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QUADRATIC ESTIMATES FOR DEGENERATE ELLIPTIC SYSTEMS ON MANIFOLDS WITH LOWER RICCI CURVATURE BOUNDS AND BOUNDARY VALUE PROBLEMS

PASCAL AUSCHER, ANDREW J. MORRIS, AND ANDREAS ROSÉN

ABSTRACT. Weighted quadratic estimates are proved for certain bisectorial first-order differential operators with bounded measurable coefficients which are (not necessarily pointwise) accretive, on complete manifolds with positive injectivity radius. As compared to earlier results, Ricci curvature is only assumed to be bounded from below, and the weight is only assumed to be locally in A_2. The Kato square root estimate is proved under this weaker assumption. On compact Lipschitz manifolds we prove solvability estimates for solutions to degenerate elliptic systems with not necessarily self-adjoint coefficients, and with Dirichlet, Neumann and Atiyah–Patodi–Singer boundary conditions.

CONTENTS

1. INTRODUCTION

This paper concerns the weighted L^2 theory for generalized singular integrals appearing in the H^∞ functional calculus of differential operators with non-smooth

coefficients. The seminal work in this area is the Kato square root estimate by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian [10], which yields estimates

$$(1.1) \qquad \|\nabla(-\mathrm{div}A\nabla)^{-1/2}u\|_2 \approx \|u\|_2$$

of Riesz transforms associated with divergence form operators $\mathrm{div}A\nabla$ on $\mathbb{R}^n$. The coefficient matrix A is assumed to be merely bounded, measurable and accretive, complex matrix-valued. Although the square root in (1.1) is in the holomorphic functional calculus of the sectorial operator $-\mathrm{div}A\nabla$, the left factor ∇ is not. The Kato estimate was subsequently put into its natural functional analytic framework, and boundedness of the H^∞ functional calculus for certain bisectorial Dirac type operators was shown. First, generalizations $\Gamma + B^{-1}\Gamma^*B$ of the Hodge–Dirac operator were considered in [15], and later perturbations DB of self-adjoint differential operators D by accretive multipliers B were considered in [8, 7] and in later works on boundary value problems discussed below. The DB operators have a simpler structure than Hodge–Dirac operators, but still the bounds for $\Gamma + B^{-1}\Gamma^*B$ operators follow from the bounds for DB operators, using Hodge splittings as shown in [7, Sec. 10.1]. Also the Kato square root estimate (1.1) is a well-known corollary, see [7, Sec. 2.1]. The main ideas in all the proofs are the following. (1) The H^∞ functional calculus bounds are reduced to proving quadratic estimates, which provide a Littlewood–Paley decomposition of L^2 adapted to the operator. (2) A local Tb theorem allows a further reduction of the problem to showing Carleson bounds for a certain multiplier. (3) This Carleson estimate is established using a stopping time argument.

The L^2 estimates for the above operators are quite sharp in the sense that L^p estimates in general only hold in a small interval around $p = 2$ for the above operators, and not for all $1 < p < \infty$ as is the case for classical Calderón–Zygmund operators. See [4, 14, 20]. For L^2 spaces, the bounds for the H^∞ functional calculus of the above Dirac type operators have been extended (1) to certain complete manifolds M in [28, 16], with earlier results for compact manifolds in [15], and (2) to weighted $L^2(\mathbb{R}^n, w)$ spaces in [13]. The main issue on manifolds, which are assumed to be complete but are allowed to be noncompact, is the geometry of the manifold towards infinity. The main issue in weighted spaces is local and the weight is assumed to be in the Muckenhoupt A_2 class. The bounds of the H^∞ functional calculus here require the double stopping time argument from [13, Sec. 3.5-7].

A first main result in this paper is the following theorem, which proves quadratic estimates for operators DB under weaker geometric assumptions than those in [15, 28, 16], and which combines those results with the weighted theory developed in [13]. See Section 2 for precise definitions. We adopt the convention for estimating $x, y \geq 0$ whereby $x \lesssim y$ means that there exists a constant $C < \infty$, which only depends on constants specified in the relevant preceding hypotheses, such that $x \leq Cy$. We write $x \approx y$ when $x \lesssim y \lesssim x$.

Theorem 1.1. *Let (M, g) denote a complete Riemannian manifold with injectivity radius $\mathrm{inj}(M, g) \geq r_E$ and Ricci curvature $\mathrm{Ric}(M, g) \geq -Kg$ in the sense of bilinear forms, where $r_E > 0$ and $K < \infty$. Let $\omega \in A_2^R(M)$ be a local Muckenhoupt weight as in Section 2.2, for some $R > 0$. Consider perturbations DB as in Section 2.2, of the self-adjoint operator D in $L^2(\mathcal{V}, \omega)$ as given by (??)-(2.4), by bounded coefficients B as in (2.5), which are accretive in the sense of (2.6). Then we have quadratic*

estimates

$$(1.2) \qquad \int_0^\infty \|tDB(I + t^2(DB)^2)^{-1}u\|_{L^2(\mathcal{V},\omega)}^2 \frac{dt}{t} \lesssim \|u\|_{L^2(\mathcal{V},\omega)}^2, \qquad u \in L^2(\mathcal{V},\omega),$$

where the implicit constant depends only on $n = \dim(M)$, r_E, K, $\|B\|_\infty$, κ_B, R *and* $[\omega]_{A_2^R(M)}$.

There are several novelties in Theorem 1.1, and its proof, as compared to earlier results. The best previous result [16, Thm. 1.1], for complete manifolds in the un-weighted setting $\omega = 1$, required an additional upper bound on the Ricci curvature. We avoid this by using a result of Anderson and Cheeger [3] to observe that the generalised bounded geometry property introduced by Bandara and McIntosh [16, Defn. 2.5], see (2.2) below and the related discussion, holds on the tangent bundle TM whenever M has a positive injectivity radius and only a lower bound on its Ricci curvature.

Our proof only uses the injectivity radius and Ricci curvature hypotheses to obtain coordinates which satisfy (2.2), so Theorem 1.1 actually holds more generally under the assumption that TM has such generalised bounded geometry. It is this property which allows us to localize the quadratic estimate to coordinate charts. For general local A_2 weights ω, we pull back the estimates to $\mathbb{R}^n$ in order to apply the Euclidean weighted quadratic estimate from [13]. In this regard, our approach is more aligned with the earlier treatment for compact manifolds in [15, Sec. 7] than with those for complete manfiolds in [16, 28], where estimates are established directly on the manifold.

In order to apply the weighted quadratic estimates from [13], we need to extend the weight ω, as well as the coefficients B, from within a euclidean ball to all of $\mathbb{R}^n$, whilst preserving the A_2 property for ω and the accretivity for B. This technical issue is resolved in Section 3.3. For our estimates to apply to general elliptic systems, we do not assume pointwise accretivity of the coefficients, but only a Gårding inequality, which is what makes the extension problem for the coefficients non-trivial.

Theorem 1.1 yields in particular the following extension of the Kato square root estimate, which we prove at the end of Section 3.

Theorem 1.2. *Let* (M,g) *denote a complete Riemannian manifold with* $\mathrm{inj}(M,g) \geq r_E$ *and* $\mathrm{Ric}(M,g) \geq -Kg$ *in the sense of bilinear forms, for some* $r_E > 0$ *and* $K < \infty$. *Let* $\omega \in A_2^R(M)$, *for some* $R > 0$. *Let*

$$J(u,v) = \int_M \Big(\langle A\nabla_M u, \nabla_M v\rangle + u\langle b, \nabla_M v\rangle + \langle c, \nabla_M \overline{u}\rangle\overline{v} + du\overline{v}\Big)\omega(x)d\mu(x)$$

be a sesqui-linear form on $L^2(M,\omega)$, *with domain* $W^{1,2}(M,\omega)$ *and with* $A \in L^\infty(\mathrm{End}(TM))$, $b, c \in L^\infty(TM)$ *and* $d \in L^\infty(M)$, *which satisfies a Gårding inequality*

$$\mathrm{Re}\, J(u,u) \gtrsim \|\nabla_M u\|_{L^2(M,\omega)}^2 + \|u\|_{L^2(M,\omega)}^2,$$

for all $u \in W^{1,2}(M,\omega)$. *Consider the associated operator*

$$L = -\mathrm{div}_{M,\omega}A\nabla_M - \mathrm{div}_{M,\omega}b + c\nabla_M + d$$

and a function $a \in L^\infty(M)$ *with* $\inf_M \mathrm{Re}\, a > 0$. *Then the weighted Kato square root estimate*

$$\|\sqrt{aL}u\|_{L^2(M,\omega)} \eqsim \|\nabla_M u\|_{L^2(M,\omega)} + \|u\|_{L^2(M,\omega)}, \qquad u \in W^{1,2}(M,\omega),$$

holds on M.

The novelty in Theorem 1.2 is that only a lower bound on the Ricci curvature of the manifold is assumed and the weight is only assumed to be a local A_2 weight. The proof involves applying Theorem 1.1 to suitable coefficients B depending on A, a, b, c, d.

An early application of the boundedness of the H^∞ functional calculus of DB operators was to elliptic non-smooth boundary value problems (BVPs) [6, 8, 5, 12]. As in the theory of smooth Dirac BVPs, see for example [17], the key idea is to write the derivative transversal to the boundary in the partial differential equation (PDE) in terms of an operator acting along the boundary. For non-smooth divergence form elliptic equations $\mathrm{div}A\nabla u = 0$, it was found in [8] that the structure of this boundary operator is precisely DB. This uses a non-linear transform $A \mapsto B$ of accretive coefficients, featuring a Schur complement of the transversal block in A, see (4.7). Theorem 1.1 applies to prove boundedness of projections onto Hardy type subspaces in particular, which is crucial for BVPs.

A second main result in this paper is the following theorem, which uses Theorem 1.1 to estimate the Neumann and Dirichlet boundary data, as a whole, for solutions to divergence form elliptic equations.

Theorem 1.3. *Let Ω be a compact manifold with Lipschitz boundary $\partial\Omega$, in the sense that we have a bilipschitz parametrization $\rho(t, x)$ of a neighbourhood, relative Ω, of $\partial\Omega$ by a cylinder $[0, \delta) \times M$ as in (4.2), with M being a closed Riemannian manifold. Let $\omega \in A_2(\Omega)$ be a Muckenhoupt weight such that $\omega_\rho = \omega \circ \rho$ is independent of t, and let $\omega_0 = \omega_\rho|_M$. Consider weak solutions u to a divergence form elliptic system $\mathrm{div}A\nabla u = 0$ in Ω, with degenerate elliptic coefficients A satisfying $A/\omega \in L^\infty(\mathrm{End}(T\Omega))$ and the accretivity conditions (4.8), (4.9). Assume that a boundary trace $A_0 = \lim_{t \to 0} A_\rho$ exists for the pulled back coefficients A_ρ on the cylinder, in the sense that we have a finite Carleson discrepancy $\|A_\rho/\omega_\rho - A_0/\omega_0\|_* < \infty$. See (4.10). Consider the conormal derivative $\partial_{\nu_{A_\rho}} u_\rho$ and the tangential gradient $\nabla_M u_\rho$ of $u_\rho = u \circ \rho$ on the cylinder. See (4.5).*

(i) If $\|\nabla u\|_{\mathcal{X}} < \infty$, with the modified non-tangential maximal function norm $\|\cdot\|_{\mathcal{X}}$ given by (4.13), then

$$(1.3) \qquad \|\partial_{\nu_{A_\rho}} u_\rho|_M\|_{L^2(M, \omega_0^{-1})} + \|\nabla_M u_\rho|_M\|_{L^2(TM, \omega_0)} \lesssim \|\nabla u\|_{\mathcal{X}},$$

where the traces exist in the L^2 average sense (4.14).

(ii) If $\|\nabla u\|_{\mathcal{Y}} < \infty$, with the square function norm $\|\cdot\|_{\mathcal{Y}}$ given by (4.17), then

$$(1.4) \qquad \|\partial_{\nu_{A_\rho}} u_\rho|_M\|_{W^{-1,2}(M, \omega_0^{-1})} + \|\nabla_M u_\rho|_M\|_{W^{-1,2}(TM, \omega_0)} \lesssim \|\nabla u\|_{\mathcal{Y}},$$

where the traces exist in the $W^{-1,2}$ sense (4.18).

Moreover, if $\|A_\rho/\omega_\rho - A_0/\omega_0\|_*$ is small enough, with smallness depending on $[\omega_0]_{A_2(M)}$, $\|A_0/\omega_0\|_\infty$ and the constant in (4.9), then the estimates $\gtrsim$ also hold in (1.3) and (1.4), with the implicit constant also depending on the constant in (4.8).

We remark that the assumed smoothness of the manifold M is not used quantitatively, and the manifold Ω may be merely a Lipschitz manifold. Note also that the small Carleson condition on $A_\rho/\omega_\rho - A_0/\omega_0$ required for the reverse estimates $\gtrsim$, in general is only satisfied for pull backs of smooth coeficients A if the Lipschitz character of $\partial\Omega$ is small, unless this boundary is locally the graph of a Lipschitz function. The above Neumann and Dirichlet data for u_ρ correspond to those for u

on $\partial\Omega$ as in (4.6). We nevertheless use the above formulation since pullback by ρ does not preserve the Sobolev space $W^{-1,2}$.

Theorem 1.3 is essentially a solvability result for the Atiyah–Patodi–Singer (APS) boundary value problem associated with the PDE $\operatorname{div} A\nabla u = 0$. The connection to the classical APS conditions is that we rewrite the second-order equation as a first-order system (4.4) for the (conormal) gradient of u. In the smooth setting, this first-order system would coincide with the Hodge–Dirac operator acting on gradient vector fields, and our APS conditions arise in the well-known way by considering spectral projections of an adapted operator on the boundary. See (4.12) for details. Under suitable identifications our spectral projection E_0^+ would correspond to the Calderón projector for the original second-order equation.

We prove such APS solvability results in Theorems 4.5 and 4.6 in Section 4.2, from which Theorem 1.3 is an immediate corollary. Theorem 1.3 extends results from [12], in that we both allow for domains Ω with arbitrary topology and for A_2 degenerate equations. Moreover, the Fredholm argument presented in Section 4 is new; the proof from [12, Sec. 16] does not generalize to domains with arbitrary topology. We limit ourselves to compact manifolds M in Theorem 1.3, for which the curvature and injectivity radius hypotheses in Theorem 1.1 hold trivially. This restriction is necessary for the Fredholm arguments involved in the proof, which localize the Carleson regularity condition to the δ-neighbourhood of $\partial\Omega$. We remark that the dependence of the implicit constant in the estimates of $\|\nabla u\|_{\mathcal{X}}$ and $\|\nabla u\|_{\mathcal{Y}}$ in Theorems 1.3, 1.4, 4.5 and 4.6 is hard to trace, as its finiteness is proved by an application of the Open Mapping Theorem.

It is important to note that the APS solvability results discussed above require only the Carleson regularity condition on the coefficients near $\partial\Omega$. The reason is that the APS boundary condition is defined in terms of operators in the H^∞ functional calculus of the boundary operator DB. However, solution operators for the classical Neumann and Dirichlet BVPs, such as the Dirichlet-to-Neumann map, are outside of this functional calculus, just like the Riesz transform in (1.1) is outside the H^∞ functional calculus of $\operatorname{div} A\nabla$. As a consequence, L^2 solvability of the Neumann and Dirichlet BVPs requires further assumptions on the coefficients, notably self-adjointness. See [8, Sec. 4]. We prove in Section 4.3 the following.

Theorem 1.4. *Let Ω, ρ, M, ω and A be as in Theorem 1.3, and assume that $\|A_\rho/\omega_\rho - A_0/\omega_0\|_*$ and $\|(A_0^* - A_0)/\omega_0\|_\infty$ are small enough, depending on $[\omega_0]_{A_2(M)}$, $\|A_0/\omega_0\|_\infty$ and the constant in (4.9). Then Neumann and Dirichlet regularity solvability estimates*

$$\|\nabla u\|_{\mathcal{X}} \lesssim \min\left(\|\partial_{\nu_{A_0}} u_\rho|_M\|_{L^2(M,\omega_0^{-1})}, \|\nabla_M u_\rho|_M\|_{L^2(TM,\omega_0)} + \sum_i \left| \int_{M_i} \partial_{\nu_{A_\rho}} u_\rho \, d\mu \right| \right)$$

hold for all weak solutions to $\operatorname{div} A\nabla u = 0$ in Ω with $\|\nabla u\|_{\mathcal{X}} < \infty$, and Dirichlet solvability estimates

$$\|\nabla u\|_{\mathcal{Y}} \lesssim \|\nabla_M u_\rho|_M\|_{W^{-1,2}(TM,\omega_0)} + \sum_i \left| \int_{M_i} \partial_{\nu_{A_\rho}} u_\rho \, d\mu \right|$$

hold for all weak solutions to $\operatorname{div} A\nabla u = 0$ in Ω with $\|\nabla u\|_{\mathcal{Y}} < \infty$. Here M_i denote the connected components of M, and the implicit constants in the estimates of $\|\nabla u\|_{\mathcal{X}}$ and $\|\nabla u\|_{\mathcal{Y}}$ also depend on the constant in (4.8).

Our solvability estimates for the Neumann and Dirichlet BVPs in Theorem 1.4 and for the Atiyah–Patodi–Singer BVP in Theorems 4.5 and 4.6 are new in that we go beyond self-adjoint coefficients, as well as allow for degenerate coefficients. Our method relies on the boundedness of the functional calculus of the boundary DB operator, the construction of which depends on the Lipschitz parametrization ρ of the boundary $\partial\Omega$. In the weighted L^2 and $W^{-1,2}$ spaces this yields not only solvability estimates, but also analytical dependence of the solution operators on the coefficients. However, at this level of generality of the coefficents there is little room for generalizations to other function spaces and boundaries rougher than Lipschitz, as considered for example in [27].

2. Setup for quadratic estimates

This section aims at making the statement of Theorem 1.1 precise. We begin with geometric preliminaries in order to define useful coverings for manifolds with a lower Ricci curvature bound and positive injectivity radius. The relevant weights and differential operators are then made precise in this context. A brief overview of the H^∞ functional calculus for bisectorial operators is also included.

