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The global structure theorem for finite groups with an abelian large p-subgroup

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For a prime p, the Local Structure Theorem [15] studies finite groups G with the property that a Sylow p-subgroup S of G is contained in at least two maximal p-local subgroups. Under the additional assumptions that G contains a so called large p-subgroup $Q \leq S$, and that composition factors of the normalizers of non-trivial p-subgroups are from the list of the known simple groups, [15] partially describes the p-local subgroups of G containing S, which are not contained in $N_G(Q)$. In the Global Structure Theorem, we extend the work of [15] and describe $N_G(Q)$ and, in almost all cases, the isomorphism type of the almost simple subgroup H generated by the p-local over-groups of S in G. Furthermore, for $p = 2$, the isomorphism type of G is determined. In this paper, we provide a reduction framework for the proof of the Global Structure Theorem and also prove the Global Structure Theorem when Q is abelian.

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1. Introduction

Let p be prime and G be a finite group. Normalizers of non-trivial p-subgroups of G are called *p-local subgroups* of G. We say that G has *characteristic* p if and only if $C_G(O_p(G)) \leq O_p(G)$ and that G has *local characteristic* p if and only if all the p-local subgroups of G have characteristic p. The group G is of *parabolic characteristic* p if and only if all the p-local subgroups containing a Sylow p-subgroup of G are of characteristic p. A group G is called a $\mathcal{CK}$-group if all composition factors are among the known simple groups: a cyclic group of prime order, an alternating group, a simple group of Lie type or one of the 26 sporadic simple groups. A group G is a $\mathcal{K}_p$-group, if any subgroup which normalizes a non-trivial p-subgroup of G is a $\mathcal{CK}$-group. This paper is part of a programme to investigate $\mathcal{K}_p$-groups of parabolic characteristic p. See [14] for an overview of this programme.

Of fundamental importance to the theory of groups of parabolic characteristic p are large subgroups: a p-subgroup Q of G is called *large* if and only if

(i) $C_G(Q) \leq Q$; and
(ii) $N_G(U) \leq N_G(Q)$ for all $1 \neq U \leq C_G(Q)$.

For example, if G is a simple group of Lie type in characteristic p and $S \in \mathrm{Syl}_p(G)$, then $Q = O_p(C_G(Z(S)))$ is almost always a large subgroup of G. Indeed this is true exactly when $Z(S)$ is a root group, that is provided G is not one of $\mathrm{Sp}_{2n}(2^a)$, $n \geq 2$, $\mathrm{F}_4(2^a)$ or $\mathrm{G}_2(3^a)$. This in part motivates the study of groups with a large subgroup.

If Q is a large subgroup of G, then $O_p(N_G(Q))$ is also a large subgroup of G [15, Lemma 1.5.2 (e)]. Thus when studying groups with a large subgroup we may and will in addition assume that the large subgroup satisfies

(iii) $Q = O_p(N_G(Q))$.

One of the consequences of having a large subgroup is that G is of parabolic characteristic p (see [15, Lemma 1.55 (c)]). In fact any p-local subgroup of G which contains Q has characteristic p. Throughout the paper we let

$$\widetilde{} : N_G(Q) \to N_G(Q)/Q$$

the natural projection homomorphism $x \mapsto xQ$.

For the remainder of this introduction we fix a prime p, a finite $\mathcal{K}_p$-group G, a Sylow p-subgroup S of G and a large subgroup $Q \leq S$ with $Q = O_p(N_G(Q))$.

For a finite group L, denote by Y_L the unique maximal elementary abelian normal p-subgroup of L with $O_p(L/C_L(Y_L)) = 1$. Such a subgroup of L exists (see for example [14, Lemma 2.0.1(a)]). The subgroup Y_L was first introduced by John Thompson.

If $\mathcal{X}$ is a set of subgroups of G, we write $\mathcal{X}^{min}$ for the set of elements in $\mathcal{X}$ which are minimal by inclusion. Further, if $T \leq G$, then

$$\mathcal{X}(T) = \{U \in \mathcal{X} \mid T \leq U\}.$$

For $H \leq G$, define

$$\mathcal{L}_H = \{L \mid L \leq H, O_p(L) \neq 1, C_H(O_p(L)) \leq O_p(L)\}.$$

For $N \in \mathcal{L}_G(S)$, we shall use the following notation

$$N^\circ = \langle Q^N \rangle, V_N = [Y_N, N^\circ]$$

Furthermore, we denote by $^{-}$ the natural homomorphism from N onto $N/C_N(Y_N)$. Note that for different members M of $\mathcal{L}_G(S)$, we still use $^{-}$ to represent the homomorphism to $M/C_M(Y_M)$. This should not lead to confusion.

We intend to investigate the groups G which have

$$\mathcal{VL} = \{L \in \mathcal{L}_G(S^g) \mid V_L \not\leq Q^g, g \in G\}$$

non-empty. In particular, we will focus on the subset $\mathcal{VL}^{min}$ of $\mathcal{VL}$. The overarching aim is to prove the Global Structure Theorem which concerns the $\mathcal{K}_p$-groups with a large subgroup Q such that $\mathcal{VL}$ is non-empty. The intention is to provide information about the subgroup $\langle \mathcal{L}_G(S) \rangle$ of G and, when $p = 2$, the isomorphism type of G.

The following subsets of $\mathcal{VL}$ play a pivotal role in our proof of the Global Structure Theorem:

$$\mathcal{VL}_{lin} = \left\{L \in \mathcal{VL} \mid \overline{L^\circ} \cong \mathrm{SL}_2(q) \text{ and } Y_L = V_L = V_{\mathrm{nat}}\right\},$$

$$\mathcal{VL}_{orthsymp} = \left\{ L \in \mathcal{VL} \mid (\overline{L^\circ}, V_L) = \begin{cases} (\Omega_n^{\pm}(q), V_{\mathrm{nat}}) & n \geq 3 \\ (\mathrm{Sp}_{2n}(q), V_{\mathrm{nat}}) & n \geq 2, p = 2 \\ (\mathrm{Sp}_4(2)', V_{\mathrm{nat}}) & \end{cases} \right\},$$

and

$$\mathcal{VL}_{wreath} = \left\{ L \in \mathcal{VL} \mid (\overline{L^\circ}, V_L) = \begin{cases} (\mathrm{O}_4^{+}(2), V_{\mathrm{nat}}) & \\ (\mathrm{SL}_2(4), V_{\mathrm{nat}}) & |Y_L : V_L| = 2 \\ (\Gamma\mathrm{L}_2(4), V_{\mathrm{nat}}) & |Y_L : V_L| \leq 2 \end{cases} \right\}.$$

Here, for a classical group X, V_{nat} denotes a natural module for X. We extend this notation to subgroups containing perfect derived subgroups of classical groups. For $L \in \mathcal{VL}_{lin} \cup \mathcal{VL}_{orthsymp}$, $q = q_L$ will be as defined for the classical groups and when $\overline{L^\circ} \cong$

$\mathrm{Sp}_4(2)'$ and V_L is the natural module, we set $q = 2$. We know from [24] that $\mathcal{VL}_{wreath} = \mathcal{VL}^{min}_{wreath}$.

Our first objective in this article is to prove the following theorem.

Theorem A. *Suppose that p is a prime, G is a finite $\mathcal{K}_p$-group, S a Sylow p-subgroup of G and $Q \leq S$ is a large subgroup of G with $Q = O_p(N_G(Q))$. Assume that $L \in \mathcal{VL}(S)$. Then there is a unique subgroup $M_L \in \mathcal{VL}^{min}(S)$ with $M_L \leq L$. Furthermore, the following hold.*

(i) *$M_L \in \mathcal{VL}_{lin}(S) \cup \mathcal{VL}_{orthsymp}(S) \cup \mathcal{VL}_{wreath}(S)$, $C_{M_L}(Y_{M_L})$ is p-closed, and $Y_{M_L} = \Omega_1(Z(O_p(M_L)))$.*

(ii) *If L is not in the wreath product case or the weak wreath product case of [15, Theorem A], then for each possibility for L, the structure of $\overline{M_L} = M_L/C_{M_L}(Y_{M_L})$, $\overline{M_L^\circ}$ and the action of the latter group on Y_{M_L} is presented in the final three columns of Table 1. In particular, in all cases V_{M_L} is a natural $\overline{M_L^\circ}$-module.*

(iii) *If $\hat{L} \in \mathcal{L}_G(M_L)$, then $\hat{L} \in \mathcal{VL}(S)$, $M_L = M_{\hat{L}}$ and*

$$\langle \hat{L}, N_G(Q)\rangle = \langle M_L, N_G(Q)\rangle = \langle L, N_G(Q)\rangle.$$

See [15, Section A.2] for a description of the modules listed in Table 1. As an immediate corollary to Theorem A we obtain the following remarkable observation.

Theorem B. *We have*

$$\mathcal{VL}^{min} = \mathcal{VL}^{min}_{lin} \cup \mathcal{VL}^{min}_{orthsymp} \cup \mathcal{VL}^{min}_{wreath}.$$

By inspecting the final three columns of Table 1, we observe that just a few possibilities occur. Namely $\overline{M_L^\circ}$ can be $\Omega_n^{\pm}(q)$ with $n \geq 3$, $\mathrm{Sp}_{2n}(q)$ with $n \geq 2$ and q even, $\mathrm{SL}_2(q)$ or $\mathrm{Sp}_4(2)'$. As the Global Structure Theorem considers the possibility that $\mathcal{VL}$ is non-empty, for the proof of the Global Structure Theorem we have $\mathcal{VL}^{min} = \mathcal{VL}^{min}_{lin} \cup \mathcal{VL}^{min}_{orthsymp} \cup \mathcal{VL}^{min}_{wreath}$ is non-empty by Theorem B and this gives a convenient division of the proof into subcases. The case $\mathcal{VL}^{min}_{wreath} \neq \emptyset$ has been studied in [24]. In this article we consider the possibility that $\mathcal{VL}^{min}_{lin} \neq \emptyset$. In a forthcoming paper we will study the case $\mathcal{VL}^{min}_{orthosymp} \neq \emptyset$.

The second main objective of this paper is to prove the Global Structure Theorem in the case that Q is abelian.

Theorem C *(Global Structure Theorem for Q abelian). Let p be a prime, G be a finite $\mathcal{K}_p$-group, S a Sylow p-subgroup of G and $Q \leq S$ be a large subgroup of G with $Q = O_p(N_G(Q))$. Suppose that Q is abelian and $N_G(Q)$ is not the unique maximal p-local subgroup of G containing S. Then one of the following holds*

Table 1
The minimal groups M_L with $Y_{M_L} \not\leq Q$.

	$\overline{L^\circ}$	p	V_{L°	$\overline{M}_L$	$\overline{M_L^\circ}$	Y_{M_L}
(1)	$\mathrm{SL}_{m_1}(q) \otimes \mathrm{SL}_{m_2}(q), m_1, m_2 \geq 2$	p	tensor product	$\Omega_4^+(q)\bar{S}$	$\Omega_4^+(q)$	$Y_{M_L} = V_{M_L}$
(2)	$\mathrm{SL}_n(q), n \geq 2$	p	V_{nat}	$\mathrm{SL}_2(q)\bar{S}$	$\mathrm{SL}_2(q)$	$Y_{M_L} = V_{M_L}$
(3)	$\mathrm{Sp}_{2n}(q), n \geq 2$	2	V_{nat}	$\mathrm{Sp}_{2n}(q)\bar{S}$	$\mathrm{Sp}_{2n}(q)$	$\lvert Y_{M_L} : V_{M_L}\rvert \leq q$
(4)	$\mathrm{Sp}_4(2)'$	2	V_{nat}	$\mathrm{Sp}_4(2)'\overline{S}$	$\mathrm{Sp}_4(2)'$	$Y_{M_L} = V_{M_L}$
(5)	$\Omega_n^\pm(q), n \geq 5$	p	V_{nat}	$\Omega_n^\pm(q)\bar{S}$	$\Omega_n^\pm(q)$	$Y_{M_L} = V_{M_L}$
(6)	$\mathrm{SL}_n(q)/\langle(-1)^{n-1}I_n\rangle, n \geq 5$	p	$\bigwedge^2(V_{\mathrm{nat}})$	$\Omega_6^+(q)\bar{S}$	$\Omega_6^+(q)$	$Y_{M_L} = V_{M_L}$
(7)	$\mathrm{SL}_n(q)/\langle(-1)^{n-1}I_n\rangle, n \geq 2$	p	$\mathrm{S}^2(V_{\mathrm{nat}})$	$\Omega_3(q)\bar{S}$	$\Omega_3(q)$	$Y_{M_L} = V_{M_L}$
(8)	$\mathrm{SL}_n(q^2)/Z, n \geq 2$	p	$\mathrm{U}^2(V_{\mathrm{nat}})$	$\Omega_4^-(q)\bar{S}$	$\Omega_4^-(q)$	$Y_{M_L} = V_{M_L}$
	$Z = \langle \lambda I_n \mid \lambda \in \mathrm{GF}(q), \lambda^n = \lambda^{q+1} = 1\rangle$					
(9)	$\mathrm{Spin}_{10}^+(q)$	p	half-spin	$\Omega_8^+(q)\bar{S}$	$\Omega_8^+(q)$	$Y_{M_L} = V_{M_L}$
(10)	$\mathrm{E}_6(q)$	p	q^{27}	$\Omega_{10}^+(q)\bar{S}$	$\Omega_{10}^+(q)$	$Y_{M_L} = V_{M_L}$
(11)	$3^{\cdot}\mathrm{Sym}(6)$	2	2^6	$\Omega_4^+(2)$	$\Omega_4^+(2)$	$Y_{M_L} = V_{M_L}$
(12)	$3^{\cdot}\mathrm{Alt}(6)$	2	2^6	$\mathrm{SL}_2(2)$	$\mathrm{SL}_2(2)$	$Y_{M_L} = V_{M_L}$
(13)	$\mathrm{Mat}(22)$	2	Golay 2^{10}	$\mathrm{O}_4^-(2)$	$\Omega_4^-(2)$	$Y_{M_L} = V_{M_L}$
(14)	$\mathrm{Mat}(24)$	2	Golay 2^{11}	$\mathrm{Sp}_4(2)$	$\mathrm{Sp}_4(2)'$	$Y_{M_L} = V_{M_L}$
(15)	$\mathrm{Mat}(24)$	2	Todd 2^{11}	$\Omega_6^+(2)$	$\Omega_6^+(2)$	$\lvert Y_{M_L} : V_{M_L}\rvert \leq 2$
(16)	$\mathrm{Mat}(11)$	3	Golay 3^5	$\Omega_4^-(3)$	$\Omega_4^-(3)$	$\lvert Y_{M_L} : V_{M_L}\rvert = 3$
(17)	$\mathrm{Aut}(\mathrm{Mat}(22))$	2	Todd 2^{10}	$\mathrm{Sp}_4(2)$	$\mathrm{Sp}_4(2)$	$\lvert Y_{M_L} : V_{M_L}\rvert = 2$
(18)	$2^{\cdot}\mathrm{Mat}(12)$	3	Golay 3^6	$\Omega_3(3)$	$\Omega_3(3)$	$Y_{M_L} = V_{M_L}$
(19)	$\Omega_4^-(3)$	3	V_{nat}	$\Omega_4^-(3)$	$\Omega_4^-(3)$	$\lvert Y_{M_L} : V_{M_L}\rvert = 3$
(20)	$\Omega_5(3)$	3	V_{nat}	$\Omega_5(3)$	$\Omega_5(3)$	$\lvert Y_{M_L} : V_{M_L}\rvert = 3$
(21)	$\Omega_6^+(2)$	2	V_{nat}	$\Omega_6^+(2)\bar{S}$	$\Omega_6^+(2)$	$\lvert Y_{M_L} : V_{M_L}\rvert = 2$

(i) *$p = 2$ and $F^*(G) \cong \mathrm{PSL}_n(2^a)$ with $a \geq 1$ and $n \geq 3$, or $F^*(G) \cong \mathrm{Mat}(22)$, $\mathrm{Mat}(23)$, $\mathrm{Mat}(24)$, $\mathrm{Alt}(6)$ or $\mathrm{Alt}(9)$;*
(ii) *p is odd and $F^*(\langle \mathcal{L}_G(S)\rangle) \cong \mathrm{PSL}_n(p^a)$ with $a \geq 1$ and $n \geq 4$; or*
(iii) *p is odd and there exists $L \in \mathcal{VL}(S)$ such that $(N_G(O_p(L)), N_G(Q))$ is a weak BN-pair of type $\mathrm{PSL}_3(p^a)$, $a \geq 1$, over $N_G(O_p(L)Q)$. Furthermore $\langle \mathcal{L}_G(S)\rangle = \langle N_G(O_p(L)), N_G(Q)\rangle$.*

Remarks

(i) Suppose that Theorem C(ii) holds and in addition assume that G is of local characteristic p. Set $H = \langle \mathcal{L}_G(S)\rangle$. Then, by [25, Theorem 1], either $G = H$ or H is a strongly p-embedded subgroup of G. Using $n \geq 4$, [21, Proposition 9.1] yields that $m_p(C_H(t)) \geq 2$ for any involution $t \in H$. Finally, under the assumption that G is a $\mathcal{K}_2$-group [21, Theorem 1.4] yields that H cannot be strongly p-embedded in G and so $F^*(G) = F^*(H) \cong \mathrm{PSL}_n(q)$.
(ii) Suppose that Theorem C(iii) holds with $q > p$. Set $H = \langle \mathcal{L}_G(S)\rangle$ and put $H^* = \langle O^{p'}(O^p(N_G(O_p(M)))), O^{p'}(O^p(N_G(Q)))\rangle$. Then $H = H^*N_G(QO_p(M))$ by Lemma 6.4. H^* is a completion of a weak BN-pair of type $\mathrm{PSL}_3(q)$. Since Q and $O_p(M)$ are both large assuming that every proper subgroup of G is a $\mathcal{K}$-group, we can apply [19, Theorem 1.6] to obtain $H^* \cong \mathrm{PSL}_3(q)$. Then [20, Main Theorem 2] or more directly [20, Proposition 9.1] and then [22] yields $F^*(G) \cong \mathrm{PSL}_3(q)$.

