Minimum codimension of eigenspaces in irreducible representations of simple classical linear algebraic groups

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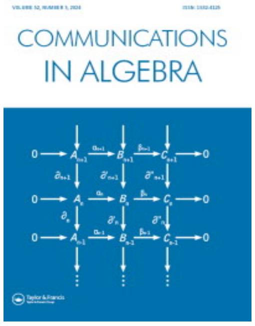

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Minimum codimension of eigenspaces in irreducible representations of simple classical linear algebraic groups

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ABSTRACT

Let k be an algebraically closed field of characteristic $p \geq 0$, let G be a simple simply connected classical linear algebraic group of rank ℓ and let T be a maximal torus in G with rational character group $\mathrm{X}(T)$. For a nonzero p-restricted dominant weight $\lambda \in \mathrm{X}(T)$, let V be the associated irreducible kG-module. We define $\nu_G(V)$ as the minimum codimension of any eigenspace on V for any non-central element of G. In this paper, we determine lower-bounds for $\nu_G(V)$ for G of type A_ℓ and $\dim(V) \leq \frac{\ell^3}{2}$, and for G of type B_ℓ, C_ℓ, or D_ℓ and $\dim(V) \leq 4\ell^3$. Moreover, we give the exact value of $\nu_G(V)$ for G of type A_ℓ with $\ell \geq 15$; for G of type B_ℓ or C_ℓ with $\ell \geq 14$; and for G of type D_ℓ with $\ell \geq 16$.

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1. Introduction

Let k be an algebraically closed field of characteristic $p \geq 0$, let V be a finite-dimensional k-vector space and let H be a group acting linearly on V. For $h \in H$ denote by $V_h(\mu)$ the eigenspace corresponding to the eigenvalue $\mu \in k^*$ of h on V. Set

$$\nu_H(V) = \min\{\dim(V) - \dim(V_h(\mu)) \mid h \in H \setminus \mathrm{Z}(H) \text{ and } \mu \in k^*\}.$$

In [2], one can find the classification of groups H acting linearly, irreducibly and primitively on a vector space V (over a field of characteristic zero) that contain an element h for which $\nu_H(V)$ is small when compared to $\dim(V)$. The following year, Hall, Liebeck and Seitz, [6], expanded on Gordeev's result by working over algebraically closed fields of arbitrary characteristic, and they proved that, in the case of linear algebraic groups, if H is classical, we have $\nu_H(V) \geq \frac{n}{8(2\ell+1)}$, where ℓ is the rank of H and V is a faithful rational irreducible kH-module of dimension n; while, if H is not of classical type, then $\nu_H(V) > \frac{\sqrt{n}}{12}$. Now, with the lower-bounds for $\nu_H(V)$ known, the following natural step was to start the classification of pairs (H, V) with bounded $\nu_H(V)$ from above, in particular the pairs (H, V) with $\nu_H(V) = 1$ or $\nu_H(V) = 2$ have been of great interest, see for example [9, 10, 23]. In [4], the irreducible subgroups H of $\mathrm{GL}(V)$, where V is a finite-dimensional k-vector space of dimension $n > 1$, which act primitively and tensor-indecomposably on V and $\nu_H(V) \leq \max\{2, \frac{\sqrt{n}}{2}\}$ have been classified.

Let G be a simple simply connected classical linear algebraic group of rank ℓ with $\ell \geq 1$ over k and let V be a nontrivial rational irreducible tensor-indecomposable kG-module. In this paper, we determine $\nu_G(V)$ in the following cases:

(a) G is of type A_ℓ with $\ell \geq 15$ and $\dim(V) \leq \frac{\ell^3}{2}$;
(b) G is of type B_ℓ with $\ell \geq 14$ and $\dim(V) \leq 4\ell^3$;

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(c) G is of type C_ℓ with $\ell \geq 14$ and $\dim(V) \leq 4\ell^3$;
(d) G is of type D_ℓ with $\ell \geq 16$ and $\dim(V) \leq 4\ell^3$.

Moreover, for the groups of smaller rank and their corresponding irreducible tensor-indecomposable modules with dimensions satisfying the above bounds, we improve the known lower-bounds for $\nu_G(V)$ (see [4, Theorem 8.4]). The origin of this paper is the PhD thesis of the author, in which the classification of pairs (G, V) with $\nu_G(V) \leq \sqrt{\dim(V)}$ was established. We now state the main results of this paper. The notation used will be introduced in Section 2.

Theorem 1.1. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected classical linear algebraic group of rank ℓ over k. When G is of type B_ℓ, we assume that $p \neq 2$. Let T be a maximal torus in G and let $V = L_G(\lambda)$, where λ is a nonzero p-restricted dominant weight. When G is of type A_ℓ with $\ell \geq 1$ assume that $\dim(V) \leq \frac{\ell^3}{2}$. In all other cases ($G$ of type C_ℓ with $\ell \geq 2$; B_ℓ with $\ell \geq 3$; D_ℓ with $\ell \geq 4$) assume that $\dim(V) \leq 4\ell^3$. The value of $\nu_G(V)$ is as given Table 15.*

In the following section, we fix the notation and terminology used throughout the text. In Section 3 we go over preliminary results, we establish an algorithm for calculating $\nu_G(V)$, and, for each type of classical group G, we determine the complete list of kG-modules that are candidates for Theorem 1.1.

Table 1. The value of $\nu_G(V)$.

Group	V	Char.	Rank	$\nu_G(V)$
A_ℓ	$L_G(\omega_1)$	$p \geq 0$	$\ell \geq 1$	1
	$L_G(\omega_2)$	$p \geq 0$	$\ell \geq 3$	$\ell - 1$
	$L_G(2\omega_1)$	$p \neq 2$	$\ell \geq 1$	ℓ
	$L_G(\omega_1 + \omega_\ell)$	$p \geq 0$	$\ell \geq 2$	$2\ell - \varepsilon_p(3)\varepsilon_\ell(2)$
	$L_G(\omega_3)$	$p \geq 0$	$\ell \geq 5$	$\binom{\ell-1}{2}$
	$L_G(3\omega_1)$	$p \neq 2, 3$	$\ell \geq 1$	$\frac{\ell^2+\ell+2}{2}$
	$L_G(\omega_1 + \omega_2)$	$p = 3$	$\ell \geq 3$	$\binom{\ell+1}{2}$
		$p \neq 3$	$\ell \geq 3$	ℓ^2
	$L_G(\omega_4)$	$p \geq 0$	$7 \leq \ell \leq 14$	$\binom{\ell-1}{3}$
	$L_G(\omega_5)$	$p \geq 0$	$9 \leq \ell \leq 10$	$\binom{\ell-1}{4}$
C_ℓ	$L_G(\omega_1)$	$p \geq 0$	$\ell \geq 2$	1
	$L_G(\omega_2)$	$p \geq 0$	$\ell \geq 2$	$2\ell - 2 - \varepsilon_\ell(2)$
	$L_G(2\omega_1)$	$p \neq 2$	$\ell \geq 2$	2ℓ
	$L_G(\omega_3)$	$p \geq 0$	$\ell = 3$	$4 - 2\varepsilon_p(2)$
			$\ell = 4$	$14 - \varepsilon_p(3)$
			$\ell \geq 5$	$2\ell^2 - 5\ell + 2 - \varepsilon_p(\ell - 1)$
	$L_G(3\omega_1)$	$p \neq 2, 3$	$\ell \geq 2$	$2\ell^2 + \ell$
	$L_G(\omega_1 + \omega_2)$	$p \geq 0$	$\ell = 2$	$8 - 2\varepsilon_p(5)$
		$p = 3$	$\ell \geq 3$	$2\ell^2 + \ell - 2$
		$p \neq 3$	$\ell \geq 3$	$\geq 4\ell^2 - 4\ell - \varepsilon_p(2\ell + 1) - 2\ell\varepsilon_p(2)^\dagger$
	$L_G(2\omega_2)$	$p \neq 2$	$\ell = 2$	4
	$L_G(\omega_1 + 2\omega_2)$	$p = 7$	$\ell = 2$	12
	$L_G(3\omega_2)$	$p = 7$	$\ell = 2$	9
	$L_G(2\omega_1 + \omega_2)$	$p = 3$	$\ell = 2$	9
	$L_G(\omega_1 + \omega_3)$	$p \geq 0$	$\ell = 3$	$30 - 5\varepsilon_p(3) - 10\varepsilon_p(2)$
	$L_G(\omega_2 + \omega_3)$	$p = 5$	$\ell = 3$	26
	$L_G(2\omega_3)$	$p = 5$	$\ell = 3$	24
	$L_G(2\omega_2)$	$p = 5$	$\ell = 3$	40
	$L_G(\omega_4)$	$p \neq 2$	$\ell = 4$	$14 - \varepsilon_p(3)$
	$L_G(\omega_1 + \omega_\ell)$	$p = 2$	$4 \leq \ell \leq 6$	$(\ell + 2) \cdot 2^{\ell-1}$
	$L_G(\omega_1 + \omega_4)$	$p = 7$	$\ell = 4$	≥ 96
	$L_G(\omega_1 + \omega_3)$	$p = 2$	$\ell = 4$	≥ 102
	$L_G(\omega_4)$	$p \geq 0$	$\ell = 5$	$\geq 48 - 8\varepsilon_p(3) - 4\varepsilon_p(2)^\dagger$
	$L_G(\omega_5)$	$p \neq 2$	$\ell = 5$	$42 - 2\varepsilon_p(3)$
	$L_G(\omega_4)$	$p \geq 0$	$\ell = 6$	$\geq 110 - 14\varepsilon_p(2)^\star$
	$L_G(\omega_5)$	$p \geq 0$	$\ell = 6$	$\geq 165 - 44\varepsilon_p(3) - 17\varepsilon_p(2)^\dagger$

(*Continued*)

Table 1. (*Continued*)

Group	V	Char.	Rank	$\nu_G(V)$
	$L_G(\omega_6)$	$p \neq 2$	$\ell = 6$	$132 - 11\varepsilon_p(3)$
	$L_G(\omega_4)$	$p \geq 0$	$\ell = 7$	$\geq 208 - 12\varepsilon_p(5) - 2\varepsilon_p(2)^{\star}$
	$L_G(\omega_6)$	$p = 3$	$\ell = 7$	364
	$L_G(\omega_7)$	$p = 3$	$\ell = 7$	364
	$L_G(\omega_5)$	$p = 2$	$\ell = 7$	≥ 340
	$L_G(\omega_4)$	$p \geq 0$	$\ell = 8$	$\geq 350 - 14\varepsilon_p(3) - 16\varepsilon_p(2)^{\circ}$
	$L_G(\omega_4)$	$p \geq 0$	$\ell = 9$	$\geq 544 - 16\varepsilon_p(7) - 2\varepsilon_p(2)^{\circ}$
	$L_G(\omega_\ell)$	$p = 2$	$4 \leq \ell \leq 13$	$2^{\ell-2}$
	$L_G(2\omega_1 + \omega_\ell)^{\ddagger}$	$p = 2$	$3 \leq \ell \leq 6$	$(\ell+2) \cdot 2^{\ell-1}$
B_ℓ	$L_G(\omega_1)$	$p \neq 2$	$\ell \geq 3$	1
	$L_G(\omega_2)$	$p \neq 2$	$\ell \geq 3$	2ℓ
	$L_G(2\omega_1)$	$p \neq 2$	$\ell \geq 3$	2ℓ
	$L_G(\omega_3)$	$p \neq 2$	$\ell \geq 3$	$2\ell^2 - \ell$
	$L_G(3\omega_1)$	$p \neq 2, 3$	$\ell \geq 3$	$2\ell^2 + \ell - \varepsilon_p(2\ell+3)$
	$L_G(\omega_1 + \omega_2)$	$p \neq 2$	$\ell \geq 3$	$4\ell^2 - 1 - \varepsilon_p(\ell) - (2\ell^2 - \ell)\varepsilon_p(3)$
	$L_G(2\omega_3)$	$p \neq 2$	$\ell = 3$	14
	$L_G(\omega_2 + \omega_3)$	$p = 3$	$\ell = 3$	≥ 50
	$L_G(\omega_2 + \omega_3)$	$p = 5$	$\ell = 3$	32
	$L_G(3\omega_3)$	$p = 5$	$\ell = 3$	52
	$L_G(2\omega_4)$	$p \neq 2$	$\ell = 4$	50
	$L_G(\omega_4)$	$p \neq 2$	$\ell = 5$	≥ 116
	$L_G(2\omega_5)$	$p \neq 2$	$\ell = 5$	183
	$L_G(\omega_4)$	$p \neq 2$	$\ell = 6$	≥ 216
	$L_G(\omega_4)$	$p \neq 2$	$\ell = 7$	≥ 360
	$L_G(\omega_1 + \omega_\ell)$	$p \neq 2$	$3 \leq \ell \leq 6$	$\geq (\ell+2) \cdot 2^{\ell-1} - 2^{\ell-2}\varepsilon_p(2\ell+1)^{\bullet}$
	$L_G(\omega_\ell)$	$p \neq 2$	$4 \leq \ell \leq 13$	$2^{\ell-2}$
D_ℓ	$L_G(\omega_1)$	$p \geq 0$	$\ell \geq 4$	2
	$L_G(\omega_2)$	$p \geq 0$	$\ell \geq 4$	$4\ell - 6$
	$L_G(2\omega_1)$	$p \neq 2$	$\ell \geq 4$	$4\ell - 4$
	$L_G(\omega_3)$	$p \geq 0$	$\ell \geq 5$	$4\ell^2 - 14\ell + 14 - 2(1 + \varepsilon_2(\ell-1))\varepsilon_p(2)$
	$L_G(3\omega_1)$	$p \neq 2, 3$	$\ell \geq 4$	$4\ell^2 - 6\ell + 4 - 2\varepsilon_p(\ell+1)$
	$L_G(\omega_1 + \omega_2)$	$p = 3$	$\ell \geq 4$	$4\ell^2 - 6\ell + 2 - 2\varepsilon_3(2\ell-1)$
		$p \neq 3$	$\ell \geq 4$	$\geq 8(\ell-1)^2 - 2\varepsilon_p(2\ell-1) - (4\ell-6)\varepsilon_p(2)^{\dagger}$
	$L_G(\omega_1 + \omega_4)$	$p \geq 0$	$\ell = 4$	$22 - 2\varepsilon_p(2)$
	$L_G(2\omega_2)$	$p = 3$	$\ell = 4$	≥ 84
	$L_G(2\omega_1 + \omega_3)$	$p \neq 2$	$\ell = 4$	$\geq 96 - 22\varepsilon_p(5) - 8\varepsilon_p(3)$
	$L_G(\omega_1 + \omega_3 + \omega_4)$	$p = 2$	$\ell = 4$	≥ 102
	$L_G(2\omega_5)$	$p \neq 2$	$\ell = 5$	≥ 46
	$L_G(\omega_4 + \omega_5)$	$p \geq 0$	$\ell = 5$	$\geq 80 - 16\varepsilon_p(2)$
	$L_G(\omega_1 + \omega_5)$	$p \geq 0$	$\ell = 5$	$\geq 52 - 4\varepsilon_p(5)^{\dagger}$
	$L_G(\omega_2 + \omega_5)$	$p = 2$	$\ell = 5$	≥ 164
	$L_G(2\omega_6)$	$p \neq 2$	$\ell = 6$	≥ 172
	$L_G(\omega_5 + \omega_6)$	$p \geq 0$	$\ell = 6$	$\geq 300 - 84\varepsilon_p(2)$
	$L_G(\omega_1 + \omega_6)$	$p \geq 0$	$\ell = 6$	$\geq 120 - 8\varepsilon_p(3) - 8\varepsilon_p(2)^{\propto}$
	$L_G(\omega_4)$	$p \geq 0$	$\ell = 6$	$\geq 184 - 40\varepsilon_p(2)^{\dagger}$
	$L_G(\omega_5)$	$p = 2$	$\ell = 7$	≥ 340
	$L_G(\omega_1 + \omega_7)$	$p \geq 0$	$\ell = 7$	$\geq 272 - 16\varepsilon_p(7)^{\circ}$
	$L_G(\omega_4)$	$p \geq 0$	$\ell = 7$	$\geq 350 - 26\varepsilon_p(2)^{\dagger}$
	$L_G(\omega_4)$	$p \geq 0$	$\ell = 8$	$\geq 596 - 56\varepsilon_p(2)^{\dagger}$
	$L_G(\omega_1 + \omega_8)$	$p \geq 0$	$\ell = 8$	$\geq 508 - 142\varepsilon_p(2)^{\wr}$
	$L_G(\omega_4)$	$p = 2$	$\ell = 9$	≥ 542
	$L_G(\omega_\ell)$	$p \geq 0$	$5 \leq \ell \leq 15$	$2^{\ell-3}$

† equality holds when $p \neq 2$.
* equality holds when $p \neq 2, 3$.
$^{\circ}$ equality holds when $p \neq 2, 3, 5$.
$^{\bullet}$ equality holds when $\varepsilon_p(2(\ell - i) + 1) = 0$ for all $0 \leq i \leq \ell - 3$.
$^{\propto}$ equality holds when $p \neq 2, 5$.
$^{\wr}$ equality holds when $p \neq 2, 3, 5, 7$.
‡ when $p = 2$ there exists an exceptional isogeny between groups of type C_ℓ and B_ℓ, see [21, *Theorem 28*]. In this case, in view of Lemma 3.3, we made the choice to only treat groups of type C_ℓ when $p = 2$. Since $\dim(L_{B_\ell}(\omega_1 + \omega_\ell)) \leq 4\ell^3$ for $3 \leq \ell \leq 6$, in order to have a complete result, we have to add the induced modules $L_{C_\ell}(2\omega_1 + \omega_\ell)$, where $3 \leq \ell \leq 6$, to the list for groups of type C_ℓ, see Remark 3.4. Furthermore, by the same result, these are the only induced modules for the groups of type C_ℓ, where the corresponding module for the group of type B_ℓ has dimension $\leq 4\ell^3$, which are not isomorphic to (a twist of) a module already listed.

The proof of Theorem 1.1 is given in Sections 4–7, where each section is dedicated to one of the types of classical groups.

2. Notation

Throughout the text k is an algebraically closed field of characteristic $p \geq 0$. Note that when we write $p \neq p_0$, for some prime p_0, we allow $p = 0$. Let G be a simple simply connected classical linear algebraic group of rank ℓ and let T be a maximal torus in G with rational character group $\mathrm{X}(T)$. Let $\mathrm{Y}(T)$ be the group of rational cocharacters of T and let $\langle -, - \rangle$ be the perfect natural pairing on $\mathrm{X}(T) \times \mathrm{Y}(T)$. We denote by Φ the root system of G corresponding to T and by $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ a set of simple roots in Φ, where we use the standard Bourbaki labeling, as given in [7, 11.4, p.58]. Let Φ^+ be the set of positive roots of G. Following [1, Section 2.1], we fix a total order $\preceq$ on Φ: for $\alpha, \beta \in \Phi$ we have $\alpha \preceq \beta$ if and only if $\alpha = \beta$, or $\beta - \alpha = \sum_{i=1}^{r} a_i\alpha_i$ with $1 \leq r \leq \ell$, $a_i \in \mathbb{Z}$, $1 \leq i \leq r$, and $a_r > 0$.

For each $\alpha \in \Phi$, let $h_\alpha \in \mathrm{Y}(T)$ be its corresponding coroot, let U_α be its corresponding root subgroup and let $x_\alpha : k \to U_\alpha$ be an isomorphism of algebraic groups with the property that $tx_\alpha(c)t^{-1} = x_\alpha(\alpha(t)c)$ for all $t \in T$ and all $c \in k$. Let G_s be the set of semisimple elements of G and let G_u be the set of unipotent elements of G. Any $s \in T$ can be written $s = \prod_{\alpha_i \in \Delta} h_{\alpha_i}(c_{\alpha_i})$, where $c_{\alpha_i} \in k^*$, respectively any $u \in G_u$ can be written as $u = \prod_{\alpha \in \Phi^+} x_\alpha(c_\alpha)$, where $c_\alpha \in k$ and the product respects $\preceq$. Lastly, let B be the positive Borel subgroup of G, $\mathcal{W}$ be the Weyl group of G corresponding to T, and $w_0 \in \mathcal{W}$ be the longest word.

The set of dominant weights of G with respect to Δ is denoted by $\mathrm{X}(T)^+$, and the set of p-restricted dominant weights by $\mathrm{X}(T)^+_p$. We adopt the usual convention that when $p = 0$, all weights are p-restricted. For $\lambda \in \mathrm{X}(T)^+$, we denote by $L_G(\lambda)$ the irreducible kG-module of highest weight λ. Further, we denote by ω_i, $1 \leq i \leq \ell$, the fundamental dominant weight of G with respect to α_i.

All representations and all modules of a linear algebraic group are assumed to be rational and nonzero. For a kG-module V we will use the notation $V = W_1 \mid W_2 \mid \cdots \mid W_m$ to express that V has a composition series $V = V_1 \supset V_2 \supset \cdots \supset V_m \supset V_{m+1} = 0$ with composition factors $W_i \cong V_i/V_{i+1}$, $1 \leq i \leq m$. Further, by V^m we denote the direct sum $V \oplus \cdots \oplus V$, in which V occurs m times. When $p > 0$, we denote by $V^{(p^i)}$ the kG-module obtained from V by twisting the action of G with the Frobenius endomorphism i times, see [16, Section 16.2]. Lastly, we will denote the natural module of G by W.

For $m \in \mathbb{Z}_{\geq 0}$, we define $\varepsilon_m : \mathbb{Z}_{\geq 0} \to \{0, 1\}$ by $\varepsilon_m(n) = 0$ if $m \nmid n$ and $\varepsilon_m(n) = 1$ if $m \mid n$.

3. Preliminary results

To begin, we prove the following result which gives us the strategy we will use for calculating $\nu_G(V)$.

Proposition 3.1. *Let G be a simple linear algebraic group and let V be an irreducible kG-module. Then:*

$$\nu_G(V) = \dim(V) - \max\{\max_{s \in T \setminus Z(G)} \dim(V_s(\mu)), \max_{u \in G_u \setminus \{1\}} \dim(V_u(1))\}.$$

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be the associated representation and let $g \in G \setminus Z(G)$. We write the Jordan decomposition of $g = g_s g_u = g_u g_s$, where $g_s \in G_s$ and $g_u \in G_u$. By [16, Theorem 2.5], $\rho(g) = \rho(g_s)\rho(g_u) = \rho(g)_s\rho(g)_u$ is the Jordan decomposition of $\rho(g)$ in $\mathrm{GL}(V)$. We choose a basis of V with the property that $\rho(g)$ is written in its Jordan normal form. Then, with respect to this basis, $\rho(g)_s$ is the diagonal matrix whose entries are just the diagonal entries of $\rho(g)$, while $\rho(g)_u$ is the unipotent matrix obtained from the Jordan normal form of $\rho(g)$ by dividing all entries of each Jordan block by the diagonal element. We distinguish the following two cases:

Case 1: Assume $g_s \in Z(G)$. First, we remark that $g_u \neq 1$, as $g \notin Z(G)$. Secondly, as $g_s \in Z(G)$, it follows that $\rho(g)_s = \mathrm{diag}(c, c, \dots, c)$ for some $c \in k^*$. Thereby, c is the sole eigenvalue of $\rho(g)$ on V and we have $\dim(V_g(c)) = \dim(V_{g_u}(1)) \leq \max_{u \in G_u \setminus \{1\}} \dim(V_u(1))$.

Case 2: Assume $g_s \notin Z(G)$. Then, since $\rho(g)_s$ is a diagonal matrix with entries the diagonal entries of $\rho(g)$, we determine that $\rho(g)$ and $\rho(g)_s$ have the same eigenvalues on V and, for any such eigenvalue $c \in k^*$ we have $\dim(V_g(c)) \leq \dim(V_{g_s}(c)) \leq \max_{s \in G_s \setminus Z(G)} \dim(V_s(\mu)) = \max_{s' \in T \setminus Z(G)} \dim(V_{s'}(\mu'))$, where the last equality follows by [16, Corollary 4.5 and Theorem 4.4]. □

3.1. Group isogenies and irreducible modules

In this section, we will assume that the simple algebraic group G is not simply connected, and we let $\tilde{G}$ be its simply connected cover. Fix a central isogeny $\phi : \tilde{G} \to G$ with $\ker(\phi) \subseteq Z(\tilde{G})$ and $d\phi \neq 0$. Let $\tilde{T}$ be a maximal torus in $\tilde{G}$ with the property that $\phi(\tilde{T}) = T$ and, similarly, let $\tilde{B}$ be the Borel subgroup of $\tilde{G}$ given by $\phi(\tilde{B}) = B$. Let $\lambda \in X(T)^+$. Since $X(T) \subseteq X(\tilde{T})$, we will denote by $\tilde{\lambda}$ the weight λ when viewing it as an element of $X(\tilde{T})$. By [8, II.2.10], as $\lambda \in X(T)^+$, it follows that $\tilde{\lambda} \in X(\tilde{T})^+$. Moreover, by the same result, we have that $L_G(\lambda)$ is a simple $k\tilde{G}$-module and $L_G(\lambda) \cong L_{\tilde{G}}(\tilde{\lambda})$ as $k\tilde{G}$-modules.

Lemma 3.2. *We have* $\max_{\tilde{s} \in \tilde{T} \setminus Z(\tilde{G})} \dim((L_{\tilde{G}}(\tilde{\lambda}))_{\tilde{s}}(\tilde{\mu})) = \max_{s \in T \setminus Z(G)} \dim((L_G(\lambda))_s(\mu))$ *and* $\max_{\tilde{u} \in \tilde{G}_u \setminus \{1\}} \dim((L_{\tilde{G}}(\tilde{\lambda}))_{\tilde{u}}(1))$ $= \max_{u \in G_u \setminus \{1\}} \dim((L_G(\lambda))_u(1))$. *In particular, we have* $\nu_{\tilde{G}}(L_{\tilde{G}}(\tilde{\lambda})) = \nu_G(L_G(\lambda))$.

Proof. Let $\tilde{g} \in \tilde{G} \setminus Z(\tilde{G})$ and let $\tilde{\mu} \in k^*$ be an eigenvalue of $\tilde{g}$ on $L_{\tilde{G}}(\tilde{\lambda})$. Let $g = \phi(\tilde{g})$ and note that $g \in G \setminus Z(G)$. Denote by $\mu \in k^*$ the eigenvalue of $\tilde{g}$ on $L_G(\lambda)$ corresponding to $\tilde{\mu}$ under $L_G(\lambda) \cong L_{\tilde{G}}(\tilde{\lambda})$. We have that:

$$\begin{aligned}\dim((L_{\tilde{G}}(\tilde{\lambda}))_{\tilde{g}}(\tilde{\mu})) &= \dim((L_G(\lambda))_{\tilde{g}}(\mu)) = \dim((L_G(\lambda))_{\phi(\tilde{g})}(\mu)) = \dim((L_G(\lambda))_g(\mu)) \\ &\leq \max_{(g',\mu') \in (G \setminus Z(G)) \times k^*} \dim((L_G(\lambda))_{g'}(\mu')).\end{aligned}$$

Lastly, let $(g^*, \mu^*) \in G \setminus Z(G) \times k^*$ be such that $\dim((L_G(\lambda))_{g^*}(\mu^*)) = \max_{(g',\mu') \in (G \setminus Z(G)) \times k^*} \dim((L_G(\lambda))_{g'}(\mu'))$. As the map $\phi : \tilde{G} \to G$ is surjective, let $\tilde{g^*}$ be an arbitrary preimage of g^* in $\tilde{G}$ and $\tilde{\mu^*} \in k^*$ be the eigenvalue of $\tilde{g^*}$ on $L_{\tilde{G}}(\tilde{\lambda})$ corresponding to μ^* under $L_G(\lambda) \cong L_{\tilde{G}}(\tilde{\lambda})$. Then:

$$\dim((L_{\tilde{G}}(\tilde{\lambda}))_{\tilde{g^*}}(\tilde{\mu^*})) = \max_{(g',\mu') \in (G \setminus Z(G)) \times k^*} \dim((L_G(\lambda))_{g'}(\mu')).$$

□

The following result justifies why we only treat groups of type B_ℓ and their respective modules over fields of characteristic different to 2.

Lemma 3.3. *Let $p = 2$. Let B, respectively C, be a simple simply connected linear algebraic group of type B_ℓ, respectively of type C_ℓ. Let ω_i^B, $1 \leq i \leq \ell$, respectively ω_i^C, $1 \leq i \leq \ell$, be the fundamental dominant weights of B, respectively of C. Then, for any 2-restricted dominant weight $\sum_{i=1}^{\ell} d_i\omega_i^B$ of B we have that:*

$$\nu_B\Big(L_B\big(\sum_{i=1}^{\ell} d_i\omega_i^B\big)\Big) = \nu_C\Big(L_C\big(\sum_{i=1}^{\ell-1} d_i\omega_i^C\big)^{(2)} \otimes L_C(\omega_\ell)\Big).$$

Proof. As $p = 2$, there exists an exceptional isogeny $\phi : C \to B$ between the two groups, see [21, Theorem 28]. Consequently, we can induce irreducible kC-modules from irreducible kB-modules by twisting with the isogeny ϕ. Therefore, we have:

$$L_B\Big(\sum_{i=1}^{\ell} d_i\omega_i^B\Big) \cong L_C\Big(2\sum_{i=1}^{\ell-1} d_i\omega_i^C + d_\ell\omega_\ell^C\Big) \cong L_C\Big(\sum_{i=1}^{\ell-1} d_i\omega_i^C\Big)^{(2)} \otimes L_C(d_\ell\omega_\ell^C).$$

□

Remark 3.4. *In view of Lemma 3.3, for any 2-restricted dominant weight $\mu = \sum_{i=1}^{\ell-1} d_i\omega_i^B$ of B, we have $\nu_B(L_B(\mu)) = \nu_C(L_C(2\lambda)) = \nu_C(L_C(\lambda)^{(2)}) = \nu_C(L_C(\lambda))$, where $\lambda = \sum_{i=1}^{\ell-1} d_i\omega_i^C$. Similarly, for the weight ω_ℓ^B, we have $\nu_B(L_B(\omega_\ell^B)) = \nu_C(L_C(\omega_\ell^C))$. Lastly, in the case of weights of the form $\mu = \sum_{i=1}^{\ell} d_i\omega_i^B$, where $d_\ell = 1$ and there exists $1 \leq i \leq \ell-1$ such that $d_i = 1$, in order to determine $\nu_B(L_B(\mu))$ it suffices to calculate $\nu_C\Big(L_C(\sum_{i=1}^{\ell-1} d_i\omega_i^C)^{(2)} \otimes L_C(\omega_\ell)\Big)$.*

3.2. Restriction to Levi subgroups

We return to the situation where G is simply connected. For each $1 \leq i \leq \ell$, let P_i be the maximal parabolic subgroup of G corresponding to $\Delta_i := \Delta \setminus \{\alpha_i\}$, and let $L_i = \langle T, U_{\pm\alpha_1}, \dots, U_{\pm\alpha_{i-1}}, U_{\pm\alpha_{i+1}}, \dots, U_{\pm\alpha_\ell}\rangle$ be a Levi subgroup of P_i. The root system of L_i is $\Phi_i = \Phi \cap (\mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_{i-1} + \mathbb{Z}\alpha_{i+1} + \cdots + \mathbb{Z}\alpha_\ell)$, in which Δ_i is a set of simple roots. Now, we have that $L_i = Z(L_i)^\circ[L_i, L_i]$, where $Z(L_i)^\circ = \Big(\bigcap_{j\neq i} \ker(\alpha_j)\Big)^\circ$ is a one-dimensional subtorus of G and $[L_i, L_i]$ is a semisimple simply connected linear algebraic group of rank $\ell - 1$, see [16, Proposition 12.14]. Lastly, let $T_i = T \cap [L_i, L_i]$ be a maximal torus in $[L_i, L_i]$, contained in the Borel subgroup $B_i = B \cap [L_i, L_i]$. We will abuse notation and denote the fundamental dominant weights of L_i corresponding to Δ_i by $\omega_1, \dots, \omega_{i-1}, \omega_{i+1}, \dots, \omega_\ell$.