2.1. **Coverings.** Let (M, g) denote a complete, not necessarily compact, Riemannian manifold M with Riemannian metric g and dimension $n = \dim(M)$, which is smooth in the sense that both the transition maps between coordinate charts on M, and the coefficients of the metric g in coordinate charts, are infinitely differentiable. We denote the geodesic distance by d and the Riemannian measure by μ. A ball $B(x, r)$ in M always refers to an open geodesic ball $B(x, r) = \{y \in M : d(x, y) < r\}$ whilst $V(x, r) = \mu(B(x, r))$ and $(\alpha B)(x, r) = B(x, \alpha r)$ for $\alpha > 0$, $r > 0$ and $x \in M$. For $E, F \subseteq M$, we set $d(E, F) := \inf\{d(E, F) : x \in E, y \in F\}$ and let 1_E denote the characteristic function of E.

We assume that the Ricci curvature on the manifold is bounded below in the sense that $\mathrm{Ric}(M, g) \geq -Kg$, as bilinear forms, for some $K < \infty$. The results of Saloff-Coste [30, Thm. 5.6.4], see also Hebey [24, Thm. 1.1] and [24, Lem. 1.1], then show that M has exponential volume growth in the sense that

$$(2.1) \qquad 0 < V(x, \alpha r) \leq \alpha^n e^{\lambda \alpha r} V(x, r) < \infty, \qquad \alpha \geq 1,\ r > 0,\ x \in M,$$

where $\lambda = \sqrt{(n-1)K}$. It is also shown there that for any $0 < T < \infty$, there exists a sequence $(x_i)_{i \in \mathbb{N}}$ of points in M such that the balls $(B(x_i, T/2))_{i \in \mathbb{N}}$ are pairwise disjoint. Moreover, for each $\alpha \geq 1$, the collection $(B(x_i, \alpha T))_{i \in \mathbb{N}}$ is a locally finite covering of M with the property that the number of balls in the collection intersecting any given ball from the collection is at most $N < \infty$, where N depends only on α, T, n and K.

Our final geometric assumption is that the manifold also has a positive injective radius $\mathrm{inj}(M, g) \geq r_E > 0$. Under this assumption, and the lower bound on Ricci curvature, the existence of charts with L^∞ control of the metric was proved by Anderson and Cheeger [3, Thm. 0.3]. Their results show that for any $\beta \in (0, 1)$ and any $C_{gbg} > 1$, there exists $r_H > 0$, depending only on n, K, r_E, β and C_{gbg}, such that at each $x \in M$ there is a (harmonic) coordinate chart $\varphi : B(x, r_H) \to \mathbb{R}^n$ with a certain $C^{0,\beta}$-control of the metric, and which implies that

$$(2.2) \qquad C_{gbg}^{-1} \leq g(y) \leq C_{gbg}, \qquad y \in B(x, r_H),$$

as a bilinear form. A typical advantage of working in such harmonic coordinates, as opposed to geodesic normal coordinates, is that Ricci curvature bounds can often supplant bounds on the full Riemann curvature tensor. This is ultimately what enables the bound (2.2). The precise formulation of this result can be found in the work by Hebey and Herzlich [25, Thm. 6 & Cor.] (see also [23, Thm. 1.3]), which also recovered the $C^{1,\beta}$-estimates for the metric obtained by Anderson in [2], and established more general $C^{k,\beta}$-estimates, all of which require higher-order bounds on the Ricci curvature that are not assumed in this paper.

2.2. **Weights and operators.** A weight ω on M refers to a non-negative measurable function $\omega : M \to [0, \infty]$. For $R > 0$, let $A_2^R(M)$ denote the set of weights satisfying

$$[\omega]_{A_2^R(M)} = \sup_{x \in M, r \leq R} \left(\fint_{B(x,r)} \omega \, d\mu \right) \left(\fint_{B(x,r)} \frac{1}{\omega} \, d\mu \right) < \infty,$$

where $\fint_B f d\mu = \mu(B)^{-1} \int_B f d\mu$ denotes the integral average with respect to the integral measure. Let $A_2(M)$ denote the set of weights such that $[\omega]_{A_2(M)} = \sup_{R>0} [\omega]_{A_2^R(M)} < \infty$.

A vector bundle $\mathcal{W}$ on M refers to a smooth complex vector bundle $\pi : \mathcal{W} \to M$ equipped with a smooth Hermitian metric $\langle \cdot, \cdot \rangle_{\mathcal{W}_x}$ on the fibres $\mathcal{W}_x$. For a weight ω on M, the Hilbert space $L^2(\mathcal{W}, \omega)$ of weighted square integrable sections of $\mathcal{W}$ has the inner product

$$\langle u, v \rangle_{L^2(\mathcal{W}, \omega)} = \int_M \langle u(x), v(x) \rangle_{\mathcal{W}_x} \, \omega(x) \, d\mu(x).$$

A linear operator T on $L^2(\mathcal{W}, \omega)$ has a domain $\mathsf{Dom}(T)$, range $\mathsf{Ran}(T)$ and null space $\mathsf{Nul}(T)$ that are subspaces of $L^2(\mathcal{W}, \omega)$. For the trivial bundle $M \times \mathbb{C}$, we often write $L^2(M, \omega)$ instead of $L^2(M \times \mathbb{C}, \omega)$. We will consider in particular the complexified tangent bundle TM and the endomorphism bundle $\mathrm{End}(TM)$, each equipped with the Hermitian metric induced by the Riemannian metric g on TM. We identify the dual cotangent bundle T^*M with TM through the metric g.

We now introduce the first-order systems DB considered in Theorem 1.1. The differential operator D uses weak gradient and divergence operators ∇_M and div_M acting on weighted L^2 spaces on M. For the definition and formulas for ∇ (exterior derivative on scalar functions) and div (interior derivative on vector fields) acting on smooth sections, we refer to [29, Sec. 11.2]. A vector field $f \in L^1_{\mathrm{loc}}(TM)$ is the weak gradient of a function $u \in L^1_{\mathrm{loc}}(M)$ if $\langle f, \phi \rangle = -\langle u, \mathrm{div} \, \phi \rangle$ for all $\phi \in C_c^\infty(TM)$, and we write $\nabla u = f$. A function $f \in L^1_{\mathrm{loc}}(M)$ is the weak divergence of a vector field $v \in L^1_{\mathrm{loc}}(TM)$ if $\langle f, \phi \rangle = -\langle v, \nabla \phi \rangle$ for all $\phi \in C_c^\infty(M)$, and we write $\mathrm{div} \, v = f$.

Lemma 2.1. Let $\rho : M \to N$ be a bi-Lipschitz map between Riemannian manifolds M and N. Denote by $d\rho \in L^\infty(\mathrm{End}(TM, TN))$ and $J_\rho \in L^\infty(M)$ the Jacobian matrix and determinant of ρ.

(i) Let $u \in L^1_{\mathrm{loc}}(N)$ have weak gradient $\nabla u \in L^1_{\mathrm{loc}}(TN)$. Then the weak gradient of $u \circ \rho \in L^1_{\mathrm{loc}}(M)$ exists and equals

$$\nabla(u \circ \rho) = \rho^*(\nabla u),$$

where $(\rho^* v)(x) = (d\rho_x)^* v(\rho(x))$, $x \in M$, denotes the pullback of a vector field v.

(ii) Let $v \in L^1_{\mathrm{loc}}(TM)$ have weak divergence $\mathrm{div}\, v \in L^1_{\mathrm{loc}}(M)$. Define the *Piola transform* $(\tilde{\rho}_* v)(\rho(x)) = J_\rho(x)^{-1}(d\rho_x)v(x)$, $x \in M$, of the vector field v. Then the weak divergence of $\tilde{\rho}_* v \in L^1_{\mathrm{loc}}(N)$ exists and equals

$$\mathrm{div}(\tilde{\rho}_* v) = J_\rho^{-1}(\mathrm{div}\, v) \circ \rho^{-1}.$$

The proof follows from a straightforward generalization of [29, Lem. 10.2.4] to locally integrable differential forms on manifolds. The identities (i) and (ii) are dual in the sense that $-\mathrm{div}$ and $\tilde{\rho}_*$ are L^2 adjoints of ∇ and ρ^* respectively. The chain rule (i), and also (ii) via Hodge duality, are instances of the well known commutation between pullback and exterior derivative.

Example 2.2. Let $\rho : M \to N$ be a bi-Lipschitz map and consider a divergence form equation $\mathrm{div}\, A\nabla u = 0$ on N, where $A \in L^\infty(\mathrm{End}(TN))$. Define $f = u \circ \rho$ on M. Lemma 2.1(i) shows that $\nabla u = (\rho^*)^{-1}\nabla f$ and Lemma 2.1(ii) applied to ρ^{-1} yields

$$0 = J_\rho(\mathrm{div}\, A\nabla u) \circ \rho = \mathrm{div}(\tilde{\rho}_*^{-1} A(\rho^*)^{-1})\nabla f.$$

We refer to the coefficents $A_\rho = \tilde{\rho}_*^{-1} A(\rho^*)^{-1} = J_\rho(d\rho)^{-1}(A \circ \rho)(d\rho^*)^{-1}$ as the *pulled back coefficients* on M.

Let $\omega \in A_2^R(M)$ for some $R > 0$. We define ∇_M to be the closed operator $\nabla_M : L^2(M, \omega) \to L^2(TM, \omega) : u \mapsto \nabla u$ with domain

$$\mathsf{Dom}(\nabla_M) = W^{1,2}(M, \omega) = \{u \in L^2(M, \omega) : \nabla u \in L^2(TM, \omega)\}.$$

More precisely, $\nabla u \in L^2(TM, \omega)$ means that the weak gradient of u exists and belongs to $L^2(TM, \omega)$. We define div_M to be the closed operator $\mathrm{div}_M : L^2(TM, \frac{1}{\omega}) \to L^2(M, \frac{1}{\omega}) : v \mapsto \mathrm{div}\, v$ with domain

$$\mathsf{Dom}(\mathrm{div}_M) = \{v \in L^2(TM, \tfrac{1}{\omega}) : \mathrm{div}\, v \in L^2(M, \tfrac{1}{\omega})\}.$$

More precisely, $\mathrm{div}\, v \in L^2(M, \frac{1}{\omega})$ means that the weak divergence of v exists and belongs to $L^2(M, \frac{1}{\omega})$. Using that multiplication $\omega : L^2(M, \omega) \to L^2(M, \frac{1}{\omega})$ and $\omega : L^2(TM, \omega) \to L^2(TM, \frac{1}{\omega})$ is an isometry, we obtain a closed operator

$$\mathrm{div}_{M,\omega} = \tfrac{1}{\omega} \mathrm{div}_M \omega : L^2(TM, \omega) \to L^2(M, \omega),$$

which is unitary equivalent to div_M.

Our definitions of $\mathsf{Dom}(\nabla_M)$ and $\mathsf{Dom}(\mathrm{div}_M)$ above use the weak distributional definition of the differential operators on the smooth manifold and the natural weighted domains. The following lemma shows that this is equivalent to the more well-known definition involving closures from smooth functions.

Lemma 2.3. Let M and ω be as in Theorem 1.1. Then the smooth compactly supported functions $C_c^\infty(M)$ are dense in $\mathsf{Dom}(\nabla_M)$, in graph norm, and $C_c^\infty(TM)$ is dense in $\mathsf{Dom}(\mathrm{div}_M)$, in graph norm. Moreover, ∇_M and $-\mathrm{div}_M$ are adjoint under

the unweighted $L^2(M)$ pairing in the sense of unbounded operators, or equivalently, ∇_M and $-\operatorname{div}_{M,\omega}$ are adjoint under the weighted $L^2(M,\omega)$ pairing in the sense of unbounded operators.

Proof. From the coverings in Section 2.1, we obtain a locally finite partition of unity $\sum_i \eta_i = 1$ on M, with $0 \leq \eta_i \leq 1$ smooth and supported on balls $B(x_i, T)$ with $0 < T < R$ and $\sup_{i,x} |\nabla \eta_i(x)| < \infty$. We first prove that $C_c^\infty(M)$ is dense in $\mathsf{Dom}(\nabla_M)$. Given $u \in \mathsf{Dom}(\nabla_M)$ and $\epsilon > 0$, we can choose a finite index set I such that

$$\int_M \left(\left| \sum_{i \notin I} \eta_i u \right|^2 + \left| \nabla \sum_{i \notin I} \eta_i u \right|^2 \right) \omega d\mu \lesssim \int_{\bigcup_{i \notin I} B(x_i, T)} \left(|u|^2 + |\nabla u|^2 \right) \omega d\mu < \epsilon^2$$

since $\int_M \left(|u|^2 + |\nabla u|^2 \right) \omega d\mu < \infty$. Therefore it suffices to find, for each $i \in I$, $\tilde{u}_i \in C_c^\infty(B(x_i, T))$ such that

$$(2.3) \qquad \int_{B(x_i, T)} \left(|\eta_i u - \tilde{u}_i|^2 + |\nabla(\eta_i u - \tilde{u}_i)|^2 \right) \omega d\mu < (\epsilon/|I|)^2.$$

Indeed, $\sum_{i \in I} \tilde{u}_i \in C_c^\infty(M)$ and $u - \sum_{i \in I} \tilde{u}_i = \sum_{i \notin I} \eta_i u + \sum_{i \in I} (\eta_i u - \tilde{u}_i)$ can be estimated by the triangle inequality. To construct $\tilde{u}_i$, we use a chart $\varphi : B(x_i, T) \to \Omega \subset \mathbb{R}^n$, and consider $v_i = (\eta_i u) \circ \varphi^{-1}$. It follows from Lemma 2.1(i), that $v_i \in L^2(\Omega, \tilde{\omega})$, $\nabla v_i \in L^2(T\Omega, \tilde{\omega})$, where $\tilde{\omega} = \omega \circ \varphi^{-1}$. Note that the weighted estimates holds since pullback by φ and φ^{-1} are bounded pointwise, and hence bounded between the weighted L^2 spaces. Next we use [13, Lem. 2.2] to construct $\tilde{v}_i \in C_c^\infty(\Omega)$ such that

$$\int_\Omega \left(|v_i - \tilde{v}_i|^2 + |\nabla(v_i - \tilde{v}_i)|^2 \right) \tilde{\omega} dx < (\epsilon/|I|)^2.$$

Note that $\tilde{\omega}$ being defined only in Ω does not pose a problem since we use a mollifier $\varphi_\epsilon \in C_c^\infty(\mathbb{R}^n)$ with support near $x = 0$ in [13, Lem. 2.2]. (Alternatively, an auxiliary extended $A_2(\mathbb{R}^n)$ weight can be constructed as in Lemma 3.4 below.) Setting $\tilde{u}_i = \tilde{v}_i \circ \varphi$ yields (2.3) in $B(x_i, T)$.

A similar argument shows that $C_c^\infty(TM)$ is dense in $\mathsf{Dom}(\operatorname{div}_M)$. We replace ∇ by div in the argument above, set $v_i = \tilde{\varphi}_*(\eta_i u)$ and use Lemma 2.1(ii). In mollifying v_i using [13, Lem. 2.2], we note that convolutions commute also with the divergence.

The duality

$$\int_M \langle \nabla_M u, v \rangle d\mu = - \int_M \langle u, \operatorname{div}_M v \rangle d\mu$$

holds by construction of ∇_M for all $u \in \mathsf{Dom}(\nabla_M)$ and all $v \in C_c^\infty(TM)$ and hence, by the above density result, for all $v \in \mathsf{Dom}(\operatorname{div}_M)$. That ∇_M and $-\operatorname{div}_M$ are adjoint in the sense of unbounded operators is then immediate from the definition of these operators. $\qquad \square$

Henceforth, we fix the vector bundle

$$(2.4) \qquad\qquad \mathcal{V} = (M \times \mathbb{C}) \oplus (M \times \mathbb{C}) \oplus TM,$$

with the induced bundle metric, so that $L^2(\mathcal{V}, \omega) = L^2(M, \omega) \oplus L^2(M, \omega) \oplus L^2(TM, \omega)$. For a weight $\omega \in A_2^R(M)$, $R > 0$, define the self-adjoint operator

$$(2.5) \qquad D = \begin{bmatrix} 0 & I & -\operatorname{div}_{M,\omega} \\ I & 0 & 0 \\ \nabla_M & 0 & 0 \end{bmatrix} : \mathsf{Dom}(D) \subseteq L^2(\mathcal{V}, \omega) \to L^2(\mathcal{V}, \omega),$$

with $\mathsf{Dom}(D) = W^{1,2}(M,\omega) \oplus L^2(M,\omega) \oplus \mathsf{Dom}(\mathrm{div}_{M,\omega})$. Note that D has closed range $\mathsf{Ran}(D)$, due to the zero order identity terms in (2.4).

