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ON SMALL TYPES IN UNIVALENT FOUNDATIONS*

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ABSTRACT. We investigate predicative aspects of constructive univalent foundations. By predicative and constructive, we respectively mean that we do not assume Voevodsky’s propositional resizing axioms or excluded middle. Our work complements existing work on predicative mathematics by exploring what cannot be done predicatively in univalent foundations. Our first main result is that nontrivial (directed or bounded) complete posets are necessarily large. That is, if such a nontrivial poset is small, then weak propositional resizing holds. It is possible to derive full propositional resizing if we strengthen nontriviality to positivity. The distinction between nontriviality and positivity is analogous to the distinction between nonemptiness and inhabitedness. Moreover, we prove that locally small, nontrivial (directed or bounded) complete posets necessarily lack decidable equality. We prove our results for a general class of posets, which includes *e.g.* directed complete posets, bounded complete posets, sup-lattices and frames. Secondly, the fact that these nontrivial posets are necessarily large has the important consequence that Tarski’s theorem (and similar results) cannot be applied in nontrivial instances. Furthermore, we explain that generalizations of Tarski’s theorem that allow for large structures are provably false by showing that the ordinal of ordinals in a univalent universe has small suprema in the presence of set quotients. The latter also leads us to investigate the inter-definability and interaction of type universes of propositional truncations and set quotients, as well as a set replacement principle. Thirdly, we clarify, in our predicative setting, the relation between the traditional definition of sup-lattice that requires suprema for all subsets and our definition that asks for suprema of all small families.

1. INTRODUCTION

We investigate predicative aspects of constructive univalent foundations. By predicative and constructive, we respectively mean that we do not assume Voevodsky’s propositional resizing axioms [Voe11, Voe15] or excluded middle and choice. Most of our work is situated in our larger programme of developing domain theory constructively and predicatively in univalent foundations. In previous work [dJE21a], we showed how to give a constructive and predicative account of many familiar constructions and notions in domain theory, such as Scott’s D_∞ model of untyped λ -calculus and the theory of continuous dcpos. The present work complements this and other existing work on predicative mathematics

Key words and phrases: univalent foundations, homotopy type theory, HoTT/UF, constructive mathematics, predicative mathematics, propositional resizing, type universes, order theory, complete posets, set quotients, propositional truncations, set replacement, ordinals.

* This is a revised and extended version of [dJE21b].

(*e.g.* [AR10, Sam87, CSSV03]) by exploring what *cannot* be done predicatively, as in [Cur10a, Cur10b, Cur15, Cur18, CR12]. We do so by showing that certain statements crucially rely on resizing axioms in the sense that they are equivalent to them. Such arguments are important in constructive mathematics. For example, the constructive failure of trichotomy on the real numbers is shown [BR87] by reducing it to a nonconstructive instance of excluded middle.

Our first main result is that nontrivial (directed or bounded) complete posets are necessarily large. In [dJE21a] we observed that all our examples of directed complete posets have large carriers. We show here that this is no coincidence, but rather a necessity, in the sense that if such a nontrivial poset is small, then weak propositional resizing holds. It is possible to derive full propositional resizing if we strengthen nontriviality to positivity in the sense of [Joh84]. The distinction between nontriviality and positivity is analogous to the distinction between nonemptiness and inhabitedness. We prove our results for a general class of posets, which includes directed complete posets, bounded complete posets and sup-lattices, using a technical notion of a δ_V -complete poset. We also show that nontrivial locally small δ_V -complete posets necessarily lack decidable equality. Specifically, we can derive weak excluded middle from assuming the existence of a nontrivial locally small δ_V -complete poset with decidable equality. Moreover, if we assume positivity instead of nontriviality, then we can derive full excluded middle.

Secondly, the fact that these nontrivial posets are necessarily large has the important consequence that Tarski's theorem (and similar results) cannot be applied in nontrivial instances. Furthermore, we explain that generalizations of Tarski's theorem that allow for large structures are provably false. Specifically, we show that the ordinal of ordinals in a univalent universe does not have a maximal element, but does have small suprema in the presence of small set quotients. The latter also leads us to investigate the inter-definability and interaction of type universes of propositional truncations and set quotients, as well as a set replacement principle. Following a construction due to Voevodsky, we construct set quotients from propositional truncations. However, while Voevodsky assumed propositional resizing rules in his construction, we show that, when propositional truncations are available, resizing is not needed to prove the universal property of the set quotient, even though the quotient will live in a higher type universe.

Finally, we clarify, in our predicative setting, the relation between the traditional definition of sup-lattice that requires suprema for all subsets and our definition that asks for suprema of all small families. This is important in practice in order to obtain workable definitions of dcpo, sup-lattice, etc. in the context of predicative univalent mathematics.

Our foundational setup is the same as in [dJE21a], meaning that our work takes place in intensional Martin-Löf Type Theory and adopts the univalent point of view [Uni13]. This means that we work with the stratification of types as singletons, propositions (or subsingletons or truth values), sets, 1-groupoids, etc., and that we work with univalence. At present, higher inductive types other than propositional truncation are not needed. Often the only consequences of univalence needed here are functional and propositional extensionality. Two exceptions are Sections 2.3 and 5.2. Full details of our univalent type theory are given at the start of Section 2.

1.1. Reasons for studying predicativity. We briefly describe some motivations for studying impredicativity in the form of propositional resizing in univalent type theory. The first reason for our interest is that, unlike the univalence axiom in cubical type theory [CCHM18], there is at present no known computational interpretation of propositional resizing axioms.

Another reason for being interested in predicativity is the fact that propositional resizing axioms fail in some models of univalent type theory. A notable example of such a model is Uemura’s cubical assembly model [Uem19]. What is particularly striking about Uemura’s model is that it does support an impredicative universe \mathcal{U} in the sense that if X is *any* type and $Y : X \rightarrow \mathcal{U}$, then $\prod_{x:X} Y(x)$ is in \mathcal{U} again even if X isn’t, but that propositional resizing fails for this universe. On the model-theoretic side, we also highlight Swan’s (unpublished) results [Swa19b, Swa19a] that show that propositional resizing axioms fail in certain presheaf (cubical) models of type theory. Interestingly, Swan’s argument works by showing that the models violate certain collection principles if we assume Brouwerian continuity principles in the metatheory.

By contrast, we should mention that propositional resizing is validated in many models when a classical metatheory is assumed. For example, this is true for any type-theoretic model topos [Shu19, Proposition 11.3]. In particular, Voevodsky’s simplicial sets model [KL21] validates excluded middle and hence propositional resizing. We note, however, that in other models it is possible for propositional resizing to hold and excluded middle to fail, as shown by [Shu15, Remark 11.24].

Another interesting aspect of impredicativity is that it is expected, by analogy to predicative and impredicative set theories, that adding resizing axioms significantly increases the proof-theoretic strength of univalent type theory [Shu19, Remark 1.2].

This paper concerns resizing *axioms*, meaning we ask a given type to be *equivalent* to one in some fixed universe \mathcal{U} of “small” types. Voevodsky [Voe11] originally introduced resizing *rules* which add judgements and hence modify the syntax of the type theory to make the given type inhabit \mathcal{U} , rather than only asking for an equivalent copy in \mathcal{U} . It is not known whether Voevodsky’s resizing rules are consistent with univalent type theory in the sense that no-one has constructed a model of univalent type theory extended with such resizing rules, or proved a contradiction in the system. It is also an open problem [CCHM18, Section 10] whether we have normalization for cubical type theory extended with resizing rules. In fact, as far as we know, this is an open problem for plain Martin-Löf Type Theory as well.

Lastly, one may have philosophical reservations regarding impredicativity. For example, some constructivists may accept predicative set theories like Aczel’s CZF and Myhill’s CST, but not Friedman’s impredicative set theory IZF. Or, paraphrasing Shulman’s narrative [Shu11], one can ask why propositions (or (-1) -types) should be treated differently, *i.e.* given that we have to take size seriously for n -types for $n > -1$, why not do the same for (-1) -types?

1.2. Related work. Curi investigated the limits of predicative mathematics in CZF [AR10] in a series of papers [Cur10a, Cur10b, Cur15, Cur18, CR12]. In particular, Curi shows (see [Cur10a, Theorem 4.4 and Corollary 4.11], [Cur10b, Lemma 1.1] and [Cur15, Theorem 2.5]) that CZF cannot prove that various nontrivial posets, including sup-lattices, dcpos and frames, are small. This result is obtained by exploiting that CZF is consistent with the anti-classical generalized uniformity principle (GUP) [vdB06, Theorem 4.3.5]. Our related Theorem 4.23 is of a different nature in two ways. Firstly, our theorem is in the spirit of reverse constructive mathematics [Ish06]: Instead of showing that GUP implies that there are no non-trivial small dcpos, we show that the existence of a non-trivial small dcpo is *equivalent* to weak propositional resizing, and that the existence of a positive small dcpo is *equivalent* to full propositional resizing. Thus, if we wish to work with small dcpos, we

are forced to assume resizing axioms. Secondly, we work in univalent foundations rather than CZF. This may seem a superficial difference, but a number of arguments in Curi’s papers [Cur15, Cur18] crucially rely on set-theoretical notions and principles such as transitive set, set-induction, and the weak regular extension axiom (wREA), which cannot even be formulated in the underlying type theory of univalent foundations. Moreover, although Curi claims that the arguments of [Cur10a, Cur10b] can be adapted to some version of Martin-Löf Type Theory, it is presently not clear whether there is any model of univalent foundations which validates GUP. However, one of the reviewers suggested that Uemura’s cubical assemblies model [Uem19] might validate it. In particular, the reviewer hinted that [Uem19, Proposition 21] may be seen as a uniformity principle.

Finally, the construction of set quotients using propositional truncations is due to Voevodsky and also appears in [Uni13, Section 6.10] and [RS15, Section 3.4]. While Voevodsky assumed resizing rules for his construction, we investigate the inter-definability of propositional truncations and set quotients in the absence of propositional resizing axioms.

1.3. Organization. *Section 2:* Foundations and size matters, including impredicativity, relation to excluded middle, univalence and closure under embedded retracts. *Section 3:* Inter-definability of set quotients and propositional truncations, and equivalence of small set quotients and set replacement. *Section 4:* Nontrivial and positive $\delta_{\mathcal{V}}$ -complete posets and reductions to impredicativity and excluded middle. *Section 5:* Predicative unavailability of Tarski’s fixed point theorem and Patarai’a’s lemma, and suprema of ordinals. *Section 6:* Comparison of completeness with respect to families and with respect to subsets. *Section 7:* Conclusion and future work.

2. FOUNDATIONS AND SMALL TYPES

We work with a subset of the type theory described in [Uni13] and we mostly adopt the terminological and notational conventions of [Uni13]. We include $+$ (binary sum), Π (dependent products), Σ (dependent sum), Id (identity type), and inductive types, including 0 (empty type), 1 (type with exactly one element $\star : 1$), \mathbb{N} (natural numbers). We assume a universe \mathcal{U}_0 and two operations: for every universe \mathcal{U} , a successor universe \mathcal{U}^+ with $\mathcal{U} : \mathcal{U}^+$, and for every two universes \mathcal{U} and \mathcal{V} another universe $\mathcal{U} \sqcup \mathcal{V}$ such that for any universe \mathcal{U} , we have $\mathcal{U}_0 \sqcup \mathcal{U} \equiv \mathcal{U}$ and $\mathcal{U} \sqcup \mathcal{U}^+ \equiv \mathcal{U}^+$. Moreover, $(-) \sqcup (-)$ is idempotent, commutative, associative, and $(-)^+$ distributes over $(-) \sqcup (-)$. We write $\mathcal{U}_1 \equiv \mathcal{U}_0^+$, $\mathcal{U}_2 \equiv \mathcal{U}_1^+$, \dots and so on. If $X : \mathcal{U}$ and $Y : \mathcal{V}$, then $X + Y : \mathcal{U} \sqcup \mathcal{V}$ and if $X : \mathcal{U}$ and $Y : X \rightarrow \mathcal{V}$, then the types $\Sigma_{x:X} Y(x)$ and $\Pi_{x:X} Y(x)$ live in the universe $\mathcal{U} \sqcup \mathcal{V}$; finally, if $X : \mathcal{U}$ and $x, y : X$, then $\text{Id}_X(x, y) : \mathcal{U}$. The type of natural numbers \mathbb{N} is assumed to be in \mathcal{U}_0 and we postulate that we have copies $0_{\mathcal{U}}$ and $1_{\mathcal{U}}$ in every universe \mathcal{U} . This has the useful consequence that while we do not assume cumulativity of universes, embeddings that lift types to higher universes are definable. For example, the map $(-) \times 1_{\mathcal{V}}$ takes a type in any universe \mathcal{U} to an equivalent type in the higher universe $\mathcal{U} \sqcup \mathcal{V}$. We assume function extensionality and propositional extensionality tacitly, and univalence explicitly when needed. Finally, we use a single higher inductive type: the propositional truncation of a type X is denoted by $\|X\|$ and we write $\exists_{x:X} Y(x)$ for $\|\Sigma_{x:X} Y(x)\|$. Apart from Section 3, we assume throughout that every universe is closed under propositional truncations, meaning that if $X : \mathcal{U}$ then $\|X\| : \mathcal{U}$ as well.

2.1. The Notion of a Small Type. We introduce the fundamental notion of a type being \mathcal{U} -small with respect to some type universe \mathcal{U} , and specify the impredicativity axioms under consideration (Section 2.2). We also note the relation to excluded middle (Section 2.2) and univalence (Section 2.3). Finally, in Section 2.4 we establish our main technical result on small types, namely that being small is closed under retracts.

Definition 2.1 (Smallness, [E⁺22, UF.Size]). A type X in any universe is said to be \mathcal{U} -small if it is equivalent to a type in the universe \mathcal{U} . That is, X is \mathcal{U} -small $\equiv \Sigma_{Y:\mathcal{U}}(Y \simeq X)$.

Definition 2.2 (Local smallness, [Rij17]). A type X is said to be *locally \mathcal{U} -small* if the type $(x = y)$ is \mathcal{U} -small for every $x, y : X$.

Examples 2.3.

- (i) Every \mathcal{U} -small type is locally \mathcal{U} -small.
- (ii) The type $\Omega_{\mathcal{U}}$ of propositions in a universe \mathcal{U} lives in \mathcal{U}^+ , but is locally \mathcal{U} -small by propositional extensionality.

2.2. Impredicativity and Excluded Middle. We consider various impredicativity axioms and their relation to (weak) excluded middle. The definitions and propositions below may be found in [Esc19, Section 3.36], so proofs are omitted here.

Definition 2.4 (Impredicativity axioms).

- (i) By *Propositional-Resizing $_{\mathcal{U},\mathcal{V}}$* we mean the assertion that every proposition P in a universe \mathcal{U} is \mathcal{V} -small.
- (ii) We write Ω -Resizing $_{\mathcal{U},\mathcal{V}}$ for the assertion that the type $\Omega_{\mathcal{U}}$ is \mathcal{V} -small.
- (iii) The type of all $\neg\neg$ -stable propositions in a universe \mathcal{U} is denoted by $\Omega_{\mathcal{U}}^{\neg\neg}$, where a proposition P is $\neg\neg$ -stable if $\neg\neg P$ implies P . By $\Omega_{\neg\neg}$ -Resizing $_{\mathcal{U},\mathcal{V}}$ we mean the assertion that the type $\Omega_{\mathcal{U}}^{\neg\neg}$ is \mathcal{V} -small.
- (iv) For the particular case of a single universe, we write Ω -Resizing $_{\mathcal{U}}$ and $\Omega_{\neg\neg}$ -Resizing $_{\mathcal{U}}$ for the respective assertions that $\Omega_{\mathcal{U}}$ is \mathcal{U} -small and $\Omega_{\mathcal{U}}^{\neg\neg}$ is \mathcal{U} -small.