Let $\lambda \in X(T)^+$, $\lambda = \sum_{i=1}^{\ell} d_i\omega_i$, let $V = L_G(\lambda)$ be the associated irreducible kG-module, and let $\Lambda(V)$ be the set of weights of V. Fix some $1 \leq i \leq \ell$. We say that a weight $\mu \in \Lambda(V)$ has *α_i-level j* if $\mu = \lambda - j\alpha_i - \sum_{r\neq i} c_r\alpha_r$, where $c_r \in \mathbb{Z}_{\geq 0}$. The *maximum α_i-level of weights* in V will be denoted by $e_i(\lambda)$. By [8, II, Proposition 2.4(b)], we have that $e_i(\lambda)$ is equal to the α_i-level of $w_0(\lambda)$. Now, consider the Levi subgroup L_i of P_i. For each $0 \leq j \leq e_i(\lambda)$, define the subspace $V^j := \bigoplus_{\gamma\in\mathbb{N}\Delta_i} V_{\lambda-j\alpha_i-\gamma}$ of V and note that V^j is invariant under L_i. Then, as a $k[L_i, L_i]$-module, V admits the following decomposition:

$$V\mid_{[L_i,L_i]} = \bigoplus_{j=0}^{e_i(\lambda)} V^j,$$

where, by [20, Proposition], $V^0 = \bigoplus_{\gamma\in\mathbb{N}\Delta_i} V_{\lambda-\gamma}$ is the irreducible $k[L_i, L_i]$-module of highest weight $\lambda\mid_{T_i}$.

Lemma 3.5. *Assume V is a self-dual kG-module. Then, for all $0 \leq j \leq \left\lfloor \frac{e_i(\lambda)}{2} \right\rfloor$, we have $V^{e_i(\lambda)-j} \cong (V^j)^*$, as $k[L_i, L_i]$-modules.*

Proof. We note that, as V is self-dual, we have $w_0(\lambda) = -\lambda$ and V is equipped with a nondegenerate bilinear form $(-,-)$. Let $\mu, \mu' \in \Lambda(V)$ be such that $\mu' \neq -\mu$. Let $v \in V_\mu$ and $v' \in V_{\mu'}$. Then

$(v, v') = (t \cdot v, t \cdot v') = (\mu(t)v, \mu'(t)v') = (\mu + \mu')(t)(v, v')$, for all $t \in T$. Therefore $(v, v') = 0$, as $\mu' \neq -\mu$, and so $V_{\mu'} \subset V_\mu^\perp$. Moreover, as $(-,-)$ is nondegenerate, it follows that $-\mu \in \Lambda(V)$ for all $\mu \in \Lambda(V)$.

Secondly, let $\mu \in \Lambda(V)$ be a weight of α_i-level j, where $0 \leq j \leq e_i(\lambda)$. We will show that $-\mu$ has α_i-level $e_i(\lambda) - j$. On one hand, we know that $e_i(\lambda)$ is equal to the α_i-level of $w_0(\lambda)$, thus $w_0(\lambda) = \lambda - e_i(\lambda)\alpha_i - \sum_{r \neq i} a_r\alpha_r$, where $a_r \in \mathbb{Z}_{\geq 0}$. On the other hand, as $\mu = \lambda - j\alpha_i - \sum_{r \neq i} c_r\alpha_r$, for $c_r \in \mathbb{Z}_{\geq 0}$, we have $-\mu = -\lambda + j\alpha_i + \sum_{r \neq i} c_r\alpha_r = \lambda - (e_i(\lambda) - j)\alpha_i - \sum_{r \neq i} b_r\alpha_r$, where $b_r \in \mathbb{Z}_{\geq 0}$ for all $r \neq i$. Thus, $-\mu$ has α_i-level equal to $e_i(\lambda) - j$. In particular, as $V_{\mu'} \subseteq (V_\mu)^\perp$ for all $\mu' \neq -\mu$, it follows that $(V^j)^\perp \supseteq \bigoplus_{r \neq e_i(\lambda) - j} V^r$.

Lastly, as $V|_{[L_i,L_i]} = \bigoplus_{j=0}^{e_i(\lambda)} V^j$ is self-dual, it follows that $V|_{[L_i,L_i]} \cong \bigoplus_{j=0}^{e_i(\lambda)} (V^j)^*$. Furthermore, as V is equipped with a nondegenerate bilinear form, we have that $(V^j)^* \cong V/(V^j)^\perp$, for all $0 \leq j \leq e_i(\lambda)$. As $(V^j)^\perp \supseteq \bigoplus_{r \neq e_i(\lambda) - j} V^r$, it follows that $\dim((V^j)^*) \leq \dim(V^{e_i(\lambda)-j})$. By the same argument, this time applied to $V^{e_i(\lambda)-j}$, we determine that $\dim((V^{e_i(\lambda)-j})^*) \leq \dim(V^j)$. Therefore, $\dim((V^j)^*) = \dim(V^{e_i(\lambda)-j})$, thus $(V^j)^\perp = \bigoplus_{r \neq e_i(\lambda) - j} V^r$, and we conclude that $(V^j)^* \cong V^{e_i(\lambda)-j}$. □

Remark 3.6. *Applying Lemma 3.5, let $V = L_G(\lambda)$, where $\lambda = \sum_{i=1}^{\ell} d_i\omega_i \in X(T)^+$. As $V^* \cong L_G(-w_0(\lambda))$, it follows that V is self-dual if $-w_0(\lambda) = \lambda$. Thus, for groups of type A_ℓ, V is self-dual if $d_i = d_{\ell+1-i}$ for all $1 \leq i \leq \ell$. For groups of type B_ℓ and C_ℓ, as $w_0 = -1$, all irreducible kG-modules are self-dual. Lastly, for groups of type D_ℓ with ℓ even, all irreducible kG-modules are self-dual, while for groups of type D_ℓ with ℓ odd, V is self-dual if $d_{\ell-1} = d_\ell$.*

In what follows, we give a formula for $e_1(\lambda)$, the maximum α_1-level of weights in $L_G(\lambda)$, for the classical linear algebraic groups. Further, for groups of type C_ℓ, we also give a formula for $e_\ell(\lambda)$.

Lemma 3.7. *Let G be of type A_ℓ and let $\lambda = \sum_{i=1}^{\ell} d_i\omega_i \in X(T)^+$. Then $e_1(\lambda) = \sum_{j=1}^{\ell} d_j$.*

Proof. In order to determine $e_1(\lambda)$ we have to calculate the α_1-level of $w_0(\lambda)$. We have that

$$w_0(\lambda) = \lambda - (\lambda - w_0(\lambda)) = \lambda - \sum_{r=1}^{\ell} d_r(\omega_r - w_0(\omega_r)) = \lambda - \sum_{r=1}^{\ell} d_r(\omega_r + \omega_{\ell-r+1}). \tag{3.1}$$

Using [7, Table 1, p. 69], we write the ω_i's, $1 \leq i \leq \ell$, in terms of the simple roots α_j, $1 \leq j \leq \ell$, and we see that for $1 \leq r \leq \lfloor \frac{\ell}{2} \rfloor$, we have $\omega_r + \omega_{\ell-r+1} = \sum_{j=1}^{r-1} j\alpha_j + r\sum_{j=r}^{\ell-r+1} \alpha_j + \sum_{j=\ell-r+2}^{\ell} (\ell+1-j)\alpha_j$; and if ℓ is odd, we have $\omega_{\frac{\ell+1}{2}} = \frac{1}{2}\left[\alpha_1 + 2\alpha_2 + \cdots + \frac{\ell-1}{2} \cdot \alpha_{\frac{\ell-1}{2}} + \frac{\ell+1}{2} \cdot \alpha_{\frac{\ell+1}{2}} + \frac{\ell-1}{2} \cdot \alpha_{\frac{\ell+1}{2}+1} + \cdots + \alpha_\ell\right]$. Substituting in (3.1), we determine that $e_1(\lambda) = \sum_{j=1}^{\ell} d_j$. □

Lemma 3.8. *Let G be of type C_ℓ and let $\lambda = \sum_{i=1}^{\ell} d_i\omega_i \in X(T)^+$. Then $e_1(\lambda) = 2\sum_{j=1}^{\ell} d_j$ and $e_\ell(\lambda) = \sum_{j=1}^{\ell} jd_j$.*

Proof. Note that we have $w_0 = -1$, hence $w_0(\lambda) = -\lambda$. We write the ω_i's, $1 \le i \le \ell$, in terms of the simple roots α_j, $1 \le j \le \ell$, see [7, Table 1, p. 69], and we get:

$$w_0(\lambda) = -\lambda = \lambda - 2\lambda = \lambda - 2(d_1 + \cdots + d_\ell)\alpha_1 - 2\Big(d_1 + 2\sum_{j=2}^{\ell} d_j\Big)\alpha_2 - \cdots - \Big(\sum_{j=1}^{\ell} jd_j\Big)\alpha_\ell.$$

We remark that the coefficient of each α_i is a nonnegative integer and the result follows. □

Lemma 3.9. *Let G be of type B_ℓ and let $\lambda = \sum_{i=1}^{\ell} d_i\omega_i \in X(T)^+$. Then $e_1(\lambda) = 2\Big[\sum_{j=i}^{\ell-1} d_j\Big] + d_\ell$.*

Proof. We have that $w_0 = -1$, hence $w_0(\lambda) = -\lambda$. Writing the fundamental dominant weights ω_i in terms of the simple roots α_j, we see that:

$$w_0(\lambda) = -\lambda = \lambda - 2\lambda = \lambda - 2\sum_{i=1}^{\ell}\Big[\sum_{j=1}^{i-1} jd_j + i\Big(\sum_{j=i}^{\ell-1} d_j + \frac{1}{2}d_\ell\Big)\Big]\alpha_i,$$

therefore $e_1(\lambda) = 2\Big[\sum_{j=i}^{\ell-1} d_j\Big] + d_\ell$. □

Lemma 3.10. *Let G be of type D_ℓ and let $\lambda = \sum_{i=1}^{\ell} d_i\omega_i \in X(T)^+$. Then $e_1(\lambda) = 2\Big[\sum_{j=1}^{\ell-2} d_j + \frac{1}{2}i(d_{\ell-1} + d_\ell)\Big]$.*

Proof. We first assume that ℓ is even. Then $w_0 = -1$, hence $w_0(\lambda) = -\lambda$, and so

$$\begin{aligned} w_0(\lambda) &= \lambda - 2\sum_{j=1}^{\ell-2} d_j\omega_j - 2d_{\ell-1}\omega_{\ell-1} - 2d_\ell\omega_\ell = \lambda - \sum_{r=1}^{\ell-2} 2\Big[\sum_{j=1}^{r-1} jd_j + r\sum_{j=r}^{\ell-2} d_j + \frac{1}{2}r(d_{\ell-1} + d_\ell)\Big]\alpha_r \\ &\quad - \Big[\sum_{j=1}^{\ell-2} jd_j + \frac{1}{2}(\ell d_{\ell-1} + (\ell-2)d_\ell)\Big]\alpha_{\ell-1} - \Big[\sum_{j=1}^{\ell-2} jd_j + \frac{1}{2}((\ell-2)d_{\ell-1} + \ell d_\ell)\Big]\alpha_\ell. \end{aligned}$$

Thus $e_1(\lambda) = 2\Big[\sum_{j=1}^{\ell-2} d_j + \frac{1}{2}(d_{\ell-1} + d_\ell)\Big]$. We now assume that ℓ is odd. We note that $w_0(\omega_j) = -\omega_j$, for all $1 \le j \le \ell - 2$, $w_0(\omega_{\ell-1}) = -\omega_\ell$ and $w_0(\omega_\ell) = -\omega_{\ell-1}$. It follows that:

$$\begin{aligned} w_0(\lambda) &= -\sum_{j=1}^{\ell-2} d_j\omega_j - d_{\ell-1}\omega_\ell - d_\ell\omega_{\ell-1} = \lambda - 2\sum_{j=1}^{\ell-2} d_j\omega_j - (d_{\ell-1} + d_\ell)(\omega_{\ell-1} + \omega_\ell) \\ &= \lambda - \sum_{r=1}^{\ell-2} 2\Big[\sum_{j=1}^{r-1} jd_j + r\sum_{j=r}^{\ell-2} d_j + \frac{1}{2}r(d_{\ell-1} + d_\ell)\Big]\alpha_r - \Big[\sum_{j=1}^{\ell-2} jd_j + \frac{1}{2}(\ell-1)(d_{\ell-1} + d_\ell)\Big](\alpha_{\ell-1} + \alpha_\ell) \end{aligned}$$

and so $e_1(\lambda) = 2\Big[\sum_{j=1}^{\ell-2} d_j + \frac{1}{2}(d_{\ell-1} + d_\ell)\Big]$. □

3.3. The algorithm for calculating $\nu_G(V)$

Let $V = L_G(\lambda)$ for some $\lambda = \sum_{i=1}^{\ell} d_i\omega_i \in X(T)^+$. Consider the restriction $V\mid_{[L_i,L_i]} = \bigoplus_{j=0}^{e_i(\lambda)} V^j$, where $1 \le i \le \ell$ and $V^j = \bigoplus_{\gamma \in \mathbb{N}\Delta_i} V_{\lambda - j\alpha_i - \gamma}$, for all $1 \le j \le e_i(\lambda)$. In view of Proposition 3.1, in order to determine $\nu_G(V)$, one has to calculate $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu))$ and $\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$. In this section we will outline an algorithm for calculating $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu))$ and $\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$.

First, let $s \in T \setminus Z(G)$. Then, in particular, $s \in L_i$ and so $s = z \cdot h$, where $z \in Z(L_i)^\circ$ and $h \in [L_i, L_i]$. As $z \in Z(L_i)^\circ$ and $Z(L_i)^\circ$ is a one-dimensional torus, there exists $c \in k^*$ and $k_r \in \mathbb{Z}$, $1 \le r \le \ell$, such that $z = \prod_{r=1}^{\ell} h_{\alpha_r}(c^{k_r})$. Moreover, we have $\alpha_j(z) = 1$ for all $1 \le j \le \ell, j \ne i$. On the other hand, as $h \in [L_i, L_i]$, we have $h = \prod_{r\ne i} h_{\alpha_r}(a_r)$, where $a_r \in k^*$ for all $r \ne i$. Now, as $z \in Z(L_i)^\circ$, z acts on each V^j, $0 \le j \le e_i(\lambda)$, as scalar multiplication by s_z^j, where:

$$s_z^j := (\lambda - j\alpha_i - \gamma)(z) = (\lambda - j\alpha_i)(\prod_{r=1}^{\ell} h_{\alpha_r}(c^{k_r})) = \prod_{r=1}^{\ell} \left(c^{k_r d_r}\right) \cdot \prod_{r=1}^{\ell} c^{-jk_r\langle\alpha_i,\alpha_r\rangle}. \tag{3.2}$$

Lastly, let $\mu_1^j, \dots, \mu_{t_j}^j$, $t_j \ge 1$, be the distinct eigenvalues of h on V^j, $0 \le j \le e_i(\lambda)$, and let $n_1^j, \dots, n_{t_j}^j$ be their respective multiplicities. Then, as $s = z \cdot h$, it follows that the eigenvalues of s on V^j are $s_z^j\mu_1^j, \dots, s_z^j\mu_{t_j}^j$ and they are distinct, as the μ_r^j's are, with respective multiplicities $n_1^j, \dots, n_{t_j}^j$. This proves the following:

Lemma 3.11. *Let $s \in T \setminus Z(G)$, $s = z \cdot h$ with $z \in Z(L_i)^\circ$ and $h \in [L_i, L_i]$. Let $\mu_1^j, \dots, \mu_{t_j}^j$, $t_j \ge 1$, be the distinct eigenvalues of h on V^j, $0 \le j \le e_i(\lambda)$, with respective multiplicities $n_1^j, \dots, n_{t_j}^j$. Then:*

(a) z acts on V^j as scalar multiplication by s_z^j, where s_z^j is given in (3.2);
(b) the distinct eigenvalues of s on V^j are $s_z^j\mu_1^j, \dots, s_z^j\mu_{t_j}^j$, with multiplicities $n_1^j, \dots, n_{t_j}^j$;
(c) the eigenvalues of s on V are $s_z^j\mu_1^j, \dots, s_z^j\mu_{t_j}^j$, $0 \le j \le e_i(\lambda)$, with respective multiplicities at least $n_1^j, \dots, n_{t_j}^j$.

An algorithm for calculating $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu))$. First, assume that s admits an eigenvalue μ on V with the property that $\dim(V_s^j(\mu)) = \dim(V^j)$ for some $0 \le j \le e_i(\lambda)$. In this case $s \in Z(L_i)^\circ \setminus Z(G)$, $s = \prod_{r=1}^{\ell} h_{\alpha_r}(c^{k_r})$, and so it acts on each V^j as scalar multiplication by s_z^j. Therefore, the eigenvalues of s on V, not necessarily distinct, are $s_z^0, \dots, s_z^{e_i(\lambda)}$. We also remark that the s_z^j's are not all equal, as $s \notin Z(G)$. We have that $\dim(V_s(\mu)) = \sum_{j\in I_\mu} \dim(V^j)$, where $I_\mu \subsetneq \{0, \dots, e_i(\lambda)\}$ is such that for any $j \in I_\mu$ we have $\mu = s_z^j$. Therefore, $\max_{s'\in Z(L_i)^\circ\setminus Z(G)} \dim(V_{s'}(\mu')) = \max_{s'\in Z(L_i)^\circ\setminus Z(G)} \sum_{j\in I_{\mu'}} \dim(V^j)$, where the calculation of the latter maximum is straightforward. Secondly, assume that $\dim(V_s^j(\mu)) < \dim(V^j)$ for all eigenvalues μ of s on V and all $0 \le j \le e_i(\lambda)$. This case is solved inductively. We write $s = z \cdot h$, where $z \in Z(L_i)^\circ$ and $h \in [L_i, L_i]$. Recall that $[L_i, L_i]$ is a semisimple simply connected group of rank $\ell - 1$. We have that $\dim(V_s(\mu)) \le \sum_{j=0}^{e_i(\lambda)} \dim(V_h^j(\mu_h^j))$, where $\mu = s_z^j\mu_h^j$ and $\dim(V_h^j(\mu_h^j)) < \dim(V^j)$ for all $0 \le j \le e_i(\lambda)$. Therefore, $\max_{s'\in T\setminus Z(L_i)^\circ} \dim(V_{s'}(\mu')) \le \sum_{j=0}^{e_i(\lambda)} \max_{h',\mu'_{h'}} \dim(V_{h'}^j(\mu'_{h'}))$, and we use induction to determine this upper-bound.

We now let $u \in G_u \setminus \{1\}$. We will denote by $k[u]$ the group algebra of $\langle u\rangle$ over k.

Lemma 3.12. *Let $u \in G$ be a unipotent element and let V be a finite-dimensional kG-module. Let $V = M_t \supseteq M_{t-1} \supseteq \cdots \supseteq M_1 \supseteq M_0 = 0$, where $t \geq 1$, be a filtration of $k[u]$-submodules of V. Then:*

$$\dim(V_u(1)) \leq \sum_{i=1}^{t} \dim((M_i/M_{i-1})_u(1)).$$

Moreover, suppose that for each i, we have a u-invariant decomposition $M_i = M_{i-1} \oplus M'_{i-1}$ with $M'_{i-1} \cong M_i/M_{i-1}$ as $k[u]$-modules. Then $\dim(V_u(1)) = \sum_{i=1}^{t} \dim((M_i/M_{i-1})_u(1))$.

Proof. For each $1 \leq i \leq t$, we fix a basis in M_i with the property that the matrix $(u)_{M_i/M_{i-1}}$ associated to the action of u on M_i/M_{i-1} is upper-triangular. Then, the matrix $(u)_V$ of the action of u on V is the block upper-triangular matrix:

$$(u)_V = \begin{pmatrix} (u)_{M_1} & \star & \star & \cdots & \star \\ 0 & (u)_{M_2/M_1} & \star & \cdots & \star \\ 0 & 0 & (u)_{M_3/M_2} & \cdots & \star \\ & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (u)_{M_t/M_{t-1}} \end{pmatrix}.$$

Using $(u)_V$, we calculate the matrix of the action of $u - \mathrm{id}_V$ on V:

$$(u-\mathrm{id}_V)_V = \begin{pmatrix} (u-\mathrm{id}_{M_1})_{M_1} & \star & \star & \cdots & \star \\ 0 & (u-\mathrm{id}_{M_2/M_1})_{M_2/M_1} & \star & \cdots & \star \\ 0 & 0 & \ddots & \cdots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (u-\mathrm{id}_{M_t/M_{t-1}})_{M_t/M_{t-1}} \end{pmatrix},$$

where $(u - \mathrm{id}_{M_i/M_{i-1}})_{M_i/M_{i-1}}$ is the matrix of the action of $u - \mathrm{id}_{M_i/M_{i-1}}$ on M_i/M_{i-1}, $1 \leq i \leq t$, with respect to the basis of M_i we have previously fixed. It follows that:

$$\mathrm{rank}(u - \mathrm{id}_V) \geq \sum_{i=1}^{t} \mathrm{rank}((u - \mathrm{id}_{M_i/M_{i-1}})_{M_i/M_{i-1}})$$

and, consequently, we have $\dim(\ker(u - \mathrm{id}_V)) \leq \sum_{i=1}^{t} \dim(\ker((u - \mathrm{id}_V) \mid_{M_i/M_{i-1}}))$. Now, as $V_u(1) = \ker(u - \mathrm{id}_V)$ we determine that $\dim(V_u(1)) \leq \sum_{i=1}^{t} \dim((M_i/M_{i-1})_u(1))$.

Lastly, for all $1 \leq i \leq t$, assume that there exists a $k[u]$-submodule M'_{i-1} of M_i such that $M_i = M_{i-1} \oplus M'_{i-1}$. Then $V \mid_{k[u]} = M'_0 \oplus \cdots \oplus M'_{t-1} \cong M_1 \oplus M_2/M_1 \oplus \cdots \oplus M_t/M_{t-1}$, and so there exists a basis of V with the property that:

$$(u-\mathrm{id}_V)_V = \begin{pmatrix} (u-\mathrm{id}_{M_1})_{M_1} & 0 & 0 & \cdots & 0 \\ 0 & (u-\mathrm{id}_{M_2/M_1})_{M_2/M_1} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (u-\mathrm{id}_{M_t/M_{t-1}})_{M_t/M_{t-1}} \end{pmatrix},$$

thereby $\mathrm{rank}(u - \mathrm{id}_V) = \sum_{i=1}^{t} \mathrm{rank}((u - \mathrm{id}_{M_i/M_{i-1}})_{M_i/M_{i-1}})$. Arguing as above, we establish that $\dim(V_u(1)) = \sum_{i=1}^{t} \dim((M_i/M_{i-1})_u(1))$. □

An algorithm for calculating $\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$. Let $P_i = L_i \cdot Q_i = \langle T, U_\beta \mid \beta \in \Phi_i\rangle \cdot \langle U_\alpha \mid \alpha \in \Phi^+ \setminus \Phi_i\rangle$ be the Levi decomposition of the maximal parabolic subgroup P_i of G. Let $u \in G_u$, $u = \prod_{\alpha\in\Phi^+} x_\alpha(c_\alpha)$, where the product respects the total order $\preceq$ on Φ and $c_\alpha \in k$. Now, as $u \in B$ and $B \subseteq P_i$, it follows that u admits a decomposition $u = \prod_{\alpha\in\Phi_i} x_\alpha(c'_\alpha) \cdot \prod_{\alpha\in\Phi^+\setminus\Phi_i} x_\alpha(c'_\alpha)$, where each of the products respects $\preceq$ and $c'_\alpha \in k$, for all $\alpha \in \Phi^+$. We set $u_{L_i} = \prod_{\alpha\in\Phi_i} x_\alpha(c'_\alpha)$ and $u_{Q_i} = \prod_{\alpha\in\Phi^+\setminus\Phi_i} x_\alpha(c'_\alpha)$, and we note that $u_{L_i} \in L_i$ and $u_{Q_i} \in Q_i$. Recall that $V = L_G(\lambda)$ for some $\lambda \in X(T)^+$ and that $V\mid_{[L_i,L_i]} = \bigoplus_{j=0}^{e_i(\lambda)} V^j$. Let $\mu \in \Lambda(V)$, with corresponding weight space V_μ, and let $\alpha \in \Phi$. As $U_\alpha V_\mu \subseteq \bigoplus_{r\in\mathbb{Z}_{\geq 0}} V_{\mu+r\alpha}$, see [16, Lemma 15.4], we have $u_{L_i} \cdot V^j \subseteq V^j$, $u_{Q_i} \cdot V^j \subseteq \bigoplus_{r=0}^{j} V^r$ and $(u_{Q_i} - 1) \cdot V^j \subseteq \bigoplus_{r=0}^{j-1} V^r$, for all $0 \leq j \leq e_i(\lambda)$. Therefore, V admits a filtration $V = M_{e_i(\lambda)} \supseteq M_{e_i(\lambda)-1} \supseteq \cdots \supseteq M_1 \supseteq M_0 \supseteq 0$ of $k[u]$-submodules, where $M_j = \bigoplus_{r=0}^{j} V^r$ for all $0 \leq j \leq e_i(\lambda)$. We see that u acts on each M^j/M^{j-1}, $1 \leq j \leq e_i(\lambda)$, as u_{L_i} and so, by Lemma 3.12, we determine that $\dim(V_u(1)) \leq \sum_{j=0}^{e_i(\lambda)} \dim(V^j_{u_{L_i}}(1)) = \dim(V_{u_{L_i}}(1))$. Therefore, if we identify the kL_i-composition factors of each V^j, $0 \leq j \leq e_i(\lambda)$, then using already proven results and Lemma 3.12, we can establish an upper-bound for each $\dim(V^j_{u_{L_i}}(1))$. Now, assuming that $u_{L_i} \neq 1$, the upper-bound we obtain for $\dim(V_{u_{L_i}}(1))$, hence for $\dim(V_u(1))$, will be strictly smaller than $\dim(V)$. Lastly, we remark that if $u = u_{L_i}$, i.e. $u_{Q_i} = 1$, then $u \cdot V^j \subseteq V^j$, for all $0 \leq j \leq e_i(\lambda)$, and thus, by Lemma 3.12, it follows that $\dim(V_u(1)) = \dim(V_{u_{L_i}}(1))$.

We end this section with two lemmas concerning the behavior of unipotent elements. The first one is due to Guralnick and Lawther, [5], and tells us which unipotent conjugacy classes in G afford the largest dimensional eigenspaces.

Lemma 3.13. [5, p.19 and Lemmas 1.4.1 and 1.4.4] *We have* $\dim(V_{u_2}(1)) \leq \dim(V_{u_1}(1))$, *if* $u_1 \in G_u$ *belongs to a unipotent conjugacy class of root elements and* $u_2 \in G_u$ *belongs to any nontrivial unipotent class, unless* $e(\Phi) > 1$ *and one of the following holds:*

(a) $G = C_\ell$, $p = 2$, u_1 *belongs to the unipotent conjugacy class of* $x_{\alpha_\ell}(1)$ *and* u_2 *belongs to the unipotent conjugacy class of* $x_{\alpha_1}(1)$.
(b) $G = C_\ell$, u_1 *belongs to the unipotent conjugacy class of* $x_{\alpha_1}(1)$ *and* u_2 *belongs to the unipotent conjugacy class of* $x_{\alpha_\ell}(1)$.
(c) $G = B_\ell$, u_1 *belongs to the unipotent conjugacy class of* $x_{\alpha_\ell}(1)$ *and* u_2 *belongs to the unipotent conjugacy class of* $x_{\alpha_1}(1)$.

The second lemma gives us $\dim(\wedge^2(V)_u(1))$, when $\mathrm{char}(k) = 2$, V is a finite-dimensional k-vector space and the unipotent element u acts as a single Jordan block on $\mathrm{GL}(V)$. For each $i \geq 0$, let V_i be the indecomposable $k[u]$-module with $\dim(V_i) = i$ and on which u acts as the full Jordan block J_i of size i. Note that $\{V_i \mid i \geq 0\}$ is a set of representatives of the isomorphisms classes of indecomposable $k[u]$-modules.

Lemma 3.14. *Let* k *be a field of characteristic* $p = 2$ *and let* V *be a vector space of dimension* $i \geq 1$ *over* k. *Let* u *be a unipotent element acting as a single Jordan block in* $\mathrm{GL}(V)$. *Then* $\dim((\wedge^2(V))_u(1)) = \lfloor \frac{i}{2} \rfloor$.