Assume that Q is abelian and $L \in \mathcal{L}_G(S)$. If $V_L \leq Q$, then, as Q is large, $L \leq N_G(V_L) \leq N_G(Q)$. Hence with the assumption of Theorem C(c), we have that $V_L \not\leq Q$ for some $L \in \mathcal{L}_G(S)$. Hence $L \in \mathcal{VL}(S)$ and so the conclusions of Theorem C form part of the Global Structure Theorem.

Among the case divisions in [15, Theorem A] there are the "wreath product case" and the "weak wreath product case". These cases are studied in [24]. In particular, it is demonstrated that, if $L \in \mathcal{VL}$ and L is in one of the wreath product cases, then either $L \in \mathcal{VL}_{wreath}^{min}$ or

$$(\overline{L^\circ}, Y_L) \in \{(\mathrm{SL}_2(q), V_{\mathrm{nat}}), (\Omega_4^+(2), V_{\mathrm{nat}})\}.$$

Hence in these cases $L \in \mathcal{VL}_{lin}^{min} \cup \mathcal{VL}_{orthsymp}^{min} \cup \mathcal{VL}_{wreath}^{min}$. Furthermore, [24, Main Theorem, Corollary] yields

Theorem 1.1. *Suppose that $\mathcal{VL}_{wreath}^{min}$ is non-empty. Then $G \cong \mathrm{Mat}(22)$, $\mathrm{Aut}(\mathrm{Mat}(22))$, $\mathrm{Sym}(8)$, $\mathrm{Sym}(9)$ or $\mathrm{Alt}(10)$.*

By inspecting the groups which appear in Theorem 1.1, we observe that, if Q is abelian, then $F^*(G) \cong \mathrm{Mat}(22)$. This case is therefore included in the conclusion as part of Theorem C(i).

Because of Theorem 1.1, we may assume that $\mathcal{VL}^{min}_{wreath}(S)$ is empty. Hence to prove the Global Structure Theorem we may assume that

$$\mathcal{VL}^{min} = \mathcal{VL}^{min}_{lin} \cup \mathcal{VL}^{min}_{orthsymp}.$$

In particular, to prove Theorem C, we may suppose that $L \in \mathcal{VL}^{min}_{lin}(S) \cup \mathcal{VL}^{min}_{orthsymp}(S)$ and, as Q is abelian, Corollary 4.6 implies $L \in \mathcal{VL}^{min}_{lin}(S)$. Conversely, if $L \in \mathcal{VL}^{mn}_{lin}(S)$, then Q is abelian by Lemma 5.2 (v). Thus Theorem C follows by combining Theorem 1.1 with

Theorem 1.2. *Suppose that p is a prime, G is a finite $\mathcal{K}_p$-group, $S \in \mathrm{Syl}_p(G)$ and $Q \leq S$ is a large subgroup of G with $Q = O_p(N_G(Q))$. If $\mathcal{VL}^{min}_{lin} \neq \emptyset$, then*

(i) *$p = 2$ and $F^*(G) \cong \mathrm{PSL}_n(2^a)$, $a \geq 1$, $n \geq 3$ or $G \cong \mathrm{Mat}(22)$, $\mathrm{Mat}(23)$, $\mathrm{Mat}(24)$, $\mathrm{Alt}(6)$ or $\mathrm{Alt}(9)$;*
(ii) *p is odd and $F^*(\langle \mathcal{L}_G(S)\rangle) \cong \mathrm{PSL}_n(p^a)$ for some $a \geq 1$ and $n \geq 4$; or*
(iii) *p is odd and, for $M \in \mathcal{VL}^{min}_{lin}(S)$, $(N_G(O_p(M)), N_G(Q))$ is a weak BN-pair of type $\mathrm{PSL}_3(p^a)$, $a \geq 1$, over $N_G(O_p(M)Q)$. Furthermore $\langle \mathcal{L}_G(S)\rangle = \langle N_G(O_p(M)), N_G(Q)\rangle$.*

The paper develops as follows. In Section 2 we gather a number of standard tools which are used in the proof. In particular, we record the theorem which gives the groups which have a strong dual F-module. We also present a proposition, Proposition 2.5, which will be used to identify the projective linear groups $\mathrm{PSL}_n(q)$ and relies on a theorem which identifies buildings from internal subgroup structure [17]. In Section 3 we gather together some basic facts about groups G with a large subgroup. We also show that in almost all cases, $O_{p'}(G) = 1$. Probably the smallest counterexample to the general statement is the non-split extension $2^6{\cdot}\mathrm{PSp}_4(3)$ which has a large 3-group and is generated by its 3-local subgroups which contain a fixed Sylow 3-subgroup. In Section 4, we prove Theorem A. However, this is not the only purpose of this section. It also lays out a number of structural results, Lemma 4.4 and Lemma 4.5 for example, which will be used in the paper following this one which proves the Global Structure Theorem in general.

As we have mentioned, to prove Theorem C, it suffices to prove Theorem 1.2 and so in Section 5 we start the proof of Theorem 1.2 by establishing results which are used to limit the structure of $\langle V_M^{N_G(Q)}\rangle$, $M \in \mathcal{VL}^{min}_{lin}(S)$, by using results about strong F-modules. Here the relatively uncomplicated structure of M plays an influential role. The bulk of the proof of Theorem 1.2 covers Section 6 and Section 7 where we cover the possibilities that $O_p(M) = V_M$ and $O_p(M) > V_M$ separately. The first case leads to the groups $\mathrm{Alt}(6)$, the Mathieu groups $\mathrm{Mat}(22)$ and $\mathrm{Mat}(23)$ and the family of weak BN-pairs of type $\mathrm{PSL}_3(q)$ and the second case uncovers the groups $\mathrm{Mat}(24)$, $\mathrm{Alt}(9)$ and, with the help of Proposition 2.5, the groups $\mathrm{PSL}_n(q)$, $n \geq 4$, $q = p^a$. Particularly in Section 7, the structure of addition members of $\mathcal{L}_G(S)$ is required and at this stage the results from Theorem A are applied. The theorem is used in two distinct ways. The uniqueness of M provided by Theorem A eventually leads in Lemma 7.8 to the fact that $\langle \mathcal{L}_G(S)\rangle = \langle M, N_G(Q)\rangle$. Theorem A is also used to restrict the structure of certain overgroups L of M. This is achieved by first showing that they are in $\mathcal{L}_G(S)$ and then by reading Table 1 from the right-hand side to the left-hand side to extract the structure of $\overline{L^\circ}$.

Our notation is mostly standard and follows [2,6,15]. For groups A and B, we write $X = A.B$ for a group with a normal subgroup N isomorphic to A and $X/N \cong B$. If it is useful to know the extension is split, we write $X = A{:}B$ and, if it is useful to know it is non-split, we write $X = A{}^{\cdot}B$. When $\mathrm{SL}_{m_1}(q) \times \mathrm{SL}_{m_2}(q)$ acts on $V_{\mathrm{nat}_1} \otimes V_{\mathrm{nat}_2}$, the kernel of the action is $Z = \langle \left(\begin{smallmatrix} \lambda I_{m_1} & 0 \\ 0 & \lambda^{-1} I_{m_2}\end{smallmatrix}\right) \mid \lambda \in \mathrm{GF}(q), \lambda^{m_1} = \lambda^{m_2} = 1\rangle$ and the notation $\mathrm{SL}_{m_1}(q) \otimes \mathrm{SL}_{m_1}(q)$ represents the group $(\mathrm{SL}_{m_1}(q) \times \mathrm{SL}_{m_2}(q))/Z$.

2. Modules and other background results

In this section we collect some facts about modules together with some other useful results, which will be required in the proof of Theorem C.

Definition 2.1. Let X be a group and V be a non-trivial module for X over $\mathrm{GF}(p)$. Assume that A is an elementary abelian p-subgroup of X with $A \not\leq C_X(V)$ and Q is a p-subgroup of X which is not normal in X. *Then*

(i) *A acts quadratically on V if and only if $[V, A, A] = 0$;*
(ii) *V is an F-module with offender A if and only if $|V/C_V(A)| \leq |A/C_A(V)|$;*
(iii) *A is a* best offender *on V if $|B||C_B(V)| \leq |A||C_V(A)|$ for all $B \leq A$;*
(iv) *an F-module V with offender A is strong if and only if $C_V(a) = C_V(A)$ for all $a \in A \setminus C_A(V)$;*
(v) *V is a* dual F-module with offender A *if and only if $[V, A, A] = 0$ and $|[V, A]| \leq |A/C_A(V)|$;*
(vi) *a dual F-module V with offender A is strong if and only if $[v, A] = [V, A]$ for all $v \in V \setminus C_V(A)$; and*
(vii) *V is a* $Q!$-module *for X with respect to Q, if $N_X(U) \leq N_X(Q)$ for all $0 \neq U \leq C_V(Q)$.*

Lemma 2.2. *Let p be a prime, X a group with $F^*(X)$ quasisimple, $F^*(X)/Z(F^*(X))$ a $\mathcal{CK}$-group, and V be an irreducible, faithful $F^*(X)$-module over $\mathrm{GF}(p)$ which is a strong dual F-module with offender A. Then for $X_1 = F^*(X)A$ one of the following holds:*

(i) *$X_1 \cong \mathrm{SL}_n(p^a)$ or $\mathrm{Sp}_{2n}(p^a)$, $a \geq 1$, and V is a natural module.*
(ii) *$p = 2$ and*
   (a) *$X_1 \cong \mathrm{Alt}(6)$ and V is one of the 4-dimensional modules over $\mathrm{GF}(2)$.*
   (b) *$X \cong \mathrm{Alt}(7)$ and V is one of the 4-dimensional modules over $\mathrm{GF}(2)$.*

In both cases (a) and (b), $|A| = |V : C_V(A)| = |[V, A]| = 4$.

(iii) $p = 2$ *and* $X = X_1 \cong \mathrm{O}^{\pm}_{2n}(2)$ *or* $\mathrm{Sym}(n)$ *and* V *is the natural module. In these cases,* $|A| = |V : C_V(A)| = |[V, A]| = 2$ *and* $A \not\le F^*(X)$.

Proof. See [13, Theorem 3.1]. □

Lemma 2.3. *Suppose that* X *is a group,* $E = O_2(X)$ *is elementary abelian of order* 16 *and* $X/E \cong \mathrm{Alt}(6)$ *induces the non-trivial irreducible part of the* 6*-point permutation module on* E*. Then* X *splits over* E.

Proof. See [24, Lemma 2.1]. □

Lemma 2.4. *Suppose that* V *is a* p*-group and* X *is a group which acts faithfully on* V *with* $O_p(X) = 1$*. Assume* $A \le X$ *is an elementary abelian* p*-subgroup of order at least* p^2 *which has the property* $C_V(A) = C_V(a)$ *for all* $a \in A^{\#}$*. If* L *is a non-trivial subgroup of* X *and* $L = [L, A]$*, then* A *acts faithfully on* L.

In particular, A *centralizes every* p'*-subgroup which it normalizes,* $[A, F(X)] = 1$, $E(X) \ne 1$ *and, if* L *is a component of* X *which is normalized but not centralized by* A*, then* A *acts faithfully on* L.

Proof. See [24, Lemma 2.8]. □

The next lemma is our main device for recognising the linear groups. It presents [17, Theorem 6.8] in a form which makes our application more straight-forward.

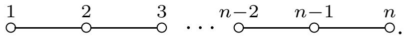
Proposition 2.5. *Suppose that* p *is a prime,* X *is a finite group,* X_1, X_2 *are subgroups of* X*,* $T \in \mathrm{Syl}_p(X_1) \cap \mathrm{Syl}_p(X_2)$ *and* $O_p(X) = 1$*. For* $i = 1, 2$*, set* $B_i = N_{X_i}(T)$*. Assume that* $\{i, j\} = \{1, 2\}$*,*

(i) $X = \langle X_1, X_2 \rangle$;
(ii) $X_1/O_p(X_1) \cong \mathrm{SL}_3(p^a)$*, and* $X_2/O_p(X_2) \cong \mathrm{SL}_n(p^a)$ *with* $a \ge 1$ *and* $n \ge 3$;
(iii) $B_i \le N_X(X_j)$ *and* $B_j \le N_X(X_i)$;
(iv) $(X_1 \cap X_2)B_i/O_p(X_i)$ *is a minimal parabolic subgroup of* $X_i/O_p(X_i)$ *corresponding to an end node of the Coxeter diagram for* $X_i/O_p(X_i)$*; and*
(v) $C_X(O_p(X_2)) \le O_p(X_2)$ *and* $|O_p(X_2)| \ge p^{an}$.

Let $M^* > B_1$ *be such that* $M^*/O_p(X_1)$ *is the parabolic subgroup of* $X_1/O_p(X_1)$ *with* $O^{p'}(M^*) \not\le X_2$ *and* $R > B_2$ *be such that* $R/O_p(X_2)$ *is the maximal parabolic subgroup of* $X_2/O_p(X_2)$ *with* $X_2 = \langle X_1 \cap X_2, R \rangle$.

If $[O^p(O^{p'}(M^*)), O^p(O^{p'}(R))] \le T$*, then* $X \cong \mathrm{PSL}_{n+1}(p^a)$.

Proof. We follow [17] for our notation regarding buildings. Let Δ be the building of type A_n over $\mathrm{GF}(p^a)$ with type set $I = \{1, \ldots, n\}$ and Coxeter diagram Π labelled by I as follows:



Assume $L = \mathrm{PSL}_{n+1}(p^a)$ acts on Δ. Define

$$D_1 = \{1, 2\} \text{ and } D_2 = \{2, 3, \ldots, n\}$$

to be subsets of I. Note that any subset of I of size at most 2 corresponding to a connected subset of Π is contained in either D_1 or D_2.

Fix a chamber c of Δ. For $J \in \{D_1, D_2\}$, $\Delta_J(c)$ denotes the residue of type J in Δ which contains c and $L_J = O^{p'}(P_J)$ where $P_J = \mathrm{Stab}_L(\Delta_J(c))/R_J$ and R_J is the subgroup of P_J which fixes every chamber in $\Delta_J(c)$. For $J_1 \subset J$, let L_{J,J_1} be the parabolic subgroup of L_J which fixes the residue of type J_1 in $\Delta_J(c)$ which contains c. Then $L_{D_1} \cong \mathrm{PSL}_3(p^a)$ and $L_{D_2} \cong \mathrm{PSL}_n(p^a)$. Set $X_{D_i} = X_i$.