Consider $B \in L^\infty(\mathrm{End}(\mathcal{V}))$ and define a bounded multiplication operator $B : L^2(\mathcal{V},\omega) \to L^2(\mathcal{V},\omega)$ by setting

(2.6) $\qquad (Bu)(x) = (B(x))(u(x)), \qquad x \in M, u \in L^2(\mathcal{V},\omega).$

In the splitting $\mathcal{V}_x = \mathbb{C} \oplus \mathbb{C} \oplus T_x M$, we write

$$u(x) = \begin{bmatrix} u_\perp(v) \\ u_0(x) \\ u_\parallel(x) \end{bmatrix} \quad \text{and} \quad B(x) = \begin{bmatrix} B_{\perp\perp}(x) & B_{\perp 0}(x) & B_{\perp\parallel}(x) \\ B_{0\perp}(x) & B_{00}(x) & B_{0\parallel}(x) \\ B_{\parallel\perp}(x) & B_{\parallel 0}(x) & B_{\parallel\parallel}(x) \end{bmatrix}, \qquad x \in M,$$

where $B_{\perp\perp}, B_{\perp 0}, B_{0\perp}, B_{00} \in L^\infty(M)$, $B_{\parallel\perp}, B_{\parallel 0} \in L^\infty(TM)$, $B_{\perp\parallel}, B_{0\parallel} \in L^\infty(T^*M) = L^\infty(TM)$ and $B_{\parallel\parallel} \in L^\infty(\mathrm{End}(TM))$. We assume that B is accretive on $\mathsf{Ran}(D)$ in the sense that

(2.7) $\qquad \kappa_B = \inf\{\mathrm{Re}\langle Bu, u\rangle_{L^2(\mathcal{V},\omega)}/\|u\|^2_{L^2(\mathcal{V},\omega)} : u \in \mathsf{Ran}(D), u \neq 0\} > 0.$

2.3. **Functional calculus.** We recall that a closed and densely defined operator T in a Hilbert space $\mathcal{H}$ is called bisectorial, if its spectrum is contained in a closed bisector

$$S_\nu = \{z \in \mathbb{C} : |\arg(z)| \leq \nu\} \cup \{0\} \cup \{z \in \mathbb{C} : |\arg(-z)| \leq \nu\},$$

for some $0 \leq \nu < \pi/2$, with resolvent bounds $\|(\lambda I - T)^{-1}\| \lesssim 1/\mathrm{dist}\,(\lambda, S_\nu)$. It follows from [8, Prop. 3.3] that the operators BD and DB from Section 2.2 are bisectorial in $L^2(\mathcal{V},\omega)$ for any coefficients B satisfying (2.6). The proof therein, which treats the case when $\omega \equiv 1$ and $\mathcal{V}$ is a trivial bundle over $\mathbb{R}^n$, extends immediately to the setting considered here. Moreover we have a topological splitting

(2.8) $\qquad L^2(\mathcal{V},\omega) = \mathsf{Ran}(DB) \oplus \mathsf{Nul}(DB),$

where in general the two closed subspaces are not orthogonal. Injective bisectorial operators T have a bounded $H^\infty(S_\theta^o)$ functional calculus for any $\nu < \theta < \pi/2$, provided quadratic estimates

(2.9) $$\int_0^\infty \|\psi_t(T)u\|^2 \frac{dt}{t} \approx \|u\|^2, \qquad u \in \mathcal{H},$$

hold. Here $\psi(\lambda) = \lambda/(1 + \lambda^2)$ and $\psi_t(\lambda) = \psi(t\lambda)$, and boundedness of the $H^\infty(S_\theta^o)$ functional calculus means that we have an estimate

$$\|b(T)\|_{\mathcal{H} \to \mathcal{H}} \lesssim \sup_{\lambda \in S_\theta^o} |b(\lambda)|$$

for all bounded holomorphic functions $b : S_\theta^o \to \mathbb{C}$, where

$$S_\theta^o = \{z : |\arg(z)| < \theta\} \cup \{z : |\arg(-z)| < \theta\}.$$

We refer to [5, Sec. 6.1] for a short proof of these well known facts, and we refer to [1, 11] for more background, where the generalization from sectorial to bisectorial operators is straightforward. In proving quadratic estimates (2.8), it suffices to prove the estimate $\lesssim$ for T and its adjoint T^*. Indeed, by a duality argument the latter implies the estimate $\gtrsim$. See [1].

For non-injective bisectorial operators T, the above applies to the restriction of T to the invariant subspace $\overline{\mathsf{Ran}(T)}$, since this gives an injective bisectorial operator,

and the splitting $\mathcal{H} = \overline{\mathsf{Ran}(T)} \oplus \mathsf{Nul}(T)$ allows an extension of the functional calculus, setting

$$b(T) = b(T|_{\overline{\mathsf{Ran}(T)}}) \oplus b(0)I_{\mathsf{Nul}(T)}$$

for $b : S_\theta^o \cup \{0\} \to \mathbb{C}$ with $b|_{S_\theta^o} \in H^\infty(S_\theta^o)$, where $T|_{\overline{\mathsf{Ran}(T)}} = T\mathbb{P}$, $I_{\mathsf{Nul}(T)} = I - \mathbb{P}$ and $\mathbb{P}$ denotes the projection onto $\overline{\mathsf{Ran}(T)}$ along $\mathsf{Nul}(T)$. We refer to the function b defining the operator $b(T)$ as the *symbol* of $b(T)$.

The following abstract argument allows us to perturb the quadratic estimates from Theorem 1.1.

Lemma 2.4. Let T be a bisectorial operator in a Hilbert space $\mathcal{H}$, with local quadratic estimates

$$\int_0^1 \|\psi_t(T)u\|^2 \frac{dt}{t} \lesssim \|u\|^2, \qquad u \in \mathcal{H}.$$

Let $V : \mathcal{H} \to \mathcal{H}$ be a bounded operator such that $T + V$, with domain $\mathsf{Dom}(T)$, is a bisectorial operator with closed range. Then

$$\int_0^\infty \|\psi_t(T+V)u\|^2 \frac{dt}{t} \lesssim \|u\|^2, \qquad u \in \mathcal{H}.$$

Proof. We have the estimate

$$\|(I + it(T+V))^{-1} - (I + itT)^{-1}\| = \|(I + it(T+V))^{-1}(tV)(I + itT)^{-1}\| \lesssim |t|$$

between resolvents, for $t \in \mathbb{R}$. Subtraction by the corresponding estimate with t replaced by $-t$, shows

$$\int_0^1 \|\psi_t(T+V)u\|^2 \frac{dt}{t} \lesssim \int_0^1 (\|\psi_t(T)u\|^2 + t^2\|u\|^2) \frac{dt}{t} \lesssim \|u\|^2, \qquad u \in \mathcal{H}.$$

Next we note that $\psi_t(T+V)u = t(T+V)\big((I + t^2(T+V)^2)^{-1}u\big) \in \mathsf{Ran}(T+V)$, which by bisectoriality is a subspace complementary to $\mathsf{Nul}(T+V)$. Assuming that $\mathsf{Ran}(T+V)$ is closed, it follows from the Open Mapping Theorem that

$$\|\psi_t(T+V)u\| \lesssim t^{-1}\|t(T+V)\psi_t(T+V)u\| \lesssim t^{-1}\|u\|, \qquad u \in \mathcal{H}.$$

This proves the estimate $\int_1^\infty \|\psi_t(T+V)u\|^2 \frac{dt}{t} \lesssim \|u\|^2$ for $u \in \mathcal{H}$. $\qquad\square$

3. Proof of quadratic estimates

We prove Theorem 1.1 in this section. The proof is split into subsections and lemmas to increase readability. For the whole proof, we fix $C_{gbg} = 4$ and $\beta = 1/2$ and let $r_H > 0$ be the harmonic radius of M, for these constants, as in Section 2.1. These choices for C_{gbg} and β are allowed, and determine the value of r_H, as per the discussion preceding (2.2). Moreover, since we will not require the maximal harmonic radius r_H, we can and will assume without loss of generality that $r_H < R$. Fix $0 < T < r_H/64$ and denote balls covering M, as in Section 2.1, by

$$B_i = B(x_i, T) \subset M,$$

and write $B_i^* = B(x_i, r_H) \subset M$ for the enlarged balls on which we have charts with the estimate (2.2). We write $\Omega \subset \mathbb{R}^n$ and $\Omega^* \subset \mathbb{R}^n$ for Euclidean balls centered at 0 with radii $16T$ and $r_H/2$ respectively. Since $C_{gbg} = 4$, length dilation is at most 2 and we have $\varphi_i(4B_i) \subset \frac{1}{2}\Omega$ and $\varphi_i^{-1}(\Omega^*) \subset B_i^*$ for the harmonic coordinate charts φ_i, where we without loss of generality assume that $\varphi_i(x_i) = 0$.

Besides the standing assumption that $T < r_H/64$, ensuring that $2\Omega \subset \Omega^*$, we require that $T > 0$ satisfies (3.4) and (3.7) below.

3.1. **Localization on** M**.** Write $\psi_t(\lambda) = t\lambda/(1 + t^2\lambda^2)$. As in Lemma 2.4 it suffices to prove that

$$(3.1) \qquad \int_0^T \|\psi_t(DB)u\|^2_{L^2(\mathcal{V},\omega)} \frac{dt}{t} \lesssim \|u\|^2_{L^2(\mathcal{V},\omega)},$$

since $\mathsf{Ran}(D)$ is closed. Moreover we may assume $u \in \mathsf{Ran}(D)$, using the splitting (2.7). We write

$$\int_0^T \|\psi_t(DB)u\|^2_{L^2(\mathcal{V},\omega)} \frac{dt}{t}$$

$$\lesssim \int_0^T \sum_{i\in\mathbb{N}} \left(\|1_{B_i}\psi_t(DB)1_{4B_i}u\|^2_{L^2(\mathcal{V},\omega)} + \|1_{B_i}\psi_t(DB)1_{M\setminus 4B_i}u\|^2_{L^2(\mathcal{V},\omega)} \right) \frac{dt}{t}$$

$$\leq \sum_{i\in\mathbb{N}} \int_0^T \|1_{4B_i}\psi_t(DB)1_{4B_i}u\|^2_{L^2(\mathcal{V},\omega)} \frac{dt}{t} + \int_0^T \sum_{i\in\mathbb{N}} \|1_{B_i}\psi_t(DB)1_{M\setminus 4B_i}u\|^2_{L^2(\mathcal{V},\omega)} \frac{dt}{t}$$

$$= I_1 + I_2.$$

The main term I_1 is estimated in the following sections. In the present section we estimate the error term I_2, using the off-diagonal decay in Lemma 3.1 to offset volume growth on the manifold. We write

$$\langle \alpha \rangle = \max\{\alpha, 1\}, \qquad \alpha > 0.$$

Lemma 3.1. There exists a constant $C_B > 0$, depending only on κ_B and $\|B\|_\infty$, such that for each $N \geq 0$ it holds that

$$\|1_E\psi_t(DB)1_F\|_{\mathcal{L}(L^2(\mathcal{V},\omega))} \lesssim \left\langle \frac{d(E,F)}{t} \right\rangle^{-N} \exp\left(-C_B \frac{d(E,F)}{t} \right),$$

for all $t > 0$ and all measurable sets $E, F \subseteq M$, where the implicit constant depends on κ_B, $\|B\|_\infty$ and N.

Proof. This follows as in [28, Prop. 5.2], which in turn combines [18, Lem. 5.3] with [15, Prop. 5.2], since DB is a bisectorial operator in $L^2(\mathcal{V}, \omega)$ and the commutator $[\eta, DB] = \eta DB - DB\eta$ satisfies the pointwise identity

$$|[\eta, DB]u(x)|_{\mathcal{V}_x} = |[\eta, D](Bu)(x)|_{\mathcal{V}_x} \leq |\nabla_M \eta(x)|_{T_xM} \|B\|_\infty |u(x)|_{\mathcal{V}_x},$$

for all $\eta \in C_c^\infty(M)$, $u \in \mathsf{Dom}(DB)$ and $x \in M$. $\qquad\square$

The following lemma can be deduced from the proof of [28, Lem. 4.4], but the short proof is included for convenience.

Lemma 3.2. Let $\lambda = \sqrt{(n-1)K}$. If $N > n$ and $m > \lambda T$, then

$$(3.2) \qquad \sup_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} \frac{\mu(B_j)}{\mu(B_i)} \left\langle \frac{t}{d(B_i,B_j)} \right\rangle^N e^{-md(B_i,B_j)/t} \lesssim 1, \qquad t \in (0,T],$$

where the implicit constant depends on n, λ, N, m and T. If in addition $\alpha > 0$, then

$$(3.3) \qquad \sup_{i\in\mathbb{N}} \sum_{\substack{j\in\mathbb{N}: \\ d(B_i,B_j)\geq\alpha}} \frac{\mu(B_j)}{\mu(B_i)} \left\langle \frac{t}{d(B_i,B_j)} \right\rangle^N e^{-md(B_i,B_j)/t} \lesssim t^N, \qquad t \in (0,T],$$

where the implicit constant depends on n, λ, N, m, T and α.

Proof. Let $T > 0$, $N > n$, $m > \lambda T$ and $\alpha > 0$. Set $\sigma = m/\lambda T > 1$ and for each $i \in \mathbb{N}$, isolate the balls intersecting concentric annuli around B_i by defining

$$\mathcal{A}_k(B_i) = \begin{cases} \{B_j : d(B_i, B_j) < \alpha\}, & \text{if} \quad k = 0, \\ \{B_j : \alpha\sigma^{k-1} \leq d(B_i, B_j) < \alpha\sigma^k\}, & \text{if} \quad k \in \mathbb{N}. \end{cases}$$

A straightforward calculation shows that

$$\bigcup_{B_j \in \mathcal{A}_k(B_i)} B_j \subseteq B(x_i, 3T + \alpha\sigma^k), \qquad k \in \mathbb{N}_0, \ i \in \mathbb{N}.$$

It follows from (2.1) and the disjointness of the balls $((1/2)B_i)_{i \in \mathbb{N}}$ that

$$\sum_{B_j \in \mathcal{A}_k(B_i)} \mu(B_j) \lesssim e^{\lambda T} \mu\left(\bigcup_{B_j \in \mathcal{A}_k(B_i)} (1/2)B_j\right) \lesssim \sigma^{kn} e^{\lambda\alpha\sigma^k} \mu(B_i), \qquad k \in \mathbb{N}_0, i \in \mathbb{N}.$$

For all $t \in (0, T]$, since $m - \sigma\lambda t \geq m - \sigma\lambda T = 0$, we then have

$$\sup_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \frac{\mu(B_j)}{\mu(B_i)} \left\langle \frac{t}{d(B_i, B_j)} \right\rangle^N e^{-md(B_i, B_j)/t}$$

$$= \sup_{i \in \mathbb{N}} \sum_{k=0}^{\infty} \sum_{B_j \in \mathcal{A}_k(B_i)} \frac{\mu(B_j)}{\mu(B_i)} \left\langle \frac{t}{d(B_i, B_j)} \right\rangle^N e^{-md(B_i, B_j)/t}$$

$$\leq \sup_{i \in \mathbb{N}} \frac{1}{\mu(B_i)} \left(\sum_{B_j \in \mathcal{A}_0(B_i)} \mu(B_j) + \sum_{k=1}^{\infty} (\alpha\sigma^{k-1}/t)^{-N} e^{-m\alpha\sigma^{k-1}/t} \sum_{B_j \in \mathcal{A}_k(B_i)} \mu(B_j) \right)$$

$$\lesssim 1 + t^N \sum_{k=1}^{\infty} \sigma^{-k(N-n)} e^{-(m-\sigma\lambda t)\alpha\sigma^{k-1}/t}$$

$$\lesssim 1 + t^N,$$

which proves (3.2), and we obtain (3.3) by excluding the term $k = 0$ in the sum. $\quad\square$

To estimate I_2, on recalling that $\lambda = \sqrt{(n-1)K}$, we require that

(3.4) $$T < C_B/\lambda,$$

where $C_B > 0$ is the constant from Lemma 3.1. In particular, choosing $N > n$ and setting $m = C_B$, we can then combine the off-diagonal estimate in Lemma 3.1 with the geometric control afforded by Lemma 3.2 to obtain

$$\sum_{i \in \mathbb{N}} \|\mathbb{1}_{B_i} \psi_t(DB) \mathbb{1}_{M \setminus 4B_i} u\|_{L^2(\mathcal{V},\omega)}^2$$

$$\leq \sum_{i \in \mathbb{N}} \left(\sum_{\substack{j \in \mathbb{N}: \\ d(B_i, B_j) \geq T}} \left(\frac{\mu(B_j)}{\mu(B_i)} \frac{\mu(B_i)}{\mu(B_j)} \right)^{1/2} \|\mathbb{1}_{B_i} \psi_t(DB) \mathbb{1}_{B_j} u\|_{L^2(\mathcal{V},\omega)} \right)^2$$

$$\leq \sum_{i \in \mathbb{N}} \left(\sum_{\substack{j \in \mathbb{N}: \\ d(B_i, B_j) \geq T}} \frac{\mu(B_j)}{\mu(B_i)} \|\mathbb{1}_{B_i} \psi_t(DB) \mathbb{1}_{B_j}\| \right)$$

$$\times \left(\sum_{j\in\mathbb{N}} \frac{\mu(B_i)}{\mu(B_j)} \|1_{B_i}\psi_t(DB)1_{B_j}\| \|1_{B_j}u\|^2_{L^2(\mathcal{V},\omega)} \right)$$

$$\lesssim \left(\sup_{i\in\mathbb{N}} \sum_{\substack{j\in\mathbb{N}:\\ d(B_i,B_j)\geq T}} \frac{\mu(B_j)}{\mu(B_i)} \left\langle \frac{t}{d(B_i,B_j)} \right\rangle^N e^{-C_B d(B_i,B_j)/t} \right)$$

$$\times \left(\sup_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} \frac{\mu(B_i)}{\mu(B_j)} \left\langle \frac{t}{d(B_i,B_j)} \right\rangle^N e^{-C_B d(B_i,B_j)/t} \right) \left(\sum_{j\in\mathbb{N}} \|1_{B_j}u\|^2_{L^2(\mathcal{V},\omega)} \right)$$

$$\lesssim t^N \|u\|^2_{L^2(\mathcal{V},\omega)}$$

for all $t \in (0,T]$. This proves that $I_2 \lesssim \|u\|^2_{L^2(\mathcal{V},\omega)}$ for all $u \in L^2(\mathcal{V},\omega)$.

