}$$

by applying Lemma 3.5, provided we choose $0 < \alpha < (m-d)/2$, and the fact that both $\big(E, \rho\big|_E, \mathscr{H}^d_{\mathscr{X},\,\rho_\#}\lfloor E\big)$ and $\big(E_Q, \rho\big|_{E_Q}, \mathscr{H}^d_{\mathscr{X},\,\rho_\#}\lfloor E_Q\big)$ are $d$-ADR spaces and that $x_0 \in E \cap E_Q$.

To estimate $I_2$, by (4.4), *(6)* in Proposition 2.3 and (4.7), with $\sigma_Q := \mathscr{H}^d_{\mathscr{X},\,\rho_\#}\lfloor E_Q$, we have

$$\begin{aligned} I_2 &= \int_{T_E(Q) \setminus E_Q} |\Theta_{E_Q} b_Q(x)|^2 \mathbf{1}_{\{z \in \mathscr{X} : C_1^{-1}\delta_E(z) \leq \delta_{E_Q}(z) \leq C_1 \delta_E(z)\}}(x)\, \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\leq C \int_{\mathscr{X} \setminus E_Q} |\Theta_{E_Q} b_Q(x)|^2\, \delta_{E_Q}(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\leq C \int_{E_Q} |b_Q|^2\, d\sigma_Q \leq C \mathscr{H}^d_{\mathscr{X},\,\rho_\#}\big(Q \cap E_Q\big) \leq C\sigma(Q), \end{aligned} \tag{4.24}$$

which combined with (4.19) and (4.23) shows that *(iii)* in Theorem 3.6 holds, as required. □

The refinement of Theorem 4.3 below follows as a corollary by a simple induction argument.

**Theorem 4.4.** *Let $0 < d < m < \infty$ denote real numbers, suppose that $(\mathscr{X}, \rho, \mu)$ is an $m$-dimensional* ADR *space, and assume that $\theta$ is as in* (4.1)-(4.3)*. If $E$ is a closed in $(\mathscr{X}, \tau_\rho)$ and has the property that there exists a Borel semiregular measure $\sigma$ on $(E, \tau_{\rho|_E})$ such that $\big(E, \rho\big|_E, \sigma\big)$ is a $d$-dimensional* ADR *space, then the following properties are equivalent:*

*(1) $E$ has* `(BP)`$^k$`SFE` *relative to $\theta$ for some $k \in \mathbb{N}$.*

*(2) $E$ has* `(BP)`$^k$`SFE` *relative to $\theta$ for every $k \in \mathbb{N}$.*

*(3) $E$ has* `(BP)`$^0$`SFE` *relative to $\theta$.*

# 5 Square Function Estimates on Uniformly Rectifiable Sets

Given an $n$-dimensional Ahlfors-David regular set $\Sigma$ in $\mathbb{R}^{n+1}$ that has so-called big pieces of Lipschitz graphs (BPLG), the inductive scheme established in the previous section allows us to deduce square function estimates for an integral operator $\Theta_\Sigma$, as in (4.4), whenever square function estimates are satisfied by $\Theta_\Gamma$ for all Lipschitz graphs $\Gamma$ in $\mathbb{R}^{n+1}$. Furthermore, induction allows us to prove the same result when the set $\Sigma$ only has $(\mathrm{BP})^k$LG for any $k \in \mathbb{N}$. The definition of $(\mathrm{BP})^k$LG is given in Definition 5.5. A recent result by J. Azzam and R. Schul (cf. [9, Corollary 1.7]) proves that uniformly rectifiable sets have $(\mathrm{BP})^2$LG (the converse implication also holds and can be found in [27, p. 16]), and this allows us to obtain square function estimates on uniformly rectifiable sets.

We work in the Euclidean codimension one setting throughout this section. In particular, fix $n \in \mathbb{N}$ and let $\mathbb{R}^{n+1}$ be the ambient space, so that in the notation of Section 4, we would have $d = n$, $m = n+1$ and $(\mathscr{X}, \rho, \mu)$ is $\mathbb{R}^{n+1}$ with the Euclidean metric and Lebesgue measure. We also restrict our attention to the following class of kernels in order to obtain square function estimates on Lipschitz graphs. Suppose that $K : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ satisfies

$$K \in C^2(\mathbb{R}^{n+1} \setminus \{0\}), \quad K(\lambda x) = \lambda^{-n} K(x) \text{ for all } \lambda > 0,\ x \in \mathbb{R}^{n+1} \setminus \{0\}, \quad K \text{ is odd.} \tag{5.1}$$

In particular, the first two properties above imply that if $C_K := \max_{0 \le j \le 2} \|\nabla^j K\|_{L^\infty(S^{n-1})}$ then

$$|\nabla^j K(x)| \le C_K |x|^{-n-j}, \quad \forall\, x \in \mathbb{R}^{n+1} \setminus \{0\}, \quad \forall\, j \in \{0, 1, 2\}. \tag{5.2}$$

For a closed subset $\Sigma$ of $\mathbb{R}^{n+1}$, let $\sigma := \mathscr{H}^n_{\mathbb{R}^{n+1}} \lfloor \Sigma$ denote the surface measure induced by the $n$-dimensional Hausdorff measure on $\Sigma$ from (2.15), and define the integral operator $\mathcal{T}$ for all functions $f \in L^p(\Sigma, \sigma)$, $1 \le p \le \infty$, by

$$\mathcal{T} f(x) := \int_\Sigma K(x - y) f(y)\, d\sigma(y), \qquad \forall\, x \in \mathbb{R}^{n+1} \setminus \Sigma. \tag{5.3}$$

In the notation of Section 4, we have $\{E, \Theta_E, \theta\} = \{\Sigma, \nabla\mathcal{T}, \nabla K\}$. We begin by proving square function estimates for $\nabla\mathcal{T}$ in the case when $\Sigma$ is a Lipschitz graph. The inductive scheme from the previous section then allows us to extend that result to the case when $\Sigma$ has $(\mathrm{BP})^k$LG for any $k \in \mathbb{N}$, and hence when $\Sigma$ is uniformly rectifiable.

## 5.1 Square function estimates on Lipschitz graphs

The main result in this subsection is the square function estimate for Lipschitz graphs contained in the theorem below. A parabolic variant of this result appears in [43], and the present proof is based on the arguments given there, and in [38].

**Theorem 5.1.** *Let $A : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function and set $\Sigma := \big\{(x, A(x)) : x \in \mathbb{R}^n\big\}$. Moreover, assume that $K$ is as in (5.1) and consider the operator $\mathcal{T}$ as in (5.3). Then there exists a finite constant $C > 0$ depending only on $\|\partial^\alpha K\|_{L^\infty(S^n)}$ for $|\alpha| \le 2$, and the Lipschitz constant of $A$ such that for each function $f \in L^2(\Sigma, \sigma)$ it holds that*

$$\int_{\mathbb{R}^{n+1} \setminus \Sigma} |(\nabla\mathcal{T} f)(x)|^2 \operatorname{dist}(x, \Sigma)\, dx \le C \int_\Sigma |f|^2\, d\sigma. \tag{5.4}$$

As a preamble to the proof of Theorem 5.1, we state and prove a couple of technical lemmas. The first has essentially appeared previously in [14], and is based upon ideas of [55].

**Lemma 5.2.** *Assume that $A : \mathbb{R}^n \to \mathbb{R}$ is a locally integrable function such that $\nabla A \in L^2(\mathbb{R}^n)$. Pick a smooth, real-valued, nonnegative, compactly supported function $\phi$ defined in $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$ and for each $t > 0$ set $\phi_t(x) := t^{-n}\phi(x/t)$ for $x \in \mathbb{R}^n$. Finally, define*

$$E_A(t, x, y) := A(x) - A(y) - \langle \nabla_x(\phi_t * A)(x), (x - y) \rangle, \qquad \forall\, x, y \in \mathbb{R}^n,\ \forall\, t > 0. \tag{5.5}$$

*Then, for some finite positive constant $C = C(\phi, n)$,*

$$\int_0^\infty t^{-n-2} \int_{\mathbb{R}^n} \int_{|x-y| \le \lambda t} |E_A(t, x, y)|^2\, dy\, dx\, \tfrac{dt}{t} \le C \lambda^{n+3} \|\nabla A\|^2_{L^2(\mathbb{R}^n)}, \qquad \forall\, \lambda \ge 1. \tag{5.6}$$

*Proof.* Applying the changes of variables $t=\lambda^{-1}\tau$, $y=x+h$ and then employing Plancherel's theorem in the variable $x$, we may write (with 'hat' denoting the Fourier transform)

$$\begin{aligned}
&\int_0^\infty t^{-n-2}\int_{\mathbb{R}^n}\int_{|x-y|\leq\lambda t}|E_A(t,x,y)|^2\,dy\,dx\,\frac{dt}{t} &(5.7)\\
&=\lambda^{n+2}\int_0^\infty \tau^{-n-2}\int_{\mathbb{R}^n}\int_{|h|\leq\tau}\big|A(x)-A(x+h)+\langle\nabla_x(\phi_{\lambda^{-1}\tau}*A)(x),h\rangle\big|^2\,dh\,dx\,\frac{d\tau}{\tau}\\
&=\lambda^{n+2}\int_0^\infty \tau^{-n-2}\int_{\mathbb{R}^n}\int_{|h|\leq\tau}\big|1-e^{i\langle\zeta,h\rangle}+i\langle\zeta,h\rangle\,\widehat{\phi}(\lambda^{-1}\tau\zeta)\big|^2\tfrac{|\widehat{\nabla A}(\zeta)|^2}{|\zeta|^2}\,dh\,d\zeta\,\frac{d\tau}{\tau}\\
&=\lambda^{n+2}\int_0^\infty\int_{\mathbb{R}^n}\int_{|w|\leq 1}\frac{\big|1-e^{i\tau\langle\zeta,w\rangle}+i\tau\langle\zeta,w\rangle\,\widehat{\phi}(\lambda^{-1}\tau\zeta)\big|^2}{\tau^2|\zeta|^2}|\widehat{\nabla A}(\zeta)|^2\,dw\,d\zeta\,\frac{d\tau}{\tau},
\end{aligned}$$

where the last equality in (5.7) is based on the change of variables $h=\tau w$.

Next we observe that for every $\zeta\in\mathbb{R}^n$ and $w\in\mathbb{R}^n$ with $|w|\leq 1$ it holds that

$$\frac{\big|1-e^{i\tau\langle\zeta,w\rangle}+i\tau\langle\zeta,w\rangle\,\widehat{\phi}(\lambda^{-1}\tau\zeta)\big|}{\tau|\zeta|}\leq C\,\min\Big\{\tau|\zeta|,\,\frac{\lambda}{\tau|\zeta|}\Big\} \tag{5.8}$$

for some $C>0$ depending only on $\phi$. To see why (5.8) is true, consider the following two cases.

*Case I:* If $\tau|\zeta|\leq\sqrt{\lambda}$, then using Taylor expansions about zero for the complex exponential function and $\widehat{\phi}$, and since $\widehat{\phi}(0)=1$, $\lambda\geq 1$ and $|w|\leq 1$, we obtain

$$\big|1-e^{i\tau\langle\zeta,w\rangle}+i\tau\langle\zeta,w\rangle+i\tau\langle\zeta,w\rangle\,(\widehat{\phi}(\lambda^{-1}\tau\zeta)-1)\big|\leq C\tau^2|\zeta|^2. \tag{5.9}$$

*Case II:* If $\tau|\zeta|>\sqrt{\lambda}$, then since $\widehat{\phi}$ is a Schwartz function, $\lambda\geq 1$ and $|w|\leq 1$, we have

$$\big|1-e^{i\tau\langle\zeta,w\rangle}+\tau\langle\zeta,w\rangle\,\widehat{\phi}(\lambda^{-1}\tau\zeta)\big|\leq 2+C\tau|\zeta|\,(1+\lambda^{-1}\tau|\zeta|)^{-1}\leq C\lambda. \tag{5.10}$$

These prove (5.8), and integrating in $\tau\in(0,\infty)$ with respect to the Haar measure then implies

$$\int_0^\infty\frac{\big|1-e^{i\tau\langle\zeta,w\rangle}+i\tau\langle\zeta,w\rangle\,\widehat{\phi}(\lambda^{-1}\tau\zeta)\big|^2}{\tau^2|\zeta|^2}\,\frac{d\tau}{\tau}\leq\int_0^\infty\min\Big\{\tau^2|\zeta|^2,\,\frac{\lambda^2}{\tau^2|\zeta|^2}\Big\}\,\frac{d\tau}{\tau}\leq C\lambda. \tag{5.11}$$

A combination of (5.7), (5.11) and Plancherel's theorem now yields (5.6), as required. □

The second lemma needed here has essentially appeared previously in [64].

**Lemma 5.3.** *Let $F:\mathbb{R}^{n+1}\setminus\{0\}\to\mathbb{R}$ be a continuous function which is even and positive homogeneous of degree $-n-1$. Then for any $a\in\mathbb{R}^n$ and any $t>0$ it holds that*

$$\int_{\mathbb{R}^n}F(y,a\cdot y+t)\,dy=\frac{1}{2t}\int_{S^{n-1}}\int_{-\infty}^{\infty}F(\omega,s)\,ds\,d\omega=\int_{\mathbb{R}^n}F(y,t)\,dy. \tag{5.12}$$

*In particular, if $F$ is some first-order partial derivative, say $F=\partial_j G$, $j\in\{1,..,n+1\}$, of a function $G\in C^1(\mathbb{R}^{n+1}\setminus\{0\})$ which is odd and homogeneous of degree $-n$, then*

$$\int_{\mathbb{R}^n}F(y,a\cdot y+t)\,dy=0\qquad\textit{for any } a\in\mathbb{R}^n \textit{ and } t>0. \tag{5.13}$$

After this preamble, we are ready to present the proof of Theorem 5.1.

*Proof of Theorem 5.1.* A moment's reflection shows that it suffices to establish (5.4) with the domain of integration $\mathbb{R}^{n+1}\setminus\Sigma$ in the left-hand side replaced by

$$\Omega := \{(x,t)\in\mathbb{R}^{n+1} : t > A(x)\}. \tag{5.14}$$

Assume that this is the case and note that by making the bi-Lipschitz change of variables $\mathbb{R}^n\times(0,\infty)\ni(x,t)\mapsto(x,A(x)+t)\in\Omega$, with Jacobian equivalent to a finite constant, the estimate (5.4) follows from the boundedness of $T^j : L^2(\mathbb{R}^n, dx)\to L^2(\mathbb{R}^{n+1}_+, \frac{dt}{t}dx)$ defined by

$$T^j f(x,t) := \int_{\mathbb{R}^n} K_t^j(x,y)f(y)\,dy \tag{5.15}$$

for $j=1,\dots,n+1$, where the family of kernels $\{K_t^j(x,y)\}_{t>0}$ is given by

$$K_t^j(x,y) := t\,(\partial_j K)(x-y, A(x)-A(y)+t), \qquad x,y\in\mathbb{R}^n,\ t>0,\ j=1,\dots,n+1. \tag{5.16}$$

The approach we present utilizes ideas developed in [16] and [38]. Based on (5.1)-(5.2) it is not difficult to check that the family $\{K_t^j(x,y)\}_{t>0}$ is standard, i.e., there hold

$$|K_t^j(x,y)| \le C\,t(t+|x-y|)^{-(n+1)} \tag{5.17}$$

$$|\nabla_x K_t^j(x,y)| + |\nabla_y K_t^j(x,y)| \le C\,t(t+|x-y|)^{-(n+2)}. \tag{5.18}$$

As such, a particular version of Theorem 3.1 gives that the operators in (5.15) are bounded as soon as we show that for each $j=1,\dots,n+1$,

$$|T^j(1)(x,t)|^2\,\tfrac{dt}{t}dx \text{ is a Carleson measure in } \mathbb{R}^{n+1}_+. \tag{5.19}$$

To this end, fix $j\in\{1,\dots,n+1\}$ and select a real-valued, nonnegative function $\phi\in C_c^\infty(\mathbb{R}^n)$, vanishing for $|x|\ge 1$, with $\int_{\mathbb{R}^n}\phi(x)\,dx = 1$ and, as usual, for every $t>0$, set $\phi_t(x) := t^{-n}\phi(x/t)$ for $x\in\mathbb{R}^n$. We write $T^j(1) = (T^j(1)-\widetilde{T}^j(1)) + \widetilde{T}^j(1)$ where

$$\widetilde{T}^j f(x,t) := \int_{\mathbb{R}^n}\widetilde{K}_t^j(x,y)f(y)\,dy, \qquad x\in\mathbb{R}^n,\ t>0, \tag{5.20}$$

with

$$\widetilde{K}_t^j(x,y) := t\,(\partial_j K)(x-y, \langle\nabla_x(\phi_t * A)(x), x-y\rangle + t), \qquad x,y\in\mathbb{R}^n,\ t>0. \tag{5.21}$$

To prove that $|(T^j-\widetilde{T}^j)(1)(x,t)|^2\,dx\frac{dt}{t}$ is a Carleson measure, fix $x_0$ in $\mathbb{R}^n$, $r>0$, and split

$$(T^j-\widetilde{T}^j)(1) = (T^j-\widetilde{T}^j)(\mathbf{1}_{B(x_0,100r)}) + (T^j-\widetilde{T}^j)(\mathbf{1}_{\mathbb{R}^n\setminus B(x_0,100r)}), \tag{5.22}$$

Using (5.17) and the fact that a similar estimate holds for $\widetilde{K}_t^j(x,y)$, we may write

$$\begin{aligned} &\int_0^r\int_{B(x_0,r)} |(T^j-\widetilde{T}^j)(\mathbf{1}_{\mathbb{R}^n\setminus B(x_0,100r)})(x,t)|^2\,dx\,\tfrac{dt}{t} \\ &\qquad\le C\int_0^r\int_{B(x_0,r)}\Big(\int_{\mathbb{R}^n\setminus B(x_0,100r)}\frac{t}{|x-y|^{n+1}}\,dy\Big)^2 dx\,\tfrac{dt}{t} \\ &\qquad= C\int_0^r\int_{B(x_0,r)}\Big(\int_{\mathbb{R}^n\setminus B(0,99r)}\frac{t}{|z|^{n+1}}\,dz\Big)^2 dx\,\tfrac{dt}{t} = C\,r^n. \end{aligned} \tag{5.23}$$

It remains to show that

$$\int_0^r \int_{B(x_0,r)} |(T^j - \widetilde{T}^j)(\mathbf{1}_{B(x_0,100r)})(x,t)|^2\,dx\,\tfrac{dt}{t} \le C\,r^n. \tag{5.24}$$

We now use Lemma 5.2 with the following adjustment. Fix a function $\Phi \in C^\infty(\mathbb{R})$ such that $0 \le \Phi \le 1$, $\operatorname{supp}\Phi \subseteq [-150r, 150r]$, $\Phi \equiv 1$ on $[-125r, 125r]$, and $\|\Phi'\|_{L^\infty(\mathbb{R})} \le c/r$. If we now set $\widetilde{A}(x) := \Phi(|x-x_0|)(A(x) - A(x_0))$ for every $x \in \mathbb{R}^n$, it follows that

$$\begin{gathered} \widetilde{A}(x) - \widetilde{A}(y) = A(x) - A(y) \;\text{ and }\; \nabla(\phi_t * \widetilde{A})(x) = \nabla(\phi_t * A)(x) \\ \text{whenever }\; x \in B(x_0,r),\; y \in B(x_0,100r),\; t \in (0,r). \end{gathered} \tag{5.25}$$

Hence, the expression $(T^j - \widetilde{T}^j)(\mathbf{1}_{B(x_0,100r)})(x,t)$ does not change for $x \in B(x_0,r)$ and $t \in (0,r)$ if we replace $A$ by $\widetilde{A}$. In addition, since $\|\nabla\widetilde{A}\|_{L^\infty(\mathbb{R}^n)} \le C\|\nabla A\|_{L^\infty(\mathbb{R}^n)}$, taking into account the support of $\widetilde{A}$ we have $\|\nabla\widetilde{A}\|_{L^2(\mathbb{R}^n)} \le Cr^{n/2}\|\nabla A\|_{L^\infty(\mathbb{R}^n)}$ for some $C > 0$ independent of $r$. Hence, there is no loss of generality in assuming that the original Lipschitz function $A$ satisfies

$$\|\nabla A\|_{L^2(\mathbb{R}^n)} \le Cr^{n/2}\|\nabla A\|_{L^\infty(\mathbb{R}^n)}. \tag{5.26}$$

Under this assumption we now return to the task of proving (5.24). To get started, recall (5.5). We claim that there exists $C = C(A,\phi) > 0$ such that

$$|K_t^j(x,y) - \widetilde{K}_t^j(x,y)| \le C\,t(t + |x-y|)^{-(n+2)}|E_A(t,x,y)|, \quad \forall\, x,y \in \mathbb{R}^n,\ \forall\, t > 0. \tag{5.27}$$

Indeed, by making use of the Mean-Value Theorem and (5.2), the claim will follow if we show that there exists $C = C(A,\phi) > 0$ with the property that

$$\sup\nolimits_{\xi \in I}\,[|\xi| + |x-y|]^{-(n+2)} \le C[t + |x-y|]^{-(n+2)}, \tag{5.28}$$

where $I$ denotes the interval with endpoints $t + A(x) - A(y)$ and $t + \langle \nabla_x(\phi_t * A)(x), (x-y)\rangle$. From the properties of $A$ and $\phi$ we see that $\xi = t + O(|x-y|)$, with constants depending only on $A$ and $\phi$. In particular, there exists some small $\varepsilon = \varepsilon(A,\phi) > 0$ such that if $|x-y| < \varepsilon t$ then $t \le C|\xi| \le C(|\xi| + |x-y|)$. On the other hand, if $|x-y| \ge \varepsilon t$ then clearly $t \le C(|\xi| + |x-y|)$. Thus, there exists $C > 0$ such that $t \le C(|\xi| + |x-y|)$ for $\xi \in I$, which implies that for some $C = C(A,\phi) > 0$ it holds that $t + |x-y| \le C(|\xi| + |x-y|)$ whenever $\xi \in I$, proving (5.28).

Next, making use of (5.27), we may write

$$\begin{aligned} &\int_0^r \int_{B(x_0,r)} |(T^j - \widetilde{T}^j)(\mathbf{1}_{B(x_0,100r)})(x,t)|^2\,dx\,\frac{dt}{t} \\ &\qquad \le C\int_0^\infty \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \frac{t}{(t+|x-y|)^{n+2}}\,|E_A(t,x,y)|\,dy\Big)^2\,dx\,\frac{dt}{t} \\ &\qquad \le C\int_0^\infty \int_{\mathbb{R}^n} \Big(t^{-n-1}\int_{B(x,t)} |E_A(t,x,y)|\,dy\Big)^2\,dx\,\frac{dt}{t} \\ &\qquad\quad + C\int_0^\infty \int_{\mathbb{R}^n} \left(\sum_{\ell=0}^\infty \int_{B(x,2^{\ell+1}t)\setminus B(x,2^\ell t)} \frac{t}{|x-y|^{n+2}}\,|E_A(t,x,y)|\,dy\right)^2\,dx\,\frac{dt}{t} \\ &\qquad \le C\int_0^\infty \int_{\mathbb{R}^n} \left(\sum_{\ell=0}^\infty 2^{-\ell}(2^\ell t)^{-n-1}\int_{B(x,2^{\ell+1}t)} |E_A(t,x,y)|\,dy\right)^2\,dx\,\frac{dt}{t}. \end{aligned} \tag{5.29}$$

Now, we apply Minkowski's inequality in order to obtain

$$\int_0^\infty \int_{\mathbb{R}^n} \left( \sum_{\ell=0}^{\infty} 2^{-\ell} (2^\ell t)^{-n-1} \int_{B(x,2^{\ell+1}t)} |E_A(t,x,y)|\, dy \right)^2 dx\, \frac{dt}{t}$$
$$\leq \left( \sum_{\ell=0}^{\infty} \left[ \int_0^\infty \int_{\mathbb{R}^n} 2^{-2\ell} (2^\ell t)^{-2n-2} \Big( \int_{B(x,2^{\ell+1}t)} |E_A(t,x,y)|\, dy \Big)^2 dx\, \frac{dt}{t} \right]^{1/2} \right)^2. \quad (5.30)$$

By the Cauchy-Schwarz inequality, the last expression above is dominated by

$$\left( \sum_{\ell=0}^{\infty} \left[ 2^{\ell(-n-4)} \int_0^\infty \int_{\mathbb{R}^n} t^{-n-2} \int_{B(x,2^{\ell+1}t)} |E_A(t,x,y)|^2\, dy\, dx\, \frac{dt}{t} \right]^{1/2} \right)^2. \quad (5.31)$$

Invoking now Lemma 5.2 with $\lambda := 2^{\ell+1} \geq 1$ for $\ell \in \mathbb{N} \cup \{0\}$, each inner triple integral in (5.31) is dominated by $C 2^{\ell(n+3)} \|\nabla A\|^2_{L^2(\mathbb{R}^n)}$ with $C > 0$ finite constant independent of $\ell$. Thus, the entire expression in (5.31) is

$$\leq C \left( \sum_{\ell=0}^{\infty} \big[ 2^{\ell(-n-4)} \cdot 2^{\ell(n+3)} \|\nabla A\|^2_{L^2(\mathbb{R}^n)} \big]^{1/2} \right)^2 = C \|\nabla A\|^2_{L^2(\mathbb{R}^n)} \leq C r^n, \quad (5.32)$$

where for the last inequality in (5.32) we have used (5.26). This finishes the proof of (5.24). In turn, when (5.24) is combined with (5.23), we obtain

$$|(T^j - \widetilde{T}^j)(1)(x,t)|^2\, \tfrac{dt}{t} dx \text{ is a Carleson measure in } \mathbb{R}^{n+1}_+. \quad (5.33)$$

At this stage, there remains to observe that, thanks to Lemma 5.3, we have

$$\widetilde{T}^j(1)(x,t) = \int_{\mathbb{R}^n} t\, (\partial_j K)\big( x-y, \langle \nabla_x (\phi_t * A)(x), (x-y) \rangle + t \big)\, dy \equiv 0,\ \forall\, x \in \mathbb{R}^n,\ \forall\, t > 0. \quad (5.34)$$

The proof of Theorem 5.1 is thus completed. □

## 5.2 Square function estimates on (BP)$^k$LG sets

We continue to work in the context of $\mathbb{R}^{n+1}$ introduced at the beginning of Section 5, and abbreviate the $n$-dimensional Hausdorff outer measure from Definition 2.2 as $\mathcal{H}^n := \mathcal{H}^n_{\mathbb{R}^{n+1}}$. We prove that square function estimates are stable under the so-called big pieces functor. Square function estimates on uniformly rectifiable sets then follow as a simple corollary. Let us begin by reviewing the concept of uniform rectifiability. In particular, following G. David and S. Semmes [26], we make the following definition.

**Definition 5.4.** *A closed set $\Sigma \subseteq \mathbb{R}^{n+1}$ is called* `uniformly rectifiable` *if it is $n$-dimensional Ahlfors-David regular and the following holds: There exist $\varepsilon$, $M \in (0,\infty)$, referred to as the* UR *constants of $\Sigma$, such that for each $x \in \Sigma$ and $r > 0$, there is a Lipschitz map $\varphi : B^n_r \to \mathbb{R}^{n+1}$, where $B^n_r$ is a ball of radius $r$ in $\mathbb{R}^n$, with Lipschitz constant at most equal to $M$, such that*

$$\mathcal{H}^n \big( \Sigma \cap B(x,r) \cap \varphi(B^n_r) \big) \geq \varepsilon r^n. \quad (5.35)$$

*If $\Sigma$ is compact, then this is only required for $r \in (0, \operatorname{diam}(\Sigma)]$.*

There are a variety of equivalent characterizations of uniform rectifiability (cf., e.g., [27, Theorem I.1.5.7, p. 22]); the version above is often specified by saying that $\Sigma$ has `Big Pieces of Lipschitz Images` (`BPLI`). Another version, in which Lipschitz maps are replaced with Bi-Lipschitz maps, is specified by saying that $\Sigma$ has `Big Pieces of Bi-Lipschitz Images` (`BPBI`). The equivalence between BPLI and BPBI can be found in [27, p. 22]. We also require the following notion of sets having big pieces of Lipschitz graphs.