Proposition 2.5.

- (i) *The principle Ω -Resizing $_{\mathcal{U},\mathcal{V}}$ implies Propositional-Resizing $_{\mathcal{U},\mathcal{V}}$ for every two universes \mathcal{U} and \mathcal{V} .*
- (ii) *The conjunction of Propositional-Resizing $_{\mathcal{U},\mathcal{V}}$ and Propositional-Resizing $_{\mathcal{V},\mathcal{U}}$ implies Ω -Resizing $_{\mathcal{U},\mathcal{V}^+}$ for every two universes \mathcal{U} and \mathcal{V} .*

It is possible to define a weaker variation of propositional resizing for the $\neg\neg$ -stable propositions only (and derive similar connections), but we don't need it in this paper.

Definition 2.6 ((Weak) excluded middle).

- (i) *Excluded middle* in a universe \mathcal{U} asserts that for every proposition P in \mathcal{U} either P or $\neg P$ holds.
- (ii) *Weak excluded middle* in a universe \mathcal{U} asserts that for every proposition P in \mathcal{U} either $\neg P$ or $\neg\neg P$ holds.

We note that weak excluded middle says precisely that $\neg\neg$ -stable propositions are decidable and is equivalent to de Morgan's Law.

Proposition 2.7. *Excluded middle implies impredicativity. Specifically,*

- (i) *Excluded middle in \mathcal{U} implies Ω -Resizing $_{\mathcal{U}, \mathcal{U}_0}$.*
- (ii) *Weak excluded middle in \mathcal{U} implies $\Omega_{\neg\neg}$ -Resizing $_{\mathcal{U}, \mathcal{U}_0}$.*

2.3. Smallness and Univalence. With univalence we can prove that the statements $\text{Propositional-Resizing}_{\mathcal{U}, \mathcal{V}}$ and $\Omega\text{-Resizing}_{\mathcal{U}, \mathcal{V}}$ are subsingletons. More generally, univalence allows us to prove that the statement that X is \mathcal{V} -small is a proposition, which is needed at the end of Section 4.4.

Proposition 2.8 [Esc19, has-size-is-subsingleton]. *If \mathcal{V} and $\mathcal{U} \sqcup \mathcal{V}$ are univalent universes, then X is \mathcal{V} -small is a proposition for every $X : \mathcal{U}$.*

The converse also holds in the following form.

Proposition 2.9. *The type X is \mathcal{U} -small is a proposition for every $X : \mathcal{U}$ if and only if the universe \mathcal{U} is univalent.*

Proof. Since X is \mathcal{U} -small $\equiv \Sigma_{Y:\mathcal{U}}(Y \simeq X)$, this follows from [Esc19, Section 3.14]. \square

2.4. Small Types and Retracts. We show our main technical result on small types here, namely that being small is closed under retracts.

Definition 2.10 (Sections and retractions). *A section is a map $s : X \rightarrow Y$ together with a left inverse $r : Y \rightarrow X$, i.e. the maps satisfy $r \circ s \sim \text{id}$. We call r the *retraction* and say that X is a *retract* of Y .*

We extend the notion of a small type to functions as follows.

Definition 2.11 (Smallness (for maps), [E⁺22, UF.Size]). *A map $f : X \rightarrow Y$ is said to be \mathcal{V} -small if every fibre is \mathcal{V} -small.*

Lemma 2.12 [E⁺22, UF.Size].

- (i) *A type X is \mathcal{V} -small if and only if the unique map $X \rightarrow \mathbf{1}_{\mathcal{U}_0}$ is \mathcal{V} -small.*
- (ii) *If Y is \mathcal{V} -small, then a map $f : X \rightarrow Y$ is \mathcal{V} -small if and only if X is.*

Proof. (i) Writing $!_X$ for the map $X \rightarrow \mathbf{1}_{\mathcal{U}_0}$ we have $\text{fib}_{!_X}(\star) \simeq X$. (ii) If X and Y are both \mathcal{V} -small, witnessed respectively by $\varphi : X' \simeq X$ and $\psi : Y' \simeq Y$, then $\text{fib}_f(y)$ is \mathcal{V} -small for every $y : Y$, because $\text{fib}_f(y) \equiv \Sigma_{x:X}(f(x) = y) \simeq \Sigma_{x':X'}(\psi^{-1}(f(\varphi(x')))) = \psi^{-1}(y)$. Conversely, if f and Y are \mathcal{V} -small, then so is X , because [Uni13, Lemma 4.8.2] tells us that $X \simeq \Sigma_{y:Y} \text{fib}_f(y)$. \square

Theorem 2.13. *Every section into a \mathcal{V} -small type is \mathcal{V} -small. In particular, its domain is \mathcal{V} -small. Hence, the \mathcal{V} -small types are closed under retracts.*

Proof. We show that the domain is \mathcal{V} -small from which it follows that the section is \mathcal{V} -small by Lemma 2.12(ii). So suppose we have a section $s : X \rightarrow Y$ with retraction $r : Y \rightarrow X$ and that Y is \mathcal{V} -small. By [Shu16, Lemma 3.6], the endomap $f \equiv r \circ s$ on Y is a quasi-idempotent [Shu16, Definition 3.5]. Hence, [Shu16, Theorem 5.3] tells us that f can be split as $Y \xrightarrow{r'} A \xrightarrow{s'} Y$ for some maps s' and r' . Now X and A are equivalent as witnessed by the maps $x \mapsto r'(s(x))$ and $a \mapsto r(s'(a))$. Finally, we recall from the proof of [Shu16, Theorem 5.3] that $A \equiv \Sigma_{\sigma:\mathbb{N} \rightarrow Y} \Pi_{n:\mathbb{N}}(f(\sigma_{n+1}) = \sigma_n)$ which is \mathcal{V} -small because Y is assumed to be. \square

Remark 2.14. In [dJE21b] we had a weaker version of Theorem 2.13 where we included the additional assumption that the section was an embedding. (Note that if every section is an embedding, then every type is a set [Shu16, Remark 3.11(2)], but that all sections into *sets* are embeddings [Esc19, 1c-maps-into-sets-are-embeddings].) We are grateful to the anonymous reviewer who proposed the above strengthening.

3. SET QUOTIENTS, PROPOSITIONAL TRUNCATIONS AND SET REPLACEMENT

We investigate the inter-definability and interaction of type universe levels of propositional truncations and set quotients in the absence of propositional resizing axioms. In particular, we will see that it is not so important if the set quotient or propositional truncation lives in a higher universe. What is paramount instead is whether the universal property applies to types in arbitrary universes. However, in some cases, like in Section 5.2, it is relevant whether set quotients are small and we show this to be equivalent to a set replacement principle in Section 3.4.

We start by recalling (the universal property of) the propositional truncation, which, borrowing terminology from category theory, we could also call the *subsingleton reflection* or *propositional reflection*.

Definition 3.1 (Propositional truncation, $\|- \|$). A *propositional truncation* of a type X , if it exists, is a proposition $\|X\|$ with a map $|-| : X \rightarrow \|X\|$ such that every function $f : X \rightarrow P$ to any proposition factors through $|-|$.

$$\begin{array}{ccc} X & \xrightarrow{|-|} & \|X\| \\ & \searrow f & \swarrow \bar{f} \\ & & P \end{array}$$

Some sources, *e.g.* [Uni13], also demand that the diagram above commutes *definitionally*: for every $x : X$, we have $f(x) \equiv \bar{f}(|x|)$. Having definitional equalities has some interesting consequences, such as being able to prove function extensionality [KECA17, Section 8]. We do not require definitional equalities, but notice that we do have $f(x) = \bar{f}(|x|)$ (up to an identification) for every $x : X$, as P is a subsingleton. In particular it follows using function extensionality that \bar{f} is the unique factorization.

Notice that if a propositional truncation exists, then it is unique up to unique equivalence.

Remark 3.2. Some remarks regarding universes are in order:

- (i) In Definition 3.1, the subsingleton P may live in an *arbitrary* universe, regardless of the universe in which X sits. The importance of this will be revisited throughout this section and in Example 3.4 in particular.
- (ii) In Definition 3.1, we haven't specified in what universe $\|X\|$ should be. When adding propositional truncations as higher inductive types, one typically assumes that $\|X\| : \mathcal{U}$ if $X : \mathcal{U}$, and indeed this is what we do in most of this paper. In this section, however, we will be more general and instead assume that $\|X\| : F(\mathcal{U})$ where F is a (meta)function on universes, so that the above case is obtained by taking F to be the identity. We will also consider $F(\mathcal{U}) = \mathcal{U}_1 \sqcup \mathcal{U}$ in the final subsection.

While in general propositional truncations may fail to exist in intensional Martin-Löf Type Theory, it is possible to construct a propositional truncation of some types in specific

cases [EX15, Section 3.1]. A particular example [KECA17, Corollary 4.4] is for a type X with a weakly constant (viz. any of its values are equal) endofunction f : the propositional truncation of X can be constructed as $\Sigma_{x:X}(x = f(x))$, the type of fixed points of f .

We review an approach by Voevodsky, who used resizing rules, to constructing propositional truncations in general in the next section.

3.1. Propositional Truncations and Propositional Resizing. Voevodsky [Voe11] introduced propositional resizing rules in order to construct propositional truncations [PVW15, Section 2.4]. Here we review Voevodsky’s construction, paying special attention to the universes involved.

NB. We do not assume the availability of propositional truncations in this section.

Definition 3.3 (Voevodsky propositional truncation, $\|X\|_v$). The *Voevodsky propositional truncation* $\|X\|_v$ of a type $X : \mathcal{U}$ is defined as

$$\|X\|_v \equiv \prod_{P:\mathcal{U}} (\text{is-subsingleton}(P) \rightarrow (X \rightarrow P) \rightarrow P).$$

Because of function extensionality, one can show that $\|X\|_v$ is indeed a proposition for every type X . Moreover, we have a map $|-|_v : X \rightarrow \|X\|_v$ given by $|x|_v \equiv (P, i, f) \mapsto f(x)$.

Observe that $\|X\|_v : \mathcal{U}^+$, so using the notation from Remark 3.2, we have $F(\mathcal{U}) = \mathcal{U}^+$. However, as we will argue for set quotients, it does not matter so much where the truncated proposition lives; it is much more important that we can eliminate into subsingletons in arbitrary universes, *i.e.* that $|-|_v$ satisfies the right universal property. Given $X : \mathcal{U}$ and a map $f : X \rightarrow P$ to a proposition $P : \mathcal{U}$ with $i : \text{is-subsingleton}(P)$, we have a map $\|X\|_v \rightarrow P$ given as $\Phi \mapsto \Phi(P, i, f)$. However, if the proposition P lives in some other universe \mathcal{V} , then we seem to be completely stuck. To clarify this, we consider the example of functoriality.

Example 3.4. If we have a map $f : X \rightarrow Y$ with $X : \mathcal{U}$ and $Y : \mathcal{U}$, then we get a lifting simply by precomposition, *i.e.* we define $|f|_v : \|X\|_v \rightarrow \|Y\|_v$ by $|f|_v(\Phi) \equiv (P, i, g) \mapsto \Phi(P, i, g \circ f)$. But obviously, we also want functoriality for maps $f : X \rightarrow Y$ with $X : \mathcal{U}$ and $Y : \mathcal{V}$, but this is impossible with the above definition of $|f|_v$, because for $\|X\|_v$ we are considering propositions in \mathcal{U} , while for $\|Y\|_v$ we are considering propositions in \mathcal{V} .

In particular, even if the types $X : \mathcal{U}$ and $Y : \mathcal{V}$ are equivalent, then it does not seem possible to construct an equivalence between $\|X\|_v$ and $\|Y\|_v$. This issue also comes up if one tries to prove that the map $|-|_v : X \rightarrow \|X\|_v$ is a surjection [Esc19, Section 3.34.1].

Proposition 3.5 [KECA17, Theorem 3.8]. *If our type theory has propositional truncations with $\|X\| : \mathcal{U}$ whenever $X : \mathcal{U}$, then $\|X\|_v$ is \mathcal{U} -small.*

Proof. We will show that $\|X\|$ and $\|X\|_v$ are logically equivalent (*i.e.* we have maps in both directions), which suffices, because both types are subsingletons. We obtain a map $\|X\| \rightarrow \|X\|_v$ by applying the universal property of $\|X\|$ to the map $|-|_v : X \rightarrow \|X\|_v$. Observe that it is essential that the universal property allows for elimination into subsingletons in universes other than \mathcal{U} , as $\|X\|_v : \mathcal{U}^+$. For the function in the other direction, simply note that $\|X\| : \mathcal{U}$, so that we can construct $\|X\|_v \rightarrow \|X\|$ as $\Phi \mapsto \Phi(\|X\|, i, |-|)$ where i witnesses that $\|X\|$ is a subsingleton. \square

Thus, as is folklore in the univalent foundations community, we can view higher inductive types as specific resizing axioms. But note that the converse to the above proposition does not appear to hold, because even if $\|X\|_v$ is \mathcal{U} -small, then it still wouldn’t have the appropriate

universal property. This is because the definition of $\|X\|_v$ is a dependent product over propositions in \mathcal{U} only, which now includes $\|X\|_v$, but still misses propositions in other universes. In the presence of resizing axioms, we could obtain the full universal property, because we would have (equivalent copies of) all propositions in a single universe:

Proposition 3.6 (see e.g. [Esc19, Section 36.5]). *If Propositional-Resizing $_{\mathcal{U}, \mathcal{U}_0}$ holds for every universe \mathcal{U} , then the Voevodsky proposition truncation satisfies the full universal property with respect to all types in all universes.*

3.2. Set Quotients from Propositional Truncations. In this section we assume to have propositional truncations with $\|X\| : F(\mathcal{U})$ when $X : \mathcal{U}$ for some (meta)function F on universes. We will be mainly interested in $F(\mathcal{U}) = \mathcal{U}$ and $F(\mathcal{U}) = \mathcal{U}_1 \sqcup \mathcal{U}$ for the reasons explained below. We prove that we can construct set quotients using propositional truncations. The construction is due to Voevodsky and also appears in [Uni13, Section 6.10] and [RS15, Section 3.4]. However, while Voevodsky assumed propositional resizing rules in his construction, the point of this section is to show that resizing is not needed to prove the universal property of the set quotient, provided propositional truncations are available. Our proof follows our earlier Agda development [Esc18] (see also [Esc19, Section 3.37]) and is fully formalized [dJE21c].

3.2.1. Images and Surjections. It will be convenient to first state and prove two lemmas on images and surjections.

Definition 3.7 (Image, $\text{im}(f)$, surjection, corestriction).