Proof. We will prove the result by induction on $i \geq 1$. First, we note that both cases $i = 1$ and $i = 2$ follow directly from the structure of $\wedge^2(V)$. Hence, we assume that $i \geq 3$ and that the result holds for all $1 \leq r < i$. Let m be the unique nonnegative integer for which $2^{m-1} < i \leq 2^m$ and set $q = 2^m$. Now, up to isomorphism, there exist exactly q indecomposable $k[u]$-modules: $V_1, V_2, \dots, V_q$, where $\dim(V_j) = j$ and u acts on V_j as J_j. Therefore, as $k[u]$-modules, we have $V \cong V_i$. Now, by [3, Theorem 2], we have $\wedge^2(V_i) = \wedge^2(V_{q-i}) \oplus (i - \frac{q}{2} - 1)V_q \oplus V_{3\frac{q}{2}-i}$, and so

$$\dim((\wedge^2(V_i))_u(1)) = \dim((\wedge^2(V_{q-i}))_u(1)) + (i - \frac{q}{2} - 1)\dim((V_q)_u(1)) + \dim((V_{3\frac{q}{2}-i})_u(1)). \quad (3.3)$$

As $3\frac{q}{2} - i < q$ and as u acts as a single Jordan block on V_q and $V_{3\frac{q}{2}-i}$, respectively, it follows that $\dim((V_q)_u(1)) = 1$ and $\dim((V_{3\frac{q}{2}-i})_u(1)) = 1$. Furthermore, we note that, as $\frac{q}{2} < i$, we have $q - i < i$ and, by applying induction, it follows that $\dim((\wedge^2(V_{q-i}))_u(1)) = \left\lfloor \frac{q-i}{2} \right\rfloor$. Substituting in (3.3) we obtain $\dim((\wedge^2(V_i))_u(1)) = \left\lfloor \frac{q-i}{2} \right\rfloor + i - \frac{q}{2} - 1 + 1 = \left\lfloor \frac{i}{2} \right\rfloor$. □

3.4. The list of modules

Lemma 3.15. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected classical linear algebraic group. Let $\lambda \in X(T)_p^+$, $\lambda \neq 0$.*

(a) *Let G be of type A_ℓ with $\ell \geq 1$ and assume that $\dim(L_G(\lambda)) \leq \frac{\ell^3}{2}$. If $\ell \geq 15$, then, up to duality of the corresponding kG-module, we have $\lambda \in \Lambda_{A_\ell} := \{\omega_1, \omega_2, 2\omega_1, \omega_1 + \omega_\ell, \omega_3, 3\omega_1, \omega_1 + \omega_2\}$. If $\ell \leq 14$, the additional λ's are given in Table 2.*

Table 2. The nonzero weights $\lambda \in X(T)_p^+ \setminus \Lambda_{A_\ell}$ with $\dim(L_G(\lambda)) \leq \frac{\ell^3}{2}$.

Rank	λ	p	$\dim(L_G(\lambda))$
$7 \leq \ell \leq 14$	ω_4	all	$\binom{\ell+1}{4}$
$9 \leq \ell \leq 10$	ω_5	all	$\binom{\ell+1}{5}$

(b) *Let G be of type C_ℓ with $\ell \geq 2$ and assume that $\dim(L_G(\lambda)) \leq 4\ell^3$. If $\ell \geq 14$, we have that $\lambda \in \Lambda_{C_\ell} := \{\omega_1, \omega_2, 2\omega_1, \omega_3, 3\omega_1, \omega_1 + \omega_2\}$. If $\ell \leq 13$, the additional λ's are given in Table 3.*

Table 3. The nonzero weights $\lambda \in X(T)_p^+ \setminus \Lambda_{C_\ell}$ with $\dim(L_G(\lambda)) \leq 4\ell^3$.

Rank	λ	p	$\dim(L_G(\lambda))$
$\ell = 2$	$2\omega_2$	$p \neq 2$	$14 - \varepsilon_p(5)$
	$\omega_1 + 2\omega_2$	$p = 7$	24
	$3\omega_2$	$p \neq 2, 3$	$30 - 5\varepsilon_p(7)$
	$2\omega_1 + \omega_2$	$p = 3$	25
$\ell = 3$	$\omega_1 + \omega_3$	all	$70 - 13\varepsilon_p(3) - 22\varepsilon_p(2)$
	$2\omega_1 + \omega_3^\ddagger$	$p = 2$	48
	$\omega_2 + \omega_3$	$p = 5$	62
	$2\omega_3$	$p \neq 2$	$84 - 21\varepsilon_p(5)$
	$2\omega_2$	$p \neq 2$	$90 - \varepsilon_p(7)$
$\ell = 4$	ω_4	$p \neq 2$	$42 - \varepsilon_p(3)$
	$\omega_1 + \omega_4$	$p = 2$	128
	$2\omega_1 + \omega_4^\ddagger$	$p = 2$	128
	$\omega_1 + \omega_4$	$p = 7$	240
	$\omega_1 + \omega_3$	$p = 2$	246
$\ell = 5$	ω_4	all	$165 - \varepsilon_p(2) - 44\varepsilon_p(3)$
	ω_5	$p \neq 2$	$132 - 10\varepsilon_p(3)$
	$\omega_1 + \omega_5$	$p = 2$	320
	$2\omega_1 + \omega_5^\ddagger$	$p = 2$	320

(*Continued*)

Table 3. (*Continued*)

Rank	λ	p	$\dim(L_G(\lambda))$
$\ell = 6$	ω_4	all	$429 - \varepsilon_p(5) - 65\varepsilon_p(2)$
	ω_6	$p \neq 2$	$429 - 64\varepsilon_p(3)$
	ω_5	all	$572 - 12\varepsilon_p(2) - 208\varepsilon_p(3)$
	$\omega_1 + \omega_6$	$p = 2$	768
	$2\omega_1 + \omega_6^{\ddagger}$	$p = 2$	768
$\ell = 7$	ω_4	all	$910 - \varepsilon_p(3) - 90\varepsilon_p(5)$
	ω_6	$p = 3$	1093
	ω_7	$p = 3$	1094
	ω_5	$p = 2$	1288
$\ell = 8$	ω_4	all	$1700 - \varepsilon_p(7) - 119\varepsilon_p(3) - 118\varepsilon_p(2)$
$\ell = 9$	ω_4	all	$2907 - 152\varepsilon_p(7) - \varepsilon_p(2)$
$4 \leq \ell \leq 13$	ω_ℓ	$p = 2$	2^ℓ

‡ Since we have made the choice to only treat groups of type C_ℓ when $p = 2$, see Lemma 3.3, and since $\dim(L_{B_\ell}(\omega_1 + \omega_\ell)) \leq 4\ell^3$ for $3 \leq \ell \leq 6$, in order to have a complete result, we have to add the weights $2\omega_1 + \omega_\ell$, where $3 \leq \ell \leq 6$, see Remark 3.4.

(c) *Let $p \neq 2$ and let G be of type B_ℓ with $\ell \geq 3$. Assume that $\dim(L_G(\lambda)) \leq 4\ell^3$. If $\ell \geq 14$, we have $\lambda \in \Lambda_{B_\ell} := \{\omega_1, \omega_2, 2\omega_1, \omega_3, 3\omega_1, \omega_1 + \omega_2\}$. If $\ell \leq 13$, the additional λ's are given in Table 4.*

Table 4. The nonzero weights $\lambda \in X(T)_p^+ \setminus \Lambda_{B_\ell}$ with $\dim(L_G(\lambda)) \leq 4\ell^3$.

Rank	λ	p	$\dim(L_G(\lambda))$
$\ell = 3$	$2\omega_3$	$p \neq 2$	35
	$\omega_2 + \omega_3$	$p = 3, 5$	$104 - 40\varepsilon_p(5)$
	$3\omega_3$	$p = 5$	104
$\ell = 4$	$2\omega_4$	$p \neq 2$	126
$\ell = 5$	$2\omega_5$	$p \neq 2$	462
$3 \leq \ell \leq 6$	$\omega_1 + \omega_\ell$	$p \neq 2$	$\ell \cdot 2^{\ell+1} - 2^\ell \varepsilon_p(2\ell + 1)$
$5 \leq \ell \leq 7$	ω_4	$p \neq 2$	$\binom{2\ell+1}{4}$
$4 \leq \ell \leq 13$	ω_ℓ	$p \neq 2$	2^ℓ

(d) *Let G be of type D_ℓ with $\ell \geq 4$ and assume that $\dim(L_G(\lambda)) \leq 4\ell^3$. If $\ell \geq 16$, we have $\lambda \in \Lambda_{D_\ell} := \{\omega_1, \omega_2, 2\omega_1, \omega_3, 3\omega_1, \omega_1 + \omega_2\}$. If $\ell \leq 15$, the additional λ's are given in Table 5 (up to duality or outer automorphisms of the corresponding kG-module).*

Table 5. The nonzero weights $\lambda \in X(T)_p^+ \setminus \Lambda_{D_\ell}$ with $\dim(L_G(\lambda)) \leq 4\ell^3$.

Rank	λ	p	$\dim(L_G(\lambda))$
$\ell = 4$	$\omega_1 + \omega_4$	all	$56 - 8\varepsilon_p(2)$
	$2\omega_2$	$p = 3$	195
	$2\omega_1 + \omega_3$	all	$224 - 56\varepsilon_p(5)$
	$\omega_1 + \omega_3 + \omega_4$	$p = 2$	246
$\ell = 5$	$2\omega_5$	$p \neq 2$	126
	$\omega_1 + \omega_5$	all	$144 - 16\varepsilon_p(5)$
	$\omega_4 + \omega_5$	all	$210 - 46\varepsilon_p(2)$
	$\omega_2 + \omega_5$	$p = 2$	416
$\ell = 6$	$\omega_1 + \omega_6$	all	$352 - 32\varepsilon_p(2) - 32\varepsilon_p(3)$
	ω_4	all	$495 - 131\varepsilon_p(2)$
	$2\omega_6$	$p \neq 2$	462
	$\omega_5 + \omega_6$	all	$792 - 232\varepsilon_p(2)$
$\ell = 7$	$\omega_1 + \omega_7$	all	$832 - 64\varepsilon_p(7)$
	ω_4	all	$1001 - 91\varepsilon_p(2)$
	ω_5	$p = 2$	1288
$\ell = 8$	ω_4	all	$1820 - 238\varepsilon_p(2)$
	$\omega_1 + \omega_8$	all	$1920 - 128\varepsilon_p(2)$
$\ell = 9$	ω_4	$p = 2$	2906
$5 \leq \ell \leq 15$	ω_ℓ	all	$2^{\ell-1}$

Proof. The result follows by [17, Theorem 1.2] and [14]. □

4. Proof of Theorem 1.1 for groups of type A_ℓ

Let G be a simple simply connected linear algebraic group of type A_ℓ with $\ell \geq 1$. In view of Proposition 3.1, it is sufficient to know $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu))$ and $\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$ in order to determine $\nu_G(V)$, where V is any irreducible kG-module. In this section we prove Theorems 4.1 and 4.2 which provide the values of $\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$ and $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu))$ for all kG-modules $V = L_G(\lambda)$ with $\lambda \in X(T)_p^+$, $\lambda \neq 0$, and $\dim(L_G(\lambda)) \leq \frac{\ell^3}{2}$. As a corollary, the part of Theorem 1.1 concerning simple simply connected linear algebraic groups of type A_ℓ will follow.

Theorem 4.1. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type A_ℓ with $\ell \geq 1$. Let $V = L_G(\lambda)$, where $\lambda \in X(T)_p^+$ and $\lambda \neq 0$, be such that $\dim(V) \leq \frac{\ell^3}{2}$. The value of $\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$ is as given in Table 6.*

Table 6. The value of $\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$ for groups of type A_ℓ.

Ref.	V	Char.	Rank	$\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$
4.1.1	$L_G(\omega_1)$	$p \geq 0$	$\ell \geq 1$	ℓ
4.1.2	$L_G(\omega_2)$	$p \geq 0$	$\ell \geq 3$	$\frac{\ell^2-\ell+2}{2}$
4.1.3	$L_G(2\omega_1)$	$p \neq 2$	$\ell \geq 1$	$\binom{\ell+1}{2}$
4.1.4	$L_G(\omega_1+\omega_\ell)$	$p \geq 0$	$\ell \geq 2$	$\ell^2 - \varepsilon_p(\ell+1)$
4.1.5	$L_G(\omega_3)$	$p \geq 0$	$\ell \geq 5$	$\binom{\ell}{3} + \ell - 1$
4.1.6	$L_G(3\omega_1)$	$p \neq 2, 3$	$\ell \geq 1$	$\binom{\ell+2}{3}$
4.1.7	$L_G(\omega_1+\omega_2)$	$p = 3$	$\ell \geq 3$	$\binom{\ell+2}{3} - \ell + 1$
		$p \neq 3$	$\ell \geq 3$	$\frac{\ell^3+2\ell}{3}$
4.1.8	$L_G(\omega_4)$	$p \geq 0$	$7 \leq \ell \leq 14$	$\binom{\ell-1}{4} + \binom{\ell-1}{3} + \binom{\ell-1}{2}$
4.1.9	$L_G(\omega_5)$	$p \geq 0$	$9 \leq \ell \leq 10$	$\binom{\ell-1}{5} + \binom{\ell-1}{4} + \binom{\ell-1}{3}$

Proof. To begin, recall that we have denoted by W the natural module of G, see end of Section 2. Let (V, p, ℓ) be a triplet featured in Table 6. In view of Lemma 3.13, for any $1 \leq i \leq \ell$, we have $\max_{u\in G_u\setminus\{1\}} \dim(V_u(1)) = \dim(V_{x_{\alpha_i}(1)}(1))$. Thus, in what follows, we will focus on calculating $\dim(V_{x_{\alpha_i}(1)}(1))$, $1 \leq i \leq \ell$. To ease notation, we reference each triplet (V, p, ℓ) featured in Table 6 by its corresponding number in column 'Ref'.
4.1.1: Follows because $V \cong W$ as kG-modules, and $x_{\alpha_1(1)}$ acts on W as $J_2 \oplus J_1^{\ell-1}$.
4.1.2: Note that $V \cong \wedge^2(W)$ by [18, Proposition 4.2.2]. We write $W = W_1 \oplus W_2$, where $\dim(W_1) = 2$ and $x_{\alpha_1}(1)$ acts on W_1 as J_2; and $\dim(W_2) = \ell - 1$ and $x_{\alpha_1}(1)$ acts trivially on W_2. Since $\wedge^2(W_1 \oplus W_2) \cong \wedge^2(W_1) \oplus [W_1 \otimes W_2] \oplus \wedge^2(W_2)$, we get $\dim((\wedge^2(W))_{x_{\alpha_1}(1)}(1)) = \dim((\wedge^2(W_1))_{x_{\alpha_1}(1)}(1)) + \dim((W_1 \otimes W_2)_{x_{\alpha_1}(1)}(1)) + \dim((\wedge^2(W_2))_{x_{\alpha_1}(1)}(1))$. As $x_{\alpha_1}(1)$ acts as a single Jordan block on $\wedge^2(W_1)$, using [15, Lemma 3.4], respectively Lemma 3.14 when $p = 2$, we get $\dim((\wedge^2(W_1))_{x_{\alpha_1}(1)}(1)) = 1$. Further, as $x_{\alpha_1}(1)$ acts trivially on W_2, we have $\dim((\wedge^2(W_2))_{x_{\alpha_1}(1)}(1)) = \frac{\ell^2-3\ell+2}{2}$. Lastly, as $x_{\alpha_1}(1)$ acts on $W_1 \otimes W_2$ as $J_2 \otimes J_1^{\ell-1}$, we have $\dim((W_1 \otimes W_2)_{x_{\alpha_1}(1)}(1)) = \ell - 1$, by [15, Lemma 3.4]. It follows that $\dim((\wedge^2(W))_{x_{\alpha_1}(1)}(1)) = \frac{\ell^2-\ell+2}{2}$.
4.1.3: Note that $V \cong S^2(W)$ by [18, Proposition 4.2.2]. We write $W = W_1 \oplus W_2$, where $\dim(W_1) = 2$ and $x_{\alpha_1}(1)$ acts on W_1 as J_2; and $\dim(W_2) = \ell - 1$ and $x_{\alpha_1}(1)$ acts trivially on W_2. Then, since $S^2(W_1 \oplus W_2) \cong S^2(W_1) \oplus [W_1 \otimes W_2] \oplus S^2(W_2)$, we argue as above to show $\dim((S^2(W))_{x_{\alpha_1}(1)}(1)) = \binom{\ell+1}{2}$.
4.1.4: Follows from [18, Proposition 4.6.10] and [11, Theorem 6.1], arguing as in 4.1.2 and 4.1.3.
4.1.5, 4.1.6, 4.1.8, and 4.1.9: Follow from [18, Proposition 4.2.2], arguing as in 4.1.2 and 4.1.3.

4.1.7: First, assume $p \neq 3$. By [18, Proposition 4.6.10], we have that $L_G(\omega_1) \otimes L_G(\omega_2) \cong V \oplus L_G(\omega_3)$, and so $\dim(V_{x_{\alpha_\ell(1)}}(1)) = \dim((L_G(\omega_1) \otimes L_G(\omega_2))_{x_{\alpha_\ell}(1)}(1)) - \dim((L_G(\omega_3))_{x_{\alpha_\ell}(1)}(1))$. As $x_{\alpha_\ell}(1)$ acts on $L_G(\omega_1)$ as $J_2 \oplus J_1^{\ell-1}$ and on $L_G(\omega_2)$ as $J_2^{\ell-1} \oplus J_1^{\frac{\ell^2-3\ell+4}{2}}$, see 4.1.2, one shows that $\dim((L_G(\omega_1) \otimes L_G(\omega_2))_{x_{\alpha_\ell}(1)}(1)) = \frac{\ell^3-\ell^2+4\ell-2}{2}$. The result now follows by 4.1.1, 4.1.2, and 4.1.5.

Now, let $p = 3$. Set $\lambda = \omega_1 + \omega_2$ and $L = L_1$. By Lemma 3.7, we have $e_1(\lambda) = 2$, therefore $V \mid_{[L,L]} = V^0 \oplus V^1 \oplus V^2$. First, by [20, Proposition], we have $V^0 \cong L_L(\omega_2)$. Secondly, the weight $(\lambda - \alpha_1) \mid_{T_1} = 2\omega_2$ admits a maximal vector in V^1, thus V^1 has a composition factor isomorphic to $L_L(2\omega_2)$. Lastly, the weight $(\lambda - 2\alpha_1 - \alpha_2) \mid_{T_1} = \omega_2 + \omega_3$ admits a maximal vector in V^2, thus V^2 has a composition factor isomorphic to $L_L(\omega_2 + \omega_3)$. By dimensional considerations, we determine that

$$V \mid_{[L,L]} \cong L_L(\omega_2) \oplus L_L(2\omega_2) \oplus L_L(\omega_2 + \omega_3).$$

By 4.1.1 and 4.1.3, we have $\dim(V_{x_{\alpha_\ell}(1)}(1)) = \ell - 1 + \binom{\ell}{2} + \dim((L_L(\omega_2 + \omega_3))_{x_{\alpha_\ell}(1)}(1))$. Recursively and using 4.1.5, for the base case of $\ell = 3$, we get $\dim(V_{x_{\alpha_\ell}(1)}(1)) = 3 + \sum_{j=2}^{\ell-1} \binom{j+1}{2} + \sum_{j=2}^{\ell-1} j = \binom{\ell+2}{3} - \ell + 1$.

□

Theorem 4.2. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type A_ℓ with $\ell \geq 1$. Let $V = L_G(\lambda)$, where $\lambda \in X(T)_p^+$ and $\lambda \neq 0$, be such that $\dim(V) \leq \frac{\ell^3}{2}$. The value of $\max_{s \in T \setminus Z(G)} \dim(V_s(\mu))$ is as given in Table 7.*

Table 7. The value of $\max_{s \in T \setminus Z(G)} \dim(V_s(\mu))$ for groups of type A_ℓ.

Ref.	V	Char.	Rank	$\max_{s \in T \setminus Z(G)} \dim(V_s(\mu))$
4.2.1	$L_G(\omega_1)$	$p \geq 0$	$\ell \geq 1$	ℓ
4.2.2	$L_G(\omega_2)$	$p \geq 0$	$\ell \geq 3$	$\frac{\ell(\ell-1)}{2} + \varepsilon_\ell(3)$
4.2.3	$L_G(2\omega_1)$	$p \neq 2$	$\ell \geq 1$	$\frac{\ell^2+\ell+2}{2}$
4.2.4	$L_G(\omega_1 + \omega_\ell)$	$p \geq 0$	$\ell \geq 2$	$\ell^2 - \varepsilon_p(\ell+1) + \varepsilon_p(3)\varepsilon_\ell(2)$
4.2.5	$L_G(\omega_3)$	$p \geq 0$	$\ell \geq 5$	$\leq \binom{\ell}{3} + 2$
4.2.6	$L_G(3\omega_1)$	$p \neq 2, 3$	$\ell \geq 1$	$\binom{\ell+2}{3} + \ell$
4.2.7	$L_G(\omega_1 + \omega_2)$	$p = 3$	$\ell \geq 3$	$\binom{\ell+2}{3}$
		$p \neq 3$	$\ell \geq 3$	$\frac{\ell^3+2\ell}{3} - \ell\varepsilon_p(2)$
4.2.8	$L_G(\omega_4)$	$p \geq 0$	$7 \leq \ell \leq 14$	$\leq \binom{\ell}{4} + 2\ell - 5$
4.2.9	$L_G(\omega_5)$	$p \geq 0$	$9 \leq \ell \leq 10$	$\leq \binom{\ell}{5} + \ell^2 - 6\ell + 9$

In order to prove Theorem 4.2, in addition to the algorithm for calculating $\max_{s \in T \setminus Z(G)} \dim(V_s(\mu))$ outlined in Section 3.3, we will sometimes use the following algorithm from [5, Section 2.2], which gives lower bounds for $\nu_G(V)$. We give a brief description of it in what follows. Let Ψ be a standard subsystem of Φ and let $\mathcal{W}(\Psi)$ be its Weyl group. We define $r_\Psi = \frac{|\mathcal{W}:\mathcal{W}(\Psi)| \cdot |\Phi \setminus \Psi|}{2|\Phi_s|}$. For $\lambda' = \sum_{i=1}^{\ell} a_i\omega_i \in X(T)^+$, set $\Psi(\lambda') = \langle \alpha_i \mid a_i = 0 \rangle$, and for $\lambda \in X(T)_p^+$, define $s_\lambda = \sum_{0 \preceq \lambda' \preceq \lambda} r_{\lambda'}$. By [5, Prop. 2.2.1], we have $\nu_G(L_G(\lambda)) \geq s_\lambda$. Now, as $s_\lambda = \sum_{0 \preceq \mu \preceq \lambda} r_\mu$, it will prove extremely useful to give a formula for r_{ω_i}, $1 \leq i \leq \ell$. To this end, we first note that $\Psi(\omega_i)$, $1 \leq i \leq \ell$, is of type $A_{i-1}A_{\ell-i}$, thus $|\Psi(\omega_i)| = \ell^2 - 2\ell i + 2i^2 + \ell - 2i$ and $|\Phi \setminus \Psi(\omega_i)| = 2i(\ell - i + 1)$. Moreover, as $|\mathcal{W}(\Psi(\omega_i))| = i!(\ell - i + 1)!$, we have $|\mathcal{W} : \mathcal{W}(\Psi(\omega_i))| = \binom{\ell+1}{i}$, therefore

$$r_{\omega_i} = \binom{\ell+1}{i} \cdot \frac{2i(\ell-i+1)}{2\ell(\ell+1)} = \binom{\ell-1}{i-1}. \tag{4.1}$$

Let $s \in T \setminus Z(G)$. To improve readability, we will use the expression "s is as in $({}^{\dagger}H_s)$" to mean that s satisfies the following: $s = \operatorname{diag}(\mu_1 \cdot I_{n_1}, \mu_2 \cdot I_{n_2}, \dots, \mu_m \cdot I_{n_m})$, where $m \geq 2$, $\mu_i \neq \mu_j$ for all $i < j$, $\ell \geq n_1 \geq \cdots \geq n_m \geq 1$ and $\prod_{i=1}^{m} \mu_i^{n_i} = 1$. Note that any $s \in T \setminus Z(G)$ is as in $({}^{\dagger}H_s)$.

Proposition 4.3. *Let $V = L_G(\omega_1)$. Then $\max_{s \in T \setminus Z(G)} \dim(V_s(\mu)) = \ell$, where the maximum is attained if and only if s is conjugate to $\operatorname{diag}(d, d, \dots, d, d^{-\ell})$ with $d^{\ell+1} \neq 1$ and $\mu = d$.*

Proof. As $V \cong W$ as kG-modules, for all $(s, \mu) \in (T \setminus Z(G)) \times k^*$ we have $\dim(V_s(\mu)) \leq \ell$. Now, equality holds if and only if s and μ are as in the statement of the result. □

Proposition 4.4. *Let $\ell \geq 3$ and let $V = L_G(\omega_2)$. Then $\max_{s \in T \setminus Z(G)} \dim(V_s(\mu)) = \frac{\ell(\ell-1)}{2} + \varepsilon_\ell(3)$, where the maximum is attained if and only if*

(1) $\ell = 3$ and s is conjugate to $\operatorname{diag}(d, d, \pm d^{-1}, \pm d^{-1})$ with $d^2 \neq \pm 1$, and $\mu = \pm 1$.
(2) $\ell = 4$ and s is conjugate to $\operatorname{diag}(d, d, d, e, e)$ with $d \neq e$ and $d^3 = e^{-2}$, and $\mu = de$.
(3) $\ell \geq 4$ and s is conjugate to $\operatorname{diag}(d, \dots, d, d^{-\ell})$ with $d^{\ell+1} \neq 1$, and $\mu = d^2$.

Proof. Let $s \in T \setminus Z(G)$ be as in $({}^{\dagger}H_s)$. As $V \cong \wedge^2(W)$, see [18, Proposition 4.2.2], we deduce that the eigenvalues of s on V, not necessarily distinct, have one of the following forms:

$$\begin{cases} \mu_i^2,\ 1 \leq i \leq m, \text{ with multiplicity at least } \frac{n_i(n_i-1)}{2}; \\ \mu_i\mu_j,\ 1 \leq i < j \leq m, \text{ with multiplicity at least } n_i n_j. \end{cases} \tag{4.2}$$

In order for s to have an eigenvalue of the form μ_j^2, there has to exist some $1 \leq i \leq m$ with $n_i \geq 2$. Suppose there exists such i, and consider the eigenvalue μ_i^2 of s on V. Now, since the μ_r's are distinct, it follows that $\mu_i^2 \neq \mu_i\mu_j$ for all $i \neq j$, hence:

$$\dim(V_s(\mu_i^2)) \leq \tfrac{\ell(\ell+1)}{2} - (\ell + 1 - n_i)n_i.$$

Let $\ell = 3$ and assume $\dim(V_s(\mu_i^2)) \geq 4$. Then $2 - (4 - n_i)n_i = (n_i - 2)^2 - 2 \geq 0$, which does not hold as $n_i \leq 3$. Thus, we let $\ell \geq 4$ and assume that $\dim(V_s(\mu_i^2)) \geq \frac{\ell(\ell-1)}{2}$. Then $(\ell - n_i)(1 - n_i) \geq 0$ and, since $\ell \geq n_i \geq 2$, the inequality holds if and only if $n_i = \ell$. Hence, $m = 2$, $n_1 = \ell$, $n_2 = 1$, $\mu_2 = \mu_1^{-\ell}$ and $\mu_1^{\ell+1} \neq 1$, as $\mu_1 \neq \mu_2$ and $\mu_1^{\ell}\mu_2 = 1$. This gives $\dim(V_s(\mu_i^2)) \leq \frac{\ell(\ell-1)}{2}$ for all $s \in T \setminus Z(G)$, where equality holds if and only if $i = 1$ and s is conjugate to $\operatorname{diag}(\mu_1, \dots, \mu_1, \mu_1^{-\ell})$ with $\mu_1^{\ell+1} \neq 1$, as in (3). This completes the case of eigenvalues of the form μ_j^2 of s on V.