**Definition 5.5.** *A set $\Sigma \subseteq \mathbb{R}^{n+1}$ is said to have* `Big Pieces of Lipschitz Graphs` (`BPLG`) *if it is $n$-dimensional Ahlfors-David regular and the following holds: There exist $\varepsilon$, $M \in (0,\infty)$, referred to as the* BPLG *constants of $\Sigma$, such that for each $x \in \Sigma$ and $r > 0$, there is an $n$-dimensional Lipschitz graph $\Gamma \subseteq \mathbb{R}^{n+1}$ with Lipschitz constant at most equal to $M$, such that*

$$\mathcal{H}^n\big(\Sigma \cap B(x,r) \cap \Gamma\big) \geq \varepsilon r^n. \tag{5.36}$$

*If $\Sigma$ is compact, then this is only required for $r \in (0, \operatorname{diam}(\Sigma)]$.*

*Let* `(BP)`$^1$`LG` *denote* `BPLG`*. For each $k \in \mathbb{N}$, a set $\Sigma \subseteq \mathbb{R}^{n+1}$ is said to have* `Big Pieces of (BP)`$^k$`LG` (`(BP)`$^{k+1}$`LG`) *if it is $n$-dimensional Ahlfors-David regular and the following holds: There exist $\delta$, $\varepsilon$, $M \in (0,\infty)$, referred to as the* (BP)$^{k+1}$LG *constants of $\Sigma$, such that for each $x \in \Sigma$ and $r > 0$, there is a set $\Omega \subseteq \mathbb{R}^{n+1}$ that has* (BP)$^k$LG *with* ADR *constant at most equal to $M$, and* (BP)$^k$LG *constants $\varepsilon$, $M$, such that*

$$\mathcal{H}^n\big(\Sigma \cap B(x,r) \cap \Omega\big) \geq \delta r^n. \tag{5.37}$$

*If $\Sigma$ is compact, then this is only required for $r \in (0, \operatorname{diam}(\Sigma)]$.*

We now combine the inductive scheme from Section 4 with the square function estimates for Lipschitz graphs from Subsection 5.1 to prove that square function estimates are stable under the so-called big pieces functor.

**Theorem 5.6.** *Let $k \in \mathbb{N}$ and suppose that $\Sigma \subseteq \mathbb{R}^{n+1}$ has* (BP)$^k$LG*. Let $K$ be a real-valued kernel satisfying* (5.1)*, and let $\mathcal{T}$ denote the integral operator associated with $\Sigma$ as in* (5.3)*. Then there exists a constant $C \in (0,\infty)$ depending only on $n$, the* (BP)$^k$LG *constants of $\Sigma$, and $\|\partial^\alpha K\|_{L^\infty(S^n)}$ for $|\alpha| \leq 2$, such that*

$$\int_{\mathbb{R}^{n+1}\setminus\Sigma} |\nabla \mathcal{T} f(x)|^2 \operatorname{dist}(x,\Sigma)\, dx \leq C \int_\Sigma |f|^2\, d\sigma, \qquad \forall\, f \in L^2(\Sigma,\sigma), \tag{5.38}$$

*where $\sigma := \mathscr{H}^n\lfloor\Sigma$ is the measure induced by the $n$-dimensional Hausdorff measure on $\Sigma$.*

*Proof.* The proof proceeds by induction on $\mathbb{N}$. For $k = 1$, suppose that $\Sigma \subseteq \mathbb{R}^{n+1}$ has BPLG with BPLG constants $\varepsilon_0$, $C_0 \in (0,\infty)$. For each $x \in \Sigma$ and $r > 0$, there is an $n$-dimensional Lipschitz graph $\Gamma \subseteq \mathbb{R}^{n+1}$ with Lipschitz constant at most equal to $C_0$, such that

$$\mathcal{H}^n\big(\Sigma \cap B(x,r) \cap \Gamma\big) \geq \varepsilon_0 r^n. \tag{5.39}$$

It follows from Theorem 5.1 that $\Sigma$ has BPSFE with BPSFE character depending only on the BPLG constants of $\Sigma$, and $\|\partial^\alpha K\|_{L^\infty(S^n)}$ for $|\alpha| \leq 2$. It then follows from Theorem 4.3 that (5.38) holds for some $C \in (0,\infty)$ depending only on $n$ and the constants just mentioned.

Now let $j \in \mathbb{N}$ and assume that the statement of the theorem holds in the case $k = j$. Suppose that $\Sigma \subseteq \mathbb{R}^{n+1}$ has (BP)$^{j+1}$LG with (BP)$^{j+1}$LG constants $\varepsilon_1$, $\varepsilon_2$, $C_1 \in (0,\infty)$. For each $x \in \Sigma$ and $r > 0$, there is a set $\Omega \subseteq \mathbb{R}^{n+1}$ that has $(\mathrm{BP})^j$LG with ADR constant at most equal to $C_1$, and $(\mathrm{BP})^j$LG constants $\varepsilon_1$, $C_1$, such that

$$\mathcal{H}^n\big(\Sigma \cap B(x,r) \cap \Omega\big) \geq \varepsilon_2 r^n. \tag{5.40}$$

It follows by the inductive assumption that $\Sigma$ has BPSFE with BPSFE character depending only on the constants specified in the theorem in the case $k = j$. Applying again Theorem 4.3 we obtain that (5.38) holds for some $C \in (0,\infty)$ depending only on $n$, the $(\mathrm{BP})^{j+1}$LG constants of $\Sigma$, and $\|\partial^\alpha K\|_{L^\infty(S^n)}$ for $|\alpha| \leq 2$. This completes the proof. □

The recent result by J. Azzam and R. Schul (cf. [9, Corollary 1.7]) that uniformly rectifiable sets have $(\mathrm{BP})^2$LG allows us to obtain the following as an immediate corollary of Theorem 5.6.

**Corollary 5.7.** *Suppose that $\Sigma \subseteq \mathbb{R}^{n+1}$ is a uniformly rectifiable set. Let $K$ be a real-valued kernel satisfying* (5.1)*, and let $\mathcal{T}$ denote the integral operator associated with $\Sigma$ as in* (5.3)*. Then there exists a constant $C \in (0,\infty)$, depending only on $n$, the UR constants of $\Sigma$, and $\|\partial^\alpha K\|_{L^\infty(S^n)}$ for $|\alpha| \leq 2$, such that*

$$\int_{\mathbb{R}^{n+1}\setminus\Sigma} |\nabla \mathcal{T} f(x)|^2 \operatorname{dist}(x,\Sigma)\, dx \leq C \int_\Sigma |f|^2\, d\sigma, \qquad \forall\, f \in L^2(\Sigma,\sigma), \tag{5.41}$$

*where $\sigma := \mathscr{H}^n\lfloor\Sigma$ is the measure induced by the $n$-dimensional Hausdorff measure on $\Sigma$.*

## 5.3 Square function estimates for integral operators with variable kernels

The square function estimates from Theorem 5.6 and Corollary 5.7 have been formulated for *convolution type* integral operators and our goal in this subsection is to prove some versions of these results which apply to integral operators with variable coefficient kernels. A first result in this regard reads as follows.

**Theorem 5.8.** *Let $k \in \mathbb{N}$ and suppose that $\Sigma \subseteq \mathbb{R}^{n+1}$ is compact and has $(\mathrm{BP})^k$LG. Then there exists a positive integer $M = M(n)$ with the following significance. Assume that $\mathcal{U}$ is a bounded, open neighborhood of $\Sigma$ in $\mathbb{R}^{n+1}$ and consider a function*

$$\mathcal{U} \times \big(\mathbb{R}^{n+1}\setminus\{0\}\big) \ni (x,z) \mapsto b(x,z) \in \mathbb{R} \tag{5.42}$$

*which is odd and* (*positively*) *homogeneous of degree $-n$ in the variable $z \in \mathbb{R}^{n+1}\setminus\{0\}$, and which has the property that*

$$\partial_x^\beta \partial_z^\alpha b(x,z) \text{ is continuous and bounded on } \mathcal{U}\times S^n \text{ for } |\alpha| \leq M \text{ and } |\beta| \leq 1. \tag{5.43}$$

*Finally, define the variable kernel integral operator*

$$\mathcal{B}f(x) := \int_\Sigma b(x,x-y) f(y)\, d\sigma(y), \qquad x \in \mathcal{U}\setminus\Sigma, \tag{5.44}$$

*where $\sigma := \mathscr{H}^n\lfloor\Sigma$ is the measure induced by the $n$-dimensional Hausdorff measure on $\Sigma$.*

*Then there exists a constant $C \in (0,\infty)$ depending only on $n$, the* $(\mathrm{BP})^k$LG *constants of $\Sigma$, the diameter of $\mathcal{U}$, and $\|\partial_x^\beta\partial_z^\alpha b\|_{L^\infty(\mathcal{U}\times S^n)}$ for $|\alpha| \leq 2$, $|\beta| \leq 1$, such that*

$$\int_{\mathcal{U}\setminus\Sigma} |\nabla \mathcal{B} f(x)|^2 \operatorname{dist}(x,\Sigma)\, dx \leq C \int_\Sigma |f|^2\, d\sigma, \qquad \forall\, f \in L^2(\Sigma,\sigma). \tag{5.45}$$

*In particular,* (5.45) *holds whenever $\Sigma$ is uniformly rectifiable* (*while retaining the other background assumptions*).

The following two geometric lemmas from [62] will be used to prove Theorem 5.8.

**Lemma 5.9.** *Let $(\mathscr{X},\rho)$ be a geometrically doubling quasi-metric space and let $\Sigma\subseteq\mathscr{X}$ be a set such that $\big(\Sigma,\rho\big|_{\Sigma},\mathcal{H}^{d}_{\mathscr{X},\rho}\lfloor\Sigma\big)$ is a d-dimensional* ADR *space for some $d>0$. Assume that $\mu$ is a Borel measure on $\mathscr{X}$ satisfying*

$$\sup_{x\in\mathscr{X},\,r>0}\frac{\mu\big(B_{\rho_{\#}}(x,r)\big)}{r^{m}}<+\infty \tag{5.46}$$

*for some $m\geq 0$. Fix a constant $c>0$ and real numbers $\alpha<m-d$ and $N<m-\max\{\alpha,0\}$. Then there exists $C\in(0,\infty)$, depending on the supremum in (5.46), the geometric doubling constant of $(\mathscr{X},\rho)$, the* ADR *constant of $\Sigma$, as well as on $N$, $\alpha$, and $c$, such that*

$$\int\limits_{B_{\rho_{\#}}(x,r)\setminus\overline{\Sigma}}\frac{\operatorname{dist}_{\rho_{\#}}(y,\Sigma)^{-\alpha}}{\rho_{\#}(x,y)^{N}}\,d\mu(y),\leq C\,r^{m-\alpha-N}\qquad\forall\,x\in\mathscr{X},\quad\forall\,r>\operatorname{dist}_{\rho_{\#}}(x,E)/c. \tag{5.47}$$

**Lemma 5.10.** *Let $(\mathscr{X},\rho)$ be a quasi-metric space. Suppose $E\subseteq\mathscr{X}$ is nonempty and $\sigma$ is a measure on $E$ such that $(E,\rho\big|_{E},\sigma)$ is a d-dimensional* ADR *space for some $d>0$. Fix a real number $0\leq N<d$. Then there exists $C\in(0,\infty)$ depending only on $N$, $\rho$, and the* ADR *constant of $E$, such that*

$$\int\limits_{E\cap B_{\rho_{\#}}(x,r)}\frac{1}{\rho_{\#}(x,y)^{N}}\,d\sigma(y)\leq C\,r^{d-N},\qquad\forall\,x\in\mathscr{X},\quad\forall\,r>\operatorname{dist}_{\rho_{\#}}(x,E). \tag{5.48}$$

We are now ready to present the proof of Theorem 5.8.

*Proof of Theorem 5.8.* Set

$$H_0:=1,\quad H_1:=n+1,\qquad\text{and}\qquad H_\ell:=\binom{n+\ell}{\ell}-\binom{n+\ell-2}{\ell-2}\qquad\text{if }\ \ell\geq 2, \tag{5.49}$$

and, for each $\ell\in\mathbb{N}_0$, let $\big\{\Psi_{i\ell}\big\}_{1\leq i\leq H_\ell}$ be an orthonormal basis for the space of spherical harmonics of degree $\ell$ on the $n$-dimensional sphere $S^n$. In particular,

$$H_\ell\leq(\ell+1)\cdot(\ell+2)\cdots(n+\ell-1)\cdot(n+\ell)\leq C_n\,\ell^n\qquad\text{for }\ \ell\geq 2 \tag{5.50}$$

and, if $\Delta_{S^n}$ denotes the Laplace-Beltrami operator on $S^n$, then for each $\ell\in\mathbb{N}_0$ and $1\leq i\leq H_\ell$,

$$\Delta_{S^n}\Psi_{i\ell}=-\ell(n+\ell-1)\Psi_{i\ell}\ \text{ on }\ S^n,\ \text{ and }\ \Psi_{i\ell}\Big(\frac{x}{|x|}\Big)=\frac{P_{i\ell}(x)}{|x|^{\ell}} \tag{5.51}$$

for some homogeneous harmonic polynomial $P_{i\ell}$ of degree $\ell$ in $\mathbb{R}^{n+1}$. Also,

$$\big\{\Psi_{i\ell}\big\}_{\ell\in\mathbb{N}_0,\,1\leq i\leq H_\ell}\ \text{ is an orthonormal basis for }\ L^2(S^n), \tag{5.52}$$

hence,

$$\|\Psi_{i\ell}\|_{L^2(S^n)}=1\ \text{ for each }\ell\in\mathbb{N}_0\text{ and }1\leq i\leq H_\ell. \tag{5.53}$$

More details on these matters may be found in, e.g., [73, pp. 137–152] and [72, pp. 68–75].

Next, fix an even integer $d > (n/2) + 2$. Sobolev's embedding theorem then gives that for each $\ell \in \mathbb{N}_0$ and $1 \le i \le H_\ell$ (with $I$ standing for the identity on $S^n$)

$$\|\Psi_{i\ell}\|_{C^2(S^n)} \le C_n \big\|(I - \Delta_{S^n})^{d/2}\Psi_{i\ell}\big\|_{L^2(S^n)} \le C_n \ell^d, \tag{5.54}$$

where the last inequality is a consequence of (5.51)-(5.53).

Fix $\ell \in \mathbb{N}_0$ and $1 \le i \le H_\ell$ arbitrary. If we now define

$$a_{i\ell}(x) := \int_{S^n} b(x,\omega)\Psi_{i\ell}(\omega)\, d\omega, \quad \text{for each} \quad x \in \mathcal{U}, \tag{5.55}$$

it follows from the last formula in (5.51) and the assumptions on $b(x,z)$ that

$$a_{i\ell} \text{ is identically zero whenever } \ell \text{ is even.} \tag{5.56}$$

Also, for each $N \in \mathbb{N}$ with $2N \le M$ and each multiindex $\beta$ of length at most 1 we have

$$\begin{aligned}
\sup_{x\in\mathcal{U}}\big|[-\ell(n+\ell-1)]^N(\partial^\beta a_{i\ell})(x)\big| &= \sup_{x\in\mathcal{U}}\Big|\int_{S^n}(\partial_x^\beta b)(x,\omega)\big(\Delta_{S^n}^N\Psi_{i\ell}\big)(\omega)\, d\omega\Big| \\
&= \sup_{x\in\mathcal{U}}\Big|\int_{S^n}\big(\partial_x^\beta\Delta_{S^n}^N b\big)(x,\omega)\Psi_{i\ell}(\omega)\, d\omega\Big| \\
&\le \sup_{x\in\mathcal{U}}\big\|\big(\partial_x^\beta\Delta_{S^n}^N b\big)(x,\cdot)\big\|_{L^2(S^n)} \\
&\le C_n \sup_{\substack{(x,z)\in\mathcal{U}\times S^n \\ |\alpha|\le M}}\big|\big(\partial_x^\beta\partial_z^\alpha b\big)(x,z)\big| =: C_b < \infty.
\end{aligned} \tag{5.57}$$

Hence, for each number $N \in \mathbb{N}$ with $2N \le M$, there exists a constant $C_{n,N}$ such that

$$\sup_{x\in\mathcal{U},\, |\beta|\le 1}\big|(\partial^\beta a_{i\ell})(x)\big| \le C_{n,N}C_b\,\ell^{-2N}, \qquad \ell \in \mathbb{N}_0, \quad 1 \le i \le H_\ell. \tag{5.58}$$

For each fixed $x \in \mathcal{U}$, expand the function $b(x,\cdot) \in L^2(S^n)$ with respect to the orthonormal basis $\big\{\Psi_{i\ell}\big\}_{\ell\in\mathbb{N}_0,\, 1\le i\le H_\ell}$ in order to obtain (in the sense of $L^2(S^n)$ in the variable $z/|z| \in S^n$)

$$b(x,z) = b\Big(x,\frac{z}{|z|}\Big)|z|^{-n} = \sum_{\ell\in 2\mathbb{N}+1}\sum_{i=1}^{H_\ell} a_{i\ell}(x)\Psi_{i\ell}\Big(\frac{z}{|z|}\Big)|z|^{-n}, \tag{5.59}$$

where the last equality is a consequence of (5.56). For each $\ell \in 2\mathbb{N}+1$ let us now set

$$k_{i\ell}(z) := \Psi_{i\ell}\Big(\frac{z}{|z|}\Big)|z|^{-n}, \quad z \in \mathbb{R}^{n+1}\setminus\{0\}, \tag{5.60}$$

so that, if $d$ is as in (5.54), then for each $|\alpha| \le 2$ we have

$$\|\partial^\alpha k_{i\ell}\|_{L^\infty(S^n)} \le C_n\|\Psi_{i\ell}\|_{C^2(S^n)} \le C_n \ell^d. \tag{5.61}$$

Also, given any $f \in L^2(\Sigma,\sigma)$, set

$$\mathcal{B}_{i\ell}f(x) := \int_\Sigma k_{i\ell}(x-y)f(y)\, d\sigma(y), \quad x \in \mathcal{U}\setminus\Sigma, \tag{5.62}$$

and note that for any compact subset $\mathcal{O}$ of $\mathcal{U} \setminus \Sigma$ and any multiindex $\alpha$ with $|\alpha| \leq 1$,

$$\sup_{x\in\mathcal{O}} \big| \big( \partial^\alpha \mathcal{B}_{i\ell} f \big)(x) \big| \leq C(n, \mathcal{O}, \Sigma) \ell^d, \tag{5.63}$$

by (5.61). On the other hand, if $N > (d+1)/2$ (a condition which we shall assume from now on) then (5.54) and (5.58) imply that the last series in (5.59) converges to $b(x, z)$ uniformly for $x \in \mathcal{U}$ and $z$ in compact subsets of $\mathbb{R}^{n+1} \setminus \{0\}$. As such, it follows from (5.60) and (5.62) that

$$\mathcal{B} f(x) = \sum_{\ell\in 2\mathbb{N}+1} \sum_{i=1}^{H_\ell} a_{i\ell}(x) \mathcal{B}_{i\ell} f(x), \quad \text{uniformly on compact subsets of } \ \mathcal{U} \setminus \Sigma. \tag{5.64}$$

Using this, (5.63) and (5.58), we may differentiate term-by-term to obtain

$$\big( \nabla \mathcal{B} f \big)(x) = \sum_{\ell\in 2\mathbb{N}+1} \sum_{i=1}^{H_\ell} a_{i\ell}(x) \big( \nabla \mathcal{B}_{i\ell} f \big)(x) + \sum_{\ell\in 2\mathbb{N}+1} \sum_{i=1}^{H_\ell} (\nabla a_{i\ell})(x) \mathcal{B}_{i\ell} f(x), \tag{5.65}$$

uniformly for all $x$ in compact subsets of $\mathcal{U} \setminus \Sigma$.

Moving on, observe that for each $\ell \in 2\mathbb{N}+1$ and $1 \leq i \leq H_\ell$, Theorem 5.6 gives

$$\int_{\mathcal{U}\setminus\Sigma} |\nabla \mathcal{B}_{i\ell} f(x)|^2 \operatorname{dist}(x, \Sigma)\, dx \leq C_{i\ell} \int_\Sigma |f|^2 \, d\sigma, \qquad \forall\, f \in L^2(\Sigma, \sigma), \tag{5.66}$$

where $C \in (0, \infty)$ depends only on $n$ and the $(\mathrm{BP})^k\mathrm{LG}$ constants of $\Sigma$, and where

$$C_{i\ell} = C \max_{|\alpha|\leq 2} \|\partial^\alpha k_{i\ell}\|_{L^\infty(S^n)} \leq C \ell^d, \tag{5.67}$$

by (5.54). If $M \in \mathbb{N}$ is odd and satisfies $M > d+1$, then we may choose $N \in \mathbb{N}$ such that $d+1 < 2N < M$, in which case (5.58) and (5.67) imply that for all $f \in L^2(\Sigma, \sigma)$, we have

$$\begin{aligned} &\sum_{\ell\in 2\mathbb{N}+1} \sum_{i=1}^{H_\ell} \Big( \int_{\mathcal{U}\setminus\Sigma} |a_{i\ell}(x)|^2 |\nabla \mathcal{B}_{i\ell} f(x)|^2 \operatorname{dist}(x, \Sigma)\, dx \Big)^{1/2} \\ &\qquad \leq C_{n,N} C_b \sum_{\ell\in 2\mathbb{N}+1} \sum_{i=1}^{H_\ell} \ell^{-2N} \Big( \int_{\mathcal{U}\setminus\Sigma} |\nabla \mathcal{B}_{i\ell} f(x)|^2 \operatorname{dist}(x, \Sigma)\, dx \Big)^{1/2} \\ &\qquad \leq C_{n,N} C_b \Big( \sum_{\ell\in 2\mathbb{N}+1} \sum_{i=1}^{H_\ell} \ell^{d/2-2N} \Big) \Big( \int_\Sigma |f|^2 \, d\sigma \Big)^{1/2} = C \Big( \int_\Sigma |f|^2 \, d\sigma \Big)^{1/2}. \end{aligned} \tag{5.68}$$

Next, if $\ell \in \mathbb{N}$, $1 \leq i \leq H_\ell$ and $f \in L^2(\Sigma, \sigma)$, then

$$\begin{aligned} \Big( \int_{\mathcal{U}\setminus\Sigma} |\nabla a_{i\ell}|^2 |\mathcal{B}_{i\ell} f(x)|^2 \operatorname{dist}(x, \Sigma)\, dx \Big)^{1/2} &\leq C_{n,N} C_b\, \ell^{-2N} \Big( \int_{\mathcal{U}\setminus\Sigma} \big| \operatorname{dist}(x, \Sigma)^{1/2} \mathcal{B}_{i\ell} f(x) \big|^2 \, dx \Big)^{1/2} \\ &= C_{n,N} C_b\, \ell^{-2N} \Big( \int_{\mathcal{U}\setminus\Sigma} \big| \mathcal{T}_{i\ell} f(x) \big|^2 \, dx \Big)^{1/2}, \end{aligned} \tag{5.69}$$

where $\mathcal{T}_{i\ell} : L^2(\Sigma, \sigma) \longrightarrow L^2(\mathcal{U} \setminus \Sigma)$ is the integral operator with integral kernel

$$K_{i\ell}(x, y) := \operatorname{dist}(x, \Sigma)^{1/2} k_{i\ell}(x - y), \qquad x \in \mathcal{U} \setminus \Sigma, \ \ y \in \Sigma. \tag{5.70}$$

Note that

$$\sup_{x\in\mathcal{U}\setminus\Sigma}\int_\Sigma \big|K_{i\ell}(x,y)\big|\,d\sigma(y) \le \|\Psi_{i\ell}\|_{L^\infty(S^n)}\sup_{x\in\mathcal{U}\setminus\Sigma}\int_\Sigma \frac{\operatorname{dist}(x,\Sigma)^{1/2}}{|x-y|^n}\,d\sigma(y)$$
$$\le C\ell^d \sup_{x\in\mathcal{U}\setminus\Sigma}\int_\Sigma \frac{1}{|x-y|^{n-1/2}}\,d\sigma(y)$$
$$\le C\ell^d\operatorname{diam}(\mathcal{U})^{1/2}, \tag{5.71}$$

by (5.54) and Lemma 5.10, and that

$$\sup_{y\in\Sigma}\int_{\mathcal{U}\setminus\Sigma}\big|K_{i\ell}(x,y)\big|\,dx \le \|\Psi_{i\ell}\|_{L^\infty(S^n)}\sup_{y\in\Sigma}\int_{\mathcal{U}\setminus\Sigma}\frac{\operatorname{dist}(x,\Sigma)^{1/2}}{|x-y|^n}\,dx$$
$$\le C\ell^d\operatorname{diam}(\mathcal{U})^{3/2}, \tag{5.72}$$

by (5.54) and Lemma 5.9. From (5.71)-(5.72) and Schur's Lemma we deduce that

$$\big\|\mathcal{T}_{i\ell}\big\|_{L^2(\Sigma,\sigma)\to L^2(\mathcal{U}\setminus\Sigma)} \le C\ell^d\operatorname{diam}(\mathcal{U}). \tag{5.73}$$

Combining (5.73) and (5.69) we conclude that

$$\Big(\int_{\mathcal{U}\setminus\Sigma}|\nabla a_{i\ell}|^2|\mathcal{B}_{i\ell}f(x)|^2\operatorname{dist}(x,\Sigma)\,dx\Big)^{1/2} \le C(\mathcal{U})\ell^d\Big(\int_\Sigma|f|^2\,d\sigma\Big)^{1/2}, \tag{5.74}$$

for all $f\in L^2(\Sigma,\sigma)$, whenever $\ell\in\mathbb{N}$ and $1\le i\le H_\ell$. Thus, there exists $C\in(0,\infty)$ such that

$$\sum_{\ell\in2\mathbb{N}+1}\sum_{i=1}^{H_\ell}\Big(\int_{\mathcal{U}\setminus\Sigma}|\nabla a_{i\ell}|^2|\mathcal{B}_{i\ell}f(x)|^2\operatorname{dist}(x,\Sigma)\,dx\Big)^{1/2} \le C\Big(\int_\Sigma|f|^2\,d\sigma\Big)^{1/2}, \tag{5.75}$$

for all $f\in L^2(\Sigma,\sigma)$.

Fix now an arbitrary compact subset $\mathcal{O}$ of $\mathcal{U}\setminus\Sigma$. Then (5.65), (5.68) and (5.75) imply

$$\Big(\int_{\mathcal{O}}|\nabla\mathcal{B}f(x)|^2\operatorname{dist}(x,\Sigma)\,dx\Big)^{1/2} \le C\Big(\int_\Sigma|f|^2\,d\sigma\Big)^{1/2}, \tag{5.76}$$

where the constant $C$ is independent of $\mathcal{O}$ and $f\in L^2(\Sigma,\sigma)$. Upon letting $\mathcal{O}\nearrow\mathcal{U}\setminus\Sigma$ in (5.76), Lebesgue's Monotone Convergence Theorem then yields (5.45). Finally, the last claim in the statement of Theorem 5.8 is justified in a similar manner, based on Corollary 5.7. □

It is also useful to treat the following variant of (5.44):

$$\widetilde{\mathcal{B}}f(x) := \int_\Sigma b(y,x-y)f(y)\,d\sigma(y), \qquad x\in\mathcal{U}\setminus\Sigma. \tag{5.77}$$

The same analysis works with $x$ replaced by $y$ in the spherical harmonic expansion (5.59) (the argument is simpler since then $a_{i\ell}$ acts as a multiplier in the $y$ variable) to prove the following.