3.2. **Pullback from M to $\mathbb{R}^n$.** To prove (3.1), and thus Theorem 1.1, it remains to prove that

$$(3.5) \quad I_1 = \sum_{i\in\mathbb{N}} \int_0^T \|1_{4B_i}\psi_t(DB)1_{4B_i}u\|^2_{L^2(\mathcal{V},\omega)} \frac{dt}{t} \lesssim \|u\|^2_{L^2(\mathcal{V},\omega)}, \qquad u \in L^2(\mathcal{V},\omega).$$

We estimate (3.5) term-wise, fix i and use the harmonic coordinate chart φ_i for the vector bundle $\mathcal{V}$ from (**??**). Suppressing i in notation, we write $\rho = \varphi_i^{-1}$. Define the pulled back weight $\omega_\rho = \omega \circ \rho$ in Ω^*. We use ρ to locally intertwine the operator DB from Section 2.2 in $L^2(\rho(2\Omega), \mathcal{V}, \omega)$ and an operator $D_\rho B_\rho$ in $L^2(2\Omega, \mathbb{C}^{2+n}, \omega_\rho)$, using Lemma 2.1. Define a pushforward operator $T_\rho : L^2(2\Omega, \mathbb{C}^{2+n}, \omega_\rho) \to L^2(\rho(2\Omega), \mathcal{V}, \omega)$ by

$$T_\rho \begin{bmatrix} u_\perp \\ u_0 \\ u_\| \end{bmatrix} (\rho(x)) = \begin{bmatrix} u_\perp(x) \\ u_0(x)/J_\rho(x) \\ (d\rho)_x u_\|(x)/J_\rho(x) \end{bmatrix}, \qquad x \in 2\Omega,$$

for $u_\perp, u_0 \in L^2(2\Omega, \mathbb{C}, \omega_\rho)$ and $u_\| \in L^2(2\Omega, \mathbb{C}^n, \omega_\rho)$. The weighted L^2 adjoint of T_ρ is $T_\rho^* : L^2(\rho(2\Omega), \mathcal{V}, \omega) \to L^2(2\Omega, \mathbb{C}^{n+1}, \omega_\rho)$ given by

$$T_\rho^* \begin{bmatrix} v_\perp \\ v_0 \\ v_\| \end{bmatrix} (x) = \begin{bmatrix} J_\rho(x)v_\perp(\rho(x)) \\ v_0(\rho(x)) \\ (d\rho_x)^* v_\|(\rho(x)) \end{bmatrix}, \qquad x \in 2\Omega,$$

for $v_\perp, v_0 \in L^2(\rho(2\Omega), \mathbb{C}, \omega)$ and $v_\| \in L^2(\rho(2\Omega), TM, \omega)$.

The Euclidean version of differential operator (2.4) is

$$D_\rho = \begin{bmatrix} 0 & I & -\operatorname{div}_{\omega_\rho} \\ I & 0 & 0 \\ \nabla & 0 & 0 \end{bmatrix},$$

where $\operatorname{div}_{\omega_\rho} = \omega_\rho^{-1} \operatorname{div} \omega_\rho$ (for now acting on functions defined on 2Ω) and ∇ and div are the weak gradient and divergence in $\mathbb{R}^n$. In Section 3.3, we extend ω_ρ and define D_ρ as a self-adjoint operator on $\mathbb{R}^n$.

Lemma 3.3. Let B be a bounded and accretive multiplication operator on $L^2(M,\omega)$ as in Section 2.2. Define the bounded multiplication operator $B_\rho = T_\rho^{-1} B T_\rho^{*-1}$ on $L^2(2\Omega, \mathbb{C}^{2+n}, \omega_\rho)$. Then

$$(3.6) \qquad\qquad T_\rho^* DBv = D_\rho B_\rho T_\rho^* v \qquad \text{on } 2\Omega,$$

for all $v \in \mathsf{Dom}(DB)$.

Proof. We note that T_ρ is an invertible multiplication operator. In particular B_ρ is a bounded multiplication operator. To prove the local intertwining formula (3.6) , it suffices to verify that $T_\rho^* D = D_\rho T_\rho^{-1}$ on $\rho(2\Omega)$. Acting on $\begin{bmatrix} v_\perp \\ v_0 \\ v_\parallel \end{bmatrix}$, this amounts to

$$\begin{bmatrix} J_\rho(v_0 \circ \rho) - J_\rho(\operatorname{div}_{M,\omega} v_\parallel) \circ \rho \\ v_\perp \circ \rho \\ d\rho^*(\nabla_M v_\perp) \circ \rho \end{bmatrix} = \begin{bmatrix} J_\rho(v_0 \circ \rho) - \operatorname{div}_{\rho,\omega_\rho} J_\rho d\rho^{-1}(v_\parallel \circ \rho) \\ v_\perp \circ \rho \\ \nabla(v_\perp \circ \rho) \end{bmatrix}.$$

We conclude using Lemma 2.1. $\qquad\square$

3.3. Extension of weights and coefficients.

In this section, we extend the weight ω_ρ and the coefficients B_ρ to $\mathbb{R}^n$, and add a lower order term $r\mathbb{P}_0$ to B_ρ, to obtain a bisectorial operator $D_\rho B_{\rho,r}$ on $\mathbb{R}^n$ for which weighted quadratic estimates are proved by tweaking the proof from [13] in Section 3.4. For the following extension argument for the weight ω_ρ, we require the more precise upper bound

(3.7) $$T < r_H/(64(1 + 2\sqrt{n})),$$

which will also suffice for our ultimate purpose of obtaining (3.1) for some $T > 0$.

Lemma 3.4. There exists a weight $\widetilde{\omega}_\rho \in A_2(\mathbb{R}^n)$ such that

$$\widetilde{\omega}_\rho = \omega_\rho \qquad \text{on } 2\Omega,$$

with $[\widetilde{\omega}_\rho]_{A_2(\mathbb{R}^n)}$ depending only $[\omega]_{A_2^R(M)}$, R, n, r_E and K.

Proof. We verify condition (ii) in [21, Thm. IV.5.6] for ω_ρ on 2Ω, which guarantees that if $\omega_\rho^{1+\epsilon} \in A_2(2\Omega)$ for some $\epsilon > 0$, then there exists $\widetilde{\omega}_\rho \in A_2(\mathbb{R}^n)$ as required by the lemma . It suffices to prove

$$\sup_Q \left(\frac{1}{|Q|} \int_{Q \cap 2\Omega} \omega_\rho^{1+\epsilon} dx \right) \left(\frac{1}{|Q|} \int_{Q \cap 2\Omega} \omega_\rho^{-1-\epsilon} dx \right) < \infty,$$

with supremum taken over all cubes $Q \subset \mathbb{R}^n$, for some $\epsilon > 0$. To this end, we first note that it suffices to consider Q with sidelength $\ell(Q) \leq 2r$ and intersecting 2Ω, where $r = 32T$ is the radius of 2Ω. (If $\ell(Q) > 2r$, we may shrink Q by moving the corner of Q farthest away from 0, without changing $Q \cap 2\Omega$.) For such Q, we estimate by increasing the domains of integration from $Q \cap 2\Omega$ to Q, since $Q \subset \Omega^*$ by (3.7). Having done this, by reverse Hölder estimate and A_p bootstrapping, see [22, Thm. 9.2.2, 9.2.5], it suffices to prove the estimate for $\epsilon = 0$. We need to show a uniform bound on

$$\left(\frac{1}{|Q|} \int_Q \omega_\rho dx \right) \left(\frac{1}{|Q|} \int_Q \omega_\rho^{-1} dx \right).$$

Changing variables to $B_i^* \subset M$, since $C_{gbg} = 4$ and we assume $r_H < R$, the bound now follows from $[\omega]_{A_2^R(M)} < \infty$. $\qquad\square$

Fix an $A_2(\mathbb{R}^n)$ weight $\widetilde{\omega}_\rho$ as in Lemma 3.4. This extension of ω_ρ from Ω to $\mathbb{R}^n$ can be constructed as in [21, Thm. IV.5.6] by factorising ω_ρ and applying maximal functions to the zero extensions of its components. Define the self-adjoint operator

$$D_\rho = \begin{bmatrix} 0 & I & -\operatorname{div}_{\widetilde{\omega}_\rho} \\ I & 0 & 0 \\ \nabla & 0 & 0 \end{bmatrix}$$

in $L^2(\mathbb{R}^n, \mathbb{C}^{2+n}, \widetilde{\omega}_\rho)$. We next turn to the extension of B_ρ to a bounded multiplication operator that is accretive on the range of D_ρ. (A related construction is in [9, App. A].) Define

$$(3.8) \qquad B_{\rho,r} = B_\rho(\eta_0^2 I) + (1 - \eta_0^2)I + r\mathbb{P}_0,$$

where

$$\mathbb{P}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The parameter r will be fixed below to ensure accretivity of $B_{\rho,r}$ on $\mathsf{Ran}(D_\rho)$, which is crucial, whilst $\eta_0 : \mathbb{R}^n \to [0,1]$ denotes a fixed smooth cut-off function such that

$$\mathrm{sppt}\, \eta_0 \subset 2\Omega \quad \text{and} \quad \eta_0 \equiv 1 \text{ on } \Omega.$$

Recall that $\widetilde{\omega}_\rho = \omega_\rho$ on 2Ω. The following lemma provides for our choice of r.

Lemma 3.5. There exists $r < \infty$, depending only on n, κ_B and $\|B\|_\infty$, such that $B_{\rho,r}$ is accretive on $\mathsf{Ran}(D_\rho)$ in the sense of (2.6) with

$$\kappa_{B_{\rho,r}} \geq \min\{\tfrac{1}{4}c^{-1}\kappa_B, 1\} > 0,$$

where $c < \infty$ is a positive constant depending only on n.

Proof. Let $u \in \mathsf{Ran}(D_\rho)$, so $u_\parallel = \nabla u_0$, where $u_\perp \in L^2(\mathbb{R}^n, \widetilde{\omega}_\rho)$ and $u_0 \in W^{1,2}(\mathbb{R}^n, \widetilde{\omega}_\rho)$. Now we use $\nabla(\eta_0 u_0) = \eta_0 \nabla u_0 + (\nabla\eta_0)u_0$ to write

$$\langle B_\rho(\eta_0^2 u), u \rangle = \langle B_\rho \begin{bmatrix} \eta_0 u_\perp \\ \eta_0 u_0 \\ \eta_0 \nabla u_0 \end{bmatrix}, \begin{bmatrix} \eta_0 u_\perp \\ \eta_0 u_0 \\ \eta_0 \nabla u_0 \end{bmatrix} \rangle$$

$$= \langle B_\rho \begin{bmatrix} \eta_0 u_\perp \\ \eta_0 u_0 \\ \nabla(\eta_0 u_0) \end{bmatrix}, \begin{bmatrix} \eta_0 u_\perp \\ \eta_0 u_0 \\ \nabla(\eta_0 u_0) \end{bmatrix} \rangle - E,$$

where the error term is

$$E = \langle B_\rho \begin{bmatrix} 0 \\ 0 \\ (\nabla\eta_0)u_0 \end{bmatrix}, \eta_0 u \rangle + \langle B_\rho \eta_0 u, \begin{bmatrix} 0 \\ 0 \\ (\nabla\eta_0)u_0 \end{bmatrix} \rangle + \langle B_\rho \begin{bmatrix} 0 \\ 0 \\ (\nabla\eta_0)u_0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ (\nabla\eta_0)u_0 \end{bmatrix} \rangle$$

and we use the $L^2(\mathbb{R}^n, \mathbb{C}^{2+n}, \widetilde{\omega}_\rho)$ inner-product. Lemma 2.1 shows that

$$T_\rho^{*-1} \begin{bmatrix} \eta_0 u_\perp \\ \eta_0 u_0 \\ \nabla(\eta_0 u_0) \end{bmatrix} = \begin{bmatrix} J_\rho^{-1}(\eta_0 u_\perp) \circ \rho^{-1} \\ (\eta_0 u_0) \circ \rho^{-1} \\ \nabla_M((\eta_0 u_0) \circ \rho^{-1}) \end{bmatrix}$$

belongs to $\mathsf{Ran}(D)$. The accretivity (2.6) of B on $\mathsf{Ran}(D)$ then implies that

$$\mathrm{Re}\langle B_\rho \begin{bmatrix} \eta_0 u_\perp \\ \eta_0 u_0 \\ \nabla(\eta_0 u_0) \end{bmatrix}, \begin{bmatrix} \eta_0 u_\perp \\ \eta_0 u_0 \\ \nabla(\eta_0 u_0) \end{bmatrix} \rangle_{L^2(\Omega, \mathbb{C}^{2+n}, \widetilde{\omega}_\rho)}$$

$$= \mathrm{Re}\langle B T_\rho^{*-1} \begin{bmatrix} \eta_0 u_\perp \\ \eta_0 u_0 \\ \nabla(\eta_0 u_0) \end{bmatrix}, T_\rho^{*-1} \begin{bmatrix} \eta_0 u_\perp \\ \eta_0 u_0 \\ \nabla(\eta_0 u_0) \end{bmatrix} \rangle_{L^2(\mathcal{V}, \omega)}$$

$$\geq \kappa_B \|T_\rho^{*-1} \begin{bmatrix} \eta_0 u_\perp \\ \eta_0 u_0 \\ \nabla(\eta_0 u_0) \end{bmatrix} \|_{L^2(\mathcal{V},\omega)}^2 \geq c^{-1}\kappa_B \| \begin{bmatrix} \eta_0 u_\perp \\ \eta_0 u_0 \\ \nabla(\eta_0 u_0) \end{bmatrix} \|_{L^2(\Omega,\mathbb{C}^{n+1},\widetilde{\omega}_\rho)}^2$$

$$\geq c^{-1}\kappa_B \left(\int_{\mathbb{R}^n} (|\eta_0 u_\perp|^2 + |\eta_0 u_0|^2)\widetilde{\omega}_\rho dx + \tfrac{1}{2}\int_{\mathbb{R}^n} |\eta_0\nabla u_0|^2\widetilde{\omega}_\rho dx - \int_{\mathbb{R}^n} |(\nabla\eta_0)u_0|^2\widetilde{\omega}_\rho dx \right)$$

$$\geq c^{-1}\kappa_B \left(\tfrac{1}{2}\int_{\mathbb{R}^n} \eta_0^2|u|^2\widetilde{\omega}_\rho dx - \int_{\mathbb{R}^n} |\nabla\eta_0|^2|u_0|^2\widetilde{\omega}_\rho dx \right),$$

for some positive constant $c < \infty$ only depending on n, due to the choice $C_{gbg} = 4$.

Meanwhile, introducing a constant $\epsilon \in (0,1)$ to be chosen, and applying Cauchy's inequality with ϵ, the remaining error term is controlled by

$$|E| \leq c\|B\|_\infty \left(2\int_{\mathbb{R}^n} |(\nabla\eta_0)u_0||\eta_0 u|\widetilde{\omega}_\rho dx + \int_{\mathbb{R}^n} |(\nabla\eta_0)u_0|^2\widetilde{\omega}_\rho dx \right)$$

$$\leq c\|B\|_\infty \left(2\epsilon\int_{\mathbb{R}^n} \eta_0^2|u|^2\widetilde{\omega}_\rho dx + (2\epsilon^{-1}+1)\int_{\mathbb{R}^n} |\nabla\eta_0|^2|u_0|^2\widetilde{\omega}_\rho dx \right).$$

Altogether, we have

$$\mathrm{Re}\langle B_{\rho,r}u, u\rangle = \mathrm{Re}\langle B_\rho(\eta_0^2 u), u\rangle + \langle(1-\eta_0^2)u, u\rangle + r\|u_0\|^2$$

$$\geq c^{-1}\kappa_B \left(\tfrac{1}{2}\int_{\mathbb{R}^n} \eta_0^2|u|^2\widetilde{\omega}_\rho dx - \int_{\mathbb{R}^n} |\nabla\eta_0|^2|u_0|^2\widetilde{\omega}_\rho dx \right) + \int_{\mathbb{R}^n}(1-\eta_0^2)|u|^2\widetilde{\omega}_\rho dx$$

$$- c\|B\|_\infty \left(2\epsilon\int_{\mathbb{R}^n} \eta_0^2|u|^2\widetilde{\omega}_\rho dx + 3\epsilon^{-1}\int_{\mathbb{R}^n} |\nabla\eta_0|^2|u_0|^2\widetilde{\omega}_\rho dx \right) + r\int_{\mathbb{R}^n}|u_0|^2\widetilde{\omega}_\rho dx$$

$$= (\tfrac{1}{2}c^{-1}\kappa_B - c\|B\|_\infty 2\epsilon)\int_{\mathbb{R}^n} \eta_0^2|u|^2\widetilde{\omega}_\rho dx + \int_{\mathbb{R}^n}(1-\eta_0^2)|u|^2\widetilde{\omega}_\rho dx$$

$$- (c^{-1}\kappa_B + c\|B\|_\infty 3\epsilon^{-1})\int_{\mathbb{R}^n} |\nabla\eta_0|^2|u_0|^2\widetilde{\omega}_\rho dx + r\int_{\mathbb{R}^n}|u_0|^2\widetilde{\omega}_\rho dx.$$

We first choose $\epsilon > 0$ sufficiently small, and then choose $r < \infty$ sufficiently large, depending only on n, κ_B and $\|B\|_\infty$ such that

$$\mathrm{Re}\langle B_{\rho,r}u, u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^{2+n},\widetilde{\omega}_\rho)} \geq \min\{\tfrac{1}{4}c^{-1}\kappa_B, 1\}\|u\|_{L^2(\mathbb{R}^n,\mathbb{C}^{2+n},\widetilde{\omega}_\rho)}^2,$$

so the result follows. $\qquad\square$

3.4. **Estimates on $\mathbb{R}^n$.** In this section, we complete the proof of Theorem 1.1 by reducing it to, and proving, the corresponding weighted quadratic estimate for $D_\rho B_{\rho,r}$ in $L^2(\mathbb{R}^n, \mathbb{C}^{n+1}, \widetilde{\omega}_\rho)$.

Theorem 3.6. *We have quadratic estimates*

$$\int_0^\infty \|\psi_t(D_\rho B_{\rho,r})u\|_{L^2(\mathbb{R}^n,\mathbb{C}^{2+n},\widetilde{\omega}_\rho)}^2 \frac{dt}{t} \lesssim \|u\|_{L^2(\mathbb{R}^n,\mathbb{C}^{2+n},\widetilde{\omega}_\rho)}^2, \qquad u \in L^2(\mathbb{R}^n, \mathbb{C}^{2+n}, \widetilde{\omega}_\rho),$$

where the implicit constant depends only on n, $\|B_{\rho,r}\|_\infty$, $\kappa_{B_{\rho,r}}$ and $[\widetilde{\omega}_\rho]_{A_2(\mathbb{R}^n)}$.