Fix $i < j$ and consider the eigenvalue $\mu_i\mu_j$ of s on V. Since the μ_r's are distinct, we remark that:

$$\begin{cases} \mu_i\mu_j \neq \mu_i^2 \text{ and } \mu_i\mu_j \neq \mu_j^2; \\ \mu_i\mu_j \neq \mu_i\mu_r, \text{ for } i < r \leq m \text{ and } r \neq j, \text{ and } \mu_i\mu_j \neq \mu_r\mu_i, \text{ for } 1 \leq r < i; \\ \mu_i\mu_j \neq \mu_r\mu_j, \text{ for } 1 \leq r < j \text{ and } r \neq i, \text{ and } \mu_i\mu_j \neq \mu_j\mu_r, \text{ for } j < r \leq m. \end{cases}$$

By (4.2), these account for at least $\frac{n_i(n_i-1)}{2} + \frac{n_j(n_j-1)}{2} + (n_i + n_j)(\ell + 1 - n_i - n_j)$ eigenvalues of s on V different to $\mu_i\mu_j$. It follows that:

$$\dim(V_s(\mu_i\mu_j)) \leq \tfrac{\ell(\ell+1)}{2} - \tfrac{n_i(n_i-1)}{2} - \tfrac{n_j(n_j-1)}{2} - (n_i + n_j)(\ell + 1 - n_i - n_j).$$

As in the previous case, we begin with $\ell = 3$. Then, since $n_i + n_j \leq 4$, we have $(n_i, n_j) \in \{(3,1), (2,2), (2,1), (1,1)\}$. Assume that $\dim(V_s(\mu_i\mu_j)) \geq 4$. Then $2 - \frac{n_i(n_i-1)}{2} - \frac{n_j(n_j-1)}{2} - (n_i + n_j)(4 - n_i - n_j) \geq 0$, which holds if and only if $n_i = n_j = 2$, i.e. $m = 2$, $n_1 = n_2 = 2$, $\mu_2 = \pm\mu_1^{-1}$, as $\mu_1^2\mu_2^2 = 1$, and $\mu_1^2 \neq \pm 1$, as $\mu_1 \neq \mu_2$. Thus $\dim(V_s(\mu_i\mu_j)) \leq 4$ for all $s \in T \setminus Z(G)$ and all $i < j$, where equality holds if and only if s is conjugate to $\mathrm{diag}(\mu_1, \mu_1, \pm\mu_1^{-1}, \pm\mu_1^{-1})$ with $\mu_1^2 \neq \pm 1$, as in (1). We now let $\ell \geq 4$ and assume $\dim(V_s(\mu_i\mu_j)) \geq \frac{\ell(\ell-1)}{2}$. Then:

$$(\ell - n_i - n_j)(1 - n_i - n_j) - \tfrac{n_i(n_i-1)}{2} - \tfrac{n_j(n_j-1)}{2} \geq 0. \tag{4.3}$$

Since $n_i \geq n_j \geq 1$, we have $\frac{n_i(n_i-1)}{2} + \frac{n_j(n_j-1)}{2} \geq 0$ and $1 - n_i - n_j < 0$, therefore, by inequality (4.3), it follows that $\ell - n_i - n_j \leq 0$. If $n_i + n_j = \ell$, then, for inequality (4.3) to hold, we must have $\frac{n_i(n_i-1)}{2} + \frac{n_j(n_j-1)}{2} = 0$, hence $n_i = n_j = 1$, contradicting $\ell \geq 4$. On the other hand, if $n_i + n_j = \ell + 1$, then $m = 2$ and, by (4.3), we determine that $n_1(3 - n_1) - (n_2 - 1)(n_2 - 2) \geq 0$. Now, the inequality holds if and only if $n_1 = 3$ and $n_2 = 2$, as $\ell \geq 4$. In this case, $\ell = 4$ and $s = \mathrm{diag}(\mu_1, \mu_1, \mu_1, \mu_2, \mu_2)$ with $\mu_1 \neq \mu_2$ and $\mu_1^3 = \mu_2^{-2}$. Therefore, $\dim(V_s(\mu_i\mu_j)) \leq \frac{\ell(\ell-1)}{2}$ for all $s \in T \setminus Z(G)$ and all $i < j$, where equality holds if and only if $\ell = 4$ and s is conjugate to $\mathrm{diag}(\mu_1, \mu_1, \mu_1, \mu_2, \mu_2)$ with $\mu_1 \neq \mu_2$ and $\mu_1^3 = \mu_2^{-2}$, as in (2). □

Proposition 4.5. *Assume $p \neq 2$ and let $V = L_G(2\omega_1)$. Then $\max_{s \in T\setminus Z(G)} \dim(V_s(\mu)) = \frac{\ell^2+\ell+2}{2}$, where the maximum is attained if and only if s is conjugate to* $\mathrm{diag}(d, \ldots, d, d^{-\ell})$ *with $d^{\ell+1} = -1$, and $\mu = d^2$.*

Proof. Let $s \in T \setminus Z(G)$ be as in $({}^{\dagger}H_s)$. Since, $V \cong S^2(W)$, see [18, Proposition 4.2.2], the eigenvalues of s on V, not necessarily distinct, have one of the following forms:

$$\begin{cases} \mu_i^2,\ 1 \leq i \leq m, \text{ with multiplicity at least } \frac{n_i(n_i+1)}{2}; \\ \mu_i\mu_j,\ 1 \leq i < j \leq m, \text{ with multiplicity at least } n_in_j. \end{cases} \tag{4.4}$$

Fix some i and consider the eigenvalue μ_i^2 of s on V. Since the μ_r's are distinct, we deduce that :

$$\dim(V_s(\mu_i^2)) \leq \tfrac{(\ell+1)(\ell+2)}{2} - n_i(\ell + 1 - n_i).$$

Assume $\dim(V_s(\mu_i^2)) \geq \frac{\ell^2+\ell+2}{2}$. Then $(\ell - n_i)(1 - n_i) \geq 0$, which holds if and only if $n_i \in \{1, \ell\}$. In both cases, we get $\dim(V_s(\mu_i^2)) \leq \frac{\ell^2+\ell+2}{2}$, where equality holds if and only if $\mu_i^2 = \mu_j^2$ for all $j \neq i$, i.e. $m = 2$ and $s = \mathrm{diag}(\mu_1, \ldots, \mu_1, \mu_1^{-\ell})$ with $\mu_1^{\ell+1} = -1$.

Fix $i < j$ and consider the eigenvalue $\mu_i\mu_j$ of s on V. Since the μ_r's are distinct, we have

$$\begin{cases} \mu_i\mu_j \neq \mu_i^2 \text{ and } \mu_i\mu_j \neq \mu_j^2; \\ \mu_i\mu_j \neq \mu_i\mu_r, \text{ for } i < r \leq m \text{ and } r \neq j, \text{ and } \mu_i\mu_j \neq \mu_r\mu_i, \text{ for } 1 \leq r < i; \\ \mu_i\mu_j \neq \mu_r\mu_j, \text{ for } 1 \leq r < j \text{ and } r \neq i, \text{ and } \mu_i\mu_j \neq \mu_j\mu_r, \text{ for } j < r \leq m. \end{cases}$$

By (4.4), these account for at least $\frac{n_i(n_i+1)}{2} + \frac{n_j(n_j+1)}{2} + (n_i + n_j)(\ell + 1 - n_i - n_j)$ eigenvalues of s on V which are different to $\mu_i\mu_j$. Hence, we have $\dim(V_s(\mu_i\mu_j)) \leq \frac{(\ell+1)(\ell+2)}{2} - \frac{n_i(n_i+1)}{2} - \frac{n_j(n_j+1)}{2} - (n_i + n_j)(\ell + 1 - n_i - n_j)$. Assume $\dim(V_s(\mu_i\mu_j)) \geq \frac{\ell^2+\ell+2}{2}$. Then:

$$(\ell - n_i - n_j)(1 - n_i - n_j) - \tfrac{n_i(n_i+1)+n_j(n_j+1)}{2} \geq 0. \tag{4.5}$$

As $n_i \geq n_j \geq 1$, we have $\frac{n_i(n_i+1)+n_j(n_j+1)}{2} \geq 2$, therefore $(\ell - n_i - n_j)(1 - n_i - n_j) - 2 \geq 0$. If $n_i + n_j \leq \ell$, then $(\ell - n_i - n_j)(1 - n_i - n_j) \leq 0$ and inequality (4.5) does not hold. If $n_i + n_j = \ell + 1$, then $m = 2$, $n_2 = \ell + 1 - n_1$ and inequality (4.5) does not hold, as $(\ell - n_i - n_j)(1 - n_i - n_j) - \frac{n_i(n_i+1)+n_j(n_j+1)}{2} = -\frac{[(\ell-n_1)^2+(n_1-1)^2+\ell+1]}{2} < 0$. Therefore, $\dim(V_s(\mu_i\mu_j)) < \frac{\ell^2+\ell+2}{2}$ for all $s \in T \setminus Z(G)$ and all $i < j$. We conclude that $\max_{s \in T\setminus Z(G)} \dim(V_s(\mu)) = \frac{\ell^2+\ell+2}{2}$. □

Proposition 4.6. *Let $\ell \geq 2$ and $V = L_G(\omega_1 + \omega_\ell)$. Then $\max_{s \in T\setminus Z(G)} \dim(V_s(\mu)) = \ell^2 - \varepsilon_p(\ell + 1) + \varepsilon_p(3)\varepsilon_\ell(2)$, where the maximum is attained if and only if*

(1) $p \neq 2$, $\ell = 2$, $\mu = -1$ *and* s *is conjugate to* $\mathrm{diag}(d, d, d^{-2})$ *with* $d^3 = -1$.
(2) $p \neq 3$, $\ell = 2$, $\mu = 1$ *and* s *is conjugate to* $\mathrm{diag}(d, d, d^{-2})$ *with* $d^3 \neq 1$.
(3) $\ell \geq 3$, $\mu = 1$ *and* s *is conjugate to* $\mathrm{diag}(d, \dots, d, d^{-\ell})$ *with* $d^{\ell+1} \neq 1$.

Proof. First, we note that if $\varepsilon_p(\ell+1) = 0$, then $W \otimes W^* \cong V \oplus L_G(0)$, while, if $\varepsilon_p(\ell+1) = 1$, then $W \otimes W^* \cong L_G(0) \mid V \mid L_G(0)$, see [18, Proposition 4.6.10]. Let $s \in T \setminus \mathrm{Z}(G)$ be as in $({}^{\dagger}H_s)$. We determine that the eigenvalues of s on V, not necessarily distinct, have one of the following forms:

$$\begin{cases} 1 \text{ with multiplicity } \sum_{i=1}^{m} n_i^2 - 1 - \varepsilon_p(\ell+1); \\ \mu_i\mu_j^{-1} \text{ and } \mu_i^{-1}\mu_j, \text{ where } 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j. \end{cases}$$

We first consider the eigenvalue 1 of s on V. Since the μ_i's are distinct, it follows that:

$$\dim(V_s(1)) = \sum_{i=1}^{m} n_i^2 - 1 - \varepsilon_p(\ell+1) = \ell^2 + 2\ell - 2\sum_{i<j} n_in_j - \varepsilon_p(\ell+1).$$

Assume $\dim(V_s(1)) \geq \ell^2 - \varepsilon_p(\ell+1) + \varepsilon_p(3)\varepsilon_\ell(2)$. Since $\ell = \sum_{i=1}^{m} n_i - 1$, we have that:

$$2(1-n_2)(n_1-1) + 2\sum_{i=3}^{m} n_i\Big(1 - \sum_{j=1}^{i-1} n_j\Big) - \varepsilon_p(3)\varepsilon_\ell(2) \geq 0, \tag{4.6}$$

which holds if and only if either $\ell = 2$, $p \neq 3$, $m = 2$, $n_2 = 1$ and $n_1 = 2$; or $\ell \geq 3$, $m = 2$, $n_2 = 1$ and $n_1 = \ell$. In both cases, as $\mu_1 \neq \mu_2$ and $\mu_1^{\ell}\mu_2 = 1$, we get $\mu_2 = \mu_1^{-\ell}$ and $\mu_1^{\ell+1} \neq 1$. Thus, $\dim(V_s(1)) \leq \ell^2 - \varepsilon_p(\ell+1) + \varepsilon_p(3)\varepsilon_\ell(2)$ for all $s \in T \setminus \mathrm{Z}(G)$ and equality holds if and only if p, ℓ, s and μ are as in (2), or (3).

We now fix $i < j$ and consider the eigenvalue $\mu_i\mu_j^{-1}$ of s on V. In the case when $\mu_i\mu_j^{-1} \neq \mu_i^{-1}\mu_j$, one shows that $\dim(V_s(\mu_i\mu_j^{-1})) \leq \frac{\ell(\ell+1)}{2} < \ell^2 - \varepsilon_p(\ell+1) + \varepsilon_p(3)\varepsilon_\ell(2)$. We thus assume that $p \neq 2$ and $\mu_i\mu_j^{-1} = -1$. Since the μ_r's are distinct, we remark that:

$$\begin{cases} \mu_i\mu_r^{-1} \neq -1 \text{ and } \mu_i^{-1}\mu_r \neq -1, \text{ where } i < r \leq m, r \neq j; \text{ and } \mu_r^{-1}\mu_i \neq -1 \text{ and } \mu_r\mu_i^{-1} \neq -1, \text{ where } 1 \leq r < i; \\ \mu_r\mu_j^{-1} \neq -1 \text{ and } \mu_r^{-1}\mu_j \neq -1, \text{ where } 1 \leq r < j, r \neq i; \text{ and } \mu_j^{-1}\mu_r \neq -1 \text{ and } \mu_j\mu_r^{-1} \neq -1, \text{ where } j < r \leq m. \end{cases}$$

By (4.6), it follows that $\dim(V_s(-1)) \leq \dim(V) - \dim(V_s(1)) - 2(n_i+n_j)(\ell+1-n_i-n_j)$. Assume $\dim(V_s(-1)) \geq \ell^2 - \varepsilon_p(\ell+1) + \varepsilon_p(3)\varepsilon_\ell(2)$. Then:

$$2(\ell-n_i-n_j)(1-n_i-n_j) - \sum_{r=1}^{m} n_r^2 + 1 + \varepsilon_p(\ell+1) - \varepsilon_p(3)\varepsilon_\ell(2) \geq 0 \tag{4.7}$$

and, for it to hold, we must have $(\ell-n_i-n_j)(1-n_i-n_j) > 0$, i.e. $m = 2$ and $n_1 + n_2 = \ell + 1$. Substituting in (4.7) gives $-(n_2-1)^2 + n_1(2-n_1) + \varepsilon_p(\ell+1) - \varepsilon_p(3)\varepsilon_\ell(2) \geq 0$, and we get $n_1 = 2$ and $n_2 \in \{1, 2\}$. One checks that the inequality holds only for $n_2 = 1$. Thus, we have $\dim(V_s(-1)) \leq \ell^2 - \varepsilon_p(\ell+1) + \varepsilon_p(3)\varepsilon_\ell(2)$, where equality holds if and only if p, ℓ, s and μ are as in (1). □

Proposition 4.7. *Let* $\ell \geq 5$ *and* $V = L_G(\omega_3)$. *Then* $\max_{s \in T \setminus \mathrm{Z}(G)} \dim(V_s(\mu)) \leq \binom{\ell}{3} + 2$.

Proof. Set $\lambda = \omega_3$ and $L = L_1$. By Lemma 3.7, we have $e_1(\lambda) = 1$, therefore $V \mid_{[L,L]} = V^0 \oplus V^1$. By [20, Proposition], we have $V^0 \cong L_L(\omega_3)$ and, since the weight $(\lambda - \alpha_1 - \alpha_2 - \alpha_3) \mid_{T_1} = \omega_4$ admits a maximal vector in V^1, by dimensional considerations, it follows that:

$$V \mid_{[L,L]} \cong L_L(\omega_3) \oplus L_L(\omega_4). \tag{4.8}$$

Let $s \in T \setminus \mathrm{Z}(G)$. If $\dim(V_s^i(\mu)) = \dim(V^i)$ for some eigenvalue μ of s on V, where $i = 0, 1$, then $s \in \mathrm{Z}(L)^\circ \setminus \mathrm{Z}(G)$ and acts on V^i as scalar multiplication by $c^{\ell-2-i(\ell+1)}$. As $c^{\ell+1} \neq 1$, we have

$\dim(V_s(\mu)) \leq \binom{\ell}{3}$ for all eigenvalues μ of s on V. We thus assume that $\dim(V_s^i(\mu)) < \dim(V^i)$ for all eigenvalues μ of s on V and for both $i = 0, 1$. We write $s = z \cdot h$, where $z \in Z(L)^\circ$ and $h \in [L, L]$. First, let $\ell = 5$. Using (4.8) and Proposition 4.4, we determine that $\dim(V_s(\mu)) \leq 12$ for all μ. Therefore, $\max_{s \in T \setminus Z(G)} \dim(V_s(\mu)) \leq 12$. We now assume that $\ell \geq 6$. By (4.8) and Proposition 4.4, we have $\dim(V_s(\mu)) \leq \frac{(\ell-1)(\ell-2)}{2} + \dim((L_L(\omega_4))_h(\mu_h))$. Recursively and using the result for $\ell = 5$, we get $\dim(V_s(\mu)) \leq \sum_{j=5}^{\ell-1} \frac{j(j-1)}{2} + 12 = \binom{\ell}{3} + 2$ for all eigenvalues μ of s on V. □

Proposition 4.8. *Assume $p \neq 2, 3$ and let $V = L_G(3\omega_1)$. Then $\max_{s \in T \setminus Z(G)} \dim(V_s(\mu)) = \binom{\ell+2}{3} + \ell$.*

Proof. We will apply the algorithm from [5, Section 2.2] (described earlier in this section) to determine a lower bound for $\nu_G(V)$. Afterwards, we will show that this bound is attained. By [14], the sub-dominant weights in V are $3\omega_1$, $\omega_1 + \omega_2$ and ω_3. Therefore, $s_{3\omega_1} = r_{3\omega_1} + r_{\omega_1+\omega_2} + r_{\omega_3}$. As $\Psi(\omega_1 + \omega_2)$ is of type $A_{\ell-2}$, we have $|\Phi \setminus \Psi(\omega_1 + \omega_2)_s| = 2(2\ell - 1)$ and $|\mathcal{W} : \mathcal{W}(\Psi(\omega_1 + \omega_2))| = \ell(\ell + 1)$, hence $r_{\omega_1+\omega_2} = 2\ell - 1$. Lastly, using (4.1), we get $s_{3\omega_1} = \frac{\ell^2+\ell+2}{2}$, therefore $\nu_G(V) \geq \frac{\ell^2+\ell+2}{2}$.

By Lemma 3.7, we have $e_1(\lambda) = 3$, therefore $V \mid_{[L_1,L_1]} = V^0 \oplus V^1 \oplus V^2 \oplus V^3$. Now, we argue as we did in the proof of Proposition 4.7 to determine the composition factors of each V^i. It will follow that $\dim(V^0) = 1$, $\dim(V^1) = \ell$, $\dim(V^2) = \binom{\ell+1}{2}$ and $\dim(V^3) = \binom{\ell+2}{3}$. Let $s \in Z(L_1)^\circ \setminus Z(G)$. We note that s acts on each V^i as scalar multiplication by $c^{3\ell - i(\ell+1)}$. For $c^{\ell+1} = -1$, we have $\dim(V_s(c^{-3})) = \binom{\ell+2}{3} + \ell$, therefore $\operatorname{codim}(V_s(c^{-3})) = \frac{\ell^2+\ell+2}{2}$. Since $\nu_G(V) \geq \frac{\ell^2+\ell+2}{2}$, it follows that $\max_{s \in T \setminus Z(G)} \dim(V_s(\mu)) = \binom{\ell+2}{3} + \ell$. □

Proof of Theorem 4.2. To ease notation, we will reference each triplet (V, p, ℓ) featured in Table 7 by its corresponding number in column "Ref". Note that 4.2.1–4.2.6 have been solved in Propositions 4.3–4.8. The remaining cases are more straightforward: one has to first determine the structure of $V \mid_{[L_1,L_1]}$ and then apply the algorithm of Section 3.3. In what follows, we will only indicate the results that are used in the inductive step of the algorithm and mention if there are small special cases to consider.
4.2.7: If $p = 3$, the proof is analogous to that of Proposition 4.8. Thus, we assume $p \neq 3$. The result follows recursively. The base case of $\ell = 3$ follows by Propositions 4.3, 4.5, and 4.6. Note that one has to treat the case when $s = z \cdot h$, where $z \in Z(L_1)^\circ$ and $h \in [L_1, L_1]$ is conjugate to $h_{\alpha_2}(d)h_{\alpha_3}(d^2)$ with $d^3 \neq 1$, separately. For $\ell \geq 4$, by Propositions 4.3, 4.4, and 4.5, we have $\dim(V_s(\mu)) \leq \ell - 1 + \left[\frac{(\ell-1)^2+\ell-1+2}{2} - \frac{(\ell-1)^2-(\ell-1)+2}{2}\varepsilon_p(2)\right] + (1 + \varepsilon_p(2))\binom{\ell-1}{2} + \dim((L_{L_1}(\omega_2 + \omega_3))_h(\mu_h))$. Recursively, it follows that $\dim(V_s(\mu)) \leq 11 - 3\varepsilon_p(2) + \sum_{j=3}^{\ell-1}[j^2 + j + 1] - \sum_{j=3}^{\ell-1} \varepsilon_p(2) = \frac{\ell^3+2\ell}{3} - \ell\varepsilon_p(2)$.
4.2.8: Follows by Proposition 4.7.
4.2.9: Follows by 4.2.8. □

4.1. Supplementary results

At this point, we have completed the proofs of Theorems 4.1 and 4.2. However, we will require additional result for the groups of type A_ℓ in the proofs of Theorems 5.1, 5.2, and 7.1, 7.2, respectively. We collect these in Theorems 4.9 and 4.10.

Theorem 4.9. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type A_ℓ with $\ell \geq 1$. Let $V = L_G(\lambda)$, where $\lambda \in X(T)_p^+$ and $\lambda \neq 0$. We have that*

Table 8. The supplementary $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu))$ for groups of type A_ℓ.

Ref.	V	Char.	Rank	$\max_{s\in T\setminus Z(G)} \dim(V_s(\mu))$
4.3.1	$L_G(m\omega_1)$ with $m\geq 4$	$p=0$ or $p\geq m$	$\ell=1$	$1+\lfloor \frac{m}{2}\rfloor$
4.3.2	$L_G(2\omega_1+\omega_2)$	$p\geq 0$	$\ell=2$	$8-4\varepsilon_p(2)$
4.3.3	$L_G(2\omega_1+2\omega_2)$	$p\neq 2$	$\ell=2$	$15-4\varepsilon_p(5)$
4.3.4	$L_G(\omega_1+\omega_2+\omega_3)$	$p\geq 0$	$\ell=3$	$\leq 36-4\varepsilon_p(5)-12\varepsilon_p(3)$
4.3.5	$L_G(2\omega_2)$	$p\neq 2$	$\ell=3$	12
4.3.6	$L_G(3\omega_2)$	$p\neq 2,3$	$\ell=3$	≤ 28
4.3.7	$L_G(\omega_1+2\omega_2)$	$p\geq 0$	$\ell=3$	$33-21\varepsilon_p(2)$
4.3.8	$L_G(2\omega_1+2\omega_3)$	$p\neq 2$	$\ell=3$	$48-\varepsilon_p(5)-9\varepsilon_p(3)$

Proof. Let (V,p,ℓ) be a triplet listed in Table 8. Let $s\in T\setminus Z(G)$ and let μ be an eigenvalue of s on V. To calculate upper-bounds for $\dim(V_s(\mu))$ in cases 4.3.2–4.3.8, one will first determine the structure of $V\mid_{[L_1,L_1]}$, and then apply the algorithm of Section 3.3. Because the proofs are this straightforward, we only indicate the results that are used in the inductive step of the algorithm and mention if there are small special cases to consider. To ease notation, we will reference each triplet (V,p,ℓ) by its corresponding number in column "Ref".

4.3.1: The eigenvalues of s on V, not necessarily distinct, are $\mu_1^m,\mu_1^{m-2},\dots,\mu_1^{-m+2},\mu_1^{-m}$, where $\mu_1\neq\pm 1$ and, consequently, $\mu_1^i\neq\mu_1^{i-2}$ for all $-m+2\leq i\leq m$.

Assume m is even. If $\mu\neq\pm1$, then $\dim(V_s(\mu))\leq\frac{\dim(V)-\dim(V_s(1))}{2}\leq\frac{m}{2}$, as V is self-dual. Let $\mu=1$. If $\mu_1^i=1$ for some $2\leq i\leq m$, then, as $\mu_1^2\neq 1$ and $\mu_1^i\neq\mu_1^{i-2}$, at most $\lfloor\frac{m-2}{4}\rfloor+\varepsilon$ of the eigenvalues $\mu_1^m,\mu_1^{m-2},\dots,\mu_1^4$ can equal 1, where $\varepsilon=1-\varepsilon_4(m-2)$. We deduce that $\dim(V_s(1))\leq\varepsilon+\frac{m}{2}$. If $\mu=-1$, then at most $\lfloor\frac{m}{4}\rfloor+\xi$ of the eigenvalues $\mu_1^m,\mu_1^{m-2},\dots,\mu_1^2$ can equal -1, where $\xi=1-\varepsilon_4(m)$. We deduce that $\dim(V_s(-1))\leq\xi+\frac{m}{2}$. The case of m odd is analogous.

To show that equality holds, consider $s=\mathrm{diag}(\mu_1,\mu_1^{-1})\in T\setminus Z(G)$, where $\mu_1^2=-1$. Then, since $\mu_1^2=-1$, we have $\dim(V_s(\mu_1^m))=1+\lfloor\frac{m}{2}\rfloor$.

4.3.2: When $p=2$ and $s=z\cdot h$, where $z\in Z(L_1)^\circ$ and $h\in[L_1,L_1]$, one shows that the eigenvalues of s on V have the form: $c^5d^{\pm1}$, c^2, $c^{-1}d^{\pm3}$, $c^{-1}d^{\pm1}$ and $c^{-4}d^{\pm2}$, where $d\neq 1$ and $c\in k^*$. Therefore $\max_{s\in T\setminus Z(G)}\dim(V_s(\mu))=4$. When $p\neq 2$, the result follows by Propositions 4.3, 4.5, and 4.8.

4.3.3: When $s=z\cdot h$ with $z\in Z(L_1)^\circ$ and $h\in[L_1,L_1]$, one has to show that $\dim(V^2(\mu_h^2))\leq 5-2\varepsilon_p(5)$ for all μ_h^2. The result then follows by Propositions 4.3, 4.5, and 4.8 and 4.3.1.

4.3.4: When $p=2$ and $s=z\cdot h$ with $z\in Z(L_1)^\circ$ and $h\in[L_1,L_1]$, one has to show that $\dim((L_{L_1}(2\omega_2+\omega_3))_h(\mu_h))\leq 4$ for all μ_h of h. The result then follows by Propositions 4.3, 4.5, 4.6, and 4.3.2.

4.3.5: Follows by Propositions 4.5 and 4.6.

4.3.6: Follows by Proposition 4.8 and 4.3.2.

4.3.7: When $s=z\cdot h$ with $z\in Z(L_1)^\circ$ and $h\in[L_1,L_1]$, one has to show that the result holds in two special cases. First case happens when $p\neq 2$ and h conjugate to $h_{\alpha_2}(d)h_{\alpha_3}(d^2)$ with $d^3=-1$. This is solved by showing that the eigenvalues of h on V^1, respectively on V^2, are ±1 with $\dim(V_h^1(1))=8$ and $\dim(V^1(-1))=10$, respectively $\pm d$ with $\dim(V_h^2(d))=10$ and $\dim(V_h^2(-d))=11$. The second case occurs when $p=2$ and, by Proposition 4.3, 4.3.2 and the structure of V^1, we have $\dim(V_s(\mu))\leq 13$. One has to show that equality does not hold. If it did, then, by Proposition 4.3, h would be conjugate to $h_{\alpha_2}(d)h_{\alpha_3}(d^2)$ with $d^3\neq 1$, and we would have $c^7d^2=c^3d^3=c^{-1}d^{-2}$, where $c\in k^*$. However, since $p=2$, we get $d^3=1$, a contradiction. Outside the two special cases, the result follows by Propositions 4.3, 4.5, 4.8, 4.6, and 4.3.2.

4.3.8: Follows by Propositions 4.3, 4.5, 4.6, 4.3.2, and 4.3.3. □

Theorem 4.10. *Let k be an algebraically closed field of characteristic $p\geq 0$ and let G be a simple simply connected linear algebraic group of type A_ℓ with $\ell\geq 1$. Let $V=L_G(\lambda)$, where $\lambda\in X(T)_p^+$ and $\lambda\neq 0$. We have that*

Table 9. The supplementary $\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$ for groups of type A_ℓ.

Ref.	V	Char.	Rank	$\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$
4.4.1	$L_G(m\omega_1)$ with $m \geq 4$	$p=0$ or $p\geq m$	$\ell=1$	1
4.4.2	$L_G(2\omega_1+\omega_2)$	$p\geq 0$	$\ell=2$	$\leq 6+\varepsilon_p(3)-\varepsilon_p(2)^{\curlyvee}$
4.4.3	$L_G(2\omega_1+2\omega_2)$	$p\neq 2$	$\ell=2$	$\leq 9+4\varepsilon_p(3)-3\varepsilon_p(5)^{\curlyvee}$
4.4.4	$L_G(\omega_1+\omega_2+\omega_3)$	$p\geq 0$	$\ell=3$	$\leq 30-4\varepsilon_p(5)-10\varepsilon_p(3)+8\varepsilon_p(2)$
4.4.5	$L_G(2\omega_2)$	$p\neq 2$	$\ell=3$	10
4.4.6	$L_G(3\omega_2)$	$p\neq 2,3$	$\ell=3$	20
4.4.7	$L_G(\omega_1+2\omega_2)$	$p\geq 0$	$\ell=3$	$\leq 26+3\varepsilon_p(3)-12\varepsilon_p(2)^{\curlyvee}$
4.4.8	$L_G(2\omega_1+2\omega_3)$	$p\neq 2$	$\ell=3$	$\leq 36-3\varepsilon_p(3)$

$^{\curlyvee}$ equality holds when $p\neq 3$.

Proof. Let (V,p,ℓ) be a triple listed in Table 9. To begin, note that, by Lemma 3.13, we have $\max_{u\in G_u\setminus\{1\}} \dim(V_u(1)) = \dim(V_{x_{\alpha_\ell}(1)}(1))$. Now, to calculate $\dim(V_{x_{\alpha_\ell}(1)}(1))$ we use the structure of $V\mid_{[L_1,L_1]}$, Lemma 3.12 and the algorithm of Section 3.3. Since the proofs are very similar to the ones of Theorem 4.1, we will only mention the results used in the inductive step of the algorithm.
4.4.1: Follows from [22, Theorem 1.9].
4.4.2: If $p=2$, we have $V\cong L_G(\omega_1)^{(2)}\otimes L_G(\omega_2)$, and $x_{\alpha_2}(1)$ acts as $J_2\oplus J_1$ on both $L_G(\omega_1)^{(2)}$ and $L_G(\omega_2)$. If $p\neq 2$, Theorem 4.1[4.1.1, 4.1.3, 4.1.6] gives the result.
4.4.3: Follows by Theorem 4.1[4.1.1, 4.1.3, 4.1.6] and the fact that $\dim((L_{L_1}(4\omega_2))_{x_{\alpha_2}(1)}) = 1+\varepsilon_p(3)$.
4.4.4: Follows by Theorem 4.1[4.1.1, 4.1.3, 4.1.7] and 4.4.2.
4.4.5: Follows by Theorem 4.1[4.1.3, 4.1.4].
4.4.6: Follows by Theorem 4.1[4.1.6] and 4.4.2.
4.4.7: When $p\neq 2$, it follows by Theorem 4.1[4.1.1, 4.1.3, 4.1.4, 4.1.6] and 4.4.2. When $p=2$, one shows that $x_{\alpha_3}(1)$ acts on $L_L(3\omega_2)$ as $J_2^4\oplus J_1$, and the result follows by Theorem 4.1[4.1.1] and 4.4.2.
4.4.8: Follows by Theorem 4.1[4.1.1, 4.1.3, 4.1.4], 4.4.2 and 4.4.3. □

5. Proof of Theorem 1.1 for groups of type C_ℓ

Let G be a simple simply connected linear algebraic group of type C_ℓ with $\ell\geq 2$. This section is dedicated to Theorems 5.1 and 5.2, which give the values of $\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$ and $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu))$ for all kG-modules $V=L_G(\lambda)$ with $\lambda\in X(T)_p^+$, $\lambda\neq 0$, and $\dim(L_G(\lambda))\leq 4\ell^3$. As a corollary, the part of Theorem 1.1 concerning simple simply connected linear algebraic groups of type C_ℓ will follow.