**Theorem 5.11.** *In the setting of Theorem 5.8 with $\widetilde{\mathcal{B}}$ given by* (5.77) *and assuming, instead of* (5.43)*, that $\partial_z^\alpha b(x,z)$ is continuous and bounded on $\mathcal{U}\times S^n$ for all $|\alpha|\le M$, it holds that*

$$\int_{\mathcal{U}\setminus\Sigma}|\nabla\widetilde{\mathcal{B}}f(x)|^2\operatorname{dist}(x,\Sigma)\,dx \le C\int_\Sigma|f|^2\,d\sigma, \qquad \forall\, f\in L^2(\Sigma,\sigma). \tag{5.78}$$

Theorem 5.8 and Theorem 5.11 also apply to the Schwartz kernels of certain pseudodifferential operators. Recall that a pseudodifferential operator $Q(x,D)$ with symbol $q(x,\xi)$ in Hörmander's class $S^m_{1,0}$ is given by the oscillatory integral

$$Q(x,D)u = (2\pi)^{-(n+1)/2}\int q(x,\xi)\hat{u}(\xi)e^{i\langle x,\xi\rangle}d\xi = (2\pi)^{-(n+1)}\iint q(x,\xi)e^{i\langle x-y,\xi\rangle}u(y)dyd\xi. \quad (5.79)$$

We define a smaller class of symbols $S^m_{\rm cl}$ by requiring that the (matrix-valued) function $q(x,\xi)$ has an asymptotic expansion of the form

$$q(x,\xi) \sim q_m(x,\xi) + q_{m-1}(x,\xi) + \cdots, \quad (5.80)$$

with $q_j$ smooth in $x$ and $\xi$ and homogeneous of degree $j$ in $\xi$ (for $|\xi| \geq 1$). We call $q_m(x,\xi)$, i.e. the leading term in (5.80), the *principal symbol* of $q(x,D)$. In fact, we shall find it convenient to work with classes of symbols which only exhibit a limited amount of regularity in the spatial variable (while still $C^\infty$ in the Fourier variable). Specifically, for each $r \geq 0$ we define

$$C^r S^m_{1,0} := \big\{q(X,\xi) : \|D^\alpha_\xi q(\cdot,\xi)\|_{C^r} \leq C_\alpha(1+|\xi|)^{m-|\alpha|}, \quad \forall\,\alpha\big\}. \quad (5.81)$$

Denote by $\mathrm{OP}C^rS^m_{1,0}$ the class of pseudodifferential operators associated with such symbols. We write $\mathrm{OP}C^rS^m_{\rm cl}$ for the subclass of *classical* pseudodifferential operators in $\mathrm{OP}C^rS^m_{1,0}$ whose symbols can be expanded as in (5.80), where $q_j(x,\xi) \in C^rS^{m-j}_{1,0}$ is homogeneous of degree $j$ in $\xi$ for $|\xi| \geq 1$, $j = m, m-1, \dots$. Finally, we set $\emptyset\mathrm{P}C^rS^m_{\rm cl}$ for the space of all formal adjoints of operators in $\mathrm{OP}C^rS^m_{\rm cl}$.

Given a classical pseudodifferential operator $Q(x,D) \in \mathrm{OP}C^rS^{-1}_{\rm cl}$, we denote by $k_Q(x,y)$ and $\mathrm{Sym}_Q(x,\xi)$ its Schwartz kernel and its principal symbol, respectively. Next, if the sets $\Sigma \subseteq \mathcal{U} \subseteq \mathbb{R}^{n+1}$ are as in Theorem 5.8, we can introduce the integral operator

$$\mathcal{B}_Q f(x) := \int_\Sigma k_Q(x,y)f(y)\,d\sigma(y), \qquad x \in \mathcal{U}\setminus\Sigma. \quad (5.82)$$

In this context, Theorem 5.8 and Theorem 5.11 yield the following result.

**Theorem 5.12.** *Let $\Sigma \subseteq \mathbb{R}^{n+1}$ be compact and uniformly rectifiable, and assume that $\mathcal{U}$ is a bounded, open neighborhood of $\Sigma$ in $\mathbb{R}^{n+1}$. Let $Q(x,D) \in \mathrm{OP}C^1S^{-1}_{\rm cl}$ be such that $\mathrm{Sym}_Q(x,\xi)$ is odd in $\xi$. Then the operator (5.82) satisfies*

$$\int_{\mathcal{U}\setminus\Sigma} |\nabla\mathcal{B}_Q f(x)|^2\,\mathrm{dist}(x,\Sigma)\,dx \leq C\int_\Sigma |f|^2\,d\sigma, \qquad \forall\, f \in L^2(\Sigma,\sigma). \quad (5.83)$$

*Moreover, a similar result is valid for a pseudodifferential operator $Q(x,D) \in \emptyset\mathrm{P}C^0S^{-1}_{\rm cl}$.*

In fact, since the main claims in Theorem 5.12 are local in nature and given the invariance of the class of domains and pseudodifferential operators (along with their Schwartz kernels and principal symbols) under smooth diffeomorphisms, these results extend naturally to domains on manifolds and pseudodifferential operators acting between vector bundles, and as such, extend results proved in [64] for Lipschitz subdomains of Riemannian manifolds.

# 6 $L^p$ Square Function Estimates

We have so far only considered $L^2$ square function estimates. We now consider $L^p$ versions for $p \in (0, \infty]$. The natural setting for the consideration of these estimates is in term of mixed norm spaces $L^{(p,q)}(\mathscr{X}, E)$, originally introduced in [60] (cf. also [10] for related matters). We begin by using the tools developed in Section 2 to analyze these spaces in the context of an ambient quasi-metric space $\mathscr{X}$ and a closed subset $E$. In the case $\mathscr{X} = \mathbb{R}^{n+1}$ and $E = \partial\mathbb{R}^{n+1}_+ \approx \mathbb{R}^n$, the mixed norm spaces correspond to the tent spaces introduced by R. Coifman, Y. Meyer and E.M. Stein in [17]. The preliminary analysis in Subsections 6.1 and 6.2 is based on the techniques developed in that paper, although we need to overcome a variety of geometric obstructions that arise outside of the Euclidean setting. We build on this in Subsection 6.3, where we prove that $L^2$ square function estimates associated with integral operators $\Theta_E$, as defined in Section 3, follow from weak $L^p$ square function estimates for any $p \in (0, \infty)$. This is achieved by combining the $T(1)$ theorem from Subsection 3.1 with a weak type John-Nirenberg lemma for Carleson measures, the Euclidean version of which appears in [5]. The theory culminates in Subsection 6.4, where we prove an extrapolation theorem for estimates associated with integral operators $\Theta_E$, as defined in Section 3. In particular, we prove that a weak $L^q$ square function estimate for any $q \in (0, \infty)$ implies that square functions are bounded from $H^p$ into $L^p$ for all $p \in (\frac{d}{d+\gamma}, \infty)$, where $H^p$ is a Hardy space, $d$ is the the dimension of $E$, and $\gamma$ is a finite positive constant depending on the ambient space $\mathscr{X}$ and the operator $\Theta_E$.

## 6.1 Mixed norm spaces

We begin by considering the mixed norm spaces $L^{(p,q)}$ from [60] (cf. also [10]) and then, following the theory of tent spaces in [17], record some extensive preliminaries that are used throughout Section 6. In particular, Theorem 6.5 contains an equivalence for the quasi-norms of the mixed norm spaces that is essential in the next subsection.

Let $(\mathscr{X}, \rho)$ be a quasi-metric space, $E$ a nonempty subset of $\mathscr{X}$, and $\mu$ a Borel measure on $(\mathscr{X}, \tau_\rho)$. Recall the regularized version $\rho_\#$ of the quasi-distance $\rho$ discussed in Theorem 2.1, and recall that we employ the notation $\delta_E(y) = \mathrm{dist}_{\rho_\#}(y, E)$ for each $y \in \mathscr{X}$. Next, let $\kappa > 0$ be arbitrary, fixed, and consider the `nontangential approach regions`

$$\Gamma_\kappa(x) := \big\{ y \in \mathscr{X} \setminus E : \ \rho_\#(x,y) < (1+\kappa)\,\delta_E(y) \big\}, \qquad \forall\, x \in E. \tag{6.1}$$

Occasionally, we shall refer to $\kappa$ as the `aperture` of the nontangential approach region $\Gamma_\kappa(x)$. Since both $\rho_\#(\cdot,\cdot)$ and $\delta_E(\cdot)$ are continuous (cf. Theorem 2.1) it follows that $\Gamma_\kappa(x)$ is an open subset of $(\mathscr{X}, \tau_\rho)$, for each $x \in E$, and that $\mathscr{X} \setminus \overline{E} = \bigcup_{x \in E} \Gamma_\kappa(x)$, where $\overline{E}$ denotes the closure of $E$ in $\tau_\rho$. If, in addition, $E$ is a proper closed subset of $(\mathscr{X}, \tau_\rho)$, then for any $\mu$-measurable function $u : \mathscr{X} \setminus E \to [0, \infty]$, both $F : E \longrightarrow [0, \infty]$ and $G : E \longrightarrow [0, \infty]$ defined by

$$F(x) := \int_{\Gamma_\kappa(x)} u(y)\, d\mu(y) \quad \text{and} \quad G(x) := \sup_{y \in \Gamma_\kappa(x)} |u(y)|, \qquad \forall\, x \in E, \tag{6.2}$$

are lower semi-continuous relative to the topology induced by $\tau_\rho$ on $E$. For each $q \in (0, \infty)$ and $\kappa \in (0, \infty)$, then define the $L^q$-based `Lusin operator`, or `area operator`, $\mathscr{A}_{q,\kappa}$ for all $\mu$-measurable functions $u : \mathscr{X} \setminus E \to \overline{\mathbb{R}} := [-\infty, +\infty]$ by

$$(\mathscr{A}_{q,\kappa} u)(x) := \Big( \int_{\Gamma_\kappa(x)} |u(y)|^q \, d\mu(y) \Big)^{\frac{1}{q}}, \qquad \forall\, x \in E. \tag{6.3}$$

The lower semi-continuity of $F$ in (6.2) implies that $\mathscr{A}_{q,\kappa}u$ is lower semi-continuous, hence

$$\{x \in E : (\mathscr{A}_{q,\kappa}u)(x) > \lambda\} \quad \text{is an open subset of } (E, \tau_\rho) \text{ for each } \lambda > 0, \tag{6.4}$$

and $\mathscr{A}_{q,\kappa}u : E \to [0, \infty]$ is $\sigma$-measurable for any Borel measure $\sigma$ on $(E, \tau_{\rho|_E})$. Also, define the `nontangential maximal operator` $\mathcal{N}_\kappa$ for all functions $u : \mathscr{X} \setminus E \to \overline{\mathbb{R}}$ by

$$(\mathcal{N}_\kappa u)(x) := \sup_{y \in \Gamma_\kappa(x)} |u(y)|, \qquad \forall\, x \in E. \tag{6.5}$$

The lower semi-continuity of $G$ in (6.2) implies that $\mathcal{N}_\kappa u$ is lower semi-continuous. We now follow [60, 10] to define the `mixed norm space of type` $(p, q)$. If $p \in (0, \infty]$ and $q \in (0, \infty)$, set

$$L^{(p,q)}(\mathscr{X}, E, \mu, \sigma; \kappa) := \{u : \mathscr{X} \setminus E \to \overline{\mathbb{R}} : \; u \; \text{ is } \mu\text{-measurable and } \; \mathscr{A}_{q,\kappa}u \in L^p(E, \sigma)\} \tag{6.6}$$

with the quasi-norm $\|u\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)} := \|\mathscr{A}_{q,\kappa}u\|_{L^p(E,\sigma)}$. If $p \in (0, \infty)$ and $q = \infty$, set

$$L^{(p,\infty)}(\mathscr{X}, E, \mu, \sigma; \kappa) := \{u : \mathscr{X} \setminus E \to \overline{\mathbb{R}} : \; \|u\|_{L^{(p,\infty)}(\mathscr{X},E,\mu,\sigma;\kappa)} := \|\mathcal{N}_\kappa u\|_{L^p(E,\sigma)} < \infty\}. \tag{6.7}$$

Also, set $L^{(\infty,\infty)}(\mathscr{X}, E, \mu, \sigma; \kappa) := L^\infty(\mathscr{X} \setminus E, \mu)$. These spaces generalize the tent spaces $T^p_q$ on $\mathbb{R}^{n+1}_+$ developed in [17], since $T^p_q = L^{(p,q)}\big(\mathbb{R}^{n+1}, \partial\mathbb{R}^{n+1}_+, \mathbf{1}_{\mathbb{R}^{n+1}_+} \frac{dx\,dt}{t^{n+1}}, dx\big)$ for all $p, q \in (0, \infty)$.

In Theorem 6.5, we clarify the dependence of the quasi-norm $\|\cdot\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)}$ on the parameter $\kappa$. The proof requires the following preliminaries. For each $A \subseteq E$ and $\kappa > 0$, define the `fan` (or `saw-tooth`) `region` $\mathcal{F}_\kappa(A)$ `above` $A$, and the `tent region` $\mathcal{T}_\kappa(A)$ `above` $A$, by

$$\mathcal{F}_\kappa(A) := \bigcup_{x \in A} \Gamma_\kappa(x) \quad \text{ and } \quad \mathcal{T}_\kappa(A) := \big(\mathscr{X} \setminus E\big) \setminus \Big(\mathcal{F}_\kappa(E \setminus A)\Big). \tag{6.8}$$

Also, let $\overline{A}$ and $A^\circ$ denote, respectively, the closure and interior of $A$ in $(E, \tau_{\rho|_E})$. Finally, for each $y \in \mathscr{X} \setminus E$, define the (reverse) `conical projection` $\pi^\kappa_y := \{x \in E : \; y \in \Gamma_\kappa(x)\}$.

**Lemma 6.1.** *Let $(\mathscr{X}, \rho)$ be a quasi-metric space and $E$ a proper, nonempty, closed subset of $(\mathscr{X}, \tau_\rho)$. If $\kappa \in (0, \infty)$, then the following properties hold for all $A \subseteq E$:*

*(1)* $\mathcal{F}_\kappa(A) = \mathcal{F}_\kappa(\overline{A})$, $\mathcal{T}_\kappa(A^\circ) = \mathcal{T}_\kappa(A)$ *and* $\mathcal{T}_\kappa(A) \subseteq \mathcal{F}_\kappa(A)$.

*(2) If $A$ is a nonempty subset of $E$, then*

$$\mathcal{F}_\kappa(A) = \{y \in \mathscr{X} \setminus E : \; \mathrm{dist}_{\rho_\#}(y, A) < (1 + \kappa)\, \delta_E(y)\}. \tag{6.9}$$

*If $A$ is a nonempty proper subset of $E$, then*

$$\begin{aligned} \mathcal{T}_\kappa(A) &= \{x \in \mathscr{X} \setminus E : \; \mathrm{dist}_{\rho_\#}(x, A) \le (1 + \kappa)^{-1} \,\mathrm{dist}_{\rho_\#}(x, E \setminus A)\} & (6.10) \\ &= \{y \in \mathscr{X} \setminus E : \; \pi^\kappa_y \subseteq A\}. & (6.11) \end{aligned}$$

*(3)* $\mathcal{F}_\kappa(E) = \mathcal{T}_\kappa(E) = \mathscr{X} \setminus E$, *and moreover, for any family $(A_j)_{j \in J}$ of subsets of $E$:*

$$\bigcup_{j \in J} \mathcal{F}_\kappa(A_j) = \mathcal{F}_\kappa\big(\cup_{j \in J} A_j\big), \qquad \bigcap_{j \in J} \mathcal{T}_\kappa(A_j) = \mathcal{T}_\kappa\big(\cap_{j \in J} A_j\big), \quad \textit{and} \tag{6.12}$$

$$A_1 \subseteq A_2 \subseteq E \implies \mathcal{F}_\kappa(A_1) \subseteq \mathcal{F}_\kappa(A_2) \quad \textit{and} \quad \mathcal{T}_\kappa(A_1) \subseteq \mathcal{T}_\kappa(A_2). \tag{6.13}$$

(4) $\mathcal{F}_\kappa(A)$ *is open and* $\mathcal{T}_\kappa(A)$ *is relatively closed in the topology induced by* $\tau_\rho$.

(5) *If* $y \in \mathscr{X} \setminus E$, *then* $\pi_y^\kappa$ *is relatively open in the topology induced by* $\tau_\rho$.

(6) $B_{\rho_\#}\big(x, C_\rho^{-1} r\big) \setminus E \subseteq \mathcal{T}_\kappa\big(E \cap B_{\rho_\#}(x,r)\big)$ *for all* $r \in (0,\infty)$ *and all* $x \in E$.

(7) *If* $(E, \rho\big|_E)$ *is geometrically doubling, then there exists a constant* $C_o \in (0,\infty)$ *such that the following property holds: If* $\mathcal{O}$ *is a nonempty, open, proper subset of* $(E, \tau_{\rho|_E})$ *with a Whitney decomposition* $\{\Delta_j\}_{j\in J}$, *where* $\Delta_j := E \cap B_\rho(x_j, r_j)$, *as in Proposition 2.5, then*

$$\mathcal{T}_\kappa(\mathcal{O}) \subseteq \bigcup_{j\in J} B_\rho(x_j, C_o r_j). \tag{6.14}$$

*Moreover, there exists* $C \in (0,\infty)$ *such that if* $x \in E$, $r > 0$ *and* $E \setminus B_\rho(x,r) \neq \emptyset$, *then*

$$\mathcal{T}_\kappa\big(E \cap B_\rho(x,r)\big) \subseteq B_\rho(x, Cr) \setminus E. \tag{6.15}$$

(8) *If* $E$ *is bounded, then there exists* $C \in (0,\infty)$ *such that if* $x_0$, $x \in E$ *and* $A$ *is a proper subset of* $E$, *then* $\mathscr{X} \setminus B_{\rho_\#}\big(x_0, C\,\mathrm{diam}_\rho(E)\big) \subseteq \Gamma_\kappa(x)$ *and* $\mathcal{T}_\kappa(A) \subseteq B_{\rho_\#}\big(x_0, C\,\mathrm{diam}_\rho(E)\big)$.

*Proof.* Except for (6.14), the above properties follow from definitions and the continuity of $\rho_\#(\cdot,\cdot)$ and $\delta_E(\cdot)$. To prove (6.14), let $x \in \mathcal{T}_\kappa(\mathcal{O}) \subseteq \mathscr{X} \setminus E$, and since $E$ is closed in $(\mathscr{X}, \tau_\rho)$, this means that $x$ does not belong to $\overline{\mathcal{O}}$ and that $\mathrm{dist}_{\rho_\#}(x, \mathcal{O}) > 0$. Thus, choose $\varepsilon > 0$ and $y \in \mathcal{O}$ so that $\rho_\#(x,y) < (1+\varepsilon)\,\mathrm{dist}_{\rho_\#}(x,\mathcal{O})$. Then there exists $j \in J$ such that $y \in \Delta_j$. We claim that $\varepsilon$ and $C_o$ can be chosen so $x \in B_\rho(x_j, C_o r_j)$, which will complete the proof. To prove the claim, select $\beta \in (0, (\log_2 C_\rho)^{-1}]$ and use (6.10) in combination with Theorem 2.1 and the fact that $y$ belongs to $\Delta_j = B_\rho(x_j, r_j) \cap E$, to write

$$\begin{aligned} \big[\rho_\#(x,y)\big]^\beta &< (1+\varepsilon)^\beta \big[\mathrm{dist}_{\rho_\#}(x,\mathcal{O})\big]^\beta \leq \Big(\frac{1+\varepsilon}{1+\kappa}\Big)^\beta \big[\mathrm{dist}_{\rho_\#}(x, E\setminus\mathcal{O})\big]^\beta \\ &\leq \Big(\frac{1+\varepsilon}{1+\kappa}\Big)^\beta \Big(\big[\rho_\#(x,y)\big]^\beta + C r_j^\beta\Big). \end{aligned} \tag{6.16}$$

Thus, setting $\varepsilon := \kappa/2$ and $C_o := C_\rho C_{\kappa,\beta}$, we have

$$\rho(x,y) \leq C_\rho^2\, \rho_\#(x,y) < C_\rho^2 C^{1/\beta} \Big(\frac{1+\kappa/2}{\big[(1+\kappa)^\beta - (1+\kappa/2)^\beta\big]^{1/\beta}}\Big) r_j =: C_{\kappa,\beta}\, r_j, \tag{6.17}$$

so $\rho(x_j, x) \leq C_\rho \max\{\rho(x_j,y), \rho(y,x)\} < C_\rho C_{\kappa,\beta} r_j$ implies $x \in B_\rho(x_j, C_o r_j)$, as required. $\square$

**Lemma 6.2.** *Let* $(\mathscr{X}, \rho)$ *be a quasi-metric space,* $E$ *a proper, nonempty, closed subset of* $(\mathscr{X}, \tau_\rho)$, $\mu$ *a Borel measure on* $(\mathscr{X}, \tau_\rho)$ *and* $\sigma$ *a Borel measure on* $(E, \tau_{\rho|_E})$. *If* $\kappa > 0$, $y \in \mathscr{X} \setminus E$,

$$\begin{aligned} &y_* \in E \text{ such that } \rho_\#(y, y_*) < (1+\eta)\delta_E(y) \text{ for some } \eta \in (0,\kappa), \text{ and} \\ &0 < \epsilon < \big[(1+\kappa)^\beta - (1+\eta)^\beta\big]^{1/\beta} \text{ for some finite } \beta \in (0, (\log_2 C_\rho)^{-1}], \end{aligned} \tag{6.18}$$

*then*

$$E \cap B_{\rho_\#}\big(y_*, \epsilon\delta_E(y)\big) \subseteq \pi_y^\kappa \subseteq E \cap B_{\rho_\#}\big(y_*, C_\rho(1+\kappa)\delta_E(y)\big). \tag{6.19}$$

*In particular, if* $(E, \rho\big|_E, \sigma)$ *is a space of homogeneous type, and* $\kappa, \kappa' > 0$, *then*

$$c_o^{-1}\, \sigma(\pi_y^\kappa) \leq \sigma(\pi_y^{\kappa'}) \leq c_o\, \sigma(\pi_y^\kappa), \qquad \forall\, y \in \mathscr{X} \setminus E, \tag{6.20}$$

*where* $c_o := C_\sigma (C_\rho^2/\epsilon)^{D_\sigma} (1 + \min\{\kappa,\kappa'\})^{D_\sigma}$, *with* $C_\sigma$ *and* $D_\sigma$ *the constants defined in* (2.19).

*Proof.* Let $\kappa > 0$, $y \in \mathscr{X} \setminus E$ and assume that (6.18) holds. If $x \in E \cap B_{\rho_\#}(y_*, \epsilon\delta_E(y))$, then

$$\rho_\#(x,y)^\beta \leq \rho_\#(x,y_*)^\beta + \rho_\#(y_*,y)^\beta < \epsilon^\beta \delta_E(y)^\beta + (1+\eta)^\beta \delta_E(y)^\beta < (1+\kappa)^\beta \delta_E(y)^\beta \tag{6.21}$$

by Theorem 2.1, thus $x \in \pi_y^\kappa$. Next, if $x \in \pi_y^\kappa$, then $\rho_\#(x,y) < (1+\kappa)\delta_E(y)$, hence

$$\rho_\#(x,y_*) \leq C_{\rho_\#} \max\{\rho_\#(x,y), \rho_\#(y,y_*)\} < C_{\rho_\#}(1+\kappa)\delta_E(y) \leq C_\rho(1+\kappa)\delta_E(y). \tag{6.22}$$

We have now proved (6.19).

Now suppose that $(E, \rho\big|_E, \sigma)$ is a space of homogeneous type. If $\kappa' \geq \kappa > 0$ and (6.18) holds, then (6.19) holds both as written and with $\kappa$ replaced by $\kappa'$, so by (2.19) we have

$$c_1^{-1}\, \sigma(\pi_y^\kappa) \leq \sigma(\pi_y^{\kappa'}) \leq c_1\, \sigma(\pi_y^\kappa), \tag{6.23}$$

where $c_1 := C_{\sigma,\rho_\#}(C_\rho(1+\kappa)/\epsilon)^{D_\sigma}$. In particular, since $C_{\sigma,\rho_\#} = C_\sigma(C_{\rho_\#}\widetilde{C}_{\rho_\#})^{D_\sigma} \leq C_\sigma(C_\rho)^{D_\sigma}$, we have $c_1 \leq C_\sigma(C_\rho^2/\epsilon)^{D_\sigma}(1+\kappa)^{D_\sigma}$. If $0 < \kappa' < \kappa$, then the same reasoning implies (6.23) with $c_1$ replaced by $c_2 := C_{\sigma,\rho_\#}(C_\rho(1+\kappa')/\epsilon)^{D_\sigma} \leq C_\sigma(C_\rho^2/\epsilon)^{D_\sigma}(1+\kappa')^{D_\sigma}$, so (6.20) holds. □

We now assume that $(E, \rho, \sigma)$ is a space of homogeneous type. For a $\sigma$-measurable set $A \subseteq E$ and $\gamma \in (0,1)$, define the set of $\gamma$-`density points` relative to $A$ by

$$A_\gamma^* := \Big\{x \in E : \inf_{r>0} \frac{\sigma\big(B_{\rho_\#}(x,r) \cap A\big)}{\sigma\big(B_{\rho_\#}(x,r)\big)} \geq \gamma\Big\}. \tag{6.24}$$

The basic properties of such sets in spaces of homogeneous type are collected below.

**Proposition 6.3.** *Let $(E, \rho, \sigma)$ be a space of homogeneous type, $\rho_\#$ the regularization of $\rho$ as in Theorem 2.1. If $\gamma \in (0,1)$ and $A \subseteq E$ is $\sigma$-measurable, then the following properties hold:*

*(1)* $E \setminus A_\gamma^* = \big\{x \in E : M_E\big(\mathbf{1}_{E\setminus A}\big)(x) > 1-\gamma\big\}$.

*(2)* $A_\gamma^*$ *is closed in* $\tau_\rho$.

*(3)* $\sigma\big(E \setminus A_\gamma^*\big) \leq \frac{C}{1-\gamma}\sigma(E \setminus A)$.

*(4) If $A$ is closed in $\tau_\rho$, then $A_\gamma^* \subseteq A$ and $\sigma(E \setminus A_\gamma^*) \approx \sigma(E \setminus A)$.*

*(5) For each $\lambda > 0$ there exist $\gamma(\lambda) \in (0,1)$ and $c(\lambda) > 0$ such that if $\gamma(\lambda) \leq \gamma < 1$, then*

$$\inf_{x \in E} \Big[\inf_{r > \operatorname{dist}_{\rho_\#}(x, A_\gamma^*)} \frac{\sigma\big(B_{\rho_\#}(x,\lambda r) \cap A\big)}{\sigma\big(B_{\rho_\#}(x,r)\big)}\Big] \geq c(\lambda). \tag{6.25}$$

*(6) If $\sigma$ is Borel semiregular, then $\sigma(A_\gamma^* \setminus A) = 0$.*

*(7) If $\widetilde{A}$ is a $\sigma$-measurable set such that $A \subseteq \widetilde{A} \subseteq E$, then $A_\gamma^* \subseteq (\widetilde{A})_\gamma^*$.*

*Proof.* If $\gamma \in (0,1)$ and $A \subseteq E$ is $\sigma$-measurable, then *(1)* follows by the definition of $A_\gamma^*$, since

$$\begin{aligned} E \setminus A_\gamma^* &= \Big\{x \in E : \exists\, r > 0 \text{ such that } \frac{\sigma\big(B_{\rho_\#}(x,r) \cap A\big)}{\sigma\big(B_{\rho_\#}(x,r)\big)} < \gamma\Big\} \\ &= \Big\{x \in E : \sup_{0 < r \leq \operatorname{diam}_{\rho_\#}(E)} \Big(⨍_{B_{\rho_\#}(x,r)} \mathbf{1}_{E\setminus A}\, d\sigma\Big) > 1-\gamma\Big\}. \end{aligned} \tag{6.26}$$

Thus, to prove *(2)*, it suffices to note that the function $M_E(\mathbf{1}_{E\setminus A}) : (E, \tau_\rho) \to [0,\infty]$ is lower semi-continuous. Indeed, this is a consequence of the continuity of $\rho_\#(\cdot,\cdot)$ from Theorem 2.1, Fatou's Lemma, and the fact that the pointwise supremum of any family of real-valued, lower semi-continuous functions defined on $E$ is itself lower semi-continuous. Also, we prove *(3)* by combining *(1)* with the weak-$(1,1)$ boundedness of $M_E$ to obtain

$$\sigma\big(E \setminus A^*_\gamma\big) \leq \frac{C}{1-\gamma}\|\mathbf{1}_{E\setminus A}\|_{L^1(E,\sigma)} = \frac{C}{1-\gamma}\sigma(E \setminus A). \tag{6.27}$$

Next, if $A$ is closed in $\tau_\rho$ and $x \in E \setminus A$, then there exists $r > 0$ such that $B_{\rho_\#}(x,r) \subseteq E \setminus A$, which implies $\sigma(B_{\rho_\#}(x,r) \cap A)) = 0$, and so $x \notin A^*_\gamma$. This shows that $A^*_\gamma \subseteq A$, hence $\sigma(E \setminus A) \leq \sigma(E \setminus A^*_\gamma)$, and *(4)* follows by combining these facts with *(3)*.