- (1) The *image* of a function $f : X \rightarrow Y$ is defined as $\text{im}(f) \equiv \Sigma_{y:Y} \exists_{x:X} (f(x) = y)$.
- (2) A function $f : X \rightarrow Y$ is a *surjection* if for every $y : Y$, there exists some $x : X$ such that $f(x) = y$.
- (3) The *corestriction* of a function $f : X \rightarrow Y$ is the function $f : X \rightarrow \text{im}(f)$ given by $x \mapsto (f(x), |x, \text{refl}|)$.

Remark 3.8. Note that if $X : \mathcal{U}$ and $Y : \mathcal{V}$ and $f : X \rightarrow Y$, then $\text{im}(f) : \mathcal{V} \sqcup F(\mathcal{U} \sqcup \mathcal{V})$, because $\Sigma_{x:X} (f(x) = y) : \mathcal{U} \sqcup \mathcal{V}$ and $\|- \|$ takes types in \mathcal{W} to subsingletons in $F(\mathcal{W})$. In case F is the identity, then we obtain the simpler $\text{im}(f) : \mathcal{U} \sqcup \mathcal{V}$.

Lemma 3.9. *Every corestriction is surjective.*

Proof. By definition of the corestriction. □

Lemma 3.10 (Image induction, [E⁺22, UF.ImageAndSurjection]). *For a surjective map $f : X \rightarrow Y$, the following induction principle holds: for every prop-valued $P : Y \rightarrow \mathcal{W}$, with \mathcal{W} an arbitrary universe, if $P(f(x))$ holds for every $x : X$, then $P(y)$ holds for every $y : Y$.*

In the other direction, for any map $f : X \rightarrow Y$, if the above induction principle holds for the specific family $P(y) \equiv \exists_{x:X} (f(x) = y)$, then f is a surjection.

Proof. Suppose that $f : X \rightarrow Y$ is a surjection, let $P : Y \rightarrow \mathcal{W}$ be subsingleton-valued and assume that $P(f(x))$ holds for every $x : X$. Now let $y : Y$ be arbitrary. We are to prove that $P(y)$ holds. Since f is a surjection, we have $\exists_{x:X} (f(x) = y)$. But $P(y)$ is a subsingleton, so we may assume that we have a specific $x : X$ with $f(x) = y$. But then $P(y)$ must hold, because $P(f(x))$ does by assumption.

For the other direction, notice that if $P(y) \equiv \exists_{x:X}(f(x) = y)$, then $P(f(x))$ clearly holds for every $x : X$. So by assuming that the induction principle applies, we get that $P(y)$ holds for every $y : Y$, which says exactly that f is a surjection. \square

3.2.2. Set Quotients. We now construct set quotients using images and specialize image induction to the set quotient.

Definition 3.11 (Equivalence relation). An *equivalence relation* on a type X is a binary type family $\approx : X \rightarrow X \rightarrow \mathcal{V}$ such that it is

- (i) subsingleton-valued, *i.e.* $x \approx y$ is a subsingleton for every $x, y : X$;
- (ii) reflexive, *i.e.* $x \approx x$ for every $x : X$;
- (iii) symmetric, *i.e.* $x \approx y$ implies $y \approx x$ for every $x, y : X$;
- (iv) transitive, *i.e.* the conjunction of $x \approx y$ and $y \approx z$ implies $x \approx z$ for every $x, y, z : X$.

Definition 3.12 (Set quotient, X/\approx). We define the *set quotient* of X by \approx to be the type $X/\approx \equiv \text{im}(e_\approx)$ where

$$\begin{aligned} e_\approx : X &\rightarrow (X \rightarrow \Omega_{\mathcal{V}}) \\ x &\mapsto (y \mapsto (x \approx y, p(x, y))) \end{aligned}$$

and p is the witness that \approx is subsingleton-valued.

Of course, we should prove that X/\approx really is the quotient of X by \approx by proving a suitable universal property. The following definition and lemmas indeed build up to this. For the remainder of this section, we will fix a type $X : \mathcal{U}$ with an equivalence relation $\approx : X \rightarrow X \rightarrow \mathcal{V}$.

Remark 3.13. By Remark 3.8, and because $\Omega_{\mathcal{V}} : \mathcal{V}^+$, we have $X/\approx : \mathcal{T} \sqcup F(\mathcal{T})$ with $\mathcal{T} \equiv \mathcal{V}^+ \sqcup \mathcal{U}$. In the particular case that F is the identity, we obtain the simpler $X/\approx : \mathcal{V}^+ \sqcup \mathcal{U}$.

Lemma 3.14. *The quotient X/\approx is a set.*

Proof. Observe that $(X/\approx) \equiv \text{im}(e_\approx)$ is a subtype of $X \rightarrow \Omega_{\mathcal{V}}$ (as $\text{pr}_1 : X/\approx \rightarrow (X \rightarrow \Omega_{\mathcal{V}})$ is an embedding), that $X \rightarrow \Omega_{\mathcal{V}}$ is a set (by function extensionality) and that subtypes of sets are sets. \square

Definition 3.15 (η). The map $\eta : X \rightarrow X/\approx$ is defined to be the corestriction of e_\approx .

Although, in general, the type X/\approx lives in another universe than X (see Remark 3.13), we can still prove the following induction principle for (subsingleton-valued) families into *arbitrary* universes.

Lemma 3.16 (Set quotient induction). *For every subsingleton-valued $P : X/\approx \rightarrow \mathcal{W}$, with \mathcal{W} any universe, if $P(\eta(x))$ holds for every $x : X$, then $P(x')$ holds for every $x' : X/\approx$.*

Proof. The map η is surjective by Lemma 3.9, so that Lemma 3.10 yields the desired result. \square

Definition 3.17 (Respect equivalence relation). A map $f : X \rightarrow A$ *respects the equivalence relation* \approx if $x \approx y$ implies $f(x) = f(y)$ for every $x, y : X$.

Observe that respecting an equivalence relation is property rather than data, when the codomain A of the map $f : X \rightarrow A$ is a set.

Lemma 3.18. *The map $\eta : X \rightarrow X/\approx$ respects the equivalence relation \approx and the set quotient is effective, i.e. for every $x, y : X$, we have $x \approx y$ if and only if $\eta(x) = \eta(y)$.*

Proof. By definition of the image and function extensionality, we have for every $x, y : X$ that $\eta(x) = \eta(y)$ holds if and only if

$$\forall z : X (x \approx z \iff y \approx z) \quad (*)$$

holds. If $(*)$ holds, then so does $x \approx y$ by reflexivity and symmetry of the equivalence relation. Conversely, if $x \approx y$ and $z : X$ is such that $x \approx z$, then $y \approx z$ by symmetry and transitivity; and similarly if $z : X$ is such that $y \approx z$. Hence, $(*)$ holds if and only if $x \approx y$ holds. Thus, $\eta(x) = \eta(y)$ if and only if $x \approx y$, as desired. \square

The universal property of the set quotient states that the map $\eta : X \rightarrow X/\approx$ is the universal function to a set preserving the equivalence relation. We can prove it using only Lemma 3.16 and Lemma 3.18, without the need to inspect the definition of the quotient.

Theorem 3.19 (Universal property of the set quotient). *For every set $A : \mathcal{W}$ in any universe \mathcal{W} and function $f : X \rightarrow A$ respecting the equivalence relation, there is a unique function $\bar{f} : X/\approx \rightarrow A$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\eta} & X/\approx \\ & \searrow f & \swarrow \bar{f} \\ & A & \end{array}$$

commutes.

Proof [dJE21c]. Let $A : \mathcal{W}$ be a set and $f : X \rightarrow A$ respect the equivalence relation. The following auxiliary type family over X/\approx will be at the heart of our proof:

$$B(x') := \Sigma_{a:A} \exists x:X ((\eta(x) = x') \times (f(x) = a)).$$

Claim. *The type $B(x')$ is a subsingleton for every $x' : X/\approx$.*

Proof of claim. By function extensionality, the type expressing that $B(x')$ is a subsingleton for every $x' : X/\approx$ is itself a subsingleton. So by set quotient induction, it suffices to prove that $B(\eta(x))$ is a subsingleton for every $x : X$. So assume that we have $(a, p), (b, q) : B(\eta(x))$. It suffices to show that $a = b$. The elements p and q witness

$$\exists x_1 : X ((\eta(x_1) = \eta(x)) \times (f(x_1) = a))$$

and

$$\exists x_2 : X ((\eta(x_2) = \eta(x)) \times (f(x_2) = b)),$$

respectively. By Lemma 3.18 and the fact that f respects the equivalence relation, we obtain $f(x) = a$ and $f(x) = b$ and hence the desired $a = b$. \square

Next, we define $k : \Pi_{x:X} B(\eta(x))$ by $k(x) = (f(x), |x, \text{refl}, \text{refl}|)$. By set quotient induction and the claim, the function k induces a dependent map $\bar{k} : \Pi_{(x':X/\approx)} B(x')$.

We then define the (nondependent) function $\bar{f} : X/\approx \rightarrow A$ as $\text{pr}_1 \circ \bar{k}$. We proceed by showing that $\bar{f} \circ \eta = f$. By function extensionality, it suffices to prove that $\bar{f}(\eta(x)) = f(x)$

for every $x : X$. But notice that:

$$\begin{aligned} \bar{f}(\eta(x)) &\equiv \text{pr}_1(\bar{k}(\eta(x))) \\ &= \text{pr}_1(k(x)) && \text{(since } \bar{k}(\eta(x)) = k(x) \text{ because of the claim)} \\ &\equiv f(x). \end{aligned}$$

Finally, we wish to show that \bar{f} is the unique such function, so suppose that $g : X/\approx \rightarrow A$ is another function such that $g \circ \eta = f$. By function extensionality, it suffices to prove that $g(x') = \bar{f}(x')$ for every $x' : X/\approx$, which is a subsingleton as A is a set. Hence, set quotient induction tells us that it is enough to show that $g(\eta(x)) = \bar{f}(\eta(x))$ for every $x : X$, but this holds as both sides of the equation are equal to $f(x)$. \square

Remark 3.20 (*cf.* [Esc19, Section 3.21]). In univalent foundations, some attention is needed in phrasing unique existence, so we pause to discuss the phrasing of Theorem 3.19 here. Typically, if we wish to express unique existence of an element $x : X$ satisfying $P(x)$ for some type family $P : \mathcal{U} \rightarrow \mathcal{V}$, then we should phrase it as $\text{is-singleton}(\Sigma_{x:X} P(x))$, where $\text{is-singleton}(Y) \equiv Y \times \text{is-subsingleton}(Y)$. That is, we require that there is a unique *pair* $(x, p) : \Sigma_{x:X} P(x)$. This becomes important when the type family P is not subsingleton-valued. However, if P is subsingleton-valued, then it is equivalent to the traditional formulation of unique existence: *i.e.* that there is an $x : X$ with $P(x)$ such that every $y : X$ with $P(y)$ is equal to x . This happens to be the situation in Theorem 3.19, because of function extensionality and the fact that A is a set.

We stress that although the set quotient increases universe levels, see Remark 3.13, it does satisfy the appropriate universal property, so that resizing is not needed.

Having small set quotients is closely related to propositional resizing, as we show now.

Proposition 3.21. *Suppose that $\|-\|$ does not increase universe levels, i.e. in the notation of Remark 3.2, the function F is the identity.*

- (i) *If $\Omega\text{-Resizing}_{\mathcal{V}\mathcal{U}}$ holds for universes \mathcal{U} and \mathcal{V} , then the set quotient X/\approx is \mathcal{U} -small for any type $X : \mathcal{U}$ and any \mathcal{V} -valued equivalence relation.*
- (ii) *Conversely, if the set quotient $2/\approx$ is \mathcal{U} -small for every \mathcal{V} -valued equivalence relation on 2 , then $\text{Propositional-Resizing}_{\mathcal{V}\mathcal{U}}$ holds.*

Proof.

- (i) If we have $\Omega\text{-Resizing}_{\mathcal{V}\mathcal{U}}$, then $\Omega_{\mathcal{V}}$ is \mathcal{U} -small, so that $X/\approx \equiv \text{im}(e_{\approx})$ is \mathcal{U} -small too when $X : \mathcal{U}$ and \approx is \mathcal{V} -valued.
- (ii) Let $P : \mathcal{V}$ be any proposition and consider the \mathcal{V} -valued equivalence relation $x \approx_P y \equiv (x = y) \vee P$ on 2 . Notice that

$$(2/\approx_P) \text{ is a subsingleton} \iff P \text{ holds,}$$

so if $2/\approx_P$ is \mathcal{U} -small, then so is the type $\text{is-subsingleton}(2/\approx_P)$ and therefore P . \square

3.3. Propositional Truncations from Set Quotients. The converse, constructing propositional truncations from set quotients, is more straightforward, although we must pay some attention to the universes involved in order to get an exact match.

Definition 3.22 (Existence of set quotients). We say that *set quotients exist* if for every type X and equivalence relation \approx on X , we have a set X/\approx with a map $\eta : X \rightarrow X/\approx$ that respects the equivalence relation such that the universal property set out in Theorem 3.19 is satisfied.

Theorem 3.23. *Any set quotient satisfies the induction principle of Lemma 3.16, i.e. the induction principle is implied by the universal property of the set quotient.*

Proof [dJ22a]. Suppose that $P : X/\approx \rightarrow \mathcal{W}$ is a proposition-valued type-family over the set quotient X/\approx and that we have $\rho : \prod_{x:X} P(\eta(x))$. We write $S \equiv \sum_{x':X/\approx} P(x')$ and define the map $f : X \rightarrow S$ by $f(x) \equiv (\eta(x), \rho(x))$. Note that f respects the equivalence relation since η does and P is proposition-valued. Moreover, S is a set, because subtypes of sets are sets and the quotient X/\approx is a set by assumption. Hence, by the universal property, f induces a map $\bar{f} : X/\approx \rightarrow S$ such that $\bar{f} \circ \eta = f$. We claim that \bar{f} is a section of $\text{pr}_1 : S \rightarrow X/\approx$. Note that this would finish the proof, because if we have $e : \prod_{x':X/\approx} \text{pr}_1(\bar{f}(x')) = x'$, then we obtain $P(x')$ for every x' by transporting $\text{pr}_2(\bar{f}(x'))$ along $e(x')$. But \bar{f} must be a section of pr_1 , because we can take both $\text{pr}_1 \circ \bar{f}$ and id for the dashed map in the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta} & X/\approx \\ & \searrow \eta & \swarrow \text{---} \\ & & X/\approx \end{array}$$

since $\text{pr}_1 \circ \bar{f} \circ \eta = \text{pr}_1 \circ f = \eta$, so $\text{pr}_1 \circ \bar{f}$ and id must be equal by the universal property of the set quotient. \square

Theorem 3.24. *If set quotients exist, then every type has a propositional truncation.*

Proof [dJ22a]. Let $X : \mathcal{U}$ be any type and consider the \mathcal{U}_0 -valued equivalence relation $x \approx_1 y \equiv 1$. To see that X/\approx_1 is a subsingleton, note that by set quotient induction it suffices to prove $\eta(x) = \eta(y)$ for every $x, y : X$. But $x \approx_1 y$ for every $x, y : X$, and η respects the equivalence relation, so this is indeed the case. Now if $P : \mathcal{V}$ is any subsingleton and $f : X \rightarrow P$ is any map, then f respects the equivalence relation \approx_1 on X , simply because P is a subsingleton. Thus, by the universal property of the quotient, we obtain the desired map $\bar{f} : X/\approx_1 \rightarrow P$ and hence, X/\approx_1 has the universal property of the propositional truncation. \square

Remark 3.25. Because the set quotients constructed using the propositional truncation live in higher universes, we embark on a careful comparison of universes. Suppose that propositional truncations of types $X : \mathcal{U}$ exist and that $\|X\| : F(\mathcal{U})$. Then by Remark 3.13, the set quotient X/\approx_1 in the proof above lives in the universe $(\mathcal{U}_1 \sqcup \mathcal{U}) \sqcup F(\mathcal{U}_1 \sqcup \mathcal{U})$.