Using (ii), define surjections

$$\phi_{D_i} : X_{D_i} \to L_{D_i}$$

to be the quotient map to X_{D_i}/K_{D_i} followed by an isomorphism where $K_{D_i}/O_p(X_{D_i}) = Z(X_{D_i}/O_p(X_{D_i}))$ and $O_p(X_{D_i}) = O_p(K_{D_i})$. For $j \in D_i$, define $X_{D_i,j}$ to be the preimage under ϕ_{D_i} of $L_{D_i,j}$. Using (iv), we can adjust the surjections by using graph automorphisms of L_{D_1} and L_{D_2} if necessary so that $\phi_{D_i}(X_{D_1} \cap X_{D_2}) \geq O^{p'}(L_{D_i,2})$ for $i = 1, 2$. Furthermore, $B_1 = X_{D_1,\emptyset}$ and $B_2 = X_{D_2,\emptyset}$. Finally define $H_{D_i,k} = O^p(O^{p'}(X_{D_i,k}))$ for $k \in D_i$, $i = 1, 2$ and notice that $H_{D_1,2} = H_{D_2,2}$. We have established the necessary notation from [17, Notation 6.1] (with X in place of G). Furthermore, [17, Hypothesis 6.2 (i), (ii), (iii), (vi) and (vi)] all hold. Assumption (iii) implies that [17, Hypothesis 6.2 (iv)] holds. We are left to show that [17, Hypothesis 6.2 (v)] is valid. This means we need to show $H_{D_1,1}H_{D_2,k} = H_{D_2,k}H_{D_1,1}$ for $k \in D_2 \setminus \{2\}$. From the definition of M^*, we have $H_{D_1,1} = O^p(O^{p'}(M^*))$ and from the definition of R we have $H_{D_2,k} \leq R$. Suppose, as in the statement, that $[O^p(O^{p'}(M^*)), O^p(O^{p'}(R))] \leq T$ and set

$$T_{1k} = [H_{D_1,1}, H_{D_2,k}] \leq [O^p(O^{p'}(M^*)), O^p(O^{p'}(R))] \leq T.$$

Then $H_{D_1,1}$ normalizes $H_{D_2,k}T_{12}$ and so also normalizes $O^p(H_{D_2,k}T_{12}) = H_{D_2,k}$. In particular, $H_{D_1,1}$ and $H_{D_2,k}$ permute. We have shown that [17, Hypothesis 6.2] holds. As $T/O_p(X_2) \in \mathrm{Syl}_p(X_2/O_p(X_2))$ and $|O_p(X_2)| \geq p^{an}$, the additional hypotheses of [17, Theorem 6.8] follow from (v). Hence $O^{p'}(X) \cong L \cong \mathrm{PSL}_{n+1}(p^a)$ by [17, Theorem 6.8]. This concludes the proof. $\square$

We shall use the next proposition in the final argument of this paper. Recall that the *Thompson subgroup*, $J(T)$, of a group T is generated by set $\mathfrak{A}_e(T)$ the maximal rank elementary abelian subgroups of T.

Proposition 2.6. *Suppose that $n \geq 4$, $X \leq \Gamma\mathrm{L}_n(2^a)$ and $F^*(X) \cong \mathrm{SL}_n(2^a)$. Let $T \in \mathrm{Syl}_2(X)$ and $T_0 = T \cap F^*(X)$. Then*

(i) *$J(T) = J(T_0)$; and*
(ii) *every involution in T_0 is X-conjugate to an element of $Z(J(T))$.*

Proof. By [7, Table 3.3.1], $F^*(X)$ has 2-rank $m_1 = \lfloor n^2/4 \rfloor a$. For involutions $x \in T \setminus T_0$, we have $C_{F^*(X)}(x) \cong \mathrm{SL}_n(2^{a/2})$ by [7, Proposition 4.9.1(a) and (d)]. Hence the 2-rank of $C_X(x)$ is $m_2 = 1 + \lfloor n^2/4 \rfloor \frac{a}{2}$. As $m_1 > m_2$, the members of $\mathfrak{A}_e(T)$ are all subgroups of T_0. Hence (i) holds.

Let V be the natural $\mathrm{GF}(2^a)F^*(X)$-module and let $V > V_{n-1} > \cdots > V_1 > 0$ be a maximal flag of V preserved by T_0. The Thompson subgroup of T_0 is described in [3, Theorem 6.1, Corollary 6.2]. From this description, we deduce that, if n is even,

$$J(T_0) = Z(J(T_0)) = C_X(V/V_{n/2}) \cap C_X(V_{n/2})$$

and, if n is odd,

$$Z(J(T_0)) = C_X(V/V_{\lfloor n/2 \rfloor}) \cap C_X(V_{\lceil n/2 \rceil}).$$

By considering the Jordan form of elements of order 2 in T_0, we see that they are $F^*(X)$-conjugate into $Z(J(T_0))$. Hence (ii) follows from (i). $\square$

3. Preliminary results about groups with a large subgroup

From here on we shall assume that G is a finite $\mathcal{K}_p$-group, S a Sylow p-subgroup of G and $Q \leq S$ is a large subgroup of G with $Q = O_p(N_G(Q))$.

We begin by drawing some facts from [15] and applying them in the case when $\mathcal{VL}$ is non-empty.

Lemma 3.1. *The following hold:*

(i) *Q is weakly closed in S with respect to G.*
(ii) *$Z(Q)$ is a trivial intersection set in G.*
(iii) *If $L \in \mathcal{L}_G(S)$, then*
  (a) *$\Omega_1(Z(S)) \leq Y_L \leq \Omega_1(Z(O_p(L)))$;*
  (b) *$L^\circ = \langle Q^{L^\circ} \rangle$; and*
  (c) *$L = L^\circ N_L(Q)$.*

(iv) *If $L \in \mathcal{L}_G(S)$ with $Q \nleq O_p(L)$, then*
  (a) *$L^\circ \nleq N_G(Q)$; and*
  (b) *$C_G(L^\circ) = 1$, $Z(L) = 1$ and $\Omega_1(Z(S)) \leq V_L$.*

(v) *If $L, M \in \mathcal{L}_G(S)$ with $L \leq M$, then $Y_L \leq Y_M$ and $V_L \leq V_M$.*

(vi) *If $L \in \mathcal{L}_G(S)$, then V_L and Y_L are $Q!$-modules for L with respect to Q.*

Proof. (i) is [15, Lemma 1.52(b)].
(ii) Suppose $Z(Q) \cap Z(Q)^g \neq 1$. Then, as Q is large,

$$Q \leq O_p(N_G(Z(Q) \cap Z(Q)^g)) \geq Q^g$$

and so $Q = Q^g$ by (i). Thus $Z(Q) = Z(Q^g) = Z(Q)^g$.

(iii) The first assertion is [15, Lemma 1.24(g)]. The second one follows from (i) and [15, Lemma 1.46(c)]. Part (c) follows from (i) by applying the Frattini argument to L°.

Assume that $L \in \mathcal{L}_G(S)$ with $Q \nleq O_p(L)$.

(iv) (a) If $L^\circ \leq N_G(Q)$, then, (iii)(b) implies $L^\circ = \langle Q^{L^\circ} \rangle = Q \leq O_p(L)$, a contradiction.

(iv) (b) $C_G(L^\circ) = 1$ is [15, Lemma 1.55(d)]. As $Z(L) \leq C_G(L^\circ)$, we have $Z(L) = 1$. That $\Omega_1(Z(S)) \leq V_L$ is an application of [15, Lemma 1.24(e)] and [9, (I.17.4)].

(v) That $Y_L \leq Y_M$ is [15, Lemma 1.24 (f)]. Hence $V_M = [Y_M, M^\circ] \leq [Y_L, L^\circ] = V_L$.

(vi) This follows from the definition for a $Q!$-module, as Q is a large subgroup of L. □

Lemma 3.2. *Assume that $L \in \mathcal{VL}(S)$ and $K \in \mathcal{L}_G(L)$. Then*

(i) *$Q \nleq O_p(L)$ and $L^\circ \nleq N_G(Q)$;*
(ii) *$K \in \mathcal{VL}(S)$;*
(iii) *$O_p(\langle L, N_G(Q) \rangle) = 1$; and*
(iv) *S is contained in at least two maximal p-local subgroups.*

Proof. (i) Assume that $Q \leq O_p(L)$. Then $V_L \leq C_G(Q) \leq Q$, a contradiction. Hence $Q \nleq O_p(L)$ and $L^\circ \nleq N_G(Q)$ by Lemma 3.1 (iv)(a).

(ii) We have $V_L \leq V_K$ by Lemma 3.1(v). As $V_L \nleq Q$, $V_K \nleq Q$ and so $K \in \mathcal{VL}(S)$.

(iii) Set $X = \langle L, N_G(Q) \rangle$. If $O_p(X) \neq 1$, then $X \in \mathcal{L}_G(S)$ and so (ii) implies $X \in \mathcal{VL}(S)$. This is impossible as $Y_X \leq O_p(X) \leq Q$. Hence $O_p(X) = 1$.

(iv) This follows from (iii). □

Lemma 3.3. *Suppose that $M \in \mathcal{L}_G(S)$ and V_M is irreducible as an $\overline{M}$-module. If N is a non-trivial normal subgroup of M, then $V_M \leq N$. In particular, V_M is contained in every non-trivial characteristic subgroup of $O_p(M)$.*

Proof. By Lemma 3.1 (iv)(b), $\Omega_1(Z(S)) \leq V_M$. By the definition of $\mathcal{L}_G(S)$, $O_{p'}(M) = 1$ and so p divides $|N|$. Hence $V_M \cap N \geq N \cap \Omega_1(Z(S)) > 1$. As $V_M \cap N$ is normalized by M and M acts irreducibly on V_M, we obtain $V_M \leq N$. □

Lemma 3.4. *Suppose that $M \in \mathcal{VL}(S)$, V_M is irreducible as an $\overline{M}$-module and $Y_M \neq O_p(M)$. Then $V_M \leq O_p(M)' \leq \Phi(O_p(M))$ and $O_p(M)$ is not abelian.*

Proof. We exploit Lemma 3.3. Consider the possibility that $O_p(M)$ is abelian. Then, as $Y_M \neq O_p(M)$, $O_p(M)$ is not elementary abelian. Hence, as $\Phi(O_p(M))$ is characteristic in $O_p(M)$, $V_M \leq \Phi(O_p(M))$ by Lemma 3.3. However, as $O_p(M)$ is abelian, $[Q, O_p(M), O_p(M)] = 1$. Hence $O_p(M)$ acts quadratically on $Q/\Phi(Q)$ and thus

$$V_M \leq \Phi(O_p(M)) \leq C_G(Q/\Phi(Q)) \leq Q,$$

which contradicts $M \in \mathcal{VL}(S)$. Hence $O_p(M)$ is not abelian and $O_p(M)'$ is a non-trivial characteristic subgroup of $O_p(M)$. Thus $V_M \leq O_p(M)' \leq \Phi(O_p(M))$ by Lemma 3.3. In addition, as $M \in \mathcal{VL}(S)$, $Q < V_M Q \leq O_p(M)'Q$ and $O_p(M)$ is not abelian. $\square$

To continue this section, we will prove two lemmas about the normal p'-subgroups of G.

Lemma 3.5. *Suppose that $Q \leq G_1 \leq G$, R is a p'-subgroup of G which is normalized G_1 and U is a p-subgroup of G_1. If U contains a non-trivial element x which is G_1-conjugate into $Z(Q)$, then $C_R(U) = 1$. In particular, if U is elementary abelian of order at least p^2 and every maximal subgroup of U contains an element which is G_1-conjugate into $Z(Q)^\#$, then $R = 1$.*

Proof. We may assume that $x \in Z(Q)$. Then $[C_R(x), Q] \leq R \cap Q = 1$, as Q is large. But then, also as Q is large, we have that $C_R(x) \leq C_G(Q) \leq Q$. Hence $C_R(x) = 1$ which is the first assertion. The second claim follows by coprime action [6, Proposition 11.13]. $\square$

Lemma 3.6. *Let $M \in \mathcal{VL}_{lin} \cup \mathcal{VL}_{orthsymp}$. Suppose that $\overline{M^\circ} \not\cong \Omega_3(p)$, p odd. Then M° does not normalize any non-trivial p'-subgroup of G. In particular $O_{p'}(G) = 1$.*

Proof. We may assume that $S \leq M$. Assume that R is a non-trivial p'-subgroup of G which is normalized by M°. By the definition of $\mathcal{VL}_{lin}$ and $\mathcal{VL}_{orthsymp}$, we have that $\overline{M^\circ} \cong \Omega^\pm_{2n}(q)$, $\Omega_{2n+1}(q)$ (q odd), $\mathrm{Sp}_{2n}(q)$ (q even), $\mathrm{Sp}_4(2)'$ or $\mathrm{SL}_2(q)$ and V_M is the corresponding natural module.

By Lemma 3.5 applied with $U = Z(Q) \cap V_M$, we have $|Z(Q) \cap V_M| = p$. In particular, as $Z(Q) \cap V_M$ contains a 1-space of V_M this implies $q = p$. Furthermore, as V_M is the natural module for $\overline{M^\circ} \cong \mathrm{SL}_2(q)$, $\mathrm{Sp}_{2n}(q)$ or $\mathrm{Sp}_4(2)'$ these cases cannot occur as all non-trivial elements of V_M are conjugate. If $\overline{M^\circ} \cong \Omega^\pm_n(p)$, $n \geq 4$, then $Z(Q) \cap V_M$ corresponds to a singular 1-space in V_M and every maximal subgroup of V_M contains an element corresponding to a singular vector for the action of M° on V_M. This contradicts Lemma 3.5 applied with $U = V_M$. We are left with $\overline{M^\circ} \cong \Omega_3(p)$ with p odd and this is the excluded case. $\square$

The special case in Lemma 3.6 with p odd, $\overline{M^\circ} \cong \Omega_3(p)$ and $O_{p'}(G) \neq 1$ really does occur. Suppose that $K \cong \mathrm{PSp}_{2n}(p)$ with p odd is a subgroup of G containing Q and $O_{p'}(G) \neq 1$. As Q is large, this means that a root element in K acts fixed-point-freely on $O_{p'}(G)$. By a result of John Thompson $O_{p'}(G)$ is nilpotent. Using a result due to Alex Zalesskii [29, Theorem 3] this implies that $2n = 4$. Furthermore, if R is a Sylow r-subgroup of $O_{p'}(G)$, then all the $\mathrm{PSp}_4(p)$-composition factors in R are isomorphic to the unipotent modules of dimension $r(r^2-1)/2$. In particular, the semidirect product of such a module with $K = \mathrm{PSp}_4(p)$ is an example of a group with a large p-subgroup Q, where $N_G(Q)$ and M are the two minimal parabolic subgroups containing S, (so $M^\circ \cong p^3.\Omega_3(p)$ and $N_G(Q)' \cong p_+^{1+2}.\mathrm{SL}_2(p)$). In this case, $\langle \mathcal{L}_G(S)\rangle$ is a complement to $O_{p'}(G)$. Selecting a non-split extension (if such exists) provides an example with $G = \langle \mathcal{L}_G(S)\rangle$. In the smallest example, $p = 3$ and we can choose G to be the non-split extension $2^6{\cdot}\mathrm{PSp}_4(3)$ (the derived group of $\mathrm{Aut}(E)$ where $E \cong 2_-^{1+6}$). In this case, as G is a non-split extension, we have $G = \langle M, N_G(Q)\rangle$ and G has parabolic characteristic 3.

Finally we prove a lemma which could have been in [15].

Lemma 3.7. *Assume that $L \in \mathcal{L}_G(S)$ with $\overline{L^\circ} \cong \mathrm{Sp}_4(2)'$. If $V_L = V_{nat}$, then $Y_L = V_L$.*

Proof. Let $X = C_{V_L}(S)$. Then $|X| = 2$ and $\overline{C_{L^\circ}(X)} \cong \mathrm{Sym}(4)$. As $X \leq Z(Q)$ and Q is large, $C_{L^\circ}(X) \leq N_G(Q)$. Since $Q \not\leq O_p(L)$ by Lemma 3.2 (i), $\overline{Q}$ is a non-trivial normal subgroup of $\overline{C_{L^\circ}(X)}$. Thus $\overline{Q} = O_2(\overline{C_{L^\circ}(X)})$. In particular, $\overline{Q}$ does not act quadratically on V_L. Hence there is $U \leq \overline{L^\circ}$, $U \cong \mathrm{Alt}(5)$, $\overline{Q} \leq U$ and V_L is the non-trivial irreducible part of the permutation module for U. As V_L is a projective U-module, $Y_L = V_L \times C_{Y_L}(U)$. In particular $[Q, C_{Y_L}(U)] = 1$. As Q is large and U does not normalize Q, we obtain $C_{Y_L}(U) = 1$, so $V_L = Y_L$. □

4. The proof of Theorem A

In this section we assume that G is a finite $\mathcal{K}_p$-group, S a Sylow p-subgroup of G and $Q \leq S$ is a large subgroup of G with $Q = O_p(N_G(Q))$ and $\mathcal{VL} \neq \emptyset$. Then, by Lemma 3.2(iv) (b), there are at least two maximal p-local subgroups containing S. Hence the conditions required to apply [15, Theorems A] are satisfied and this provides us with the structure of $\overline{L^\circ}$ and V_L for $L \in \mathcal{VL}$. To interpret what we require from [15] we need some notation.