Theorem 5.1. *Let k be an algebraically closed field of characteristic $p\geq 0$ and let G be a simple simply connected linear algebraic group of type C_ℓ with $\ell\geq 2$. Let $V=L_G(\lambda)$, where $\lambda\in X(T)_p^+$ and $\lambda\neq 0$, be such that $\dim(V)\leq 4\ell^3$. The value of $\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$ is as given in Table 10.*

Table 10. The value of $\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$ for groups of type C_ℓ.

Ref.	V	Char.	Rank	$\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$
5.1.1	$L_G(\omega_1)$	$p\geq 0$	$\ell\geq 2$	$2\ell-1$
5.1.2	$L_G(\omega_2)$	$p\geq 0$	$\ell\geq 2$	$2\ell^2-3\ell+1-\varepsilon_p(\ell)+\varepsilon_\ell(2)\varepsilon_p(2)$
5.1.3	$L_G(2\omega_1)$	$p\neq 2$	$\ell\geq 2$	$2\ell^2-\ell$
			$\ell=3$	$9-3\varepsilon_p(2)$
5.1.4	$L_G(\omega_3)$	$p\geq 0$	$\ell=4$	$34-7\varepsilon_p(3)$
			$\ell\geq 5$	$\binom{2\ell-1}{3}-1-(2\ell-1)\varepsilon_p(\ell-1)$

(*Continued*)

Table 10. (*Continued*)

Ref.	V	Char.	Rank	$\max_{u\in G_u\setminus\{1\}} \dim(V_u(1))$
5.1.5	$L_G(3\omega_1)$	$p\neq 2,3$	$\ell\geq 2$	$\binom{2\ell+1}{3}$
		$p\geq 0$	$\ell=2$	$8-3\varepsilon_p(5)$
5.1.6	$L_G(\omega_1+\omega_2)$	$p=3$	$\ell\geq 3$	$\frac{4\ell^3-7\ell+6}{3}$
		$p\neq 3$	$\ell\geq 3$	$\leq 2\binom{2\ell}{3}-(2\ell-1)\varepsilon_p(2\ell+1)+2\ell\varepsilon_p(2)^{\dagger}$
5.1.7	$L_G(2\omega_2)$	$p\neq 2$	$\ell=2$	$\leq 6-\varepsilon_p(5)+2\varepsilon_p(3)^{\curlyvee}$
5.1.8	$L_G(\omega_1+2\omega_2)$	$p=7$	$\ell=2$	7
5.1.9	$L_G(3\omega_2)$	$p=7$	$\ell=2$	7
5.1.10	$L_G(2\omega_1+\omega_2)$	$p=3$	$\ell=2$	≤ 11
5.1.11	$L_G(\omega_1+\omega_3)$	$p\geq 0$	$\ell=3$	$\leq 40-9\varepsilon_p(3)-12\varepsilon_p(2)^{人}$
5.1.12	$L_G(\omega_2+\omega_3)$	$p=5$	$\ell=3$	25
5.1.13	$L_G(2\omega_3)$	$p=5$	$\ell=3$	25
5.1.14	$L_G(2\omega_2)$	$p=5$	$\ell=3$	≤ 50
5.1.15	$L_G(\omega_4)$	$p\neq 2$	$\ell=4$	$28-\varepsilon_p(3)$
5.1.16	$L_G(\omega_1+\omega_\ell)$	$p=2$	$4\leq\ell\leq 6$	$(3\ell-2)\cdot 2^{\ell-1}$
5.1.17	$L_G(\omega_1+\omega_4)$	$p=7$	$\ell=4$	142
5.1.18	$L_G(\omega_1+\omega_3)$	$p=2$	$\ell=4$	≤ 144
5.1.19	$L_G(\omega_4)$	$p\geq 0$	$\ell=5$	$\leq 117-36\varepsilon_p(3)+3\varepsilon_p(2)^{\dagger}$
5.1.20	$L_G(\omega_5)$	$p\neq 2$	$\ell=5$	$90-9\varepsilon_p(3)$
5.1.21	$L_G(\omega_4)$	$p\geq 0$	$\ell=6$	$\leq 319-\varepsilon_p(5)-51\varepsilon_p(2)^{\star}$
5.1.22	$L_G(\omega_5)$	$p\geq 0$	$\ell=6$	$\leq 407-164\varepsilon_p(3)+5\varepsilon_p(2)^{\dagger}$
5.1.23	$L_G(\omega_6)$	$p\neq 2$	$\ell=6$	$297-54\varepsilon_p(3)$
5.1.24	$L_G(\omega_4)$	$p\geq 0$	$\ell=7$	$\leq 702-78\varepsilon_p(5)-\varepsilon_p(3)+2\varepsilon_p(2)^{\star}$
5.1.25	$L_G(\omega_6)$	$p=3$	$\ell=7$	729
5.1.26	$L_G(\omega_7)$	$p=3$	$\ell=7$	729
5.1.27	$L_G(\omega_5)$	$p=2$	$\ell=7$	≤ 948
5.1.28	$L_G(\omega_4)$	$p\geq 0$	$\ell=8$	$\leq 1350-\varepsilon_p(7)-105\varepsilon_p(3)-102\varepsilon_p(2)^{\circ}$
5.1.29	$L_G(\omega_4)$	$p\geq 0$	$\ell=9$	$\leq 2363-136\varepsilon_p(7)+\varepsilon_p(2)^{\circ}$
5.1.30	$L_G(\omega_\ell)$	$p=2$	$4\leq\ell\leq 13$	$3\cdot 2^{\ell-2}$
5.1.31	$L_G(2\omega_1+\omega_\ell)$	$p=2$	$3\leq\ell\leq 6$	$(3\ell-2)\cdot 2^{\ell-1}$

† equality holds when $p\neq 2$.
$^{\curlyvee}$ equality holds when $p\neq 3$.
^{人} equality holds when $p\neq 5$.
* equality holds when $p\neq 2,3$.
$^{\circ}$ equality holds when $p\neq 2,3,5$.

Proof. Recall that we have denoted by W the natural module of G. To begin, let (V,p,ℓ) be a triplet listed in Table 10. Remark that, by Lemma 3.13, we have $\max_{u\in G_u\setminus\{1\}}\dim(V_u(1))=\max_{i=j,\ell}\dim(V_{x_{\alpha_i}(1)}(1))$, where $j\neq\ell$. For 5.1.7 - 5.1.31, to calculate $\max_{i=j,\ell}\dim(V_{x_{\alpha_i}(1)}(1))$, $j\neq\ell$, we use: the structure of either one, or both of $V\mid_{[L_1,L_1]}$ and $V\mid_{[L_\ell,L_\ell]}$; Lemma 3.12 and the algorithm of Section 3.3. When the proofs are very similar to ones of Theorem 4.1, we will only mention the results used in the inductive step.
5.1.1: Note that $x_{\alpha_1(1)}$, respectively $x_{\alpha_\ell}(1)$, acts on W as $J_2^2\oplus J_1^{2\ell-4}$, respectively as $J_2\oplus J_1^{2\ell-2}$.
5.1.2: Note that $\wedge^2(W)\cong L_G(0)\oplus V$, if $\varepsilon_p(\ell)=0$, while if $\varepsilon_p(\ell)=1$, we have $\wedge^2(W)\cong L_G(0)\mid V\mid L_G(0)$, see [18, Lemma 4.8.2]. By the proof of Theorem 4.1[4.1.2] we get $\dim((\wedge^2(W))_{x_{\alpha_\ell}(1)}(1))=2\ell^2-3\ell+2$. To calculate $\dim((\wedge^2(W))_{x_{\alpha_1}(1)}(1))$, we write $W=W_1\oplus W_2$, where $\dim(W_1)=4$ and $x_{\alpha_1}(1)$ acts as J_2^2 on W_1; and $\dim(W_2)=2\ell-4$ and $x_{\alpha_1}(1)$ acts trivially on W_2. Using [15, Lemma 3.4] and Lemma 3.14, we determine that $\dim((\wedge^2(W))_{x_{\alpha_1}(1)}(1))=2\ell^2-5\ell+6$. Lastly, by [18, Lemma 4.8.2] and [11, Corollary 6.2], or [12, Theorem B] if $p=2$, we determine that $\dim(V_{x_{\alpha_\ell}(1)}(1))=2\ell^2-3\ell+1-\varepsilon_p(\ell)$ and $\dim(V_{x_{\alpha_1}(1)}(1))=2\ell^2-5\ell+5-\varepsilon_p(\ell)+\varepsilon_p(2)\varepsilon_\ell(2)$.
5.1.3: Follows by [18, Proposition 4.2.2] and Theorem 4.1[4.1.3].
5.1.4: The result for $\ell=3$ follows by 5.1.1 and 5.1.2. Assume $\ell\geq 4$. We note that if $\varepsilon_p(\ell-1)=0$, we have $\wedge^3(W)\cong V\oplus L_G(\omega_1)$, while, if $\varepsilon_p(\ell-1)=1$, then $\wedge^3(W)\cong L_G(\omega_1)\mid V\mid L_G(\omega_1)$, as kG-modules, see [18, Lemma 4.8.2]. For $x_{\alpha_1}(1)$, we argue as in 5.1.2 to calculate $\dim((\wedge^3(W))_{x_{\alpha_1}(1)}(1))$, and we deduce

$\dim(V_{x_{\alpha_1}(1)}(1)) \geq \binom{2\ell-1}{3} - 2\ell^2 + 9\ell - 11 - (2\ell-2)\varepsilon_p(\ell-1)$. For $x_{\alpha_\ell}(1)$, using Theorem 4.1[4.1.5], one shows that $\dim(V_{x_{\alpha_\ell}(1)}(1)) \geq \binom{2\ell-1}{3} - 1 - (2\ell-1)\varepsilon_p(\ell-1)$, where equality holds for $\varepsilon_p(\ell-1) = 0$. To show that equality also holds for $\varepsilon_p(\ell-1) = 1$, we determine the structure of $V\mid_{[L_1,L_1]}$ and use 5.1.2.
5.1.5: Follows by [18, Proposition 4.2.2], arguing as in 5.1.2.
5.1.6: Let $\ell = 2$. When $p \neq 2$, the result follows by Theorem 4.1[4.1.1, 4.1.3]. When $p = 2$, we have $V = L_G(\omega_1) \otimes L_G(\omega_2)$, as kG-modules, see [19, (1.6)]. Using the Jordan form of $x_{\alpha_i}(1)$, $i = 1, 2$, on $L_G(\omega_1)$ and [12, Theorem B], one shows that $\max_{u \in G_u \setminus \{1\}} \dim(V_u(1)) = 8$.

Let $\ell \geq 3$. If $p = 3$, the result follows by 5.1.1, 5.1.3 and the result for $\ell = 2$. Assume $p \neq 3$. First, we argue recursively, using 5.1.1, 5.1.2, 5.1.3 and the result for $\ell = 2$, to show that $\dim(V_{x_{\alpha_i}(1)}(1)) \leq 2\binom{2\ell}{3} - (2\ell-1)\varepsilon_p(2\ell+1) + 2\ell\varepsilon_p(2)$, $i = \ell-1, \ell$. Lastly, assume that $p \neq 2$. By [18, Lemmas 4.9.1 and 4.9.2], we know that V is a composition factor of the kG-module $W \otimes \wedge^2(W)$, where $\mathrm{ch}(W \otimes \wedge^2(W)) = \chi(\omega_1 + \omega_2) + \chi(\omega_3) + 2\chi(\omega_1)$. Note that by $\chi(\lambda')$ we understand the character of the Weyl kG-module of highest weight $\lambda' \in X(T)$. Therefore, in view of [18, Lemmas 4.8.2 and 4.9.2], we have $\dim(V_u(1)) \geq \dim((W \otimes \wedge^2(W))_u(1)) - \dim((L_G(\omega_3))_u(1)) - (2 + \varepsilon_p(\ell-1) + \varepsilon_p(2\ell+1))\dim((L_G(\omega_1))_u(1))$ for all $u \in G_u$. For $x_{\alpha_\ell}(1)$, it follows that $\dim(V_{x_{\alpha_\ell}(1)}(1)) \geq 2\binom{2\ell}{3} - (2\ell-1)\varepsilon_p(2\ell+1)$.
5.1.7: Follows by Theorem 4.1[4.1.1, 4.1.3, 4.1.6].
5.1.8: Follows by Theorems 4.1[4.1.1, 4.1.3, 4.1.6] and 4.10[4.4.1].
5.1.9: Because we will need to know $\max_{u \in G_u \setminus \{1\}} \dim((L_G(3\omega_2))_u(1))$ also when $p \neq 7$, we will not limit ourselves to the case $p = 7$ and, instead, we will only assume that $p \neq 2, 3$. In this case, by Theorems 4.1[4.1.3, 4.1.6] and 4.10[4.4.1], we get $\max_{u \in G_u \setminus \{1\}} \dim((L_G(3\omega_2))_u(1)) \leq 10 + 2\varepsilon_p(5) - 3\varepsilon_p(7)$, where equality holds for $p \neq 5$.
5.1.10: As above, we will only assume that $p \neq 2$. Then, by Theorems 4.1[4.1.1, 4.1.3, 4.1.6] and 4.10[4.4.1], we deduce that $\max_{u \in G_u \setminus \{1\}} \dim(V_u(1)) \leq 15 - 4\varepsilon_p(3)$.
5.1.11: Follows by 5.1.1, 5.1.2, 5.1.3, and 5.1.6.
5.1.12: Follows by 5.2.6 and 5.2.7.
5.1.13: As in 5.1.9, we only assume $p \neq 2$. Then, by 5.1.3, 5.1.6, and 5.1.7, we determine that $\max_{u \in G_u \setminus \{1\}} \dim((L_G(2\omega_3))_u(1)) \leq 40 - 15\varepsilon_p(5) + 4\varepsilon_p(3)$, where equality holds for $p \neq 3$.
5.1.14: Again, we only assume $p \neq 2$. Then, by 5.1.1, 5.1.2, 5.1.3, 5.1.6, and 5.1.7, we determine that $\max_{u \in G_u \setminus \{1\}} \dim(V_u(1)) \leq 50 - \varepsilon_p(7)$.
5.1.15: Follows by 5.1.2 and 5.1.4.
5.1.16: Follows recursively, using 5.1.11 and 5.1.30.
5.1.17: Follows by 5.1.4, 5.1.6 and 5.1.11.
5.1.18: Follows by 5.1.1, 5.1.2, 5.1.6, and 5.1.11.
5.1.19: Follows by 5.1.2, 5.1.4, 5.1.15, and 5.1.30.
5.1.20: Follows by 5.1.4 and 5.1.15.
5.1.21: Follows by 5.1.2, 5.1.4 and 5.1.19.
5.1.22: Follows by 5.1.4, 5.1.19, 5.1.20, and 5.1.30.
5.1.23: Follows by 5.1.19 and 5.1.20.
5.1.24: Follows by 5.1.2, 5.1.4 and 5.1.21.
5.1.25 and 5.1.26: Follow by 5.1.22 and 5.1.23.
5.1.27: Follows by 5.1.21 and 5.1.22.
5.1.28: Follows by 5.1.2, 5.1.4, and 5.1.24.
5.1.29: Follows by 5.1.2, 5.1.4, and 5.1.28.
5.1.30: Follows recursively, using 5.1.4.
5.1.31: Note that $V \cong L_G(\omega_1)^{(2)} \otimes L_G(\omega_\ell)$ and $L_G(\omega_1 + \omega_\ell) \cong L_G(\omega_1) \otimes L_G(\omega_\ell)$, by [19, (1.6)]. The result follows by 5.1.11 and 5.1.16. □

Theorem 5.2. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a simple simply connected linear algebraic group of type C_ℓ with $\ell \geq 2$. Let $V = L_G(\lambda)$, where $\lambda \in X(T)^+_p$ and $\lambda \neq 0$, be such that $\dim(V) \leq 4\ell^3$. The value of $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu))$ is as given in Table 11.*

Table 11. The value of $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu))$ for groups of type C_ℓ.

Ref.	V	Char.	Rank	$\max_{s\in T\setminus Z(G)} \dim(V_s(\mu))$
5.2.1	$L_G(\omega_1)$	$p \geq 0$	$\ell \geq 2$	$2\ell - 2$
5.2.2	$L_G(\omega_2)$	$p \geq 0$	$\ell \geq 2$	$2\ell^2 - 5\ell + 3 - \varepsilon_p(\ell) + (3 - \varepsilon_p(2))\varepsilon_\ell(2) + (2 + \varepsilon_p(3))\varepsilon_\ell(3) + (1 - \varepsilon_p(2))\varepsilon_\ell(4)$
5.2.3	$L_G(2\omega_1)$	$p \neq 2$	$\ell \geq 2$	$2\ell^2 - 3\ell + 4$
5.2.4	$L_G(\omega_3)$	$p \geq 0$	$\ell = 3$	$10 - 6\varepsilon_p(2)$
			$\ell = 4$	$28 - 2\varepsilon_p(3) - 8\varepsilon_p(2)$
			$\ell \geq 5$	$\leq \binom{2\ell-2}{3} + 10 - (2\ell - 2)\varepsilon_p(\ell - 1) + 4\varepsilon_p(3) - 10\varepsilon_p(2)$
5.2.5	$L_G(3\omega_1)$	$p \neq 2, 3$	$\ell \geq 2$	$\binom{2\ell}{3} + 3(2\ell - 2)$
5.2.6	$L_G(\omega_1 + \omega_2)$	$p \geq 0$	$\ell = 2$	$8 - 2\varepsilon_p(5) - 2\varepsilon_p(2)$
		$p = 3$	$\ell \geq 3$	$\leq \frac{4\ell^3 - 6\ell^2 + 14\ell - 12}{3}$
		$p \neq 3$	$\ell \geq 3$	$\leq 16\binom{\ell}{3} + 8\ell + 4 - 2(\ell - 1)\varepsilon_p(2\ell + 1) - (4\ell - 2)\varepsilon_p(2) - (6 - 6\varepsilon_p(2))\varepsilon_\ell(3) - (2 - 2\varepsilon_p(2))\varepsilon_\ell(4)$
5.2.7	$L_G(2\omega_2)$	$p \neq 2$	$\ell = 2$	$10 - \varepsilon_p(5)$
5.2.8	$L_G(\omega_1 + 2\omega_2)$	$p = 7$	$\ell = 2$	12
5.2.9	$L_G(3\omega_2)$	$p = 7$	$\ell = 2$	16
5.2.10	$L_G(2\omega_1 + \omega_2)$	$p = 3$	$\ell = 2$	16
5.2.11	$L_G(\omega_1 + \omega_3)$	$p \geq 0$	$\ell = 3$	$40 - 8\varepsilon_p(3) - 20\varepsilon_p(2)$
5.2.12	$L_G(\omega_2 + \omega_3)$	$p = 5$	$\ell = 3$	36
5.2.13	$L_G(2\omega_3)$	$p = 5$	$\ell = 3$	39
5.2.14	$L_G(2\omega_2)$	$p = 5$	$\ell = 3$	50
5.2.15	$L_G(\omega_4)$	$p \neq 2$	$\ell = 4$	28
5.2.16	$L_G(\omega_1 + \omega_\ell)$	$p = 2$	$4 \leq \ell \leq 6$	$(2\ell - 1) \cdot 2^{\ell-1}$
5.2.17	$L_G(\omega_1 + \omega_4)$	$p = 7$	$\ell = 4$	≤ 144
5.2.18	$L_G(\omega_1 + \omega_3)$	$p = 2$	$\ell = 4$	≤ 130
5.2.19	$L_G(\omega_4)$	$p \geq 0$	$\ell = 5$	$\leq 100 - 20\varepsilon_p(3) - 24\varepsilon_p(2)$
5.2.20	$L_G(\omega_5)$	$p \neq 2$	$\ell = 5$	$84 - 2\varepsilon_p(3)$
5.2.21	$L_G(\omega_4)$	$p \geq 0$	$\ell = 6$	$\leq 260 - \varepsilon_p(5) + 16\varepsilon_p(3) - 88\varepsilon_p(2)$
5.2.22	$L_G(\omega_5)$	$p \geq 0$	$\ell = 6$	$\leq 350 - 108\varepsilon_p(3) - 86\varepsilon_p(2)$
5.2.23	$L_G(\omega_6)$	$p \neq 2$	$\ell = 6$	$\leq 268 - 24\varepsilon_p(3)^{\bowtie}$
5.2.24	$L_G(\omega_4)$	$p \geq 0$	$\ell = 7$	$\leq 565 - 66\varepsilon_p(5) + 23\varepsilon_p(3) - 65\varepsilon_p(2)$
5.2.25	$L_G(\omega_6)$	$p = 3$	$\ell = 7$	728
5.2.26	$L_G(\omega_7)$	$p = 3$	$\ell = 7$	730
5.2.27	$L_G(\omega_5)$	$p = 2$	$\ell = 7$	≤ 608
5.2.28	$L_G(\omega_4)$	$p \geq 0$	$\ell = 8$	$\leq 1091 - \varepsilon_p(7) - 59\varepsilon_p(3) - 175\varepsilon_p(2)$
5.2.29	$L_G(\omega_4)$	$p \geq 0$	$\ell = 9$	$\leq 1930 - 119\varepsilon_p(7) + 40\varepsilon_p(3) - 106\varepsilon_p(2)$
5.2.30	$L_G(\omega_\ell)$	$p = 2$	$4 \leq \ell \leq 13$	$2^{\ell-1}$
5.2.31	$L_G(2\omega_1 + \omega_\ell)$	$p = 2$	$3 \leq \ell \leq 6$	$(2\ell - 1) \cdot 2^{\ell-1}$

$^{\bowtie}$ equality holds when $p = 3$.

Let $s \in T \setminus Z(G)$. As in Section 4, to improve readability, we will say "s is as in $({}^\dagger H_s)$" to express the fact that s satisfies the following: $s = \mathrm{diag}(\mu_1 \cdot I_{n_1}, \dots, \mu_m \cdot I_{n_m}, \mu_m^{-1} \cdot I_{n_m}, \dots, \mu_1^{-1} \cdot I_{n_1})$, where $\mu_i \neq \mu_j^{\pm 1}$ for all $i < j$, $\sum_{i=1}^{m} n_i = \ell$ and $n_1 \geq \cdots \geq n_m \geq 1$; and if $m = 1$, then $\mu_1 \neq \pm 1$. Note that any $s \in T \setminus Z(G)$ is as in $({}^\dagger H_s)$.

Proposition 5.3. *Let $V = L_G(\omega_1)$. Then $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu)) = 2\ell - 2$, where the maximum is attained if and only if $\mu = \pm 1$ and s is conjugate to $\mathrm{diag}(\pm 1, \dots, \pm 1, d, d^{-1}, \pm 1, \dots, \pm 1)$ with $d \neq \pm 1$.*

Proof. The proof is analogous to that of Proposition 4.3. □

Proposition 5.4. *Let $V = L_G(\omega_2)$. Then*

(1) $\ell = 2$ *and* $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu)) = 4 - 2\varepsilon_p(2)$, *where the maximum is attained if and only if*

(1.1) $p \neq 2$, $\mu = -1$ *and s is conjugate to* $\mathrm{diag}(1, -1, -1, 1)$.
(1.2) $p = 2$, $\mu = 1$ *and s is conjugate to* $\mathrm{diag}(d, d, d^{-1}, d^{-1})$ *with* $d \neq 1$.
(1.3) $p = 2$, $\mu = d^{\pm 1}$ *and s is conjugate to* $\mathrm{diag}(d, 1, 1, d^{-1})$ *with* $d \neq 1$.

(2) $\ell = 3$ *and* $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu)) = 8$, *where the maximum is attained if and only if*

(2.1) $p \neq 3$, $\mu = 1$ *and s is conjugate to* $\mathrm{diag}(d, d, d, d^{-1}, d^{-1}, d^{-1})$ *with* $d^2 \neq 1$.
(2.2) $p \neq 2$, $\mu = -1$ *and s is conjugate to* $\pm\,\mathrm{diag}(1, 1, -1, -1, 1, 1)$.

(3) $\ell = 4$ *and* $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu)) = 16 - 2\varepsilon_p(2)$, *where the maximum is attained if and only if*

(3.1) $p \neq 2$, $\mu = -1$ *and s is conjugate to* $\mathrm{diag}(1, 1, -1, -1, -1, -1, 1, 1)$.
(3.2) $p = 2$, $\mu = 1$ *and s is conjugate to* $\mathrm{diag}(d, d, d, d, d^{-1}, d^{-1}, d^{-1}, d^{-1})$ *with* $d \neq 1$.
(3.3) $p = 2$, $\mu = 1$ *and s is conjugate to* $\mathrm{diag}(1, 1, 1, d, d^{-1}, 1, 1, 1)$ *with* $d \neq 1$.

(4) $\ell \geq 5$ *and* $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu)) = 2\ell^2 - 5\ell + 3 - \varepsilon_p(\ell)$, *where the maximum is attained if and only if* $\mu = 1$ *and s is conjugate to* $\pm\,\mathrm{diag}(1, \dots, 1, d, d^{-1}, 1, \dots, 1)$ *with* $d \neq 1$.

Proof. To ease notation, define B_ℓ, $\ell \geq 2$, in the following way: $B_2 = 4 - 2\varepsilon_p(2)$; $B_3 = 8$; $B_4 = 16 - 2\varepsilon_p(2)$; and $B_\ell = 2\ell^2 - 5\ell + 3 - \varepsilon_p(\ell)$ for all $\ell \geq 5$. Let $s \in T \setminus Z(G)$ be as in $({}^\dagger H_s)$. Using the structure of $\wedge^2(W)$ as a kG-module, see [18, Lemma 4.8.2], we deduce that the eigenvalues of s on V, not necessarily distinct, have one of the following forms:

$$\begin{cases} \mu_i^2 \text{ and } \mu_i^{-2},\ 1 \leq i \leq m, \text{ each with multiplicity at least } \frac{n_i(n_i-1)}{2}; \\ \mu_i\mu_j \text{ and } \mu_i^{-1}\mu_j^{-1},\ 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j; \\ \mu_i\mu_j^{-1} \text{ and } \mu_i^{-1}\mu_j,\ 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j; \\ 1 \text{ with multiplicity at least } \sum_{i=1}^{m} n_i^2 - 1 - \varepsilon_p(\ell). \end{cases} \tag{5.1}$$

Let $\mu \in k^*$ be an eigenvalue of s on V. If $\mu \neq \mu^{-1}$, then, as V is self-dual, we have $\dim(V_s(\mu)) \leq \frac{\dim(V)-\dim(V_s(1))}{2} \leq \frac{\dim(V)-(\ell-1-\varepsilon_p(\ell))}{2} \leq B_\ell$, where equality holds if and only if ℓ, p, s and μ are as in (1.3). Thus, for the remainder of the proof, we assume that $\mu = \pm 1$.

Let $m = 1$, i.e. $n_1 = \ell$ and $\mu_1^2 \neq 1$. Using (5.1), we show that $\dim(V_s(\pm 1)) \leq B_\ell$, where equality holds if and only if ℓ, p, s and μ are as in (1.2), (2.1), and (3.2). Thus, for the remainder of the proof, we assume that $m \geq 2$.

If $\ell = 2$, then $m = 2$, i.e. $n_1 = n_2 = 1$. Using (5.1), we determine that $\dim(V_s(\pm 1)) \leq B_2$, where equality holds if and only if p, s and μ are as in (1.1). If $\ell = 3$, then $m \leq 3$. If $m = 2$, i.e. $n_1 = 2$ and $n_2 = 1$, then $\dim(V_s(\pm 1)) \leq B_3$ where equality holds if and only if p, s and μ are as in (2.2). If $m = 3$, by (5.1), we get $\dim(V_s(1)) = 2 - \varepsilon_p(\ell)$; and $\dim(V_s(-1)) \leq 12$. Further, as -1 can equal at most one eigenvalue of the form $\mu_i\mu_j$ and at most one of the form $\mu_i\mu_j^{-1}$, we deduce that $\dim(V_s(-1)) \leq 4$. For the remainder of the proof, we assume that $\ell \geq 4$.

For the eigenvalue $\mu = 1$, as $\mu_i \neq \mu_j^{\pm 1}$ for all $i < j$, we have $\mu_i^{\pm 1}\mu_j^{\pm 1} \neq 1$ for all $i < j$, thus:

$$\dim(V_s(1)) \leq 2\ell^2 - \ell - 1 - \varepsilon_p(\ell) - 4\sum_{i<j} n_in_j. \tag{5.2}$$

Let $\ell = 4$ and assume that $\dim(V_s(1)) \geq B_4$. It follows that, $11 + \varepsilon_p(\ell) - 4\sum_{i<j} n_i n_j \geq 0$ and, as $\sum_{r=1}^{m} n_r = 4$, $m \geq 2$ and $n_r \geq n_q$ for all $r < q$, we get $\sum_{i<j} n_i n_j \geq 3$. Thus, $11 + \varepsilon_p(\ell) - 4\sum_{i<j} n_i n_j \geq 0$ holds if and only if $p = 2$, $m = 2$, $n_1 = 3$ and $n_2 = 1$. Substituting in (5.2), gives $\dim(V_s(1)) \leq 14$, where equality holds if and only if s is as in (3.3). We now let $\ell \geq 5$ and assume that $\dim(V_s(1)) \geq B_\ell$. By (5.2), we get:

$$\ell - 1 - \sum_{i<j} n_i n_j \geq 0. \tag{5.3}$$

As $\ell = \sum_{i=1}^{m} n_i$, it follows that $\sum_{i=1}^{m-2} n_i(1 - \sum_{i<j} n_j) + (n_{m-1} - 1)(1 - n_m) \geq 0$. But $\sum_{i=1}^{m-2} n_i(1 - \sum_{i<j} n_j) \leq 0$ and $(n_{m-1} - 1)(1 - n_m) \leq 0$, as $n_i \geq 1$ for all $1 \leq i \leq m$, therefore inequality (5.3) holds if and only if $m = 2$, $n_2 = 1$ and $n_1 = \ell - 1$. Substituting in (5.2), gives $\dim(V_s(1)) \leq B_\ell$, where equality holds if and only if all eigenvalues of s on V different to $\mu_1^{\pm 1}\mu_2^{\pm 1}$ are equal to 1, i.e. if and only if s is as in (4).

