# Exploring a result by Ghilardi

van der Giessen, Iris



*Citation for published version (Harvard):*
van der Giessen, I 2021, Exploring a result by Ghilardi: projective formulas vs. the extension property. in M Blicha & I Sedlar (eds), *The Logica Yearbook 2020.* The Logica Yearbook, College Publications, pp. 97-114. <https://www.collegepublications.co.uk/logica/?00034>

# Exploring a Result by Ghilardi: Projective Formulas vs. the Extension Property

IRIS VAN DER GIESSEN[1]

**Abstract:** Ghilardi (1999, 2000) presents in his papers on unification in IPC and several classical modal logics an important theorem that is used in the study of admissible rules. The theorem connects the extension property of Kripke models to projective formulas. In this paper, we investigate Ghilardi's bisimulation proof method used for classical modal logics and we present a small simplification of the solution. Our investigation of the key elements of Ghilardi's proof provides an explanation of the close relationship between bisimulation and the extension property via so-called extension structures.

**Keywords:** modal logic, extension property, projective formulas, admissible rules

## 1 Introduction

In this paper we examine an important result established by Ghilardi that provides a robust connection between *projective formulas* and the *extension property*. This is a characterization of a syntactic property of formulas in terms of a semantic property of Kripke models. Ghilardi first develops this characterization for IPC (Ghilardi, 1999, Theorem 5 of Section 2). Later he proves it for many well-known classical modal logics extending K4, among them S4 and GL (Ghilardi, 2000, Theorem 2.2). He used this result in the study of unification in logic. Ghilardi shows that unification in IPC and several modal logics is finitary, which means that the set of 'maximal' unifiers is finite.

The purpose of this paper is to provide a new explanation of Ghilardi's proof of the connection between projectivity and the extension property. We hope that our investigation will help researchers who are not familiar

[1]I would like to thank Rosalie Iemhoff, Amir Tabatabai and Raheleh Jalali for a lot of helpful discussions. Support by the Netherlands Organisation for Scientific Research under grant 639.073.807 is gratefully acknowledged.

with the result. In his first paper, Ghilardi provides a strategy that works for IPC. Here we are interested in the more general method that is based on bisimulation which is used for classical modal logic. Among researchers, the paper is considered to be both beautiful and difficult. The current paper is an attempt to clarify the ideas of Ghilardi. We indicate the key elements of Ghilardi's proof which provides an explanation of the close relationship between bisimulation and the extension property. We introduce so-called *extension structures* to explain this relationship.

Surprisingly, our analysis reveals an additional benefit in terms of a shortening of the solution. To prove projectivity from the extension property, Ghilardi constructs a unifier that is a concatenation of substitutions. We will argue that in the classical modal case, the concatenation of substitutions can be shortened. We would like to stress that this is a minor simplification and is still strongly based on Ghilardi's proof strategy. This paper does not reveal new big results, it rather provides a new explanation of Ghilardi's proof and it provides some examples, which will hopefully help the reader to understand the result.

An important reason to investigate Ghilardi's papers is that the equivalence between projectivity and the extension property is a very useful tool in the field of admissible rules. Admissible rules are those rules under which the set of theorems of a logic is closed. More precisely, a rule $A/B$ is said to be admissible if every unifier of $A$ is also a unifier for $B$. For example, the rule $\Box A/A$ is an admissible rule in many modal logics. Admissible rules are interesting to study, because they give insight in the structure of the logic in terms of consequence relations (see Iemhoff, 2016, for an introduction).

Projective formulas play an important role in the study of admissible rules. Projective formulas are formulas $A$ for which admissibility and derivability are the same in the sense that rule $A/B$ is admissible if and only if it is derivable. It can be complicated to show that a certain formula is projective directly from its definition. It is often easier to prove the extension property of a class of Kripke models. Ghilardi's result provides the useful semantic characterization for projectivity in terms of the extension property.

Ghilardi's result is successfully applied in constructing bases for admissible rules. See for examples the papers of Iemhoff (2001), Jeřábek (2005) and Iemhoff and Metcalfe (2009) for logics including K4, GL, S4 and IPC. In addition, Ghilardi points out that an algorithm (provided in his papers) computing the finitely many maximal unifiers yields a new solution to Friedman's problem (Friedman, 1975): admissibility in IPC is decidable. This was first proved by Rybakov (1984).

The paper is structured as follows. In Section 2 we introduce semantic terminology, including the definitions of the extension property and bisimulation. Section 3 treats the definition of projective formulas. Section 4 describes the key elements in Ghilardi's method, introduces the notion of extension structures and explains how the substitutions can be shortened. We end with a short conclusion.

## 2 Kripke semantics and the extension property

Following Ghilardi (2000), we consider classical modal logics that are sound and complete with regard to finite Kripke models. We consider the modal language with constant $\bot$, propositional variables $p, q, \ldots$, connectives $\wedge, \vee, \to$ and modal operator $\Box$. We often use the term *atoms* to mean propositional variables. If $A$ is a formula, $\neg A$, $\Diamond A$ and $\boxdot A$ are defined as $A \to \bot$, $\neg\Box\neg A$ and $A \wedge \Box A$, respectively. $F(p_1, \ldots, p_m)$ denotes the set of all formulas built from proposition letters $p_1, \ldots, p_m$. We consider *normal modal logics* $L$, which is a set of formulas containing all classical tautologies, $K$-axiom $\Box(A \to B) \to \Box A \to \Box B$, and is closed under modus ponens (if $A$ in $L$ and $A \to B$ in $L$, then $B$ in $L$), uniform substitution and necessitation (if $A$ in $L$ then also $\Box A$ in $L$). Following Ghilardi, we write $A \vdash_L B$ to mean that $A \to B \in L$. We are interested in normal extensions of $\mathsf{K4}$.