We assume that the reader is familiar with the action of the simple groups on small modules as can be found in [15, Appendix A and B] and, in particular, in the section [15, Naming Modules]. For a classical group X defined over $\mathrm{GF}(q)$, $q = p^a$, V_{nat} denotes a natural module for $\mathrm{GF}(q)X$ considered as a $\mathrm{GF}(p)X$-module. We extend this notation to subgroups of classical groups which contain the derived subgroup when this group is perfect. For $X = \mathrm{SL}_n(q)$, we denote by $\bigwedge^2(V_{\mathrm{nat}})$, $\mathrm{S}^2(V_{\mathrm{nat}})$ and $\mathrm{U}^2(V_{\mathrm{nat}})$, the alternating square, symmetric square and unitary square of a natural module $\mathrm{GF}(q)X$-

module considered as a $\mathrm{GF}(p)X$-module. For clarity this means that when $|V_{\mathrm{nat}}| = p^{an}$, $|\bigwedge^2(V_{\mathrm{nat}})| = p^{a(n-1)n/2}$, $|\mathrm{S}^2(V_{\mathrm{nat}})| = p^{a(n+1)n/2}$ and $|\mathrm{U}^2(V_{\mathrm{nat}})| = p^{an^2/2}$.

We also need to consider some exceptional cases. In this context $X = \mathrm{Spin}_{10}^+(q)$ appears acting on one of the half-spin modules, again considered as a $\mathrm{GF}(p)X$-module and simply denoted *half-spin*. For X the quasisimple group $\mathrm{E}_6(q)$, there are two natural modules over $\mathrm{GF}(p)$, both of dimension $27a$ where $q = p^a$. In this situation, we denote these modules by q^{27}. Similarly, 2^6 denotes one of the two faithful $\mathrm{GF}(2)$-modules of dimension 6 for $3^{\cdot}\mathrm{Alt}(6)$. We also call the dual module of a natural module a natural module. The same principle applies for all the modules above. See the discussion about this in [15, Section A.2].

One of the main objectives of this section is to prove Theorem A. As we have already remarked in the introduction, the theorem has already been proved for $L \in \mathcal{VL}$ which satisfies the wreath product case [15, Theorem A (3)]. We now will prove a little bit more. We will show that Theorem A holds if $\mathcal{VL}_{wreath} \neq \emptyset$. Recall that the *unambiguous wreath product* cases and corresponding *ambiguous wreath product* cases are explained just before the statement of the main theorem in [24].

Lemma 4.1. *Assume there exists $L \in \mathcal{VL}(S)$ with L in the unambiguous wreath product case. Then*

$$\mathcal{VL}(S) = \mathcal{VL}_{wreath}(S)$$

and Theorem A holds.

Proof. The main theorem in [24] characterises the groups which have $L \in \mathcal{VL}(S)$ with L in the unambiguous wreath product case. By [24, Proposition 3.5], we have that $\overline{L^\circ} \cong \mathrm{O}_4^+(2)$, $\Gamma\mathrm{L}_2(4)$ or $\mathrm{SL}_2(4)$ and $L \in \mathcal{VL}_{wreath}(S)$. In all cases $L^\circ S \in \mathcal{VL}_{wreath}^{min}(S)$. Thus Theorem A holds for L.

We now will show $\mathcal{VL}(S) = \mathcal{VL}_{wreath}(S)$, which then proves the lemma.

Assume that $\overline{L^\circ} \cong \mathrm{O}_4^+(2)$, then by [24, Proposition 4.1] we have $G \cong \mathrm{Sym}(8)$, $\mathrm{Sym}(9)$ or $\mathrm{Alt}(10)$ and $\overline{Q} \cong \mathrm{Dih}(8)$. If $G \cong \mathrm{Sym}(8)$ or $\mathrm{Sym}(9)$, then there is just one maximal element $M \in \mathcal{L}_G(S)$ with $M \neq L$. This group is the normalizer of Q and so the claim holds in this case. In case of $G \cong \mathrm{Alt}(10)$ again we have just one maximal group $M \in \mathcal{L}_G(S) \setminus \{L\}$, $M \cong 2^4\mathrm{O}_4^-(2)$, where Y_M is the natural module. Further $N_G(Q) \leq M$ and so $M \notin \mathcal{VL}(S)$. Thus $\mathcal{VL}_{wreath}(S) = \mathcal{VL}(S)$ in this case as well.

Assume that $\overline{L^\circ} \cong \Gamma\mathrm{L}_2(4)$. Then by [24, Proposition 5.1] we have $G \cong \mathrm{Mat}(22)$ or $\mathrm{Aut}(\mathrm{Mat}(22))$. By [7, Table 5.3c] all elements in $\mathcal{L}_G(S)$, which are not contained in L are contained in the subgroup $M \cong 2^4.\mathrm{Alt}(6)$ or $2^4.\mathrm{Sym}(6)$, respectively. In both case $N_G(Q) \leq M$ and so $M \not\leq \mathcal{VL}(S)$. Hence we have also $\mathcal{VL}(S) = \mathcal{VL}_{wreath}(S)$ holds in these cases.

Assume finally that $\overline{L^\circ} \cong \mathrm{SL}_2(4)$. By [24, Proposition 6.2], $|Y_L : V_L| = 2$ and $G \cong \mathrm{Aut}(\mathrm{Mat}(22))$ again. Furthermore we have that $N_G(Q) \cong 2^4.\mathrm{Sym}(6)$, which again shows that $\mathcal{VL}_{wreath}(S) = \mathcal{VL}(S)$ holds. $\square$

Table 2
The possibilities for $(\overline{L^\circ}, p, V_L)$.

	$\overline{L^\circ}$	p	V_L	LST Thm. A
(1)	$\mathrm{SL}_n(q)$, $n \geq 2$	p	V_{nat}	(1), (3), (6)
(2)	$\mathrm{Sp}_{2n}(q), n \geq 2$ or $\mathrm{Sp}_4(2)'$	2	V_{nat}	(2)
(3)	$\Omega_n^{\pm}(q), n \geq 5$ or $(n, \pm) = (4, +)$	p	V_{nat}	(3), (5), (10)(5, 6, 7)
(4)	$\mathrm{SL}_n(q)/\langle(-I_n)^{n-1}\rangle, n \geq 5$	p	$\bigwedge^2(V_{\mathrm{nat}})$	(7)(1)
(5)	$\mathrm{SL}_n(q)/\langle(-I_n)^{n-1}\rangle, n \geq 2$	p odd	$\mathrm{S}^2(V_{\mathrm{nat}})$	(7)(2)
(6)	$\mathrm{SL}_n(q^2)/Z, n \geq 2$	p	$\mathrm{U}^2(V_{\mathrm{nat}})$	(7)(3)
	$Z = \langle \lambda I_n \mid \lambda \in \mathrm{GF}(q^2), \lambda^n = \lambda^{q+1} = 1\rangle$			
(7)	$\mathrm{Spin}_{10}^+(q)$	p	half-spin	(8)(1)
(8)	$\mathrm{E}_6(q)$	p	q^{27}	(8)(2)
(9)	$3^{\cdot}\mathrm{Alt}(6)$	2	2^6	(9)(1)
(10)	$3^{\cdot}\mathrm{Sym}(6)$	2	2^6	(9)(1)
(11)	Mat(22)	2	Golay 2^{10}	(9)(2)
(12)	Mat(24)	2	Todd 2^{11}	(9)(3), (10)(8)
(13)	Mat(24)	2	Golay 2^{11}	(9)(3)
(14)	Mat(11)	3	Golay 3^5	(9)(4)
(15)	Aut(Mat(22))	2	Todd 2^{10}	(10)(2)
(16)	$2^{\cdot}\mathrm{Mat}(12)$	3	Golay 3^6	(10)(3)
(17)	$(\mathrm{SL}_{m_1}(q) \otimes \mathrm{SL}_{m_2}(q))/Z$	p	$V_{\mathrm{nat}_1} \otimes V_{\mathrm{nat}_2}$	(6)

Assume that there is $L \in \mathcal{VL}(S)$, which is in the ambiguous wreath product case. From [24], this means that $\overline{L^\circ} \cong \mathrm{SL}_2(q)$ or $q = p = 2$ and $\overline{L^\circ} \cong \Omega_4^+(2)$ and in both cases, Y_L is the natural $\overline{L^\circ}$-module and so $L \in \mathcal{VL}_{lin} \cup \mathcal{VL}_{orthsymp}$.

Thus from here on, to prove Theorem A, we may assume that no member of $\mathcal{VL}(S)$ is in the unambiguous wreath product case. In particular, this has the consequence that no member of $\mathcal{VL}$ satisfies the weak wreath product case [15, Theorem A (4)].

We now list the triples $(\overline{L^\circ}, p, V_L)$ that the Local Structure Theorem requires that we examine in order to prove Theorem A.

Proposition 4.2 *(Case Division for the Global Structure Theorem). Assume that p is a prime, G is a finite $\mathcal{K}_p$-group which contains a large subgroup and $L \in \mathcal{VL}$. Then the triple $(\overline{L^\circ}, p, V_L)$ and its location in the Local Structure Theorem (LST) is as described in Table 2. In particular, in each case V_L is an irreducible L°-module.*

Proof. We use our standard notation and apply [15, Theorem A] to check which cases specify that $V_L \not\leq Q$ ($Q^\bullet$ in [15, Theorem A]). In Table 2, the final column indicates to which of the 10 outcomes of [15, Theorem A] a particular line in the table corresponds. After observing that case [15, Theorem A (10)(1)] does not appear as there $Y_L \leq Q$ since Y_L is not characteristic p-tall in G, there are just three cases which require further discussion. These are the wreath product cases [15, Theorem A(3) and (4)] and the tensor product case [15, Theorem A (6)]. We have already mentioned that in the wreath product case, we have repositioned two families of possibilities. The first is that $\overline{L^\circ} \cong \mathrm{SL}_2(q)$ and V_L is the natural module, this is included in line (1) here (and hence the requirement $n \geq 2$) and the other is that $\overline{L^\circ} \cong \Omega_4^+(2)$ and V_L is the natural module and this case is now included in row (3). For the tensor product case which is listed as (6) in [15, Theorem A] we adopt their notation. In particular, $\overline{L} = \overline{L_1 L_2}$ with $L_1/C_L(Y_L) \cong \mathrm{SL}_{m_1}(q)$, $L_2/C_L(Y_L) \cong \mathrm{SL}_{m_2}(q)$, $m_1, m_2 \geq 2$ and Y_L is the tensor product of two natural modules. The first possibility listed in [15, Theorem A (6)(c)] is that $\overline{L^\circ} \cong \mathrm{O}_4^+(2)$ and Y_L is the natural module. This case can also be regarded as being in the wreath product case and in that guise is handled in [24] and so is not included here. The next possibility is that $L^\circ C_L(Y_L) = L_1$ (or L_2). In this case L_1 is a normal subgroup of L and we set $L_0 = L^\circ S = L_1 S$. Then $Y_{L_0} = C_{Y_L}(S \cap L_2)$ is the natural module for $L_0^\circ C_{L_1}(Y_{L_1})/C_{L_1}(Y_{L_1})$. Since $[L_1, L_2] \leq C_L(Y_L)$, and $L^\circ C_L(Y_L) = L_1$, we obtain $L_2 \leq N_G(Q)$ as Q is weakly closed in S with respect to L. So Y_L is just the normal closure of Y_{L_1} under L_2 and therefore $Y_L \not\leq Q$ if and only if $Y_{L_1} \not\leq Q$. Thus this configuration is the one described in item (1) of Table 2. The final possibility is listed in line (17) of Table 2 if $(t_1, t_2) \neq (2, 2)$ and otherwise it is included in line (3) as $\overline{L} \cong \Omega_4^+(q)$ and V_L is the natural module. □

Lemma 4.3. *Suppose that $K \in \mathcal{VL}(S)$ and $L \in \mathcal{L}_K(S)$ with $L \not\leq N_G(Q)$. If $V_L \leq Q$, then $\overline{L^\circ} \cong \mathrm{SL}_n(q)$ for some n and q and $V_L = V_{\mathrm{nat}}$.*

Proof. As $V_L \leq Q$ and $K \in \mathcal{VL}(S)$, we have $V_L \neq V_K$. Since the statement concerns L°, and $Y_{L^\circ S} \leq Y_L \leq Q$ by Lemma 3.1 (v), we may replace L by $L^\circ S$.

We apply [15, Lemma 1.56(a)] first to K. Thus there exists $M \in \mathfrak{M}_G(S)$ and $K^* \leq M$ such that

$$S \leq K^*, Y_K = Y_{K^*}, KC_G(Y_K) = K^* C_G(Y_K) \text{ and } K^\circ = (K^*)^\circ.$$

Since $L = L^\circ S$ and $L^\circ \leq K^\circ = (K^*)^\circ$, we have $L \leq M$. In particular, [15, Theorem A] applies to L and M. As $Y_K = Y_{K^*}$ and $(K^*)^\circ = K^\circ$, we obtain $Y_K \leq Y_M$ and $V_K \leq V_M$. Hence $M \in \mathcal{VL}(S)$. We consider each of the cases in [15, Theorem A].

In the linear case, [15, Theorem A(1)], we have the claimed outcome.

In the symplectic case, [15, Theorem A(2)], $\overline{L^\circ} \cong \mathrm{Sp}_{2n}(q)$ and $n \geq 2$ or $\overline{L^\circ} \cong \mathrm{Sp}_4(2)'$. As $V_L \neq V_M$, [15, Theorem A(2)(d)] applies to give $V_L \not\leq Q$, a contradiction.

Similarly, in the wreath product case, [15, Theorem A(3)(b) and (c)] yields $\overline{L^\circ} \cong \mathrm{SL}_2(q)$ and $Y_L = V_L$ is the natural module.

In the weak wreath product case, [15, Theorem A(4)], M satisfies the wreath product case with $\overline{M}^\circ \ncong \mathrm{SL}_2(q)$. However, we have already remarked that this case cannot occur.

In conclusions [15, Theorem A(5)–(9)], $V_L \not\leq Q$. In case [15, Theorem A (10)], then either $V_L = V_M$ or $V_L \not\leq Q$, both of which are impossible

This concludes the inspection of the cases in [15, Theorem A] and proves the lemma. □

Lemma 4.4. *Suppose that $L \in \mathcal{VL}_{lin}(S) \cup \mathcal{VL}_{orthsymp}(S)$. Then*

$$L^\circ \cap N_G(Q) = N_{L^\circ}(C_{V_L}(S \cap L^\circ))$$

Table 3
The structure of $N_{\overline{L^\circ}}(C_{V_L}(S \cap L^\circ))$.

$\overline{L^\circ}$	Structure of $N_{\overline{L^\circ}}(C_{V_L}(S \cap L^\circ))$
$\Omega_n^\pm(q), n \geq 4, (n, \pm) \neq (4, +)$	$q^{n-2}.\Omega_{n-2}^\pm(q).(q-1)$
$\Omega_4^+(q)$	$q^2.(q-1)^2/\gcd(q-1, 2)$
$\Omega_3(q)$, q odd	$q.(q-1)/2$
$\mathrm{Sp}_{2n}(q), n \geq 2$ and q even	$q^{2n-1}.\mathrm{Sp}_{2n-2}(q).(q-1)$
$\mathrm{Sp}_4(2)'$	$\mathrm{Sym}(4)$
$\mathrm{SL}_2(q)$	$q.(q-1)$

Table 4
The action of $L^\circ \cap N_G(Q)$ on V_L and $\overline{Q}$.