Proof. The proof follows closely that presented in [13, Sec. 3], and we only point out the differences which arise due to the presence of zero order identity blocks in D_ρ. Since $\mathsf{Ran}(D_\rho)$ is closed, it suffices to bound the integral $\int_0^1$. As in the proof of [13, Prop. 3.7], we approximate $Q_t^B = \psi_t(D_\rho B_{\rho,r})$ with a dyadic paraproduct $\gamma_t E_t$ by writing

$$Q_t^B u = Q_t^B(I - P_t)u + (Q_t^B - \gamma_t E_t)P_t u + \gamma_t E_t(P_t - I)u + \gamma_t E_t u.$$

By [13, Prop. 3.1(i)], we may assume that $u \in \mathsf{Ran}(D_\rho)$. Note that $\mathsf{Ran}(D_\rho)$ is now a closed subspace, as compared to [13]. As in [13, Defn. 3.5], the principal part γ_t of Q_t^B is defined to be the multiplication operator $\gamma_t(x)z = (Q_t^B z)(x)$, $x \in \mathbb{R}^n$, where we view $z \in \mathbb{C}^{2+n}$ as a constant function on $\mathbb{R}^n$.

The mollification operators P_t and E_t, on scale t, are

$$P_t u = \begin{bmatrix} (I - t^2 \Delta_{\widetilde{\omega}_\rho})^{-1} u_\perp \\ (I - t^2 \Delta)^{-1} u_0 \\ (I - t^2 \Delta)^{-1} u_\| \end{bmatrix},$$

where $\Delta_{\widetilde{\omega}_\rho} = \mathrm{div}_{\widetilde{\omega}_\rho} \nabla$, and with $\Delta = \mathrm{div}\, \nabla$ acting componentwise, and

$$E_t u(x) = \begin{bmatrix} \fint_Q u_\perp \widetilde{\omega}_\rho dx \\ \fint_Q u_0 dx \\ \fint_Q u_\| dx \end{bmatrix},$$

where Q denotes the dyadic cube on scale t which contains x. See [13, Sec. 2.2]. Note the new u_0 component as compared to [13].

The first term $Q_t^B (I - P_t)u$, assuming $u \in \mathsf{Ran}(D_\rho)$, is estimated as in [13, Lem. 3.10], now using the identity

$$(I - P_t)u = tD_\rho \begin{bmatrix} t(-\Delta)^{1/2}(I - t^2 \Delta)^{-1}(-\Delta)^{1/2} u_0 \\ 0 \\ t\nabla(I - t^2 \Delta_{\widetilde{\omega}_\rho})^{-1} u_\perp \end{bmatrix}, \qquad u = \begin{bmatrix} u_\perp \\ u_0 \\ \nabla u_0 \end{bmatrix}.$$

The second term $(Q_t^B - \gamma_t E_t)P_t u$ is estimated as in [13, Lem. 3.11], using polynomial off-diagonal estimates for $D_\rho B_{\rho,r}$ on $\mathbb{R}^n$. The third term $\gamma_t E_t(P_t - I)u$ is estimated as in [13, Lem. 3.12-14], with the new u_0 component estimated in the same way as $u_\|$.

The last term $\gamma_t E_t u$ is estimated using the double stopping time argument in [13, Sec. 3.5-7], and this estimate requires that $t < 1$ (which we may assume since $\mathsf{Ran}(D_\rho)$ is closed). The only adaption needed in this proof is in [13, Lem. 3.16], where now

$$E_{Q_1}(f_{Q_1}^\xi) - \xi = \begin{bmatrix} \fint_Q (u_0 - \mathrm{div}_{\widetilde{\omega}_\rho} u_\|) \widetilde{\omega}_\rho dx \\ \fint_Q u_\perp dx \\ \fint_Q \nabla u_\perp dx \end{bmatrix},$$

using the notation of [13, Lem. 3.16]. Besides [13, Lem. 3.12-13], we require the following two estimates. For u_0, we have

$$\left| \fint_{Q_1} u_0 \widetilde{\omega}_\rho dx \right| \leq \left(\fint_{Q_1} |u_0|^2 \widetilde{\omega}_\rho dx \right)^{1/2}$$

$$\lesssim \sigma_3 \ell(Q_1) \left(\int_{Q_1} \widetilde{\omega}_\rho dx \right)^{-1/2} \left(\int_{\mathbb{R}^n} |1_{Q_1} \xi|^2 \widetilde{\omega}_\rho dx \right)^{1/2} \lesssim \sigma_3,$$

provided $\ell(Q_1) \lesssim 1$. For $u_\perp$, we have

$$\left| \fint_{Q_1} u_\perp dx \right| \leq \frac{1}{|Q_1|} \left(\int_{Q_1} |u_\perp|^2 \widetilde{\omega}_\rho dx \right)^{1/2} \left(\int_{Q_1} \widetilde{\omega}_\rho^{-1} dx \right)^{1/2}$$

$$\lesssim \left(\fint_{Q_1} |u_\perp|^2 \widetilde{\omega}_\rho dx \right)^{1/2} \lesssim \sigma_3,$$

using the A_2 condition on $\widetilde{\omega}_\rho$ and estimating as for u_0 at the end.

Replacing $\mathbb{C}^{1+n}$ by $\mathbb{C}^{2+n}$, the rest of the argument from [13, Sec. 3] goes through, and completes the proof. $\qquad\square$

Proof of Theorem 1.1. To compare $\psi_t(BD)$ and $\psi_t(D_\rho B_{\rho,r})$, we use a fixed smooth cut-off function $\eta : \mathbb{R}^n \to [0,1]$ such that

$$\mathrm{sppt}\,\eta \subset \Omega \quad \text{and} \quad \eta \equiv 1 \text{ on } \tfrac{1}{2}\Omega.$$

Denote the resolvents by $R_t = (I + itDB)^{-1}$ and $R_t^{\rho,r} = (I + itD_\rho B_{\rho,r})^{-1}$. Consider $u \in L^2(\mathcal{V},\omega)$ with $\mathrm{sppt}\,u \subseteq 4B_i$. We have

$$\begin{aligned}
R_t u &= R_t T_\rho^{*-1} T_\rho^*((\eta \circ \rho^{-1})u) = R_t T_\rho^{*-1} \eta T_\rho^* u \\
&= [R_t T_\rho^{*-1}\eta - T_\rho^{*-1}\eta R_t^{\rho,r}]T_\rho^* u + T_\rho^{*-1}\eta R_t^{\rho,r} T_\rho^* u \\
&= R_t T_\rho^{*-1}[\eta(I + itD_\rho B_{\rho,r}) - T_\rho^*(I + itDB)T_\rho^{*-1}\eta]R_t^{\rho,r} T_\rho^* u + T_\rho^{*-1}\eta R_t^{\rho,r} T_\rho^* u \\
&= R_t T_\rho^{*-1} it(\eta D_\rho B_{\rho,r} - D_\rho B_\rho \eta)R_t^{\rho,r} T_\rho^* u + T_\rho^{*-1}\eta R_t^{\rho,r} T_\rho^* u \\
&= R_t T_\rho^{*-1} it([\eta, D_\rho]B_\rho \eta_0^2 + r\eta D_\rho \mathbb{P}_0)R_t^{\rho,r} T_\rho^* u + T_\rho^{*-1}\eta R_t^{\rho,r} T_\rho^* u,
\end{aligned}$$

where in the second last line we used Lemma 3.3 and in the final line we used (3.8) to obtain

$$
(3.9) \quad
\begin{aligned}
\eta D_\rho &B_{\rho,r} - D_\rho B_\rho \eta \\
&= \eta D_\rho[B_\rho(\eta_0^2 I) + (1 - \eta_0^2)I + r\mathbb{P}_0] - D_\rho B_\rho \eta \\
&= [\eta, D_\rho]B_\rho \eta_0^2 + D_\rho B_\rho \eta(\eta_0^2 - 1) + \eta D_\rho(1 - \eta_0^2) + r\eta D_\rho \mathbb{P}_0 \\
&= [\eta, D_\rho]B_\rho \eta_0^2 + r\eta D_\rho \mathbb{P}_0,
\end{aligned}
$$

since $\eta_0 = 1$ on $\mathrm{sppt}\,\eta$.

We now note the identity $\psi_t(DB) = \frac{1}{2i}(R_{-t} - R_t)$, the pointwise commutator estimate $|[\eta, D_\rho]u(x)| \leq |\nabla\eta(x)||u(x)|$, and that $D_\rho\mathbb{P}_0(u_\perp, u_0, u_\parallel) = (u_0, 0, 0)$ is a bounded operator. This shows that

$$\|1_{4B_i}\psi_t(DB)1_{4B_i}u\|^2_{L^2(\mathcal{V},\omega)} \lesssim t^2\|1_{4B_i}u\|^2_{L^2(\mathcal{V},\omega)} + \|\psi_t(D_\rho B_{\rho,r})T_\rho^*(1_{4B_i}u)\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^{2+n},\widetilde{\omega}_\rho)}$$

for all $t \in (0,T)$, all $u \in L^2(\mathcal{V},\omega)$ and all $i \in \mathbb{N}$. Theorem 3.5 therefore provides the estimate

$$\begin{aligned}
I_1 &\lesssim \sum_{i\in\mathbb{N}}\left(\int_0^T t\|1_{4B_i}u\|^2_{L^2(\mathcal{V},\omega)}\,dt + \int_0^T \|\psi_t(D_\rho B_{\rho,r})(T_\rho^* 1_{4B_i}u)\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^{2+n},\widetilde{\omega}_\rho)}\frac{dt}{t}\right) \\
&\lesssim \sum_{i\in\mathbb{N}}\left(\|1_{4B_i}u\|^2_{L^2(\mathcal{V},\omega)} + \|T_\rho^* 1_{4B_i}u\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^{2+n},\widetilde{\omega}_\rho)}\right) \\
&\lesssim \|u\|^2_{L^2(\mathcal{V},\omega)},
\end{aligned}$$

which completes the proof of Theorem 1.1. $\qquad\square$

Proof of Theorem 1.2. In the splitting

$$L^2(\mathcal{V},\omega) = L^2(M,\omega) \oplus L^2((M\times\mathbb{C}) \oplus TM,\omega),$$

we consider operators

$$D = \begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & 0 \\ 0 & \tilde{A} \end{bmatrix},$$

where $S = \begin{bmatrix} I \\ \nabla_M \end{bmatrix}$ and $\tilde{A} = \begin{bmatrix} d & c \\ b & A \end{bmatrix}$. It follows from the accretivity hypotheses that B is accretive on $\mathsf{Ran}(D)$. Moreover, $\mathsf{Ran}(DB) = \mathsf{Ran}(D)$ is closed since $\mathsf{Ran}(S)$ is closed. Theorem 1.1 proves quadratic estimates for DB. The H^∞ functional calculus bounds then yield

$$(3.10) \qquad \|\sqrt{(BD)^2}f\|_{L^2(\mathcal{V},\omega)} \eqsim \|BDf\|_{L^2(\mathcal{V},\omega)} \eqsim \|Df\|_{L^2(\mathcal{V},\omega)},$$

since the symbol $\lambda/\sqrt{\lambda^2}$ is self-inverse and belongs to $H^\infty(S_\nu)$. Since $(BD)^2 = \begin{bmatrix} aS^*\tilde{A}S & 0 \\ 0 & \tilde{A}SaS^* \end{bmatrix}$, the Kato square root estimate in Theorem 1.2 is the special case $f = \begin{bmatrix} u \\ 0 \end{bmatrix}$ of (3.10). $\qquad \square$

Remark 3.7. We remark that the full strength of the double stopping time argument for weights and test functions from [13, Sec. 3.5-7], that Theorem 1.1 ultimately builds on, is not needed to prove Theorem 1.2. The reason is that Theorem 1.2 only uses B with a block form structure. See [13, Sec. 3.8].

4. Degenerate boundary value problems

4.1. **The boundary DB operator.** We consider second-order elliptic divergence form systems

$$(4.1) \qquad \mathrm{div}A\nabla u = 0$$

on a compact manifold Ω, with Lipschitz boundary $\partial\Omega$. We assume without loss of generality that Ω is connected, but not necessarily so for $\partial\Omega$, thus allowing for geometries such as annuli. In general u may take values in some $\mathbb{C}^N$, but to simplify the presentation we suppress this in notation. As in [13], we only assume that the coefficents are degenerate elliptic in the sense that $A/\omega \in L^\infty(\mathrm{End}(T\Omega))$ is accretive as in (4.8) and (4.9) below, for a given weight $\omega \in A_2(\Omega)$. By u being a weak solution to (4.1) we mean that $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \omega)$, so in particular $A\nabla u \in L^1_{\mathrm{loc}}(\Omega)$, and that $\mathrm{div}(A\nabla u) = 0$ holds in distributional sense.

We let M be a closed Riemannian manifold (smooth in the sense of Section 2.1) and consider the cylinder $\mathbb{R} \times M$. We assume that there exists a bi-Lipschitz map

$$(4.2) \qquad \rho : [0,\delta) \times M \to U \subset \Omega$$

between a finite part of the cylinder $\mathbb{R} \times M$ and a neighbourhood U of $\partial\Omega$, for some $\delta > 0$, with $\rho(\{0\} \times M) = \partial\Omega$ so that in particular $\partial\Omega$ is bi-Lipschitz homeomorphic to M. Write $M_\rho = (0,\delta) \times M$. For example, in [12] we used $\rho(t,x) = e^{-t}x$, M being the unit sphere. As in Example 2.2, the function $u_\rho = u \circ \rho$ is a weak solution to the elliptic divergence form system

$$(4.3) \qquad \mathrm{div}A_\rho\nabla u_\rho = 0$$

on the cylinder M_ρ, where $A_\rho = J_\rho(d\rho)^{-1}A(d\rho^*)^{-1}$. In M_ρ, we assume that

$$\omega_\rho = \omega \circ \rho$$

is independent of $t \in (0,\delta)$. We write $\omega_0 = \omega_\rho|_M$ and note that $\omega_0 \in A_2(M)$ by t-independence. This is natural since even in the uniformly elliptic case $\omega = 1$, it is well known that some regularity of $t \to A_\rho$ is necessary for the results that we aim

at. In Ω, we use the measure $d\omega = \omega d\nu$ and on M we use the measure $d\omega_0 = \omega_0 d\mu$, where $d\nu$ and $d\mu$ denote the Riemannian measures on Ω and M respectively.

Following [5, 13], we analyze u_ρ on M_ρ by rewriting (4.3) as a generalized Cauchy–Riemann system

$$(4.4) \qquad \partial_t f + D_0 B f = 0, \qquad 0 < t < \delta, x \in M,$$

for the conormal gradient f of u_ρ at layer t, with $D_0 = \begin{bmatrix} 0 & \mathrm{div}_{M,\omega_0} \\ -\nabla_M & 0 \end{bmatrix}$ having domain $\mathsf{Dom}(D_0) = W^{1,2}(M, \omega_0) \oplus \mathsf{Dom}(\mathrm{div}_{M,\omega_0})$. Writing $A_\rho = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in the splitting $T(M_\rho) \simeq \mathbb{C} \times TM$, we define the conormal gradient of u_ρ to be

$$(4.5) \qquad f = (f_t)_{0<t<\delta} = \begin{bmatrix} \omega_\rho^{-1} \partial_{\nu_{A_\rho}} u_\rho \\ \nabla_M u_\rho \end{bmatrix},$$

where $\partial_{\nu_{A_\rho}} u_\rho = a \partial_t u_\rho + b \nabla_M u_\rho$ is the conormal derivative of u_ρ. We write f_t for the function $f(t, \cdot)$ and sometimes, abusing notation, also for the function f. Note that the boundary data $\partial_{\nu_{A_\rho}} u_\rho$ and $\nabla_M u_\rho$ on M essentially are the same as $\partial_{\nu_A} u$ and $\nabla_{\partial\Omega} u$ on $\partial\Omega$. Indeed, assuming that these boundary traces exist, we have

$$(4.6) \qquad \partial_{\nu_{A_\rho}} u_\rho = J_{\rho|_M} \partial_{\nu_A} u \circ \rho \quad \text{and} \quad \nabla_M u_\rho = (d\rho|_M)^* \nabla_{\partial\Omega} u \circ \rho,$$

where $J_{\rho|_M}$ and $(d\rho|_M)^*$ are invertible. The transformed accretive coefficients $B = (B_t)_{0<t<\delta}$ appearing in (4.4) are

$$(4.7) \qquad B = \begin{bmatrix} \omega_\rho a^{-1} & -a^{-1} b \\ ca^{-1} & \omega_\rho^{-1}(d - ca^{-1}b) \end{bmatrix}.$$

We remark that $d - ca^{-1}b$ is a well known object in matrix theory called *the Schur complement* of a in the matrix A_ρ, and is denoted by A_ρ/a. All this follows as in [5, Sec. 4], replacing $(0, \infty)$ and $\mathbb{R}^n$ by $(0, \delta)$ and M, with modifications as in [13] in the degenerate case. In particular, the scaling of f by ω_ρ means that weak solutions $u \in W^{1,2}(\Omega, \omega)$ have conormal gradients $f \in L^2_{\mathrm{loc}}((0, \delta), \overline{L}^2)$ on M_ρ, where $\overline{L}^2 = L^2(M, \omega_0) \oplus L^2(TM, \omega_0)$. Below in this section, $\|f\| = (\int_M |f|^2 d\omega_0)^{1/2}$ always refer to this weighted $\overline{L}^2$ norm, unless otherwise indicated. By $f \in L^2_{\mathrm{loc}}((0, \delta), \overline{L}^2)$ being a weak solution to (4.4), we mean that this equation holds in distributional sense on M_ρ. Note that f and Bf in particular belong to $L^1_{\mathrm{loc}}((M_\rho \times \mathbb{C}) \oplus TM_\rho)$.

The accretivity assumptions on A that we need are

$$(4.8) \qquad \mathrm{Re} \int_\Omega \langle A\nabla u, \nabla u \rangle d\nu \gtrsim \int_\Omega |\nabla u|^2 d\omega, \quad u \in W^{1,2}(\Omega, \omega),$$

$$(4.9) \qquad \int_M \langle A_\rho(t, x) \nabla v(t, x), \nabla v(t, x) \rangle d\mu(x) \gtrsim \|\nabla v(t, x)\|^2, \quad v \in W^{1,2}(M_\rho, \omega_\rho),$$

for almost every $t \in (0, \delta)$. The second stronger assumption near M is needed to obtain a bisectorial boundary operator. In particular it ensures that the matrix element a_t in A_ρ is invertible, since it follows from (4.9) that a_t is accretive on all $L^2(M)$, and that B_t are bounded and accretive. See [12, Sec. 3] for this argument.