We deal with Kripke models that are defined on the basis of a finite transitive *frame*, which are structures $(W, R)$ where $W$ is a finite set of worlds equipped with a *transitive* relation $R$. We assume that those frames have a minimal element $\rho$, called a *root*, satisfying $\rho R w$ for each $w \neq \rho$. This minimal element does not have to be unique, but from now on we work with pointed frames $(W, R, \rho)$ where the root is specifically specified. The *cluster* $cl(w)$ of a point $w$ is the equivalence class of $w$ under the equivalence relation $\sim_R$ defined as

$$w \sim_R v \text{ iff } wR^+v \text{ and } vR^+w,$$

where $R^+$ stands for the relation $R \cup id$. Note that for irreflexive frames, $\#cl(w) = 1$ for all $w \in W$. We also define the relation

$$wR^> v \text{ iff } wRv \text{ and not } vRw.$$

A *Kripke model* is a triple $(W, R, V)$ where $(W, R)$ is a frame and $V$ is the *valuation*, which is a function $V : W \times \mathit{Atoms} \to \{0, 1\}$. We use letters

$K, M$ to indicate Kripke models. We usually implicitly restrict the domain of the valuation to atoms which play a role in question and say that $K$ is a model *over atoms* $\{p_1, \ldots, p_m\}$ if the domain is restricted to those atoms. We often write $w \in K$ to mean $w \in W$ when $K = (W, R, V)$. We write $K(w) := \{p \mid V(w,p) = 1\}$, the set of all atoms that hold in $w$. We extend the valuation to a forcing relation $\Vdash$ as usual:

$$
\begin{array}{ll}
K, w \Vdash p & \text{iff } V(w,p) = 1, \\
K, w \Vdash \bot & \text{never}, \\
K, w \Vdash A \wedge B & \text{iff } K, w \Vdash A \text{ and } K, w \Vdash B, \\
K, w \Vdash A \vee B & \text{iff } K, w \Vdash A \text{ or } K, w \Vdash B, \\
K, w \Vdash A \to B & \text{iff } K, w \Vdash A \text{ implies } K, w \Vdash B, \\
K, w \Vdash \Box A & \text{iff for all } v \text{ such that } wRv;\ K, v \Vdash A.
\end{array}
$$

We write $K \models A$ to mean $K, w \Vdash A$ for every $w \in K$ and say that $K$ *satisfies* $A$. Since we consider rooted transitive models, $K, \rho \Vdash \boxdot A$ if and only if $K \models \boxdot A$. We denote $K_v$ for the submodel of $K$ generated by $v$. We let $v$ be the root of $K_v$. We say that model $K$ *almost satisfies* $A$ if $K_w \models A$ for all $w$ except for $w \in cl(\rho)$.

A frame is said to be an *$L$-frame* if for every model $K$ based on that frame and every formula $A \in L$, we have that $K \models A$. We call $K$ an *$L$-model* if $K$ is based on an $L$-frame. We write $Mod_L$ to be the set of all $L$-models and $Mod_L(A)$ to be the set of all $L$-models that satisfy $A$ in the root.

Ghilardi makes two assumptions about the logic $L$. For the purpose of this paper we only have to require the first assumption, which is completeness with respect to the described finite models:

**Assumption 1** *For all formulas $A, B$, we have $A \vdash_L B$ if and only if $Mod_L(A) \subseteq Mod_L(B)$.*

Ghilardi's second assumption is needed for the unification results (Ghilardi, 1999, 2000). This assumption is known as $L$ being *extensible*, where the construction of attaching a new root to $L$-frames again yields an $L$-frame. This assumption is also crucial in the field of admissible rules (Iemhoff & Metcalfe, 2009; Jeřábek, 2005). Examples of logics satisfying these assumptions are K4, S4, GL and S4.Grz.

A *variant* of an $L$-model $K$ is an $L$-model $K'$, such that they have the same frame and their valuation agree on all worlds except for possibly worlds $w \in cl(\rho)$.

**Definition 1** *A class $\mathcal{K}$ of L-models over $\{p_1, \ldots, p_n\}$ is said to have the* extension property *if for every L-model $K$, if $K_w \in \mathcal{K}$ for each $w \notin cl(\rho)$, then there is a variant $K'$ of $K$ such that $K' \in \mathcal{K}$.*

We are interested in the extension property of classes $Mod_L(\boxdot A)$. The extension property states that we can turn models that almost satisfy $\boxdot A$ into a model of $\boxdot A$. In Section 4, we will see that there is a close relationship between the extension property and bisimulation. Here we only give the definitions.

**Definition 2** *Let $K, M$ be Kripke models. The notion for two models $K, M$ together with points $k \in K$ and $m \in M$ being $n$-*bisimilar *is defined recursively and we denote it by $K_k \sim_n M_m$.*

| | | |
|---|---|---|
| $K_k \sim_0 M_m$ | *iff* | $K(k) = M(m)$ *($k$ and $m$ satisfy the same atoms).* |
| $K_k \sim_{n+1} M_m$ | *iff* | $K(k) = M(m)$ *and for all $k'$ such that $kRk'$ there exists an $m'$ such that $mRm'$ and $K_{k'} \sim_n M_{m'}$, and vice versa.* |

Note that $K_k \sim_l M_m$ implies $K_k \sim_n M_m$ for all $l \geq n$. For each $n$, $\sim_n$ is an equivalence relation. We denote the equivalence classes by $[K_k]_n$. For fixed $n$, the number of equivalence classes is bounded.

**Proposition 1** *Consider models over $\{p_1, \ldots, p_m\}$. Define $N(0) := 2^m$ and $N(n+1) := 2^{N(n)+m}$. The number $N$ of possible $\sim_n$ equivalence classes is smaller or equal to $N(n)$.*

*Proof.* See (Visser, 1996). □

The *modal degree* $d(A)$ of a formula $A$ is defined inductively as follows: $d(\bot) = d(p) = 0$, for atoms $p$, $d(A_1 \wedge A_2) = d(A_1 \vee A_2) = d(A_1 \to A_2) = \max\{d(A_1), d(A_2)\}$ and $d(\Box A) = d(A) + 1$. The relation between bisimilar models and modal degree is explained in the following theorem.