$\overline{L^\circ}$	$\lvert C_{V_L}(Q)\rvert$	$\lvert (V_L \cap Q)/C_{V_L}(Q)\rvert$	irr.	$\overline{Q}$ irr.
$\Omega_n^\pm(q), (n, \pm) \neq (4, +)$	q	q^{n-2}	yes	yes
$\Omega_4^+(q)$	q	q^2	no	no
$\mathrm{Sp}_{2n}(q), n \geq 2$	q	q^{2n-2}	yes	no
$\mathrm{Sp}_4(2)'$	2	2^2	yes	yes
$\mathrm{SL}_2(q)$	q	1	-	yes

and the structure of $N_{\overline{L^\circ}}(C_{V_L}(S \cap L^\circ))$ is as described in Table 3. In particular, $\overline{Q}$ is normalized by $N_{\overline{L^\circ}}(C_{V_L}(S \cap L^\circ))$ and

$$O_p(N_{\overline{L^\circ}}(C_{V_L}(S \cap L^\circ))) \geq \overline{Q} > 1.$$

Proof. Since $Q \leq L^\circ$, $C_{V_L}(S \cap L^\circ) \leq C_{V_L}(Q) \leq Z(Q)$ and so, as Q is large, then $N_{L^\circ}(C_{V_L}(S \cap L^\circ)) \leq N_G(Q)$. Since V_L is a natural $\overline{L^\circ}$-module, we have $N_{L^\circ}(C_{V_L}(S \cap L^\circ))$ is a maximal parabolic subgroup of L°. As $L \not\leq N_G(Q)$ it follows that $L^\circ \cap N_G(Q) = N_{L^\circ}(C_{V_L}(S \cap L^\circ))$. The information in Table 3 follows from the structure of point stabilisers when acting on natural modules as given in [15, Lemma B.28] for example. Note here that $\Omega_n^\pm(q)$ for $n \geq 4$ is transitive on singular vectors where as $\Omega_3(q)$ with q odd, has two orbits on singular vectors. $\square$

For each possibility for $\overline{L^\circ}$ and V_L in Table 3, columns 2 and 3 of Table 4 give $|C_{V_L}(Q)|$ and $|(V_L \cap Q)/C_{V_L}(Q)|$. In addition, column 4 of Table 4 indicates whether or not $L^\circ \cap N_G(Q)$ acts irreducibly on $(V_L \cap Q)/C_{V_L}(Q)$ and the final column gives the same information for the action of $\overline{L^\circ \cap N_G(Q)}$ on $\overline{Q}$.

Lemma 4.5. *Assume that $L \in \mathcal{VL}_{lin}(S) \cup \mathcal{VL}_{orthsymp}(S)$ and set $X = C_{V_L}(S \cap L^\circ)$. Then $\overline{Q} = O_p(N_{\overline{L^\circ}}(X)$, $C_{V_L}(Q) = X$ has order q, $V_L \cap Q = [V_L, Q]$, $|\widetilde{V_L}| = q$ and $N_{\overline{L^\circ}}(X))$ acts irreducibly on $\widetilde{V_L}$. In particular, the information presented in Table 4 is correct.*

Proof. Observe that $QO_p(L)$ is normal in $L \cap N_G(Q) = N_{L^\circ}(X)$ and so $\overline{Q}$ is normal in $N_{\overline{L^\circ}}(X)$ and $\overline{Q} \neq 1$ by Lemma 4.4.

Recall from Lemma 3.1 (vi) that V_L is a Q!-module for L with respect to Q. We begin by considering $L \in \mathcal{VL}_{orthsymp}(S)$ and $|V_L| \geq q^4$. Then, as V_L is a natural L°-module, we may apply [15, Lemma B.37]. We note that the fact that $\overline{L}^\circ \cong \Omega_n^\pm(q)$, $\mathrm{Sp}_{2n}(q)$ or $\mathrm{Sp}_4(2)'$ implies that $\overline{L}^\circ \leq Cl^\diamond(V_M)^\diamond = H^\diamond$ where we have temporarily adopted the notation of [15, Page 268] to ease the application of [15, Lemma B.37]. In particular, $\overline{Q} \leq H^\diamond \cong \Omega_{2n}^\pm(q)$, $\mathrm{Sp}_{2n}(q)$. This means that the exceptional cases [15, Lemma B.37 (3),(4), (5) and (6)] do not occur as in these instances $\overline{Q}$ is not contained in $H^\diamond$ or $\overline{L}^\circ \leq \overline{Q}^{H^\diamond}$ is too small. Thus [15, Lemma B.37 (1) or (2)] hold and this shows that $C_V(Q) = X$ and $\overline{Q} = O_p(N_{\overline{L^\circ}}(X))$ in all these cases. Furthermore, $|V_L/[V_L, Q]| = q$ and $|[V_L, Q]/X| = q^{n-2}$ if $\overline{L^\circ}$ is orthogonal and q^{2n-2} if $\overline{L^\circ}$ is symplectic or $\mathrm{Sp}_4(2)'$.

If $\overline{L}^\circ \cong \Omega_3(q)$, or $L \in \mathcal{VL}_{lin}(S)$, then $N_{\overline{L^\circ}}(X) = N_{\overline{L^\circ}}(S \cap L^\circ)$ acts irreducibly on $\overline{S \cap L^\circ}$ and so $\overline{Q} = \overline{S \cap L^\circ} = O_p(N_{\overline{L^\circ}}(X))$. Furthermore, in both cases we have $|X| = q = |V_L : [V_L, Q]|$ with $[V_L, Q] : X| = q$ when $\overline{L}^\circ \cong \Omega_3(q)$.

Since $N_{\overline{L}^\circ}(\overline{Q})$ acts irreducibly on $V_L/[V_L, Q]$ which has order q and $V_L \cap Q$ is normalized by $N_{L^\circ}(\overline{Q})$, $V_L > V_L \cap Q \geq [V_L, Q]$ implies $|\widetilde{V_L}| = q$. This establishes the main statements of the lemma and so it just remains confirm the details in the fourth column of Table 4. By [15, Lemmas B.28 (b) and (c)] if $\overline{L^\circ} \cong \mathrm{Sp}_{2n}(q)$, $n \geq 2$, or $\Omega_n^\pm(q)$ with $(n, \pm) \neq (4, +)$, $[V_L, Q]/C_{V_L}(Q)$ is an irreducible natural module for $N_{\overline{L^\circ}}(C_{V_L}(S \cap L^\circ))/\overline{Q}$. Since $C_{V_L}(Q) = [V_L, Q]$ for $L^\circ \cong \mathrm{SL}_2(q)$, we have established row 1,3 and 5 of Table 4. If $\overline{L^\circ} \cong \Omega_4^+(q)$, $[V_L, Q]/C_{V_L}(Q)$ is the natural module for $\Omega_2^+(q)$ by [15, Lemmas B.28 (d)] and, as this group does not act irreducibly on the natural module, the information provided in row 2 of Table 4 is correct. To provide the information in row 4, we may calculate explicitly with $\mathrm{Sp}_4(2)'$ acting on V_L. □

Corollary 4.6. *Suppose that $L \in \mathcal{VL}_{orthsymp}(S)$. Then $[V_L, Q] = V_L \cap Q$, and $[V_L, Q, Q] = C_{V_L}(Q)$. In particular, Q is not abelian and Q does not act quadratically on V_L.*

Proof. By Lemma 4.5, $\overline{Q} = O_p(N_{\overline{L^\circ}}(C_{V_L}(S \cap L^\circ)))$ and $[V_L, Q] = V_L \cap Q$. Since V_L is the natural $\overline{L^\circ}$-module, we have $[V_L \cap Q, Q] = C_{V_L}(Q)$ by applying [15, Lemma B.28] or by calculation if $\overline{L^\circ} \cong \mathrm{Sp}_4(2)'$. This proves the claim. □

Lemma 4.7. *Suppose that $L \in \mathcal{VL}(S)$. Then $L^\circ S \in \mathcal{VL}(S)$.*

Proof. We know from Lemma 3.1(iii)(c) and (v) that $L = L^\circ N_L(Q)$ and $V_{L^\circ S} \leq V_L$. Therefore, as V_L is an irreducible L°-module by Proposition 4.2, we have

$$V_L = \langle V_{L^\circ S}^L \rangle = \langle (V_{L^\circ S})^{L^\circ N_L(Q)} \rangle = \langle (V_{L^\circ S})^{N_L(Q)} \rangle.$$

Since $L \in \mathcal{VL}(S)$, we conclude that $V_{L^\circ S} \not\leq Q$ which means that $L^\circ S \in \mathcal{VL}(S)$. □

Lemma 4.8. *If $M \in \mathcal{VL}^{min}(S)$, then $M = M^\circ S = N_M(S \cap C_M(Y_M))$, $S \cap C_M(Y_M) = O_p(M)$, $C_M(Y_M)$ is p-closed and $Y_M = \Omega_1(Z(O_p(M)))$.*

Proof. As $M \in \mathcal{VL}^{min}(S)$, Lemma 4.7 implies $M = M^\circ S$. In particular, $C_M(Y_M)$ is p-closed. Employing [15, Lemma 1.24 (k)] now gives $Y_M = \Omega_1(Z(O_p(M)))$. □

Lemma 4.9. *Suppose that $L \in \mathcal{VL}(S)$ and that there exists a unique $U \le L$ normalized by S and containing $(S \cap L^\circ)C_L(Y_L)$ minimal with the property that $US \in \mathcal{VL}(S)$. Then $U^\circ S$ is the unique member of $\mathcal{VL}^{min}(S)$ contained in L.*

Proof. Suppose that $M \in \mathcal{VL}^{min}(S)$, $M \le L$. Then Lemma 4.8 implies that $M = M^\circ S \le L^\circ S$. Let $U^* = M^\circ(S \cap L^\circ)C_L(Y_L)$. Then U^* is normalized by S, contains $(S \cap L^\circ)C_L(Y_L)$ and $Y_M \le Y_{U^*S}$ by [15][Lemma 1.26(f)]. Hence $U^*S \in \mathcal{VL}(S)$ and so $U \le U^*$ by the uniqueness of U. It follows that

$$U^\circ \le (U^*)^\circ = (M^\circ)^\circ = M^\circ.$$

Since $U^\circ S \in \mathcal{VL}(S)$ by Lemma 4.7 and $M = M^\circ S \in \mathcal{VL}^{min}(S)$, we have $M = U^\circ S$. This proves the lemma. □

Lemma 4.10. *Let $L \in \mathcal{VL}_{lin}(S) \cup \mathcal{VL}_{orthsymp}(S)$. If $M \in \mathcal{VL}(S)$ with $M \le L$, then $M^\circ = L^\circ$. In particular, $L^\circ S \in \mathcal{VL}^{min}_{lin}(S) \cup \mathcal{VL}^{min}_{orthsymp}(S)$ and $L^\circ S$ is the unique member of $\mathcal{VL}^{min}(S)$ contained in L.*

Proof. Since $M \le L$, $M^\circ \le L^\circ$ and $V_M \le V_L$ by Lemma 3.1 (v). Assume $M^\circ < L^\circ$. Then $M^\circ(S \cap L^\circ)/O_p(L^\circ)$ is a proper subgroup of $L^\circ/O_p(L^\circ)$. As $L^\circ/O_p(L^\circ)$ is a group of Lie type in characteristic p, the Borel-Tits Theorem [7, Theorem 2.6.7] implies $O_p(M^\circ(S \cap L^\circ))O_p(L^\circ)/O_p(L^\circ) \ne 1$. Therefore $V_M \le C_{V_L}(O_p(M^\circ(S \cap L^\circ))) < V_L$ with $C_{V_L}(O_p(M^\circ(S \cap L^\circ)))$ a GF(q)-subspace of V_L which is normalized by Q. But then Lemma 4.5 and Table 4 imply

$$V_M \le C_{V_L}(O_p(M^\circ(S \cap L^\circ))) \le [V_L, Q] = V_L \cap Q \le Q.$$

This contradicts $M \in \mathcal{VL}(S)$ and so we conclude that $M^\circ = L^\circ$. □

We are now going to prove Theorem A and at the same time verify Table 1.

Proof of Theorem A. We prove parts (i) and (ii) together. Recall that by Lemma 4.1 we may assume that $\mathcal{VL}_{wreath} = \emptyset$. By Lemma 4.8, $C_{M_L}(Y_{M_L})$ is p-closed and $Y_{M_L} = \Omega_1(Z(O_p(M_L)))$. Hence the corresponding parts of Theorem A (i) hold once we have found M_L.

Set

$$S_0 = S \cap L^\circ C_L(Y_L),$$
$$J = N_{L^\circ C_L(Y_L)}(C_{Y_L}(S_0))).$$

We investigate the cases described in Proposition 4.2. We first consider the tensor product case described in the final line of Table 2. Thus $\overline{L^\circ} = \overline{L_1 L_2}$ with $\overline{L_i} \cong \mathrm{SL}_{m_i}(q)$ for $i = 1, 2$. As $Y_L = V_L$ is the tensor product module for $\overline{L^\circ}$, we have $V_L = W_1 \otimes W_2$ where W_i is a natural $\overline{L_i}$-module. We calculate that $O^{p'}(\overline{J})$ has shape

$$(q^{m_1-1}.\mathrm{SL}_{m_1-1}(q)) \times (q^{m_2-1}.\mathrm{SL}_{m_2-1}(q))$$

and, as $\overline{L^\circ} = \overline{L_1 L_2}$, $\overline{Q} = O_p(\overline{J})$.

Select subgroups P_1 of L_1 containing $S_0 \cap L_1$ and P_2 of L_2 containing $S_0 \cap L_2$ such that $P_1 \not\leq J$, $P_2 \not\leq J$ and, for $i = 1, 2$, $\overline{P_i}/O_p(\overline{P_i}) \cong \mathrm{SL}_2(q)$. Notice that these subgroups are uniquely determined. Furthermore, as S normalizes J, P_1P_2 is normalized by S. Set $M_L = (P_1P_2)^\circ S$. By construction $(M_L/C_{M_L}(Y_{M_L}))^\circ \cong \Omega_4^+(q)$ and $Y_{M_L} = V_{M_L}$ is the corresponding natural module. Hence $M_L \in \mathcal{VL}_{orthsymp}(S)$ by Lemma 4.3. Therefore Lemma 4.9 implies that M_L is the unique member of $\mathcal{VL}^{min}(S)$ contained in L. This completes the analysis of the tensor product case in Table 2.

Assume that Table 2 (1) holds. Then $\overline{L^\circ} \cong \mathrm{SL}_m(q)$ and V_L is the natural module. Furthermore, $V_L = Y_L$ by [15, Theorem A(1)(b)]. We may take $m > 2$ as otherwise we define $M_L = \overline{L^\circ} S$ and the conclusion of the lemma holds by Lemma 4.10. As V_L is the natural $\overline{L^\circ}$-module, $O^{p'}(\overline{J})$ has shape $q^{m-1}.\mathrm{SL}_{m-1}(q)$. Hence there is a unique subgroup U of $L^\circ C_L(Y_L)$ containing $S_0 C_L(Y_L)$ with $\overline{U}/O_p(\overline{U}) \cong \mathrm{SL}_2(q)$ such that U is not contained in $N_G(Q)$. As S normalizes J and $N_G(Q)$, we know S normalizes U. Because J acts irreducibly on $V_L/C_{V_L}(S_0)$, we have

$$V_L = \langle (V_{US})^J \rangle = \langle (V_{US})^{N_L(Q)} \rangle.$$

In particular, as $V_L \not\leq Q$, $V_{US} \not\leq Q$ which means that $US \in \mathcal{VL}(S)$. Thus setting $M_L = US$, Lemma 4.9 applies to give M_L is the unique member of $\mathcal{VL}^{min}(S)$ contained in L. Furthermore, as $V_{M_L} \leq V_L$, we have that $V_{M_L} = Y_{ML}$ is the natural $\overline{M}$-module where $\overline{M_L^\circ} \cong \mathrm{SL}_2(q)$. That is $M_L \in \mathcal{VL}_{lin}^{min}$. This establishes line (2) of Table 1.

Lines (2) and (3) of Table 2 are the symplectic and orthogonal cases and these have been investigated in Lemma 4.10. In particular, in the case of line (2) of Table 2, we obtain line (3) of Table 1 and, when $\overline{L^\circ} \cong \mathrm{Sp}_4(2)'$, Lemma 3.7 implies that $V_{M_L} = Y_{M_L}$ and this is line (4) of Table 1. If (3) of Table 2 holds and $V_L = Y_L$, we have line (5) of Table 1 whereas the cases in which $Y_L > V_L$ are listed separately in lines (19), (20) and (21) of Table 1.

We next explore the non-natural $\mathrm{SL}_n(q)$-cases and the exceptional cases. These possibilities are enumerated in lines (4) to (8) of Table 2. In each case, $\overline{J}$ is a maximal parabolic subgroup of $\overline{L^\circ}$ and so there is a unique $P \in \mathcal{L}_L(S)$ with $P \not\leq N_G(Q)$ and $\overline{P}/O_p(\overline{P}) \cong \mathrm{SL}_2(q)$.