To prove *(5)*, let $\lambda > 0$ and $x \in E$, and select $r > 0$ such that $\mathrm{dist}_{\rho_\#}(x, A^*_\gamma) < r$. Then there exists $x_0 \in A^*_\gamma$ such that $\rho_\#(x,x_0) < r$, which forces

$$B_{\rho_\#}(x,\lambda r) \subseteq B_{\rho_\#}(x_0, C_{\rho_\#}(1+\lambda)r) \subseteq B_{\rho_\#}(x, C^2_{\rho_\#}(1+\lambda)r). \tag{6.28}$$

Consequently, since $x_0 \in A^*_\gamma$, we obtain

$$\begin{aligned} \gamma\sigma\big(B_{\rho_\#}(x_0, C_{\rho_\#}(1+\lambda)r)\big) &\leq \sigma\big(B_{\rho_\#}(x_0, C_{\rho_\#}(1+\lambda)r) \cap A\big) \\ &\leq \sigma\big(B_{\rho_\#}(x_0, C_{\rho_\#}(1+\lambda)r) \setminus B_{\rho_\#}(x,\lambda r)\big) + \sigma\big(B_{\rho_\#}(x,\lambda r) \cap A\big), \end{aligned} \tag{6.29}$$

which further implies that

$$\sigma\big(B_{\rho_\#}(x,\lambda r)\big) - (1-\gamma)\sigma\big(B_{\rho_\#}(x_0, C_{\rho_\#}(1+\lambda)r)\big) \leq \sigma\big(B_{\rho_\#}(x,\lambda r) \cap A\big). \tag{6.30}$$

Recalling the second inclusion in (6.28) and (2.19), we obtain

$$\sigma\big(B_{\rho_\#}(x_0, C_{\rho_\#}(1+\lambda)r)\big) \leq C_{\sigma,\rho_\#}(C^2_{\rho_\#}(1+\lambda)/\lambda)^{D_\sigma}\sigma\big(B_{\rho_\#}(x,\lambda r)\big), \tag{6.31}$$

where $C_{\sigma,\rho_\#}$, $D_\sigma$ are associated with $\sigma$, $\rho_\#$ as in (2.19). Together, (6.30) and (6.31) yield

$$\sigma\big(B_{\rho_\#}(x,\lambda r)\big)\Big[1 - C_{\sigma,\rho_\#}(1-\gamma)(C^2_{\rho_\#}(1+\lambda)/\lambda)^{D_\sigma}\Big] \leq \sigma\big(B_{\rho_\#}(x,\lambda r) \cap A\big). \tag{6.32}$$

Also, by (2.19), if $\lambda \in (0,1)$, then $\sigma\big(B_{\rho_\#}(x,r)\big) \leq C_{\sigma,\rho_\#}\lambda^{-D_\sigma}\sigma\big(B_{\rho_\#}(x,\lambda r)\big)$, thus

$$\sigma\big(B_{\rho_\#}(x,\lambda r)\big) \geq \min\Big\{1, \frac{\lambda^{D_\sigma}}{C_{\sigma,\rho_\#}}\Big\}\sigma\big(B_{\rho_\#}(x,r)\big), \qquad \forall\, \lambda > 0. \tag{6.33}$$

We now complete the proof of *(5)* by setting

$$\gamma(\lambda) := 1 - \frac{1}{2C_{\sigma,\rho_\#}}\Big(\frac{\lambda}{C^2_{\rho_\#}(1+\lambda)}\Big)^{D_\sigma} \in (0,1) \quad \text{ and } \quad c(\lambda) := \frac{1}{2}\min\Big\{1, \frac{\lambda^{D_\sigma}}{C_{\sigma,\rho_\#}}\Big\} > 0, \tag{6.34}$$

since then (6.32)-(6.33) imply that $\sigma\big(B_{\rho_\#}(x,\lambda r) \cap A\big) \geq c(\lambda)\sigma\big(B_{\rho_\#}(x,r)\big)$ for all $\gamma \in \big[\gamma(\lambda), 1\big)$.

If $\sigma$ is Borel semiregular, then by Lebesgue's Differentiation Theorem (see [2]), there exists $F \subseteq E$ such that $\sigma(F) = 0$ and $\lim_{r\to 0^+} \fint_{B_{\rho_\#}(x,r)} \mathbf{1}_A\, d\sigma/\sigma(B_{\rho_\#}(x,r)) = \mathbf{1}_A(x)$ for all $x \in E \setminus F$. In particular, if $x \in A^*_\gamma \setminus F$, then $\mathbf{1}_A(x) = \lim_{r\to 0^+} \sigma(B_{\rho_\#}(x,r) \cap A)/\sigma(B_{\rho_\#}(x,r)) \geq \gamma > 0$, which implies that $A^*_\gamma \setminus F \subseteq A$, thus $A^*_\gamma \setminus A \subseteq F$. Then, since $A^*_\gamma \setminus A$ is $\sigma$-measurable, we must have $\sigma(A^*_\gamma \setminus A) = 0$, which proves *(6)*. Finally, property *(7)* is a direct consequence of (6.24). □

The lemma below contains the final auxiliary results required for Theorem 6.5.

**Lemma 6.4.** *Let $(\mathscr{X},\rho)$ be a quasi-metric space, $\mu$ a Borel measure on $(\mathscr{X},\tau_\rho)$, $E$ a proper, nonempty, closed subset of $(\mathscr{X},\tau_\rho)$ and $\sigma$ a Borel measure on $(E,\tau_{\rho|_E})$ such that $(E,\rho\big|_E,\sigma)$ is a space of homogeneous type. The following properties hold for all $\mu$-measurable functions $u:\mathscr{X}\setminus E\to[0,\infty]$, all $\sigma$-measurable sets $A\subseteq E$ and all $\sigma$-measurable functions $f:E\to[0,\infty]$:*

*(1) If $\kappa>0$, then*

$$\int_A\Big(\int_{\Gamma_\kappa(x)}u(y)\,d\mu(y)\Big)d\sigma(x)=\int_{\mathscr{X}\setminus E}u(y)\sigma\big(A\cap\pi_y^\kappa\big)d\mu(y)=\int_{\mathcal{F}_\kappa(A)}u(y)\sigma\big(A\cap\pi_y^\kappa\big)d\mu(y).\tag{6.35}$$

*(2) For each $\kappa,\kappa'>0$, there exist $\gamma\in(0,1)$ and $C\in(0,\infty)$ such that*

$$\int_{A_\gamma^*}\Big(\int_{\Gamma_\kappa(x)}u(y)\,d\mu(y)\Big)\,d\sigma(x)\leq C\int_A\Big(\int_{\Gamma_{\kappa'}(x)}u(y)\,d\mu(y)\Big)\,d\sigma(x).\tag{6.36}$$

*(3) For each $\kappa,\kappa'>0$, there exists $C\in(0,\infty)$ such that*

$$\int_E\Big(\int_{\Gamma_\kappa(x)}u(y)\,d\mu(y)\Big)f(x)\,d\sigma(x)\leq C\int_E\Big(\int_{\Gamma_{\kappa'}(x)}u(y)\,d\mu(y)\Big)(M_Ef)(x)\,d\sigma(x).\tag{6.37}$$

*Proof.* The identities in (6.35) follow by Fubini's Theorem and the fact that if $y\in\mathscr{X}\setminus E$ and $A\cap\pi_y^\kappa\neq\emptyset$, then $y\in\mathcal{F}_\kappa(A)$. To prove (6.36), we recall the notation in (2.4) and claim that for each $\kappa,\kappa'>0$, there exist $\gamma\in(0,1)$ and $c>0$ such that

$$\sigma\big(A\cap\pi_y^{\kappa'}\big)\geq c\,\sigma\big(A_\gamma^*\cap\pi_y^\kappa\big),\qquad\forall\,y\in\mathcal{F}_\kappa(A_\gamma^*).\tag{6.38}$$

If (6.38) holds, then by (6.35) and the fact that $\mathcal{F}_\kappa(A_\gamma^*)\subseteq\mathscr{X}\setminus E$, we have

$$\int_A\int_{\Gamma_{\kappa'}(x)}u(y)\,d\mu(y)d\sigma(x)\geq c\int_{\mathcal{F}_\kappa(A_\gamma^*)}u(y)\sigma\big(A_\gamma^*\cap\pi_y^\kappa\big)\,d\mu(y)=c\int_{A_\gamma^*}\int_{\Gamma_\kappa(x)}u(y)\,d\mu(y)d\sigma(x).\tag{6.39}$$

Hence, to prove (6.36), it suffices to prove (6.38).

To prove (6.38), let $\kappa,\kappa'>0$ and fix $\gamma\in(0,1)$ to be chosen later. Fix $\eta\in(0,\min\{\kappa,\kappa'\})$, and for each $y\in\mathcal{F}_\kappa(A_\gamma^*)$, choose $y_*\in E$ and $\epsilon>0$ such that (6.18) holds (for the chosen value of $\eta$). Lemma 6.2 then implies that the inclusions in (6.19) hold for both $\kappa$ and $\kappa'$. Also, the fact that $y\in\mathcal{F}_\kappa(A_\gamma^*)$ entails $\pi_y^\kappa\cap A_\gamma^*\neq\emptyset$, which when combined with (6.19), implies that $B_{\rho_\#}\big(y_*,C_\rho(1+\kappa)\delta_E(y)\big)\cap A_\gamma^*\neq\emptyset$ and so $\operatorname{dist}_{\rho_\#}(y_*,A_\gamma^*)<C_\rho(1+\kappa)\delta_E(y)$. Property *(5)* in Proposition 6.3 with $\lambda:=\epsilon/(C_\rho(1+\kappa))$, $x:=y_*$ and $r:=C_\rho(1+\kappa)\delta_E(y)$, then guarantees the existence of $\gamma_0=\gamma_0(\lambda)\in(0,1)$ and $c=c(\kappa)>0$ such that

$$\frac{\sigma\big(B_{\rho_\#}(y_*,\epsilon\delta_E(y))\cap A\big)}{\sigma\big(B_{\rho_\#}(y_*,C_\rho(1+\kappa)\delta_E(y))\big)}\geq c\qquad\forall\,\gamma\in(\gamma_0,1).\tag{6.40}$$

Hence, if we initially select $\gamma\in(\gamma_0,1)$, then (6.40) and (6.19) imply that

$$\sigma\big(B_{\rho_\#}(y_*,\epsilon\delta_E(y))\cap A\big)\geq c\,\sigma\big(B_{\rho_\#}(y_*,C_\rho(1+\kappa)\delta_E(y))\big)\geq c\,\sigma\big(\pi_y^\kappa\big)\geq c\,\sigma\big(A_\gamma^*\cap\pi_y^\kappa\big).\tag{6.41}$$

Since (6.19) also holds with $\kappa$ replaced by $\kappa'$, we also obtain

$$\sigma\big(A\cap\pi_y^{\kappa'}\big)\geq\sigma\big(B_{\rho_\#}(y_*,\epsilon\delta_E(y))\cap A\big)\geq c\,\sigma\big(A_\gamma^*\cap\pi_y^{\kappa}\big),\tag{6.42}$$

which proves (6.38) and thus completes the proof of (6.36).

To prove (6.37), since

$$\int_E\Big(\int_{\Gamma_\kappa(x)}u(y)\,d\mu(y)\Big)f(x)\,d\sigma(x)=\int_{\mathscr{X}\setminus E}u(y)\sigma\big(\pi_y^\kappa\big)\Big(\fint_{\pi_y^\kappa}f\,d\sigma\Big)\,d\mu(y),\tag{6.43}$$

it suffices to show that there exists $C_1\in(0,\infty)$ such that

$$\fint_{\pi_y^\kappa}f\,d\sigma\leq C_1\fint_{\pi_y^{\kappa'}}M_Ef\,d\sigma,\qquad\forall\,y\in\mathscr{X}\setminus E.\tag{6.44}$$

To this end, fix $y\in\mathscr{X}\setminus E$, $y_*\in E$ and $\epsilon>0$ such that (6.18) holds for some $\eta\in(0,\min\{\kappa,\kappa'\})$, hence (6.19) holds for $\kappa$ and $\kappa'$. In particular, if $z\in\pi_y^{\kappa'}$, then $\rho_\#(z,y_*)<C_\rho(1+\kappa')\delta_E(y)$, so

$$\begin{aligned}B_{\rho_\#}\big(y_*,C_\rho(1+\kappa)\delta_E(y)\big)&\subseteq B_{\rho_\#}\big(z,C_\rho^2(1+\max\{\kappa,\kappa'\})\delta_E(y)\big)\\&\subseteq B_{\rho_\#}\big(y_*,C_\rho^3(1+\max\{\kappa,\kappa'\})\delta_E(y)\big),\qquad\forall\,z\in\pi_y^{\kappa'}.\end{aligned}\tag{6.45}$$

Using (6.19), (6.45) and (2.19), we obtain

$$\begin{aligned}\fint_{\pi_y^\kappa}f\,d\sigma&\leq\frac{1}{\sigma(B_{\rho_\#}(y_*,\epsilon\delta_E(y)))}\int_{B_{\rho_\#}\left(y_*,C_\rho(1+\kappa)\delta_E(y)\right)}f\,d\sigma\\&\leq C_{\sigma,\rho_\#}\big(C_\rho^3\epsilon^{-1}(1+\max\{\kappa,\kappa'\})\big)^{D_\sigma}\fint_{B_{\rho_\#}(z,C_\rho^2(1+\max\{\kappa,\kappa'\})\delta_E(y))}f\,d\sigma\\&\leq C_{\sigma,\rho_\#}\big(C_\rho^3\epsilon^{-1}(1+\max\{\kappa,\kappa'\})\big)^{D_\sigma}M_Ef(z),\qquad\forall\,z\in\pi_y^{\kappa'}.\end{aligned}\tag{6.46}$$

Thus, setting $C_1:=C_{\sigma,\rho_\#}\big(C_\rho^3\epsilon^{-1}(1+\max\{\kappa,\kappa'\})\big)^{D_\sigma}$, we have

$$\fint_{\pi_y^\kappa}f\,d\sigma\leq C_1\inf_{z\in\pi_y^{\kappa'}}[M_Ef(z)]\leq C_1\fint_{\pi_y^{\kappa'}}M_Ef\,d\sigma,\tag{6.47}$$

which proves (6.44) and finishes the proof of the lemma. □

We now turn to the following equivalence for the quasi-norms of the mixed norm spaces.

**Theorem 6.5.** *Let $(\mathscr{X},\rho)$ be a quasi-metric space, $\mu$ a Borel measure on $(\mathscr{X},\tau_\rho)$, $E$ a proper, nonempty, closed subset of $(\mathscr{X},\tau_\rho)$, and $\sigma$ a Borel measure on $(E,\tau_{\rho|_E})$ such that $(E,\rho\big|_E,\sigma)$ is a space of homogeneous type. If $\kappa,\kappa'>0$, and $(p,q)\in(0,\infty)\times(0,\infty]$ or $p=q=\infty$, then*

$$\|u\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)}\approx\|u\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa')}\tag{6.48}$$

*for all $\mu$-measurable functions $u:\mathscr{X}\setminus E\to\overline{\mathbb{R}}$.*

*Proof.* Let $\kappa, \kappa' > 0$. There is nothing to prove when $p = q = \infty$, so now suppose that $(p, q) \in (0, \infty) \times (0, \infty]$. It suffices to find some $C = C(\kappa, \kappa') \in (0, \infty)$ such that

$$\|u\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa')} \leq C\|u\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)} \tag{6.49}$$

for all $\mu$-measurable functions $u : \mathscr{X} \setminus E \to \overline{\mathbb{R}}$. We prove this below by considering four cases.

*Case I:* $0 < p < q < \infty$. Let $\lambda > 0$ and set $A := \big\{x \in E : (\mathscr{A}_{q,\kappa}u)(x) \leq \lambda\big\}$. It follows from (6.4) and *(4)* in Proposition 6.3 that $A$ is closed in $(E, \tau_{\rho|_E})$ and $A^*_\gamma \subseteq A$ for all $\gamma \in (0, 1)$. Let $\gamma = \gamma(\kappa, \kappa') \in (0, 1)$ be such that (6.36) holds, so then by *(3)* in Proposition 6.3, we have

$$\begin{aligned}
\sigma\big(\{x \in E : (\mathscr{A}_{q,\kappa'}u)(x) > \lambda\}\big) &\leq \sigma\big(E \setminus A^*_\gamma\big) + \sigma\big(\{x \in A^*_\gamma : (\mathscr{A}_{q,\kappa'}u)(x) > \lambda\}\big)\\
&\leq \frac{C}{1-\gamma}\sigma(E \setminus A) + \frac{1}{\lambda^q}\int_{A^*_\gamma}(\mathscr{A}_{q,\kappa'}u)(x)^q\, d\sigma(x)\\
&\leq \frac{C}{1-\gamma}\sigma\big(\{x \in E : (\mathscr{A}_{q,\kappa}u)(x) > \lambda\}\big) + \frac{C}{\lambda^q}\int_A(\mathscr{A}_{q,\kappa}u)(x)^q\, d\sigma(x).
\end{aligned} \tag{6.50}$$

Multiplying by $p\lambda^{p-1}$ and integrating in $\lambda \in (0, \infty)$, we then obtain

$$\|\mathscr{A}_{q,\kappa'}u\|^p_{L^p(E,\sigma)} \leq \frac{C}{1-\gamma}\|\mathscr{A}_{q,\kappa}u\|^p_{L^p(E,\sigma)} + C\int_0^\infty \lambda^{p-q-1}\Big(\int_{\{\mathscr{A}_{q,\kappa}u \leq \lambda\}}(\mathscr{A}_{q,\kappa}u)^q\, d\sigma\Big)\, d\lambda. \tag{6.51}$$

By Fubini's Theorem, and since we are assuming that $0 < p < q < \infty$, we further have

$$\begin{aligned}
\int_0^\infty \lambda^{p-q-1}\Big(\int_{\{\mathscr{A}_{q,\kappa}u \leq \lambda\}}(\mathscr{A}_{q,\kappa}u)^q\, d\sigma\Big)\, d\lambda &= \int_E\Big(\int_{(\mathscr{A}_{q,\kappa}u)(x)}^\infty \lambda^{p-q-1}\, d\lambda\Big)(\mathscr{A}_{q,\kappa}u)(x)^q\, d\sigma(x)\\
&= (q-p)^{-1}\|\mathscr{A}_{q,\kappa}u\|^p_{L^p(E,\sigma)}.
\end{aligned} \tag{6.52}$$

In concert, (6.51) and (6.52) now yield (6.49) when $0 < p < q < \infty$.

*Case II:* $p = q \in (0, \infty)$. We obtain (6.49) by combining (6.35) with (6.20) to write

$$\|\mathscr{A}_{p,\kappa'}u\|^p_{L^p(E,\sigma)} = \int_{\mathscr{X}\setminus E}|u(y)|^p\sigma(\pi^{\kappa'}_y)\, d\mu(y) \approx \int_{\mathscr{X}\setminus E}|u(y)|^p\sigma(\pi^\kappa_y)\, d\mu(y) = \|\mathscr{A}_{p,\kappa}u\|^p_{L^p(E,\sigma)}. \tag{6.53}$$

*Case III:* $0 < q < p < \infty$. Let $q/p + 1/r = 1$, so $r \in (1, \infty)$, thus duality and (6.37) imply that

$$\begin{aligned}
\|\mathscr{A}_{q,\kappa'}u\|_{L^p(E,\sigma)} &= \Big\|\int_{\Gamma_{\kappa'}(x)}|u|^q\, d\mu\Big\|_{L^{p/q}_x(E,\sigma)}\\
&= \sup_{\|f\|_{L^r(E,\sigma)}=1}\int_E\Big(\int_{\Gamma_{\kappa'}(x)}|u(y)|^q\, d\mu(y)\Big)f(x)\, d\sigma(x)\\
&\leq C\sup_{\|f\|_{L^r(E,\sigma)}=1}\int_E\Big(\int_{\Gamma_\kappa(x)}|u(y)|^q\, d\mu(y)\Big)(M_Ef)(x)\, d\sigma(x)\\
&\leq C\sup_{\|f\|_{L^r(E,\sigma)}=1}\|\mathscr{A}_{q,\kappa}u\|_{L^p(E,\sigma)}\|M_Ef\|_{L^r(E,\sigma)} \leq C\|\mathscr{A}_{q,\kappa}u\|_{L^p(E,\sigma)},
\end{aligned} \tag{6.54}$$

which proves (6.49) when $0 < q < p < \infty$.

*Case IV:* $0 < p < \infty$, $q = \infty$. Let $\lambda > 0$ and set

$$\mathcal{O}_\kappa := \big\{x \in E : (\mathcal{N}_\kappa u)(x) > \lambda\big\}, \qquad \mathcal{O}_{\kappa'} := \big\{x \in E : (\mathcal{N}_{\kappa'}u)(x) > \lambda\big\}. \tag{6.55}$$

To prove (6.49), it suffices to show that $\sigma(\mathcal{O}_{\kappa'}) \le C\sigma(\mathcal{O}_\kappa)$, thus by *(3)* in Proposition 6.3, it suffices to find $\gamma \in (0,1)$ such that $\mathcal{O}_{\kappa'} \subseteq E \setminus (E \setminus \mathcal{O}_\kappa)^*_\gamma$. To this end, let $x \in \mathcal{O}_{\kappa'}$, so there exists $y \in \Gamma_{\kappa'}(x)$ with $|u(y)| > \lambda$. Fix $\eta \in (0, \min\{\kappa, \kappa'\})$ and select $y_* \in E$ and $\epsilon \in (0,1)$ such that (6.18) holds. In particular, $\rho_\#(y, y_*) < (1+\eta)\delta_E(y)$. Observe from (6.19) and (6.55) that

$$E \cap B_{\rho_\#}\big(y_*, \epsilon\delta_E(y)\big) \subseteq \pi^\kappa_y \subseteq \mathcal{O}_\kappa. \tag{6.56}$$

We also claim that

$$E \cap B_{\rho_\#}\big(y_*, \epsilon\delta_E(y)\big) \subseteq E \cap B_{\rho_\#}\big(x, C_\rho(1+\kappa')\delta_E(y)\big). \tag{6.57}$$

To see this, recall that $\epsilon \in (0,1)$ and note that if $z \in E$ satisfies $\rho_\#(z, y_*) < \delta_E(y)$, then

$$\begin{aligned}
\rho_\#(x,z) &\le C_\rho \max\{\rho_\#(x,y), \rho_\#(y,z)\} \\
&\le C_\rho \max\Big\{(1+\kappa')\delta_E(y),\, C_\rho \max\{(1+\eta)\delta_E(y), \delta_E(y)\}\Big\} \\
&= C_\rho(1+\kappa')\delta_E(y),
\end{aligned} \tag{6.58}$$

which proves (6.57). In concert, (6.56) and (6.57) yield

$$E \cap B_{\rho_\#}\big(y_*, \delta_E(y)\big) \subseteq \mathcal{O}_\kappa \cap B_{\rho_\#}\big(x, C_\rho(1+\kappa')\delta_E(y)\big). \tag{6.59}$$

Let us also observe that

$$\rho_\#(x, y_*) \le C_\rho \max\{(1+\kappa')\delta_E(y), (1+\eta)\delta_E(y)\} = C_\rho(1+\kappa')\delta_E(y). \tag{6.60}$$

Setting $r := C_\rho(1+\kappa')\delta_E(y)$, then by (6.59), (6.60) and the fact that $(E, \rho\big|_E, \sigma)$ is a space of homogeneous type, there exists $c \in (0,1)$, depending only on $\kappa, \kappa'$ and geometry, such that

$$\sigma(\mathcal{O}_\kappa \cap B_{\rho_\#}(x,r)) \ge \sigma(E \cap B_{\rho_\#}(y_*, \epsilon\delta_E(y))) \ge c\sigma(E \cap B_{\rho_\#}(x,r)). \tag{6.61}$$

In particular, we have $\sigma((E\setminus\mathcal{O}_\kappa)\cap B_{\rho_\#}(x,r))/\sigma(E\cap B_{\rho_\#}(x,r)) \le 1-c$. Thus, selecting $\gamma$ such that $1-c < \gamma < 1$ implies that $x \notin (E \setminus \mathcal{O}_\kappa)^*_\gamma$, hence $\mathcal{O}_{\kappa'} \subseteq E \setminus (E \setminus \mathcal{O}_\kappa)^*_\gamma$, as required. □

**Remark 6.6.** *We expand on the comment at the bottom of page 183 in [72], and present an example where* (6.48) *fails in the limiting case $p = \infty$, $q \in (0,\infty)$, i.e. where*

$$\sup_{x\in E}\Big(\int_{\Gamma_\kappa(x)} |u(y)|^q \, d\mu(y)\Big)^{1/q} \approx \sup_{x\in E}\Big(\int_{\Gamma_{\kappa'}(x)} |u(y)|^q \, d\mu(y)\Big)^{1/q} \tag{6.62}$$

*fails. In particular, consider when $\mathscr{X} := \mathbb{R}^2$, $E := \mathbb{R} \equiv \partial\mathbb{R}^2_+$, $\kappa := \sqrt{2}-1$ and $\kappa' \in (0, \sqrt{2}-1)$. Also, without loss of generality, assume that $q = 1$ and consider $u : \mathscr{X} \setminus E \to \overline{\mathbb{R}}$ given by*

$$u(x,y) := \begin{cases} x^{-2} & \text{if } x > 0 \text{ and } x < y < 2x, \\ 0 & \text{otherwise.} \end{cases} \tag{6.63}$$

*Then*

$$\sup_{z\in\mathbb{R}}\Big(\int_{\Gamma_\kappa(z)} |u(x,y)| \, dxdy\Big) \ge \int_{|x|<y} |u(x,y)| \, dxdy = \int_0^\infty x^{-2}\Big(\int_x^{2x} 1 \, dy\Big) \, dx = \infty, \tag{6.64}$$

*whereas, for all $z \in (0,\infty)$, elementary geometry implies that*

$$\int_{\Gamma_{\kappa'}(z)} |u(x,y)| \, dxdy \le Cz^{-2} \cdot \text{Area}\{(x,y) \in \Gamma_{\kappa'}(z) : \, 0 < x < y < 2x\} \le C, \tag{6.65}$$

*where $C$ only depends on $\kappa'$, hence $\sup_{z\in\mathbb{R}}\Big(\int_{\Gamma_{\kappa'}(z)} |u(x,y)| \, dxdy\Big) < \infty$, and* (6.62) *fails.*

## 6.2 Estimates relating the Lusin and Carleson operators

We now introduce a Carleson operator $\mathfrak{C}$ to provide an equivalent quasi-norm for the mixed norm spaces. This is essential in Subsection 6.4, and it is achieved by combining Theorem 6.5 with a good-$\lambda$ inequality originating in [17]. In particular, the theorem below extends the result on $\mathbb{R}^{n+1}_{+}$ obtained in [17, Theorem 3]. We first dispense with the following preliminaries.

Let $(\mathscr{X},\rho)$ be a quasi-metric space, $\mu$ a Borel measure on $(\mathscr{X},\tau_\rho)$, $E$ a nonempty, proper, closed subset of $(\mathscr{X},\tau_\rho)$, and $\sigma$ a measure on $E$ such that $(E,\rho\big|_E,\sigma)$ is a space of homogeneous type. We say that $\mu$ is `locally finite relative to` $(\mathscr{X},E,\rho)$ provided

$$\mu\big(\{y\in B_\rho(x,R)^\circ:\,\delta_E(y)>r\}\big)<+\infty\ \text{ for all }\ x\in\mathscr{X}\ \text{ and }\ R,r\in(0,\infty), \tag{6.66}$$

where the interior is taken in the topology $\tau_\rho$.

For $q\in(0,\infty)$ and $\kappa\in(0,\infty)$, define the $L^q$-based `Carleson operator` $\mathfrak{C}_{q,\kappa}$ for all $\mu$-measurable functions $u:\mathscr{X}\setminus E\to\overline{\mathbb{R}}:=[-\infty,+\infty]$ by

$$(\mathfrak{C}_{q,\kappa}u)(x):=\sup_{\Delta\subseteq E,\,x\in\Delta}\Big(\frac{1}{\sigma(\Delta)}\int_{\mathcal{T}_\kappa(\Delta)}|u(y)|^q\sigma(\pi^\kappa_y)\,d\mu(y)\Big)^{\frac{1}{q}},\qquad\forall\,x\in E, \tag{6.67}$$

where the supremum is taken over all `surface balls` $\Delta$ containing $x$ and defined by

$$\Delta:=\Delta(y,r):=E\cap B_{\rho_\#}(y,r),\qquad y\in E,\quad r>0, \tag{6.68}$$

and where the conical projection $\pi^\kappa_y$ and the tent $\mathcal{T}_\kappa(\Delta)$ over $\Delta$ are from (6.8).