In particular, if F is the identity and the propositional truncation of $X : \mathcal{U}$ lives in \mathcal{U} , then the quotient X/\approx_1 lives in $\mathcal{U}_1 \sqcup \mathcal{U}$, which simplifies to \mathcal{U} whenever \mathcal{U} is at least \mathcal{U}_1 . In other words, the universes of $\|X\|$ and X/\approx_1 match up for types X in every universe, *except* the first universe \mathcal{U}_0 .

If we always wish to have X/\approx_1 in the same universe as $\|X\|$, then we can achieve this by assuming $F(\mathcal{V}) \equiv \mathcal{U}_1 \sqcup \mathcal{V}$, which says that the propositional truncations stay in the same universe, *except* when the type is in the first universe \mathcal{U}_0 in which case the truncation will be in the second universe \mathcal{U}_1 .

Theorem 3.26. *All set quotients are effective, i.e. $\eta(x) = \eta(y)$ implies $x \approx y$.*

Proof. If we have set quotients, then we have propositional truncations by Theorem 3.24 which we can use to construct effective set quotients following Section 3.2. But any two set quotients of a type by an equivalence relation must be equivalent, so the original set quotients are effective too. \square

3.4. Set Replacement. In this section, we return to our running assumption that universes are closed under propositional truncations, *i.e.* the function F above is assumed to be the identity. We study the equivalence of a set replacement principle and the existence of small set quotients. These principles will find application in Section 5.2.

Definition 3.27 (Set replacement, [E⁺22, UF.Size]). The *set replacement* principle asserts that the image of a map $f : X \rightarrow Y$ is $\mathcal{U} \sqcup \mathcal{V}$ -small if X is \mathcal{U} -small and Y is locally \mathcal{V} -small set.

In particular, if \mathcal{U} and \mathcal{V} are the same, then the image is \mathcal{U} -small. The name “set replacement” is inspired by [BBC⁺22, Section 2.19], but is different in two ways: In [BBC⁺22], replacement is not restricted to maps into sets, and the universe parameters \mathcal{U} and \mathcal{V} are taken to be the same. Rijke [Rij17] shows that the replacement of [BBC⁺22] is provable in the presence of a univalent universe closed under pushouts.

We show that set replacement is logically equivalent to having small set quotients, where the latter means that the quotient of a type $X : \mathcal{U}$ by a \mathcal{V} -valued equivalence relation lives in $\mathcal{U} \sqcup \mathcal{V}$.

Definition 3.28 (Existence of small set quotients). We say that *small set quotients exist* if set quotients exists in the sense of Definition 3.22, and moreover, the quotient X/\approx of a type $X : \mathcal{U}$ by a \mathcal{V} -valued equivalence relation lives in $\mathcal{U} \sqcup \mathcal{V}$.

Note that we would get small set quotients if we added set quotients as a primitive higher inductive type. Also, if one assumes Ω -Resizing $_{\mathcal{V}}$, then the construction of set quotients in Section 3.2.2 yields a quotient X/\approx in $\mathcal{U} \sqcup \mathcal{V}$ when $X : \mathcal{U}$ and \approx is a \mathcal{V} -valued equivalence relation on X .

Theorem 3.29. *Set replacement is logically equivalent to the existence of small set quotients.*

Proof [dJ22a, dJ22b]. Suppose set replacement is true and that a type $X : \mathcal{U}$ and a \mathcal{V} -valued equivalence relation \approx are given. Using the construction laid out in Section 3.2.2, we construct a set quotient X/\approx in $\mathcal{U} \sqcup \mathcal{V}^+$ as the image of a map $X \rightarrow (X \rightarrow \Omega_{\mathcal{V}})$. But by propositional extensionality $\Omega_{\mathcal{V}}$ is locally \mathcal{V} -small and by function extensionality so is $X \rightarrow \Omega_{\mathcal{V}}$. Hence, X/\approx is $(\mathcal{U} \sqcup \mathcal{V})$ -small by set replacement, so X/\approx is equivalent to a type $Y : \mathcal{U} \sqcup \mathcal{V}$. It is then straightforward to show that Y satisfies the properties of the set quotient as well, finishing the proof of one implication.

Conversely, let $f : X \rightarrow Y$ be a map from a \mathcal{U} -small type to a locally \mathcal{V} -small set. Since X is \mathcal{U} -small, we have $X' : \mathcal{U}$ such that $X' \simeq X$. And because Y is locally \mathcal{V} -small, we have a \mathcal{V} -valued binary relation $=_{\mathcal{V}}$ on Y such that $(y =_{\mathcal{V}} y') \simeq (y = y')$ for every $y, y' : Y$. We now define the \mathcal{V} -valued equivalence relation \approx on X' by $(x \approx x') := (f'(x) =_{\mathcal{V}} f'(x'))$, where f' is the composite $X' \simeq X \xrightarrow{f} Y$. By assumption, the quotient X'/\approx lives in $\mathcal{U} \sqcup \mathcal{V}$. But it is straightforward to work out that $\text{im}(f)$ is equivalent to this quotient. Hence, $\text{im}(f)$ is $(\mathcal{U} \sqcup \mathcal{V})$ -small, as desired. \square

The left-to-right implication of the theorem above is similar to [Rij17, Corollary 5.1], but our theorem generalizes the universe parameters and restricts to maps into sets. The latter is the reason why the converse also holds.

4. LARGENESS OF COMPLETE POSETS

A well-known result of Freyd in classical mathematics says that every complete small category is a preorder [Fre64, Exercise D of Chapter 3]. In other words, complete categories are necessarily large and only complete preorders can be small, at least impredicatively. Predicatively, by contrast, we show that many weakly complete posets (including directed complete posets, bounded complete posets and sup-lattices) are necessarily large. We capture these structures by a technical notion of a $\delta_{\mathcal{V}}$ -complete poset in Section 4.1. In Section 4.2 we define when such structures are nontrivial and introduce the constructively stronger notion of positivity. Section 4.3 and Section 4.4 contain the two fundamental technical lemmas and the main theorems, respectively. Finally, we consider alternative formulations of being nontrivial and positive that ensure that these notions are properties rather than data and shows how the main theorems remain valid, assuming univalence.

4.1. $\delta_{\mathcal{V}}$ -complete Posets. We start by introducing a class of weakly complete posets that we call $\delta_{\mathcal{V}}$ -complete posets. The notion of a $\delta_{\mathcal{V}}$ -complete poset is a technical and auxiliary notion sufficient to make our main theorems go through. The important point is that many familiar structures (dcpos, bounded complete posets, sup-lattices) are $\delta_{\mathcal{V}}$ -complete posets (see Examples 4.3).

Definition 4.1 ($\delta_{\mathcal{V}}$ -complete poset, $\delta_{x,y,P}, \bigvee \delta_{x,y,P}$). A *poset* is a type X with a subsingleton-valued binary relation \sqsubseteq on X that is reflexive, transitive and antisymmetric. It is not necessary to require X to be a set, as this follows from the other requirements. A poset (X, \sqsubseteq) is $\delta_{\mathcal{V}}$ -complete for a universe \mathcal{V} if for every pair of elements $x, y : X$ with $x \sqsubseteq y$ and every subsingleton P in \mathcal{V} , the family

$$\begin{aligned} \delta_{x,y,P} : 1 + P &\rightarrow X \\ \text{inl}(\star) &\mapsto x; \\ \text{inr}(p) &\mapsto y; \end{aligned}$$

has a supremum $\bigvee \delta_{x,y,P}$ in X .

Remark 4.2 (Classically, every poset is $\delta_{\mathcal{V}}$ -complete). Consider a poset (X, \sqsubseteq) and a pair of elements $x \sqsubseteq y$. If $P : \mathcal{V}$ is a decidable proposition, then we can define the supremum of $\delta_{x,y,P}$ by case analysis on whether P holds or not. For if it holds, then the supremum is y , and if it does not, then the supremum is x . Hence, if excluded middle holds in \mathcal{V} , then the family $\delta_{x,y,P}$ has a supremum for every $P : \mathcal{V}$. Thus, if excluded middle holds in \mathcal{V} , then every poset (with carrier in any universe) is $\delta_{\mathcal{V}}$ -complete.

The above remark naturally leads us to ask whether the converse also holds, *i.e.* if every poset is $\delta_{\mathcal{V}}$ -complete, does excluded middle in \mathcal{V} hold? As far as we know, we can only get weak excluded middle in \mathcal{V} , as we will later see in Proposition 4.6. This proposition also shows that in the absence of excluded middle, the notion of $\delta_{\mathcal{V}}$ -completeness isn't trivial. For now, we focus on the fact that, also constructively and predicatively, there are many examples of $\delta_{\mathcal{V}}$ -complete posets.

Examples 4.3.

- (i) Every \mathcal{V} -sup-lattice is $\delta_{\mathcal{V}}$ -complete. That is, if a poset X has suprema for all families $I \rightarrow X$ with I in the universe \mathcal{V} , then X is $\delta_{\mathcal{V}}$ -complete.
- (ii) The \mathcal{V} -sup-lattice $\Omega_{\mathcal{V}}$ is $\delta_{\mathcal{V}}$ -complete. The type $\Omega_{\mathcal{V}}$ of propositions in \mathcal{V} is a \mathcal{V} -sup-lattice with the order given by implication and suprema by existential quantification. Hence, $\Omega_{\mathcal{V}}$ is $\delta_{\mathcal{V}}$ -complete. Specifically, given propositions Q, R and P , the supremum of $\delta_{Q,R,P}$ is given by $Q \vee (R \times P)$.
- (iii) The \mathcal{V} -powerset $\mathcal{P}_{\mathcal{V}}(X) \equiv X \rightarrow \Omega_{\mathcal{V}}$ of a type X is $\delta_{\mathcal{V}}$ -complete. Note that $\mathcal{P}_{\mathcal{V}}(X)$ is another example of a \mathcal{V} -sup-lattice (ordered by subset inclusion and with suprema given by unions) and hence $\delta_{\mathcal{V}}$ -complete. We will sometimes employ familiar set-theoretic notation when using elements of $\mathcal{P}_{\mathcal{V}}(X)$, *e.g.* given $A : \mathcal{P}_{\mathcal{V}}(X)$, we might write $x \in A$ for the assertion that $A(x)$ holds.
- (iv) Every \mathcal{V} -bounded complete poset is $\delta_{\mathcal{V}}$ -complete. That is, if (X, \sqsubseteq) is a poset with suprema for all bounded families $I \rightarrow X$ with I in the universe \mathcal{V} , then (X, \sqsubseteq) is $\delta_{\mathcal{V}}$ -complete. A family $\alpha : I \rightarrow X$ is bounded if there exists some $x : X$ with $\alpha(i) \sqsubseteq x$ for every $i : I$. For example, the family $\delta_{x,y,P}$ is bounded by y .
- (v) Every \mathcal{V} -directed complete poset (dcpo) is $\delta_{\mathcal{V}}$ -complete, since the family $\delta_{x,y,P}$ is directed. We note that [dJE21a] provides a host of examples of \mathcal{V} -dcpo.

4.2. Nontrivial and Positive Posets. In Remark 4.2 we saw that if we can decide a proposition P , then we can define $\bigvee \delta_{x,y,P}$ by case analysis. What about the converse? That is, if $\delta_{x,y,P}$ has a supremum and we know that it equals x or y , can we then decide P ? Of course, if $x = y$, then $\bigvee \delta_{x,y,P} = x = y$, so we don't learn anything about P . But what if add the assumption that $x \neq y$? It turns out that constructively we can only expect to derive decidability of $\neg P$ in that case. This is due to the fact that $x \neq y$ is a negated proposition, which is rather weak constructively, leading us to later define (see Definition 4.8) a constructively stronger notion for elements of $\delta_{\mathcal{V}}$ -complete posets.

Definition 4.4 (Nontriviality). A poset X is *nontrivial* if we have designated $x, y : X$ with $x \sqsubseteq y$ and $x \neq y$.

Lemma 4.5. *For a nontrivial poset (X, \sqsubseteq, x, y) and a proposition $P : \mathcal{V}$, we have the following two implications:*

- (i) *if the supremum of $\delta_{x,y,P}$ exists and $x = \bigvee \delta_{x,y,P}$, then $\neg P$ is the case;*
- (ii) *if the supremum of $\delta_{x,y,P}$ exists and $y = \bigvee \delta_{x,y,P}$, then $\neg\neg P$ is the case.*

Proof.

- (i) Suppose that $x = \bigvee \delta_{x,y,P}$ and assume for a contradiction that we have $p : P$. Then $y \equiv \delta_{x,y,P}(\text{inr}(p)) \sqsubseteq \bigvee \delta_{x,y,P} = x$, which is impossible by antisymmetry and our assumptions that $x \sqsubseteq y$ and $x \neq y$.
- (ii) Suppose that $y = \bigvee \delta_{x,y,P}$ and assume for a contradiction that $\neg P$ holds. Then $x = \bigvee \delta_{x,y,P} = y$, contradicting our assumption that $x \neq y$. \square

Proposition 4.6 [dJE21a, Section 4]. *If the poset 2 with exactly two elements $0 \sqsubseteq 1$ is $\delta_{\mathcal{V}}$ -complete, then weak excluded middle in \mathcal{V} holds.*

Proof. Suppose that 2 were $\delta_{\mathcal{V}}$ -complete and let $P : \mathcal{V}$ be an arbitrary subsingleton. We must show that $\neg P$ is decidable. Since 2 has exactly two elements, the supremum $\bigvee \delta_{0,1,P}$ must be 0 or 1 . But then we apply Lemma 4.5 to get decidability of $\neg P$. \square

Combining Remark 4.2 and Proposition 4.6 yields that excluded middle implies that every poset is $\delta_{\mathcal{V}}$ -complete, which in turns implies weak excluded middle. We do not know whether these implications can be reversed. That the conclusion of the implication in Lemma 4.5(ii) cannot be strengthened to say that P is the case is shown by the following observation.

Proposition 4.7. *Recall Examples 4.3, which show that $\Omega_{\mathcal{V}}$ is $\delta_{\mathcal{V}}$ -complete. If for every two propositions Q and R with $Q \sqsubseteq R$ and $Q \neq R$ we have that the equality $R = \bigvee \delta_{Q,R,P}$ in $\Omega_{\mathcal{V}}$ implies P for every proposition $P : \mathcal{V}$, then excluded middle in \mathcal{V} follows.*

Proof. Assume the hypothesis in the proposition. We show that $\neg\neg P \rightarrow P$ for every proposition $P : \mathcal{V}$, from which excluded middle in \mathcal{V} follows. Let P be a proposition in \mathcal{V} and assume that $\neg\neg P$. This yields $0 \neq P$, so by assumption the equality $P = \bigvee \delta_{0,P,P}$ implies P . But this equality holds, because $\bigvee \delta_{0,P,P} = 0 \vee (P \times P) = P$, as described in Examples 4.3(ii). \square

Thus, having a pair of elements $x \sqsubseteq y$ with $x \neq y$ is rather weak constructively in that we can only derive $\neg\neg P$ from $y = \bigvee \delta_{x,y,P}$. As promised in the introduction of this section, we now introduce and motivate a constructively stronger notion.