If (5) or (6) of Table 2 holds. Then $V_{PS} = Y_{PS} \leq Y_{L_0S}$ is the natural $PS/C_{PS}(Y_{PS})$-module where $\overline{P^\circ} \cong \Omega_3(q)$, p odd in the first case and $\overline{P^\circ} \cong \Omega_4^-(q)$ in the second case. Since $L \in \mathcal{VL}(S)$, Lemma 4.3 implies $V_{PS} \not\leq Q$. Hence, by Lemma 4.10, $P^\circ S \in$

$\mathcal{VL}_{orthsymp}(S)$ and so we have $M_L = P^\circ S$ in these cases and furthermore M_L is unique by Lemma 4.9. These possibilities are listed in lines (7) and (8) of Table 1 respectively.

In case (4), (7) and (8) of Table 2, we have $Y_L = V_L$ (see [15, Theorem A(7)(1), (8)]). In this case, V_{PS} is the natural $\overline{P^\circ}$-module and $\overline{P^\circ} \cong \mathrm{SL}_2(q)$. The subgroup P corresponds to the second node of the A_{n-1} Coxeter diagram in case (4), to one of the end nodes on the short arms in the D_5 Coxeter diagram in case (7) and to one of the end nodes on the long arms in the E_6 Coxeter diagram in case (8). Let $M \geq P$ be such $M \leq L$ and

$$\overline{M^\circ} \cong \begin{cases} \Omega_6^+(q) & \text{case (4)} \\ \Omega_8^+(q) & \text{case (7)} \\ \Omega_{10}^+(q) & \text{case (8).} \end{cases}$$

Notice that this choice of M is unique and minimal subject to the requirement the $\overline{M^\circ}/O_p(\overline{M^\circ})$ is an orthogonal group. In each case $Y_M \leq Y_L$ is the natural $\overline{M^\circ}$-module. By Lemma 4.3, $V_{MS} \not\leq Q$ and so $MS \in \mathcal{VL}(S)$. Now Lemma 4.10 implies that $M^\circ S \in \mathcal{VL}^{min}_{orthsymp}(S)$ is the unique member of $\mathcal{VL}^{min}(S)$ in L. These cases are listed in Table 1 lines (6), (9) and (10).

Consider lines (9) and (10) of Table 2. Then $\overline{L^\circ} \cong 3{}^{\cdot}\mathrm{Alt}(6)$ or $3{}^{\cdot}\mathrm{Sym}(6)$, $Y_L = V_L$ is an irreducible GF(2)-module of dimension 6. On restriction to $O^2(L^\circ)$, V_L can be regarded as a 3-dimensional module over GF(4). There are subgroups N_1 and N_2 in $O^2(L^\circ)$ containing $S \cap L^\circ$, such that N_1 is the normalizer of a 1-dimensional and N_2 is the normalizer of a 2-dimensional GF(4)-subspace of Y_L. We have

$$N_1C_L(Y_L)/C_L(Y_L) \cong N_2C_L(Y_L)/C_L(Y_L) \cong 3 \times \mathrm{Sym}(4).$$

Furthermore, $O^{2'}(N_1)$ centralizes $C_{Y_L}(S)$ and

$$L^\circ S = \langle N_1, N_2\rangle S = \langle O^{2'}(N_1), N_2\rangle S.$$

Since $O^{2'}(N_1) \leq N_G(Q)$, $N_2 \not\leq N_G(Q)$. In addition, have $Y_{N_2} \leq Y_L$ and Y_{N_2} has order 2^4.

If $\overline{L_0^\circ} \cong 3{}^{\cdot}\mathrm{Sym}(6)$, then, as

$$\overline{Q} \not\leq O_2(N_1/C_{L_0}(Y_{L_0})) \cap O_2(N_2/C_{L_0}(Y_{L_0})),$$

we have $\overline{Q} \not\leq Z(\overline{S})$. Hence $\overline{Q} = O_2(N_1/C_{L_0}(Y_{L_0}))$. In particular,

$$N_2^\circ C_{N_2}(Y_{N_2})/C_{N_2}(Y_{N_2}) \cong \Omega_4^+(2)$$

and Y_{N_2} is the corresponding natural module. Hence $N_2S \in \mathcal{VL}_{orthsymp}(S)$ and $N_2^\circ S \in \mathcal{VL}^{min}_{orthsymp}(S)$ by Lemma 4.10.

If $\overline{L_0^\circ} \cong 3{\cdot}\mathrm{Alt}(6)$, then $\overline{L_0} \cong 3{\cdot}\mathrm{Sym}(6)$ by [15, Theorem A (9)(1)] and so Y_{N_2} is a natural module for

$$N_2^\circ C_{N_2}(Y_{N_2})/C_{N_2}(Y_{N_2}) \cong \mathrm{SL}_2(2).$$

Thus $N_2^\circ S \in \mathcal{VL}_{lin}^{min}(S)$, which gives line (12) of Table 1.

Next consider Table 2 line (11). Then $\overline{L^\circ} \cong \mathrm{Mat}(22)$ and Y_L is the Golay module of order 2^{10}. We argue as in [15, Chapter 10.2, page 237, Case 10]. The structure of Y_L restricted to subgroups of $\mathcal{L}_{\overline{L^\circ}}(S \cap L^\circ)$ can be found in [16, Lemma 3.3]. In particular, this shows that $\overline{J} \cong 2^4{:}\mathrm{Alt}(6)$ and that there is a unique subgroup U with $\overline{U} \cong 2^4{:}\Gamma\mathrm{L}_2(4)$ such that Y_U is the $\mathrm{O}_4^-(2)$-module. Furthermore, $U^\circ C_U(Y_U)/C_U(Y_U) \cong \Omega_4^-(2)$. By Lemma 4.3, $Y_U \not\leq Q$ and so $U^\circ S \in \mathcal{VL}_{orthsymp}^{min}(S)$ is unique in L. This case is listed in Table 1 line (13).

If we have Table 2 line (12) or (13), then $\overline{L^\circ} \cong \mathrm{Mat}(24)$. If Y_L is the simple Golay code module, then arguing as in [15, Chapter 10.2, Case 11] we obtain line (14) in Table 1. If V_{L_0} is the Todd module, then the argument in [15, Chapter 10.2, Case 12] applies to give line (15) in Table 1 with $|Y_{M_L} : V_{M_L}| \leq 2$.

If Table 2 line (14) holds, then the justification in [15, Chapter 10.2, Case 13] yields Table 1 line (16).

Finally, in the case of line (15) of Table 2, arguing as in [15, Chapter 10.2, Case 14] delivers Table 1 line (17) and, if line (16) of Table 2 holds, then [15, Chapter 10.2, Case 15] gives line (18) of Table 1.

This completes the investigation of Table 2 and proves that for each such $L \in \mathcal{VL}(S)$ there is a unique minimal $M_L \in \mathcal{VL}_{lin}^{min}(S) \cup \mathcal{VL}_{orthsymp}^{min}(S)$. In particular, both Theorem A (i) and (ii) hold.

(iii) By the construction above we have that $L_0 = \langle M_L, N_G(Q) \cap L_0\rangle$. As $M_L \leq \hat{L}$, we have that $M_L = M_{\hat{L}}$. This now shows that

$$\langle L, N_G(Q)\rangle = \langle M_L, N_G(Q)\rangle = \langle M_{\hat{L}}, N_G(Q)\rangle = \langle \hat{L}, N_G(Q)\rangle. \quad \square$$

From Theorem A we now get the following corollary, which is Theorem B:

Corollary 4.11. $\mathcal{VL}^{min} = \mathcal{VL}_{lin}^{min} \cup \mathcal{VL}_{orthsymp}^{min} \cup \mathcal{VL}_{wreath}^{min}$.

5. Preliminary results for the proof of Theorem 1.2

As explained in the introduction, to prove Theorem C it suffices to prove Theorem 1.2. Hence for the remainder of this article we assume

Hypothesis 5.1. *G is a finite $\mathcal{K}_p$-group, S a Sylow p-subgroup of G, $Q \leq S$ is a large subgroup of G with $Q = O_p(N_G(Q))$ and $M \in \mathcal{VL}_{lin}^{min}(S)$.*

This means $M \in \mathcal{VL}_{lin}^{min}(S)$ corresponds to the minimal configurations listed in Table 1 lines (2) and (12). So, by definition, we have $\overline{M^\circ} \cong \mathrm{SL}_2(q)$ and $Y_M = V_M$ is the natural $\overline{M^\circ}$-module. We begin by enumerating some fundamental facts about the configuration, which have mostly already been established. If $z \in Z(Q)^\#$ and $x = z^g$ for some $g \in G$, then Q^g is the unique conjugate of Q which is normalized by $C_G(x)$ by Lemma 3.1(i). For this reason we will write $Q_x = Q^g$ and observe that $Q = Q_z$. We also define

$$Q_M = O_p(M).$$

Lemma 5.2. *The following hold:*

(i) $M = M^\circ S$ *and* $M^\circ \cap N_G(Q) = N_{M^\circ}(C_{V_M}(S \cap M^\circ)) = N_{M^\circ}(QQ_M)$.
(ii) $QQ_M \in \mathrm{Syl}_p(M^\circ Q_M)$ *and* $|Q : Q \cap Q_M| = q$.
(iii) $V_M = Y_M = \Omega_1(Z(Q_M))$ *and* $|C_{V_M}(Q)| = |[V_M, Q]| = |\widetilde{V_M}| = q$.
(iv) *If* $x \in M^\circ \setminus N_G(Q)$, $\langle Q, Q^x\rangle Q_M = M^\circ Q_M$.
(v) Q *is elementary abelian and a trivial intersection set in* G.
(vi) Q *is a strong dual* F*-module with strong dual offender* $\widetilde{V_M}$.
(vii) Q *is a strong* F*-module with strong offender* $\widetilde{V_M}$.
(viii) $N_G(V_M)^\circ = M^\circ$.

Proof. That $M = M^\circ S$ and $V_M = \Omega_1(Z(Q_M))$ is included in Lemma 4.8. Part (i) now follows from Lemma 4.4. Part (ii) is stated in Lemma 4.5. In particular, $V_M \cap Q$ is normalized by $M^\circ \cap N_G(Q)$ and so $V_M \cap Q = C_{V_M}(Q)$ has order q and $|\widetilde{V_M}| = q$. Hence (iii) holds.

Since $\mathrm{SL}_2(q)$ is generated by any two of its Sylow 2-subgroups, (iv) follows from (i) and (ii).

By Lemma 4.5 and Table 4, we have that $[Q, V_M] = C_{V_M}(Q) \le Z(Q)$ and $\Phi(Q) \le Q_M$. If $\Phi(Q) \ne 1$, then, as $Z(S) \cap \Phi(Q) \ne 1$, we have that $\Phi(Q) \cap V_M \ne 1$ by Lemma 3.1(iv)(b). Since $N_M(QQ_M)$ normalizes Q by Lemma 3.1(i), the irreducible action of $M^\circ \cap N_G(Q)$ on $C_{V_M}(Q)$ yields $\Phi(Q) \cap V_M = Z(Q) \cap V_M$. Therefore $[V_M, Q] \le \Phi(Q)$. However, this means

$$V_M \le C_{N_G(Q)}(Q/\Phi(Q)) \le O_p(N_G(Q)) = Q,$$

which is a contradiction as $V_M \not\le Q$. Therefore Q is elementary abelian. As $Q = Z(Q)$ is large it is a trivial intersection set in G by Lemma 3.1 (ii). This proves (v).

As V_M is the natural $\overline{M^\circ}$-module, we have $|[Q, V_M]| = |\widetilde{V_M}| = q$ by (iii) and $[Q, v] = [Q, V_M]$ for all $v \in V_M \setminus Q$. Since $[Q, V_M, V_M] = 1$, Q is a strong dual F-module with offender $\widetilde{V_M}$ in the sense of Definition 2.1. This is (vi).

Recall from (iii) that $|[Q, V_M]| = |\widetilde{V_M}|$. Hence [12, Lemma 1.5 (d)] and part (vi) yield $\widetilde{V_M}$ is a strong offender on Q. This is (vii).

Finally, as M acts transitively on $V_M^\#$, (vii) follows by [15, Lemma 1.57(c)]. This proves (viii). □

An important role in the proof is played by the following subgroup

$$U_M = \langle (Q \cap Q_M)^{M^\circ} \rangle.$$

From here on we fix $x \in M^\circ \setminus N_G(Q)$. With this fixed we have $\langle Q, Q^x \rangle Q_M = M^\circ Q_M$ by Lemma 5.2 (iv).

Lemma 5.3. *The subgroup $U_M = (Q_M \cap Q^x)(Q_M \cap Q)$ is elementary abelian and contains all the non-central M°-chief factors in Q_M. Furthermore, $Q^x \cap Q_M \cap Q = 1$ and $[Q, Q_M] = C_Q(U_M) = Q \cap Q_M$.*

Proof. We have that M° is Q-minimal as defined in [15, Definition 1.36]. Further, by Lemma 5.2 (v), Q is an elementary abelian trivial intersection set in G and, by Lemma 3.1 (iv)(b), $Z(M^\circ) = 1$. We intend to apply [15, Lemma 1.43] with $Y = Q$, $U_M = A$ and $M^\circ = L$. Then [15, Lemma 1.43(e)] implies $U_M = (Q \cap Q_M)(Q^x \cap Q_M)$ and by [15, Lemma 1.43(a)] U_M is elementary abelian. Further, as Q is a trivial intersection set, also $Q^x \cap Q_M \cap Q = 1$. As $M^\circ = \langle Q, Q^x \rangle$, we have $[Q_M, M^\circ] \le U_M$, so all non-central chief factors of M° in Q_M are contained in U_M. In particular, $[Q, Q_M] \le U_M$. By [15, Lemma 1.43(g)] we have that $[U_M, Q] = Q \cap U_M$, which now implies $[Q_M, Q] = U_M \cap Q = C_Q(U_M)$. □

Lemma 5.4. *Suppose that $[V_M, O_{p'}(\widetilde{N_G(Q)})] \neq 1$. Then $q = p \in \{2, 3\}$ and $V_M = Q_M$ has order p^2 and S is extraspecial of order p^3.*

Proof. By Lemma 5.2(vii), we may apply Lemma 2.4 to $\widetilde{N_G(Q)}$ and $\widetilde{V_M}$. If $|\widetilde{V_M}| \ge p^2$, then by Lemma 2.4 $\widetilde{V_M}$ centralizes $O_{p'}(\widetilde{N_G(Q)})$. Thus $|V_M : C_{V_M}(O_{p'}(\widetilde{N_G(Q)}))| = p$. As Lemma 5.2 (iii) gives $q = |\widetilde{V_M}|$, we have

$$q = |\widetilde{V_M}| = p.$$

Hence $\overline{M^\circ} \cong \mathrm{SL}_2(p)$ and V_M induces $\mathrm{GF}(p)$-transvections on Q. Suppose that $r \neq p$ is a prime and $R \in \mathrm{Syl}_r(O_{p'}(\widetilde{N_G(Q)}))$. By coprime action [2, 18.7], we may assume that R is normalized by $\widetilde{Q_M}$. As $\widetilde{V_M}$ does not centralize $O_{p'}(\widetilde{N_G(Q)})$, we may additionally assume that $\widetilde{V_M}$ does not centralize R. By [2, 23.3], $[R, \widetilde{V_M}]\widetilde{V_M} \cong \mathrm{SL}_2(p)$ with $p \in \{2, 3\}$. Since Q_M acts on $[R, \widetilde{V_M}]\widetilde{V_M}$, we see that $\widetilde{Q_M} = \widetilde{V_M} C_{Q_M}([R, \widetilde{V_M}]\widetilde{V_M})$. Therefore $V_M \not\le \Phi(Q_M)$ and so Lemma 3.4 with Lemma 5.2 (iii) implies that $V_M = Q_M$. It follows that $|Q_M| = p^2$ and S is extraspecial of order p^3. □

Lemma 5.5. *Suppose $\widetilde{L}$ is a component of $\widetilde{N_G(Q)}$ with $[\widetilde{V_M}, \widetilde{L}] \neq 1$ and let $L \ge Q$ denote its preimage. Then $\widetilde{L}$ is the unique component of $\widetilde{N_G(Q)}$ which admits $\widetilde{V_M}$ non-trivially, $M \cap N_G(Q)$ normalizes L and Q is irreducible as an L-module.*

Proof. By Lemma 5.2(vi) $\widetilde{V_M}$ is a strong dual offender on Q. Application of [12, Lemma 1.5(c)] implies that $\widetilde{V_M}$ is a best offender on Q, which then by [12, Lemma 2.3(a)] yields that $\widetilde{V_M}$ normalizes $\widetilde{L}$. By [12, Lemma 2.6] $\widetilde{V_M}$ normalizes any perfect L-submodule $X = [X, L]$ in Q. Now [12, Lemma 1.4(b)] implies that $[Q, V_M] \leq X$ and so

$$[Q, L] \leq [Q, \langle V_M^L \rangle] = \langle [Q, V_M]^L \rangle \leq X = [X, L] \leq [Q, L].$$

Therefore

$$[Q, L]/C_{[Q,L]}(L) \text{ is irreducible and } [Q, V_M] \leq \langle [Q, V_M]^L \rangle = [Q, L]. \tag{$*$}$$

Assume $\widetilde{K} \neq \widetilde{L}$ is a component of $\widetilde{N_G(Q)}$ which is not centralized by $\widetilde{V_M}$. Then, by ($*$), $[Q, K] = \langle [Q, V_M]^K \rangle \leq [Q, L]$ and similarly $[Q, L] \leq [Q, K]$. Hence $[Q, K] = [Q, L]$. Since $[Q, K, L] = 0$ by [12, Theorem 1(b)], we now have $[Q, L, L] = 0$, which is a contradiction as L is not a p-group. Hence $\widetilde{L}$ is the unique component of $\widetilde{N_G(Q)}$ which is normalized and not centralized by V_M.