For $p\in(0,\infty)$ and $r\in(0,\infty]$, let $L^{p,r}(E,\sigma)$ denote the Lorentz space with quasi-norm

$$\|f\|_{L^{p,r}(E,\sigma)}:=\bigg(\int_0^\infty\lambda^r\sigma\big(\{x\in E:\,|f(x)|>\lambda\}\big)^{r/p}\,\frac{d\lambda}{\lambda}\bigg)^{1/r},\quad\text{ if }\ r<\infty, \tag{6.69}$$

$$\|f\|_{L^{p,\infty}(E,\sigma)}:=\sup_{\lambda>0}\Big[\lambda\,\sigma\left(\{x\in E:\,|f(x)|>\lambda\}\right)^{1/p}\Big]. \tag{6.70}$$

Note that $L^{p,p}(E,\sigma)=L^p(E,\sigma)$ for each $p\in(0,\infty)$.

**Theorem 6.7.** *Let $(\mathscr{X},\rho)$ be a quasi-metric space, $E$ a proper, nonempty, closed subset of $(\mathscr{X},\tau_\rho)$, $\mu$ a Borel measure on $(\mathscr{X},\tau_\rho)$ that is locally finite relative to $(\mathscr{X},E,\rho)$, and $\sigma$ a measure on $E$ such that $(E,\rho\big|_E,\sigma)$ is a space of homogeneous type. If $\kappa>0$ and $q\in(0,\infty)$, then following estimates hold for every $\mu$-measurable function $u:\mathscr{X}\setminus E\to\overline{\mathbb{R}}$:*

*(1) If $p\in(0,\infty)$, then $\|\mathscr{A}_{q,\kappa}u\|_{L^p(E,\sigma)}\leq C\|\mathfrak{C}_{q,\kappa}u\|_{L^p(E,\sigma)}$.*

*(2) If $p\in(q,\infty)$ and $r\in(0,\infty]$, then $\|\mathfrak{C}_{q,\kappa}u\|_{L^{p,r}(E,\sigma)}\leq C\|\mathscr{A}_{q,\kappa}u\|_{L^{p,r}(E,\sigma)}$.*

*(3) If $p=q$ or $p=\infty$ in (2), then*

$$\|\mathfrak{C}_{q,\kappa}u\|_{L^{q,\infty}(E,\sigma)}\leq C\|\mathscr{A}_{q,\kappa}u\|_{L^q(E,\sigma)}\ \ \textit{and}\ \ \|\mathfrak{C}_{q,\kappa}u\|_{L^\infty(E,\sigma)}\leq C\|\mathscr{A}_{q,\kappa}u\|_{L^\infty(E,\sigma)}. \tag{6.71}$$

*(4) If $p\in(q,\infty)$, then $\|\mathscr{A}_{q,\kappa}u\|_{L^p(E,\sigma)}\approx\|\mathfrak{C}_{q,\kappa}u\|_{L^p(E,\sigma)}$.*

*In each case, the comparability constants depend only on $\kappa$, $q$, $p$, $r$ and geometric constants.*

*Proof.* Suppose that $u : \mathscr{X} \setminus E \to \overline{\mathbb{R}}$ is $\mu$-measurable. Fix $q \in (0, \infty)$ and define $c_q := 2^{(1/q)-1}$, if $q \in (0, 1)$, and $c_q := 1$, if $q \geq 1$. To prove *(1)*, we will prove the following good-$\lambda$ inequality:

$$\begin{aligned}&\forall\, \kappa > 0,\ \exists\, \kappa' > \kappa \text{ and } \exists\, c \in (0, \infty) \text{ such that } \forall\, \gamma \in (0, 1] \text{ and } \forall\, \lambda \in (0, \infty), \text{ it holds that}\\ &\sigma\big(\{x \in E : (\mathscr{A}_{q,\kappa}u)(x) > 2c_q\lambda, (\mathfrak{C}_{q,\kappa}u)(x) \leq \gamma\lambda\}\big) \leq c\,\gamma^q \sigma\big(\{x \in E :\, (\mathscr{A}_{q,\kappa'}u)(x) > \lambda\}\big).\end{aligned} \tag{6.72}$$

Assume for now that this holds. Fix $\kappa$, $\kappa'$ and $c$ as in (6.72). If $\gamma \in (0, 1]$ and $\lambda > 0$, then

$$\begin{aligned}\sigma\big(\{x \in E :\, &(\mathscr{A}_{q,\kappa}u)(x) > 2c_q\lambda\}\big)\\ &\leq \sigma\big(\{x \in E :\, (\mathfrak{C}_{q,\kappa}u)(x) > \gamma\lambda\}\big) + c\,\gamma^q \sigma\big(\{x \in E :\, (\mathscr{A}_{q,\kappa'}u)(x) > \lambda\}\big).\end{aligned} \tag{6.73}$$

Multiplying by $p\lambda^{p-1}$, integrating in $\lambda \in (0, \infty)$ and applying Theorem 6.5, we obtain

$$(2c_q)^{-p}\|\mathscr{A}_{q,\kappa}u\|^p_{L^p(E,\sigma)} \leq \gamma^{-p}\|\mathfrak{C}_{q,\kappa}u\|^p_{L^p(E,\sigma)} + c\, C^p\, \gamma^q\|\mathscr{A}_{q,\kappa}u\|^p_{L^p(E,\sigma)}, \quad \forall\, \gamma \in (0, 1]. \tag{6.74}$$

To justify subtracting the last term above, fix $x_0 \in E$ and, for each $j \in \mathbb{N}$, set

$$u_j := \min\{|u|, j\} \cdot \mathbf{1}_{B_{\rho_\#}(x_0,j)\setminus\{x\in E:\, \delta_E(x)\leq 1/j\}} \qquad \text{on } \ \mathscr{X} \setminus E. \tag{6.75}$$

The support of $\mathscr{A}_{q,\kappa}u_j$ and the assumption that $\mu$ is locally finite relative to $(\mathscr{X}, E, \rho)$ imply

$$\|\mathscr{A}_{q,\kappa}u_j\|_{L^p(E,\sigma)} \leq j \cdot \mu\big(B_{\rho_\#}(x_0, j)\big)^{1/q}\sigma\big(E \cap B_{\rho_\#}(x_0, C_\rho(1 + \kappa)j)\big)^{1/p} < \infty. \tag{6.76}$$

Thus, choosing $\gamma \in (0, 1]$ so that $(2c_q)^{-p} > 2c\, C^p\, \gamma^q$, we have $\|\mathscr{A}_{q,\kappa}u_j\|^p_{L^p(E,\sigma)} \leq \widetilde{C}\|\mathfrak{C}_{q,\kappa}u_j\|^p_{L^p(E,\sigma)}$ for all $j \in \mathbb{N}$ and some $\widetilde{C} \in (0, \infty)$ independent of $j$. We then use Lebesgue's Monotone Convergence Theorem and Fatou's Lemma to conclude that *(1)* holds.

To prove *(1)*, it remains to establish (6.72). To this end, fix $\kappa' > \kappa > 0$, $\gamma \in (0, 1]$ and set

$$\mathcal{O}_\lambda := \big\{x \in E :\, (\mathscr{A}_{q,\kappa'}u)(x) > \lambda\big\}, \quad \forall \lambda > 0, \tag{6.77}$$

which is open in $(E, \tau_{\rho|_E})$ by (6.4). Also, since $\mathscr{A}_{q,\kappa'}u \geq \mathscr{A}_{q,\kappa}u$ pointwise in $E$, we have

$$\{x \in E :\, (\mathscr{A}_{q,\kappa}u)(x) > 2c_q\lambda\} \subseteq \mathcal{O}_\lambda. \tag{6.78}$$

If $\mathcal{O}_\lambda = \emptyset$, then the second line of (6.72) is trivially satisfied. Now assume that $\mathcal{O}_\lambda \neq \emptyset$ and

$$\begin{aligned}&\text{the } \mu\text{-measurable function } u : \mathscr{X} \setminus E \to \overline{\mathbb{R}} \text{ is such that}\\ &\mathcal{O}_\lambda \text{ from (6.77) is a proper subset of } E \text{ for each } \lambda > 0.\end{aligned} \tag{6.79}$$

The assumption in (6.79) will be eliminated *a posteriori*. For a fixed, suitably chosen $\lambda_o > 1$, we have a Whitney covering of $\mathcal{O}_\lambda$ by balls, relative to $(E, \rho|_E)$, of the form $B_j := E \cap B_{\rho_\#}(x_j, r_j)$, $j \in \mathbb{N}$, satisfying *(1)-(4)* in Proposition 2.5 for some $\Lambda > \lambda_o$. It then suffices to prove that

$$\begin{aligned}&\exists\, \kappa' > \kappa \text{ and } \exists\, c \in (0, \infty) \text{ such that } \forall\, \gamma \in (0, 1] \text{ and } \forall\, \lambda \in (0, \infty), \text{ it holds that}\\ &\sigma\big(\{x \in B_j : (\mathscr{A}_{q,\kappa}u)(x) > 2c_q\lambda,\ (\mathfrak{C}_{q,\kappa}u)(x) \leq \gamma\lambda\}\big) \leq c\,\gamma^q\sigma(B_j) \text{ for every } j \in \mathbb{N},\end{aligned} \tag{6.80}$$

since combining (6.80) with (6.78) and properties *(1)-(2)* in Proposition 2.5, we obtain

$$\begin{aligned}\sigma\big(\{x \in E :&(\mathscr{A}_{q,\kappa}u)(x) > 2c_q\lambda, (\mathfrak{C}_{q,\kappa}u)(x) \leq \gamma\lambda\}\big)\\ &\leq \sum_{j=1}^{\infty} \sigma\big(\{x \in B_j : (\mathscr{A}_{q,\kappa}u)(x) > 2c_q\lambda, (\mathfrak{C}_{q,\kappa}u)(x) \leq \gamma\lambda\}\big) \leq C\gamma^q\sigma(\mathcal{O}_\lambda),\end{aligned} \tag{6.81}$$

which implies (6.72).

We now turn to the proof of (6.80). Fix $j \in \mathbb{N}$ and assume, without loss of generality, that

$$\big\{x \in B_j : (\mathscr{A}_{q,\kappa}u)(x) > 2c_q\lambda,\ (\mathfrak{C}_{q,\kappa}u)(x) \leq \gamma\lambda\big\} \neq \emptyset, \tag{6.82}$$

since otherwise there is nothing to prove. Write $u = u\mathbf{1}_{\{\delta_E \geq r_j\}} + u\mathbf{1}_{\{\delta_E < r_j\}} =: u_1 + u_2$ and choose $z_j \in E \setminus \mathcal{O}_\lambda$ such that $\rho_\#(x_j, z_j) \leq \Lambda r_j$ by *(3)* in Proposition 2.5. We claim that

$$\begin{array}{c}\text{there exists } \ \kappa' > \kappa \ \text{ independent of } j \in \mathbb{N} \text{ with the property that} \\ \text{if } \ x \in B_j \ \text{ and } \ y \in \Gamma_\kappa(x) \ \text{ is such that } \ \delta_E(y) \geq r_j \ \text{ then } \ y \in \Gamma_{\kappa'}(z_j).\end{array} \tag{6.83}$$

Indeed, if $x \in B_j$, $y \in \Gamma_\kappa(x)$ and $\delta_E(y) \geq r_j$, then

$$\rho_\#(y, z_j) \leq C_{\rho_\#} \max\{\rho_\#(y,x), \rho_\#(x,z_j)\} \leq C_\rho \max\{(1+\kappa), C_\rho\Lambda\}\delta_E(y). \tag{6.84}$$

Choosing $\kappa' > C_\rho \max\{(1+\kappa), C_\rho\Lambda\} - 1 \geq \kappa$, so $\kappa'$ is independent of $j$, we obtain (6.83) and

$$(\mathscr{A}_{q,\kappa}u_1)(x)^q \leq \int_{\Gamma_{\kappa'}(z_j)} |u(y)|^q \, d\mu(y) = (\mathscr{A}_{q,\kappa'}u)(z_j)^q \leq \lambda^q, \quad \forall\, x \in B_j. \tag{6.85}$$

Next, we use (6.35) to obtain

$$\int_{B_j} (\mathscr{A}_{q,\kappa}u_2)(x)^q \, d\sigma(x) \leq \int_{y \in \mathcal{F}_\kappa(B_j),\, \delta_E(y) < r_j} |u(y)|^q \sigma(\pi_y^\kappa)\, d\mu(y). \tag{6.86}$$

In order to complete the proof of (6.80), we now prove, using the notation in (6.68), that

$$\begin{array}{c}\text{there exists a finite constant } c_o > 0 \text{ such that for every } r > 0 \text{ and every } x_0 \in E \\ \text{if } \ y \in \mathcal{F}_\kappa(\Delta(x_0,r)) \ \text{ and } \ \delta_E(y) < r \ \text{ then } \ y \in \mathcal{T}_\kappa\big(E \cap B_{\rho_\#}(w, c_o r)\big)\ \forall\, w \in \Delta(x_0, r).\end{array} \tag{6.87}$$

Let $y \in \mathcal{F}_\kappa(\Delta(x_0,r))$ with $\delta_E(y) < r$, so there exists $x \in \Delta(x_0,r)$ with $\rho_\#(y,x) < (1+\kappa)\delta_E(y) < (1+\kappa)r$. Let $c_o > C_\rho$ be arbitrary. If $w \in \Delta(x_0,r)$, then $x \in E \cap B_{\rho_\#}(w, c_o r)$ and

$$\mathrm{dist}_{\rho_\#}\big(y, E \cap B_{\rho_\#}(w, c_o r)\big) \leq \rho_\#(y,x) < (1+\kappa)\delta_E(y). \tag{6.88}$$

Also, if $w \in \Delta(x_0,r)$ and $z \in E \setminus B_{\rho_\#}(w, c_o r)$, since $\rho_\#(y,w) \leq C_\rho \max\{C_\rho, 1+\kappa\}r$, we have

$$c_o r \leq C_\rho \max\big\{\rho_\#(z,y),\, C_\rho \max\{C_\rho, 1+\kappa\}r\big\} = C_\rho\, \rho_\#(z,y), \tag{6.89}$$

where the last equality holds by now restricting $c_o > C_\rho^2 \max\{C_\rho, 1+\kappa\}$. This implies that $C_\rho \rho_\#(y,z) \geq c_o r > c_o \delta_E(y)$ for all $z \in E \setminus B_{\rho_\#}(w, c_o r)$, so restricting $c_o \geq C_\rho(1+\kappa)^2$, we have

$$\mathrm{dist}_{\rho_\#}\big(y, E \setminus B_{\rho_\#}(w, c_o r)\big) \geq (1+\kappa)^2 \delta_E(y). \tag{6.90}$$

Finally, choosing $c_o > \max\big\{C_\rho(1+\kappa)^2, C_\rho^3, C_\rho^2(1+\kappa)\big\}$, we obtain from (6.88) and (6.90) that

$$\mathrm{dist}_{\rho_\#}\big(y, E \cap B_{\rho_\#}(w, c_o r)\big) \leq (1+\kappa)^{-1} \mathrm{dist}_{\rho_\#}\big(y, E \setminus B_{\rho_\#}(w, c_o r)\big), \tag{6.91}$$

which, by (6.10), implies that $y \in \mathcal{T}_\kappa\big(E \cap B_{\rho_\#}(w, c_o r)\big)$ and proves (6.87).

We can now complete the proof of (6.80). Combining (6.86) with (6.87), we obtain

$$\begin{aligned}\frac{1}{\sigma(B_j)}\int_{B_j}(\mathscr{A}_{q,\kappa}u_2)(x)^q\,d\sigma(x) &\le \frac{C}{\sigma(B_j)}\int_{y\in\mathcal{F}_\kappa(B_j),\,\delta_E(y)<r_j}|u(y)|^q\sigma(\pi_y^\kappa)\,d\mu(y)\\ &\le \frac{C}{\sigma\big(E\cap B_{\rho_\#}(w,c_o r_j)\big)}\int_{\mathcal{T}_\kappa\left(E\cap B_{\rho_\#}(w,c_o r_j)\right)}|u(y)|^q\sigma(\pi_y^\kappa)\,d\mu(y)\\ &\le C\,\inf_{w\in B_j}\big[(\mathfrak{C}_{q,\kappa}u)(w)\big]^q \le C\gamma^q\lambda^q, \qquad (6.92)\end{aligned}$$

where we used the equivalence $\sigma(B_j)\approx\sigma\big(E\cap B_{\rho_\#}(w,c_o r_j)\big)$ for all $j\in\mathbb{N}$ and $w\in B_j$, implied by (2.19), and assumption (6.82). Tschebyshev's inequality then implies that

$$\sigma\big(\{x\in B_j:\,(\mathscr{A}_{q,\kappa}u_2)(x)>\lambda\}\big)\le C\gamma^q\sigma(B_j), \qquad (6.93)$$

for some $C\in(0,\infty)$ independent of $\gamma\in(0,1]$ and $j\in\mathbb{N}$. Also, in view of (6.85), we obtain

$$\big\{x\in B_j:\,(\mathscr{A}_{q,\kappa}u)(x)>2c_q\lambda\big\}\subseteq\big\{x\in B_j:\,(\mathscr{A}_{q,\kappa}u_2)(x)>\lambda\big\}, \qquad (6.94)$$

since pointwise on $E$ we have $\mathscr{A}_{q,\kappa}u\le c_q\big(\mathscr{A}_{q,\kappa}u_1+\mathscr{A}_{q,\kappa}u_2\big)$, where $c_q$ is from above. Combined with (6.93), this proves (6.80) and thus completes the proof of *(1)* when (6.79) holds.

We now complete the proof of *(1)* by removing assumption (6.79) in the following two cases:

*Case I:* $\operatorname{diam}_\rho(E)=\infty$. An inspection of the proof reveals that (6.74) has only been utilized with $u_j$ (from (6.75)) in place of $u$. Thus, it suffices to show that $\big\{x\in E:\,(\mathscr{A}_{q,\kappa'}u_j)(x)>\lambda\big\}$ is a proper subset of $E$ for each $j\in\mathbb{N}$ and each $\lambda>0$. This follows by observing that $\sigma(E)=\infty$ by (2.20), whilst $\sigma\big(\big\{x\in E:\,(\mathscr{A}_{q,\kappa'}u_j)(x)>\lambda\big\}\big)<\infty$ by (6.76) and Tschebyshev's inequality.

*Case II:* $\operatorname{diam}_\rho(E)<\infty$. We note that $\sigma(E)<\infty$ by (2.20). Set $R:=\operatorname{diam}_{\rho_\#}(E)$, let $\varepsilon_o>0$ to be specified later, and write $|u|=|u|\mathbf{1}_{\{\delta_E(\cdot)<\varepsilon_o R\}}+|u|\mathbf{1}_{\{\delta_E(\cdot)\ge\varepsilon_o R\}}=:u'+u''$. Note that $u',u''$ are $\mu$-measurable with $0\le u',u''\le|u|$ and for each $x\in E$, we have

$$\begin{aligned}(\mathfrak{C}_{q,\kappa}u'')(x) &\ge \Big(\frac{1}{\sigma(E)}\int_{\mathscr{X}\setminus E}u''(y)^q\sigma(\pi_y^\kappa)\,d\mu(y)\Big)^{1/q}\\ &\ge c\Big(\int_{y\in\mathscr{X}\setminus E,\,\delta_E(y)\ge\varepsilon_o R}u''(y)^q\,d\mu(y)\Big)^{1/q}\ge c(\mathscr{A}_{q,\kappa}u'')(x), \qquad (6.95)\end{aligned}$$

by taking $r>R$ in (6.68), recalling (3) in Lemma 6.1, and noting that there exists $C\in(0,\infty)$ such that for each $y\in\mathscr{X}\setminus E$ (with $y_*$ and $\epsilon$ as in Lemma 6.2), we have

$$\sigma(\pi_y^\kappa)\ge\sigma\big(E\cap B_{\rho_\#}(y_*,\epsilon\delta_E(y))\big)\ge\sigma\big(E\cap B_{\rho_\#}(y_*,\epsilon\varepsilon_o R)\big)\ge C\sigma(E) \qquad (6.96)$$

by the doubling property of $\sigma$. The monotonicity of the Carleson operator then implies that

$$\|\mathscr{A}_{q,\kappa}u''\|_{L^p(E,\sigma)}\le C\|\mathfrak{C}_{q,\kappa}u''\|_{L^p(E,\sigma)}\le C\|\mathfrak{C}_{q,\kappa}u\|_{L^p(E,\sigma)}. \qquad (6.97)$$

To proceed, set $\varepsilon_o:=1/4C_\rho(1+\kappa')$ and fix $x_1,x_2\in E$ satisfying $\rho_\#(x_1,x_2)>R/2$. Then

$$\Gamma_{\kappa'}(x_1)\cap\Gamma_{\kappa'}(x_2)\subseteq\{x\in\mathscr{X}\setminus E:\,\delta_E(x)>\varepsilon_0 R\}. \qquad (6.98)$$

Indeed, if $y\in\Gamma_{\kappa'}(x_1)\cap\Gamma_{\kappa'}(x_2)$ then $R/2<\rho_\#(x_1,x_2)\le C_\rho(1+\kappa')\delta_E(y)=\delta_E(y)/4\varepsilon_o$. Next, write $u'=u'\mathbf{1}_{\Gamma_{\kappa'}(x_1)}+u'\big(1-\mathbf{1}_{\Gamma_{\kappa'}(x_1)}\big)=:u_1'+u_2'$. By (6.98) and the fact that $u_1$ is supported in $\{\delta_E(\cdot) < \varepsilon_o R\}$, we have $(\mathscr{A}_{q,\kappa'} u_1')(x_2) = 0$ and $(\mathscr{A}_{q,\kappa'} u_2')(x_1) = 0$. Thus, hypothesis (6.79) holds for $u_1'$ and $u_2'$, so the first part of the proof applies to $u_1'$ and $u_2'$, which implies that

$$\begin{aligned}\|\mathscr{A}_{q,\kappa} u'\|_{L^p(E,\sigma)} &\leq C\|\mathscr{A}_{q,\kappa} u_1'\|_{L^p(E,\sigma)} + C\|\mathscr{A}_{q,\kappa} u_2'\|_{L^p(E,\sigma)} \\ &\leq C\|\mathfrak{C}_{q,\kappa} u_1'\|_{L^p(E,\sigma)} + C\|\mathfrak{C}_{q,\kappa} u_2'\|_{L^p(E,\sigma)} \leq C\|\mathfrak{C}_{q,\kappa} u\|_{L^p(E,\sigma)}.\end{aligned} \tag{6.99}$$

Together with (6.97), this proves *(1)* for $u$ in Case II. The proof of *(1)* is thus complete.

To prove *(2)*, first note that the pointwise estimate

$$(\mathfrak{C}_{q,\kappa} u)(x_0) \leq C\big[M_E(\mathscr{A}_{q,\kappa} u)^q(x_0)\big]^{\frac{1}{q}}, \qquad \forall\, x_0 \in E, \tag{6.100}$$

holds for some $C \in (0,\infty)$ depending only on $\kappa, p, q$ and geometric constants. Indeed, if $\Delta$ is a ball in $(E, (\rho|_E)_\#)$, then (6.35), (1) in Lemma 6.1 and (6.11) imply

$$\int_\Delta (\mathscr{A}_{q,\kappa} u)(x)^q\, d\sigma(x) = \int_{\mathcal{F}_\kappa(\Delta)} |u(y)|^q \sigma\big(\Delta \cap \pi_y^\kappa\big)\, d\mu(y) \geq \int_{\mathcal{T}_\kappa(\Delta)} |u(y)|^q \sigma\big(\pi_y^\kappa\big)\, d\mu(y) \tag{6.101}$$

and (6.100) follows. Next, if $p \in (q,\infty)$ and $r \in (0,\infty]$, then $M_E$ is bounded on $L^{p/q,r/q}(E,\sigma)$, so by the general fact that $\||f|^\alpha\|_{L^{p,r}(E,\sigma)} = C(p,r,\alpha)\|f\|^\alpha_{L^{p\alpha,r\alpha}(E,\sigma)}$ for all $\alpha > 0$, we have

$$\|\mathfrak{C}_{q,\kappa} u\|_{L^{p,r}(E,\sigma)} \leq C\|M_E(\mathscr{A}_{q,\kappa} u)^q\|^{1/q}_{L^{p/q,r/q}(E,\sigma)} \leq C\|\mathscr{A}_{q,\kappa} u\|_{L^{p,r}(E,\sigma)}, \tag{6.102}$$

as required to prove *(2)*. This also proves *(4)*, since it is a combination of *(1)* and *(2)*. To prove *(3)*, we use a computation similar to (6.102) based on (6.100), the weak-$(1,1)$ boundedness of $M_E$, and the boundedness of $M_E$ on $L^\infty(E,\sigma)$. This finishes the proof of the theorem. □

**Remark 6.8.** *The case $p = q = r$ of part (2) of Theorem 6.7, which corresponds to the estimate $\|\mathfrak{C}_{p,\kappa} u\|_{L^p(E,\sigma)} \leq C\|\mathscr{A}_{p,\kappa} u\|_{L^p(E,\sigma)}$, fails in general. A counterexample in Euclidean space when $p = 2$ is given in the remarks stated below Theorem 3 of [17].*

## 6.3 Weak $L^p$ square function estimates imply $L^2$ square function estimates

We are now in a position to consider $L^p$ versions of the $L^2$ square function estimates considered in Section 3 for integral operators $\Theta_E$. The main result, stated in Theorem 6.9, is that $L^2$ square function estimates follow from weak $L^p$ square function estimates for any $p \in (0,\infty)$. The result is proved by combining the $T(1)$ theorem in Theorem 3.1 with a weak type John-Nirenberg lemma for Carleson measures based on Lemma 2.14 in [5] (cf. [27, Lemma IV.1.12]).

**Theorem 6.9.** *Let $0 < d < m < \infty$. Assume that $(\mathscr{X}, \rho, \mu)$ is an $m$-dimensional* ADR *space, $E$ is a closed subset of $(\mathscr{X}, \tau_\rho)$, and $\sigma$ is a Borel semiregular measure on $(E, \tau_{\rho|_E})$ such that $(E, \rho\big|_E, \sigma)$ is a $d$-dimensional* ADR *space. Suppose that $\Theta$ is an integral operator with kernel $\theta$ satisfying* (3.1)-(3.4). *If there exist $\kappa$, $p$, $C_o \in (0,\infty)$ such that*

$$\sigma\Big(\Big\{x \in E : \int_{\Gamma_\kappa(x)} |(\Theta \mathbf{1}_\Delta)(y)|^2\, \delta_E(y)^{2\upsilon - m}\, d\mu(y) > \lambda^2\Big\}\Big) \leq C_o \lambda^{-p} \sigma(\Delta), \quad \forall\, \lambda > 0, \tag{6.103}$$

*for all surface balls $\Delta \subseteq E$, as in* (6.68)*, then there exists $C \in (0,\infty)$, depending only on $\kappa$, $p$, $C_o$ and finite positive geometric constants* (*including* $\operatorname{diam}_\rho(E)$ *when $E$ is bounded*)*, such that*

$$\int_{\mathscr{X}\setminus E} |(\Theta f)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \leq C \int_E |f(x)|^2\, d\sigma(x), \qquad \forall\, f \in L^2(E,\sigma). \tag{6.104}$$

*Proof.* We set $q = 2$ in Proposition 6.11 below to obtain the Carleson measure estimate in (6.107) and then apply the $T(1)$ theorem for square functions in Theorem 3.1. $\square$

**Remark 6.10.** *The requirement in* (6.103) *is less restrictive than a weak $L^p$ square function estimate. In particular, it is satisfied whenever the weak $L^p$ square function estimate*

$$\sup_{\lambda>0}\left[\lambda\cdot\sigma\Big(\Big\{x\in E:\int_{\Gamma_\kappa(x)}|(\Theta f)(y)|^2\delta_E(y)^{2\upsilon-m}\,d\mu(y)>\lambda^2\Big\}\Big)^{1/p}\right]\leq C_o\|f\|_{L^p(E,\sigma)} \tag{6.105}$$

*holds for all $f\in L^p(E,\sigma)$, since then* (6.103) *follows by setting $f=\mathbf{1}_\Delta$.*

The remainder of this subsection concerns the proposition below used to prove Theorem 6.9.