Definition 4.8 (Strictly below, $x \sqsubset y$). We say that x is *strictly below* y in a $\delta_{\mathcal{V}}$ -complete poset if $x \sqsubseteq y$ and, moreover, for every $z \sqsupseteq y$ and every proposition $P : \mathcal{V}$, the equality $z = \bigvee \delta_{x,z,P}$ implies P .

Note that with excluded middle, $x \sqsubset y$ is equivalent to the conjunction of $x \sqsubseteq y$ and $x \neq y$. But constructively, the former is much stronger, as the following examples and proposition illustrate.

Examples 4.9 (Strictly below in $\Omega_{\mathcal{V}}$ and $\mathcal{P}_{\mathcal{V}}(X)$).

- (i) Recall from Examples 4.3 that $\Omega_{\mathcal{V}}$ is $\delta_{\mathcal{V}}$ -complete. Let $P : \mathcal{V}$ be an arbitrary proposition. Observe that $0_{\mathcal{V}} \neq P$ holds precisely when $\neg\neg P$ does. However, $0_{\mathcal{V}}$ is strictly below P if and only if P holds. More generally, for any two propositions $Q, P : \mathcal{V}$, we have $(Q \sqsubseteq P) \times (Q \neq P)$ if and only if $\neg Q \times \neg\neg P$ holds. But, $Q \sqsubset P$ holds if and only if $\neg Q \times P$ holds.
- (ii) Another example (see Examples 4.3) of a $\delta_{\mathcal{V}}$ -complete-poset is the powerset $\mathcal{P}_{\mathcal{V}}(X)$ of a type $X : \mathcal{V}$. If we have two subsets $A \sqsubseteq B$ of X , then $A \neq B$ if and only if $\neg(\forall x : X(x \in B \rightarrow x \in A))$.

However, if $A \sqsubset B$ and $y \in A$ is decidable for every $y : X$, then we get the stronger $\exists x : X(x \in B \times x \notin A)$. For we can take $P : \mathcal{V}$ to be $\exists x : X(x \in B \times x \notin A)$ and observe that $\bigvee \delta_{A,B,P} = B$, because if $x \in B$, either $x \in A$ in which case $x \in \bigvee \delta_{A,B,P}$, or $x \notin A$ in which case P must hold and $x \in B = \bigvee \delta_{A,B,P}$.

Conversely, if we have $A \sqsubseteq B$ and an element $x \in B$ with $x \notin A$, then $A \sqsubset B$. For if $C \sqsupseteq B$ is a subset and $P : \mathcal{V}$ a proposition such that $\bigvee \delta_{A,C,P} = C$, then $x \in C = \bigvee \delta_{A,C,P} = A \cup \{y \in C \mid P\}$, so either $x \in A$ or P must hold. But $x \notin A$ by assumption, so P must be true, proving $A \sqsubset B$.

Proposition 4.10. *For elements x and y of a $\delta_{\mathcal{V}}$ -complete poset, we have that $x \sqsubset y$ implies both $x \sqsubseteq y$ and $x \neq y$. However, if the conjunction of $x \sqsubseteq y$ and $x \neq y$ implies $x \sqsubset y$ for every $x, y : \Omega_{\mathcal{V}}$, then excluded middle in \mathcal{V} holds.*

Proof. Note that $x \sqsubset y$ implies $x \sqsubseteq y$ by definition. Now suppose that $x \sqsubset y$. Then the equality $y = \bigvee \delta_{x,y,0_{\mathcal{V}}}$ implies that $0_{\mathcal{V}}$ holds. But if $x = y$, then this equality holds, so $x \neq y$, as desired.

For $P : \Omega_{\mathcal{V}}$ we observed that $0_{\mathcal{V}} \neq P$ is equivalent to $\neg\neg P$ and that $0_{\mathcal{V}} \sqsubset P$ is equivalent to P , so if we had $((x \sqsubseteq y) \times (x \neq y)) \rightarrow x \sqsubset y$ in general, then we would have $\neg\neg P \rightarrow P$ for every proposition P in \mathcal{V} , which is equivalent to excluded middle in \mathcal{V} . \square

Lemma 4.11. *The following transitivity properties hold for all elements x, y and z of a $\delta_{\mathcal{V}}$ -complete poset:*

- (i) *if $x \sqsubseteq y \sqsubset z$, then $x \sqsubset z$;*
- (ii) *if $x \sqsubset y \sqsubseteq z$, then $x \sqsubset z$.*

Proof.

- (i) Assume $x \sqsubseteq y \sqsubset z$, let P be an arbitrary proposition in \mathcal{V} and suppose that $z \sqsubseteq w$. We must show that $w = \bigvee \delta_{x,w,P}$ implies P . But $y \sqsubset z$, so we know that the equality $w = \bigvee \delta_{y,w,P}$ implies P . Now observe that $\bigvee \delta_{x,w,P} \sqsubseteq \bigvee \delta_{y,w,P}$, so if $w = \bigvee \delta_{x,w,P}$, then $w = \bigvee \delta_{y,w,P}$, finishing the proof.
- (ii) Assume $x \sqsubset y \sqsubseteq z$, let P be an arbitrary proposition in \mathcal{V} and suppose that $z \sqsubseteq w$. We must show that $w = \bigvee \delta_{x,w,P}$ implies P . But $x \sqsubset y$ and $y \sqsubseteq w$, so this follows immediately. \square

Proposition 4.12. *The following are equivalent for an element y of a \mathcal{V} -sup-lattice X :*

- (i) *the least element of X is strictly below y ;*
- (ii) *for every family $\alpha : I \rightarrow X$ with $I : \mathcal{V}$, if $y \sqsubseteq \bigvee \alpha$, then I is inhabited;*
- (iii) *there exists some $x : X$ with $x \sqsubset y$.*

Proof. Write \perp for the least element of X . By Lemma 4.11 we have:

$$\perp \sqsubset y \iff \exists_{x:X}(\perp \sqsubseteq x \sqsubset y) \iff \exists_{x:X}(x \sqsubset y),$$

which proves the equivalence of (i) and (iii). It remains to prove that (i) and (ii) are equivalent. Suppose that $\perp \sqsubset y$ and let $\alpha : I \rightarrow X$ with $y \sqsubseteq \bigvee \alpha$. Using $\perp \sqsubset y \sqsubseteq \bigvee \alpha$ and Lemma 4.11, we have $\perp \sqsubset \bigvee \alpha$. Hence, we only need to prove $\bigvee \alpha \sqsubseteq \bigvee \delta_{\perp, \bigvee \alpha, \exists i:I}$, but $\alpha_j \sqsubseteq \bigvee \delta_{\perp, \bigvee \alpha, \exists i:I}$ for every $j : I$, so this is true indeed. For the converse, assume that y satisfies (ii), suppose $z \sqsupseteq y$ and let $P : \mathcal{V}$ be a proposition such that $z = \bigvee \delta_{\perp, z, P}$. We must show that P holds. But notice that $y \sqsubseteq z = \bigvee \delta_{\perp, z, P} = \bigvee((p : P) \mapsto z)$, so P must be inhabited as y satisfies (ii). \square

Item (ii) in Proposition 4.12 says exactly that y is a positive element in the sense of [Joh84, p. 98]. Observe that (ii) makes sense for any poset, not just \mathcal{V} -sup-lattices: we don't need to assume the existence of suprema to formulate condition (ii), because we can rephrase $y \sqsubseteq \bigvee \alpha$ as “for every $x : X$, if x is an upper bound of α and x is below any other upper bound of α , then $y \sqsubseteq x$ ”. Similarly, the strictly-below relation makes sense for any poset. What Proposition 4.12 shows is that the strictly-below relation generalizes Johnstone's notion of positivity from a *unary* relation to a *binary* one. Another binary generalization of positivity in a different direction is that of a positivity relation in formal topology [Sam03, CS18, CV16]. For a formal topology S , one considers a binary relation \times between S and its powerclass. Then $a \times S$ implies that a is positive [CS18, p. 764], while sets of the form $\{a \in S \mid a \times U\}$ are thought of as formal closed subsets [CV16].

Looking to strengthen the notion of a nontrivial poset, we make the following definitions.

Definition 4.13 (Positivity; *cf.* [Joh84, p. 98]).

- (i) An element of a $\delta_{\mathcal{V}}$ -complete poset is *positive* if it satisfies Proposition 4.12(iii).
- (ii) A $\delta_{\mathcal{V}}$ -complete poset X is *positive* if we have designated $x, y : X$ with x strictly below y .

Examples 4.14 (Nontriviality and positivity in $\Omega_{\mathcal{V}}$ and $\mathcal{P}_{\mathcal{V}}(X)$).

- (i) Consider an element P of the $\delta_{\mathcal{V}}$ -complete poset $\Omega_{\mathcal{V}}$. The pair $(0_{\mathcal{V}}, P)$ witnesses nontriviality of $\Omega_{\mathcal{V}}$ if and only if $\neg\neg P$ holds, while it witnesses positivity if and only if P holds.
- (ii) Consider the \mathcal{V} -powerset $\mathcal{P}_{\mathcal{V}}(X)$ on a type X as a $\delta_{\mathcal{V}}$ -complete poset (recall Examples 4.3). We write $\emptyset : \mathcal{P}_{\mathcal{V}}(X)$ for the map $x \mapsto 0_{\mathcal{V}}$. Say that a subset $A : \mathcal{P}_{\mathcal{V}}(X)$ is nonempty if $A \neq \emptyset$ and inhabited if there exists some $x : X$ such that $A(x)$ holds. The pair (\emptyset, A) witnesses nontriviality of $\mathcal{P}_{\mathcal{V}}(X)$ if and only if A is nonempty, while it witnesses positivity if and only if A is inhabited.

In domain theory the *way-below* relation is of fundamental importance. It will be instructive to see how it relates to the strictly-below relation.

Definition 4.15 (Way-below relation, compactness; [dJE21a, Definition 44]). Let x and y be elements of a \mathcal{V} -dcpo D .

- (i) We say that x is *way below* y , written $x \ll y$, if for every directed family $\alpha : I \rightarrow D$ with $y \sqsubseteq \bigvee \alpha$, there exists $i : I$ such that $x \sqsubseteq \alpha_i$ already.
- (ii) An element x is said to be *compact* if it is way below itself.

Proposition 4.16. *If $x \sqsubseteq y$ are unequal elements of a \mathcal{V} -dcpo D and y is compact, then $x \sqsubset y$ without the need to assume excluded middle. In particular, a compact element x of a \mathcal{V} -dcpo with a least element \perp is positive if and only if $x \neq \perp$.*

Proof. Suppose that $x \sqsubseteq y$ are unequal and that y is compact. We are to show that $x \sqsubset y$. So assume we have $z \sqsupseteq y$ and a proposition $P : \mathcal{V}$ such that $y \sqsubseteq z = \bigvee \delta_{x,z,P}$. By compactness of y , there exists $i : 1 + P$ such that $y \sqsubseteq \delta_{x,z,P}(i)$ already. But i can't be equal to $\text{inl}(\star)$, since $x \neq y$ is assumed. Hence, $i = \text{inr}(p)$ and P must hold. \square

Note that $x \sqsubset y$ does not imply $x \ll y$ in general, because with excluded middle, $x \sqsubset y$ is simply the conjunction of $x \sqsubseteq y$ and $x \neq y$, which does not imply $x \ll y$ in general. Also, the conjunction of $x \ll y$ and $x \neq y$ does not imply $x \sqsubset y$, as far as we know.

We end this section by summarizing why we consider the strictly-below relation to be suitable in our constructive framework. First of all, $x \sqsubset y$ coincides with $(x \sqsubseteq y) \times (x \neq y)$ in the presence of excluded middle, so it is compatible with classical logic. Secondly, we've seen in Examples 4.9 that the strictly-below relation works well in the poset of truth values and in powersets, yielding familiar constructive strengthenings. Thirdly, the strictly-below relation generalizes Johnstone's notion of positivity from a unary to a binary relation. And finally, as we will see shortly, the derived notion of positive poset is exactly what we need to derive Ω -Resizing $_{\mathcal{V}}$ rather than the weaker $\Omega_{\neg\neg}$ -Resizing $_{\mathcal{V}}$ in Theorem 4.23.

4.3. Retract Lemmas. We show that the type of propositions in \mathcal{V} is a retract of any positive $\delta_{\mathcal{V}}$ -complete poset and that the type of $\neg\neg$ -stable propositions in \mathcal{V} is a retract of any nontrivial $\delta_{\mathcal{V}}$ -complete poset.

Definition 4.17 ($\Delta_{x,y}$). For a nontrivial $\delta_{\mathcal{V}}$ -complete poset (X, \sqsubseteq, x, y) , we define the map $\Delta_{x,y} : \Omega_{\mathcal{V}} \rightarrow X$ by the assignment $P \mapsto \bigvee \delta_{x,y,P}$.

We will often omit the subscripts in $\Delta_{x,y}$ when it is clear from the context.

Definition 4.18 (Locally smallness). A $\delta_{\mathcal{V}}$ -complete poset (X, \sqsubseteq) is *locally small* if its order has \mathcal{V} -small values, *i.e.* we have $\sqsubseteq_{\mathcal{V}} : X \rightarrow X \rightarrow \mathcal{V}$ with $(x \sqsubseteq y) \simeq (x \sqsubseteq_{\mathcal{V}} y)$ for every $x, y : X$.

Examples 4.19.

- (i) The \mathcal{V} -sup-lattices $\Omega_{\mathcal{V}}$ and $\mathcal{P}_{\mathcal{V}}(X)$ (for $X : \mathcal{V}$) are locally small.
- (ii) All examples of \mathcal{V} -dcpos in [dJE21a] are locally small.

Lemma 4.20. *A locally small $\delta_{\mathcal{V}}$ -complete poset (X, \sqsubseteq) is nontrivial, witnessed by elements $x \sqsubseteq y$, if and only if the composite $\Omega_{\mathcal{V}}^{\neg\neg} \hookrightarrow \Omega_{\mathcal{V}} \xrightarrow{\Delta_{x,y}} X$ is a section.*

Proof. Suppose first that (X, \sqsubseteq, x, y) is nontrivial and locally small. We define

$$\begin{aligned} r : X &\rightarrow \Omega_{\mathcal{V}}^{\neg\neg} \\ z &\mapsto z \not\sqsubseteq_{\mathcal{V}} x. \end{aligned}$$

Note that negated propositions are $\neg\neg$ -stable, so r is well-defined. Let $P : \mathcal{V}$ be an arbitrary $\neg\neg$ -stable proposition. We want to show that $r(\Delta_{x,y}(P)) = P$. By propositional extensionality, establishing logical equivalence suffices. Suppose first that P holds. Then $\Delta_{x,y}(P) \equiv \bigvee \delta_{x,y,P} = y$, so $r(\Delta_{x,y}(P)) = r(y) \equiv (y \not\sqsubseteq_{\mathcal{V}} x)$ holds by antisymmetry and our assumptions that $x \sqsubseteq y$ and $x \neq y$. Conversely, assume that $r(\Delta_{x,y}(P))$ holds, *i.e.* that we have $\bigvee \delta_{x,y,P} \not\sqsubseteq_{\mathcal{V}} x$. Since P is $\neg\neg$ -stable, it suffices to derive a contradiction from $\neg P$. So assume $\neg P$. Then $x = \bigvee \delta_{x,y,P}$, so $r(\Delta_{x,y}(P)) = r(x) \equiv x \not\sqsubseteq_{\mathcal{V}} x$, which is false by reflexivity.