Since $M \cap N_G(Q)$ normalizes V_M and permutes the components of $\widetilde{N_G(Q)}$ which are not centralized by $\widetilde{V_M}$, $M \cap N_G(Q)$ normalizes L. In particular, Q_M normalizes L and so also normalizes $C_{[Q,L]}(L)$. If $C_{[Q,L]}(L) \neq 1$, then

$$V_M \cap C_{[Q,L]}(L) = \Omega_1(Z(Q_M Q)) \cap C_{[Q,L]}(L) \neq 1$$

by Lemma 5.2 (ii). As $M \cap N_G(Q)$ acts irreducibly on $[V_M, Q]$, we deduce $[V_M, Q] \leq C_{[Q,L]}(L)$. Since, by ($*$), $[Q, L] = \langle [V_M, Q]^L \rangle$, this is impossible. Hence $C_{[Q,L]}(L) = 1$ and $[Q, L]$ is irreducible as an L-module. Since $[Q, L]$ is normalized by $M \cap N_G(Q)$ and $[Q, L] \not\leq Q_M$, $QQ_M = [Q, L]Q_M$ and $Q = [Q, L](Q \cap Q_M)$. Hence, as U_M is abelian by Lemma 5.3, $Q \cap Q_M = [Q, Q_M] = [[Q, L], U_M]$. Thus $Q = (Q_M \cap Q)[Q, L] = [Q, L]$ and this proves the claim. $\square$

Lemma 5.6. *Suppose that $\widetilde{V_M}$ acts faithfully on a component $\widetilde{L}$ of $\widetilde{N_G(Q)}$. Assume that $\widetilde{L}$ is isomorphic to $\mathrm{SL}_n(p^f)$ or $\mathrm{Sp}_{2n}(p^f)$ with $n \geq 2$. Then $\widetilde{L} = \langle \widetilde{V_M}^{\widetilde{N_G(Q)}} \rangle \cong \mathrm{SL}_n(q)$, Q is the natural $\widetilde{L}$-module, $\widetilde{V_M}$ is a long root group of L and $\widetilde{U_M} = C_{\widetilde{L}}(Q \cap Q_M)$.*

Proof. As L is quasisimple we have that

$$(n, p^f) \neq (2, 2). \tag{$*$}$$

Lemma 5.5 states that Q is an irreducible L-module and L is normalized by $M \cap N_G(Q)$. By Lemma 5.2 (vi), Q is a strong dual F-module with offender $\widetilde{V_M}$. Hence Lemma 2.2(i) and ($*$) imply that Q is a natural module for $\widetilde{L}$ and $\widetilde{V_M} \leq \widetilde{L}$.

Let $wQ \in (\widetilde{V_M} \cap Z(\widetilde{S \cap L}))^\#$. Then $[Q, w] = [Q, V_M]$ has order q, and so, as Q is the natural module, $|[Q, w]| \in \{p^f, p^{2f}\}$. Thus $\widetilde{L} \cong \mathrm{SL}_n(q)$, $\mathrm{Sp}_{2n}(q)$ or $\mathrm{Sp}_{2n}(\sqrt{q})$ and in the latter case q is a power of 2.

Aiming for a contradiction, suppose that $\widetilde{L} \cong \mathrm{Sp}_{2n}(\sqrt{q})$. As $\widetilde{Q_M}$ centralizes $[Q, V_M]$ and $[Q, V_M]$ contains the 1-space $C_Q(S \cap L) \leq [Q, V_M]$, no element of Q_M induces a non-trivial field automorphisms on $\widetilde{L}$. In particular Q_M and U_M act linearly on Q. By Lemma 5.3 $[Q, Q_M] = Q \cap Q_M = C_Q(U_M)$ and we deduce from [18, Lemma 2.53] that $C_Q(U_M) = [Q, Q_M]$ is an isotropic subspace of Q. Since $|Q : C_Q(U_M)| = q$, this now implies $|Q| = q^2$ and

$$\widetilde{L} \cong \mathrm{Sp}_4(\sqrt{q}). \tag{+}$$

Since $M = SM^\circ$, we have $S/(S \cap M^\circ Q_M) = S/QQ_M$ embeds into $\mathrm{Out}(M^\circ Q_M/Q_M)$ and so is cyclic. Set $N = N_L(Q \cap Q_M)$ Then $\widetilde{N} \cong q^{3/2} : \mathrm{SL}_2(\sqrt{q})$ and so $(S \cap N)/QQ_M$ is elementary abelian of order $\sqrt{q}$. As $\sqrt{q} > 2$ by $(*)$ and $S/Q_M Q$ is cyclic, this is impossible. Hence $\widetilde{L} \not\cong \mathrm{Sp}_{2n}(\sqrt{q})$.

Suppose that $\widetilde{L} \cong \mathrm{SL}_n(q)$ or $\mathrm{Sp}_{2n}(q)$. Then V_M induces a group of $\mathrm{GF}(q)$-transvections to a point and a hyperplane. As $\widetilde{V_M}$ is normalized by $\widetilde{S \cap L}$ this means that $\widetilde{V_M}$ is contained in a root group of $\widetilde{L}$. As $\widetilde{U_M}$ centralizes $Q \cap Q_M \geq [Q, V_M]$ and $|\widetilde{U_M}| = |Q \cap Q_M|$ by Lemma 5.3 we also have $\widetilde{U_M} \leq \widetilde{L}$. Hence $\widetilde{L} \cong \mathrm{SL}_n(q)$ and $C_{\widetilde{L}}(Q \cap Q_M) = \widetilde{U_M}$. This proves the lemma. □

6. The proof of Theorem 1.2 when $Q_M = V_M$

In this section we will consider the possibility that $Q_M = V_M$ when Hypothesis 5.1 is satisfied. Thus we have

$$\overline{M^\circ} \cong \mathrm{SL}_2(q)$$

and

$$Q_M = V_M \text{ is the natural } \overline{M^\circ}\text{-module.}$$

This hypothesis will reveal the conclusion in Theorem C(iv) and also provide some of the examples in parts of (i) and (iii) of Theorem C. More precisely we will prove

Proposition 6.1. *Assume that Hypothesis 5.1 holds and that $Q_M = V_M$. Then either*

(i) $G \cong \mathrm{Mat}(22)$ *or* $\mathrm{Mat}(23)$*; or*

(ii) $(N_G(Q_M), N_G(Q))$ *is a weak BN-pair over* $\widehat{B} = N_G(Q_M Q)$ *of type* $\mathrm{PSL}_3(q)$*. Moreover,*
   - (a) *if q is even, then* $F^*(G) \cong \mathrm{PSL}_3(q)$ *or* $\mathrm{Alt}(6)$*; and*
   - (b) *if q is odd, then* $\langle \mathcal{L}_G(S) \rangle = \langle N_G(Q_M), N_G(Q) \rangle$*.*

We start by providing details about the structure of M.

Lemma 6.2. *Suppose $Q_M = V_M$. Then one of the following holds*

(i) $M^\circ \cong q^2{:}\mathrm{SL}_2(q) \cong \langle V_M^{N_G(Q)}\rangle$; *or*
(ii) $\overline{M^\circ} \cong \mathrm{SL}_2(4)$, V_M *is the natural* $\mathrm{SL}_2(4)$*-module and* $\overline{M} \cong \mathrm{Sym}(5)$ *and either*
  (a) $N_G(Q) \cong 2^4.\mathrm{Alt}(6)$; *or*
  (b) $N_G(Q) \cong 2^4.\mathrm{Alt}(7)$.

Proof. If $q = 2$ or 3, then $S \cong \mathrm{Dih}(8)$ or 3_+^{1+2} and $M^\circ \cong q^2 : \mathrm{SL}_2(q)$. Since Q is abelian and large, $|Q| = q^2$ and so we also have $\langle V_M^{N_G(Q)}\rangle \cong q^2 : \mathrm{SL}_2(q)$. So (i) holds when $q \le 3$.

Assume that $q > 3$. Then application of Lemma 5.4 gives

$$[\widetilde{V_M}, O_{p'}(\widetilde{N_G(Q)})] = 1.$$

Hence $[E(\widetilde{N_G(Q)}), \widetilde{V_M}] \ne 1$ and Lemma 5.5, implies $\widetilde{N_G(Q)}$ has a unique component which is not centralized by $\widetilde{V_M}$ and Lemma 2.4 implies that $\widetilde{V_M}$ acts faithfully on $\widetilde{L}$. Furthermore Q is irreducible as an L-module. By Lemma 5.2 (vi), Q is a strong dual F-module for $\widetilde{LV_M}$ and so Lemma 2.2 yields that either

(a) $q = 4$ and $\widetilde{L} \cong \mathrm{Alt}(6)$ or $\mathrm{Alt}(7)$; or
(b) $\widetilde{L} \cong \mathrm{SL}_n(p^f)$ or $\mathrm{Sp}_{2n}(p^f)$ with $n \ge 2$.

If (a) holds, then $|Q_M| = 2^4$ and $|\widetilde{S}| \le 2^3$. Thus $|S| \in \{2^6, 2^7\}$ and consequently $\widetilde{N_G(Q)} \cong \mathrm{Alt}(6)$ or $\mathrm{Alt}(7)$, $|Q| = 2^4$ and $|S| = 2^7$. Furthermore, $|S/Q_M| = 8$ and so $M/Q_M \cong \mathrm{Sym}(5)$ and (ii)(a) or (b) holds.

We are left to consider case (b). By Lemma 5.6, $\widetilde{L} \cong \mathrm{SL}_n(q)$, Q is the natural module and $\widetilde{V_M}$ induces $\mathrm{GF}(q)$-transvections. As $|Q : Q \cap Q_M| = |Q : Q \cap V_M| = q = |Q \cap V_M|$, we have $|Q| = q^2$ and obtain $L \cong \mathrm{SL}_2(q)$, which is the configuration in (i). □

The next proposition proves most of Proposition 6.1.

Proposition 6.3. *Suppose $V_M = Q_M$. Then either*

(i) $G \cong \mathrm{Mat}(22)$ *or* $\mathrm{Mat}(23)$ *and Lemma 6.2(ii) holds; or*
(ii) $N_G(QQ_M) = N_G(V_M) \cap N_G(Q)$, *and* $(N_G(V_M), N_G(Q))$ *is a weak BN-pair over* $\widehat{B} = N_G(Q_M Q)$ *of type* $\mathrm{PSL}_3(q)$ *and Lemma 6.2(i) holds.*

Proof. Suppose that Lemma 6.2(ii) holds. In case (ii)(a), Lemma 2.3 states that $N_G(Q)$ splits over Q and so, as Q is large, the centralizer of a 2-central involution is a split extension of an elementary abelian group of order 16 by $\mathrm{Sym}(4)$. Thus $G \cong \mathrm{Mat}(22)$ by [10].

In case Lemma 6.2(ii)(b), we have $N_G(Q) \cong 2^4.\mathrm{Alt}(7)$ contains $2^4.\mathrm{Alt}(6)$ and so this extension also splits by Lemma 2.3. Hence in case (ii)(b), we have that a 2-central involution has a centralizer which is a split extension of 2^4 by $\mathrm{SL}_3(2)$. Application of [11] gives $G \cong \mathrm{Mat}(23)$. We have now proved (i).

Suppose Lemma 6.2(i) holds. Set $D = \langle V_M^{N_G(Q)}\rangle$, $N = N_G(V_M)$, $B_0 = N_G(Q) \cap N$, $T = V_M Q$ and $\widehat{B} = N_G(T)$. By construction, we have D is normal in $N_G(Q)$ and, by Lemma 5.2 (viii), $M^\circ = N^\circ$ is normalized by N. We claim $\widehat{B} = B_0$. By its definition, $B_0 \le \widehat{B}$. Since Q is weakly closed in T, $\widehat{B} \le N_G(Q)$. It remains to show $\widehat{B} \le N$. If $p = 2$, then, by [18, Lemma 2.9], V_M and Q are the only elementary abelian subgroups of T of order q^2. Hence, as $\widehat{B}$ normalizes Q, it also normalizes V_M. Thus $\widehat{B} \le N$ in this case.

Assume that p is odd. Let $t \in M^\circ$ be an involution with $[t, M^\circ] \le Q_M$. Then $t \in N_G(Q)$ as Q is large. Further t inverts $Z = V_M \cap Q = [V_M, Q] = T'$ and centralizes $Q/Z \cong QQ_M/Q_M$. As $\widehat{B}$ normalizes Q and Z, this gives $\widehat{B} \le N_G(tQ)$ and $\widehat{B} = C_{\widehat{B}}(t)Q$. As $\widehat{B}$ normalizes T we see that $C_{\widehat{B}}(t)$ normalizes $[T, t] = V_M$. Therefore $\widehat{B}$ normalizes V_M and so $\widehat{B} = B_0$ as claimed.

This proves that $(N, N_G(Q))$ is a weak BN-pair over $\widehat{B}$ and then by [5, Theorem A] it is of type $\mathrm{PSL}_3(q)$. □

Lemma 6.4. *If Proposition 6.3(ii) holds, then Q_M is large.*

Proof. From Proposition 6.3(ii) it follows that $Q_M = V_M$ and we have the situation of Lemma 6.2(i). Set $D = \langle V_M^{N_G(Q)}\rangle$. Since $C_G(Q_M) = Q_M$, to prove Q_M is large, we only need to show that $N_G(U) \le N_G(Q_M)$ for all $1 \ne U \le Q_M$. So assume that $1 \ne U \le Q_M$. Since M° acts transitively on $Q_M^\#$ by Lemma 6.2, we may assume that there exists $z \in (U \cap Q)^\#$. Thus $C_G(z) \le N_G(Q)$. Since D acts transitively on $Q^\#$, we have $N_G(Q) = C_G(z)D$ and $C_G(z) \cap D = QQ_M$ as $\widetilde{D} \cong \mathrm{SL}_2(q)$. Hence $C_G(z) \le N_G(QQ_M) \le N_G(Q_M)$ by Proposition 6.3(ii).

Assume that $x \in N_G(U)$. Then, as M° acts transitively on $Q_M^\#$, there exists $g \in M^\circ \le N_G(Q_M)$ such that $z^g = z^x$. Hence $xg^{-1} \in C_G(z) \le N_G(Q_M)$. Thus $x \in N_G(Q_M)$ and so $N_G(U) \le N_G(Q_M)$ as required. □

The conclusion of the next lemma also holds when $p = 2$, but we shall prove much more in that case.