**Theorem 1** *Let $K, M$ be models over $\{p_1, \ldots, p_m\}$. We have $K_k \sim_n M_m$ if and only if for each formula $B$ with atoms in $\{p_1, \ldots, p_m\}$ with $d(B) \leq n$ we have $K, k \Vdash B \Leftrightarrow M, m \Vdash B$.*

*Proof.* See (Ghilardi, 2000). □

## 3 Projective formulas

In this section we introduce substitutions and the notion of projective formula. This can also be read in (Ghilardi, 2000), but we use other notation. A *substitution* is a function $\sigma : \{p_1, \ldots, p_m\} \to F(q_1, \ldots, q_l)$. This function can be extended to a function with domain $F(p_1, \ldots, p_m)$ by

$$\sigma(A(p_1, \ldots, p_m)) = A(\sigma(x_1)/x_1, \ldots, \sigma(x_m)/x_m).$$

From now on, we identify $\sigma$ with this extension. The composition of $\sigma : F(p_1, \ldots, p_m) \to F(q_1, \ldots, q_l)$ and $\tau : F(q_1, \ldots, q_l) \to F(r_1, \ldots, r_k)$ is defined by $\tau\sigma(p) = \tau(\sigma(p))$.

A *unifier* for a formula $A$ built from atoms $p_1, \ldots, p_m$ is a substitution $\sigma : F(p_1, \ldots, p_m) \to F(q_1, \ldots, q_l)$ such that

$$\vdash_L \sigma(A).$$

We are only interested in unifiers where domain and codomain are the same.

**Definition 3** *A formula of the form $\boxdot A$ with proposition letters $p_1, \ldots, p_m$ is* projective in $L$ *if there exists a unifier $\sigma$ for it such that*

$$\boxdot A \vdash_L p \leftrightarrow \sigma(p) \tag{1}$$

*for all proposition letters $p_i$. We call $\sigma$ a* projective unifier.

Using the substitution axiom, it is easy to prove that condition (1) is equivalent to

$$\boxdot A \vdash_L B \leftrightarrow \sigma(B) \tag{2}$$

for all formulas $B$ in proposition letters $\{p_1, \ldots, p_m\}$.

Ghilardi builds suitable substitutions adopting this property in the following way. Let $\{p_1, \ldots, p_m\}$ be the atoms occurring in $A$. Let $a$ be a subset of those atoms; the substitution $\sigma_a : F(p_1, \ldots, p_m) \to F(p_1, \ldots, p_m)$ is defined as:

$$\sigma_a^{\boxdot A}(p) = \begin{cases} \boxdot A \to p & \text{if } p \in a, \\ \boxdot A \wedge p & \text{if } p \notin a. \end{cases}$$

From now, we omit the superscript and just write $\sigma_a$ when $\boxdot A$ is clear from the context. It is easy to see that $\boxdot A \vdash_L \sigma_a(p) \leftrightarrow p$. We sometimes call those substitutions *simple*. Ghilardi defines substitution $\theta := \sigma_{a_1} \cdots \sigma_{a_s}$ where $a_1, \ldots, a_s$ is any fixed ordering on the subsets of $\{p_1, \ldots, p_m\}$. Since the simple substitutions are closed under condition (2), we know that $\theta$ also satisfies condition (2).

## 4 Connecting projectivity to the extension property

In this section we investigate the important theorem that connects projectivity to the extension property. Recall that $L$ is a logic extending $\mathsf{K4}$ that satisfies the finite model property (Assumption 1).

**Theorem 2** (Ghilardi, 2000) *Formula $\boxdot A$ is projective in $L$ if and only if $Mod_L(\boxdot A)$ has the extension property.*

We are interested in the difficult direction of this theorem, which is from right to left. For a proof for the other direction we refer to (Ghilardi, 2000). We give an analysis of Ghilardi's proof and we will identify key elements of his method.

We fix some notation that we use in the rest of the paper. Let $\boxdot A$ be a formula with atoms from $\{p_1, \ldots, p_m\}$. Assume that $Mod_L(\boxdot A)$ has the extension property. Suppose that $d(A) \leq n$. Let $N$ be the number of different equivalence classes of $n$-bisimilar models and let $N'$ be the number for $(n-1)$-bisimilar models. The goal is to prove that $\boxdot A$ is projective.

In short, Ghilardi proves that $\theta^{2N}$ is a projective unifier for $\boxdot A$. Number $N$ belongs to $n$-bisimilar equivalence classes, but we will show that it suffices to use $(n-1)$-bisimilar classes. Number $N'$ is smaller than $N$, so this results in the shorter concatenation $\theta^{2N'}$. If we carefully read the proof of Theorem 3, we actually conclude that $\theta^{2(N'+1)}$ is the projective unifier for $\boxdot A$.

The first ingredient in the proof makes a bridge between substitutions in syntax and semantic operations in models. Ghilardi gives the following definition of the semantic operator $\sigma^*$ on models based on substitution $\sigma$:

$$\sigma^*(K), w \Vdash p \iff K, w \Vdash \sigma(p).$$

Note that $\sigma^*$ only changes the valuation in the model. From now on we abuse terminology and call $\sigma^*$ a substitution on models. This is a first step to connect the extension property to projectivity because the first is a property of semantics and the latter of syntax. We give some properties of $\sigma^*$.

**Lemma 1** *Let $A$ be a formula and let $\sigma$ be a substitution. For every Kripke model $K$, we have*

*(i)* $\sigma^*(K) \models A$ *iff* $K \models \sigma(A)$,

*(ii)* *and for every substitution* $\tau$, $(\tau\sigma)^*(K) = \sigma^*(\tau^*(K))$.

Point (ii) shows that the order of substitutions $\sigma$ and $\tau$ reverses.

Ghilardi defines the useful substitutions $\sigma_a$, which we already defined in Section 3. We already saw that they are closed under condition (2), a key condition for $\boxdot A$ being projective. Now we only have to search for a suitable combination of those $\sigma_a$'s and prove that this is a unifier for $\boxdot A$. However, finding the right concatenation is the hard part of the proof.