**Proposition 6.11.** *Assume the hypotheses of Theorem 6.9. Let $\mathbb{D}(E)$ denote a dyadic cube structure on $E$ and consider a Whitney covering $\mathbb{W}_\lambda(\mathscr{X}\setminus E)$ of $\mathscr{X}\setminus E$ as in Lemma 2.17 with corresponding dyadic Carleson tents from (2.97). If there exists $\kappa$, $p$, $q$, $C_o\in(0,\infty)$ such that*

$$\sigma\Big(\Big\{x\in E:\int_{\Gamma_\kappa(x)}|(\Theta\mathbf{1}_\Delta)(y)|^q\,\delta_E(y)^{q\upsilon-m}\,d\mu(y)>\lambda^q\Big\}\Big)\leq C_o\lambda^{-p}\sigma(\Delta),\quad \forall\,\lambda>0, \tag{6.106}$$

*for all surface balls $\Delta\subseteq E$, as in (6.68), then there exists $C\in(0,\infty)$, depending only on $\kappa$, $p$, $q$, $C_o$ and finite positive geometric constants such that*

$$\sup_{Q\in\mathbb{D}(E)}\frac{1}{\sigma(Q)}\int_{T_E(Q)}|(\Theta 1)(x)|^q\delta_E(x)^{q\upsilon-(m-d)}\,d\mu(x)\leq C. \tag{6.107}$$

We require the following auxiliary results to prove Proposition 6.11. The first such result is a variation of the Whitney decomposition in Proposition 2.5.

**Lemma 6.12.** *Let $(E,\rho,\sigma)$ be a space of homogeneous type with a Borel measure $\sigma$ and a dyadic cube structure $\mathbb{D}(E)$. Suppose that $\mathcal{O}$ is an open subset of $(E,\tau_\rho)$ such that $\big(\mathcal{O},\rho\big|_{\mathcal{O}},\sigma\lfloor\mathcal{O}\big)$ is a space of homogeneous type. If $\lambda\in(1,\infty)$ and $\Omega$ is an open, proper, non-empty subset of $\mathcal{O}$, then there exist $\varepsilon\in(0,1)$, $N\in\mathbb{N}$, $\Lambda\in(\lambda,\infty)$ and a subset $\mathcal{W}\subseteq\mathbb{D}(E)$ such that the following properties hold:*

*(1)* $\sigma\big(\Omega\setminus\bigcup_{Q\in\mathcal{W}}Q\big)=0$.

*(2) If $Q$, $Q'\in\mathcal{W}$ and $Q\neq Q'$, then $Q\cap Q'=\emptyset$.*

*(3) If $x\in\Omega$, then $\#\big\{Q\in\mathcal{W}:B_\rho\big(x,\varepsilon\,\mathrm{dist}_\rho(x,\mathcal{O}\setminus\Omega)\big)\cap Q\neq\emptyset\big\}\leq N$.*

*(4) If $Q\in\mathcal{W}$, then $\lambda Q\subseteq\Omega$ and $\Lambda Q\cap[\mathcal{O}\setminus\Omega]\neq\emptyset$.*

*(5) $\ell(Q)\approx\ell(Q')$ uniformly for all $Q$, $Q'\in\mathcal{W}$ such that $\lambda Q\cap\lambda Q'\neq\emptyset$.*

*(6)* $\sum_{Q\in\mathcal{W}}\mathbf{1}_{\lambda Q}\leq N$.

*Proof.* Given $\lambda\in(1,\infty)$ and an open, proper, non-empty subset $\Omega$ of the space of homogeneous type $\big(\mathcal{O},\rho\big|_{\mathcal{O}},\sigma\lfloor\mathcal{O}\big)$, we obtain $\varepsilon\in(0,1)$, $N\in\mathbb{N}$, $\Lambda\in(\lambda,\infty)$ and a covering of $\Omega$ with balls such that $\Omega=\bigcup_{j\in\mathbb{N}}\big(\mathcal{O}\cap B_\rho(x_j,r_j)\big)$ by applying Proposition 2.5. For each $j\in\mathbb{N}$, set

$$I_j:=\big\{Q\in\mathbb{D}(E):\,\ell(Q)\approx r_j\text{ and }Q\cap B_\rho(x_j,r_j)\neq\emptyset\big\}. \tag{6.108}$$

Then *(1)-(6)* hold for any maximal disjoint subcollection $\mathcal{W}$ of $\bigcup_{j\in\mathbb{N}}I_j$ by the properties of the dyadic cube structure $\mathbb{D}(E)$ and the covering $\{B_\rho(x_j,r_j)\}_{j\in\mathbb{N}}$ in Propositions 2.11 and 2.5. $\square$

We now state the aforementioned weak type John-Nirenberg lemma for Carleson measures, cf. [5, Lemma 2.14] for a result similar in spirit in the Euclidean setting.

**Lemma 6.13.** *Assume the hypotheses of Proposition 6.11. Let $\kappa$, $q$, $N \in (0,\infty)$ and $\beta \in (0,1)$. There exists $\eta_o \in (0,\infty)$, depending only on geometric constants, such that if $\eta \in [\eta_o,\infty)$ and*

$$\sigma(\{x \in Q : S_Q(x) > N\}) < (1-\beta)\sigma(Q), \qquad \forall\, Q \in \mathbb{D}(E), \tag{6.109}$$

*where $S_Q(x) := \left(\int_{y\in\Gamma_\kappa(x):\, \rho_\#(x,y)<\eta\ell(Q)} |(\Theta 1)(y)|^q \delta_E(y)^{qv-m} d\mu(y)\right)^{1/q}$, for all $x \in E$, then there exists $C \in (0,\infty)$, depending only on $\kappa$, $\eta$, finite positive geometric constants and the constants in the kernel estimates for $\theta$ in* (3.1)-(3.4)*, such that*

$$\sup_{Q\in\mathbb{D}(E)} \frac{1}{\sigma(Q)} \int_{T_E(Q)} |(\Theta 1)(x)|^q \delta_E(x)^{qv-(m-d)}\, d\mu(x) \le C\beta^{-1}(1+N^q). \tag{6.110}$$

*Proof.* Fix $\eta_0 \in (0,\infty)$, to be specified, and suppose that (6.109) holds for some $\eta \in [\eta_0,\infty)$, $\kappa$, $q$, $N \in (0,\infty)$ and $\beta \in (0,1)$. For each $i \in \mathbb{N}$, let $\Theta_i$ be as in (3.77) and associate the function $S_Q^i$ to $\Theta_i$ in the sense that $S_Q$ is associated to $\Theta$. We fix $\widetilde{\kappa} \in (0,\kappa)$, to be specified, and use the notation $S^i_{Q,\widetilde{\kappa}}$ for the function defined similarly to $S_Q^i$ but with $\widetilde{\kappa}$ in place of $\kappa$. The set $\Omega_Q^{N,i} := \{x \in Q : S_Q^i(x) > N\}$ is an open, proper subset of $Q$ by (6.109), the pointwise inequality $S_Q^i \le S_Q$, and since $S_Q^i$ is lower semi-continuous by (6.4). We also define

$$A^i := \sup_{Q\in\mathbb{D}(E)} \frac{1}{\sigma(Q)} \int_Q (S^i_{Q,\widetilde{\kappa}}(x))^q\, d\sigma(x), \qquad \forall\, i \in \mathbb{N}. \tag{6.111}$$

To show that $A^i < \infty$, let $x_Q$ denote the center of $Q$ and apply (6.35) to obtain

$$\begin{aligned}\int_Q (S^i_{Q,\widetilde{\kappa}}(x))^q\, d\sigma(x) &\le \int_{y\in\mathcal{F}_{\widetilde{\kappa}}(Q):\, \mathrm{dist}_{\rho_\#}(y,Q)\le c\ell(Q)} |(\Theta_i 1)(y)|^q \delta_E(y)^{qv-m} \sigma\big(Q\cap\pi_y^{\widetilde{\kappa}}\big)\, d\mu(y)\\ &\le C\int_{B_{\rho_\#}(x_Q,c\ell(Q))} |(\Theta_i 1)(y)|^q \delta_E(y)^{qv-(m-d)}\, d\mu(y),\end{aligned} \tag{6.112}$$

for some $c \in (0,\infty)$ depending on $\eta$ and geometry, where we used $\sigma(Q\cap\pi_y^{\widetilde{\kappa}}) \le C\delta_E(y)^d$, which follows from Lemma 6.2, the fact that $(E,\rho\big|_E,\sigma)$ is $d$-dimensional ADR, and that $\delta_E(y) \le \mathrm{dist}_{\rho_\#}(y,Q) \le C\mathrm{diam}_\rho(E)$ when $y \in B_{\rho_\#}(x_Q,c\ell(Q))$. Also, as in (3.81), we have

$$\int_{B_{\rho_\#}(x_Q,C\ell(Q))} |(\Theta_i 1)(y)|^q \delta_E(y)^{qv-(m-d)}\, d\mu(y) \le Ci^{2qv}\ell(Q)^d \le Ci^{2qv}\sigma(Q), \tag{6.113}$$

for all $Q \in \mathbb{D}(E)$, hence $A^i < \infty$ for each $i \in \mathbb{N}$.

We claim that there exists $C \in (0,\infty)$, with the dependency stated in the theorem, such that $\sup_{i\in\mathbb{N}} A^i \le \beta^{-1}(C+N^q)$. To prove this, let $i \in \mathbb{N}$ and $Q \in \mathbb{D}(E)$. It suffices to consider when $\Omega_Q^{N,i} \neq \emptyset$, since otherwise $S^i_{Q,\widetilde{\kappa}} \le S^i_Q \le N \le \beta^{-1}N^q$ on $Q$. We noted above that $\Omega_Q^{N,i}$ is an open proper subset of $Q$ whilst $\big(Q,\rho|_Q,\sigma\lfloor Q\big)$ is a space of homogeneous type with doubling constant independent of $Q$ by Proposition 2.11. Thus, by Lemma 6.12, we have a Whitney decomposition of $\Omega_Q^{N,i}$, relative to $Q$, of dyadic cubes $\{Q_k\}_{k\in I_Q^{N,i}}$. Let $F_Q^{N,i} := Q\setminus\Omega_Q^{N,i}$ to write

$$\int_Q (S^i_{Q,\widetilde{\kappa}}(x))^q\,d\sigma(x) = \int_{F_Q^{N,i}} (S^i_{Q,\widetilde{\kappa}}(x))^q\,d\sigma(x) + \sum\nolimits_{k\in I_Q^{N,i}} \int_{Q_k} (S^i_{Q,\widetilde{\kappa}}(x))^q\,d\sigma(x) =: I + II. \tag{6.114}$$

Since $\widetilde{\kappa}\in(0,\kappa)$, we have $S^i_{Q,\widetilde{\kappa}} \le S^i_Q \le N$ on $F_Q^{N,i}$, so $I \le N^q\sigma(Q)$. To estimate $II$, we write

$$\begin{aligned} II = &\sum\nolimits_{k\in I_Q^{N,i}} \int_{Q_k} (S^i_{Q_k,\widetilde{\kappa}}(x))^q\,d\sigma(x) \qquad (6.115)\\ &+ \sum\nolimits_{k\in I_Q^{N,i}} \int_{Q_k}\int_{y\in\Gamma_{\widetilde{\kappa}}(x):\,\eta\ell(Q_k)\le\rho_\#(x,y)<\eta\ell(Q)} |(\Theta_i 1)(y)|^q\delta_E(y)^{q\upsilon-m}d\mu(y)d\sigma(x) =: III + IV. \end{aligned}$$

To estimate $III$, recall (6.111), the fact that the family $\{Q_k\}_{k\in I_Q^{N,i}}$ consists of pairwise disjoint cubes from $\mathbb{D}(E)$ contained in $\Omega_Q^{N,i}$, and assumption (6.109), to obtain

$$III \le \sum\nolimits_{k\in I_Q^{N,i}} A^i\sigma(Q_k) \le A^i\sigma\big(\Omega_Q^{N,i}\big) \le A^i(1-\beta)\sigma(Q). \tag{6.116}$$

To estimate $IV$, fix $C_0>0$, to be specified later, and note that $|(\Theta_i 1)(y)| \le C/\delta_E(y)^\upsilon$ for all $y\in\mathscr{X}\setminus E$ by (3.2) and (3.12). Thus, if $k\in I_Q^{N,i}$ and $x\in Q_k$, then

$$\begin{aligned} &\int_{y\in\Gamma_{\widetilde{\kappa}}(x):\,\eta\ell(Q_k)\le\rho_\#(x,y)\le C_0\ell(Q_k)} |(\Theta_i 1)(y)|^q\delta_E(y)^{q\upsilon-m}\,d\mu(y)\\ &\qquad\le C\ell(Q_k)^{-m}\mu(\{y\in\Gamma_{\widetilde{\kappa}}(x):\,\eta\ell(Q_k)\le\rho_\#(x,y)\le C_0\ell(Q_k)\}) \le C, \qquad (6.117)\end{aligned}$$

and similarly

$$\int_{y\in\Gamma_{\widetilde{\kappa}}(x):\,\eta C_\rho^{-1}\ell(Q)\le\rho_\#(x,y)\le\eta\ell(Q)} |(\Theta_i 1)(y)|^q\delta_E(y)^{q\upsilon-m}\,d\mu(y) \le C, \tag{6.118}$$

for some $C\in(0,\infty)$ independent of $x$, $k$, $Q$ and $i$. Next, by the properties of the Whitney decomposition, for each $k\in I_Q^{N,i}$, there exists $x_k\in F_Q^{N,i}$ such that $\mathrm{dist}_{\rho_\#}(x_k,x)\le c\,\ell(Q_k)$ for all $x\in Q_k$ and some $c\in(0,\infty)$ independent of $k$, $Q$ and $i$. We claim that there exits $\widetilde{\kappa}\in(0,\kappa)$, depending only on $\kappa$, $C_0$ and geometric constants, but independent of $k$, $Q$ and $i$, such that

$$x\in Q_k,\ \ y\in\Gamma_{\widetilde{\kappa}}(x)\ \text{ and }\ C_0\ell(Q_k)<\rho_\#(x,y) \implies y\in\Gamma_\kappa(x_k). \tag{6.119}$$

Indeed, if $\widetilde{\kappa}\in(0,\kappa)$, $x\in Q_k$, $y\in\Gamma_{\widetilde{\kappa}}(x)$ and $C_0\ell(Q_k)<\mathrm{dist}_{\rho_\#}(y,Q)$, then

$$C_0\ell(Q_k) < \rho_\#(y,x) < (1+\widetilde{\kappa})\delta_E(y) < (1+\kappa)\delta_E(y), \tag{6.120}$$

so choosing $\vartheta\in\big(0,(\log_2 C_\rho)^{-1}\big]$ such that $(\rho_\#)^\vartheta$ is a genuine distance by Theorem 2.1, we have

$$\rho_\#(y,x_k)^\vartheta \le \rho_\#(y,x)^\vartheta + \rho_\#(x,x_k)^\vartheta < (1+\widetilde{\kappa})^\vartheta\delta_E(y)^\vartheta + c^\vartheta(1+\kappa)^\vartheta\delta_E(y)^\vartheta/C_0^\vartheta, \tag{6.121}$$

and (6.119) holds by choosing $C_0 > c[1-(1+\kappa)^{-\vartheta}]^{-1/\vartheta}$ and $0<\widetilde{\kappa}<(1+\kappa)[1-(c/C_0)^\vartheta]^{1/\vartheta}-1$. Next, we restrict $\eta_0\in[c\,C_\rho,\infty)$, so that $\rho_\#(x_k,y)\le\eta\ell(Q)$ when $\rho_\#(x,y)<\eta C_\rho^{-1}\ell(Q)$ and $x\in Q_k$. In combination with (6.119), if $k\in I_Q^{N,i}$ and $x\in Q_k$, we then have

$$\int_{y\in\Gamma_{\widetilde{\kappa}}(x):\,C_0\ell(Q_k)<\rho_\#(x,y)<\eta C_\rho^{-1}\ell(Q)} |(\Theta_i 1)(y)|^q\delta_E(y)^{q\upsilon-m}\,d\mu(y) \le (S^i_Q(x_k))^q \le N^q. \tag{6.122}$$

We complete the estimate for $IV$ by combining (6.117), (6.118) and (6.122) to obtain

$$IV \leq (C + N^q) \sum\nolimits_{k \in I_Q^{N,i}} \sigma(Q_k) \leq (C + N^q)\sigma(Q). \tag{6.123}$$

We now combine (6.123) with (6.114)-(6.116) to conclude that

$$\int_Q (S^i_{Q,\widetilde{\kappa}}(x))^q d\sigma \leq A^i(1-\beta)\sigma(Q) + (C + N^q)\sigma(Q), \qquad \forall\, Q \in \mathbb{D}(E), \tag{6.124}$$

hence $A^i \leq A^i(1-\beta) + C + N^q$ for each $i \in \mathbb{N}$, and $\sup_{i\in\mathbb{N}} A^i \leq \beta^{-1}(C + N^q)$, as required.

Now define $S_{Q,\widetilde{\kappa}}$ as $S_Q$ but with $\widetilde{\kappa}$ instead of $\kappa$. Since $\lim_{i\to\infty} S^i_{Q,\widetilde{\kappa}} = S_{Q,\widetilde{\kappa}}$ pointwise in $E$ and $\sup_{i\in\mathbb{N}} A^i \leq \beta^{-1}(C + N^q)$, Lebesgue's Monotone Convergence Theorem implies that

$$\sup_{Q\in\mathbb{D}(E)} \frac{1}{\sigma(Q)} \int_Q (S_{Q,\widetilde{\kappa}}(x))^q \, d\sigma(x) \leq \beta^{-1}(C + N^q). \tag{6.125}$$

Next, since $\rho_\#(x,y) \leq \eta\ell(Q)$ for all $x, y \in B_{\rho_\#}\big(x_Q, \eta C_\rho^{-1}\ell(Q)\big)$, based on (6.35), property (1) in Lemma 6.1, (6.11) and the fact that $\big(E, \rho\big|_E, \sigma\big)$ is a $d$-dimensional ADR space, we have

$$\begin{aligned}
&\int_{\Delta(x_Q,\eta C_\rho^{-1}\ell(Q))} (S_{Q,\widetilde{\kappa}}(x))^q \, d\sigma(x) \\
&\quad \geq \int_{\Delta(x_Q,\eta C_\rho^{-1}\ell(Q))} \int_{\Gamma_{\widetilde{\kappa}}(x)} \mathbf{1}_{B_{\rho_\#}(x_Q,\eta C_\rho^{-1}\ell(Q))}(y)\, |(\Theta 1)(y)|^q \delta_E(y)^{qv-m} \, d\mu(y) d\sigma(x) \\
&\quad \geq \int_{\mathcal{T}_{\widetilde{\kappa}}(\Delta(x_Q,\eta C_\rho^{-1}\ell(Q)))} \mathbf{1}_{B_{\rho_\#}(x_Q,\eta C_\rho^{-1}\ell(Q))}(y)\, |(\Theta 1)(y)|^q \delta_E(y)^{qv-m} \sigma\big(\pi_y^{\widetilde{\kappa}}\big) \, d\mu(y) \\
&\quad \approx \int_{\mathcal{T}_{\widetilde{\kappa}}(\Delta(x_Q,\eta C_\rho^{-1}\ell(Q)))} \mathbf{1}_{B_{\rho_\#}(x_Q,\eta C_\rho^{-1}\ell(Q))}(y)\, |(\Theta 1)(y)|^q \delta_E(y)^{qv-(m-d)} \, d\mu(y),
\end{aligned} \tag{6.126}$$

for all $Q \in \mathbb{D}(E)$. Also, there exists $M \in \mathbb{N}$, depending only on geometric constants, such that for every $Q \in \mathbb{D}(E)$, the ball $\Delta(x_Q, \eta C_\rho^{-1}\ell(Q))$ is covered by at most $M$ cubes $\widetilde{Q} \in \mathbb{D}(E)$ with $\ell(\widetilde{Q}) = \ell(Q)$ and $S_{\widetilde{Q}} = S_Q$. Therefore, by (6.125), (6.126) and *(6)* in Lemma 6.1, we obtain

$$\sup_{Q\in\mathbb{D}(E)} \frac{1}{\sigma(Q)} \int_{B_{\rho_\#}(x_Q,\,\eta C_\rho^{-2}\ell(Q))\setminus E} |(\Theta 1)(x)|^q \delta_E(x)^{qv-(m-d)} \, d\mu(x) \leq C\beta^{-1}(1 + N^q). \tag{6.127}$$

Then (6.110) follows by using (2.108) to choose $\eta_0 \in [c\, C_\rho, \infty)$ large enough, depending only on geometric constants, so that $T_E(Q) \subseteq B_{\rho_\#}(x_Q,\, \eta C_\rho^{-2}\ell(Q)) \setminus E$ for all $Q \in \mathbb{D}(E)$. $\square$

The result below is a geometric estimate on nontangential approach regions from [62].

**Lemma 6.14.** *Let $0 < d < m < \infty$. Assume that $(\mathscr{X}, \rho, \mu)$ is an $m$-dimensional* ADR *space, $E$ is a closed subset of $(\mathscr{X}, \tau_\rho)$, and there exists a Borel measure $\sigma$ on $(E, \tau_{\rho|_E})$ such that $(E, \rho\big|_E, \sigma)$ is a $d$-dimensional* ADR *space. For each $\kappa > 0$, $\beta < m$ and $M > m - \beta$, there exists $C \in (0,\infty)$, depending on $\kappa$, $M$, $\beta$, and the* ADR *constants of $\mathscr{X}$ and $E$, such that*

$$\int_{\Gamma_\kappa(z)} \frac{\delta_E(x)^{-\beta}}{\rho_\#(x,y)^M} \, d\mu(x) \leq C\rho(y,z)^{m-\beta-M}, \quad \textit{for all } \ z, y \in E \textit{ with } z \neq y. \tag{6.128}$$

Now we are ready to proceed with the proof of Proposition 6.11.

*Proof of Proposition 6.11.* Assume the hypotheses of Proposition 6.11. Lemma 6.13 shows that it suffices to find some $N \in (0, \infty)$, $\beta \in (0, 1)$ and $\eta \in [\eta_o, \infty)$ such that (6.109) holds. To this end, fix $N, c \in (0, \infty)$, to be specified, and $\eta \in [\eta_o, \infty)$. For each $Q \in \mathbb{D}(E)$, we have

$$\begin{aligned}
&\sigma(\{x \in Q : S_Q(x) > N\}) \qquad (6.129)\\
&\quad \leq \sigma\big(\big\{x \in Q : \int_{\Gamma_\kappa(x) \cap B_{\rho_\#}(x, \eta\ell(Q))} |(\Theta \mathbf{1}_{cQ})(y)|^q \delta_E(y)^{qv-m}\, d\mu(y) > (N/2)^q\big\}\big)\\
&\qquad + \sigma\big(\big\{x \in Q : \int_{\Gamma_\kappa(x) \cap B_{\rho_\#}(x, \eta\ell(Q))} |(\Theta \mathbf{1}_{E \setminus cQ})(y)|^q \delta_E(y)^{qv-m}\, d\mu(y) > (N/2)^q\big\}\big) =: I + II,
\end{aligned}$$

where $c\,Q := E \cap B_{\rho_\#}(x_Q, c\,\ell(Q))$. Assumption (6.106) and doubling imply that $I \leq CN^{-p}\sigma(Q)$.

To estimate $II$, we fix a finite constant $c_o \geq \sup_{Q' \in \mathbb{D}(E)} \big(\mathrm{diam}_{\rho_\#}(Q')/\ell(Q')\big)$ and let $x \in Q$. If $y \in B_{\rho_\#}(x, \eta\ell(Q))$ and $z \in E \setminus c\,Q$, then

$$\begin{aligned}
\rho_\#(z, x_Q) &\leq C_\rho \rho_\#(z, y) + C_\rho \max\{\eta, c_o\}\, \ell(Q)\\
&\leq C_\rho \rho_\#(z, y) + c^{-1} C_\rho^2 \max\{\eta, c_o\}\, \rho_\#(z, x_Q),
\end{aligned} \tag{6.130}$$

so restricting $c \in (0, \infty)$ such that $c^{-1} C_\rho^2 \max\{\eta, c_o\} < 1/2$, we have $\rho_\#(z, x_Q) \leq 2C_\rho \rho_\#(z, y)$, and then by (3.2) and (3.13), we obtain

$$|(\Theta \mathbf{1}_{E \setminus cQ})(y)| \leq C \int_{z \in E:\, \rho_\#(z, x_Q) > c\,\ell(Q)} \frac{d\sigma(z)}{\rho_\#(z, x_Q)^{d+v-a}} \leq C \frac{\delta_E(y)^{-a}}{\ell(Q)^{v-a}}. \tag{6.131}$$

Moreover, since $(E, \rho|_E, \sigma)$ is ADR, we now choose $c \in (0, \infty)$ as above and such that there exists $w \in c_2 Q \setminus c_1 Q$ for some $1 < c_1 < c_2 < c$. Then $\rho_\#(x, w) \approx \ell(Q)$ and we claim that

$$\rho_\#(y, w) \approx \ell(Q), \qquad \text{uniformly for all } y \in \Gamma_\kappa(x) \cap B_{\rho_\#}(x, \eta\ell(Q)). \tag{6.132}$$

Indeed, if $\rho_\#(y, x) < \eta\ell(Q)$, then $\rho_\#(y, w) \leq C_\rho \max\{\rho_\#(y, x), \rho_\#(x, w)\} \leq C\ell(Q)$, and if in addition $y \in \Gamma_\kappa(x)$, then $\rho_\#(y, x) < (1+\kappa)\delta_E(y) \leq (1+\kappa)\rho_\#(y, w)$, hence

$$C\ell(Q) \leq \rho_\#(x, w) \leq C_\rho(1+\kappa)\rho_\#(y, w) \leq C\ell(Q). \tag{6.133}$$

Now choosing $M > q(v-a)$, and using (6.131), (6.132) and Lemma 6.14, we obtain

$$\begin{aligned}
&\int_{y \in \Gamma_\kappa(x):\, \rho_\#(x,y) < \eta\ell(Q)} |(\Theta \mathbf{1}_{E \setminus cQ})(y)|^q \delta_E(y)^{qv-m}\, d\mu(y)\\
&\qquad \leq C\ell(Q)^{M - q(v-a)} \int_{\Gamma_\kappa(x)} \frac{\delta_E(y)^{-[m - q(v-a)]}}{\rho_\#(y, w)^M}\, d\mu(y) \leq C, \quad \forall\, x \in Q.
\end{aligned} \tag{6.134}$$

Applying Tschebyshev's inequality, we then obtain

$$II \leq \frac{C}{N} \int_Q \bigg(\int_{y \in \Gamma_\kappa(x):\, \rho_\#(x,y) < \eta\ell(Q)} |(\Theta \mathbf{1}_{E \setminus cQ})(y)|^q \frac{d\mu(y)}{\delta_E(y)^{m-qv}}\bigg)^{1/q} d\sigma(x) \leq \frac{C}{N}\sigma(Q). \tag{6.135}$$

Now combining (6.129) and (6.135), we have

$$\sigma(\{x \in Q : S_Q(x) > N\}) \leq CN^{-\min\{1,p\}}\sigma(Q), \quad \forall\, Q \in \mathbb{D}(E), \tag{6.136}$$

so (6.109) holds for any $\beta \in (0, 1)$ by choosing $N \in (0, \infty)$ such that $CN^{-\min\{1,p\}} < 1 - \beta$. $\square$

## 6.4 Extrapolating square function estimates

We now use Theorems 6.7 and 6.9 to prove the extrapolation results in Theorem 6.17 below for square function estimates associated with integral operators $\Theta_E$, as defined in Section 3. Let us first digress to clarify terminology and background results concerning the scale of Hardy spaces $H^p(E,\rho,\sigma)$ for $p\in(0,\infty)$ in the context of a $d$-dimensional ADR space $(E,\rho,\sigma)$. In particular, we record an atomic characterization for these spaces based on the work of R.R. Coifman and G. Weiss in [20], and a maximal function characterization based on the work of R.A. Macías and C. Segovia in [58]. The theory of Hardy spaces in this context has also been developed by D. Mitrea, I. Mitrea, M. Mitrea and S. Monniaux in [61] and subseqetly refined by R. Alvarado and M. Mitrea in [2].