For the converse, assume that $\Omega_{\mathcal{V}}^{\neg\neg} \hookrightarrow \Omega_{\mathcal{V}} \xrightarrow{\Delta_{x,y}} X$ has a retraction $r : \Omega_{\mathcal{V}}^{\neg\neg} \rightarrow X$. Then $0_{\mathcal{V}} = r(\Delta_{x,y}(0_{\mathcal{V}})) = r(x)$ and $1_{\mathcal{V}} = r(\Delta_{x,y}(1_{\mathcal{V}})) = r(y)$, where we used that $0_{\mathcal{V}}$ and $1_{\mathcal{V}}$ are $\neg\neg$ -stable. Since $0_{\mathcal{V}} \neq 1_{\mathcal{V}}$, we get $x \neq y$, so (X, \sqsubseteq, x, y) is nontrivial, as desired. \square

The appearance of the double negation in the above lemma is due to the definition of nontriviality. If we instead assume a positive poset X , then we can exhibit all of $\Omega_{\mathcal{V}}$ as a retract of X .

Lemma 4.21. *A locally small $\delta_{\mathcal{V}}$ -complete poset (X, \sqsubseteq) is positive, witnessed by elements $x \sqsubset y$, if and only if for every $z \sqsupseteq y$, the map $\Delta_{x,z} : \Omega_{\mathcal{V}} \rightarrow X$ is a section.*

Proof. Suppose first that (X, \sqsubseteq, x, y) is positive and locally small and let $z \sqsupseteq y$ be arbitrary. We define

$$\begin{aligned} r_z : X &\mapsto \Omega_{\mathcal{V}} \\ w &\mapsto z \sqsubseteq_{\mathcal{V}} w. \end{aligned}$$

Let $P : \mathcal{V}$ be arbitrary proposition. We want to show that $r_z(\Delta_{x,z}(P)) = P$. Because of propositional extensionality, it suffices to establish a logical equivalence between P and $r_z(\Delta_{x,z}(P))$. Suppose first that P holds. Then $\Delta_{x,z}(P) = z$, so $r_z(\Delta_{x,z}(P)) = r_z(z) \equiv (z \sqsubseteq_{\mathcal{V}} z)$ holds as well by reflexivity. Conversely, assume that $r_z(\Delta_{x,z}(P))$ holds, *i.e.* that we have $z \sqsubseteq_{\mathcal{V}} \bigvee \delta_{x,z,P}$. Since $\bigvee \delta_{x,z,P} \sqsubseteq z$ always holds, we get $z = \bigvee \delta_{x,z,P}$ by antisymmetry. But by assumption and Lemma 4.11, the element x is strictly below z , so P must hold.

For the converse, assume that for every $z \sqsupseteq y$, the map $\Delta_{x,z} : \Omega_{\mathcal{V}} \rightarrow X$ has a retraction $r_z : X \rightarrow \Omega_{\mathcal{V}}$. We must show that the equality $z = \Delta_{x,z}(P)$ implies P for every $z \sqsupseteq y$

and proposition $P : \mathcal{V}$. Assuming $z = \Delta_{x,z}(P)$, we have $1_{\mathcal{V}} = r_z(\Delta_{x,z}(1_{\mathcal{V}})) = r_z(z) = r_z(\Delta_{x,z}(P)) = P$, so P must hold indeed. Hence, (X, \sqsubseteq, x, y) is positive, as desired. \square

4.4. Small Completeness with Resizing. We present our main theorems here, which show that, constructively and predicatively, nontrivial $\delta_{\mathcal{V}}$ -complete posets are necessarily large and necessarily lack decidable equality.

Definition 4.22 (Smallness). A $\delta_{\mathcal{V}}$ -complete poset is *small* if it is locally small and its carrier is \mathcal{V} -small.

Theorem 4.23.

- (i) *There is a nontrivial small $\delta_{\mathcal{V}}$ -complete poset if and only if $\Omega_{\neg\neg}$ -Resizing $_{\mathcal{V}}$ holds.*
- (ii) *There is a positive small $\delta_{\mathcal{V}}$ -complete poset if and only if Ω -Resizing $_{\mathcal{V}}$ holds.*

Proof.

- (i) Suppose that (X, \sqsubseteq, x, y) is a nontrivial small $\delta_{\mathcal{V}}$ -complete poset. By Lemma 4.20, we can exhibit $\Omega_{\mathcal{V}}^{\neg\neg}$ as a retract of X . But X is \mathcal{V} -small by assumption, so by Theorem 2.13 the type $\Omega_{\mathcal{V}}^{\neg\neg}$ is \mathcal{V} -small as well. For the converse, note that $(\Omega_{\mathcal{V}}^{\neg\neg}, \rightarrow, 0_{\mathcal{V}}, 1_{\mathcal{V}})$ is a nontrivial locally small \mathcal{V} -sup-lattice with $\bigvee \alpha$ given by $\neg\neg\exists_{i:I}\alpha_i$. And if we assume $\Omega_{\neg\neg}$ -Resizing $_{\mathcal{V}}$, then it is small.
- (ii) Suppose that (X, \sqsubseteq, x, y) is a positive small poset. By Lemma 4.21, we can exhibit $\Omega_{\mathcal{V}}$ as a retract of X . But X is \mathcal{V} -small by assumption, so by Theorem 2.13 the type $\Omega_{\mathcal{V}}$ is \mathcal{V} -small as well. For the converse, note that $(\Omega_{\mathcal{V}}, \rightarrow, 0_{\mathcal{V}}, 1_{\mathcal{V}})$ is a positive locally small \mathcal{V} -sup-lattice. And if we assume Ω -Resizing $_{\mathcal{V}}$, then it is small. \square

Lemma 4.24 [E⁺22, TypeTopology.DiscreteAndSeparated].

- (i) *Types with decidable equality are closed under retracts.*
- (ii) *Types with $\neg\neg$ -stable equality are closed under retracts.*

Examples 4.25 (Types with $\neg\neg$ -stable equality). The simple types \mathbb{N} , $\mathbb{N} \rightarrow \mathbb{N}$, $\mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}$, etc. [E⁺22, TypeTopology.SimpleTypes], and the type of Dedekind real numbers [E⁺22, Various.Dedekind] all have $\neg\neg$ -stable equality, as does the type $\Omega_{\mathcal{U}}^{\neg\neg}$ of $\neg\neg$ -stable propositions in any universe \mathcal{U} .

Theorem 4.26. *There is a nontrivial locally small $\delta_{\mathcal{V}}$ -complete poset with decidable equality if and only if weak excluded middle in \mathcal{V} holds.*

Proof. Suppose that (X, \sqsubseteq, x, y) is a nontrivial locally small $\delta_{\mathcal{V}}$ -complete poset with decidable equality. Then by Lemmas 4.20 and 4.24, the type $\Omega_{\mathcal{V}}^{\neg\neg}$ must have decidable equality too. But negated propositions are $\neg\neg$ -stable, so this yields weak excluded middle in \mathcal{V} . For the converse, note that $(\Omega_{\mathcal{V}}^{\neg\neg}, \rightarrow, 0_{\mathcal{V}}, 1_{\mathcal{V}})$ is a nontrivial locally small \mathcal{V} -sup-lattice that has decidable equality if and only if weak excluded middle in \mathcal{V} holds. \square

Theorem 4.27. *The following are equivalent:*

- (i) *there is a positive locally small $\delta_{\mathcal{V}}$ -complete poset with $\neg\neg$ -stable equality;*
- (ii) *there is a positive locally small $\delta_{\mathcal{V}}$ -complete poset with decidable equality;*
- (iii) *excluded middle in \mathcal{V} holds.*

Proof. Note that (ii) \Rightarrow (i), so we are left to show that (iii) \Rightarrow (ii) and that (i) \Rightarrow (iii). For the first implication, note that $(\Omega_{\mathcal{V}}, \rightarrow, 0_{\mathcal{V}}, 1_{\mathcal{V}})$ is a positive locally small \mathcal{V} -sup-lattice that has decidable equality if and only if excluded middle in \mathcal{V} holds. To see that (i) implies (iii), suppose that (X, \sqsubseteq, x, y) is a positive locally small $\delta_{\mathcal{V}}$ -complete poset with $\neg\neg$ -stable equality. Then by Lemmas 4.21 and 4.24 the type $\Omega_{\mathcal{V}}$ must have $\neg\neg$ -stable equality. But this implies that $\neg\neg P \rightarrow P$ for every proposition P in \mathcal{V} which is equivalent to excluded middle in \mathcal{V} . \square

In particular, Theorem 4.27(i) shows that, constructively, none of the types from Examples 4.25 can be equipped with the structure of a positive $\delta_{\mathcal{V}}$ -complete poset. In particular, we cannot expect the type of Dedekind reals to be a positive bounded complete poset.

Lattices, bounded complete posets and dcpos are necessarily large and necessarily lack decidable equality in our predicative constructive setting. More precisely:

Corollary 4.28.

- (i) *There is a nontrivial small \mathcal{V} -sup-lattice (or \mathcal{V} -bounded complete poset or \mathcal{V} -dcpo) if and only if $\Omega_{\neg\neg}$ -Resizing $_{\mathcal{V}}$ holds.*
- (ii) *There is a positive small \mathcal{V} -sup-lattice (or \mathcal{V} -bounded complete poset or \mathcal{V} -dcpo) if and only if Ω -Resizing $_{\mathcal{V}}$ holds.*
- (iii) *There is a nontrivial locally small \mathcal{V} -sup-lattice (or \mathcal{V} -bounded complete poset or \mathcal{V} -dcpo) with decidable equality if and only if weak excluded middle in \mathcal{V} holds.*
- (iv) *There is a positive locally small \mathcal{V} -sup-lattice (or \mathcal{V} -bounded complete poset or \mathcal{V} -dcpo) with decidable equality if and only if excluded middle in \mathcal{V} holds.*

The above notions of non-triviality and positivity are data rather than property. Indeed, a nontrivial poset (X, \sqsubseteq) is (by definition) equipped with two designated points $x, y : X$ such that $x \sqsubseteq y$ and $x \neq y$. It is natural to wonder if the propositionally truncated versions of these two notions yield the same conclusions. We show that this is indeed the case if we assume univalence. The need for the univalence assumption comes from the fact that smallness is a property precisely if univalence holds, as shown in Propositions 2.8 and 2.9.

Definition 4.29 (Nontrivial/positive in an unspecified way). A poset (X, \sqsubseteq) is *nontrivial in an unspecified way* if there exist some elements $x, y : X$ such that $x \sqsubseteq y$ and $x \neq y$, i.e. $\exists_{x,y:X}((x \sqsubseteq y) \times (x \neq y))$. Similarly, we can define when a poset is *positive in an unspecified way* by truncating the notion of positivity.

Theorem 4.30. *Suppose that the universes \mathcal{V} and \mathcal{V}^+ are univalent.*

- (i) *There is a small $\delta_{\mathcal{V}}$ -complete poset that is nontrivial in an unspecified way if and only if $\Omega_{\neg\neg}$ -Resizing $_{\mathcal{V}}$ holds.*
- (ii) *There is a small $\delta_{\mathcal{V}}$ -complete poset that is positive in an unspecified way if and only if Ω -Resizing $_{\mathcal{V}}$ holds.*

Proof.

- (i) Suppose that (X, \sqsubseteq) is a $\delta_{\mathcal{V}}$ -complete poset that is nontrivial in an unspecified way. By Proposition 2.8 and univalence of \mathcal{V} and \mathcal{V}^+ , the type “ $\Omega_{\mathcal{V}^{\neg\neg}}$ is \mathcal{V} -small” is a proposition. By the universal property of the propositional truncation, in proving that the type $\Omega_{\mathcal{V}^{\neg\neg}}$ is \mathcal{V} -small we can therefore assume that are given points $x, y : X$ with $x \sqsubseteq y$ and $x \neq y$. The result then follows from Theorem 4.23.
- (ii) By reduction to item (ii) of Theorem 4.23. \square

Similarly, we can prove the following theorems by reduction to Theorems 4.26 and 4.27.

Theorem 4.31.

- (i) *There is a locally small $\delta_{\mathcal{V}}$ -complete poset with decidable equality that is nontrivial in an unspecified way if and only if weak excluded middle in \mathcal{V} holds.*
- (ii) *There is a locally small $\delta_{\mathcal{V}}$ -complete poset with decidable equality that is positive in an unspecified way if and only if excluded middle in \mathcal{V} holds.*

5. MAXIMAL POINTS AND FIXED POINTS

As is well known, in impredicative mathematics, a poset has suprema of all subsets if and only if it has infima of all subsets. Perhaps counter-intuitively, this “duality” theorem can be proved predicatively. However, in the absence of impredicativity, it is not possible to fulfil its hypotheses when trying to apply it, because there are no nontrivial examples.

To explain this, we first have to make the statement of the duality theorem precise. A single universe formulation is “every \mathcal{V} -small \mathcal{V} -sup-lattice has all infima of families indexed by types in \mathcal{V} ”. The usual proof, adapted from subsets to families, shows that this formulation is predicatively provable, but in our predicative setting Theorem 4.23 tells us that there are no nontrivial examples to apply it to.

It is natural to wonder whether the single universe formulation can be generalized to *locally* small \mathcal{V} -sup-lattices (with necessarily large carriers), resulting in a predicatively useful result. However, as one of the anonymous reviewers pointed out that this generalization gives rise to a false statement and suggested the ordinals as a counterexample in a set-theoretic setting: it is a class with suprema for all subsets but has no greatest element. This led us to prove (Section 5.2) in our type-theoretic context that the locally small, but large type of ordinals in a univalent universe \mathcal{V} is a \mathcal{V} -sup-lattice. But this is not a \mathcal{V} -inf-lattice, because the unique family indexed by the empty type does not have a greatest lower bound since the type of ordinals has no greatest element.

Similarly, consider a generalized formulation of Tarski’s theorem [Tar55] that allows for multiple universes, *i.e.* we define *Tarski’s-Theorem $_{\mathcal{V},\mathcal{U},\mathcal{T}}$* as the assertion that every monotone endofunction on a \mathcal{V} -sup-lattice with carrier in a universe \mathcal{U} and order taking values in a universe \mathcal{T} has a greatest fixed point. Then *Tarski’s-Theorem $_{\mathcal{V},\mathcal{V},\mathcal{V}}$* corresponds to the original formulation and, moreover, is provable predicatively, but not useful predicatively because Theorem 4.24 shows that its hypotheses can only be fulfilled for trivial posets. On the other hand, *Tarski’s-Theorem $_{\mathcal{V},\mathcal{V}^+,\mathcal{V}}$* is provably false because the identity map on the \mathcal{V} -sup-lattice of ordinals in \mathcal{V} is a counterexample. Analogous considerations can be made for a lemma due to Pataria [Pat97, Esc03] saying that every dcpo has a greatest monotone inflationary endofunction.