The extension property will guide us in the right direction for finding the correct concatenation of $\sigma_a$'s. The method consists of several steps. We start with two relatively simple lemmas. For proofs see (Ghilardi, 2000, Lemma 2.1 and 2.3).

**Lemma 2** *Let $\boxdot A$ be a formula with atoms in $\{p_1, \ldots, p_m\}$ and let $K$ be an L-model. Suppose $a \subseteq \{p_1, \ldots, p_m\}$. We have*

*(i)* $(\sigma_a^*(K))(w) = K(w)$ *if* $K_w \models \boxdot A$,

*(ii)* $(\sigma_a^*(K))(w) = a$ *if* $K_w \not\models \boxdot A$, *and*

*(iii)* $\sigma_a^* \sigma_a^* = \sigma_a^*$.

In words, the first two points of the lemma say that the atoms forced in a world $w$ stay the same (in case $K_w \models \boxdot A$), or become exactly the atoms in $a$ (in case $K_w \not\models \boxdot A$).

**Lemma 3** *Let $\boxdot A$ be a formula with atoms in $\{p_1, \ldots, p_m\}$ and suppose that $Mod_L(\boxdot A)$ has the extension property. Let $K$ be a model that almost satisfies $\boxdot A$. Then there is a set $a \subseteq \{p_1, \ldots, p_m\}$ such that $\sigma_a^*(K) \models \boxdot A$.*

We combine the ingredients so far and sketch a proof idea to find a unifier for $\boxdot A$. We will see that this naive idea is not sufficient and that we need more. For simplicity, one can think of models without any clusters. We want to find a unifier $\theta$ that is a concatenation of $\sigma_a$'s. In other words, we want to show that $\vdash_L \theta(\boxdot A)$. Using the completeness theorem and Lemma 1, we want to show that $\theta^*(K) \models \boxdot A$ for each $L$-model $K$. Let $K$ be an $L$-model. We start at the leafs of the model and work our way down to the root. In each step we want to apply a $\sigma_a$ that gives us a model in which more nodes validate $\boxdot A$. Consider a world $w$ that almost satisfies $\boxdot A$, i.e., $K, w \nVdash \boxdot A$ and $K, v \Vdash \boxdot A$ for all $wR^{>}v$. By Lemma 3 there is a valuation $a$ such that $w$ satisfies the atoms from $a$ and $\sigma_a^*(K), w \Vdash \boxdot A$. We pick $\sigma_a$ and apply it to our model. This strategy sounds promising, because we can go through all the nodes and apply a substitution that works for that node. Define $\theta$ on the basis of all those substitutions to yield $\theta^*(K) \models \boxdot A$. The big problem is that the definition of our $\theta$ depends on $K$, so we cannot define a good sequence of $\sigma$'s that works *for all* models $K$. Doing induction on the depth of the model will not solve the problem, because the depth is not bounded.

The key idea is to connect the extension property to bisimulation of models. Ghilardi defines $\theta := \sigma_{a_1} \cdots \sigma_{a_s}$ where $a_1, \ldots, a_s$ is any fixed ordering on the subsets of $\{p_1, \ldots, p_m\}$. He shows that $\theta^{2N}$ is a projective unifier, where $N$ is the number of the different $n$-bisimilar models. Ghilardi defines four important ingredients: frontier points, a rank, homogeneous models and the minimal rank (the last is our terminology).

- $f_K[\boxdot A] := \{w \in K \mid K_w \not\models \boxdot A$ and $\forall v(wR^{>}v \Rightarrow K_v \models \boxdot A)\}$ is the set of *frontier points*.
- The *rank* of a model $K$ is defined as

$$r(K) := \#\{[K_w]_n \mid \rho R w \text{ and } K_w \models \boxdot A\}.$$

- Model $K$ is *homogeneous* if $r(K_w) = r(K_v)$ for each $w, v$ with $K, w \nVdash \boxdot A$ and $K, v \nVdash \boxdot A$.
- $\mu(K) := \min\{r(K_w) \mid K_w \not\models \boxdot A\}$, which we call the *minimal rank*.

Frontier points are the points $w$ such that $K_w$ almost satisfies $\boxdot A$. As observed above, for each frontier point we can use the extension property (Lemma 3) to find a $\sigma_a$ such that $\boxdot A$ becomes true in that frontier point. For different frontier points there can be different $\sigma_a$'s that work. However, after one application of $\theta$, all frontier points are turned into points that satisfy $\boxdot A$. The next step is to find the new frontier points and apply $\theta$ again. Ghilardi shows that after two applications of $\theta$, the minimal rank grows strictly. One $\theta$ covers irreflexive nodes and the other $\theta$ reflexive nodes. The minimal rank is bounded by $N$, therefore $K \models \theta^{2N}(\boxdot A)$ for all models $K$.

Figure 1 sketches the idea of the frontier points in a model. Each curved line represents the set of frontier points, which lowers after two applications of $\theta^*$. There are at most $N$ steps of $\theta^*\theta^*$ in the picture.

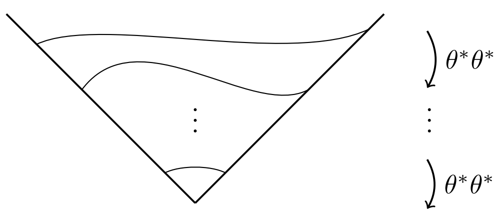



Figure 1: Lines of frontier points.

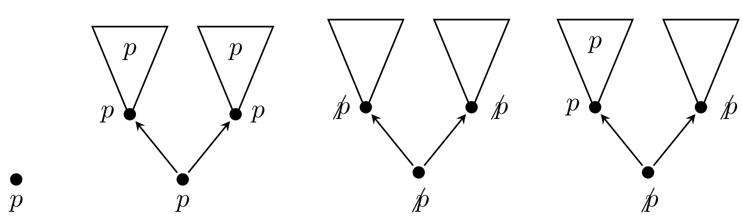
We keep the same idea in mind, but we propose to change the definition of the rank and give another approach for the homogeneous models. Those elements are highly based on Ghilardi's method. With our investigation, we want to address the important role of the frontier points and the link between bisimilar models and the extension property. The idea is to identify different so-called *extension structures* in the extension property of $Mod(\boxdot A)$. Those extension structures are identified using bisimulation. In turn, each extension structure will correspond to a simple substitution $\sigma_a$ which are again the building blocks for $\theta$. We will see that $2(N'+1)$ applications of $\theta$ is enough, where again $N'$ is the number of different $(n-1)$-bisimulation equivalence classes.