We begin by defining, for each $\beta\in(0,\infty)$, the homogeneous Hölder space

$$\dot{\mathscr{C}}^{\beta}(E,\rho):=\big\{f:E\to\mathbb{R}:\ \|f\|_{\dot{\mathscr{C}}^{\beta}(E,\rho)}:=\sup_{x,y\in E,\,x\neq y}\frac{|f(x)-f(y)|}{\rho(x,y)^{\beta}}<\infty\big\}. \tag{6.137}$$

Let $\dot{\mathscr{C}}^{\beta}_c(E,\rho)$ denote the subspace of functions in $\dot{\mathscr{C}}^{\beta}(E,\rho)$ that vanish identically outside a bounded set. The class of `test functions` $\mathscr{D}(E,\rho):=\bigcap_{0<\beta<\mathrm{ind}(E,\rho)}\dot{\mathscr{C}}^{\beta}_c(E,\rho)$, with $\mathrm{ind}(E,\rho)$ defined as in (2.6), is then equipped with the topology $\tau_{\mathscr{D}}$, defined as follows: Fix a nested family $\{K_n\}_{n\in\mathbb{N}}$ of $\rho$-bounded subsets of $E$ such that every $\rho$-ball is contained some $K_n$. For each $n\in\mathbb{N}$, let $\mathscr{D}_n(E,\rho)$ denote the collection of functions in $\mathscr{D}(E,\rho)$ that vanish on $E\setminus K_n$, which becomes a Frechét space when equipped with the topology $\tau_n$ induced by the family of norms

$$\{\|\cdot\|_{\infty}+\|\cdot\|_{\dot{\mathscr{C}}^{\beta}(E,\rho)}:\ \beta \text{ is a rational number such that } 0<\beta<\mathrm{ind}(E,\rho)\}. \tag{6.138}$$

For each $n\in\mathbb{N}$, the topology induced by $\tau_{n+1}$ on $\mathscr{D}_n(E,\rho)$ coincides with $\tau_n$, so we define $\tau_{\mathscr{D}}$ as the the associated strict inductive limit topology on $\mathscr{D}(E,\rho)$. We then define the `space of distributions` $\mathscr{D}'(E,\rho)$ on $E$ as the (topological) dual of $\mathscr{D}(E,\rho)$ and let $\langle\cdot,\cdot\rangle$ denote the natural duality pairing between distributions in $\mathscr{D}'(E,\rho)$ and test functions in $\mathscr{D}(E,\rho)$.

For each $\gamma\in(0,\mathrm{ind}(E,\rho))$, define the `grand maximal function` of $f\in\mathscr{D}'(E,\rho)$ by

$$f^*_{\rho,\gamma}(x):=\sup_{\psi\in\mathcal{B}^{\gamma}_{\rho}(x)}\big|\langle f,\psi\rangle\big|,\qquad \forall\,x\in E, \tag{6.139}$$

where $\mathcal{B}^{\gamma}_{\rho}(x)$ is the set of all $(\rho,\gamma)$-`normalized bump-functions supported near` $x$, that is, $\psi\in\mathscr{D}(E,\rho)$ such that $\psi=0$ on $E\setminus B_\rho(x,r)$ and $\|\psi\|_\infty+r^{\gamma}\|\psi\|_{\dot{\mathscr{C}}^{\gamma}(E,\rho)}\leq r^{-d}$ for some $r>0$. For $d/(d+\mathrm{ind}(E,\rho))<p<\infty$, define the `Hardy space`

$$H^p(E,\rho,\sigma):=\{f\in\mathscr{D}'(E,\rho):f^*_{\rho_\#,\gamma}\in L^p(E,\sigma)\text{ for all } d(\tfrac{1}{p}-1)<\gamma<\mathrm{ind}(E,\rho)\} \tag{6.140}$$

and the closely related space

$$\widetilde{H}^p(E,\rho,\sigma):=\{f\in\mathscr{D}'(E,\rho):f^*_{\rho_\#,\gamma}\in L^p(E,\sigma)\text{ for some } d(\tfrac{1}{p}-1)<\gamma<\mathrm{ind}(E,\rho)\}. \tag{6.141}$$

For $d/(d+\mathrm{ind}(E,\rho))<p\leq1$, a function $a\in L^\infty(E,\sigma)$ is called a $p$-`atom` if there exist $x_0\in E$ and a real number $r>0$ such that

$$\mathrm{supp}\,a\subseteq E\cap B_\rho(x_0,r),\quad \|a\|_{L^\infty(E,\sigma)}\leq r^{-d/p},\quad \int_E a\,d\sigma=0, \tag{6.142}$$

(when $E$ is bounded, the constant function $\sigma(E)^{-1/p}$ is also called a $p$-atom), and we define

$$H^p_{at}(E,\rho,\sigma) := \big\{f \in \big(\dot{\mathscr{C}}^{d(1/p-1)}(E,\rho)\big)^* : \text{ there exist } \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N}) \text{ and } p\text{-atoms } \{a_j\}_{j\in\mathbb{N}} \text{ such that } \textstyle\sum_{j\in\mathbb{N}} \lambda_j a_j \text{ converges to } f \text{ in } \big(\dot{\mathscr{C}}^{d(1/p-1)}(E,\rho)\big)^*\big\}, \tag{6.143}$$

with the quasi-norm $\|f\|_{H^p_{at}(E,\rho,\sigma)} := \inf\big\{\big(\sum_{j\in\mathbb{N}}|\lambda_j|^p\big)^{1/p} : f = \sum_{j\in\mathbb{N}}\lambda_j a_j \text{ as in (6.143)}\big\}$. The following characterizations of these spaces from [2] extend work in [58] and [61].

**Theorem 6.15.** *Let $d>0$. Assume that $(E,\rho,\sigma)$ is $d$-dimensional* ADR. *If $p\in(1,\infty)$, then*

$$H^p(E,\rho,\sigma) = \widetilde{H}^p(E,\rho,\sigma) = L^p(E,\sigma). \tag{6.144}$$

*If $d/(d+\operatorname{ind}(E,\rho)) < p \le 1$ and $d(\frac{1}{p}-1) < \gamma < \operatorname{ind}(E,\rho)$, then*

$$\|f\|_{H^p_{at}(E,\rho,\sigma)} \approx \|(\widetilde{f})^*_{\rho_\#,\gamma}\|_{L^p(E,\sigma)}, \quad \forall\, f \in H^p_{at}(E,\rho,\sigma), \tag{6.145}$$

*where $\widetilde{f} \in \mathscr{D}'(E,\rho)$ denotes the restriction of $f$ to $\mathscr{D}(E,\rho)$. Moreover, the assignment $f \mapsto \widetilde{f}$ provides an injective linear mapping from $H^p_{at}(E,\rho,\sigma)$ onto $\widetilde{H}^p(E,\rho,\sigma)$, and thus provides a natural identification of $H^p(E,\rho,\sigma)$ and $\widetilde{H}^p(E,\rho,\sigma)$ with $H^p_{at}(E,\rho,\sigma)$. Furthermore, there exists $C\in(0,\infty)$, depending on $p$, $\rho$, $\gamma$, such that if $f\in\mathscr{D}'(E,\rho)$ and $f^*_{\rho_\#,\gamma}\in L^p(E,\sigma)$, then there exist $\{\lambda_j\}_{j\in\mathbb{N}}\in\ell^p(\mathbb{N})$ and $p$-atoms $\{a_j\}_{j\in\mathbb{N}}$ such that $\sum_{j\in\mathbb{N}}\lambda_j a_j$ converges to $f$ in $\mathscr{D}'(E,\rho)$ with $\|\{\lambda_j\}_{j\in\mathbb{N}}\|_{\ell^p} \le C\|f^*_{\rho_\#,\gamma}\|_{L^p(E,\sigma)}$. Conversely, for $\{\lambda_j\}_{j\in\mathbb{N}}\in\ell^p(\mathbb{N})$ and $p$-atoms $\{a_j\}_{j\in\mathbb{N}}$, if $\sum_{j\in\mathbb{N}}\lambda_j a_j$ converges to some $f$ in $\mathscr{D}'(E,\rho)$, then $\|f^*_{\rho_\#,\gamma}\|_{L^p(E,\sigma)} \le C\|\{\lambda_j\}_{j\in\mathbb{N}}\|_{\ell^p}$.*

We will also need the following estimate from [62] for a Marcinkiewicz-type integral.

**Lemma 6.16.** *Let $d>0$. Assume that $(E,\rho,\sigma)$ is $d$-dimensional* ADR. *If $\alpha>0$, then there exists $C\in(0,\infty)$ such that for all nonempty closed subsets $F$ of $(E,\tau_\rho)$, it holds that*

$$\int_F\int_E \frac{\operatorname{dist}_{\rho_\#}(y,F)^\alpha}{\rho_\#(x,y)^{d+\alpha}}\,d\sigma(y)\,d\sigma(x) \le C\sigma(E\setminus F). \tag{6.146}$$

We will consider an integral operator $\Theta$ with kernel $\theta$ as in (3.1)-(3.2) for which, instead of (3.3), there exists $\alpha\in(0,\infty)$ such that for all $x\in\mathscr{X}\setminus E$, $y\in E$ and $\widetilde{y}\in E$ with $\rho(y,\widetilde{y}) \le \frac{1}{2}\rho(x,y)$, the following holds:

$$|\theta(x,y)-\theta(x,\widetilde{y})| \le C_\theta \frac{\rho(y,\widetilde{y})^\alpha}{\rho(x,y)^{d+\upsilon+\alpha}}\Big(\frac{\operatorname{dist}_\rho(x,E)}{\rho(x,y)}\Big)^{-a}. \tag{6.147}$$

In particular, setting $\gamma := \min\{\alpha, \operatorname{ind}(E,\rho|_E)\}$, for $\kappa>0$, $q\in(1,\infty)$ and $p\in(\frac{d}{d+\gamma},\infty)$, the $(p,q)$-`Square Function Estimate` $(\texttt{SFE})_{p,q}$ is said to hold when

$$\Big[\int_E\Big(\int_{\Gamma_\kappa(x)}|(\Theta f)(y)|^q\,\frac{d\mu(y)}{\delta_E(y)^{m-q\upsilon}}\Big)^{p/q}d\sigma(x)\Big]^{1/p} \le C\|f\|_{H^p(E,\rho|_E,\sigma)} \tag{6.148}$$

holds for all $f\in H^p(E,\rho|_E,\sigma)$, or equivalently, when $\delta_E^{\upsilon-m/q}\Theta : H^p(E,\rho|_E,\sigma) \longrightarrow L^{(p,q)}(\mathscr{X},E)$ is a well-defined bounded linear operator, where

$$\|f\|_{H^p(E,\rho|_E,\sigma)} := \begin{cases} \|f\|_{L^p(E,\sigma)} & \text{if } p\in(1,\infty), \\ \|f\|_{H^p_{at}(E,\rho|_E,\sigma)} & \text{if } p\in(\frac{d}{d+\gamma},1]. \end{cases} \tag{6.149}$$

Also, for $p \in (0,\infty)$, the `weak` $(p,q)$-`Square Function Estimate` $(\texttt{wSFE})_{p,q}$ is said to hold when

$$\sup_{\lambda>0}\left[\lambda\cdot\sigma\Big(\Big\{x\in E:\int_{\Gamma_\kappa(x)}|(\Theta f)(y)|^q\,\frac{d\mu(y)}{\delta_E(y)^{m-qv}}>\lambda^q\Big\}\Big)^{1/p}\right]\leq C\|f\|_{L^p(E,\sigma)} \tag{6.150}$$

holds for all $f \in L^p(E,\sigma)$. We now present the extrapolation result.

**Theorem 6.17.** *Let $0<d<m<\infty$. Assume that $(\mathscr{X},\rho,\mu)$ is an $m$-dimensional* ADR *space, $E$ is a closed subset of $(\mathscr{X},\tau_\rho)$, and $\sigma$ is a Borel measure on $(E,\tau_{\rho|_E})$ such that $(E,\rho\big|_E,\sigma)$ is a $d$-dimensional* ADR *space. Suppose that $\Theta$ is an integral operator with kernel $\theta$ satisfying* (3.1)*,* (3.2)*,* (3.4) *and* (6.147) *for some $\alpha\in(0,\infty)$. Let $\kappa>0$ and set $\gamma:=\min\{\alpha,\mathrm{ind}(E,\rho|_E)\}$. The following properties hold:*

*(1) If $q\in(1,\infty)$, $p_o\in(1,\infty)$ and $(\mathrm{wSFE})_{p_o,q}$ holds, then $(\mathrm{wSFE})_{1,q}$ holds and $(\mathrm{SFE})_{p,q}$ holds for each $p\in(1,p_o)$.*

*(2) If $q\in(1,\infty)$ and either $(\mathrm{SFE})_{q,q}$ holds or $(\mathrm{wSFE})_{p_o,q}$ holds for some $p_o\in(q,\infty)$, then $(\mathrm{SFE})_{p,q}$ holds for each $p\in(\frac{d}{d+\gamma},\infty)$.*

*(3) If $\sigma$ is a Borel semiregular measure on $(E,\tau_{\rho|_E})$, $p_o\in(0,\infty)$ and $(\mathrm{wSFE})_{p_o,2}$ holds, then $(\mathrm{SFE})_{p,2}$ holds for each $p\in(\frac{d}{d+\gamma},\infty)$.*

*Proof.* We successively prove *(1)*, *(2)* and *(3)* below. The proof of *(2)* is split into *Parts (a)-(e)*.

*Proof of (1).* Let $q\in[1,\infty)$, $p_o\in(1,\infty)$ and suppose that $(\mathrm{wSFE})_{p_o,q}$ holds. It suffices to prove that $\mathcal{A}_{q,\kappa}\circ(\delta_E^{v-m/q}\Theta)$ is weak type $(1,1)$, that is, there exists $C\in(0,\infty)$ such that

$$\sigma\big(\big\{x\in E:\,\mathcal{A}_{q,\kappa}(\delta_E^{v-m/q}(\Theta f))(x)>\lambda\big\}\big)\leq C\lambda^{-1}\|f\|_{L^1(E,\sigma)},\ \forall\,\lambda>0,\ \forall\,f\in L^1(E,\sigma), \tag{6.151}$$

since this is the statement that $(\mathrm{wSFE})_{1,q}$ holds, and then since $\mathcal{A}_{q,\kappa}\circ(\delta_E^{v-m/q}\Theta)$ is subadditive and weak type $(p_o,p_o)$ by the $(\mathrm{wSFE})_{p_o,q}$ assumption, the Marcinkiewicz interpolation theorem implies that $\mathcal{A}_{q,\kappa}\circ(\delta_E^{v-m/q}\Theta)$ is strong type $(p,p)$, hence $(\mathrm{SFE})_{p,q}$ holds, for each $p\in(1,p_o)$.

To prove (6.151), let $f\in L^1(E,\sigma)$. If $0<\lambda\leq\|f\|_{L^1(E,\sigma)}/\sigma(E)<\infty$, then $E$ is bounded and

$$\sigma\big(\big\{x\in E:\,\mathcal{A}_{q,\kappa}\big(\delta_E^{v-m/q}(\Theta f)\big)(x)>\lambda\big\}\big)\leq\sigma(E)\leq\lambda^{-1}\|f\|_{L^1(E,\sigma)}, \tag{6.152}$$

so (6.151) holds. Now assume that $\lambda>\|f\|_{L^1(E,\sigma)}/\sigma(E)$ and, without loss of generality, that $f$ has bounded support. We now introduce a Calderón-Zygmund decomposition of $f$ at level $\lambda$. More precisely (cf., e.g., [19]), there exist $C\in(0,\infty)$, $N\in\mathbb{N}$, depending only on geometry, balls $Q_j:=B_\rho(x_j,r_j)$, $j\in J\subseteq\mathbb{N}$, and functions $g,b:E\to\mathbb{R}$ such that $f=g+b$ on $E$ with

$$g\in L^1(E,\sigma)\cap L^\infty(E,\sigma),\quad \|g\|_{L^1(E,\sigma)}\leq C\|f\|_{L^1(E,\sigma)},\quad |g(x)|\leq C\lambda,\ \forall\,x\in E, \tag{6.153}$$

$$b=\sum\nolimits_{j\in J}b_j,\quad \operatorname{supp}b_j\subseteq Q_j,\quad \int_E b_j\,d\sigma=0,\quad \fint_{Q_j}|b_j|\,d\sigma\leq C\lambda,\ \forall\,j\in J, \tag{6.154}$$

and setting $\mathcal{O}:=\bigcup_{j\in J}Q_j\subseteq E$ and $F:=E\setminus\mathcal{O}$, it holds that

$$\sum\nolimits_{j\in J}\mathbf{1}_{Q_j}\leq N,\quad \sigma(\mathcal{O})\leq C\lambda^{-1}\|f\|_{L^1(E,\sigma)},\quad \mathrm{dist}_\rho(Q_j,F)\approx r_j,\ \forall\,j\in J. \tag{6.155}$$

The series in (6.154) converges absolutely in $L^1(E,\sigma)$, since $\sum_{j\in J}\|b_j\|_{L^1(E,\sigma)}\leq C\|f\|_{L^1(E,\sigma)}$.

To prove (6.151), since $(\mathrm{wSFE})_{p_o,q}$ holds and $p_o > 1$, we first use (6.153) to obtain

$$\sigma\big(\big\{x \in E : \mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta g)\big)(x) > \lambda/2\big\}\big) \leq C\lambda^{-p_o}\|g\|_{L^{p_o}(E,\sigma)}^{p_o} \leq C\lambda^{-1}\|f\|_{L^1(E,\sigma)}. \quad (6.156)$$

We then use (6.155) to obtain

$$\sigma\big(\big\{x \in \mathcal{O} : \mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta b)\big)(x) > \lambda/2\big\}\big) \leq \sigma(\mathcal{O}) \leq C\lambda^{-1}\|f\|_{L^1(E,\sigma)} \quad (6.157)$$

whilst it is immediate that

$$\sigma\big(\big\{x \in F : \mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta b)\big)(x) > \lambda/2\big\}\big) \leq \lambda^{-1}\int_F \mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta b)\big)\,d\sigma. \quad (6.158)$$

Thus, to prove (6.151), since $\mathcal{A}_{q,\kappa}\circ(\delta_E^{\upsilon-m/q}\Theta)$ is quasi-subadditive, it remains to prove that

$$\int_F \mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta b)\big)\,d\sigma \leq C\|f\|_{L^1(E,\sigma)}. \quad (6.159)$$

To prove (6.159), let $j \in J$, $Q_j = B_\rho(x_j, r_j)$, $x \in F$ and $y \in \Gamma_\kappa(x)$. We use (6.154) to write

$$|(\Theta b_j)(y)| \leq \int_{Q_j} |\theta(y,z) - \theta(y,x_j)||b_j(z)|\,d\sigma(z) =: I_1 + I_2, \quad (6.160)$$

where $I_1$ and $I_2$ are the integrals over $B_{\rho_\#}(x_j, \epsilon\rho_\#(y,x_j))$ and $Q_j \setminus B_{\rho_\#}(x_j, \epsilon\rho_\#(y,x_j))$ for some $\epsilon > 0$. We choose $0 < \epsilon < 2^{-1}\widetilde{C}_\rho^{-1}C_\rho^{-2}$ so that by Theorem 2.1, if $\rho_\#(z,x_j) < \epsilon\rho_\#(y,x_j)$, then

$$\rho(z,x_j) \leq C_\rho^2\rho_\#(z,x_j) < \epsilon C_\rho^2\rho_\#(y,x_j) \leq \epsilon\widetilde{C}_\rho C_\rho^2\rho_\#(y,x_j) < \tfrac{1}{2}\rho(y,x_j). \quad (6.161)$$

It then follows from assumption (6.147) and (6.154) that

$$I_1 \leq C\int_{Q_j} \frac{\rho_\#(z,x_j)^\alpha\delta_E(y)^{-a}}{\rho_\#(y,x_j)^{d+\upsilon+\alpha-a}}|b_j(z)|\,d\sigma(z) \leq C\lambda\,\frac{r_j^\alpha\delta_E(y)^{-a}\sigma(Q_j)}{\rho_\#(y,x_j)^{d+\upsilon+\alpha-a}}. \quad (6.162)$$

To estimate $I_2$, let $z \in Q_j \setminus B_{\rho_\#}(x_j, \epsilon\rho_\#(y,x_j))$. We have

$$r_j \approx \mathrm{dist}_\rho(Q_j, F) \leq \widetilde{C}_\rho\rho(x,x_j) \leq C(1+\kappa)\delta_E(y) + C\rho(y,x_j) \leq C\rho(y,x_j), \quad (6.163)$$

whilst $\rho(z,x_j) < r_j$ implies that $\rho(y,x_j) \leq C\epsilon^{-1}r_j$, hence $r_j \approx \rho(y,x_j)$ uniformly for all $j \in J$, $y \in \Gamma_\kappa(x)$ and $x \in F$. Also, replacing $x_j$ with $z$ in (6.163) shows that $r_j \leq C\rho(y,z)$, hence $\rho(y,x_j) \leq C\rho(y,z) + Cr_j \leq C\rho(y,z)$ uniformly for all $j \in J$, $y \in \Gamma_\kappa(x)$, $x \in F$ and $z \in Q_j \setminus B_{\rho_\#}(x_j, \epsilon\rho_\#(y,x_j))$. Together with (3.2), on the domain of integration in $I_2$, we have

$$\big|\theta(y,z) - \theta(y,x_j)\big| \leq \frac{C\delta_E(y)^{-a}}{\rho_\#(y,z)^{d+\upsilon-a}} + \frac{C\delta_E(y)^{-a}}{\rho_\#(y,x_j)^{d+\upsilon-a}} \leq \frac{Cr_j^\alpha\delta_E(y)^{-a}}{\rho_\#(y,x_j)^{d+\upsilon+\alpha-a}}. \quad (6.164)$$

Together with (6.154), this allows us to estimate

$$I_2 \leq C\lambda\,\frac{r_j^\alpha\delta_E(y)^{-a}\sigma(Q_j)}{\rho_\#(y,x_j)^{d+\upsilon+\alpha-a}}. \quad (6.165)$$

Cumulatively, (6.160), (6.162) and (6.165) prove that

$$\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta b_j)\big)(x) \le C\lambda\, r_j^{\alpha}\sigma(Q_j)\left(\int_{\Gamma_\kappa(x)} \frac{\delta_E(y)^{q(\upsilon-a)-m}}{\rho_\#(y,x_j)^{q(d+\upsilon+\alpha-a)}}\,d\mu(y)\right)^{1/q} \tag{6.166}$$

uniformly for all $j \in J$ and $x \in F$.

For all $j \in J$ and $x \in F$, using (6.166) and Lemma 6.14 (recall that $\nu - a > 0$), we obtain

$$\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta b_j)\big)(x) \le C\lambda\, r_j^{\alpha}\sigma(Q_j)\rho(x,x_j)^{-d-\alpha} \le C\lambda \int_{Q_j} \frac{\operatorname{dist}_{\rho_\#}(z,F)^{\alpha}}{\rho_\#(x,z)^{d+\alpha}}\,d\sigma(z), \tag{6.167}$$

where in the last inequality we used that $\operatorname{dist}_{\rho_\#}(z,F) \approx r_j$, uniformly for all $z \in Q_j$, and that

$$\rho(x,z) \le C\rho(x,x_j) + Cr_j \le C\rho(x,x_j) + C\operatorname{dist}_\rho(Q_j,F) \le C\rho(x,x_j), \quad \forall\, z \in Q_j. \tag{6.168}$$

Summing over $j \in J$ and using the sublinearity of $\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}\Theta(\cdot)\big)$ (recall that $q \ge 1$), and the finite overlap property in (6.155), we obtain

$$\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta b)\big)(x) \le C\lambda \int_{\mathcal{O}} \frac{\operatorname{dist}_{\rho_\#}(z,F)^{\alpha}}{\rho_\#(x,z)^{d+\alpha}}\,d\sigma(z), \quad \forall\, x \in F. \tag{6.169}$$

Consequently, from (6.169), Lemma 6.16 and (6.155), we deduce that

$$\int_F \mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta b)\big)(x)\,dx \le C\lambda\,\sigma(E\setminus F) = C\lambda\,\sigma(\mathcal{O}) \le C\|f\|_{L^1(E,\sigma)}. \tag{6.170}$$

This proves (6.159), which in turn proves (6.151) and property *(1)* in the theorem.

*Proof of (2).* The proof of *(2)* is divided into *Parts (a)-(e)* below.

*Part (a):* Let $q \in (1,\infty)$, $p \in (1,q]$ and assume that $(\mathrm{SFE})_{q,q}$ holds in order to prove that $(\mathrm{SFE})_{p,q}$ holds. If $p = q$, then there is nothing to prove. If $p \in (1,q)$, then since property *(1)* implies that $(\mathrm{SFE})_{r,q}$ holds for each $r \in (1,q)$, it follows that $(\mathrm{SFE})_{p,q}$ holds.

*Part (b):* Let $q \in (1,\infty)$, $p \in (q,\infty)$ and assume that $(\mathrm{SFE})_{q,q}$ holds in order to prove that $(\mathrm{SFE})_{p,q}$ holds. In particular, by the equivalence in Theorem 6.7, it suffices to show that

$$\|\mathfrak{C}_{q,\kappa}(\delta_E^{\upsilon-m/q}(\Theta f))\|_{L^p(E,\sigma)} \le C\|f\|_{L^p(E,\sigma)}. \tag{6.171}$$

Moreover, by the boundedness of the Hardy-Littlewood maximal operator on $L^p(E,\sigma)$ and $L^{p/q}(E,\sigma)$, since $p > \max\{q,1\}$, it suffices to show, for all $f \in L^p(E,\sigma)$, that

$$\mathfrak{C}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta f)\big)(x_0) \le C\big[\big(M_E(|f|^q)(x_0)\big)^{1/q} + (M_E(M_E(f)))(x_0)\big], \quad \forall\, x_0 \in E. \tag{6.172}$$

To this end, let $r > 0$ and $x_0 \in E$. Fix $c \in (0,\infty)$, to be specified, set $\Delta := E \cap B_{\rho_\#}(x_0,r)$, $c\Delta := E \cap B_{\rho_\#}(x_0,cr)$, and write $f = f\mathbf{1}_{c\Delta} + f\mathbf{1}_{E\setminus c\Delta} =: f_1 + f_2$. Using (6.35), Lemma 6.2, the fact that $(E,\rho|_E,\sigma)$ is $d$-dimensional ADR, and $(\mathrm{SFE})_{q,q}$, we obtain

$$\begin{aligned}
\frac{1}{\sigma(\Delta)} \int_{\mathcal{T}_\kappa(\Delta)} &|(\Theta f_1)(x)|^q \delta_E(x)^{q\upsilon-(m-d)}\,d\mu(x) \\
&\le \frac{C}{\sigma(\Delta)} \int_E \Big(\int_{\Gamma_\kappa(x)} |(\Theta f_1)(y)|^q\,\delta_E(y)^{q\upsilon-m}\,d\mu(y)\Big)\,d\sigma(x) \\
&\le \frac{C}{\sigma(\Delta)} \int_{c\Delta} |f|^q\,d\sigma \le C\,M_E(|f|^q)(x_0).
\end{aligned} \tag{6.173}$$

To treat the contribution from $f_2$, we now choose $c > C_\rho$. If $y \in E \setminus c\Delta$ and $w \in \Delta$, then $cr < \rho_\#(y,x_0) \le C_\rho \max\{\rho_\#(y,w), \rho_\#(w,x_0)\} \le C_\rho \rho_\#(y,w)$, hence $E \setminus c\Delta \subseteq \{y \in E : \rho_\#(y,w) > r\}$ and $\rho_\#(y,x_0) \approx \rho_\#(y,w)$. Therefore, if $z \in \mathcal{T}_\kappa(\Delta)$, then since $\rho_\#(y,x_0) \le C_\rho \max\{\rho_\#(y,z), (1+\kappa)\delta_E(z)\} \le C\rho_\#(y,z)$, by (3.2) and (3.13), we have

$$|(\Theta f_2)(z)| \le C \int_{E\setminus c\Delta} \frac{\delta_E(z)^{-a}}{\rho_\#(z,y)^{d+\upsilon-a}} |f(y)|\, d\sigma(y) \le \frac{C\delta_E(z)^{-a}}{r^{\upsilon-a}} (M_E f)(w), \quad \forall\, w \in \Delta. \tag{6.174}$$

Thus, in concert with (6.15) and Lemma 3.5 (which uses $\upsilon - a > 0$), we obtain

$$\begin{aligned}
&\Big(\frac{1}{\sigma(\Delta)} \int_{\mathcal{T}_\kappa(\Delta)} |(\Theta f_2)(z)|^q \delta_E(z)^{q\upsilon-(m-d)}\, d\mu(z)\Big)^{1/q} \\
&\qquad \le \frac{C}{r^{\upsilon-a}} \inf_{w\in\Delta} (M_E f)(w) \Big(\frac{1}{\sigma(\Delta)} \int_{B_{\rho_\#}(x_0,Cr)\setminus E} \delta_E(z)^{q(\upsilon-a)-(m-d)}\, d\mu(z)\Big)^{1/q} \\
&\qquad \le C \inf_{w\in\Delta} (M_E f)(w) \le C \fint_\Delta M_E f\, d\sigma \le C M_E(M_E f)(x_0).
\end{aligned} \tag{6.175}$$

Now (6.172), and thus $(\mathrm{SFE})_{p,q}$, follows from (6.173) and (6.175) in view of (6.67) and the fact that $\mathfrak{C}_{q,\kappa} \circ \big(\delta_E^{\upsilon-m/q}\Theta\big)$ is sub-linear in the current context.