The subspace

$$\mathcal{H} = \mathsf{Ran}(D_0) = \mathsf{Ran}(\mathrm{div}_{M,\omega_0}) \oplus \mathsf{Ran}(\nabla_M) \subset \overline{L}^2$$

plays a central role in our analysis.

Proposition 4.1. The subspace $\mathcal{H}$ is closed. Moreover, the restriction of the self-adjoint operator $D_0 : \mathsf{Dom}(D_0) \subset \overline{L}^2 \to \overline{L}^2$ to $\mathcal{H}$ has a compact inverse $D_0^{-1} : \mathcal{H} \to \mathcal{H}$.

Proof. Recall from Lemma 2.3 that $\mathrm{div}_{M,\omega_0} = -(\nabla_M)^*$. It also follows from weighted Poincaré inequalities in [19, Thm. 1.5] that ∇_M has a compact left Fredholm inverse. In particular $\mathsf{Ran}(\nabla_M)$ is closed and it follows from the closed range theorem, see [26, Thm. IV.5.13], that $\mathsf{Ran}(\mathrm{div}_{M,\omega_0})$ is closed. Hence $\mathsf{Ran}(D_0)$ is closed. Since compactness is preserved when taking adjoints, this also shows that $D_0^{-1} : \mathcal{H} \to \mathcal{H}$ is compact. $\qquad\square$

All the main estimates of the operators below build on the following quadratic estimate for the boundary operator $D_0 B_0$, where $B_0 = \lim_{t \to 0} B_t$. The precise definition of this limit involves the Carleson norm

$$(4.10) \qquad \|\mathcal{E}\|_* = \|C(W_\infty(|\mathcal{E}|^2/t))\|_\infty^{1/2},$$

where

$$Cg(x) = \sup_{r<\delta} \left(\int_{B(x,r)} d\omega_0 \right)^{-1} \int_{(0,r) \times B(x,r)} |g(t,y)| dt d\omega_0(y)$$

and $(W_\infty g)(t,x) = \mathrm{ess}\,\sup_{W(t,x)} |g|$, using Whitney regions $W(t,x) = (t/c_0, c_0 t) \times B(x, c_1 t)$, for fixed $c_0 > 1, c_1 > 0$. We assume that there exists $B_0 \in L^\infty(M; \mathrm{End}(\mathbb{C} \times TM))$ such that $\|B_t - B_0\|_* < \infty$. As in [5, Lem. 2.2] (where $\|\cdot\|_C$ equals $\|\cdot\|_*$, but without the restriction $r < \delta$), such trace B_0 is unique determined by $B = (B_t)_{0<t<\delta}$ and (4.9) implies the accretivity (2.6) for B_0, with $D, B, (\mathcal{V}, \omega)$ replaced by D_0, B_0, $(\mathbb{C} \times TM, \omega_0)$. Also let $A_0 = \lim_{t \to 0} A_\rho$, in the sense that $\|A_\rho/\omega_\rho - A_0/\omega_0\|_* < \infty$.

Proposition 4.2. For any bisectorial operator in $\overline{L}^2$ of the form $D_0 B_0$, where D_0 is the operator appearing in (4.4) and B_0 is any bounded multiplication operator which is accretive on $\mathsf{Ran}(D_0)$, we have

$$\int_0^\infty \|t D_0 B_0 (I + t^2 (D_0 B_0)^2)^{-1} f\|^2 \frac{dt}{t} \lesssim \|f\|^2, \qquad \text{for all } f \in \overline{L}^2.$$

Proof. In the splitting

$$L^2(M, \omega) \oplus L^2(M, \omega) \oplus L^2(TM, \omega)$$

we consider the operator D from Theorem 1.1, given by (2.4). Given $B_0 = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$,

we define $\widetilde{B} = \begin{bmatrix} a' & 0 & b' \\ 0 & I & 0 \\ c' & 0 & d' \end{bmatrix}$. It follows that $\widetilde{B}$ is accretive on $\mathsf{Ran}(D)$ in the sense

(2.6). Since M is compact, the geometric hypotheses in Theorem 1.1 hold, giving quadratic estimates for $D\widetilde{B}$. We have

$$\begin{bmatrix} 0 & 0 & -\mathrm{div}_{M,\omega_0} \\ 0 & 0 & 0 \\ \nabla_M & 0 & 0 \end{bmatrix} \begin{bmatrix} a' & 0 & b' \\ 0 & 0 & 0 \\ c' & 0 & d' \end{bmatrix} = D\widetilde{B} + V$$

for some bounded operator V. The left hand side is unitary equivalent to $-D_0 B_0 \oplus 0$, so Proposition 4.1 shows that $\mathsf{Ran}(D\widetilde{B} + V)$ is closed. Lemma 2.4 now proves that quadratic estimates also hold for $D_0 B_0$. $\qquad\square$

As in Section 2.3, it follows from Proposition 4.2 that $D_0 B_0$ has a bounded H^∞ functional calculus (the quadratic estimates for the adjoint $B_0^* D_0$ hold by similarity, since $B_0^* D_0 = B_0^* (D_0 B_0^*)(B_0^*)^{-1}$ on $B_0^* \mathsf{Ran}(D_0) = \mathsf{Ran}(B_0^* D_0)$). In particular we use the spectral projections $E_0^+ = \chi^+(D_0 B_0)$ and $E_0^- = \chi^-(D_0 B_0)$ for the right and left half planes, given by the symbols

$$\chi^\pm(\lambda) = \begin{cases} 1, & \pm \operatorname{Re} \lambda > 0, \\ 0, & \pm \operatorname{Re} \lambda \le 0. \end{cases}$$

The spectral subspaces $\mathsf{Ran}(E_0^+)$ and $\mathsf{Ran}(E_0^-)$ are closed complementary subspaces of $\mathcal{H}$, which generalize the classical Hardy subspaces from complex analysis, and we denote them $E_0^\pm \mathcal{H}$. We set $E_0^0 = I - E_0^+ - E_0^-$, which is the projection onto $\mathsf{Nul}(D_0 B_0)$ along $\mathcal{H}$. We also use the bounded Poisson semigroup $e^{-t\Lambda}$, with $\Lambda = |D_0 B_0|$, which are the operators given by the symbols $e^{-t|\lambda|}$, where $|\lambda| = \lambda \operatorname{sgn}(\lambda)$ and $\operatorname{sgn}(\lambda) = \chi^+(\lambda) - \chi^-(\lambda)$.

Similar to [12, Prop. 3.3] it follows that, for each fixed $0 < t < \delta$, the conormal gradient f of u_ρ almost belongs to $\mathcal{H}$. The following lemma takes care of this technical issue.

Lemma 4.3. Consider a weak solution $f = (f_t)_{0<t<\delta} \in L^2_{\mathrm{loc}}((0,\delta), \overline{L}^2)$ to (4.4). Then $E_0^0 f_t \in \overline{L}^2$ is independent of t and belongs to a finite dimensional subspace of $\overline{L}^2$. Denoting this constant value by $E_0^0 f_0$, we have

$$\| E_0^0 f_0 \| \simeq \sum_i \Big| \int_{M_i} \partial_{\nu_{A_0}} u_\rho \, d\mu \Big|,$$

where M_i denote the connected components of M.

A simple example showing that $E_0^0 f_0$ may be non-trivial is $u(t,x) = t$ on $\Omega = (0,1) \times S^1$.

Proof. Applying E_0^0 to (4.4) immediately shows that $\partial_t E_0^0 f_t = 0$. To see the norm estimate, we consider the orthogonal projection P onto $\mathcal{H}^\perp$ in $\overline{L}^2$. Since $\mathsf{Nul}(E_0^0) = \mathsf{Nul}(P) = \mathcal{H}$ it follows that $\| E_0^0 f_t \| \simeq \| P f_t \|$. Indeed $\| E_0^0 f_t \| = \| E_0^0 (P f_t + (I - P) f_t) \| = \| E_0^0 P f_t \| \lesssim \| P f_t \|$, and the reverse estimate is proved similarly. Since $\nabla_M u_\rho \in \mathsf{Ran}(\nabla_M)$ and $\mathsf{Ran}(\mathrm{div}_{M,\omega_0})^\perp = \mathsf{Nul}(\nabla_M)$, it follows that $P f_t = \begin{bmatrix} c \\ 0 \end{bmatrix}$, where c is the locally constant function given by the average of $\partial_{\nu_{A_\rho}} u_\rho$ on each component M_i. This shows the estimate of $E_0^0 f_t = E_0^0 P f_t$. Since the locally constant functions c form a finite dimensional space (a basis is given by 1_{M_i}) , this completes the proof. $\square$

Besides the $\overline{L}^2$ topology for f_0, we also obtain results for f_0 in a larger trace space $\overline{W}^{-1,2}$ which we now define. We use test functions v from the weighted Sobolev space $W^{1,2}(M, \omega_0)$ with norm

$$\| v \|_{W^{1,2}}^2 = \int_M (|\nabla v|^2 + |v|^2) \, d\omega_0.$$

Define dual spaces

$$W^{-1,2}(M, \omega_0^{-1}) = (W^{1,2}(M, \omega_0))^*,$$
$$W^{-1,2}(TM, \omega_0) = (W^{1,2}(TM, \omega_0^{-1}))^*.$$

Using the unweighted $L^2(M, d\mu)$ pairing, we realize $L^2(M, \omega_0^{-1})$ as a subspace of $W^{-1,2}(M, \omega_0^{-1})$ and $L^2(TM, \omega_0)$ as a subspace of $W^{-1,2}(TM, \omega_0)$. The weighted Sobolev $W^{-1,2}$ trace space that we use for f is

$$(4.11) \qquad \overline{W}^{-1,2} = \omega_0^{-1}W^{-1,2}(M, \omega_0^{-1}) \oplus W^{-1,2}(TM, \omega_0),$$

that is $\partial_{\nu_A}u \in W^{-1,2}(M, \omega_0^{-1})$ and $\nabla_M u \in W^{-1,2}(TM, \omega_0)$. Here $\omega_0^{-1}W^{-1,2}(M, \omega_0^{-1})$ denotes the space dual to $\omega_0 W^{1,2}(M, \omega_0)$. Note that the $W^{1,2}$ and $W^{-1,2}$ analogues of $\omega_0^{-1}L^2(M, \omega_0^{-1}) = L^2(M, \omega_0)$ do not hold since in general ω_0 is not differentiable.

Lemma 4.4. The Sobolev space $\overline{W}^{-1,2}$ contains $\overline{L}^2$ as a dense subspace. Denote by $\mathcal{H}^{-1}$ the closure of $\mathcal{H}$ in $\overline{W}^{-1,2}$. The operators D_0, D_0B_0 and Λ all have $\overline{L}_2$ domains which are dense in $\mathcal{H}$. They all extend uniquely to isomorphisms $\mathcal{H} \to \mathcal{H}^{-1}$.

Proof. The denseness follows from a localization and mollification argument as in Lemma 2.3, and duality. To prove the mapping properties $\mathcal{H} \to \mathcal{H}^{-1}$, consider first the operator D_0.

We have a semi-Fredholm map $\nabla_M : W^{1,2}(M, \omega_0) \to L^2(TM, \omega_0)$, by weighted Poincaré inequalities [19, Thm. 1.5]. By duality this yields a semi-Fredholm map $L^2(TM, \omega_0) \to \omega_0^{-1}W^{-1,2}(M, \omega_0^{-1})$, which by Lemma 2.3 is seen to coincide with div_{M,ω_0} on the dense subspace $\text{Dom}(\text{div}_{M,\omega_0}) \subset L^2(TM, \omega_0)$. We continue to write div_{M,ω_0} for this unique continuous extension to $L^2(TM, \omega_0)$.

Similarly we have a bounded operator $\text{div}_M : W^{1,2}(TM, \omega_0^{-1}) \to L^2(M, \omega_0^{-1})$. This is also a semi-Fredholm map, which can be proved through weighted estimates of regularized Poincaré maps for the divergence. See [29, Sec. 10.4] for the unweighted case, which can be extended to the weighted case, similar to [19, Thm. 1.5]. By duality this yields a semi-Fredholm map $L^2(M, \omega_0) \to W^{-1,2}(TM, \omega_0)$, which by Lemma 2.3 is seen to coincide with ∇_M on the dense subspace $W^{1,2}(M, \omega_0) \subset L^2(M, \omega_0)$. We continue to write ∇_M for this unique continuous extension to $L^2(M, \omega_0)$.

This yields a unique continuous extension of D_0 to a semi-Fredholm operator $\overline{L}^2 \to \overline{W}^{-1,2}$, with null space $\mathcal{H}^\perp \subset \overline{L}^2$ and range being the closure of $\mathcal{H}$ in $\overline{W}^{-1,2}$. Hence the restriction $D_0 : \mathcal{H} \to \mathcal{H}^{-1}$ is an isomorphism. For the operator D_0B_0 we note that the accretivity of B_0 on $\mathcal{H}$ shows that $PB_0 : \mathcal{H} \to \mathcal{H}$ is an isomorphism, where P denotes the orthogonal projection $\overline{L}^2 \to \mathcal{H}$. Therefore $D_0B_0 = D_0(PB_0) : \mathcal{H} \to \mathcal{H}^{-1}$ is an isomorphism. For the operator $\Lambda = D_0B_0 \text{sgn}(D_0B_0)$ we note that $\text{sgn}(D_0B_0) : \mathcal{H} \to \mathcal{H}$ is an isomorphism, which yields the isomorphism $\Lambda : \mathcal{H} \to \mathcal{H}^{-1}$. $\qquad \square$

We want to define $b(D_0B_0)$ in the functional calculus of D_0B_0 as operators on the closed subspace $\mathcal{H}^{-1}$ and to show that these are bounded in the $\overline{W}^{-1,2}$ norm. Using Lemma 4.4, we extend operators $b(D_0B_0) : \mathcal{H} \to \mathcal{H}$ in the functional calculus of D_0B_0, for $b \in H^\infty(S_\theta^o)$, to bounded operators $b(D_0B_0) : \mathcal{H}^{-1} \to \mathcal{H}^{-1}$, by setting

$$b(D_0B_0)f = \Lambda(b(D_0B_0)\Lambda^{-1}f), \qquad f \in \mathcal{H}^{-1}.$$

(Λ can be replaced by D_0B_0 in this definition.) It appears problematic to extend the operators $b(D_0B_0)$ to all of $\overline{W}^{-1,2}$, but we shall not need to do so. We only need to act with $E_0^\pm$ on solutions $f_0 \in \overline{W}^{-1,2}$ to (4.4). To this end, we recall from Lemma 4.3 that $E_0^0 f_0 \in \overline{L}^2$, even if $f_0 \in \overline{W}^{-1,2}$, since $t \mapsto E_0^0 f_t$ is constant. With this in mind,

$$E_0^\pm f_0 = E_0^\pm(f_0 - E_0^0 f_0)$$

is well defined, since $f_0 - E_0^0 f_0 \in \mathcal{H}^{-1}$ and $E_0^{\pm} E_0^0 = 0$.

4.2. **Atiyah-Patodi-Singer conditions.** Given a weak solution u to $\operatorname{div} A \nabla u = 0$ in Ω, with conormal gradient f on M_ρ, we prove in this section existence and estimates of a trace f_0 on M as $t \to 0$. The spectral projections for the bisectorial operator $D_0 B_0$ give the splitting

$$(4.12) \qquad f_0 = E_0^+ f_0 + E_0^0 f_0 + E_0^- f_0.$$

The part $E_0^+ f_0$ constitutes the APS boundary datum in our setting, and we prove below equivalences of norms between the APS datum $E_0^+ f_0$ and the solution ∇u, modulo the part $E_0^0 f_0$, which takes values in a finite dimensional space. We summarize this in the next two statements.

For boundary L^2 topology for f_0, we use the following non-tangential maximal function for ∇u. Given a function f on the cylinder, supported on M_ρ, we define the L^2 Whitney averaged non-tangential maximal function

$$\widetilde{N}_* f(x) = \sup_{0 < t < c_0 \delta} (W_2 f)(t, x), \qquad x \in M,$$

where $(W_2 f)(t, x) = \left(\fint_{W(t,x)} |f|^2 ds d\omega_0 \right)^{1/2}$.

Let also $\eta_t = \max(0, \min(1, 2 - 2t/\delta))$. With slight abuse of notation we also consider η as a function on M_ρ, and on Ω by zero extension. Our APS solvability estimate for the $\overline{L}^2$ boundary function space is the following.

Theorem 4.5. *Let Ω, ρ, M, ω, A and B_t be as in Section 4.1. Assume that there exists t-independent coefficients $B_0 = B_0(x)$ on M_ρ, so that $\|B_t - B_0\|_* < \infty$. Consider a weak solution u to (4.1), with conormal gradient f of u_ρ as in (4.5), where ∇u belongs to the Banach space $\mathcal{X} \subset L^2(T\Omega, \omega)$ with norm*

$$(4.13) \qquad \|g\|_{\mathcal{X}}^2 = \int_M |\widetilde{N}_*(\eta g \circ \rho))|^2 \omega_0 d\mu + \int_\Omega |(1 - \eta) g|^2 \omega d\nu.$$

Then the trace $\nabla u|_{\partial \Omega}$ exists in the sense that

$$(4.14) \qquad \lim_{t \to 0} t^{-1} \int_t^{2t} \|f_s - f_0\|^2 ds = 0,$$

for some $f_0 \in \overline{L}^2$ with

$$(4.15) \qquad \|f_0\| \lesssim \|\nabla u\|_{\mathcal{X}}.$$

If $\|B_t - B_0\|_*$ is small enough, for some $\delta > 0$, depending on $[\omega_0]_{A_2(M)}$, $\|A|_M/\omega_0\|_\infty$ and the constant in (4.9), then we also have the estimate

$$(4.16) \qquad \|\nabla u\|_{\mathcal{X}} \lesssim \|E_0^+ f_0\| + \|E_0^0 f_0\|,$$

with the implicit constant also depending on the constant in (4.8).