5.1. A Predicative Counterexample. Because the type of ordinals in \mathcal{V} is not \mathcal{V} -small even impredicatively, the above does not rule out the possibility that a \mathcal{V} -sup-lattice X has all \mathcal{V} -infima provided X is \mathcal{V} -small impredicatively. To address this, we present an example of a \mathcal{V} -sup-lattice, parameterized by a proposition, that is \mathcal{V} -small impredicatively, but predicatively does not necessarily have a maximal element. In particular, it need not have a greatest element or all \mathcal{V} -infima.

Definition 5.1 (Lifting, cf. [EK17]). Fix a proposition $P_{\mathcal{U}}$ in a universe \mathcal{U} . Lifting $P_{\mathcal{U}}$ with respect to a universe \mathcal{V} is defined by

$$\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}}) \equiv \sum_{Q:\Omega_{\mathcal{V}}} (Q \rightarrow P_{\mathcal{U}}).$$

This is a subtype of $\Omega_{\mathcal{V}}$ (the map $\text{pr}_1 : \mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}}) \rightarrow \Omega_{\mathcal{V}}$ is an embedding) and it is closed under \mathcal{V} -suprema (in particular, it contains the least element).

Examples 5.2.

- (i) If $P_{\mathcal{U}} \equiv 0_{\mathcal{U}}$, then $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}}) \simeq (\Sigma_{Q:\Omega_{\mathcal{V}}} \neg Q) \simeq (\Sigma_{Q:\Omega_{\mathcal{V}}} (Q = 0_{\mathcal{V}})) \simeq 1$.
- (ii) If $P_{\mathcal{U}} \equiv 1_{\mathcal{U}}$, then $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}}) \equiv (\Sigma_{Q:\Omega_{\mathcal{V}}} (Q \rightarrow 1_{\mathcal{U}})) \simeq \Omega_{\mathcal{V}}$.

What makes $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ useful is the following observation.

Lemma 5.3. *Suppose that the poset $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ has a maximal element $Q : \Omega_{\mathcal{V}}$. Then $P_{\mathcal{U}}$ is equivalent to Q , which is the greatest element of $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$. In particular, $P_{\mathcal{U}}$ is \mathcal{V} -small. Conversely, if $P_{\mathcal{U}}$ is equivalent to a proposition $Q : \Omega_{\mathcal{V}}$, then Q is the greatest element of $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$.*

Proof. Suppose that $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ has a maximal element $Q : \Omega_{\mathcal{V}}$. We wish to show that $Q \simeq P_{\mathcal{U}}$. By definition of $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$, we already have that $Q \rightarrow P_{\mathcal{U}}$. So only the converse remains. Therefore suppose that $P_{\mathcal{U}}$ holds. Then, $1_{\mathcal{V}}$ is an element of $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$. Obviously $Q \rightarrow 1_{\mathcal{V}}$, but Q is maximal, so actually $Q = 1_{\mathcal{V}}$, that is, Q holds, as desired. Thus, $Q \simeq P_{\mathcal{U}}$. It is then straightforward to see that Q is actually the greatest element of $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$, since $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}}) \simeq \Sigma_{Q':\Omega_{\mathcal{V}}} (Q' \rightarrow Q)$. For the converse, assume that $P_{\mathcal{U}}$ is equivalent to a proposition $Q : \Omega_{\mathcal{V}}$. Then, as before, $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}}) \simeq \Sigma_{Q':\Omega_{\mathcal{V}}} (Q' \rightarrow Q)$, which shows that Q is indeed the greatest element of $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$. \square

Corollary 5.4. *The \mathcal{V} -sup-lattice $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ has all \mathcal{V} -infima if and only if $P_{\mathcal{U}}$ is \mathcal{V} -small.*

Proof. Suppose first that $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ has all \mathcal{V} -infima. Then it must have a infimum for the empty family $0_{\mathcal{V}} \rightarrow \mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$. But this infimum must be the greatest element of $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$. So by Lemma 5.3 the proposition $P_{\mathcal{U}}$ must be \mathcal{V} -small.

Conversely, suppose that $P_{\mathcal{U}}$ is equivalent to a proposition $Q : \mathcal{V}$. Then the infimum of a family $\alpha : I \rightarrow \mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ with $I : \mathcal{V}$ is given by $(Q \times \prod_{i:I} \alpha_i) : \mathcal{V}$. \square

In [dJE21b] we used Lemma 5.3 to conclude that a version of Zorn's lemma that says that every pointed dcpo has a maximal element is predicatively unavailable, as $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ is a pointed \mathcal{V} -dcpo, but has a maximal element if and only if $P_{\mathcal{U}}$ is \mathcal{V} -small. But, as in our discussion above of the duality theorem and Tarski's theorem, we must pay attention to the universes here. Zorn's lemma restricted to \mathcal{V} -small \mathcal{V} -sup-lattices is, assuming excluded middle [Bel97], equivalent to the axiom of choice, as usual. Disregarding its constructive status for a moment, the predicative issue is that there are no nontrivial \mathcal{V} -small \mathcal{V} -sup-lattices (Theorem 4.23). But the generalization of Zorn's lemma to *locally* small \mathcal{V} -sup-lattices is false (even if we assume the axiom of choice and hence, excluded middle), because the \mathcal{V} -sup-lattice of ordinals in \mathcal{V} , having no maximal element, is a counterexample.

5.2. Small Suprema of Small Ordinals in Univalent Foundations. We now show that the ordinal $\text{Ord}_{\mathcal{V}}$ of ordinals in a fixed *univalent* universe \mathcal{V} has suprema for all families indexed by types in \mathcal{V} and that it has no maximal element. The latter is implied by [Uni13, Lemma 10.3.21], but we were not able to find a proof of the former in the literature: Theorem 9 of [KNFX21] only proves $\text{Ord}_{\mathcal{V}}$ to have joins of increasing sequences, while [Uni13, Lemma 10.3.22] shows that every family indexed by a type in \mathcal{V} has some upper bound, but does not prove it to be the least (although least upper bounds are required for [Uni13, Exercise 10.17(ii)]). We present two proofs: one based on [Uni13, Lemma 10.3.22] using small set quotients and an alternative one using small images.

Following [Uni13, Section 10.3], we define an ordinal to be a type equipped with a proposition-valued, transitive, extensional and (inductive) well-founded relation. In [Uni13] the underlying type of an ordinal is required to be a set, but this actually follows from the other axioms, see [E⁺22, `Ordinals.Type`]. The type of ordinals, denoted by $\text{Ord}_{\mathcal{V}}$, in a given *univalent* universe \mathcal{V} can itself be equipped with such a relation [Uni13, Theorem 10.3.20] and thus is an ordinal again. However, it is not an ordinal in \mathcal{V} , but rather in the next universe \mathcal{V}^+ , and this is necessary, because it is contradictory for $\text{Ord}_{\mathcal{V}}$ to be isomorphic to an ordinal in \mathcal{V} , see [BCDE20].

Before we prove that $\text{Ord}_{\mathcal{V}}$ has \mathcal{V} -suprema, we need to recall a few facts. The well-order on $\text{Ord}_{\mathcal{V}}$ is given by: $\alpha < \beta$ if and only if we can find a (necessarily) unique $y : \beta$ such that α and $\beta \downarrow y$ are isomorphic ordinals. Here $\beta \downarrow y$ denotes the ordinal of elements $b : \beta$ satisfying $b < y$.

Lemma 5.5 [E⁺22, `Ordinals.OrdinalOfOrdinals`]. *For every two points x and y of an ordinal α , we have $x < y$ in α if and only if $\alpha \downarrow x < \alpha \downarrow y$ as ordinals.*

Proof. If $x < y$, then we can consider $\alpha \downarrow y \downarrow x$ which is easily seen to be isomorphic to $\alpha \downarrow x$, so that $\alpha \downarrow x < \alpha \downarrow y$. Conversely, if $\alpha \downarrow x < \alpha \downarrow y$, then $\alpha \downarrow x$ is isomorphic to $\alpha \downarrow y \downarrow z$ for some unique $z < y$. But now $\alpha \downarrow x$ and $\alpha \downarrow z$ are isomorphic which implies that $x = z < y$. \square

Definition 5.6 (Simulation, [Uni13, Section 10.3]). A *simulation* between two ordinals α and β is a map $f : \alpha \rightarrow \beta$ satisfying the following conditions:

- (i) for every $x, y : \alpha$, if $x < y$, then $f(x) < f(y)$;
- (ii) for every $x : \alpha$ and $y : \beta$, if $y < f(x)$, then we can find a (necessarily unique) $x' : \alpha$ such that $x' < x$ and $f(x') = y$.

Lemma 5.7 [E⁺22, `Ordinals.OrdinalOfOrdinals`]. *For ordinals α and β , the following are equivalent:*

- (i) we can find a (necessarily unique) simulation from α to β ;
- (ii) for every ordinal γ , if $\gamma < \alpha$, then $\gamma < \beta$.

We write $\alpha \preceq \beta$ in case the equivalent conditions above hold.

Proof. Given a simulation $f : \alpha \rightarrow \beta$ and an ordinal $\gamma < \alpha$, we have $x : \alpha$ such that γ and $\alpha \downarrow x$ are isomorphic. We claim that $\alpha \downarrow x$ and $\beta \downarrow f(x)$ are isomorphic, which entails $\gamma < \beta$, as desired. The forward direction of the isomorphism is given by f , while in the other direction we map $y < f(x)$ to the unique $x' : \alpha$ with $f(x') = y$ given by the fact that f is a simulation.

Conversely, if $\gamma < \alpha$ implies $\gamma < \beta$, then for every $x : \alpha$, we have a unique $y : \beta$ such that $\alpha \downarrow x$ and $\beta \downarrow y$ are isomorphic. This defines a map $f : \alpha \rightarrow \beta$ which is easily seen to be

monotone. Moreover, if $y \prec f(x)$, then $\beta \downarrow y \prec \alpha \downarrow x$, so that we get $x' \prec x$ with $y = f(x')$, and f is thus a simulation. \square

Recall from Definition 3.28 what it means to have small set quotients. If these are available, then the type of ordinals has all small suprema.

Theorem 5.8 (Extending [Uni13, Lemma 10.3.22]). *Assuming small set quotients, the large ordinal $\text{Ord}_{\mathcal{V}}$ has suprema of families indexed by types in \mathcal{V} .*

Proof [dJE22]. Given $\alpha : I \rightarrow \text{Ord}_{\mathcal{V}}$, define $\hat{\alpha}$ as the quotient of $\Sigma_{i:I} \alpha_i$ by the \mathcal{V} -valued equivalence relation \approx where $(i, x) \approx (j, y)$ if and only if $\alpha_i \downarrow x$ and $\alpha_j \downarrow y$ are isomorphic as ordinals. By our assumption, the quotient $\hat{\alpha}$ lives in \mathcal{V} . Next, [Uni13, Lemma 10.3.22] tells us that $(\hat{\alpha}, \prec)$ with

$$[(i, x)] \prec [(j, y)] \equiv (\alpha_i \downarrow x) \prec (\alpha_j \downarrow y).$$

is an ordinal that is an upper bound of α . So we show that $\hat{\alpha}$ is a lower bound of upper bounds of α . To this end, suppose that $\beta : \text{Ord}_{\mathcal{V}}$ is such that $\alpha_i \preceq \beta$ for every $i : I$. In light of Lemma 5.7, this assumption yields two things:

- (1) for every $i : I$ and $x : \alpha_i$ there exists a unique $b_i^x : \beta$ such that $\alpha_i \downarrow x = \beta \downarrow b_i^x$;
- (2) for every $i : I$, a simulation $f_i : \alpha_i \rightarrow \beta$ such that for every $x : \alpha_i$, we have $f_i(x) = b_i^x$.

We are to prove that $\hat{\alpha} \preceq \beta$. We start by defining

$$\begin{aligned} f : (\Sigma_{i:I} \alpha_i) &\rightarrow \beta \\ (i, x) &\mapsto b_i^x \end{aligned}$$

Observe that f respects \approx , for if $(i, x) \approx (j, y)$, then by univalence,

$$(\beta \downarrow b_i^x) = (\alpha_i \downarrow x) = (\alpha_j \downarrow y) = (\beta \downarrow b_j^y),$$

so $b_i^x = b_j^y$ by uniqueness of b_i^x . Thus, f induces a map $\hat{f} : \hat{\alpha} \rightarrow \beta$ satisfying the equality $\hat{f}([(i, x)]) = f(i, x)$ for every $(i, x) : \Sigma_{j:I} \alpha_j$.

It remains to prove that \hat{f} is a simulation. Because the defining properties of a simulation are propositions, we can use set quotient induction and it suffices to prove the following two things:

- (I) If $\alpha_i \downarrow x \prec \alpha_j \downarrow y$, then $b_i^x \prec b_j^y$.
- (II) If $b \prec b_i^x$, then there exists $j : I$ and $y : \alpha_j$ such that $\alpha_i \downarrow y \prec \alpha_j \downarrow x$ and $b_j^y = b$.

For (I), observe that if $\alpha_i \downarrow x \prec \alpha_j \downarrow y$, then $\beta \downarrow b_i^x \prec \beta \downarrow b_j^y$, from which $b_i^x \prec b_j^y$ follows using Lemma 5.5. For (II), suppose that $b \prec b_i^x$. Because f_i (see item (2) above) is a simulation, there exists $y : \alpha_i$ with $y \prec x$ and $f_i(y) = b$. By Lemma 5.5, we get $\alpha_i \downarrow y \prec \alpha_i \downarrow x$. Moreover, $b_i^y = f_i(y) = b$, finishing the proof of (II). \square

In Section 3.4 we saw that set replacement is equivalent to the existence of small set quotients, so the following result immediately follows from the theorem above. But the point is that an alternative construction without set quotients is available, if set replacement is assumed.