Lastly, let $\mu = -1$. We remark that $\dim(V_s(-1)) \leq \dim(V) - \dim(V_s(1)) \leq 2\ell^2 - \ell - \sum_{r=1}^{m} n_r^2$, see (5.1). If $\mu_i\mu_j \neq -1$ for all $i < j$, we have $\dim(V_s(-1)) \leq 2\ell^2 - \ell - \sum_{r=1}^{m} n_r^2 - 2\sum_{i<j} n_i n_j = \ell^2 - \ell < B_\ell$. We thus assume that there exist $i < j$ such that $\mu_i\mu_j = -1$. Then $\mu_i^{-1}\mu_j^{-1} = -1$ and, since the μ_i's are distinct, we have that:

$$\begin{cases} \mu_i^2 \neq -1 \text{ and } \mu_j^2 \neq -1, \text{ hence } \mu_i^{-2} \neq -1 \text{ and } \mu_j^{-2} \neq -1; \\ \mu_i\mu_r \neq -1 \text{ and } \mu_i^{-1}\mu_r^{-1} \neq -1, \text{ where } i < r \leq m, r \neq j; \text{ and } \mu_r\mu_i \neq -1 \text{ and } \mu_r^{-1}\mu_i^{-1} \neq -1, \text{ where } 1 \leq r < i; \\ \mu_r\mu_j \neq -1 \text{ and } \mu_r^{-1}\mu_j^{-1} \neq -1, \text{ where } 1 \leq r < j, r \neq i; \text{ and } \mu_j\mu_r \neq -1 \text{ and } \mu_j^{-1}\mu_r^{-1} \neq -1, \text{ where} j < r \leq m. \end{cases} \tag{5.4}$$

By (5.1), all of the above account for at least $n_i(n_i - 1) + n_j(n_j - 1) + 2(n_i + n_j)(\ell - n_i - n_j)$ additional eigenvalues of s on V different to -1. This gives:

$$\dim(V_s(-1)) \leq 2\ell^2 - \ell - \sum_{r=1}^{m} n_r^2 - n_i(n_i - 1) - n_j(n_j - 1) - 2(n_i + n_j)(\ell - n_i - n_j). \tag{5.5}$$

Let $\ell = 4$ and assume $\dim(V_s(-1)) \geq B_4$. Then $(n_i, n_j) \in \{(3,1), (2,2), (2,1), (1,1)\}$ and we have $12 - \sum_{r\neq i,j} n_r^2 - 7(n_i + n_j) + 4n_i n_j \geq 0$. For $(n_i, n_j) \in \{(3,1), (2,1)\}$ the inequality does not hold. For $(n_i, n_j) = (2,2)$ we get $\dim(V_s(-1)) \leq 16$, see (5.5), where equality holds if and only if s is as in (3.1). For $(n_i, n_j) = (1,1)$ we have $\dim(V_s(-1)) \leq 18 - \sum_{r\neq i,j} n_r^2$, see (5.5). Since we are assuming $\dim(V_s(-1)) \geq B_4$, we must have $n_r = 1$ for all $r \neq i, j$, therefore $m = 4$ and $n_i = 1$ for all $1 \leq i \leq 4$. Substituting in (5.5) gives $\dim(V_s(-1)) \leq 16$, where equality holds if and only if all eigenvalues of s on V different to 1 and the ones listed in (5.4) are equal to -1. However, as at most one eigenvalue of the form $\mu_i\mu_r^{-1}$, $r \neq i, j$ can equal -1, we see that the condition for equality cannot be satisfied. This completes the case of $\ell = 4$. Thus, let $\ell \geq 5$ and assume $\dim(V_s(-1)) \geq B_\ell$. It follows that:

$$\ell(4 - n_i - n_j) - 3 + \varepsilon_p(\ell) - \sum_{r\neq i,j} n_r^2 - (n_i - n_j)^2 - (n_i + n_j)(\ell - n_i - n_j - 1) \geq 0. \tag{5.6}$$

If $\ell - n_i - n_j - 1 < 0$, then, as $\sum_{r=1}^{m} n_r = \ell$, we have $m = 2$ and so $\ell = n_1 + n_2$. Substituting in (5.6) gives $\ell(5 - \ell) - 3 + \varepsilon_p(\ell) - \sum_{r\neq i,j} n_r^2 - (2n_1 - \ell)^2 \geq 0$, which does not hold as $\ell \geq 5$. If $\ell - n_i - n_j - 1 \geq 0$, then, for (5.6) to hold, we must have $\ell(4 - n_i - n_j) > 0$, hence $(n_i, n_j) \in \{(2,1), (1,1)\}$. If $(n_i, n_j) = (2,1)$, inequality (5.6) does not hold. If $(n_i, n_j) = (1,1)$, substituting in (5.6) gives $3 + \varepsilon_p(\ell) - \sum_{r\neq i,j} n_r^2 \geq 0$. One checks that this inequality holds if and only if $\ell \in \{5,6\}$, $n_r = 1$ for all $1 \leq r \leq \ell$, and $\varepsilon_p(6) = 1$ when $\ell = 6$. In both cases, we can assume without loss of generality that $\mu_1\mu_2 = -1$. As the μ_r's are distinct, at most one eigenvalue of each of the forms $\mu_1\mu_r^{-1}$, $\mu_1^{-1}\mu_r$, $\mu_2\mu_r^{-1}$ and $\mu_2^{-1}\mu_r$, $3 \leq r \leq \ell$, can equal -1. This gives an additional $4(\ell-3)$ eigenvalues of s on V that are different to -1. Consequently, we have $\dim(V_s(-1)) \leq 2\ell^2 - 10\ell + 20 < B_\ell$. This completes the proof. □

Proposition 5.5. *Assume $p \neq 2$ and let $V = L_G(2\omega_1)$. Then* $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu)) = 2\ell^2 - 3\ell + 4$, *where the maximum is attained if and only if*

(1) $\ell = 2$, $\mu = -1$ and s is conjugate to $\mathrm{diag}(d,d,d^{-1},d^{-1})$ *with $d^2 = -1$.*
(2) $\ell \geq 2$, $\mu = 1$ and s is conjugate to $\pm\,\mathrm{diag}(1,\dots,1,-1,-1,1,\dots,1)$.

Proof. Let $s \in T \setminus Z(G)$ be as in $({}^\dagger H_s)$. We note that $V \cong S^2(W)$, see [18, Proposition 4.2.2], therefore the eigenvalues of s on V, not necessarily distinct, have one of the following forms:

$$\begin{cases} \mu_i^2 \text{ and } \mu_i^{-2},\ 1 \leq i \leq m, \text{ each with multiplicity at least } \frac{n_i(n_i+1)}{2}; \\ \mu_i\mu_j \text{ and } \mu_i^{-1}\mu_j^{-1},\ 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j; \\ \mu_i\mu_j^{-1} \text{ and } \mu_i^{-1}\mu_j,\ 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j; \\ 1 \text{ with multiplicity at least } \sum_{i=1}^m n_i^2. \end{cases} \tag{5.7}$$

The cases of $\mu \neq \mu^{-1}$; $s \in T \setminus Z(G)$ with $m = 1$; and $s \in T \setminus Z(G)$ with $m \geq 2$ and $\mu = 1$ are handled as in the proof of Proposition 5.4. The only case we have left to consider is $s \in T \setminus Z(G)$ with $m \geq 2$ and $\mu = -1$. We note that $\dim(V_s(-1)) \leq 2\ell^2 + \ell - \sum_{i=1}^m n_i^2$. If $\mu_i\mu_j \neq -1$ for all $i < j$, then, by (5.7), there are at least $2\sum_{i<j} n_in_j$ additional eigenvalues of s on V different to -1. This gives:

$$\dim(V_s(-1)) \leq 2\ell^2 + \ell - \sum_{i=1}^m n_i^2 - 2\sum_{i<j} n_in_j = 2\ell^2 + \ell - (\sum_{i=1}^m n_i)^2 = \ell^2 + \ell. \tag{5.8}$$

Therefore $\dim(V_s(-1)) \leq 2\ell^2 - 3\ell + 4$, where equality holds if and only if $\ell = 2$ and all eigenvalues of s on V different to 1, $\mu_1\mu_2$ and $\mu_1^{-1}\mu_2^{-1}$ are equal to -1. But then, by (5.7), we must have $\mu_1^2 = \mu_1\mu_2^{-1}$, a contradiction. We thus assume that there exist $i < j$ such that $\mu_i\mu_j = -1$. Then $\mu_i^{-1}\mu_j^{-1} = -1$, and:

$$\begin{cases} \mu_i^2 \neq -1,\ \mu_i^{-2} \neq -1 \text{ and } \mu_j^2 \neq -1,\ \mu_j^{-2} \neq -1; \\ \mu_i\mu_r \neq -1 \text{ and } \mu_i^{-1}\mu_r^{-1} \neq -1, \text{ where } i < r \leq m,\ r \neq j; \text{ and } \mu_r\mu_i \neq -1 \text{ and } \mu_r^{-1}\mu_i^{-1} \neq -1, \text{ where } 1 \leq r < i; \\ \mu_j\mu_r \neq -1 \text{ and } \mu_j^{-1}\mu_r^{-1} \neq -1, \text{ where } j < r \leq m; \text{ and } \mu_r\mu_j \neq -1 \text{ and } \mu_r^{-1}\mu_j^{-1} \neq -1, \text{ where } 1 \leq r < j,\ r \neq i. \end{cases}$$

By (5.7), these amount to at least $n_i(n_i+1) + n_j(n_j+1) + 2(n_i+n_j)(\ell - n_i - n_j)$ additional eigenvalues of s on V different to -1. This gives:

$$\dim(V_s(-1)) \leq 2\ell^2 + \ell - \sum_{r=1}^m n_r^2 - n_i(n_i+1) - n_j(n_j+1) - 2(n_i+n_j)(\ell - n_i - n_j). \tag{5.9}$$

If $\dim(V_s(-1)) \geq 2\ell^2 - 3\ell + 4$, it follows that

$$\ell(4 - n_i - n_j) - \sum_{r\neq i,j} n_r^2 - (n_i - n_j)^2 - (n_i + n_j)(\ell + 1 - n_i - n_j) - 4 \geq 0,$$

which does not hold. This completes the proof of the proposition. □

Proposition 5.6. *Let $\ell = 4$ and $V = L_G(\omega_3)$. Then* $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu)) = 28 - 2\varepsilon_p(3) - 8\varepsilon_p(2)$.

Proof. First, one determines the structure of $V\mid_{[L_1,L_1]}$ and then applies the algorithm from Section 3.3, using Theorem 5.2[5.2.4 for $\ell = 3$] and Propositions 5.4 and 5.5. For $p = 3$, it follows that $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu)) = 26$, while, for $p \neq 3$, we get $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu)) \leq 30 - 2\varepsilon_p(2)$ and that there exist $(s,\mu) \in (Z(L)^\circ \setminus Z(G)) \times k^*$ with $\dim(V_s(\mu)) = 28 - 8\varepsilon_p(2)$. In what follows we will show that $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu)) = 28 - 8\varepsilon_p(2)$.

Assume there exist $(s,\mu) \in (T\setminus Z(G))\times k^*$ with $\dim(V_s(\mu)) > 28-8\varepsilon_p(2)$. Then $s \notin Z(L_1)^\circ$, and we write $s = z\cdot h$ with $z \in Z(L_1)^\circ$ and $h \in [L_1,L_1]$. If $p \neq 2$, by Proposition 5.4, h must be conjugate to one of $\{\mathrm{diag}(1,1,1,-1,-1,1,1,1), \mathrm{diag}(1,-1,-1,1,1,-1,-1,1), \mathrm{diag}(1,d,d,d,d^{-1},d^{-1},d^{-1})$ with $d^2 \neq 1\}$. In all cases, one shows that $\dim(V_s(\mu)) \leq 28$ for all μ. Let $p = 2$. In view of Proposition 5.4, assume h is conjugate to one of $\{\mathrm{diag}(1,1,1,d,d^{-1},1,1,1)$ with $d \neq 1, \mathrm{diag}(1,d,d,d,d^{-1},d^{-1},d^{-1},1)$ with $d \neq 1, \mathrm{diag}(1,d,d,e,e^{-1},d^{-1},d^{-1},1)$ with $d \neq 1, e^{\pm 1}\}$. Using the weight structure of $V\mid_{[L_1,L_1]}$, one shows that $\dim(V_s(\mu)) \leq 20$ for all μ. On the other hand, if h belongs to a different conjugacy class, then by Propositions 5.4, 5.5 and the weight structure of V^0, it follows that $\dim(V_s(\mu)) \leq 18$ for all μ. □

Proposition 5.7. *Let $\ell \geq 5$ and $V = L_G(\omega_3)$. Then* $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu)) \leq \binom{2\ell-2}{3} + 10 - 2(\ell-1)\varepsilon_p(\ell-1) + 4\varepsilon_p(3) - 10\varepsilon_p(2)$.

Proof. To prove the result, we determine the structure of $V\mid_{[L_1,L_1]}$ and apply the algorithm of Section 3.3. Note that the case when $s \notin Z(L_1)^\circ$ is handled recursively. We write $s = z\cdot h$, where $z \in Z(L_1)^\circ$ and $h \in [L_1,L_1]$. For $\ell = 5$, we use Propositions 5.3, 5.4, and 5.6, to show that $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu)) \leq 66 + 4\varepsilon_p(3) - 18\varepsilon_p(2)$. For $\ell \geq 6$, using the result for $\ell = 5$, one shows that:

$$\begin{aligned}\dim(V_s(\mu)) &\leq 4(\ell-1)^2 - 8(\ell-1) + 4 - 2(\ell-1)\varepsilon_p(\ell-1) + (2(\ell-1)-2)\varepsilon_p(\ell-2) + \dim((L_L(\omega_4))_h(\mu_h))\\ &\leq 4\sum_{j=5}^{\ell-1} j^2 - 8\sum_{j=5}^{\ell-1} j + \sum_{j=5}^{\ell-1} 4 - \sum_{j=5}^{\ell-1} 2j\varepsilon_p(j) + \sum_{j=5}^{\ell-1} 2(j-1)\varepsilon_p(j-1) + 66 + 4\varepsilon_p(3) - 18\varepsilon_p(2)\\ &= \binom{2\ell-2}{3} + 10 - 2(\ell-1)\varepsilon_p(\ell-1) + 4\varepsilon_p(3) - 10\varepsilon_p(2).\end{aligned}$$

□

Proof of Theorem 5.2. The proofs of 5.2.1, 5.2.2, and 5.2.3 are covered in Propositions 5.3, 5.4, and 5.5. We have made this choice as they require more in-depth analysis. The proofs of the remaining results are much more straightforward: for each triplet (V,p,ℓ) we first determine the structure of $V\mid_{[L_i,L_i]}$, $i = 1$ or $i = 2$, and then apply the algorithm of Section 3.3.
5.2.4: Propositions 5.3 and 5.4 give the result for $\ell = 3$, while Proposition 5.6, respectively 5.7, gives the result for $\ell = 4$, respectively $\ell \geq 5$.
5.2.5: The proof is analogous to that of Proposition 5.7, using Propositions 5.3 and 5.5.
5.2.6: Let $\ell = 2$. If $p = 5$, the result follows from Propositions 4.3, 4.5, and 4.8. If $p = 2$, one identifies the eigenvalues of s on V and determines that $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu)) = 6$. If $p \neq 2,5$, one uses the weight structure of V^1 and Proposition 4.3 to prove the result. Assume $\ell \geq 3$. If $p = 3$, the result follows recursively, using Propositions 5.3, 5.5, and 5.2.6. If $p \neq 3$, the proof is analogous to that of Proposition 5.7 and uses 5.2.4 and 5.2.30.
5.2.7: Follows by Propositions 4.3, 4.5, and 4.8.
5.2.8: Follows by Propositions 4.5 and 4.8. Note that when $s \notin Z(L_1)^\circ$, one has to show that there do not exist μ with $\dim(V_s(\mu)) > 12$. If this were the case, then, as V is self-dual, we would have $\mu = \pm 1$ and $\dim(V_s^i(\pm 1)) = 2$ for all $0 \leq i \leq 6, i \neq 3$. We write $s = z\cdot h$ with $z \in Z(L_1)^\circ$ and $h \in [L_1,L_1]$. Then $\dim(V_h^i(\mu_h^i)) = 2$ for all $i \neq 3$, where $\mu_h^i = \mu\cdot c^{i-3}$. In particular, by Propositions 4.5 and 4.8, we have $\mu_h^2 = -1$, thereby $c^2 = 1$, and $\mu_h^1 = d^{\pm 1}$ with $d^2 = -1$, thereby $\mu = d^{\pm 1}$, contradicting $\mu = \pm 1$.

5.2.9: Because we will need to know $\max_{s\in T\setminus Z(G)} \dim((L_G(3\omega_2))_s(\mu))$ also when $p \neq 7$, we will not limit ourselves to the case $p = 7$ and, instead, only assume that $p \neq 2, 3$. Using Propositions 4.5, 4.8, and Theorem 4.9[4.3.1], we deduce that $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu)) = 20 - \varepsilon_p(7)$.
5.2.10: As above, we only assume $p \neq 2$. Using Proposition 4.5 we deduce that $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu)) = 20 - 4\varepsilon_p(3)$. Note that when $s \notin Z(L_2)^\circ$ ($s = z' \cdot h'$, where $z' \in Z(L_2)^\circ$ and $h' \in [L_2, L_2]$), one has to show that $\dim(V^1_{h'}(\mu^1_{h'})) \leq 5 - \varepsilon_p(3)$ and $\dim(V^2_{h'}(\mu^2_{h'})) \leq 6 - 2\varepsilon_p(3)$ for all $\mu^i_{h'}$, $i = 1, 2$. Lastly, equality is shown to hold for $s = \mathrm{diag}(1, -1, -1, 1) \in T \setminus Z(G)$ and $\mu = -1$.
5.2.11: Follows by Propositions 5.3, 5.4, 5.5, and 5.2.6. Note that when $s \notin Z(L_1)^\circ$, one has to show that $\dim(V^2_h(\mu^2_h)) \leq 12 - 4\varepsilon_p(3) - 8\varepsilon_p(2)$ for all μ^2_h, where $s = z \cdot h$ with $z \in Z(L_1)^\circ$ and $h \in [L_1, L_1]$.
5.2.12: Follows by 5.2.6 and 5.2.7.
5.2.13: As in 5.2.9, we only assume $p \neq 2$. Using Proposition 5.5, 5.2.6, and 5.2.7, we deduce that $\max_{s\in T\setminus Z(G)} \dim((L_G(2\omega_3))_s(\mu)) = 52 - 13\varepsilon_p(5)$.
5.2.14: We only assume $p \neq 2$. Using Propositions 5.3, 5.4, 5.5, 5.2.6, and 5.2.7, we deduce that $\max_{s\in T\setminus Z(G)} \dim(V_s(\mu)) = 50 - \varepsilon_p(7)$. Note that one has to show that there do not exist $(s, \mu) \in (T\setminus Z(G)) \times k^*$ with $\dim(V_s(\mu)) > 50 - \varepsilon_p(7)$. If this were the case, then $s \notin Z(L_1)^\circ$ ($s = z \cdot h$ with $z \in Z(L_1)^\circ$ and $h \in [L_1, L_1]$) and h would be conjugate to one of $\{\mathrm{diag}(1, 1, -1, -1, 1, 1), \mathrm{diag}(1, d, d, d^{-1}, d^{-1}, 1)$ with $d^2 = -1\}$. In both cases, one shows that $\dim(V_s(\mu)) \leq 50 - \varepsilon_p(7)$.
5.2.15: Follows by Proposition 5.4 and 5.2.4.
5.2.16: Follows recursively, using 5.2.11 and 5.2.30.
5.2.17: Follows by 5.2.4, 5.2.6, and 5.2.11.
5.2.18: Follows by Propositions 5.3, 5.4, 5.2.6, and 5.2.11.
5.2.19: Follows by Propositions 5.4, 5.6, 5.2.30, and 5.2.15.
5.2.20: Follows by Proposition 5.6 and 5.2.15.
5.2.21: Follows by Propositions 5.4, 5.7 and 5.2.19.
5.2.22: Follows by Proposition 5.7, 5.2.19, 5.2.20, and 5.2.30.
5.2.23: Follows by 5.2.19 and 5.2.20.
5.2.24: Follows by Propositions 5.4, 5.7, and 5.2.21.
5.2.25 and 5.2.26: Follow by 5.2.22 and 5.2.23.
5.2.27: Follows by 5.2.21 and 5.2.22.
5.2.28: Follows by Propositions 5.4, 5.7, and 5.2.24.
5.2.29: Follows by Propositions 5.4, 5.7, and 5.2.28.
5.2.30: Follows by 5.2.4.
5.2.31: Proof is analogous to that of Proposition 5.7, and uses 5.2.4 and 5.2.30 . □

6. Proof of Theorem 1.1 for groups of type B_ℓ

Let k be an algebraically closed field of characteristic $p \neq 2$ and let G, respectively $\tilde{G}$, be a simple adjoint, respectively simply connected, linear algebraic group of type B_ℓ with $\ell \geq 3$. We fix a central isogeny $\phi : \tilde{G} \to G$ with $\ker(\phi) \subseteq Z(\tilde{G})$ and $d\phi \neq 0$, and let $\tilde{T}$, respectively $\tilde{B}$, be a maximal torus, respectively a Borel subgroup, in $\tilde{G}$ with the property that $\phi(\tilde{T}) = T$, respectively $\phi(\tilde{B}) = B$. We denote by $X(\tilde{T})$, $Z(\tilde{G})$, $\tilde{G}_u$, $\tilde{\Delta} = \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_\ell\}$ and $\tilde{\omega}_1, \ldots, \tilde{\omega}_\ell$ the character group of $\tilde{T}$, the center of $\tilde{G}$, the set of unipotent elements in $\tilde{G}$, the set of simple roots in $\tilde{G}$ corresponding to $\tilde{B}$, and the fundamental dominant weights of $\tilde{G}$ corresponding to $\tilde{\Delta}$. We denote by $\tilde{L}_i$ a Levi subgroup of the maximal parabolic subgroup $\tilde{P}_i$ of $\tilde{G}$ corresponding to $\tilde{\Delta}_i = \tilde{\Delta} \setminus \{\tilde{\alpha}_i\}$, $1 \leq i \leq \ell$, and we let $\tilde{T}_i = \tilde{T} \cap [\tilde{L}_i, \tilde{L}_i]$. Now, for $\tilde{\lambda} \in X(\tilde{T})$ a p-restricted dominant weight, we let $\tilde{V} = L_{\tilde{G}}(\tilde{\lambda})$ and we have $\tilde{V} \mid_{[\tilde{L}_i, \tilde{L}_i]} = \bigoplus_{j=0}^{e_i(\tilde{\lambda})} \tilde{V}^j$, where $\tilde{V}^j = \bigoplus_{\tilde{\gamma}\in\mathbb{N}_{\geq 0}\tilde{\Delta}_i} \tilde{V}_{\tilde{\lambda} - j\tilde{\alpha}_i - \tilde{\gamma}}$, $0 \leq j \leq e_i(\tilde{\lambda})$.

This section is dedicated to Theorems 6.1 and 6.2 which give the values of $\max_{\tilde{u}\in\tilde{G}_u\setminus\{1\}} \dim(\tilde{V}_{\tilde{u}}(1))$ and $\max_{\tilde{s}\in\tilde{T}\setminus Z(\tilde{G})} \dim(\tilde{V}_{\tilde{s}}(\tilde{\mu}))$ for all $k\tilde{G}$-modules $\tilde{V} = L_{\tilde{G}}(\tilde{\lambda})$, with $\tilde{\lambda} \in X(\tilde{T})$ a nonzero p-restricted dominant weight, and $\dim(L_{\tilde{G}}(\tilde{\lambda})) \leq 4\ell^3$. As a corollary, the part of Theorem 1.1 concerning simple simply connected linear algebraic groups of type B_ℓ will follow.

Theorem 6.1. *Let k be an algebraically closed field of characteristic $p \neq 2$ and let $\tilde{G}$ be a simple simply connected linear algebraic group of type B_ℓ with $\ell \geq 3$. Let $\tilde{V} = L_{\tilde{G}}(\tilde{\lambda})$, where $\tilde{\lambda} \in X(\tilde{T})$ is a nonzero p-restricted dominant weight, be such that $\dim(\tilde{V}) \leq 4\ell^3$. The value of $\max_{\tilde{u}\in\tilde{G}_u\setminus\{1\}} \dim(\tilde{V}_{\tilde{u}}(1))$ is as given in Table 12.*

Table 12. The value of $\max_{\tilde{u}\in\tilde{G}_u\setminus\{1\}} \dim(\tilde{V}_{\tilde{u}}(1))$ for groups of type B_ℓ.

Ref.	$\tilde{V}$	Char.	Rank	$\max_{\tilde{u}\in\tilde{G}_u\setminus\{1\}} \dim(\tilde{V}_{\tilde{u}}(1))$
6.1.1	$L_{\tilde{G}}(\tilde{\omega}_1)$	$p \neq 2$	$\ell \geq 3$	$2\ell - 1$
6.1.2	$L_{\tilde{G}}(\tilde{\omega}_2)$	$p \neq 2$	$\ell \geq 3$	$2\ell^2 - 3\ell + 4$
6.1.3	$L_{\tilde{G}}(2\tilde{\omega}_1)$	$p \neq 2$	$\ell \geq 3$	$2\ell^2 - \ell - \varepsilon_p(2\ell+1)$
6.1.4	$L_{\tilde{G}}(\tilde{\omega}_3)$	$p \neq 2$	$\ell \geq 3$	$\frac{4\ell^3-12\ell^2+29\ell-27}{3}$
6.1.5	$L_{\tilde{G}}(3\tilde{\omega}_1)$	$p \neq 2,3$	$\ell \geq 3$	$\binom{2\ell+1}{3} - (2\ell-1)\varepsilon_p(2\ell+3)$
6.1.6	$L_{\tilde{G}}(\tilde{\omega}_1+\tilde{\omega}_2)$	$p \neq 2$	$\ell \geq 3$	$2[\binom{2\ell}{3} + 3\ell + 5] - (2\ell-1)\varepsilon_p(\ell) - (\frac{4\ell^3-12\ell^2+29\ell-30}{3})\varepsilon_p(3)$
6.1.7	$L_{\tilde{G}}(2\tilde{\omega}_3)$	$p \neq 2$	$\ell = 3$	21
6.1.8	$L_{\tilde{G}}(\tilde{\omega}_2+\tilde{\omega}_3)$	$p = 3$	$\ell = 3$	≤ 54
6.1.9	$L_{\tilde{G}}(\tilde{\omega}_2+\tilde{\omega}_3)$	$p = 5$	$\ell = 3$	30
6.1.10	$L_{\tilde{G}}(3\tilde{\omega}_3)$	$p = 5$	$\ell = 3$	50
6.1.11	$L_{\tilde{G}}(2\tilde{\omega}_4)$	$p \neq 2$	$\ell = 4$	76
6.1.12	$L_{\tilde{G}}(\tilde{\omega}_4)$	$p \neq 2$	$\ell = 5$	202
6.1.13	$L_{\tilde{G}}(2\tilde{\omega}_5)$	$p \neq 2$	$\ell = 5$	279
6.1.14	$L_{\tilde{G}}(\tilde{\omega}_4)$	$p \neq 2$	$\ell = 6$	453
6.1.15	$L_{\tilde{G}}(\tilde{\omega}_4)$	$p \neq 2$	$\ell = 7$	897
6.1.16	$L_{\tilde{G}}(\tilde{\omega}_1+\tilde{\omega}_\ell)$	$p \neq 2$	$3 \leq \ell \leq 6$	$\leq (3\ell-2)\cdot 2^{\ell-1} - 3\cdot 2^{\ell-2}\varepsilon_p(2\ell+1)^\bullet$
6.1.17	$L_{\tilde{G}}(\tilde{\omega}_\ell)$	$p \neq 2$	$4 \leq \ell \leq 13$	$3\cdot 2^{\ell-2}$

$^\bullet$ equality holds when $\varepsilon_p(2(\ell-i)+1) = 0$ for all $0 \leq i \leq \ell-3$.

Proof. For 6.1.1–6.1.5 we will deduce the result for $\tilde{V} = L_{\tilde{G}}(\tilde{\lambda})$ by calculating $\max_{u\in G_u\setminus\{1\}} \dim((L_G(\lambda))_u(1)) = \max_{i=1,\ell} \dim((L_G(\lambda))_{x_{\alpha_i}(1)}(1))$, see Section 3.1 and Lemmas 3.2 and 3.13. The proofs for 6.1.6 - 6.1.17 are much more straightforward: first one determines the structure of $\tilde{V}\mid_{[\tilde{L}_1,\tilde{L}_1]}$, and then applies the algorithm of Section 3.3. Once more, by Lemma 3.13, we have $\max_{\tilde{u}\in\tilde{G}_u\setminus\{1\}} \dim(\tilde{V}_{\tilde{u}}(1)) = \max_{i=\ell-1,\ell} \dim(\tilde{V}_{x_{\tilde{\alpha}_j}(1)}(1))$.

6.1.1: Note that $\tilde{V} \cong L_G(\omega_1)$ as $k\tilde{G}$-modules and $L_G(\omega_1) \cong W$ as kG-modules. We have that $x_{\alpha_1}(1)$, respectively $x_{\alpha_\ell}(1)$, acts on W as $J_3 \oplus J_1^{2\ell-2}$, respectively as $J_2^2 \oplus J_1^{2\ell-3}$.

6.1.2: Note that $\tilde{V} \cong L_G(\omega_2)$ as $k\tilde{G}$-modules and $L_G(\omega_2) \cong \wedge^2(W)$ as kG-modules, see [18, Proposition 4.2.2]. For $x_{\alpha_1}(1)$, we write $W = W_1 \oplus W_2$, where $\dim(W_1) = 3$ and $x_{\alpha_1}(1)$ acts as J_3 on W_1; and $\dim(W_2) = 2\ell - 2$ and $x_{\alpha_1}(1)$ acts trivially on W_2. Using [15, Lemma 3.4], we show that $\dim((L_G(\omega_2))_{x_{\alpha_1}(1)}(1)) = 2\ell^2 - 3\ell + 2$. Similarly, for $x_{\alpha_\ell}(1)$, we write $W = W_1' \oplus W_2'$, where $\dim(W_1') = 4$ and $x_{\alpha_\ell}(1)$ acts as J_2^2 on W_1'; and $\dim(W_2') = 2\ell - 3$ and $x_{\alpha_\ell}(1)$ acts trivially on W_2'. One shows that $\dim((L_G(\omega_2))_{x_{\alpha_\ell}(1)}(1)) = 2\ell^2 - 3\ell + 4$.