Our APS solvability estimate for the $\overline{W}^{-1,2}$ boundary function space is the following.

Theorem 4.6. *Let Ω, ρ, M, ω, A and B_t be as in Section 4.1. Assume that there exists t-independent coefficients $B_0 = B_0(x)$ on M_ρ, so that $\|B_t - B_0\|_* < \infty$.*

Consider a weak solution u to (4.1), with conormal gradient f of u_ρ as in (4.5), where ∇u belongs to the Hilbert space $\mathcal{Y} \supset L^2(T\Omega, \omega)$, with norm

$$(4.17) \qquad \|g\|_{\mathcal{Y}}^2 = \int_M \int_0^\delta |\eta_t g(\rho(t,x))|^2 \, t dt \, \omega_0 d\mu(x) + \int_\Omega |(1-\eta)g|^2 \omega d\nu.$$

Then the trace $\nabla u|_{\partial\Omega}$ exists in the sense that

$$(4.18) \qquad \lim_{t \to 0} \|f_t - f_0\|_{\overline{W}^{-1,2}} = 0,$$

for some $f_0 \in \overline{W}^{-1,2}$ with

$$(4.19) \qquad \|f_0\|_{\overline{W}^{-1,2}} \lesssim \|\nabla u\|_{\mathcal{Y}}.$$

If $\|B_t - B_0\|_*$ is small enough, for some $\delta > 0$, depending on $[\omega_0]_{A_2(M)}$, $\|A|_M/\omega_0\|_\infty$ and the constant in (4.9), then then we also have the estimate

$$(4.20) \qquad \|\nabla u\|_{\mathcal{Y}} \lesssim \|E_0^+ f_0\|_{\overline{W}^{-1,2}} + \|E_0^0 f_0\|_{\overline{W}^{-1,2}},$$

with the implicit constant also depending on the constant in (4.8).

We recall that in both Theorems 4.5 and 4.6, the norm of the finite dimensional part $E_0^0 f_0$ is equivalent to the sum of the integrals of the conormal derivative over the connected components of M, by Lemma 4.3.

The main computation for the proofs of Theorem 4.5 and 4.6 is the following. Consider a conormal gradient $f \in L^2_{\text{loc}}((0,\delta), \overline{L}^2)$ solving (4.4), which we multiply by η_t and rewrite as

$$(4.21) \qquad (\partial_t + D_0 B_0)(\eta_t f_t) = D_0 B_0 \mathcal{E}_t \eta_t f_t + \eta'_t f_t, \qquad 0 < t < \delta,$$

with the Carleson multiplier $\mathcal{E}_t = B_0^{-1}(B_0 - B_t)$. (Our accretivity assumptions do in general only imply the existence of B_0^{-1} on $\mathcal{H}$. This technical issue is fixed as in [5, Eqn. (22)].) We integrate (4.21) using operators in the H^∞ functional calculus of $D_0 B_0$, bounds of which are supplied by Proposition 4.2. Splitting (4.21) with the projections E_0^+, E_0^- and E_0^0, we get

$$(\partial_t + \Lambda)(\eta_t E_0^+ f_t) = \Lambda E_0^+ \mathcal{E}_t \eta_t f_t + \eta'_t E_0^+ f_t,$$
$$(\partial_t - \Lambda)(\eta_t E_0^- f_t) = -\Lambda E_0^- \mathcal{E}_t \eta_t f_t + \eta'_t E_0^- f_t,$$
$$\partial_t(\eta_t E_0^0 f_t) = \eta'_t E_0^0 f_t.$$

Integrating each of these equation and adding the obtained integral equations yields

$$(4.22) \qquad \eta_t f_t = e^{-t\Lambda} h^+ + S(\mathcal{E}_t \eta_t f_t) + \widetilde{S}(\eta'_t f_t) + \eta_t E_0^0 f_t, \qquad 0 < t < \delta.$$

See the proof of [5, Thm. 8.2] for the detailed calculations for the $E_0^\pm$ equations. In view of Lemma 4.3, the addition of the E_0^0 equation is a trivial modification.

In the third term, $\eta' f$ is supported on $\delta/2 \le t \le \delta$, and we show below compactness properties of the operator

$$\widetilde{S} g_t = \int_0^t e^{-(t-s)\Lambda} E_0^+ g_s ds - \int_t^\delta e^{-(s-t)\Lambda} E_0^- g_s ds, \qquad 0 < t < \delta.$$

The second term accounts for the t-variation of the coeffients and uses the operator

$$S g_t = \int_0^t \Lambda e^{-(t-s)\Lambda} E_0^+ g_s ds + \int_t^\delta \Lambda e^{-(s-t)\Lambda} E_0^- g_s ds, \qquad 0 < t < \delta.$$

As in [5, Prop. 7.1], Proposition 4.2 implies boundedness of $S\mathcal{E}$ on both $\mathcal{X}$ and $\mathcal{Y}$, with norm bounded by $\|\mathcal{E}\|_*$. In the first term, we obtain the existence of the boundary function h^+ on M, belong to the range of E_0^+, by a limiting argument in the equation (4.22). See [5, Thm. 8.2] for the space $\mathcal{X}$ and [5, Thm. 9.2] for the space $\mathcal{Y}$. In the fourth term we recall that $E_0^0 f_t = E_0^0 f_0$ is independent of t and represents a finite rank operator, by Lemma 4.3.

Letting $t = 0$ in (4.22), we note the formula

$$(4.23) \qquad f_0 = h^+ + E_0^- \int_0^\delta e^{-s\Lambda}(\Lambda\mathcal{E}_s\eta_s f_s - \eta'_s f_s)ds + E_0^0 f_0$$

for the full Cauchy trace f_0. Since $h^+ \in \mathsf{Ran}(E_0^+)$, this gives the splitting (4.12) of f_0.

Proof of Theorem 4.5. (i) Consider a weak solution u with $\nabla u \in \mathcal{X}$, so that $\|\widetilde{N}_*(\eta f)\| < \infty$. To prove the existence of the trace f_0, we use (4.22). The first and third terms converge in $\overline{L}^2$ as $t \to 0$, as in [5, Prop. 6.4]. The second term is seen to converge in the average L^2 sense (4.14) as in part (iii) of the proof of [5, Thm. 8.2]. The estimate $\|f_0\| \lesssim \|\nabla u\|_{\mathcal{X}}$ follows as in that proof. We conclude that there is a well defined trace f_0 satisfying (4.23) in $\overline{L}^2$.

To prove the reverse estimate (4.16), by the Open Mapping Theorem it suffices to show that $\nabla u \mapsto (E_0^+ + E_0^0)f_0$ is an injective semi-Fredholm operator, as a bounded map from the closed subspace of $\mathcal{X}$ consisting of gradients of solutions, to $\overline{L}^2$, which we prove next.

(ii) To prove injectivity, we define a cutoff function $\hat{\eta}_\epsilon = \max(0, \min(1, t/\epsilon - 1))$ on M_ρ and extend by $\eta_\epsilon = 1$ to the interior of Ω. For a weak solution to $\operatorname{div} A\nabla u = 0$, we have $\int_\Omega \langle A\nabla u, \nabla u\rangle\hat{\eta}_\epsilon d\nu = -\epsilon^{-1}\int_\epsilon^{2\epsilon}\int_M (\partial_{\nu_{A_\rho}} u_\rho)\overline{u}_\rho d\mu dt$. Letting $\epsilon \to 0$ and taking real parts, accretivity (4.8) shows that

$$(4.24) \qquad \int_\Omega |\nabla u|^2 d\omega \lesssim -\operatorname{Re}\int_M (\partial_{\nu_{A_0}} u_\rho)\overline{u}_0 d\mu$$

if $f_0 = \begin{bmatrix} \omega_0^{-1}\partial_{\nu_{A_0}} u_\rho|_M \\ \nabla_M u_0 \end{bmatrix}$. This uses (4.14) (which we established in (i) above) and $C([0,\delta), \overline{L}^2)$ continuity for u_t, which is proved as in [5, Prop. 7.2].

Assuming that $E_0^+ f_0 = 0$ and $E_0^0 f_0 = 0$, we have $f_0 = E_0^- f_0$. Defining $f_t = e^{-t\Lambda}f_0$ for $t < 0$, it follows that $\partial_t f_t + D_0 B_0 f_t = 0$ for $t < 0$ and that f_t is the conormal gradient of a solution u^- on $(-\infty, 0) \times M$, with L^2 limit f_0 as $t \to 0^-$. By suitable choices of constants of integration, for each connected component M_i, we may also assume that $u^- = u_0$ at $t = 0$. It follows from Proposition 4.1 that $\|f_t\| \lesssim |t|^{-N}$ for any $N < \infty$, as $t \to -\infty$. Another application of the Divergence Theorem shows that

$$(4.25) \qquad \operatorname{Re}\int_M (\partial_{\nu_{A_0}} u^-)\overline{u^-}d\mu = \operatorname{Re}\int_{-\infty}^0 \int_M \langle A_0\nabla u^-, \nabla u^-\rangle d\mu dt \geq 0,$$

where the inequality follows from accretivity (4.9). Matching the right hand side in (4.24) and the left hand side in (4.25), the continuity at $t = 0$ shows that $\int_\Omega |\nabla u|^2 d\omega \leq 0$, and it follows that $\nabla u = 0$ in Ω.

(iii) To show that the range of $\nabla u \mapsto (E_0^+ + E_0^0)f_0$ is closed, we note from (4.22) that

(4.26) $\quad \|1_{(0,\delta)}(I - S\mathcal{E}_t)(\eta_t f_t)\|_{\mathcal{X}} \leq \|1_{(0,\delta)}e^{-t\Lambda}h^+\|_{\mathcal{X}} + \|1_{(0,\delta)}\widetilde{S}(\eta_t' f_t)\|_{\mathcal{X}} + \|\eta_t E_0^0 f_t\|_{\mathcal{X}}.$

If $\|\mathcal{E}\|_*$ is small enough, then the left hand side is $\gtrsim \|\eta_t f_t\|_{\mathcal{X}}$. Weighted non-tangential maximal estimates, proved as in [5, Thm. 5.2] but using weighted estimates [13, Lem. 6.1] and Proposition 4.2, shows that the first term on the right hand side is $\lesssim \|h^+\| = \|E_0^+ f_0\|$. The last term is a finite rank operator, as discussed above, and is $\lesssim \|E_0^0 f_0\|$. For the second term, we claim that

$$1_{(0,\delta)}\widetilde{S}\eta_t' = 1_{(0,\delta/4)}\widetilde{S}\eta_t' + 1_{(\delta/4,\delta)}\widetilde{S}\eta_t' = I + II$$

is a compact operator on $\mathcal{X}$. For II, it suffices to show that

$$f_t \mapsto \int_{\delta/4}^t e^{-(t-s)\Lambda}E_0^+ g_s ds - \int_t^\delta e^{-(s-t)\Lambda}E_0^- g_s ds$$

is compact on $L^2((\delta/4,\delta), \mathcal{H})$. This follows from Proposition 4.1 by operational calculus as in [5, Prop. 6.4], since the symbols $F(\lambda) : L^2((\delta/4,\delta), \mathbb{C}) \to L^2((\delta/4,\delta), \mathbb{C})$ (replacing Λ by $\lambda \in \mathbb{C}$) are Hilbert–Schmidt operators on $L^2((\delta/4,\delta), \mathbb{C})$ with $\lim_{\lambda \to \infty} \|F(\lambda)\| = 0$. For I, we use the support of η_t' and compute

$$\int_{\delta/2}^\delta e^{-(s-t)\Lambda}E_0^- g_s ds$$

$$= \int_{\delta/2}^\delta e^{-(s-t)\Lambda}(I - e^{-2t\Lambda})E_0^- g_s ds + e^{-t\Lambda}\int_{\delta/2}^\delta e^{-s\Lambda}E_0^- g_s ds = I_1 + I_2.$$

For the first term, since $\mathcal{X} \supset L^2((0,\delta/4), t^{-1}dt, \mathcal{H})$, it suffices to show that $I_1 : L^2((\delta/2,\delta), \mathcal{H}) \to L^2((0,\delta/4), t^{-1}dt, \mathcal{H})$ is compact. Estimating $|e^{-(s-t)\lambda}(1-e^{-2t\lambda})| \lesssim t|\lambda|e^{-(\delta/4)\operatorname{Re}\lambda}$, this follows by operational calculus as for II. For I_2 we write

$$I_2 = e^{-t\Lambda}e^{-(\delta/4)\Lambda}\int_{\delta/2}^\delta e^{-(s-\delta/4)\Lambda}E_0^- g_s ds,$$

where the left factor is bounded $\mathcal{H} \to \mathcal{X}$, the middle factor is compact $\mathcal{H} \to \mathcal{H}$ as a consequence of Proposition 4.1, and the right factor is bounded $L^2((\delta/2,\delta), \mathcal{H}) \to \mathcal{H}$.

(iv) Summarizing, we obtain from (4.26) the estimate

(4.27) $$\|\eta_t f_t\|_{\mathcal{X}} \lesssim \|E_0^+ f_0\| + \|K f_t\|_{\mathcal{X}}$$

with a compact operator K. To estimate ∇u on the interior of Ω, we use the Caccioppoli inequality

$$\int_\Omega |\nabla u|^2 \tilde{\eta}^2 d\omega \lesssim \int_\Omega |u|^2 |\nabla \tilde{\eta}|^2 d\omega,$$

where $\tilde{\eta} \in C_c^\infty(\Omega)$ with $\tilde{\eta} = 1$ on $\Omega \setminus \rho((0,\delta/2) \times M)$. Applying weighted Poincaré inequalities [19, Thm. 1.5] to u on $\operatorname{supp}\nabla\tilde{\eta}$, and combining with (4.27), proves a lower estimate $\|\nabla u\|_{\mathcal{X}} \lesssim \|E_0^+ f_0\| + \|K(\nabla u)\|_{\mathcal{X}}$, with a compact operator K, in the case when $\operatorname{supp}\nabla\tilde{\eta}$ is connected. The case when $\operatorname{supp}\nabla\tilde{\eta}$ is disconnected does not cause a problem, since the difference between the mean values of u on different components can be added as a finite rank operator to K. This implies that the range of $\nabla u \mapsto (E_0^+ + E_0^0)f_0$ is closed, see [26, Thm. IV.5.26], and completes the proof of Theorem 4.5. $\qquad\square$

Proof of Theorem 4.6. (i) Consider a weak solution u with $\nabla u \in \mathcal{Y}$, so that $\int_0^\delta \|\eta_t f_t\|^2 t dt < \infty$. Similar to [5, Thm. 8.2], we write the first term in (4.22) as

$$e^{-t\Lambda} h^+ = \Lambda e^{-t\Lambda} \tilde{h}^+,$$

with $\tilde{h}^+ \in E_0^+ \mathcal{H}$ and $h^+ = \Lambda \tilde{h}^+ \in \mathcal{H}^{-1} \subset \overline{W}^{-1,2}$, using Lemma 4.4. Existence of the trace f_0 and the estimate $\|f_0\|_{\overline{W}^{-1,2}} \lesssim \|\nabla u\|_{\mathcal{Y}}$ follows from strong convergence $e^{-t\Lambda} \to I$ on $\overline{L}^2$, Lemma 4.4 and estimating as in [5, Prop. 7.2] for the second term in (4.22). Note that the estimate of the trace of $E_0^0 f_t$ is trivial, since it is a locally constant function on M_ρ and $\mathcal{Y} \subset L^2_{\mathrm{loc}}((0,\delta), \overline{L}^2)$. We conclude that there is a well defined trace f_0 satisfying (4.23) in $\overline{W}^{-1,2}$.

To prove the reverse estimate (4.20), it suffices to show that $\nabla u \mapsto (E_0^+ + E_0^0) f_0$ is an injective semi-Fredholm operator, as a bounded map from the closed subspace of $\mathcal{Y}$ consisting of gradients of solutions, to $\overline{W}^{-1,2}$.

(ii) To prove injectivity, assume that $\nabla u \in \mathcal{Y}$ and $E_0^+ f_0 = 0$, $E_0^0 f_0 = 0$. It follows from (4.22) that

$$1_{(0,\delta)}(I - S\mathcal{E}_t)(\eta_t f_t) = 1_{(0,\delta)} \widetilde{S}(\eta_t' f_t).$$

It follows from step (iii) in the proof of Theorem 4.5 that in fact the right hand side belongs to $\mathcal{X}$. By hypothesis $1_{(0,\delta)} S\mathcal{E}_t 1_{(0,\delta)}$ is bounded with small norm on both $\mathcal{Y}$ and on $\mathcal{X}$, where $\mathcal{X} \subset \mathcal{Y}$. Using a Neumann series for the inverse shows that in fact $\nabla u \in \mathcal{X}$, and $\nabla u = 0$ now follows as in step (ii) in the proof of Theorem 4.5.

(iii) To prove that the range is closed, we note that

$$(4.28) \quad \|1_{(0,\delta)}(I - S\mathcal{E}_t)(\eta_t f_t)\|_{\mathcal{Y}} \leq \|1_{(0,\delta)} \Lambda e^{-t\Lambda} \tilde{h}^+\|_{\mathcal{Y}} + \|1_{(0,\delta)} \widetilde{S}(\eta_t' f_t)\|_{\mathcal{Y}} + \|\eta_t E_0^0 f_t\|_{\mathcal{Y}}.$$

If $\|\mathcal{E}\|_*$ is small enough, then the left hand side is $\gtrsim \|\eta_t f_t\|_{\mathcal{Y}}$. Quadratic estimates, proved as in [5, Thm. 5.2] but building on Proposition 4.2, show that the first term on the right hand side is $\lesssim \|\tilde{h}^+\| = \|E_0^+ f_0\|_{\overline{W}^{-1,2}}$, and the last term is a finite rank operator. The second term maps $L^2((\delta/2, \delta), \mathcal{H}) \to L^2((0,\delta), \mathcal{H}) \subset \mathcal{Y}$, where the inclusion is continuous, and the L^2 map is shown to be compact, arguing as for terms II and I_1 in the proof of Theorem 4.5. Estimating u on the interior of Ω as in part (iv) of that proof, it follows that the range of $\nabla u \mapsto (E_0^+ + E_0^0) f_0$ is closed. $\qquad\square$

Proof of Theorem 1.3. The estimate (1.3) is the same as (4.15), which has been proved above. The estimate (1.4) is the same as (4.19), which has been proved above. In Theorem 4.5, the reverse estimate $\|\nabla u\|_{\mathcal{X}} \lesssim \|E_0^+ f_0\| + \|E_0^0 f_0\|$ was also proved. In Theorem 4.6, the corresponding $\overline{W}^{-1,2}$ estimate was shown. It remains only to note that $\|E_0^+ f_0\| + \|E_0^0 f_0\| \lesssim \|f_0\|$, and the corresponding estimate for $\overline{W}^{-1,2}$ norms, which hold due to the boundedness of the operators and Lemma 4.3. $\qquad\square$

4.3. Neumann and Dirichlet conditions.

In this section we further assume that the coefficients A_ρ have trace A_0 which is L^∞ close to a self-adjoint coefficient matrix, and we prove Theorem 1.4. The main ingredient in the proof is the following Rellich identity.