Before we explore the new method, we give some examples to see that in many cases a short substitution suffices to act as a projective unifier for $\boxdot A$ and that this depends on the nature of the extension property of $Mod(\boxdot A)$.

**Example 1** Let $A$ be of the form $p \to B$ for some formula $B$ and atom $p$. Formula $\boxdot A$ has the extension property, because for each model $K$ that almost satisfies $\boxdot A$, we can find a variant $K'$ in which no atom is forced in the root. This works independently of the shape of $K$. So $K' \models \boxdot A$. This means that $\sigma_\emptyset^*(K) \models \boxdot A$ for each $K$, so $\sigma_\emptyset$ is a projective unifier of $\boxdot A$.

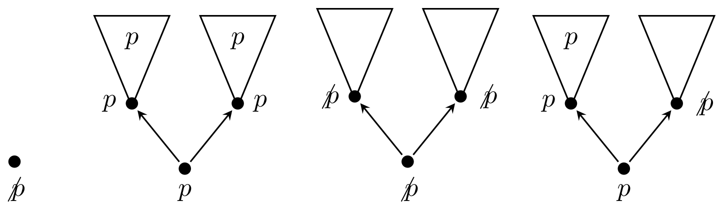
Also for box-free formulas with the extension property, one $\sigma$ suffices as projective unifier. In general, if the extension property does not depend on the models above the root, one $\sigma$ suffices.

**Example 2** Consider formula $A = (\Box p \to p) \wedge (p \to \Box p)$. For simplicity, we work with tree-like models. There are multiple cases of the extension property. If all nodes above the root satisfy $p$, extend it with a node where $p$ also holds. This is illustrated below in the first two pictures, where in the first picture there are no submodels above the root. Note that $A$ is true for each reflexive leaf in the tree. If there is at least one node in which $p$ does not hold, extend the models with a node where $p$ does not hold, illustrated in the last two models.



We want to know which sequence of $\sigma$'s turns each model into a model that satisfies $\boxdot A$. Let $K$ be a model. We can first apply $\sigma_p^*$ that belongs to the left two pictures. By Lemma 2, if $K_w \models \boxdot A$, then the atoms forced in $w$ in model $\sigma_p^*(K)$ stay the same, and if $K_w \not\models \boxdot A$, then the only atom forced in $w$ is $p$. Moreover, for each world $w$ in $\sigma_p^*(K)$ such that $\sigma_p^*(K_w) \not\models \boxdot A$ we have that there is at least one node $v$ above $w$ such that $\sigma_p^*(K_v) \not\models p$. So all these nodes belong to the third or fourth picture. Now we can take $\sigma_\emptyset^*$ to conclude $\sigma_\emptyset^* \sigma_p^*(K) \models \boxdot A$. Hence, $\sigma_p \sigma_\emptyset$ is a projective unifier for $\boxdot A$.

**Example 3** Formula $B = (\Box\neg p \to \neg p) \wedge (\neg p \to \Box\neg p)$ is the substitution instance of $A$ from the previous example where $\neg p$ is substituted for $p$. The corresponding extensions are now as follows:



Now we see that $\sigma_p^* \sigma_\emptyset^*$ turns each model in a model that satisfies $\boxdot B$. Therefore $\sigma_\emptyset \sigma_p$ is a projective unifier for $\boxdot B$. Note that here the $\sigma$'s depend on $B$, so now $\sigma_p$ means $\sigma_p^{\boxdot B}$.

The examples illustrate that the set $a$ of atoms forced in the extended root depends on the structure of the models above it. In addition, we distinguish between the extended root being reflexive or irreflexive. This results in different *extension structures* defined in Definition 4.

We will formalise our method. Recall that we work with formula $\boxdot A$ with atoms from $\{p_1, \ldots, p_m\}$ and $d(A) \leq n$. Let $N'$ be the number of $(n-1)$-bisimilar equivalent models. Let us introduce our ingredients.

- We keep the same notion of *frontier points*.
- Define the *bisimulation set of* $K$ as

$$B(K) = \{[K_w]_{n-1} \mid \rho R w \text{ and } K_w \models \boxdot A\}.$$

- The *rank* $r(K)$ is the cardinality of $B(K)$.
- We call a frontier point $w$ *B-minimal* in $K$, if $r(K_w) \leq r(K_v)$, for all other frontier points $v$ in $K$.

The bisimulation set of $K$ is a subset of the set of all equivalence classes of $(n-1)$-bisimilar models that satisfy $\boxdot A$. Because we work with transitive models, we have the following important fact: $B(K) \subseteq B(\sigma_a^*(K))$ for every $a \subseteq \{p_1, \ldots, p_m\}$. And so $r(K) \leq r(\sigma_a^*(K))$. The rank is bounded by $N'$.

**Example 4** Consider Example 2 and Example 3. The box-depth of formulas $A$ and $B$ is 1. So the different bisimulation sets depend on 0-bisimulation. Therefore we have to look at the atoms that are forced in the nodes above the root. There are four bisimulation sets which we write as $\emptyset$, $\{p\}$, $\{\not p\}$ and $\{p, \not p\}$. They correspond from left to right to the pictures in the examples.

The applied substitution in the examples is the same for reflexive and irreflexive roots, but this is not the case in general.

**Definition 4** *Let $K$ be an $L$-model almost satisfying $\boxdot A$ and suppose that $Mod_L(\boxdot A)$ has the extension property. The* extension structure *of $K$ (with respect to $\boxdot A$) is the pair $(B(K), \cdot)$ of its bisimulation set and $\cdot = \mathrm{i}$ if the root of $K$ is irreflexive and $\cdot = \mathrm{r}$ if the root is reflexive.*

Each bisimulation set may define two extension structures, depending on the (ir)reflexivity of the root. The following lemma shows that the same substitutions work for models with the same extension structure. We will see that each extension structure gives rise to a *corresponding substitution*.