Theorem 5.9. *Assuming set replacement, the large ordinal $\text{Ord}_{\mathcal{V}}$ has suprema of families indexed by types in \mathcal{V} .*

Proof [dJE22]. Given $\alpha : I \rightarrow \text{Ord}_{\mathcal{V}}$, consider the image of the map $e : \Sigma_{i:I} \alpha_i \rightarrow \text{Ord}_{\mathcal{V}}$ given by $e(i, x) \equiv \alpha_i \downarrow x$. The image of e is equivalent to the type $\Sigma_{\gamma:\text{Ord}_{\mathcal{V}}} \exists_{i:I} \gamma \prec \alpha_i$, *i.e.* the type of ordinals that are initial segments of some α_i . One can prove that $\text{im}(e)$ with the induced order from $\text{Ord}_{\mathcal{V}}$ is again a well-order and that for every $i : I$, the canonical map $\alpha_i \rightarrow \text{im}(e)$ is a simulation. Moreover, if β is an ordinal such that $\alpha_i \preceq \beta$ for every $i : I$, then for every $i : I$ and every $x : \alpha_i$ there exists a unique $b_i^x : \beta$ such that $\alpha_i \downarrow x = \beta \downarrow b_i^x$. Now observe that for every $\gamma : \text{Ord}_{\mathcal{V}}$, the map $(\Sigma_{i:I} \Sigma_{x:\alpha_i} (\alpha_i \downarrow x = \gamma)) \rightarrow \beta$ defined by the assignment $(i, x, p) \mapsto b_i^x$ is a constant function to a set. Hence, by [KECA17, Theorem 5.4], this map factors through the propositional truncation $\exists_{i:I} \Sigma_{x:\alpha_i} (\alpha_i \downarrow x = \gamma)$. This yields a map $\text{im}(e) \rightarrow \beta$ which can be proved to be a simulation, as desired. Finally, we use set replacement and the fact that $\text{Ord}_{\mathcal{V}}$ is locally \mathcal{V} -small (by univalence) to get an ordinal in \mathcal{V} equivalent to $\text{im}(e)$, finishing the proof. \square

6. FAMILIES AND SUBSETS

In traditional impredicative foundations, completeness of posets is usually formulated using subsets. For instance, dcpos are defined as posets D such that every directed subset of D has a supremum in D . Examples 4.3 are all formulated using small families instead of subsets. While subsets are primitive in set theory, families are primitive in type theory, so this could be an argument for using families above. However, that still leaves the natural question of how the family-based definitions compare to the usual subset-based definitions, especially in our predicative setting, unanswered. This section addresses this question. We first study the relation between subsets and families predicatively and then clarify our definitions in the presence of impredicativity. In our answers we will consider sup-lattices, but similar arguments could be made for posets with other sorts of completeness, such as dcpos.

We first show that simply asking for completeness with respect to all subsets is not satisfactory from a predicative viewpoint. In fact, we will now see that even asking for completeness with respect to all elements of $\mathcal{P}_{\mathcal{T}}(X)$ for some fixed universe \mathcal{T} is problematic from a predicative standpoint, where we recall that $\mathcal{P}_{\mathcal{T}}(X) \equiv (X \rightarrow \Omega_{\mathcal{T}})$.

Definition 6.1 (\mathcal{T} -valued subsets). For a universe \mathcal{T} and a type X in any universe, we refer to the elements of $\mathcal{P}_{\mathcal{T}}(X)$ as \mathcal{T} -valued subsets of X .

Theorem 6.2. *Let \mathcal{U} and \mathcal{V} be universes, fix a proposition $P_{\mathcal{U}} : \mathcal{U}$ and recall $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ from Definition 5.1, which has \mathcal{V} -suprema. If $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ has suprema for all \mathcal{T} -valued subsets, then $P_{\mathcal{U}}$ is \mathcal{V} -small independently of the choice of the type universe \mathcal{T} .*

Proof. Let \mathcal{T} be a type universe and consider the subset S of $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ given by $Q \mapsto 1_{\mathcal{T}}$. Note that S has a supremum in $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ if and only if $\mathcal{L}_{\mathcal{V}}(P_{\mathcal{U}})$ has a greatest element, but by Lemma 5.3, the latter is equivalent to $P_{\mathcal{U}}$ being \mathcal{V} -small. \square

The proof above illustrates that if we have a subset $S : \mathcal{P}_{\mathcal{T}}(X)$, then there is no reason why the total space $\Sigma_{x:X} (x \in S)$ should be \mathcal{T} -small. In fact, for $S(x) \equiv 1_{\mathcal{T}}$ as above, the latter is equivalent to asking that X is \mathcal{T} -small.

Definition 6.3 (Total space of a subset, \mathbb{T}). The *total space* of a \mathcal{T} -valued subset S of a type X is defined as $\mathbb{T}(S) \equiv \Sigma_{x:X} (x \in S)$.

In an attempt to solve the problem described in Theorem 6.2, we look to impose size restrictions on the total space of a subset. There are two natural such restrictions and they are reminiscent of Bishop and Kuratowski finite subsets.

Definition 6.4 (\mathcal{V} -small and \mathcal{V} -covered subsets). An element $S : \mathcal{P}_{\mathcal{T}}(X)$ is

- (i) \mathcal{V} -small if its total space is \mathcal{V} -small, and
- (ii) \mathcal{V} -covered if we have $I : \mathcal{V}$ with a surjection $e : I \rightarrow \mathbb{T}(S)$.

Observe that every \mathcal{V} -small subset is \mathcal{V} -covered, because every equivalence is a surjection. But the converse does not hold: We can emulate the well-known argument used to show that, constructively, Kuratowski finiteness does not necessarily imply Bishop finiteness to show that, predicatively, being \mathcal{V} -covered does not necessarily imply being \mathcal{V} -small.

Proposition 6.5. *For every two universes \mathcal{U} and \mathcal{V} , if every \mathcal{V} -covered element of $\mathcal{P}_{\mathcal{U}}(\Omega_{\mathcal{U}})$ is \mathcal{V} -small, then Propositional-Resizing $_{\mathcal{U},\mathcal{V}}$ holds.*

Proof. Suppose that every \mathcal{V} -covered \mathcal{U} -valued subset of $\Omega_{\mathcal{U}}$ is \mathcal{V} -small and let $P : \mathcal{U}$ be an arbitrary proposition. Consider the subset $S_P : \Omega_{\mathcal{U}} \rightarrow \Omega_{\mathcal{U}}$ given by $S_P(Q) := (Q = P) \vee (Q = 1_{\mathcal{U}})$. Notice that this is \mathcal{V} -covered as witnessed by

$$\begin{aligned} (1_{\mathcal{V}} + 1_{\mathcal{V}}) &\rightarrow \mathbb{T}(S_P) \\ \text{inl}(\star) &\mapsto (P, |\text{inl}(\text{refl})|) \\ \text{inr}(\star) &\mapsto (1_{\mathcal{U}}, |\text{inr}(\text{refl})|), \end{aligned}$$

so by assumption $\mathbb{T}(S_P)$ is \mathcal{V} -small. But observe that P holds if and only if $\mathbb{T}(S_P)$ is a subsingleton, but the latter type is \mathcal{V} -small by assumption, hence so is P . \square

In the case where we restrict our attention to a single universe \mathcal{V} and a locally \mathcal{V} -small set X , the two notions coincide if and only if we have set replacement for maps into X with \mathcal{V} -small domain.

Proposition 6.6. *If X is locally \mathcal{V} -small set, then every \mathcal{V} -covered element of $\mathcal{P}_{\mathcal{V}}(X)$ is \mathcal{V} -small if and only if the image of any map into X with \mathcal{V} -small domain is \mathcal{V} -small.*

Proof. Suppose first that every \mathcal{V} -covered subset $S : X \rightarrow \Omega_{\mathcal{V}}$ is \mathcal{V} -small and let $f : I \rightarrow X$ be map such that I is \mathcal{V} -small. Without loss of generality, we may assume that $I : \mathcal{V}$, because we can always precompose f with the equivalence witnessing that I is \mathcal{V} -small. Now consider the subset $S : X \rightarrow \Omega_{\mathcal{V}}$ given by $S(x) := \exists_{i:I}(f(i) =_{\mathcal{V}} x)$, where $=_{\mathcal{V}}$ has values in \mathcal{V} and is provided by our assumption that X is locally \mathcal{V} -small. Then S is \mathcal{V} -covered, because we have $I \rightarrow \text{im}(f) \simeq \mathbb{T}(S)$, where the first map is the corestriction of f . So by assumption $\mathbb{T}(S)$ is \mathcal{V} -small, which means that $\text{im}(f)$ must be \mathcal{V} -small too.

Conversely, assume the set replacement principle and let $S : X \rightarrow \Omega_{\mathcal{V}}$ be \mathcal{V} -covered by $e : I \rightarrow \mathbb{T}(S)$. Define the subset $S' : X \rightarrow \Omega_{\mathcal{V}}$ by $S'(x) := \exists_{i:I}(x =_{\mathcal{V}} \text{pr}_1(e_i))$. By the assumed set replacement principle for X , the subset S' is a \mathcal{V} -small since $\mathbb{T}(S') \simeq \text{im}(\text{pr}_1 \circ e)$. Finally, it follows from the surjectivity of e that S and S' are equal as subsets, and therefore that $\mathbb{T}(S) \simeq \mathbb{T}(S')$. Hence, S is a \mathcal{V} -small subset, as desired. \square

So, predicatively, and in the absence of a set replacement principle, the notion of a \mathcal{V} -small subset is strictly stronger than that of a \mathcal{V} -covered subset. Hence, in this setting, having suprema for all \mathcal{V} -small subsets is strictly weaker than having suprema for all \mathcal{V} -covered subsets. Meanwhile, Corollary 6.8 will imply that there are plenty of examples of

posets with suprema for all \mathcal{V} -covered subsets, even predicatively. So we prefer the stronger, but predicatively reasonable requirement of asking for suprema of all \mathcal{V} -covered subsets.

Form a practical viewpoint, \mathcal{V} -covered subsets also give us an easy handle on examples like the following: Let X be a poset with suprema for all (directed) \mathcal{U}_0 -covered subsets. Then the least fixed point of a Scott continuous endofunction f on X can be computed as the supremum of the subset $\{\perp, f(\perp), f^2(\perp), \dots\}$, which is covered by \mathbb{N} . But it is not clear that this subset is \mathcal{U}_0 -small, at least not in the absence of set replacement.

Our preference for \mathcal{V} -covered subsets over \mathcal{V} -small subsets also makes it clear why we do not impose an injectivity condition on families, because for every type $X : \mathcal{U}$ there is an equivalence between embeddings $I \hookrightarrow X$ with $I : \mathcal{V}$ and $(\mathcal{U} \sqcup \mathcal{V})$ -valued subsets of X whose total spaces are \mathcal{V} -small, cf. [E⁺22, `Slice.Slice`].

Theorem 6.7. *For $X : \mathcal{U}$ and any universe \mathcal{V} we have an equivalence between \mathcal{V} -covered $(\mathcal{U} \sqcup \mathcal{V})$ -valued subsets of X and families $I \rightarrow X$ with $I : \mathcal{V}$.*

Proof. The forward map φ is given by $(S, I, e) \mapsto (I, \text{pr}_1 \circ e)$. In the other direction, we define ψ by mapping (I, α) to the triple (S, I, e) where S is the subset of X given by $S(x) \equiv \exists_{i:I} (x = \alpha(i))$ and $e : I \rightarrow \mathbb{T}(S)$ is defined as $e(i) \equiv (\alpha(i), |(i, \text{refl})|)$. The composite $\varphi \circ \psi$ is easily seen to be equal to the identity. To show that $\psi \circ \varphi$ equals the identity, we need the following intermediate result, which is proved using function extensionality and path induction.

Claim. *Let $S, S' : X \rightarrow \Omega_{\mathcal{U} \sqcup \mathcal{V}}$, $e : I \rightarrow \mathbb{T}(S)$ and $e' : I \rightarrow \mathbb{T}(S')$. If $S = S'$ and $\text{pr}_1 \circ e \sim \text{pr}_1 \circ e'$, then $(S, e) = (S', e')$.*

The result follows from the claim using function and propositional extensionality. \square

Corollary 6.8. *A poset with carrier in \mathcal{U} has suprema for all \mathcal{V} -covered $(\mathcal{U} \sqcup \mathcal{V})$ -valued subsets if and only if it has suprema for all families indexed by types in \mathcal{V} .*

Proof. This is because the supremum of a \mathcal{V} -covered subset equals the supremum of the corresponding family and vice versa by inspecting the proof of Theorem 6.7. \square

We conclude by comparing our family-based approach to the subset-based approach in the presence of impredicativity.

Theorem 6.9. *Assuming $\Omega\text{-Resizing}_{\mathcal{T}, \mathcal{U}_0}$ for every universe \mathcal{T} , the following are equivalent for a poset with carrier in a universe \mathcal{U} :*

- (i) *the poset has suprema for all subsets;*
- (ii) *the poset has suprema for all \mathcal{U} -covered subsets;*
- (iii) *the poset has suprema for all \mathcal{U} -small subsets;*
- (iv) *the poset has suprema for all families indexed by types in \mathcal{U} .*

Proof. Clearly (i) \Rightarrow (ii) \Rightarrow (iii). We show that (iii) implies (i), which proves the equivalence of (i)–(iii). Assume that a poset X has suprema for all \mathcal{U} -small subsets and let $S : X \rightarrow \Omega_{\mathcal{T}}$ be any subset of X . Using $\Omega\text{-Resizing}_{\mathcal{T}, \mathcal{U}_0}$, the total space $\mathbb{T}(S)$ is \mathcal{U} -small. So X has a supremum for S by assumption, as desired. Finally, (ii) and (iv) are equivalent in the presence of $\Omega\text{-Resizing}_{\mathcal{T}, \mathcal{U}_0}$ by Corollary 6.8. \square

If condition (iv) of Theorem 6.9 holds, then the poset has suprema for all families indexed by types in \mathcal{V} provided that $\mathcal{V} \sqcup \mathcal{U} \equiv \mathcal{U}$. Typically, in the examples of [dJE21a] for instance,

$\mathcal{U} := \mathcal{U}_1$ and $\mathcal{V} := \mathcal{U}_0$, so that $\mathcal{V} \sqcup \mathcal{U} \equiv \mathcal{U}$ holds. Thus, our \mathcal{V} -families-based approach generalizes the traditional subset-based approach.

7. CONCLUSION

Firstly, we have shown, constructively and predicatively, that nontrivial dcpos, bounded complete posets and sup-lattices are all necessarily large and necessarily lack decidable equality. We did so by deriving a weak impredicativity axiom or weak excluded middle from the assumption that such nontrivial structures are small or have decidable equality, respectively. Strengthening nontriviality to the (classically equivalent) positivity condition, we derived a strong impredicativity axiom and full excluded middle.

Secondly, we showed that Tarski's greatest fixed point theorem cannot be applied in nontrivial instances in our predicative setting, while generalizations of Tarski's theorem that allow for large structures are provably false. Specifically, we showed that the ordinal of ordinals in a univalent universe does not have a maximal element, but does have small suprema in the presence of small set quotients, or equivalently, set replacement. More generally, we investigated the inter-definability and interaction of type universes of propositional truncations and set quotients in the absence of propositional resizing axioms. In particular, we showed that in the presence of propositional truncations, but without assuming propositional resizing, it is possible to construct set quotients that happen to live in higher type universes but that do satisfy the appropriate universal properties with respect to sets in arbitrary type universes.

Finally, we clarified, in our predicative setting, the relation between the traditional definition of a lattice that requires completeness with respect to subsets and our definition that asks for completeness with respect to small families.

In future work, it would be interesting to study the predicative validity of Pataraia's theorem and Tarski's *least* fixed point theorem. Curi [Cur15, Cur18] develops predicative versions of Tarski's fixed point theorem in some extensions of CZF. It is not clear whether these arguments could be adapted to univalent foundations, because they rely on the set-theoretical principles discussed in the introduction. The availability of such fixed-point theorems might be useful for application to inductive sets [Acz77], which we might otherwise introduce in univalent foundations using higher inductive types [Uni13]. In another direction, we have developed a notion of apartness [BV11] for continuous dcpos [GHK⁺03] that is related to the strictly-below relation introduced in this paper. Namely, if $x \sqsubseteq y$ are elements of a continuous dcpo, then x is strictly below y if x is apart from y . In [dJ21], we give a constructive analysis of the Scott topology [GHK⁺03] using this notion of apartness.

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