Lemma 4.7. Consider the operator $D_0 \widetilde{B}$ acting in $\overline{L}^2$ on M, where $\widetilde{B} \in L^\infty(\mathrm{End}(\mathbb{C} \oplus TM))$ is accretive on $\mathrm{Ran}(D_0)$, see (2.6), and $\widetilde{B}$ is the transform (4.7) of self-adjoint

coefficients $\tilde{A}$. Then we have the Rellich identity

$$\int_M \langle h_\perp, (\widetilde{B}h)_\perp\rangle d\omega_0 = \int_M \langle h_\parallel, (\widetilde{B}h)_\parallel\rangle d\omega_0$$

for all $h = \begin{bmatrix} h_\perp \\ h_\parallel \end{bmatrix} \in \chi^+(D_0\widetilde{B})\overline{L}^2$. The same identity holds also for all $h \in \chi^-(D_0\widetilde{B})\overline{L}^2$.

Proof. Introduce the auxiliary operator $N = \begin{bmatrix} -1 & 0 \\ 0 & I \end{bmatrix}$ and note that $ND_0 + D_0N = 0$. Furthermore we observe that the self-adjointness of $\tilde{A}$ translates to $\widetilde{B}^* = N\widetilde{B}N$ with (4.7). Define the Cauchy extension $h_t = e^{-t\Lambda}h$, $t > 0$, of h on the half-cylinder $(0,\infty) \times M$. Since $h \in \chi^+(D_0\widetilde{B})\overline{L}^2$, we have

$$\partial_t h_t = -\Lambda h_t = -D_0\widetilde{B}h_t$$

and $h_t \to h$ in $\overline{L}^2$ as $t \to 0$. Proposition 4.1 and the boundedness of the H^∞ functional calculus yields $\|h_t\| \lesssim \|(D_0\widetilde{B})^N e^{-t\Lambda}h\| = t^N\|(tD_0\widetilde{B})^N e^{-t\Lambda}h\| \lesssim t^{-N}$ for any $N < \infty$ as $t \to \infty$. We obtain

$$\begin{aligned}
\langle Nh, \widetilde{B}h\rangle_{\overline{L}^2} &= -\int_0^\infty \partial_t\langle Nh_t, \widetilde{B}h_t\rangle_{\overline{L}^2} dt \\
&= \int_0^\infty \left(\langle N(D_0\widetilde{B}h_t), \widetilde{B}h_t\rangle_{\overline{L}^2} + \langle Nh_t, \widetilde{B}(D_0\widetilde{B}h_t)\rangle_{\overline{L}^2}\right) dt \\
&= \int_0^\infty \langle (ND_0\widetilde{B} + D_0\widetilde{B}^*N)h_t, \widetilde{B}h_t\rangle_{\overline{L}^2} dt = 0,
\end{aligned}$$

from which the stated Rellich identity follows.

The proof for $h \in \chi^-(D_0\widetilde{B})\overline{L}^2$ is similar, using h_t on the lower half cylinder $t < 0$. $\square$

Definition 4.8. Define the subspaces

$$N^+\mathcal{H} = \{0\} \oplus \mathsf{Ran}(\nabla_M), \quad N^-\mathcal{H} = \mathsf{Ran}(\mathrm{div}_{M,\omega_0}) \oplus \{0\}$$

of tangential and normal vector fields on M, respectively, with the projections

$$N^+f = f_\parallel, \quad N^-f = f_\perp, \quad f = \begin{bmatrix} f_\perp \\ f_\parallel \end{bmatrix}.$$

Using Rellich identities, we prove that these subspaces are transversal to the subspaces $E_0^\pm\mathcal{H}$. More precisely, we have the following.

Proposition 4.9. Let $\widetilde{B}$ be the transform of self-adjoint coefficients as in Lemma 4.7. Then there exists $\epsilon > 0$ such that whenever coefficients B satisfy $\|B - \widetilde{B}\|_\infty < \epsilon$, the four restricted projections

$$E_0^\pm\mathcal{H} \to N^\pm\mathcal{H} : h \mapsto N^\pm h,$$

as well as the four restricted projections

$$N^\pm\mathcal{H} \to E_0^\pm\mathcal{H} : h \mapsto E_0^\pm h,$$

are all invertible isomorphisms. Here $E_0^\pm = \chi^\pm(D_0B)$. The same hold if $\mathcal{H} \subset \overline{L}^2$ is replaced by $\mathcal{H}^{-1} \subset \overline{W}^{-1,2}$.

Proof. (i) Consider first transforms of self-adjoint coefficients, that is, $B = \widetilde{B}$. Lemma 4.7 shows that

$$\|h\|^2 \lesssim \mathrm{Re} \int_M \langle h_\perp, (\widetilde{B}h)_\perp \rangle d\omega_0 + \mathrm{Re} \int_M \langle h_\parallel, (\widetilde{B}h)_\parallel \rangle d\omega_0$$

$$\lesssim \|h_\perp\| \|(\widetilde{B}h)_\perp\| \lesssim \|h_\perp\| \|h\|,$$

where the first estimate follows from the accretivity of $\widetilde{B}$. Therefore $\|h\| \lesssim \|h_\perp\|$ for all $h \in E_0^\pm \mathcal{H}$. A similar argument, instead keeping the $\langle h_\parallel, (\widetilde{B}h)_\parallel \rangle$ term, proves that $\|h\| \lesssim \|h_\parallel\|$. This shows that $N^\pm : E_0^\pm \mathcal{H} \to N^\pm \mathcal{H}$ are all injective semi-Fredholm operators. This entails that the four restricted projections $E_0^\pm : N^\pm \mathcal{H} \to E_0^\pm \mathcal{H}$ also are injective semi-Fredholm operators. We prove this for $E_0^+ : N^+ \mathcal{H} \to E_0^+ \mathcal{H}$; the proofs for the other three maps are similar. Let $h \in N^+ \mathcal{H}$ and consider $E_0^- h$. For the map $N^- : E_0^- \mathcal{H} \to N^- \mathcal{H}$, we have proved above that

$$\|E_0^- h\| \lesssim \|N^-(E_0^- h)\|.$$

This yields $\|h\| - \|E_0^+ h\| \lesssim \|N^-(E_0^+ h)\| \lesssim \|E_0^+ h\|$, by the reverse triangle inequality and since $E_0^- h = h - E_0^+ h$ and $N^- h = 0$. This estimate proves that $E_0^+ : N^+ \mathcal{H} \to E_0^+ \mathcal{H}$ is an injective semi-Fredholm operator.

(ii) We note that the projections $\chi^+(D_0 B)$ depend continuously (in fact, analytically) on B. This follows from quadratic estimates as in [15, Sec. 6], and allows us to apply perturbation theory for Fredholm operators. We reduce to fixed spaces as in [8, Lem. 4.3]. To show surjectivity of the restricted projections, let $\widetilde{B}$ be the transform of self-adjoint coefficients $\tilde{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Consider self-adjoint coefficients $\tilde{A}_t = \begin{bmatrix} a & tb \\ tc & d \end{bmatrix}$ for $t \in [0, 1]$, with transform $\widetilde{B}_t$. For $t = 0$, we have explicit inverses for the restricted projections. To see this, we write $N^\pm = (I \pm N)/2$, with N as in the proof of Lemma 4.7. We always have $N D_0 + D_0 N = 0$, but for $B = \widetilde{B}_0$ we also have $N \widetilde{B}_0 = \widetilde{B}_0 N$. As a consequence $N E_0^\pm = E_0^\mp N$ when $E_0^\pm = \chi^\pm(D_0 \widetilde{B}_0)$. For example, consider $h \in E_0^+ \mathcal{H}$. Then

$$E_0^+ N^+ h = \tfrac{1}{2}(E_0^+ + E_0^+ N h) = \tfrac{1}{2}(h + N E_0^- h) = \tfrac{1}{2} h.$$

This shows that, modulo a factor 2, the restricted projections $E_0^+ : N^+ \mathcal{H} \to E_0^+ \mathcal{H}$ and $N^+ : E_0^+ \mathcal{H} \to N^+ \mathcal{H}$ are inverse maps. Similarly, the other six restricted projections among $N^\pm|_{E_0^\pm \mathcal{H}}$ and $E_0^\pm|_{N^\pm \mathcal{H}}$ are seen to be pairwise inverse, modulo a factor 2.

By stability of the index for Fredholm operators, it follows that all eight restricted projections are invertible also for $\widetilde{B} = \widetilde{B}_1$. By the continuity of $B \mapsto \chi^\pm(D_0 B)$, this continues to hold for all non-selfadjoint B such that $\|B - \widetilde{B}\|_\infty < \epsilon$.

(iii) Finally, we prove the result for $\mathcal{H}^{-1}$. By perturbation theory as in (ii), it suffices to show that the eight restricted projections are injective semi-Fredholm maps in the $\mathcal{H}^{-1}$ topology, for coefficients $\widetilde{B}$. Given $h^+ \in E_0^\pm \mathcal{H}^{-1}$, we write $h^+ = D_0 \widetilde{B} \tilde{h}^+$ with $\tilde{h}^+ \in E_0^+ \mathcal{H}$. From Lemma 4.7 we obtain the estimate $\|\tilde{h}^+\|_{\overline{L}^2} \lesssim \|(\widetilde{B}\tilde{h}^+)_\perp\|_{\overline{L}^2}$. By Lemma 4.4, we equivalently have $\|h^+\|_{\overline{W}^{-1,2}} \lesssim \|(h^+)_\parallel\|_{\overline{W}^{-1,2}}$ since $D_0(\widetilde{B}\tilde{h}^+)_\perp = (h^+)_\parallel$, as D_0 swaps normal and tangential parts. We also have the estimate $\|h^+\|_{\overline{W}^{-1,2}} \simeq \|\tilde{h}^+\|_{\overline{L}^2} \lesssim \|(\widetilde{B}\tilde{h}^+)_\parallel\|_{\overline{L}^2} \approx \|(h^+)_\perp\|_{\overline{W}^{-1,2}}$. Given these two $\mathcal{H}^{-1}$

Rellich estimates, invertibility of the eight restricted projections follows as in the space $\mathcal{H}$. □

Proof of Theorem 1.4. (i) We first establish the Neumann solvability estimate $\|\nabla u\|_{\mathcal{X}} \lesssim \|\partial_{\nu_{A_\rho}} u_\rho|_M\|_{L^2(M,\omega_0^{-1})}$. It suffices to show that $\nabla u \mapsto (f_0)_\perp$ is an injective semi-Fredholm operator, as a bounded map from the closed subspace of $\mathcal{X}$ consisting of gradients of solutions, to $\overline{L}^2$, which we prove next. Theorem 4.5 shows that

$$\|\nabla u\|_{\mathcal{X}} \lesssim \|E_0^+ f_0\| + \|E_0^0 f_0\|.$$

Proposition 4.9 shows that $N^- : E_0^+ \mathcal{H} \to N^- \mathcal{H}$ is an isomorphism, and hence

$$\|E_0^+ f_0\| \lesssim \|(E_0^+ f_0)_\perp\|.$$

From (4.22), where $h^+ = E_0^+ f_0$, it now follows that

$$\|(h^+)_\perp\| \le \|(f_0)_\perp\| + \epsilon\|\eta_t f_t\|_{\mathcal{X}} + \|K f_t\|_{\mathcal{X}},$$

with a compact operator K and $\epsilon > 0$ small depending on $\|A_\rho/\omega_\rho - A_0/\omega_0\|_*$. Combining the above three estimates, we obtain $(1 - C\epsilon)\|\nabla u\|_{\mathcal{X}} \lesssim \|(f_0)_\perp\| + \|K(\nabla u)\|_{\mathcal{X}}$, which proves that $\nabla u \mapsto (f_0)_\perp$ has closed range, provided that $\epsilon < 1/C$. Injectivity is immediate from (4.24).

(ii) The proof of the Dirichlet regularity solvability estimate

$$\|\nabla u\|_{\mathcal{X}} \lesssim \|\nabla_M u_\rho|_M\|_{L^2(TM,\omega_0)} + \sum_i \left| \int_{M_i} \partial_{\nu_{A_\rho}} u_\rho\, d\mu \right|$$

is similar. It suffices to show that $\nabla u \mapsto (f_0)_\parallel + E_0^0 f_0$ is an injective semi-Fredholm operator, as a bounded map from the closed subspace of $\mathcal{X}$ consisting of gradients of solutions, to $\overline{L}^2$. That the range is closed follows as in (i), now using the invertibility of $N^+ : E_0^+ \mathcal{H} \to N^+ \mathcal{H}$ shown in Proposition 4.9. To see injectivity, we note that $(f_0)_\parallel = 0$ entails that u_ρ is locally constant on M and that $E_0^0 f_0 = 0$ entails that $\int_{M_i} \partial_{\nu_{A_\rho}} u_\rho\, d\mu = 0$ for each connected component M_i of M. Therefore $\nabla u = 0$ follows from (4.24).

(iii) To prove the Dirichlet solvability estimate

$$\|\nabla u\|_{\mathcal{Y}} \lesssim \|\nabla_M u_\rho|_M\|_{W^{-1,2}(TM,\omega_0)} + \sum_i \left| \int_{M_i} \partial_{\nu_{A_\rho}} u_\rho\, d\mu \right|,$$

it suffices to show that $\nabla u \mapsto (f_0)_\parallel + E_0^0 f_0$ is an injective semi-Fredholm operator, as a bounded map from the closed subspace of $\mathcal{Y}$ consisting of gradients of solutions, to $\overline{W}^{-1,2}$. Theorem 4.6 shows that $\|\nabla u\|_{\mathcal{Y}} \lesssim \|E_0^+ f_0\|_{\overline{W}^{-1,2}} + \|E_0^0 f_0\|$. We now use the invertibility of $N^+ : E_0^+ \mathcal{H}^{-1} \to N^+ \mathcal{H}^{-1}$, provided by Proposition 4.9. As in (ii), we conclude from (4.22) an estimate $(1 - C\epsilon)\|\nabla u\|_{\mathcal{Y}} \lesssim \|(f_0)_\parallel\|_{\overline{W}^{-1,2}} + \|K(\nabla u)\|_{\mathcal{Y}}$, with some compact operator K, from which it follows that the range of $\nabla u \mapsto (f_0)_\parallel + E_0^0 f_0$ is closed.

To prove injectivity, assume that $\operatorname{div} A\nabla u = 0$ with $\nabla u \in \mathcal{Y}$, $(f_0)_\parallel = 0$ and $E_0^0 f_0 = 0$. It suffices to show regularity $\nabla u \in \mathcal{X}$. Indeed, as in (ii) it then follows from (4.24) that $\nabla u = 0$. To show that $\nabla u \in \mathcal{X}$, we use (4.23) to write

$$f_0 = h^+ + h_{\mathcal{Y}}^- + h_{\mathcal{X}}^-,$$

where

$$h_{\mathcal{Y}}^- = S(\mathcal{E}_t \eta_t f_t)|_{t=0},$$

$$h_{\mathcal{X}}^- = \widetilde{S}(\eta_t' f_t)|_{t=0}.$$

We have $h_{\mathcal{Y}}^- \in E_0^- \mathcal{H}^{-1}$, and from part (iii) of the proof of Theorem 4.5 it follows that $h_{\mathcal{X}}^- \in E_0^- \mathcal{H}$ and that $\widetilde{S}(\eta_t' f_t) \in \mathcal{X}$. Now note that $f_0 \in N^- \mathcal{H}^{-1}$, $h^+ \in E_0^+ \mathcal{H}^{-1}$ and that $E_0^+ : N^- \mathcal{H}^{-1} \to E_0^+ \mathcal{H}^{-1}$ and $E_0^- : N^- \mathcal{H}^{-1} \to E_0^- \mathcal{H}^{-1}$ are isomorphisms by Proposition 4.9. Define the Hankel type operator

$$H : E_0^- \mathcal{H}^{-1} \to E_0^+ \mathcal{H}^{-1} : h^- \to h^+$$

if $f \in N^- \mathcal{H}^{-1}$ with $h^{\pm} = E_0^{\pm} f$. We apply H to $h_{\mathcal{Y}}^- + h_{\mathcal{X}}^-$, with $f = f_0$, to get $h^+ = h_{\mathcal{Y}}^+ + h_{\mathcal{X}}^+$ with $h_{\mathcal{Y}}^+ = H(h_{\mathcal{Y}}^-)$ and $h_{\mathcal{X}}^+ = H(h_{\mathcal{X}}^-)$. Rewrite (4.22) as

$$(I - (I + e^{-t\Lambda} H\gamma)S\mathcal{E}_t)(\eta_t f_t) = e^{-t\Lambda} h_{\mathcal{X}}^+ + \widetilde{S}(\eta_t' f_t), \qquad 0 < t < \delta,$$

where γ denotes trace at $t = 0$. Since the right hand side belongs to $\mathcal{X}$ and the operator $(I + e^{-t\Lambda} H\gamma)S\mathcal{E}_t$ is bounded with small norm on both $\mathcal{Y}$ and $\mathcal{X}$, using a Neumann series for the inverse it follows that $\nabla u \in \mathcal{X}$ as required. $\qquad \square$

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