**Lemma 4** *Let $K_1, K_2$ be two models that almost satisfy $\boxdot A$. Assume that they have the same extension structure. Then for each $a$, $\sigma_a^*(K_1) \models \boxdot A$ if and only if $\sigma_a^*(K_2) \models \boxdot A$.*

*Proof.* Let $\rho_1$ and $\rho_2$ be the roots of $K_1$ and $K_2$. The models have the same extension structure, so $B(K_1) = B(K_2)$ and $\rho_1$ and $\rho_2$ are both irreflexive or reflexive. By Lemma 2.3 of (Ghilardi, 2000), it is enough to consider models $K_i$ with $cl(\rho_i)$ being a singleton. Suppose $\sigma_a^*(K_1) \models \boxdot A$. We will show that $\sigma_a^*(K_1) \sim_n \sigma_a^*(K_2)$. From this it follows from Theorem 1 that $\sigma_a^*(K_2) \models A$, since $\sigma_a^*(K_1) \models A$ and $d(A) \leq n$. Then also $\sigma_a^*(K_2) \models \Box A$, hence $\sigma_a^*(K_2) \models \boxdot A$.

Now we prove $\sigma_a^*(K_1) \sim_n \sigma_a^*(K_2)$. We have $\sigma_a^*(K_1) \sim_0 \sigma_a^*(K_2)$ by Lemma 2 (ii). First assume that $\rho_1$ and $\rho_2$ are irreflexive. Suppose $\rho_1 R_1 w$. Root $\rho_1$ is irreflexive so $w \neq \rho_1$ and therefore $K_{1,w} \models \boxdot A$. So

$$[K_{1,w}]_{n-1} \in B(K_1) = B(K_2).$$

Hence there is a $v$ such that $\rho_2 R_2 v$, $K_{2,v} \models \boxdot A$ and $K_{2,v} \sim_{n-1} K_{1,w}$. By Lemma 2 (i) we have $\sigma_a^*(K_{2,v}) = K_{2,v}$ and $\sigma_a^*(K_{1,w}) = K_{1,w}$ and so

$\sigma_a^*(K_{2,v}) \sim_{n-1} \sigma_a^*(K_{1,w})$. The other direction is analogous. Therefore $\sigma_a^*(K_1) \sim_n \sigma_a^*(K_2)$.

Now suppose that $\rho_1$ and $\rho_2$ are reflexive. We show by induction $\sigma_a^*(K_1) \sim_k \sigma_a^*(K_2)$, for $k = 0, \ldots, n$. We have $\sigma_a^*(K_1) \sim_0 \sigma_a^*(K_2)$ by Lemma 2 (ii). Take $w$ such that $\rho_1 R_1 w$. If $w \neq \rho_1$ do the same as in the irreflexive case. If $w = \rho_1$, define $v = \rho_2$. By induction hypothesis we have $\sigma_a^*(K_1) \sim_{k-1} \sigma_a^*(K_2)$. Now pick $v$ such that $\rho_2 R_2 v$. This case is symmetric of the previous one, so we can apply a similar argument. Hence $\sigma_a^*(K_1) \sim_k \sigma_a^*(K_2)$ for each $k = 0, \ldots, n$. $\square$

Note that there can be multiple substitutions that can correspond to an extension structure, but there is at least one by Lemma 3. For each extension structure we fix such a substitution and call it the *corresponding substitution* to that extension structure. Note that different extension structures can be identified by the same substitution $\sigma_a$. We write $\sigma_{\mathrm{i}}$ and $\sigma_{\mathrm{r}}$ denoting the corresponding substitutions to the irreflexive and, respectively, reflexive extension structure of some bisimulation set.

Lemma 5 shows the connection between extensions of reflexive and irreflexive nodes under certain criteria. Informally, the substitution $\sigma_{\mathrm{r}}$ corresponding to a reflexive extension also works for the irreflexive extension with the same bisimulation set under these criteria.

**Lemma 5** *Let $K_1, K_2$ be two models, with roots $\rho_1, \rho_2$, that almost satisfy $\boxdot A$. Let $\rho_1$ be reflexive and $\rho_2$ irreflexive. Suppose $B(K_1) = B(K_2)$. If $\sigma_a^*(K_1) \models \boxdot A$ and $B(K_1) = B(\sigma_a^*(K_1))$, then also $\sigma_a^*(K_2) \models \boxdot A$.*

*Proof.* Similarly to the proof of the previous lemma, we will show that $\sigma_a^*(K_1) \sim_n \sigma_a^*(K_2)$. From this it follows that $\sigma_a^*(K_2) \models \boxdot A$.

By Lemma 2.3 of (Ghilardi, 2000), it is enough to consider models $K_i$ with $cl(\rho_i)$ being a singleton. We have $\sigma_a^*(K_1) \sim_0 \sigma_a^*(K_2)$ by Lemma 2 (ii). We must show that for all $w$ such that $\rho_1 R_1 w$ there exists $v$ such that $\rho_2 R_2 v$ and $\sigma_a^*(K_{1,w}) \sim_{n-1} \sigma_a^*(K_{2,v})$ and vice versa. First take $w$ such that $\rho_1 R_1 w$. If $w \neq \rho_1$, we proceed in the same way as for the irreflexive case in the proof of Lemma 4. If $w = \rho_1$, we use the assumption $\sigma_a^*(K_1) \models \boxdot A$ to see that

$$[\sigma_a^*(K_1)]_{n-1} \in B(\sigma_a^*(K_1)) = B(K_1) = B(K_2).$$

There is a $v$ such that $\rho_2 R_2 v$, $K_{2,v} \models \boxdot A$ and $K_{2,v} \sim_{n-1} \sigma_a^*(K_{1,\rho_1})$. By Lemma 2 (i), we have $K_{2,v} = \sigma_a^*(K_{2,v})$ and so $\sigma_a^*(K_{2,v}) \sim_{n-1} \sigma_a^*(K_{1,\rho_1})$.