6.1.3: Note that $\tilde{V} \cong L_G(2\omega_1)$ as $k\tilde{G}$-modules. Moreover, by [18, Proposition 4.7.3], we have $S^2(W) \cong L_G(2\omega_1) \oplus L_G(0)$ if $\varepsilon_p(2\ell+1) = 0$, and $S^2(W) \cong L_G(0) \mid L_G(2\omega_1) \mid L_G(0)$ if $\varepsilon_p(2\ell+1) = 1$. We argue

as in 6.1.2 to show that $\dim((S^2(W))_{x_{\alpha_i}(1)}(1)) = 2\ell^2 - \ell + 1, i = 1, \ell$. Then, $\dim((L_G(2\omega_1))_{x_{\alpha_i}(1)}(1)) = 2\ell^2 - \ell - \varepsilon_p(2\ell+1), i = 1, \ell$, by [11, Corollary 6.3].
6.1.4: For $\ell = 3$, we use Theorem 5.1[5.1.1]. Assume $\ell \geq 4$. Note that $\tilde{V} \cong L_G(\omega_3)$ as $k\tilde{G}$-modules and that $\wedge^3(W) \cong L_G(\omega_3)$ as kG-modules, see [18, Proposition 4.2.2]. We now argue as in 6.1.2 to obtain the result.
6.1.5: Note that $\tilde{V} \cong L_G(3\omega_1)$ as $k\tilde{G}$-modules and that $S^3(W) \cong L_G(3\omega_1) \oplus W$ if $\varepsilon_p(2\ell+3) = 0$, respectively $S^3(W) \cong W \mid L_G(3\omega_1) \mid W$ if $\varepsilon_p(2\ell+3) = 1$, see [18, Propositions 4.7.4]. We argue in 6.1.2 to show that $\max_{u \in G_u \setminus \{1\}} \dim((L_G(3\omega_1))_u(1)) \geq \frac{4\ell^3-\ell}{3} - (2\ell-1)\varepsilon_p(2\ell+3)$. Now, one has to establish the structure of $L_G(3\omega_1) \mid_{[L_1,L_1]}$ and apply the algorithm of Section 3.3. Equality will follow recursively, using 6.1.1 and 6.1.3.
6.1.6: For $\ell = 3$, it follows by Theorem 5.1[5.1.2, 5.1.3, 5.1.7, 5.1.10]. For $\ell \geq 4$, it follows recursively by 6.1.1, 6.1.2, 6.1.3 and the result for $\ell = 3$.
6.1.7: Follows by Theorem 5.1[5.1.2, 5.1.3].
6.1.8, 6.1.9, and 6.1.10: Follow by Theorem 5.1[5.1.5, 5.1.10].
6.1.11: Follows by 6.1.2, 6.1.4, and 6.1.7.
6.1.12 and 6.1.13: Follow by 6.1.2, 6.1.4, and 6.1.11.
6.1.14: Follows by 6.1.2, 6.1.4, and 6.1.12.
6.1.15: Follows by 6.1.2, 6.1.4, and 6.1.14.
6.1.16: Follows recursively, by 6.1.17 and Theorem 5.1[5.1.1, 5.1.6] for the base case.
6.1.17: Follows recursively, using 6.1.4 to prove the base case. □

Theorem 6.2. *Let k be an algebraically closed field of characteristic $p \neq 2$ and let $\tilde{G}$ be a simple simply connected linear algebraic group of type B_ℓ with $\ell \geq 3$. Let $\tilde{V} = L_{\tilde{G}}(\tilde{\lambda})$, where $\tilde{\lambda} \in X(\tilde{T})$ is a nonzero p-restricted dominant weight, be such that $\dim(\tilde{V}) \leq 4\ell^3$. The value of $\max_{\tilde{s} \in \tilde{T} \setminus Z(\tilde{G})} \dim(\tilde{V}_{\tilde{s}}(\tilde{\mu}))$ is as given in Table 13.*

Table 13. The value of $\max_{\tilde{s} \in \tilde{T} \setminus Z(\tilde{G})} \dim(\tilde{V}_{\tilde{s}}(\tilde{\mu}))$ for groups of type B_ℓ.

Ref.	$\tilde{V}$	Char.	Rank	$\max_{\tilde{s} \in \tilde{T} \setminus Z(\tilde{G})} \dim(\tilde{V}_{\tilde{s}}(\tilde{\mu}))$
6.2.1	$L_{\tilde{G}}(\tilde{\omega}_1)$	$p \neq 2$	$\ell \geq 3$	2ℓ
6.2.2	$L_{\tilde{G}}(\tilde{\omega}_2)$	$p \neq 2$	$\ell \geq 3$	$2\ell^2 - \ell$
6.2.3	$L_{\tilde{G}}(2\tilde{\omega}_1)$	$p \neq 2$	$\ell \geq 3$	$2\ell^2 + \ell - \varepsilon_p(2\ell+1)$
6.2.4	$L_{\tilde{G}}(\tilde{\omega}_3)$	$p \neq 2$	$\ell \geq 3$	$\frac{4\ell^3-6\ell^2+2\ell}{3}$
6.2.5	$L_{\tilde{G}}(3\tilde{\omega}_1)$	$p \neq 2,3$	$\ell \geq 3$	$\binom{2\ell+1}{3} + 2\ell^2 + \ell - 2\ell\varepsilon_p(2\ell+3)$
6.2.6	$L_{\tilde{G}}(\tilde{\omega}_1 + \tilde{\omega}_2)$	$p \neq 2$	$\ell \geq 3$	$2\binom{2\ell+1}{3} - 2\ell\varepsilon_p(\ell) - \binom{2\ell}{3}\varepsilon_p(3)$
6.2.7	$L_{\tilde{G}}(2\tilde{\omega}_3)$	$p \neq 2$	$\ell = 3$	20
6.2.8	$L_{\tilde{G}}(\tilde{\omega}_2 + \tilde{\omega}_3)$	$p = 3$	$\ell = 3$	52
6.2.9	$L_{\tilde{G}}(\tilde{\omega}_2 + \tilde{\omega}_3)$	$p = 5$	$\ell = 3$	32
6.2.10	$L_{\tilde{G}}(3\tilde{\omega}_3)$	$p = 5$	$\ell = 3$	52
6.2.11	$L_{\tilde{G}}(2\tilde{\omega}_4)$	$p \neq 2$	$\ell = 4$	≤ 74
6.2.12	$L_{\tilde{G}}(\tilde{\omega}_4)$	$p \neq 2$	$\ell = 5$	≤ 214
6.2.13	$L_{\tilde{G}}(2\tilde{\omega}_5)$	$p \neq 2$	$\ell = 5$	≤ 278
6.2.14	$L_{\tilde{G}}(\tilde{\omega}_4)$	$p \neq 2$	$\ell = 6$	≤ 499
6.2.15	$L_{\tilde{G}}(\tilde{\omega}_4)$	$p \neq 2$	$\ell = 7$	≤ 1005
6.2.16	$L_{\tilde{G}}(\tilde{\omega}_1 + \tilde{\omega}_\ell)$	$p \neq 2$	$3 \leq \ell \leq 6$	$\ell \cdot 2^\ell - 2^{\ell-1}\varepsilon_p(2\ell+1)$
6.2.17	$L_{\tilde{G}}(\tilde{\omega}_\ell)$	$p \neq 2$	$4 \leq \ell \leq 13$	$2^{\ell-1}$

Before we begin, we recall that G is a simple adjoint linear algebraic group of type B_ℓ, and that we have denoted by W the natural module of G. Let $s \in T \setminus Z(G)$. As in Section 4, we will say"s is as in $({}^\dagger H_s)$"

to mean that s satisfies the following: $s = \mathrm{diag}(\mu_1 \cdot \mathrm{I}_{n_1}, \ldots, \mu_m \cdot \mathrm{I}_{n_m}, 1 \cdot \mathrm{I}_n, \mu_m^{-1} \cdot \mathrm{I}_{n_m}, \ldots, \mu_1^{-1} \cdot \mathrm{I}_{n_1})$ with $\mu_i \neq \mu_j^{\pm 1}$, $i < j$, $\mu_i \neq 1$, $1 \leq i \leq m$ and $n + 2\sum_{i=1}^{m} n_i = 2\ell + 1$, where $1 \leq n \leq 2\ell - 1$ and $\ell \geq n_1 \geq \cdots \geq n_m \geq 1$. Note that every $s \in T \setminus Z(G)$ is as in $({}^{\dagger}H_s)$.

Proposition 6.3. *Let $\tilde{V} = L_{\tilde{G}}(\tilde{\omega}_1)$. Then* $\max_{\tilde{s} \in \tilde{T} \setminus Z(\tilde{G})} \dim(\tilde{V}_{\tilde{s}}(\tilde{\mu})) = 2\ell$.

Proof. Set $V = L_G(\omega_1)$. We note that $\tilde{V} \cong V$ as $k\tilde{G}$-modules and that $V \cong W$ as kG-modules. The proof now follows that of Proposition 4.3. □

Proposition 6.4. *Let $\tilde{V} = L_{\tilde{G}}(\tilde{\omega}_2)$. Then* $\max_{\tilde{s} \in \tilde{T} \setminus Z(\tilde{G})} \dim(\tilde{V}_{\tilde{s}}(\tilde{\mu})) = 2\ell^2 - \ell$.

Proof. Set $V = L_G(\omega_2)$ and note that $\tilde{V} \cong V$ as $k\tilde{G}$-modules. Further, by [18, Proposition 4.2.2], we have $V \cong \wedge^2(W)$ as kG-modules. Let $s \in T \setminus Z(G)$ be as in $({}^{\dagger}H_s)$. The eigenvalues of s on V, not necessarily distinct, have one of the following forms:

$$\begin{cases} \mu_i^2 \text{ and } \mu_i^{-2}, \text{ where } 1 \leq i \leq m, \text{ each with multiplicity at least } \frac{n_i(n_i-1)}{2}; \\ \mu_i\mu_j \text{ and } \mu_i^{-1}\mu_j^{-1}, \text{ where } 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j; \\ \mu_i\mu_j^{-1} \text{ and } \mu_i^{-1}\mu_j, \text{ where } 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j; \\ \mu_i \text{ and } \mu_i^{-1}, \text{ where } 1 \leq i \leq m, \text{ each with multiplicity at least } nn_i; \\ 1 \text{ with multiplicity at least } \frac{n(n-1)}{2} + \sum_{r=1}^{m} n_r^2. \end{cases} \tag{6.1}$$

Let $\mu \in k^*$ be an eigenvalue of s on V. If $\mu \neq \mu^{-1}$, one shows that $\dim(V_s(\mu)) \leq \frac{2\ell^2+\ell}{2} < 2\ell^2 - \ell$. We thus assume for the remainder of the proof that $\mu = \pm 1$.

Assume $m = 1$. Then $\mu_1 \neq 1$, as $s \notin Z(G)$. By (6.1), the eigenvalues of s on V, not necessarily distinct, are $\mu_1^{\pm 2}$ each with multiplicity at least $\frac{n_1(n_1-1)}{2}$; $\mu_1^{\pm 1}$ each with multiplicity at least nn_1; and 1 with multiplicity at least $\frac{n(n-1)}{2} + n_1^2$. Let $\mu = 1$. Since $n = 2\ell + 1 - 2n_1$, we have $\dim(V_s(1)) \leq 2\ell^2 - \ell - 2(\ell - n_1)(2n_1 - 1) \leq 2\ell^2 - \ell$, where equality holds if and only if $n_1 = \ell$ and $\mu_1 = -1$. Now, let $\mu = -1$. If $\mu_1 = -1$, then $\dim(V_s(-1)) = 2nn_1$, while, if $\mu_1^2 = -1$, then $\dim(V_s(-1)) = n_1(n_1-1)$, therefore $\dim(V_s(-1)) < 2\ell^2 - \ell$ for all s with $m = 1$. We thus assume that $m \geq 2$.

Let $\mu = 1$. Since $\mu_i \neq \mu_j^{\pm 1}$ for all $i < j$, by (6.1), we determine that $\dim(V_s(1)) \leq 2\ell^2 + \ell - 4\sum_{i<j} n_in_j - 2n\sum_{i=1}^{m} n_i$. Assume $\dim(V_s(1)) \geq 2\ell^2 - \ell$. Then, as $2\sum_{i=1}^{m} n_i = 2\ell + 1 - n$, it follows that

$$(2\ell - n)(1 - n) - 4\sum_{i<j} n_in_j \geq 0, \tag{6.2}$$

which does not hold. Therefore $\dim(V_s(1)) < 2\ell^2 - \ell$ for all $s \in T \setminus Z(G)$ with $m \geq 2$.

Lastly, let $\mu = -1$. If $\mu_i \neq -1$ for all i, then $\dim(V_s(-1)) \leq 2\ell^2 + \ell - \dfrac{n(n-1)}{2} - \sum_{r=1}^{m} n_r^2 - 2n\sum_{r=1}^{m} n_r$. Suppose that $\dim(V_s(-1)) \geq 2\ell^2 - \ell$. Then $2\ell - \dfrac{n(n-1)}{2} - \sum_{r=1}^{m} n_r^2 - 2n\sum_{r=1}^{m} n_r \geq 0$. Since $2\sum_{r=1}^{m} n_r = 2\ell + 1 - n$, we must have $(2\ell - n)(1 - n) - \dfrac{n(n-1)}{2} - \sum_{r=1}^{m} n_r^2 \geq 0$, which does not hold. We thus assume there exist i such that $\mu_i = -1$. Then, since the μ_i's are distinct and different to 1, by (6.1), we determine that $\dim(V_s(-1)) \leq 2\ell^2 + \ell - \dfrac{n(n-1)}{2} - \sum_{r=1}^{m} n_r^2 - 2n\sum_{r \neq i} n_r - 4n_i\sum_{r \neq i} n_r - n_i(n_i - 1)$. Suppose

$\dim(V_s(-1)) \geq 2\ell^2 - \ell$. Then:

$$2\ell - \frac{n(n-1)}{2} - \sum_{r=1}^{m} n_r^2 - 2n\sum_{r\neq i} n_r - 4n_i \sum_{r\neq i} n_r - n_i(n_i-1) \geq 0. \tag{6.3}$$

We have that $\sum_{r=1}^{m} n_r^2 \geq \sum_{r=1}^{m} n_r$, as $n_r \geq 1$, and that $2\ell = 2\sum_{r=1}^{m} n_r + n - 1$. Substituting in (6.3) gives $\sum_{r\neq i} n_r(1-2n) + n_i(2 - n_i - 4\sum_{r\neq i} n_r) - \frac{(n-1)(n-2)}{2} \geq 0$, which does not hold. Therefore $\dim(V_s(-1)) < 2\ell^2 - \ell$ for all $s \in T \setminus Z(G)$ with $m \geq 2$. □

Proposition 6.5. *Let $\tilde{V} = L_{\tilde{G}}(2\tilde{\omega}_1)$. Then $\max_{\tilde{s}\in\tilde{T}\setminus Z(\tilde{G})} \dim(\tilde{V}_{\tilde{s}}(\tilde{\mu})) = 2\ell^2 + \ell - \varepsilon_p(2\ell+1)$.*

Proof. Set $V = L_G(2\omega_1)$. Note that $\tilde{V} \cong V$ as $k\tilde{G}$-modules and that $S^2(W) \cong V \oplus L_G(0)$ if $\varepsilon_p(2\ell+1) = 0$; while $S^2(W) \cong L_G(0) \mid V \mid L_G(0)$ if $\varepsilon_p(2\ell+1) = 1$, see [18, Proposition 4.7.3]. Let $s \in T \setminus Z(G)$ be as in $({}^{\dagger}H_s)$. The eigenvalues of s on V, not necessarily distinct, have one of the following forms:

$$\begin{cases} \mu_i^2 \text{ and } \mu_i^{-2},\ 1 \leq i \leq m, \text{ each with multiplicity at least } \frac{n_i(n_i+1)}{2}; \\ \mu_i\mu_j \text{ and } \mu_i^{-1}\mu_j^{-1},\ 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j; \\ \mu_i\mu_j^{-1} \text{ and } \mu_i^{-1}\mu_j,\ 1 \leq i < j \leq m, \text{ each with multiplicity at least } n_in_j; \\ \mu_i \text{ and } \mu_i^{-1},\ 1 \leq i \leq m, \text{ each with multiplicity at least } nn_i; \\ 1 \text{ with multiplicity at least } \sum_{r=1}^{m} n_r^2 + \frac{n(n+1)}{2} - 1 - \varepsilon_p(2\ell+1). \end{cases} \tag{6.4}$$

The cases $\mu = \mu^{-1}$ and $s \in T \setminus Z(G)$ with $m = 1$ are handled as in the proof of Proposition 6.4. The only case we have left is that of $s \in T \setminus Z(G)$ with $m \geq 2$. For $\mu = 1$, as $\mu_i^{\pm 1}\mu_j^{\pm 1} \neq 1$ for all $i < j$, by (6.4), we determine that $\dim(V_s(1)) \leq 2\ell^2 + 3\ell - \varepsilon_p(2\ell+1) - 2n\sum_{i=1}^{m} n_i - 4\sum_{i<j} n_in_j < 2\ell^2 + \ell - \varepsilon_p(2\ell+1)$, see (6.2). Thus, let $\mu = -1$. Suppose that $\mu_i \neq -1$ for all i. Then, by (6.4), we have $\dim(V_s(-1)) \leq 2\ell^2 + 3\ell + 1 - \sum_{r=1}^{m} n_r^2 - \frac{n(n+1)}{2} - 2n\sum_{r=1}^{m} n_r$. One shows, as in the corresponding case in the proof of Proposition 6.4, that $\dim(V_s(-1)) < 2\ell^2 + \ell - \varepsilon_p(2\ell+1)$. Lastly assume that there exists i such that $\mu_i = -1$. Then $\mu_r^{\pm 1} \neq -1$ for all $r \neq i$ and, by (6.4), we get $\dim(V_s(-1)) \leq 2\ell^2 + 3\ell + 1 - \sum_{r=1}^{m} n_r^2 - \frac{n(n+1)}{2} - 2n\sum_{r\neq i} n_r - n_i(n_i+1)$. Once more, as in the proof of Proposition 6.4, we show that $\dim(V_s(-1)) < 2\ell^2 + \ell - \varepsilon_p(2\ell+1)$. □

Proof of Theorem 6.2. The results 6.2.1, 6.2.2, and 6.2.3 are covered in Propositions 6.3, 6.4, and 6.5, respectively. The proofs for 6.2.4–6.1.17 are much more straightforward: first one determines the structure of $\tilde{V}\mid_{[\tilde{L}_1,\tilde{L}_1]}$ and then applies the algorithm of Section 3.3.
6.2.4: The case $\ell = 3$ follows by Proposition 5.3, while the case $\ell = 4$ follows by Propositions 6.3, 6.4 and 6.2.7. Note that in the latter one has to show that for $\tilde{s} = h_{\tilde{\alpha}_1}(-1)h_{\tilde{\alpha}_3}(-1)h_{\tilde{\alpha}_4}(d)$ with $d^2 = 1$, we have $\dim(\tilde{V}_{\tilde{s}}(-1)) = 56$. The case $\ell \geq 5$ follows recursively, by Propositions 6.3, 6.4 and the result for $\ell = 4$. Moreover, one shows that for ℓ even and $\tilde{s} = h_{\tilde{\alpha}_1}(-1)h_{\tilde{\alpha}_3}(-1)\cdots h_{\tilde{\alpha}_{\ell-1}}(-1)h_{\tilde{\alpha}_\ell}(c)$ with $c^2 = 1$, respectively ℓ odd and $\tilde{s} = h_{\tilde{\alpha}_1}(-1)h_{\tilde{\alpha}_3}(-1)\cdots h_{\tilde{\alpha}_{\ell-2}}(-1)h_{\tilde{\alpha}_\ell}(c)$ with $c^2 = -1$, we have $\dim(\tilde{V}_{\tilde{s}}(-1)) = \frac{4\ell^3 - 6\ell^2 + 2\ell}{3}$.
6.2.5: For $\ell = 3$, it follows by Proposition 5.4 and Theorem 5.2[5.2.7, 5.2.9]. Also, one shows that equality holds for $\tilde{s} = h_{\tilde{\alpha}_1}(-1)h_{\tilde{\alpha}_3}(c)$ with $c^2 = -1$ and $\tilde{\mu} = -1$. For $\ell \geq 4$, the result is obtained recursively, using Propositions 6.3, 6.5 and the result for $\ell = 3$. Moreover, one shows that equality holds

for $\tilde{\mu} = -1$, ℓ even and $\tilde{s} = h_{\tilde{\alpha}_1}(-1)h_{\tilde{\alpha}_3}(-1)\cdots h_{\tilde{\alpha}_{\ell-1}}(-1)h_{\tilde{\alpha}_\ell}(c)$ with $c^2 = 1$, respectively ℓ odd and $\tilde{s} = h_{\tilde{\alpha}_1}(-1)h_{\tilde{\alpha}_3}(-1)\cdots h_{\tilde{\alpha}_{\ell-2}}(-1)h_{\tilde{\alpha}_\ell}(c)$ with $c^2 = -1$.
6.2.6: For $\ell = 3$, it follows by Propositions 5.4, 5.5 and Theorem 5.2[5.2.7, 5.2.10]. For $\ell \geq 4$, it follows recursively from 6.3, 6.4, 6.5 and the result for $\ell = 3$.
6.2.7: Follows by Propositions 5.4 and 5.5. Note that when $\tilde{s} = \tilde{z}\cdot\tilde{h}$, where $\tilde{z} \in Z(\tilde{L}_1)^\circ$ and $\tilde{h} \in [\tilde{L}_1, \tilde{L}_1]$, one has to first show that $\dim(\tilde{V}^1_{\tilde{h}}(\tilde{\mu}^1_{\tilde{h}})) \leq 8$ for all $\tilde{\mu}^1_{\tilde{h}}$.
6.2.8: Follows by Proposition 5.3 and Theorem 5.2[5.2.10].
6.2.9 and 6.2.10: Follow by Theorem 5.2[5.2.5, 5.2.10].
6.2.11: Follows by Proposition 6.4 and 6.2.7. Moreover, one shows that there do not exist $(\tilde{s}, \tilde{\mu}) \in (\tilde{T} \setminus Z(\tilde{G})) \times k^*$ with $\dim(\tilde{V}_{\tilde{s}}(\tilde{\mu})) = 75$.
6.2.12 and 6.2.13: Follow by Proposition 6.4, 6.2.4, and 6.2.11.
6.2.14: Follows by Proposition 6.4, 6.2.4, and 6.2.12.
6.2.15: Follows by Proposition 6.4, 6.2.4, and 6.2.14.
6.2.16: Follows recursively, by 6.2.17, Proposition 5.3 and Theorem 5.2[5.2.6] for the base case.
6.2.17: Follows recursively, using 6.2.4 to prove the base case. □

7. Proof of Theorem 1.1 for groups of type D_ℓ

Let k be an algebraically closed field of characteristic $p \geq 0$, let W be a 2ℓ-dimensional k-vector space equipped with a nondegenerate quadratic form Q and let $G = SO(W, Q)$. Note that G is a simple algebraic group of type D_ℓ. We let $\tilde{G}$ be a simply connected cover of G, and we fix a central isogeny $\phi : \tilde{G} \to G$ with $\ker(\phi) \subseteq Z(\tilde{G})$ and $d\phi \neq 0$. As in Section 6, we will denote by $\tilde{X}$ the object in $\tilde{G}$ corresponding to the object X in G under ϕ. For example, $\tilde{T}$ is a maximal torus in $\tilde{G}$ with $\phi(\tilde{T}) = T$.

Let a be a nondegenerate alternating bilinear form on W and let $H = Sp(W, a)$. Note that H is a simple simply connected linear algebraic group of type C_ℓ. Let T_H be a maximal torus in H. Note that we can choose a symplectic basis and an orthogonal basis in W such that $T = T_H$. Lastly, we denote by $\omega^H_1, \ldots, \omega^H_\ell$ the fundamental dominant weights of H.

This section is dedicated to Theorems 7.1 and 7.2 which give the values of $\max_{\tilde{u}\in\tilde{G}_u\setminus\{1\}} \dim(\tilde{V}_{\tilde{u}}(1))$ and $\max_{\tilde{s}\in\tilde{T}\setminus Z(\tilde{G})} \dim(\tilde{V}_{\tilde{s}}(\tilde{\mu}))$ for all $k\tilde{G}$-modules $\tilde{V} = L_{\tilde{G}}(\tilde{\lambda})$, with $\tilde{\lambda} \in X(\tilde{T})$ a nonzero p-restricted dominant weight, and $\dim(L_{\tilde{G}}(\tilde{\lambda})) \leq 4\ell^3$. As a corollary, the part of Theorem 1.1 concerning simple simply connected linear algebraic groups of type D_ℓ will follow.

Theorem 7.1. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let $\tilde{G}$ be a simple simply connected linear algebraic group of type D_ℓ with $\ell \geq 4$. Let $\tilde{V} = L_{\tilde{G}}(\tilde{\lambda})$, where $\tilde{\lambda} \in X(\tilde{T})$ is a nonzero p-restricted dominant weight, be such that $\dim(\tilde{V}) \leq 4\ell^3$. The value of $\max_{\tilde{u}\in\tilde{G}_u\setminus\{1\}} \dim(\tilde{V}_{\tilde{u}}(1))$ is as given in Table 14*

Table 14. The value of $\max_{\tilde{u}\in\tilde{G}_u\setminus\{1\}} \dim(\tilde{V}_{\tilde{u}}(1))$ for groups of type D_ℓ.

Ref.	$\tilde{V}$	Char.	Rank	$\max_{\tilde{u}\in\tilde{G}_u\setminus\{1\}} \dim(\tilde{V}_{\tilde{u}}(1))$
7.1.1	$L_{\tilde{G}}(\tilde{\omega}_1)$	$p \geq 0$	$\ell \geq 4$	$2\ell - 2$
7.1.2	$L_{\tilde{G}}(\tilde{\omega}_2)$	$p \geq 0$	$\ell \geq 4$	$2\ell^2 - 5\ell + 6 - (1 + \varepsilon_2(\ell))\varepsilon_p(2)$
7.1.3	$L_{\tilde{G}}(2\tilde{\omega}_1)$	$p \neq 2$	$\ell \geq 4$	$2\ell^2 - 3\ell + 1 - \varepsilon_p(\ell)$
7.1.4	$L_{\tilde{G}}(\tilde{\omega}_3)$	$p \geq 0$	$\ell \geq 5$	$\binom{2\ell-2}{3} + 6\ell - 10 - [2\ell - 2 + (2\ell - 2)\varepsilon_2(\ell - 1)]\varepsilon_p(2)$
7.1.5	$L_{\tilde{G}}(3\tilde{\omega}_1)$	$p \neq 2, 3$	$\ell \geq 4$	$\binom{2\ell}{3} - (2\ell - 2)\varepsilon_p(\ell + 1)$

(*Continued*)

Table 14. (*Continued*)

Ref.	$\tilde{V}$	Char.	Rank	$\max_{\tilde{u}\in\tilde{G}_u\setminus\{1\}} \dim(\tilde{V}_{\tilde{u}}(1))$
7.1.6	$L_{\tilde{G}}(\tilde{\omega}_1+\tilde{\omega}_2)$	$p=3$	$\ell\geq 4$	$\leq \binom{2\ell}{3}-2\ell+6-(2\ell-2)\varepsilon_3(2\ell-1)$
		$p\neq 3$	$\ell\geq 4$	$\leq 2^4\binom{\ell}{3}+8\ell-10-(2\ell-2)(\varepsilon_p(2\ell-1)-2\varepsilon_p(2))$
7.1.7	$L_{\tilde{G}}(\tilde{\omega}_1+\tilde{\omega}_4)$	$p\geq 0$	$\ell=4$	$34-6\varepsilon_p(2)$
7.1.8	$L_{\tilde{G}}(2\tilde{\omega}_2)$	$p=3$	$\ell=4$	≤ 93
7.1.9	$L_{\tilde{G}}(2\tilde{\omega}_1+\tilde{\omega}_3)$	$p\neq 2$	$\ell=4$	$\leq 114-34\varepsilon_p(5)+6\varepsilon_p(3)$
7.1.10	$L_{\tilde{G}}(\tilde{\omega}_1+\tilde{\omega}_3+\tilde{\omega}_4)$	$p=2$	$\ell=4$	≤ 144
7.1.11	$L_{\tilde{G}}(2\tilde{\omega}_5)$	$p\neq 2$	$\ell=5$	76
7.1.12	$L_{\tilde{G}}(\tilde{\omega}_4+\tilde{\omega}_5)$	$p\geq 0$	$\ell=5$	$\leq 128-28\varepsilon_p(2)$
7.1.13	$L_{\tilde{G}}(\tilde{\omega}_1+\tilde{\omega}_5)$	$p\geq 0$	$\ell=5$	$\leq 92-12\varepsilon_p(5)^{\dagger}$
7.1.14	$L_{\tilde{G}}(\tilde{\omega}_2+\tilde{\omega}_5)$	$p=2$	$\ell=5$	≤ 252
7.1.15	$L_{\tilde{G}}(2\tilde{\omega}_6)$	$p\neq 2$	$\ell=6$	280
7.1.16	$L_{\tilde{G}}(\tilde{\omega}_5+\tilde{\omega}_6)$	$p\geq 0$	$\ell=6$	$\leq 484-140\varepsilon_p(2)$
7.1.17	$L_{\tilde{G}}(\tilde{\omega}_1+\tilde{\omega}_6)$	$p\geq 0$	$\ell=6$	$\leq 232-24\varepsilon_p(6)^{\propto}$
7.1.18	$L_{\tilde{G}}(\tilde{\omega}_4)$	$p\geq 0$	$\ell=6$	$\leq 311-91\varepsilon_p(2)^{\dagger}$
7.1.19	$L_{\tilde{G}}(\tilde{\omega}_5)$	$p=2$	$\ell=7$	≤ 948
7.1.20	$L_{\tilde{G}}(\tilde{\omega}_1+\tilde{\omega}_7)$	$p\geq 0$	$\ell=7$	$\leq 560-48\varepsilon_p(7)^{\circ}$
7.1.21	$L_{\tilde{G}}(\tilde{\omega}_4)$	$p\geq 0$	$\ell=7$	$\leq 651-65\varepsilon_p(2)^{\dagger}$
7.1.22	$L_{\tilde{G}}(\tilde{\omega}_4)$	$p\geq 0$	$\ell=8$	$\leq 1224-182\varepsilon_p(2)^{\dagger}$
7.1.23	$L_{\tilde{G}}(\tilde{\omega}_1+\tilde{\omega}_8)$	$p\geq 0$	$\ell=8$	$\leq 1312-96\varepsilon_p(2)^{\wr}$
7.1.24	$L_{\tilde{G}}(\tilde{\omega}_4)$	$p=2$	$\ell=9$	≤ 2364
7.1.25	$L_{\tilde{G}}(\tilde{\omega}_\ell)$	$p\geq 0$	$5\leq\ell\leq 15$	$3\cdot 2^{\ell-3}$

† equality holds when $p\neq 2$.
$^{\propto}$ equality holds when $p\neq 2,5$.
$^{\circ}$ equality holds when $p\neq 2,3,5$.
$^{\wr}$ equality holds when $p\neq 2,3,5,7$.