*Part (c):* Let $q \in (1,\infty)$, $p \in (\frac{d}{d+\gamma}, 1]$ and assume that $(\mathrm{SFE})_{q,q}$ holds in order to prove that

$$\sup\big\{\|\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta a)\big)\|^p_{L^p(E,\sigma)} : a \text{ is a } p\text{-atom}\big\} \le C. \tag{6.176}$$

To do this, let $a$ denote a $p$-atom with $x_0 \in E$ and $r > 0$ as in (6.142), so $\operatorname{supp} a \subseteq B_{\rho_\#}(x_0, \widetilde{C}_\rho r)$. Fix $c \in (1,\infty)$, to be specified, and set $\Delta := E \cap B_{\rho_\#}(x_0, cr)$ to write

$$\begin{aligned}
\big\|\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta a)\big)\big\|^p_{L^p(E,\sigma)} &= \int_\Delta \Big(\int_{\Gamma_\kappa(x)} |(\Theta a)(y)|^q \delta_E(y)^{q\upsilon-m}\, d\mu(y)\Big)^{p/q} d\sigma(x) \\
&\quad + \int_{E\setminus\Delta} \Big(\int_{\Gamma_\kappa(x)} |(\Theta a)(y)|^q \delta_E(y)^{q\upsilon-m}\, d\mu(y)\Big)^{p/q} d\sigma(x) =: I_1 + I_2.
\end{aligned} \tag{6.177}$$

Using Hölder's inequality, the fact that $(E, \rho|_E, \sigma)$ is $d$-ADR, $(\mathrm{SFE})_{q,q}$ and (6.142), we obtain

$$\begin{aligned}
I_1 &\le C\Big(\int_\Delta \int_{\Gamma_\kappa(x)} |(\Theta a)(y)|^q \delta_E(y)^{q\upsilon-m}\, d\mu(y) d\sigma(x)\Big)^{p/q} r^{d(1-p/q)} \\
&\le C\|\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta a)\big)\|^p_{L^q(E,\sigma)}\, r^{d(1-p/q)} \le C\|a\|^p_{L^q(E,\sigma)}\, r^{d(1-p/q)} \le C,
\end{aligned} \tag{6.178}$$

for some $C \in (0,\infty)$ independent of $a$.

It remains to estimate $I_2$. Let $x \in E \setminus \Delta$ and $y \in \Gamma_\kappa(x)$. If $z \in E \cap B_{\rho_\#}(x_0, \widetilde{C}_\rho r)$, then

$$\rho_\#(x_0,z) \le \widetilde{C}_\rho r \le \frac{1}{c}\rho_\#(x,x_0) \le \frac{C}{c}\max\{\rho_\#(x,y), \rho_\#(y,x_0)\} \le \frac{C}{c}(1+\kappa)\rho_\#(y,x_0), \tag{6.179}$$

so now choosing $c \in (1,\infty)$ sufficiently large, depending only on geometry, we have

$$\rho(z,x_0) \le \tfrac{1}{2}\rho(y,x_0), \quad \forall\, z \in E \cap B_{\rho_\#}(x_0, \widetilde{C}_\rho r). \tag{6.180}$$

Thus, using (6.147), (6.142) and that $\operatorname{supp} a \subseteq B_{\rho_\#}(x_0, \widetilde{C}_\rho r)$ and $(E, \rho|_E, \sigma)$ is $d$-ADR, we have

$$
\begin{aligned}
|(\Theta a)(y)| &\leq \int_{E\cap B_{\rho_\#}(x_0,\widetilde{C}_\rho r)} |\theta(y,z)-\theta(y,x_0)||a(z)|\,d\sigma(z)\\
&\leq C\int_{E\cap B_{\rho_\#}(x_0,\widetilde{C}_\rho r)} \frac{\rho_\#(z,x_0)^\alpha\delta_E(y)^{-a}}{\rho_\#(y,x_0)^{d+\upsilon+\alpha-a}}\,|a(z)|\,d\sigma(z)\\
&\leq C2^{\gamma-\alpha}\delta_E(y)^{-a}\int_{E\cap B_{\rho_\#}(x_0,\widetilde{C}_\rho r)} \frac{\rho_\#(z,x_0)^\gamma}{\rho_\#(y,x_0)^{d+\upsilon-a+\gamma}}\,|a(z)|\,d\sigma(z)\\
&\leq C\frac{\delta_E(y)^{-a}r^{\gamma+d(1-1/p)}}{\rho_\#(y,x_0)^{d+\upsilon-a+\gamma}}, \qquad \forall\, y\in\Gamma_\kappa(x),\ \forall\, x\in E\setminus\Delta. 
\end{aligned}
\tag{6.181}
$$

Furthermore, applying Lemma 6.14, we obtain

$$
\int_{\Gamma_\kappa(x)} |(\Theta a)(y)|^q \frac{d\mu(y)}{\delta_E(y)^{m-q\upsilon}} \leq C\frac{r^{q\gamma+qd(1-1/p)}}{\rho_\#(x,x_0)^{qd+q\gamma}}, \qquad \forall\, x\in E\setminus\Delta. \tag{6.182}
$$

Finally, applying (3.13) (with $f\equiv 1$) and noting that $p(d+\gamma)>d$, we obtain

$$
I_2 \leq Cr^{p\gamma+pd(1-1/p)}\int_{E\setminus\Delta}\frac{d\sigma(x)}{\rho_\#(x,x_0)^{pd+p\gamma}} \leq C\frac{r^{p\gamma+pd(1-1/p)}}{r^{pd+p\gamma-d}} = C. \tag{6.183}
$$

The uniform bound on $p$-atoms in (6.176) now follows from (6.177), (6.178) and (6.183).

*Part (d):* Let $q\in(1,\infty)$, $p\in(\frac{d}{d+\gamma},1]$ and assume that $(\mathrm{SFE})_{q,q}$ holds in order to prove that $(\mathrm{SFE})_{p,q}$ holds. We begin by defining the sets

$$
\dot{\mathscr{C}}^\gamma_{b,0}(E,\rho|_E) := \{f\in\dot{\mathscr{C}}^\gamma(E,\rho|_E) : f \text{ has bounded support and } \textstyle\int_E f\,d\sigma = 0\}. \tag{6.184}
$$

$$
\mathcal{F}(E) := \begin{cases} \dot{\mathscr{C}}^\gamma_{b,0}(E,\rho|_E) & \text{if } E \text{ is unbounded;}\\ \dot{\mathscr{C}}^\gamma_{b,0}(E,\rho|_E)\cup\{\mathbf{1}_E\} & \text{if } E \text{ is bounded.}\end{cases} \tag{6.185}
$$

$$
\mathcal{D}_0(E) := \text{the finite linear span of functions in } \mathcal{F}(E). \tag{6.186}
$$

We now prove that $\mathcal{D}_0(E)$ is dense in $H^p(E,\rho|_E,\sigma)$. First, recall the approximation to the identity $\{\mathcal{S}_l\}_{l\in\mathbb{Z},\,l\geq\kappa_E}$ of order $\gamma$ from Proposition 2.13. It follows from Definition 2.12 that $\mathcal{S}_l a\in\mathcal{D}_0(E)$ for every $p$-atom $a$ and each $l\in\mathbb{N}$. Also, it is proved in [49, Lemma 3.2(iii), p. 108], that $\{\mathcal{S}_l\}_{l\in\mathbb{N}}$ is uniformly bounded from $H^p(E,\rho|_E,\sigma)$ to $H^p(E,\rho|_E,\sigma)$ and that $\mathcal{S}_l f\to f$ in $H^p(E,\rho|_E,\sigma)$ as $l\to+\infty$ for all $f\in H^p(E,\rho|_E,\sigma)$. Together, these facts prove that individual $p$-atoms may be approximated in $H^p(E,\rho|_E,\sigma)$ with functions from $\mathcal{D}_0(E)$, and since finite linear spans of $p$-atoms are dense in $H^p(E,\rho|_E,\sigma)$ by Theorem 6.15, it follows that $\mathcal{D}_0(E)$ is dense in $H^p(E,\rho|_E,\sigma)$.

To prove that $(\mathrm{SFE})_{p,q}$ holds, it now suffices to find $C\in(0,\infty)$ such that

$$
\|\delta_E^{\upsilon-m/q}\Theta f\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)} \leq C\|f\|_{H^p(E,\rho|_E,\sigma)}, \quad \forall\, f\in\mathcal{D}_0(E), \tag{6.187}
$$

since then the density of $\mathcal{D}_0(E)$ in $H^p(E,\rho|_E,\sigma)$ and the fact that the mixed-norm spaces $L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)$ are quasi-Banach spaces (see [63, 10]), imply that the bounded linear operator $\delta_E^{\upsilon-m/q}\Theta : \mathcal{D}_0(E)\to L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)$ extends to a bounded linear operator from

$H^p(E,\rho|_E,\sigma)$ into $L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)$, as required. In particular, since the mixed-norm spaces are only quasi-normed, the fact that the linear extension is bounded relies on the following property of a general quasi-normed vector space $(X,\|\cdot\|)$ (see [61, Theorem 1.5(6)]): There exists $C\in[1,\infty)$ such that for any sequence $x_j\to x$ in $X$ as $j\to\infty$ in the topology induced by the quasi-norm, it holds that $C^{-1}\|x\|\le\liminf_{j\to\infty}\|x_j\|\le\limsup_{j\to\infty}\|x_j\|\le C\|x\|$.

To prove (6.187), let $f\in\dot{\mathscr{C}}^{\gamma}_{b,0}(E,\rho|_E)$, so by [49, Proposition 3.1, p. 112], there exist $(\lambda_j)_{j\in\mathbb{N}}\in\ell^p$ and a sequence of $p$-atoms $(a_j)_{j\in\mathbb{N}}$ such that $\sum_{j\in\mathbb{N}}\lambda_j a_j$ converges to $f$ in both $H^p(E,\rho|_E,\sigma)$ and $L^q(E,\sigma)$, and $\sum_{j\in\mathbb{N}}|\lambda_j|^p\le C\|f\|^p_{H^p(E,\rho|_E,\sigma)}$ for some $C\in(0,\infty)$ independent of $f$. The $(\mathrm{SFE})_{q,q}$ assumption implies that $\delta_E^{\upsilon-m/q}\Theta:L^q(E,\sigma)\to L^{(q,q)}(\mathscr{X},E,\mu,\sigma;\kappa)$ is a bounded linear operator, hence $\lim_{N\to\infty}\sum_{j=1}^N\lambda_j\delta_E^{\upsilon-m/q}\Theta a_j$ converges to $\delta_E^{\upsilon-m/q}\Theta f$ in $L^{(q,q)}(\mathscr{X},E,\mu,\sigma;\kappa)$. We may then apply [63, Theorem 1.5] to obtain a subsequence of these partial sums, corresponding to an increasing sequence $(N_k)_{k\in\mathbb{N}}\subseteq\mathbb{N}$, such that

$$\lim_{k\to\infty}F_k(x):=\lim_{k\to\infty}\sum_{j=1}^{N_k}\lambda_j\delta_E^{\upsilon-m/q}(\Theta a_j)(x)=\delta_E^{\upsilon-m/q}(\Theta f)(x),\quad\text{for }\mu\text{-a.e. }x\in\mathscr{X}\setminus E.\tag{6.188}$$

Since we are assuming that $0<p\le1<q<\infty$, it holds that $\|\cdot\|^p_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)}$ is subadditive, so for each $k\in\mathbb{N}$, by the uniform estimate on $p$-atoms from (6.176) in *Part (c)*, we have

$$\|F_k\|^p_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)}=\left\|\sum_{j=1}^{N_k}\lambda_j\delta_E^{\upsilon-m/q}\Theta a_j\right\|^p_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)}\le C\sum_{j=1}^{N_k}|\lambda_j|^p\le C\|f\|^p_{H^p(E,\rho|_E,\sigma)}.\tag{6.189}$$

We now combine the general fact that $\|u\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)}=\|\,|u|\,\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)}$ with Fatou's Lemma in $L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)$, (6.188) and (6.189) to obtain

$$\begin{aligned}\|\delta_E^{\upsilon-m/q}\Theta f\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)}&=\|\liminf_{k\to\infty}|F_k|\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)}\\&\le\liminf_{k\to\infty}\|F_k\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)}\le C\|f\|_{H^p(E,\rho|_E,\sigma)}.\end{aligned}\tag{6.190}$$

We established (6.190) for any $f\in\dot{\mathscr{C}}^{\gamma}_{b,0}(E,\rho|_E)$, which proves (6.187) for all $f\in\mathcal{D}_0(E)$ except in the case when $E$ is bounded and $f=\mathbf{1}_E$. In that case, since $E$ is $d$-ADR, it holds that $\sigma(E)<\infty$, so by Hölder's inequality and the $(\mathrm{SFE})_{q,q}$ assumption, we have

$$\|\delta_E^{\upsilon-m/q}\Theta\mathbf{1}_E\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)}\le\sigma(E)^{\frac1p-\frac1q}\|\mathscr{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}\Theta\mathbf{1}_E\big)\|_{L^q(E,\sigma)}\le C\sigma(E)^{\frac1p}=C.\tag{6.191}$$

Now (6.187) follows by (6.190) and (6.191), which completes the proof that $(\mathrm{SFE})_{p,q}$ holds.

*Part (e):* Let $q\in(1,\infty)$, $p\in(\frac{d}{d+\gamma},\infty)$ and assume that $(\mathrm{wSFE})_{p_o,q}$ holds for some $p_o\in(q,\infty)$ in order to prove that $(\mathrm{SFE})_{p,q}$ holds. In this case, property *(1)* implies that $(\mathrm{SFE})_{r,q}$ holds for each $r\in(1,p_o)$, hence $(\mathrm{SFE})_{q,q}$ holds and $(\mathrm{SFE})_{p,q}$ follows by *Parts (a)*, *(b)* and *(d)*.

*Proof of (3).* Now suppose that $\sigma$ is a Borel semiregular measure on $(E,\tau_{\rho|_E})$. If $p_o\in(0,\infty)$ and $(\mathrm{wSFE})_{p_o,2}$ holds, then Theorem 6.9 implies that $(\mathrm{SFE})_{2,2}$ holds, so then property *(2)* implies that $(\mathrm{SFE})_{p,2}$ holds for each $p\in(\frac{d}{d+\gamma},\infty)$. □

# 7 Conclusion

Theorem 1.1 asserts the equivalence of a number of the properties encountered in the manuscript.

*Proof of Theorem 1.1.* The fact that *(1)* $\Rightarrow$ *(2)* is a consequence of Theorem 3.1. It is easy to see that if *(2)* holds, then *(7)* holds by taking $b_Q := \mathbf{1}_Q$ for each $Q \in \mathbb{D}(E)$, hence *(2)* $\Rightarrow$ *(7)*. The implication *(7)* $\Rightarrow$ *(1)* is proved in Theorem 3.6. The implication *(9)* $\Rightarrow$ *(1)* is proved in Theorem 4.3. Moreover, *(1)* $\Leftrightarrow$ *(9)* $\Leftrightarrow$ *(10)* by Theorem 4.4. The implication *(11)* $\Rightarrow$ *(12)* is proved in Theorem 6.17. Clearly *(12)* $\Rightarrow$ *(11)*, while *(11)* $\Rightarrow$ *(1)* is contained in Theorem 6.9. To show that *(1)* $\Rightarrow$ *(11)*, suppose that *(1)* holds, let $f \in L^2(E,\sigma)$ and $\lambda > 0$, and estimate

$$\begin{aligned}
\lambda^2 \cdot \sigma\Big(\Big\{x \in E : \int_{\Gamma_\kappa(x)} |(\Theta f)(y)|^2 \,\frac{d\mu(y)}{\delta_E(y)^{m-2\upsilon}} > \lambda^2\Big\}\Big) &\leq \int_E \int_{\Gamma_\kappa(x)} |(\Theta f)(y)|^2 \,\frac{d\mu(y)}{\delta_E(y)^{m-2\upsilon}} d\sigma(x) \\
&\leq \int_{\mathscr{X}\setminus E} \frac{|(\Theta f)(y)|^2}{\delta_E(y)^{m-2\upsilon}}\, \sigma(\pi_y^\kappa)\, d\mu(y) \\
&\leq C \int_{\mathscr{X}\setminus E} |(\Theta f)(y)|^2 \delta_E(y)^{2\upsilon-(m-d)}\, d\mu(y) \\
&\leq C\|f\|^2_{L^2(E,\sigma)},
\end{aligned} \tag{7.1}$$

where the first inequality uses Tschebyshev's inequality, the second uses (6.35), the third uses (6.19) in Lemma 6.2 and the fact that $(E, \rho\big|_E, \sigma)$ is $d$-ADR, and the last uses (1.25). Thus, *(1)* $\Rightarrow$ *(11)* as desired. Since (1.35) is a rewriting of (1.34), it is immediate that *(12)* $\Leftrightarrow$ *(13)*. In summary, so far we have shown that *(1)*, *(2)*, *(7)* and *(9)*-*(13)* are equivalent.

The implication *(6)* $\Rightarrow$ *(4)* is trivial and, based on (2.108), we have that *(4)* $\Rightarrow$ *(2)*. We focus next on *(1)* $\Rightarrow$ *(6)*. Suppose *(1)* holds and fix $f \in L^\infty(E,\sigma)$, $x \in E$, and $r \in (0,\infty)$ arbitrary. Then, using the notation $B_{cr} := B_{\rho_\#}(x, cr)$ for $c > 0$, we may write

$$\int_{B_r\setminus E} |\Theta f|^2 \delta_E^{2\upsilon-(m-d)}\, d\mu \leq \int_{B_r\setminus E} \big(|\Theta(f\mathbf{1}_{E\cap B_{2rC_\rho}})|^2 + |\Theta(f\mathbf{1}_{E\setminus B_{2rC_\rho}})|^2\big)\delta_E^{2\upsilon-(m-d)}\, d\mu =: I + II. \tag{7.2}$$

To estimate $I$ we apply (1.25) and the property of $E$ being $d$-dimensional ADR to obtain

$$I \leq C \int_{E\cap B_{2rC_\rho}} |f|^2\, d\sigma \leq C\|f\|^2_{L^\infty(E,\sigma)}\sigma(E \cap B_r). \tag{7.3}$$

As regards $II$, we first note that if $z \in B_r \setminus E$ and $y \in E \setminus B_{2rC_\rho}$ are arbitrary points then $\rho_\#(x,y) \leq C_\rho(\rho_\#(x,z) + \rho_\#(z,y)) < C_\rho r + C_\rho \rho_\#(z,y) \leq \frac{1}{2}\rho_\#(x,y) + C_\rho\rho_\#(z,y)$ which implies $\rho_\#(z,y) \geq \rho_\#(x,y)/(2C_\rho)$. This fact, in combination with (1.22) and (3.13), yields

$$|\Theta(f\mathbf{1}_{E\setminus B_{2rC_\rho}})(z)| \leq C\|f\|_{L^\infty(E,\sigma)} \int_{E\setminus B_{2rC_\rho}} \frac{1}{\rho_\#(x,y)^{d+\upsilon}}\, d\sigma(y) \leq C\|f\|_{L^\infty(E,\sigma)}\, r^{-\upsilon}, \tag{7.4}$$

for all $z \in B_r \setminus E$, so applying (3.15) (with $R := r$ and $\gamma := m - d - 2\upsilon$) we obtain

$$II \leq C\|f\|^2_{L^\infty(E,\sigma)} r^{-2\upsilon} r^{d+2\upsilon} \leq C\|f\|^2_{L^\infty(E,\sigma)}\sigma(E \cap B_r). \tag{7.5}$$

At this point, (1.30) follows from (7.2), (7.3), and (7.5), completing the proof of *(1)* $\Rightarrow$ *(6)*. Based on (2.108) we have that *(6)* $\Rightarrow$ *(3)* while *(3)* $\Rightarrow$ *(2)* is trivial.

Next, we show that *(8)* ⇒ *(7)*. Suppose that *(8)* holds and set $\varepsilon_o := \min\{\varepsilon, a_0\}$, where $\varepsilon$ is as in Lemma 2.18 and $a_0$ is as in (2.30). Let $Q \in \mathbb{D}(E)$ and set $\Delta_Q := B_{\rho_\#}(x_Q, \varepsilon_o \ell(Q)/2C\rho) \cap E$. Then (2.30), (2.109) and the fact that $E$ is $d$-dimensional ADR imply

$$\Delta_Q \subseteq Q, \quad B_{\rho_\#}(x_Q, \varepsilon_o \ell(Q)) \setminus E \subseteq T_E(Q), \quad \sigma(\Delta_Q) \approx \sigma(Q) = C\ell(Q)^d. \tag{7.6}$$

Hence, if we now define $b_Q := b_{\Delta_Q}$, where $b_{\Delta_Q}$ is the function associated to $\Delta_Q$ as in *(8)*, then $b_{\Delta_Q}$ satisfies (1.32) which, when combined with the support condition of $b_{\Delta_Q}$ and the last condition in (7.6), implies that $b_Q$ satisfies the first two conditions in (1.31) (with $\widetilde{Q} = Q$). In order to show that $b_Q$ also verifies the last condition in (1.31), we write

$$\begin{aligned}\int_{T_E(Q)} |(\Theta\, b_Q)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) &= \int_{T_E(Q)\setminus B_{\rho_\#}(x_Q, \varepsilon_o \ell(Q))} |(\Theta\, b_Q)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &+ \int_{B_{\rho_\#}(x_Q, \varepsilon_o \ell(Q))} |(\Theta\, b_Q)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) =: I_1 + I_2. \end{aligned} \tag{7.7}$$

To estimate $I_1$, note that if $x \in T_E(Q) \setminus B_{\rho_\#}\big(x_Q, \varepsilon_o \ell(Q)\big)$ and $y \in \Delta_Q$, then $\rho_\#(x, y) \geq \frac{\varepsilon_o}{2C_\rho} \ell(Q)$, so (1.22), (1.31) and (7.6) imply that $|(\Theta\, b_Q)(x)| \leq C\ell(Q)^{-\upsilon}$, and with (2.108) we have

$$\begin{aligned} I_1 &\leq C\ell(Q)^{-2\upsilon} \int_{T_E(Q)\setminus B_{\rho_\#}(x_Q, \varepsilon_o \ell(Q))} \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\leq C\ell(Q)^{-2\upsilon} \int_{B_{\rho_\#}(x_Q, C\ell(Q))\setminus E} \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\leq C\ell(Q)^{-2\upsilon} \ell(Q)^{d+2\upsilon} \leq C\sigma(Q), \end{aligned} \tag{7.8}$$

where the third inequality uses (3.15) (with $R = r = \ell(Q)$ and $\gamma := m - d - 2\upsilon$). To estimate $I_2$, first note that by (1.32) and (7.6), it is immediate that $I_2 \leq C_0 \sigma(\Delta_Q) \leq C\sigma(Q)$. This fact, (7.7) and (7.8) show that $b_Q$ also satisfies the last condition in (1.31), since the constants in our estimates are independent of $Q$. This completes the proof of *(8)* ⇒ *(7)*.

To see that *(1)* ⇒ *(8)*, set $b_\Delta := \mathbf{1}_\Delta$ for each surface ball $\Delta$. The first two estimates in (1.32) are immediate while the third is a consequence of (1.25) with $f := b_\Delta$.

It is trivial that *(2)* ⇒ *(5)*. Now assume that *(5)* holds. If we set $b_Q := b\mathbf{1}_Q$ for each $Q \in \mathbb{D}(E)$, then (1.31) holds for the family $\{b_Q\}_{Q\in\mathbb{D}(E)}$, by (1.29) and the fact that $b$ is para-accretive. This shows that *(5)* ⇒*(7)*, and so the proof of Theorem 1.1 is complete. □

We conclude the manuscript with the proof of Theorem 1.2.

*Proof of Theorem 1.2.* The idea is to apply Theorem 6.17 in the setting $\mathscr{X} := E \times [0, \infty)$ and $E \equiv E \times \{0\}$ (i.e., we identify $(y, 0) \equiv y$ for every $y \in E$). Moreover, we set

$$\rho((x,t),(y,s)) := \max\{|x-y|, |t-s|\} \text{ for every } (x,t),(y,s) \in E \times [0,\infty), \tag{7.9}$$

and set $\mu := \sigma \otimes \mathcal{L}^1$, where $\mathcal{L}^1$ is the one-dimensional Lebesgue measure on $[0, \infty)$, and consider the integral operator $\Theta$ in (3.4) with integral kernel $\theta : (\mathscr{X} \setminus E) \times E \to \mathbb{R}$ defined by

$$\theta((x,t), y) := 2^{-k} \psi_k(x - y) \;\text{ if } x, y \in E,\ t > 0 \text{ and } k \in \mathbb{Z},\ 2^k \leq t < 2^{k+1}. \tag{7.10}$$

It is not difficult to verify that $(\mathscr{X}, \rho, \mu)$ is a $(d+1)$-ADR space, that $\alpha_\rho = 1$, that $\theta$ satisfies (3.1)-(3.3) for $a := 0$, $\alpha := 1$, $\upsilon := 1$, and that $\delta_E(x,t) = t$ for every $x \in E$ and $t \in [0,\infty)$. In particular, $\gamma$ defined in Theorem 6.17 now equals 1. Fix some $\kappa > 0$ and observe that

$$\Gamma_\kappa(x) = \big\{(y,t) \in E \times (0,\infty) : |x-y| < (1+\kappa)t\big\}, \quad \forall\, x \in E. \tag{7.11}$$

If $f \in L^2(E,\sigma)$, then by Fubini's Theorem, the fact that $E$ is $d$-ADR, and (7.10), we have

$$\begin{aligned}
\int_E \int_{\Gamma_\kappa(x)} |(\Theta f)(y,t)|^2 \frac{d\mu(y,t)}{\delta_E(y,t)^{d-1}} d\sigma(x) &= \int_E \int_{\Gamma_\kappa(x)} |(\Theta f)(y,t)|^2\, t^{1-d} d\mu(y,t) d\sigma(x) \\
&= \int_{E\times(0,\infty)} |(\Theta f)(y,t)|^2\, t^{1-d} \sigma(E \cap B(y,(1+\kappa)t))\, d\mu(y,t) \\
&\approx C \int_0^\infty \int_E |(\Theta f)(y,t)|^2\, t\, d\sigma(y)\, dt \\
&= C \sum\nolimits_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \int_E \left| \int_E 2^{-k} \psi_k(y-z) f(z)\, d\sigma(z)\right|^2 d\sigma(y)\, t\, dt \\
&= C \sum\nolimits_{k\in\mathbb{Z}} \int_E \left| \int_E \psi_k(y-z) f(z)\, d\sigma(z)\right|^2 d\sigma(y).
\end{aligned} \tag{7.12}$$

It was proved in [26, Theorem, p. 10] that there exists $C \in (0,\infty)$ such that

$$\sum\nolimits_{k\in\mathbb{Z}} \int_E \left| \int_E \psi_k(x-y) f(y)\, d\sigma(y)\right|^2 d\sigma(x) \leq C \int_E |f|^2\, d\sigma, \quad \forall\, f \in L^2(E,\sigma). \tag{7.13}$$

Therefore, the $(\mathrm{SFE})_{2,2}$ estimate from (6.148) holds, so Theorem 6.17 implies that $(\mathrm{SFE})_{p,2}$ holds for each $p \in \big(\frac{d}{d+1}, \infty\big)$, which by reasoning as in (7.12), is equivalent to the estimate

$$\int_E \left( \sum\nolimits_{k\in\mathbb{Z}} ⨍_{\Delta(x,(1+\kappa)2^k)} \left| \int_E \psi_k(z-y) f(z)\, d\sigma(z)\right|^2 d\sigma(y)\right)^{p/2} d\sigma(x) \leq C' \|f\|^p_{H^p(E,\sigma)}, \tag{7.14}$$

for all $f \in H^p(E,\sigma)$, which implies (1.37). In fact, estimate (7.14) is equivalent to (1.37). To prove the nontrivial implication, choose $N \in \mathbb{N}$ sufficiently large and consider (7.14) with $\widetilde{\psi}(x) := \psi(x/2^N)$ in place of $\psi$, and note that if $\psi \in C_0^\infty(\mathbb{R}^{n+1})$ is odd, then $\widetilde{\psi}(x)$ is also odd, smooth and compactly supported, and satisfies $\widetilde{\psi}_k = 2^{dN} \psi_{k+N}$ for every $k \in \mathbb{Z}$. □

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