Now pick $v$ such that $\rho_2 R_2 v$. This case is easier than the previous one and is left to the reader. Therefore $\sigma_a^*(K_1) \sim_n \sigma_a^*(K_2)$. $\square$

Now we present the key lemma. Recall that $\theta := \sigma_{a_1} \cdots \sigma_{a_s}$ where $a_1, \ldots, a_s$ is any fixed ordering on the subsets of $\{p_1, \ldots, p_m\}$. The lemma states that after two applications of $\theta^*$, the $B$-minimal rank of the new frontier points increases. Intuitively, one $\theta$ covers the corresponding irreflexive substitutions $\sigma_{\mathrm{i}}$'s and the other the corresponding reflexive substitutions $\sigma_{\mathrm{r}}$'s. In the following we use the notation $\theta_j^* := \sigma_{a_j}^* \sigma_{a_{j-1}}^* \cdots \sigma_{a_1}^*$, where we define $\theta_0^*$ to be the empty substitution, i.e., $\theta_0^*(K) = K$ for each model $K$.

**Lemma 6** *Let $K$ be a model and let $w$ be a $B$-minimal frontier point in $K$. Then for each frontier point $v$ in $\theta^*\theta^*(K)$ below $w$ we have that $B(K_w) \subset B(\theta^*\theta^*(K_v))$. Consequently, $r(K_w) < r(\theta^*\theta^*(K_v))$.*

*Proof.* Let $K$ be a model with $B$-minimal frontier point $w$. Let $v$ be a frontier point in $\theta^*\theta^*(K)$ below $w$. Note that $B(K_w) \subseteq B(\theta^*\theta^*(K_v))$. Suppose $B(K_w) = B(\theta^*\theta^*(K_v))$. We will prove that it implies $\theta^*\theta^*(K_v) \models \boxdot A$, and so $v$ cannot be a frontier point in model $\theta^*\theta^*(K)$.

Observe that $B(K_w) \subseteq B(K_v) \subseteq B(\theta^*\theta^*(K_v))$, so $B(K_w) = B(K_v)$. Consider all $v'$ above $v$ such that $K_{v'} \not\models \boxdot A$ (this includes $w$ itself). Since $w$ is $B$-minimal and $B(K_{v'}) \subseteq B(K_v)$, these $v'$'s satisfy $B(K_w) = B(K_{v'})$ as well. Also note that for each such $v'$ and each index $j$ we have

$$B(K_w) = B(K_{v'}) \subseteq B(\theta_j^*(K_{v'})) \subseteq B(\theta_j^*\theta^*(K_{v'})) \subseteq B(\theta^*\theta^*(K_v)).$$

Therefore, $B(K_w) = B(\theta_j^*(K_{v'})) = B(\theta_j^*\theta^*(K_{v'}))$ for each $j$. We have two cases: all $v'$ are irreflexive or there is at least one that is reflexive.

Let start with the first case. Here $w$ is irreflexive. By Lemma 3 we have, $\sigma_{a_j}^*(K_w) \models \boxdot A$ for some $j$. Note that also $\theta_j^*(K_w) \models \boxdot A$. This $\sigma_{a_j}$ is the irreflexive substitution $\sigma_{\mathrm{i}}$ corresponding to $B(K_w)$. For each $v'$ above $v$ we will prove $\theta_j^*(K_{v'}) \models \boxdot A$. We proceed by induction on the maximal length of sequences $v'Rx_1R\ldots Rx_k$ where $x_k$ is a frontier point in $K$. If the length equals 1, then $v'$ is a frontier point in $K$. If $\theta_{j-1}^*(K_{v'}) \models \boxdot A$, then also $\theta_j^*(K_{v'}) \models \boxdot A$ by Lemma 2 (i). If $\theta_{j-1}^*(K_{v'}) \not\models \boxdot A$, we know that $v'$ is a frontier point in $\theta_{j-1}^*(K)$. Since $B(K_w) = B(\theta_{j-1}^*(K_{v'}))$, we can apply Lemma 4 to conclude $\theta_j^*(K_{v'}) \models \boxdot A$. Suppose now the length is $l > 1$. By induction hypothesis we know that $v'$ is an irreflexive point for which all its successors satisfy $\boxdot A$ in $\theta_j^*(K)$. If $\theta_j^*(K_{v'}) \models \boxdot A$ we are done. If not, since $B(K_w) = B(\theta_j^*(K_{v'}))$ we know by Lemma 4 that $\sigma_{a_j}^*\theta_j^*(K_{v'}) \models \boxdot A$. Hence, by Lemma 2 (iii), $\theta_j^*(K_{v'}) \models \boxdot A$. Therefore by Lemma 2 (i), we have $\theta^*\theta^*(K_v) \models \boxdot A$.