Proof. For 7.1.1–7.1.5, 7.1.7, 7.1.10, 7.1.19, and 7.1.24 we will deduce the result for $\tilde{V}=L_{\tilde{G}}(\tilde{\lambda})$ by calculating $\max_{u\in G_u\setminus\{1\}}\dim((L_G(\lambda))_u(1))=\dim((L_G(\lambda))_{x_{\alpha_\ell}(1)}(1))$, see Section 3.1 and Lemmas 3.2 and 3.13. The proofs for the remaining results are more straightforward: first one determines the structure of $\tilde{V}\mid_{[\tilde{L}_1,\tilde{L}_1]}$, and then applies the algorithm of Section 3.3. Once more, we will use Lemma 3.13, by which we have $\max_{\tilde{u}\in\tilde{G}_u\setminus\{1\}}\dim(\tilde{V}_{\tilde{u}}(1))=\dim(\tilde{V}_{x_{\tilde{\alpha}_\ell}(1)}(1))$.

7.1.1: Note that $\tilde{V}\cong L_G(\omega_1)$ as $k\tilde{G}$-modules and $L_G(\omega_1)\cong W$ as kG-modules. Moreover, $x_{\alpha_\ell}(1)$ acts on W as $J_2^2\oplus J_1^{2\ell-4}$.

7.1.2: Note that $\tilde{V}\cong L_G(\omega_2)$ as $k\tilde{G}$-modules. The result follows by [18, Proposition 4.2.2], respectively by Lemma 7.3 if $p=2$, the proof of Theorem 5.1[5.1.2] and, when $\varepsilon_2(\ell)=1$, by [12, Theorem B].

7.1.3: Note that $\tilde{V}\cong L_G(2\omega_1)$ as $k\tilde{G}$-modules. The result follows by [18, Propositions 4.7.3], the proof of Theorem 5.1[5.1.3] and [11, Corollary 6.2].

7.1.4: Note that $\tilde{V}\cong L_G(\omega_3)$ as $k\tilde{G}$-modules. If $p\neq 2$, the result follows by [18, Proposition 4.2.2] and Theorem 5.1[5.1.4]. Assume $p=2$. Then $L_H(\omega_3^H)\cong L_G(\omega_3)$ as kG-modules, see [19, Table 1]. Moreover, $\wedge^3(W)\cong L_H(\omega_3^H)\oplus L_H(\omega_1^H)$ if $\varepsilon_2(\ell-1)=0$ and $\wedge^3(W)\cong L_H(\omega_1^H)\mid L_H(\omega_3^H)\mid L_H(\omega_1^H)$ if $\varepsilon_2(\ell-1)=1$, see [18, Lemma 4.8.2]. Then $\max_{\tilde{u}\in\tilde{G}_u\setminus\{1\}}\dim(\tilde{V}_{\tilde{u}}(1))\geq\frac{4\ell^3-18\ell^2+44\ell-42}{3}-(1+\varepsilon_2(\ell-1))(2\ell-2)$. Equality is shown recursively, using the structure of $L_G(\omega_3)\mid_{[L_1,L_1]}$, 7.1.1 and 7.1.2.

7.1.5: Note that $\tilde{V}\cong L_G(3\omega_1)$ as $k\tilde{G}$-modules. If $\varepsilon_p(\ell+1)=0$, then $S^3(W)\cong L_G(3\omega_1)\oplus L_G(\omega_1)$, while, if $\varepsilon_p(\ell+1)=1$, then $S^3(W)\cong L_G(\omega_1)\mid L_G(3\omega_1)\mid L_G(\omega_1)$ as kG-modules, see [18, Proposition 4.7.4]. Then $\max_{\tilde{u}\in\tilde{G}_u\setminus\{1\}}\dim(\tilde{V}_{\tilde{u}}(1))\geq\binom{2\ell}{3}-(2\ell-2)\varepsilon_p(\ell+1)$. Equality is shown recursively, using the structure of $L_G(3\omega_1)\mid_{[L_1,L_1]}$, 7.1.1, 7.1.3 and Theorem 5.1[5.1.2, 5.1.9, 5.1.14].

7.1.6: The result follows recursively, using 7.1.1 and 7.1.3 (and 7.1.2 when $p \neq 3$). To prove the base case of $\ell = 4$, we need Theorems 4.1[4.1.2] (and [4.1.4] when $p \neq 3$) and 4.10[4.3.4, 4.3.5].
7.1.7: Note that $\tilde{V} \cong L_G(\omega_3 + \omega_4)$ as $k\tilde{G}$-modules. If $p = 2$, then $\Lambda^3(W) \cong L_G(\omega_3 + \omega_4) \oplus W$ as kG-modules, see [19, Table 1] and [18, Lemma 4.8.2]. The result now follows by Theorem 5.1[5.1.4]. If $p \neq 2$, it follows by the structure of $\tilde{V}\mid_{[\tilde{L}_1,\tilde{L}_1]}$ and Theorem 4.1[4.1.2, 4.1.3, 4.1.4].
7.1.8: Follows by Theorem 4.10[4.4.4, 4.4.5, 4.4.8].
7.1.9: Follows by Theorems 4.1[4.1.1, 4.1.7] and 4.10[4.4.7].
7.1.10: Follows by [19, Table 1] and Theorem 5.1[5.1.18].
7.1.11: Follows by 7.1.3 and 7.1.7.
7.1.12: Follows by 7.1.2, 7.1.3, and 7.1.7.
7.1.13: Follows by 7.1.1 and 7.1.7.
7.1.14: Follows by 7.1.7.
7.1.15: Follows by 7.1.11 and 7.1.12.
7.1.16: Follows by 7.1.4, 7.1.11, 7.1.12, and 7.1.25.
7.1.17: Follows by 7.1.13 and 7.1.25.
7.1.18: Follows by 7.1.2, 7.1.4, and 7.1.12.
7.1.19: Follows by [19, Table 1] and Theorem 5.1[5.1.19].
7.1.20: Follows by 7.1.17 and 7.1.25.
7.1.21: Follows by 7.1.2, 7.1.4, and 7.1.18.
7.1.22: Follows by 7.1.2, 7.1.4, and 7.1.21.
7.1.23: Follows by 7.1.20 and 7.1.25.
7.1.24: Follows by [19, Table 1] and Theorem 5.1[5.1.29].
7.1.25: Follows recursively, using 7.1.1. □

Theorem 7.2. *Let k be an algebraically closed field of characteristic $p \geq 0$ and let $\tilde{G}$ be a simple simply connected linear algebraic group of type D_ℓ with $\ell \geq 4$. Let $\tilde{V} = L_{\tilde{G}}(\tilde{\lambda})$, where $\tilde{\lambda} \in X(\tilde{T})$ is a nonzero p-restricted dominant weight, be such that $\dim(\tilde{V}) \leq 4\ell^3$. The value of $\max_{\tilde{s}\in\tilde{T}\setminus Z(\tilde{G})} \dim(\tilde{V}_{\tilde{s}}(\tilde{\mu}))$ is as given in Table 15.*

Table 15. The value of $\max_{\tilde{s}\in\tilde{T}\setminus Z(\tilde{G})} \dim(\tilde{V}_{\tilde{s}}(\tilde{\mu}))$ for groups of type D_ℓ.

Ref.	$\tilde{V}$	Char.	Rank	$\max_{\tilde{s}\in\tilde{T}\setminus Z(\tilde{G})} \dim(\tilde{V}_{\tilde{s}}(\tilde{\mu}))$
7.2.1	$L_{\tilde{G}}(\tilde{\omega}_1)$	$p \geq 0$	$\ell \geq 4$	$2\ell - 2$
7.2.2	$L_{\tilde{G}}(\tilde{\omega}_2)$	$p \geq 0$	$\ell \geq 4$	$2\ell^2 - 5\ell + 4 - (1 + \varepsilon_2(\ell))\varepsilon_p(2)$
7.2.3	$L_{\tilde{G}}(2\tilde{\omega}_1)$	$p \neq 2$	$\ell \geq 4$	$2\ell^2 - 3\ell + 3 - \varepsilon_p(\ell)$
7.2.4	$L_{\tilde{G}}(\tilde{\omega}_3)$	$p \geq 0$	$\ell \geq 5$	$\leq \binom{2\ell-2}{3} + 2\ell + 6 - (2\ell + 6 + (2\ell - 2)\varepsilon_2(\ell - 1))\varepsilon_p(2)$
7.2.5	$L_{\tilde{G}}(3\tilde{\omega}_1)$	$p \neq 2, 3$	$\ell \geq 4$	$\binom{2\ell}{3} + 4\ell - 4 - (2\ell - 2)\varepsilon_p(\ell + 1)$
7.2.6	$L_{\tilde{G}}(\tilde{\omega}_1 + \tilde{\omega}_2)$	$p = 3$	$\ell \geq 4$	$\binom{2\ell}{3} + (2\ell - 2)(1 - \varepsilon_3(2\ell - 1))$
		$p \neq 3$	$\ell \geq 4$	$2^4\binom{\ell}{3} + (2\ell - 2)(4 - \varepsilon_p(2\ell - 1) - 2\varepsilon_p(2))$
7.2.7	$L_{\tilde{G}}(\tilde{\omega}_1 + \tilde{\omega}_4)$	$p \geq 0$	$\ell = 4$	$\leq 34 - 14\varepsilon_p(2)$
7.2.8	$L_{\tilde{G}}(2\tilde{\omega}_2)$	$p = 3$	$\ell = 4$	≤ 111
7.2.9	$L_{\tilde{G}}(2\tilde{\omega}_1 + \tilde{\omega}_3)$	$p \neq 2$	$\ell = 4$	$\leq 128 - 34\varepsilon_p(5) + 8\varepsilon_p(3)$
7.2.10	$L_{\tilde{G}}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)$	$p = 2$	$\ell = 4$	≤ 130
7.2.11	$L_{\tilde{G}}(2\tilde{\omega}_5)$	$p \neq 2$	$\ell = 5$	≤ 80
7.2.12	$L_{\tilde{G}}(\tilde{\omega}_4 + \tilde{\omega}_5)$	$p \geq 0$	$\ell = 5$	$\leq 130 - 50\varepsilon_p(2)$
7.2.13	$L_{\tilde{G}}(\tilde{\omega}_1 + \tilde{\omega}_5)$	$p \geq 0$	$\ell = 5$	$\leq 92 - 12\varepsilon_p(5) - 16\varepsilon_p(2)$
7.2.14	$L_{\tilde{G}}(\tilde{\omega}_2 + \tilde{\omega}_5)$	$p = 2$	$\ell = 5$	≤ 192
7.2.15	$L_{\tilde{G}}(2\tilde{\omega}_6)$	$p \neq 2$	$\ell = 6$	≤ 290

(*Continued*)

Table 15. (*Continued*)

Ref.	$\tilde{V}$	Char.	Rank	$\max_{\tilde{s}\in\tilde{T}\setminus Z(\tilde{G})} \dim(\tilde{V}_{\tilde{s}}(\tilde{\mu}))$
7.2.16	$L_{\tilde{G}}(\tilde{\omega}_5+\tilde{\omega}_6)$	$p\geq 0$	$\ell=6$	$\leq 492-216\varepsilon_p(2)$
7.2.17	$L_{\tilde{G}}(\tilde{\omega}_1+\tilde{\omega}_6)$	$p\geq 0$	$\ell=6$	$\leq 224-4\varepsilon_p(5)-20\varepsilon_p(3)-52\varepsilon_p(2)$
7.2.18	$L_{\tilde{G}}(\tilde{\omega}_4)$	$p\geq 0$	$\ell=6$	$\leq 303-127\varepsilon_p(2)$
7.2.19	$L_{\tilde{G}}(\tilde{\omega}_5)$	$p=2$	$\ell=7$	≤ 608
7.2.20	$L_{\tilde{G}}(\tilde{\omega}_1+\tilde{\omega}_7)$	$p\geq 0$	$\ell=7$	$\leq 528-40\varepsilon_p(7)-8\varepsilon_p(5)-64\varepsilon_p(2)$
7.2.21	$L_{\tilde{G}}(\tilde{\omega}_4)$	$p\geq 0$	$\ell=7$	$\leq 625-119\varepsilon_p(2)$
7.2.22	$L_{\tilde{G}}(\tilde{\omega}_4)$	$p\geq 0$	$\ell=8$	$\leq 1172-250\varepsilon_p(2)$
7.2.23	$L_{\tilde{G}}(\tilde{\omega}_1+\tilde{\omega}_8)$	$p\geq 0$	$\ell=8$	$\leq 1216-16\varepsilon_p(5)-208\varepsilon_p(2)$
7.2.24	$L_{\tilde{G}}(\tilde{\omega}_4)$	$p=2$	$\ell=9$	≤ 1824
7.2.25	$L_{\tilde{G}}(\tilde{\omega}_\ell)$	$p\geq 0$	$5\leq\ell\leq 15$	$5\cdot 2^{\ell-4}$

Before we begin, recall that $H=\mathrm{Sp}(W,a)$ is a simple simply connected linear algebraic group of type C_ℓ with maximal torus T_H with the property that $T=T_H$. Thus, any $s\in T$ is conjugate in H to an element $s_H\in T_H$ of the form $s_H=\mathrm{diag}(\mu_1\cdot \mathrm{I}_{n_1},\mu_2\cdot\mathrm{I}_{n_2},\dots,\mu_m\cdot\mathrm{I}_{n_m},\mu_m^{-1}\cdot\mathrm{I}_{n_m},\dots,\mu_2^{-1}\cdot\mathrm{I}_{n_2},\mu_1^{-1}\cdot\mathrm{I}_{n_1})$ with $\mu_i\neq\mu_j^{\pm1}$ for all $i<j$, $\sum_{i=1}^m n_i=\ell$ and $\ell\geq n_1\geq\cdots\geq n_m\geq 1$.

Lemma 7.3. *Let $p=2$. If $\varepsilon_2(\ell)=0$, then $\wedge^2(W)\cong L_G(\omega_2)\oplus L_G(0)$, as kG-modules. If $\varepsilon_2(\ell)=1$, then the kG-module $\wedge^2(W)$ has three composition factors one isomorphic to $L_G(\omega_2)$ and two to $L_G(0)$.*

Proof. By [19, 1.15], the kG-module $\wedge^2(W)$ admits a unique nontrivial composition factor of highest weight ω_2. Since $\dim(L_G(\omega_2))=2\ell^2-\ell-\gcd(2,\ell)$, see [13, Table 2], we determine that, if $\varepsilon_2(\ell)=0$, then $\wedge^2(W)$ has two composition factors: one isomorphic to $L_G(\omega_2)$ and one to $L_G(0)$, hence $\wedge^2(W)\cong L_G(\omega_2)\oplus L_G(0)$, by [8, II.2.14]. If $\varepsilon_2(\ell)=1$, then $\wedge^2(W)$ has three composition factors: one isomorphic to $L_G(\omega_2)$ and two to $L_G(0)$. □

Proposition 7.4. *Let $\tilde{V}=L_{\tilde{G}}(\tilde{\omega}_2)$. Then $\max_{\tilde{s}\in\tilde{T}\setminus Z(\tilde{G})}\dim(\tilde{V}_{\tilde{s}}(\tilde{\mu}))=2\ell^2-5\ell+4-(1+\varepsilon_2(\ell))\varepsilon_p(2)$.*

Proof. Let $s\in T\setminus Z(G)$ and note that, in particular, we have $s\in T_H\setminus Z(H)$. Let μ be an eigenvalue of s on V. Now, when $p\neq 2$, as $V\cong\wedge^2(W)$, we have $\dim(V_s(\mu))=\dim((\wedge^2(W))_s(\mu))$. Further, by [18, Lemma 4.8.2] which gives the structure of $\wedge^2(W)$ as a kH-module, we have $\dim(V_s(\mu))=\dim((L_H(\omega_2^H))_s(\mu))$ for $\mu\neq 1$, and $\dim(V_s(1))=\dim((L_H(\omega_2^H))_s(1))+1+\varepsilon_p(\ell)$. We now use Proposition 5.4 to get the result. Similarly, when $p=2$, by the structure of $\wedge^2(W)$ as a kG-module, we determine that $\dim(V_s(\mu))=\dim((\wedge^2(W))_s(\mu))$ for $\mu\neq 1$, and $\dim(V_s(1))=\dim((\wedge^2(W))_s(1))-1-\varepsilon_2(\ell)$. Arguing as in the previous case, we determine that $\dim(V_s(\mu))=\dim((L_H(\omega_2^H))_s(\mu))$ for all μ, and the result follows by Proposition 5.4. □

Proposition 7.5. *Let $p\neq 2$ and $\tilde{V}=L_{\tilde{G}}(2\tilde{\omega}_1)$. Then $\max_{\tilde{s}\in\tilde{T}\setminus Z(\tilde{G})}\dim(\tilde{V}_{\tilde{s}}(\tilde{\mu}))=2\ell^2-3\ell+3-\varepsilon_p(\ell)$.*

Proof. Set $V=L_G(2\omega_1)$ and note that $\tilde{V}\cong V$ as $k\tilde{G}$-modules. Further, if $\varepsilon_p(\ell)=0$, then $\mathrm{S}^2(W)=V\oplus L_G(0)$, while, if $\varepsilon_p(\ell)=1$, then $\mathrm{S}^2(W)=L_G(0)\mid V\mid L_G(0)$, see [18, Propositions 4.7.3].

Let $s\in T\setminus Z(G)$ and note that, in particular, $s\in T_H\setminus Z(H)$. Let μ be an eigenvalue of s on V. Arguing as in the proof of Proposition 7.4, we show that $\dim(V_s(\mu))=\dim((L_H(2\omega_1^H))_s(\mu))$ for $\mu\neq 1$; and $\dim(V_s(1))=\dim((L_H(2\omega_1^H))_s(1))-1-\varepsilon_p(\ell)$. The result for $\mu=1$ is given by Proposition 5.5. For μ such that $\mu\neq\mu^{-1}$ we have $\dim(V_s(\mu))\leq\frac{\dim(V)}{2}<2\ell^2-3\ell+3-\varepsilon_p(\ell)$, as V is self-dual. Thus, for the remainder of the proof, we let $\mu=-1$.

As $s \in T_H \setminus Z(H)$, we write $s = \mathrm{diag}(\mu_1 \cdot I_{n_1}, \dots, \mu_m \cdot I_{n_m}, \mu_m^{-1} \cdot I_{n_m}, \dots, \mu_1^{-1} \cdot I_{n_1})$, where $\mu_i \neq \mu_j^{\pm 1}$ for all $i < j$, $\sum_{i=1}^{m} n_i = \ell$ and $n_1 \geq \cdots \geq n_m \geq 1$. If $\mu_i\mu_j \neq -1$ for all $i < j$, then $\dim(V_s(-1)) \leq \ell^2 + \ell < 2\ell^2 - 3\ell + 3 - \varepsilon_p(\ell)$, see inequality (5.8). If there exists $i < j$ such that $\mu_i\mu_j = -1$, then $\dim(V_s(-1)) \leq 2\ell^2 + \ell - \sum_{r=1}^{m} n_r^2 - n_i(n_i+1) - n_j(n_j+1) - 2(n_i+n_j)(\ell - n_i - n_j)$, see inequality (5.9). Assume that $\dim(V_s(-1)) \geq 2\ell^2 - 3\ell + 3 - \varepsilon_p(\ell)$. Then:

$$\ell(4 - n_i - n_j) - 3 + \varepsilon_p(\ell) - \sum_{r \neq i,j} n_r^2 - (n_i - n_j)^2 - (n_i + n_j)(\ell + 1 - n_i - n_j) \geq 0.$$

As $\ell + 1 > n_i + n_j$, we must have $(n_i, n_j) \in \{(1,1), (2,1)\}$. For $(n_i, n_j) = (1,1)$, we get $-1 + \varepsilon_p(\ell) - \sum_{r \neq i,j} n_r^2 \geq 0$, which does not hold. Similarly, if $(n_i, n_j) = (2,1)$, then $-2\ell + 2 + \varepsilon_p(\ell) - \sum_{r \neq i,j} n_r^2 \geq 0$, which does not hold. Thus, $\dim(V_s(-1)) < 2\ell^2 - 3\ell + 3 - \varepsilon_p(\ell)$ for all $s \in T \setminus Z(G)$. □

Proposition 7.6. *Let $\ell \geq 4$ and let $\tilde{V} = L_{\tilde{G}}(\tilde{\omega}_1 + \tilde{\omega}_2)$. If $p = 3$, then $\max_{\tilde{s} \in \tilde{T} \setminus Z(\tilde{G})} \dim(\tilde{V}_{\tilde{s}}(\tilde{\mu})) = \binom{2\ell}{3} + (2\ell - 2)(1 - \varepsilon_3(2\ell - 1))$; while if $p \neq 3$, then $\max_{\tilde{s} \in \tilde{T} \setminus Z(\tilde{G})} \dim(\tilde{V}_{\tilde{s}}(\tilde{\mu})) = 2^4\binom{\ell}{3} + (2\ell - 2)(4 - \varepsilon_p(2\ell - 1) - 2\varepsilon_p(2))$.*

Proof. The result follows recursively, using the algorithm of Section 3.3, the structure of $\tilde{V} \mid_{[\tilde{L}_1, \tilde{L}_1]}$, Proposition 7.5, Theorem 7.2[7.2.1] and, additionally, Proposition 7.4 when $p \neq 3$. In what follows we prove that the base case holds.

Let $\ell = 4$. If $p = 3$, we use Propositions 4.4 and Theorem 4.9[4.3.4, 4.3.5] to prove the result. Note that when $\tilde{s} = \tilde{z} \cdot \tilde{h}$ with $\tilde{z} \in Z(\tilde{L}_1)^\circ$ and $\tilde{h} \in [\tilde{L}_1, \tilde{L}_1]$, one has to treat the case when $\tilde{h}$ is conjugate to $h_{\tilde{\alpha}_2}(\pm 1) h_{\tilde{\alpha}_3}(d^{-1}) h_{\tilde{\alpha}_4}(d)$ with $d^2 \neq \pm 1$ separately. Thus, assume $p \neq 3$. The composition factors of $\tilde{V} \mid_{[\tilde{L}_1, \tilde{L}_1]}$ are as follows: one isomorphic to $L_{\tilde{L}_1}(\tilde{\omega}_2 + \tilde{\omega}_3 + \tilde{\omega}_4)$, $4 - \varepsilon_p(7) + \varepsilon_p(5)$ to $L_{\tilde{L}_1}(\tilde{\omega}_2)$, two to $L_{\tilde{L}_1}(2\tilde{\omega}_2)$, $2 + 2\varepsilon_p(2)$ to $L_{\tilde{L}_1}(\tilde{\omega}_3 + \tilde{\omega}_4)$ and $2 - 2\varepsilon_p(7) + 2\varepsilon_p(2)$ to $L_{\tilde{L}_1}(0)$.

When $s \notin Z(\tilde{L}_1)^\circ$, i.e. $\tilde{s} = \tilde{z} \cdot \tilde{h}$ with $\tilde{z} \in Z(\tilde{L}_1)^\circ$ and $\tilde{h} \in [\tilde{L}_1, \tilde{L}_1]$, one has to eliminate the cases when there exists $\tilde{\mu}_{\tilde{h}}$ with $\dim((L_{\tilde{L}_1}(\tilde{\omega}_2))_{\tilde{h}}(\tilde{\mu}_{\tilde{h}})) = 4$ and $\dim((L_{\tilde{L}_1}(\tilde{\omega}_2))_{\tilde{h}}(\tilde{\mu}_{\tilde{h}})) = 3$. If $p \neq 2$, the result then follows from Proposition 4.6 and Theorem 4.9[4.3.4, 4.3.5]. Thus, assume $p = 2$ and let $\tilde{L}_1^1$ be a Levi subgroup of the maximal parabolic subgroup $\tilde{P}_1^1$ of $\tilde{L}_1$ corresponding to $\tilde{\Delta}_1^1 = \tilde{\Delta}_1 \setminus \{\tilde{\alpha}_3\}$. We once again abuse notation and denote by $\tilde{\omega}_2$ and $\tilde{\omega}_4$ the fundamental dominant weights of $\tilde{L}_1^1$ corresponding to the simple roots $\tilde{\alpha}_2$ and $\tilde{\alpha}_4$. One shows that if $\tilde{h} \in Z(\tilde{L}_1^1)^\circ \setminus Z(\tilde{L}_1)$, then $\dim((L_{\tilde{L}_1}(\tilde{\omega}_2 + \tilde{\omega}_3 + \tilde{\omega}_4))_{\tilde{h}}(\tilde{\mu}_{\tilde{h}})) \leq 24$ for all $\tilde{\mu}_{\tilde{h}}$. On the other hand, if $\tilde{h} \notin Z(\tilde{L}_1^1)^\circ$, then $\dim((L_{\tilde{L}_1}(\tilde{\omega}_2 + \tilde{\omega}_3 + \tilde{\omega}_4))_{\tilde{h}}(\tilde{\mu}_{\tilde{h}})) \leq 2\dim((L_{\tilde{L}_1^1}(\tilde{\omega}_2 + \tilde{\omega}_4))_{\tilde{h}_1^1}(\tilde{\mu}_{\tilde{h}_1^1})) + 2\dim((L_{\tilde{L}_1^1}(2\tilde{\omega}_2 + \tilde{\omega}_4))_{\tilde{h}_1^1}(\tilde{\mu}_{\tilde{h}_1^1})) + 10\dim((L_{\tilde{L}_1^1}(\tilde{\omega}_2))_{\tilde{h}_1^1}(\tilde{\mu}_{\tilde{h}_1^1}))$, where $\tilde{h} = \tilde{z}_1^1 \cdot \tilde{h}_1^1$ with $\tilde{z}_1^1 \in Z(\tilde{L}_1^1)^\circ$ and $\tilde{h}_1^1 \in [\tilde{L}_1^1, \tilde{L}_1^1]$. One treats the case when there exists $\tilde{\mu}_{\tilde{h}_1^1}$ such that $\dim((L_{\tilde{L}_1^1}(\tilde{\omega}_2))_{\tilde{h}_1^1}(\tilde{\mu}_{\tilde{h}_1^1})) = 2$ separately, and afterwards concludes that $\dim(\tilde{V}_{\tilde{s}}(\tilde{\mu})) \leq 70$. □

Proof of Theorem 7.2. The proofs of 7.2.2, 7.2.3, and 7.2.6 are given in Propositions 7.4–7.6, respectively.
7.2.1: We argue as in the proofs of Theorem 7.1[7.1.1] and Proposition 4.3.
7.2.4: If $p = 2$, we argue as we did in the proof of Theorem 7.1[7.1.4] and use Theorem 5.2[5.2.4]. If $p \neq 2$, the result follows recursively, using [18, Proposition 4.2.2], Theorem 5.2[5.2.4], Proposition 7.4, 7.2.1, and 7.2.7 for the base case of $\ell = 5$.
7.2.5: It follows recursively, using Proposition 7.5, 7.2.1, and, for the base case of $\ell = 4$, Proposition 4.4 and Theorem 4.9[4.3.5, 4.3.6].

7.2.7: If $p = 2$, we argue as in the proof of Theorem 7.1[7.1.7] and use Theorem 5.2[5.2.4]. When $p \neq 2$, we use Propositions 4.4–4.6, to prove the result. Note that when $\tilde{s} = \tilde{z} \cdot \tilde{h}$ with $\tilde{z} \in Z(\tilde{L}_1)^\circ$ and $\tilde{h} \in [\tilde{L}_1, \tilde{L}_1]$, one has to treat the case when $\tilde{h}$ is conjugate to $h_{\tilde{\alpha}_2}(d)h_{\tilde{\alpha}_3}(d^2)h_{\tilde{\alpha}_4}(d^3)$ with $d^4 \neq 1$ separately.
7.2.8: Follows by Theorem 4.9[4.3.4, 4.3.5, 4.3.8].
7.2.9: Follows by Proposition 4.3 and Theorems 4.2[4.2.7] and 4.9[4.3.7].
7.2.10: Follows by [19, Table 1] and Theorem 5.2[5.2.18].
7.2.11: Follows by Proposition 7.5 and 7.2.7.
7.2.12: Follows by Propositions 7.4, 7.5, and 7.2.7.
7.2.13: Follows by 7.2.1 and 7.2.7.
7.2.14: Follows by 7.2.7.
7.2.15: Follows by 7.2.11 and 7.2.12.
7.2.16: Follows by 7.2.4, 7.2.11, 7.2.12, and 7.2.25.
7.2.17: Follows by 7.2.13 and 7.2.25.
7.2.18: Follows by Proposition 7.4, 7.2.4, and 7.2.12.
7.2.19: Follows by [19, Table 1] and Theorem 5.2[5.2.19].
7.2.20: Follows by 7.2.17 and 7.2.25.
7.2.21: Follows by Proposition 7.4, 7.2.4, and 7.2.18.
7.2.22: Follows by Proposition 7.4, 7.2.4, and 7.2.21.
7.2.23: Follows by 7.2.20 and 7.2.25.
7.2.24: Follows by [19, Table 1] and Theorem 5.2[5.2.29].
7.2.25: Follows recursively, using 7.2.1. Note that when $\ell = 5$ and $\tilde{s} \notin Z(\tilde{L}_1)^\circ$, one has to prove that: there do not exist $\tilde{\mu}$ such that $\dim(\tilde{V}_{\tilde{s}}(\tilde{\mu})) > 10$; and that for $\tilde{s} = h_{\tilde{\alpha}_4}(d)h_{\tilde{\alpha}_5}(d)$ with $d \neq 1$ we have $\dim(\tilde{V}_{\tilde{s}}(1)) = 10$. □

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