Now we turn to the second case. We consider model $\theta_j^*(K)$, where $j$ is defined in such a way that $\sigma_{a_j}$ is the irreflexive substitution $\sigma_{\mathrm{i}}$ corresponding to $B(K_w)$. In case there is no corresponding irreflexive substitution, we define $j = 0$. If $\theta_j^*(K_v) \models \boxdot A$ we are done. If not, we will see further in the proof that all frontier points in $\theta_j^*(K)$ above $v$ are reflexive. Fix such a frontier point $w'$. Let $\sigma_{a_h}$ be the corresponding reflexive substitution $\sigma_{\mathrm{r}}$ to $B(K_w)$. Note that $B(K_w) = B(\theta_j^*(K_{w'}))$, so $\sigma_{a_h}^*\theta_j^*(K_{w'}) \models \boxdot A$. We prove for all $v'$ above $v$ that $\theta_h^*\theta^*(K_{v'}) \models \boxdot A$. We do so by induction on the maximal length of $v'Rx_1 \ldots x_{k-1}Rx_k$ where $x_i$'s do not belong to the same cluster and $x_k$ is a frontier point in $\theta_j^*(K)$. If the length equals 1, then $v'$ is a frontier point in $\theta_j^*(K)$ ($v'$ may equal $w'$). Frontier point $v'$ must be reflexive, because suppose $v'$ was irreflexive. Recall that $B(K_w) = B(\theta_j^*(K_{v'}))$. By Lemma 4 and Lemma 2 (iii) it would follow that $\theta_j^*(K_{v'}) = \sigma_{a_j}^*\theta_j^*(K_{v'}) \models \boxdot A$. And so $v'$ would not be a frontier point in $\theta_j^*(K)$. Thus $v'$ is reflexive. If $\theta_{h-1}^*\theta^*(K_{v'}) \models \boxdot A$, we are done. If not, since $B(\theta_j^*(K_{w'})) = B(\theta_{h-1}^*\theta^*(K_{v'}))$, we can apply Lemma 4 to conclude $\theta_h^*\theta^*(K_{v'}) \models \boxdot A$. Now suppose the length is $l > 1$. By induction hypothesis, all the successors of $v'$ not in the cluster of $v'$ satisfy $\boxdot A$ in $\theta_h^*\theta^*(K)$. Again, if $\theta_h^*\theta^*(K_{v'}) \models \boxdot A$, we are done. If not, we have two cases. If $v'$ is reflexive we can apply Lemma 4, because $B(\theta_j^*(K_{w'})) = B(\theta_h^*\theta^*(K_{v'}))$. If $v'$ is irreflexive, we apply Lemma 5, because $B(\theta_j^*(K_{w'})) = B(\theta_h^*\theta^*(K_{v'}))$ and $B(\theta_j^*(K_{w'})) = B(\sigma_{a_h}^*\theta_j^*(K_{w'}))$. In both cases we obtain $\sigma_{a_h}^*\theta_h^*\theta^*(K_{v'}) \models \boxdot A$, hence $\theta_h^*\theta^*(K_{v'}) \models \boxdot A$. This concludes $\theta^*\theta^*(K_v) \models \boxdot A$. □

Consider again Figure 1 illustrating the frontier lines. Lemma 6 shows that the $B$-minimal rank of the frontier lines increases after each step of $\theta^*\theta^*$ in the picture. We show in the final theorem that there are at most $N' + 1$ of these steps. And so a concatenation of $2(N' + 1)$ $\theta$'s forms a projective unifier for $\boxdot A$. As mentioned before, Ghilardi uses $2N$ $\theta$'s. From a close look at the induction proof of Lemma 2.8 from (Ghilardi, 2000), we think that he would conclude $2(N + 1)$ instead of $2N$ $\theta$'s. The rank is indeed bounded by $N$, but it may start at 0, which contributes to an extra application of $\theta$. However, this is not so important. We even think that a more clever proof can show that $2N'$ applications is sufficient in our case.

Note that if $d(A) = n = 0$, and thus $A$ is box-free, $(n - 1)$-bisimulation is undefined. In that case one $\theta$ will suffice. More precisely, only one $\sigma_a$ will be enough, namely its classical propositional valuation making $A$ true (compare to Example 1).

**Theorem 3** $(\theta^*)^{2(N'+1)}(K) \models \boxdot A$ *for all models* $K$.

*Proof.* From Lemma 6 it follows with induction on $l \leq N'$, that the rank of the $B$-minimal frontier points in $(\theta^*)^{2l}$ is greater than or equal to $l$. Note that the rank can be 0, so the $B$-minimal rank can start at 0. Since the rank is bounded by $N'$, we have that $(\theta^*)^{2(N'+1)}(K)$ does not contain any frontier points. Therefore $(\theta^*)^{2(N'+1)}(K) \models \boxdot A$. $\square$

# 5 Conclusion

This paper provides an extensive examination of Ghilardi's proof method of the connection between projective formulas and the extension property for several modal logics extending K4. The result plays an important role in the fields of unification and admissible rules. We provide an explanation of the close relationship between bisimulation and the extension property on the basis of extension structures.

It should be mentioned that the method only works for transitive models. For instance, it follows from (Jeřábek, 2015) that it is not possible to establish the same property for modal logic K. In terms of admissibility there are a lot of open questions for K.

We hope that this study will give more insight into the beautiful work of Ghilardi. It may clarify some aspects of Ghilardi's work which may be helpful for further research in the field of unification and admissibility. An interesting direction would be to establish a similar result for intuitionistic modal logics.

# References

Friedman, H. (1975). One hundred and two problems in mathematical logic. *The Journal of Symbolic Logic*, *40*(2), 113–129.

Ghilardi, S. (1999). Unification in intuitionistic logic. *The Journal of Symbolic Logic*, *64*(2), 859–880.

Ghilardi, S. (2000). Best solving modal equations. *Annals of Pure and Applied Logic*, *102*(3), 183–198.

Iemhoff, R. (2001). On the admissible rules of intuitionistic propositional logic. *The Journal of Symbolic Logic*, *66*(1), 281–294.

Iemhoff, R. (2016). Consequence relations and admissible rules. *Journal of Philosophical Logic*, *45*, 327–348.

Iemhoff, R., & Metcalfe, G. (2009). Proof theory for admissible rules. *Annals of Pure and Applied Logic*, *159*(1–2), 171–186.

Jeřábek, E. (2005). Admissible rules of modal logics. *Journal of Logic and Computation*, *15*(4), 411–431.

Jeřábek, E. (2015). Blending margins: The modal logic K has nullary unification type. *Journal of Logic and Computation*, *25*(5), 1231–1240.

Rybakov, V. (1984). A criterion for admissibility of rules in the model system S4 and the intuitionistic logic. *Algebra and Logic*, *23*, 369–384.

Visser, A. (1996). Uniform interpolation and layered bisimulation. In P. Hájek (Ed.), *Gödel '96: Logical foundations of mathematics, computer science and physics—Kurt Gödel's legacy, Brno, Czech Republic, August 1996, proceedings* (Vol. 6, pp. 139–164). Berlin: Springer-Verlag.

Iris van der Giessen
Utrecht University, Department of Philosophy and Religious Studies
The Netherlands
E-mail: `i.vandergiessen@uu.nl`