Lemma 6.5. *Suppose that $Q_M = V_M$ and that p is odd. Then $\langle \mathcal{L}_G(S)\rangle = \langle N_G(Q_M), N_G(Q)\rangle$.*

Proof. Set $H = \langle N_G(Q_M), N_G(Q)\rangle$. Since p is odd, Lemma 6.2 implies that Proposition 6.3(ii) holds. Suppose that $K \in \mathcal{L}_G(S)$ and $K \not\le H$. Since Q is abelian, $Y_K \not\le Q$ for otherwise $K = N_K(Y_K) \le N_G(Q) \le H$ as Q is large. Hence $K \in \mathcal{VL}(S)$. Let $M_K \in \mathcal{VL}^{min}(S)$ be as in Theorem A. If $M_K \le H$, then

$$K \le \langle K, N_G(Q)\rangle = \langle M_K, N_G(Q)\rangle \le H$$

by Theorem A (iii) and this is impossible. Hence $M_K \not\leq H$ and $M_K \in \mathcal{VL}^{min}_{lin}(S) \cup \mathcal{VL}^{min}_{orthsymp}(S)$ by Theorem 1.1 (recall that in the wreath product case of Theorem 1.1 we have $p = 2$). By Corollary 4.6, $M_K \in \mathcal{VL}^{min}_{lin}(S)$ and so $M_K^\circ/O_p(M_K^\circ) \cong \mathrm{SL}_2(r)$, $r = p^b$, and V_{M_K} is a natural module. In particular, Hypothesis 5.1 applies with M_K in place of M. Set $Q_{M_K} = O_p(M_K)$ and put $D = \langle V_M^{N_G(Q)}\rangle$. Then Q is the natural $\widetilde{D} \cong \mathrm{SL}_2(q)$-module and, as V_{M_K} is normalized by S, $V_{M_K} \cap D \not\leq Q$. Hence $[V_{M_K}, Q] \geq [Q_M, Q]$ and

$$\widetilde{V_{M_K}} \leq \widetilde{Q_{M_K}} \leq C_{\widetilde{S}}([Q_M, Q]) = \widetilde{V_M}.$$

We deduce that $r = |[V_{M_K}, Q]| = |[Q_M, Q]| = q$. Therefore $V_{M_K}Q = V_MQ$ and $Q_{M_K} = V_{M_K}$. Now Q_{M_K}, Q_M and Q are all large subgroups contained in QQ_M by Lemma 6.4. In particular, they are all normal in $N_G(QQ_M)$. Let $t_D \in N_G(Q)$ be an involution which inverts Q. Then t_D centralizes $\widetilde{Q_M} = \widetilde{Q_{M_K}}$. Hence $C_{QQ_M}(t_D)$ has order q and, as Q_M and Q_{M_K} are both t_D-invariant,

$$Q_M = (Q_M \cap Q)C_{QQ_M}(t_D) = Q_{M_K},$$

which is a contradiction as $M_K \not\leq N_G(Q_M) \leq H$. Hence $H = \langle \mathcal{L}_G(S)\rangle$ as claimed. □

To complete the proof of Proposition 6.1, we just have to investigate Proposition 6.3(ii) when q even and prove

Proposition 6.6. *Assume that q is even and that Proposition 6.3(ii) holds. Then $F^*(G) \cong \mathrm{PSL}_3(q)$ or $\mathrm{Alt}(6)$.*

Hence until the beginning of Section 7 we will assume that we are in the situation described in Proposition 6.3(ii) with q even. We start with the smallest possible cases.

Lemma 6.7. *If Proposition 6.3(ii) holds with $q = 2$, then $G \cong \mathrm{PSL}_3(2)$ or $\mathrm{Alt}(6)$.*

Proof. In this configuration we have that $C_G(Z(Q)) \cong \mathrm{Dih}(8)$. Furthermore, using Lemma 3.6, we easily deduce that G is simple. Now application of [26, Example 2, page 231] gives the assertion. □

We now turn to the general case when $q > 2$ is a power of 2. Set

$$T = QQ_M$$

and continue with $D = \langle V_M^{N_G(Q)}\rangle$.

Lemma 6.8. *We have T is a Sylow 2-subgroup of both D and M°. Furthermore, if $K \geq \langle D, M^\circ\rangle$, then all the involutions in $M^\circ \cup D$ are K-conjugate.*

Proof. We have that $M^\circ/Q_M \cong \widetilde{D} \cong \mathrm{SL}_2(q)$ by Lemma 6.2 (i). Thus $T \in \mathrm{Syl}_2(M^\circ) \cap \mathrm{Syl}_2(D)$. As q is even, V_M and Q are the only elementary abelian subgroups of T of order q^2 by [18, Lemma 2.9]. Hence every involution of T is in $Q \cup Q_M$. As all the involutions in Q are D-conjugate and all the involutions in Q_M are M°-conjugate, the claim follows. □

Set

$$G_1 = G^{(\infty)}.$$

Lemma 6.9. *Assume that Proposition 6.3(ii) holds with $q > 2$. Then $\langle M^\circ, D\rangle \le G_1$, $T \in \mathrm{Syl}_2(G_1)$ and $(N_{G_1}(Q_M), N_{G_1}(Q))$ is a weak BN-pair over $N_{G_1}(T)$ of type $\mathrm{PSL}_3(q)$.*

Proof. Let $S_1 = S \cap G_1$. As M° and D are perfect, $\langle M^\circ, D\rangle \le G_1$. Assume that $T \notin \mathrm{Syl}_2(G_1)$. Then $S_1 > T$ and T is normalized by S_1. As $\widetilde{S_1}$ acts faithfully on $\widetilde{D}$, we have that S_1/T is isomorphic to a cyclic group of outer automorphisms of $\widetilde{D}$. Let R be a Hall $2'$-subgroup of $N_D(T)$. Then, as $q > 2$, $R = C_D(R) = N_{RT}(R)$. By a Frattini argument, $S_1D = N_{S_1RT}(R)D$ and $S_1D/D = N_{S_1R}(R)D/D$ is cyclic. We may suppose that R is chosen so that $N_{S_1R}(R) = N_{S_1}(R)R$ and so $N_{S_1}(R)$ is cyclic. Let $x \in N_{S_1}(R)$ be an involution. As $N_G(T) = N_G(Q_M) \cap N_G(Q)$ by Proposition 6.3, x normalizes both Q and Q_M. As x is an involution and $|Q| = |Q_M| = q^2$, $|C_{Q_M}(x)| \ge q$ and $|C_Q(x)| \ge q$. Notice that x induces a field automorphism on M°/Q_M and so $T = C_{S_1}(Q_M \cap Q_M)$. Thus $C_{Q_M}(x) \nleq Q$ and $C_Q(x) \nleq Q_M$. Therefore $C_{Q_M}(C_Q(x)) = Q \cap Q_M$ and so $C_{Q_M}(x)C_Q(x)$ is non-abelian. Hence

$$1 \ne (\Omega_1(C_{S_1}(x)))' \le Z(T) = Q \cap Q_M \le Q.$$

Since Q is large, $N_G(C_{S_1}(x)) \le N_G(Q)$. In particular, $C_{S_1}(x) \in \mathrm{Syl}_2(C_G(x))$ and so x is not G-conjugate to an element of T by Lemma 6.8. Now application of [6, Proposition 15.15] yields that G_1 has a subgroup of index 2, a contradiction. Hence $T \in \mathrm{Syl}_2(G_1)$. Since Q is large in G_1, we can now apply Proposition 6.3 to G_1 and obtain the statement of the lemma. □

We now define $B = N_G(T)$, $P_1 = DB \cap G_1$ and $P_2 = M^\circ B \cap G_1$. Then $O^{2'}(P_1) = D$, $O^{2'}(P_2) = M^\circ$ and $P_1 \cap P_2 = B \cap G_1 = N_{G_1}(T)$.

Lemma 6.10. *For $i = 1, 2$, we have $N_{G_1}(P_i) = P_i$.*

Proof. We have that $N_{G_1}(P_1)$ normalizes D and $N_G(P_2)$ normalizes M°. Hence by the Frattini argument

$$P_1 \le N_{G_1}(P_1) = DN_{N_{G_1}(P_1)}(T) \le D(P_1 \cap P_2) \le P_1$$

and

$$P_2 \leq N_{G_1}(P_2) = M^\circ N_{N_{G_1}(P_2)}(T) \leq M^\circ(P_1 \cap P_2) \leq P_2$$

Hence $N_{G_1}(P_i) = P_i$ for $i = 1, 2$. $\square$

We have that (P_1, P_2) is a weak BN-pair over $P_1 \cap P_2$. Set $P = \langle P_1, P_2\rangle$ and denote by $\Gamma = \Gamma(P; P_1, P_2, P_1 \cap P_2)$ the corresponding *coset graph*. For elementary properties of this coset graph and the amalgam method see [15, Appendix E]. In particular, the vertices of Γ are the right cosets of P_1 and P_2 in P and two vertices form an edge if and only if they are distinct and have non-empty intersection. The graph Γ is connected and the set of right cosets of P_1 and the set of right cosets of P_2 forms a bipartition of Γ. The group P acts faithfully on Γ by right multiplication. This action is transitive on the parts of the bipartition and on the edges of Γ. If ν is a vertex of Γ, then we denote by P_ν the stabilizer of ν in P. The general theory tells us that P_ν is P-conjugate to either P_1 or P_2. In addition, we denote the vertex stabilized by P_1 by 1 and the vertex stabilized by P_2 by 2. The action of P_i on the neighbours of the vertex i, is exactly the action of P_i on the cosets on $P_1 \cap P_2$ and so, as $|P_i : P_1 \cap P_2| = q + 1$, this can be identified with the action on the projective line. We denote the distance between vertices $\alpha, \beta \in \Gamma$ by $\mathrm{d}(\alpha, \beta)$. The action of P on Γ is called *locally n-path transitive* if and only if for $i \in \{1, 2\}$, P_i acts transitively on all paths of length n emanating from the vertex i. If $\gamma = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ is a path of length 3, then we say γ is *regular* if $P_{\alpha_0} \cap P_{\alpha_1} \cap P_{\alpha_2} \cap P_{\alpha_3}$ acts transitively on the neighbours of α_0 other than α_1 on the neighbours of α_3 other than α_2.

Lemma 6.11. *The group P acts locally 4-path transitively on Γ, there are regular 3-paths, and Γ has no cycles of length less than 6.*

Proof. This is elementary and is taken from [5, Table on page 98]. $\square$

We have the following

Lemma 6.12. *Suppose that $\nu \in \Gamma \setminus \{1, 2\}$. Then $P_1 \neq P_\nu \neq P_2$.*

Proof. We may as well suppose that $\nu = P_1 g$ where $g \in P$. So $P_\nu = P_1^g$. If $P_\nu = P_1$, then $g \in P_1$ by Lemma 6.10. But then $\nu = P_1 g = P_1$ which is the vertex 1, a contradiction. If $P_\nu = P_2$, then $P_1^g = P_2$ and so P_1 and P_2 are conjugate by an element of $N_{G_1}(T)$ by Sylow's Theorem. However, Proposition 6.3(ii) gives $N_{G_1}(T)$ normalizes both Q and Q_M. Hence $P_\nu \neq P_2$. $\square$

Let F be a complement to T in $N_{G_1}(T)$. Then, for $i = 1, 2$, $F \cap O^{2'}(P_i)$ is a complement to T in $N_{O^{2'}(P_i)}(T)$. As $q > 2$, F is non-trivial. The group F will be fixed in the remainder of this section.

We denote by Γ^F the subgraph of Γ which consists of the fixed points of F on Γ. Since $F \leq P_1 \cap P_2$, we obviously have $\{1, 2\} \subseteq \Gamma^F$. Let $\mathcal{A}$ denote the connected component of Γ^F which contains $\{1, 2\}$.

Lemma 6.13. *We have that $\mathcal{A}$ is a circuit.*

Proof. Suppose that $i \in \{1, 2\}$. Then $P_i = O^{2'}(P_i)F$ and $F \cap O^{2'}(P_i)$ normalizes exactly two Sylow 2-subgroups of $O^{2'}(P_i)$. Because F has odd order, we deduce that F normalizes exactly two Sylow 2-subgroups of P_i. Since edge stabilisers are precisely normalizers of Sylow 2-subgroups, the result follows. $\square$

Lemma 6.14. *Suppose Proposition 6.3(ii) with $q > 2$ holds. Then there is an involution $t \in Z(Q)$ and a P-conjugate z of t, such that z normalizes F and acts as a reflection on $\mathcal{A}$. Furthermore, $z \in O_2(P_\beta)$ where $\mathrm{d}(\beta, 1) = 2$ and $|[F, z]| = q - 1$.*

Proof. All the involutions of P_1 are conjugate to t by Lemma 6.8. Furthermore, $D = O^{2'}(P_1)$ contains an involution z which inverts $F \cap D$ and so we may suppose that z inverts $F \cap D$. Then $N_{FD}(F \cap D) = \langle z \rangle F$ and $[F, z] = F \cap D$ has order $q - 1$. As $z \in P_1$ and z normalize F, z leaves $\mathcal{A}$ invariant. Since z exchanges the two neighbours of 1 fixed by F, we conclude that z acts as a reflection on $\mathcal{A}$. Let $T \in \mathrm{Syl}_2(P_1)$ be such that $z \in T$. Then $T = T^g$ for some $g \in P_1$. As every involution in T is in $Q \cup Q_M^g$ and $z \notin Q$, $z \in Q_M^g \setminus Q$. Now $Q_M^g = \bigcap_{h \in P_2^g} (Q \cap Q_M^g)^h$ and so we conclude that there exists $h \in P_2^g$ such that $z \in Q_M^g \cap Q^h$. Set $\beta = P_1 h$. Then $z \in O_2(P_\beta)$ and $\mathrm{d}(1, \beta) = 2$. $\square$

Lemma 6.15. *Suppose Proposition 6.3(ii) with $q > 2$ and let z be as in Lemma 6.14. Then z fixes a vertex α opposite to 1 in $\mathcal{A}$.*

Proof. The assertion follows directly from Lemma 6.14 as the circuit $\mathcal{A}$ has even length. $\square$

We refer to [27, page 15] for the definition of a generalized polygon and recall that a generalized 3-gon is just the incidence graph of a projective plane.

Lemma 6.16. *Suppose Proposition 6.3(ii) with $q > 2$ holds. Then Γ is a generalized 3-gon.*

Proof. We first show that

$$\text{there are 6-cycles in } \Gamma. \tag{1}$$

Suppose that $\alpha \in \mathcal{A}$ is as in Lemma 6.15 and β as in Lemma 6.14. Then $z \in O^{2'}(P_\alpha)$ as z fixes α. By Lemma 6.8 we have a neighbour δ of α not in $\mathcal{A}$ such that $z \in O_2(P_\delta)$. Furthermore $z \in O_2(P_\gamma)$, where γ is the common neighbour of β and 1.

Recall that by Lemma 6.4, Q_M is also large in P. In particular, Q and Q_M are both trivial intersection sets in P by Lemma 5.2 (v).

If $\alpha = P_1 g$ for some $g \in P$, then, as $z \in O_2(P_\gamma) \cap O_2(P_\delta)$ and γ and δ are cosets of P_2, the trivial intersection property implies $O_2(P_\delta) = O_2(P_\gamma)$. But then $P_\gamma = N_P(O_2(P_\gamma)) = N_P(O_2(P_\delta)) = P_\delta$ and so by Lemma 6.12 $\gamma = \delta$. This means that $d(1, \alpha) = 2$. Now we move the path $(1, \delta, \alpha)$ by applying an element of F, and this provides us with a 4-circuit in Γ, contradicting Lemma 6.11.

Thus we have that α is of type 2 and so δ is of type 1. Now $z \in O_2(P_\beta) \cap O_2(P_\delta)$ and as $O_2(P_\delta)$ is a conjugate of Q, we have $O_2(P_\delta) = O_2(P_\beta)$. This time Lemma 6.12 yields $\delta = \beta$. This shows $d(1, \alpha) = 3$ and acting with an element of F yields a 6-circuit in Γ, which is (1).

Now the assertion follows from (1), Lemma 6.11 and the definition of a generalized polygon. □

Lemma 6.17. *Assume that Proposition 6.3(ii) holds with $q > 2$. Then*

$$F^*(P) \cong \mathrm{PSL}_3(q).$$

Proof. A generalized 3-gon is just the incidence graph of a projective plane. We call the vertices of which are cosets of P_1 points and those which are cosets of P_2 lines. There are exactly $q^2 + q + 1$ points and $q^2 + q + 1$ lines. We have that Q stabilizes a line and all the points on it. Hence it acts regularly on the remaining points. This means that the projective plane is Moufang and by [28, Theorem A] it is Desarguesian and contains $\mathrm{PSL}_3(q)$ as a normal subgroup. This proves the lemma. □

Proof of Proposition 6.6. If (P_1, P_2) has type $\mathrm{PSL}_3(2)$, the proposition follows by Lemma 6.7. Hence we may assume that $q > 2$.

As G/G_1 is soluble and $O(G) = 1$ by Lemma 3.6, $F^*(G) = F^*(G_1)$. By Lemma 6.9, QQ_M is a Sylow 2-subgroup of G_1. Set $H = N_{G_1}(QQ_M)P$. Then $H \geq \langle D, M^\circ \rangle$ and so H has one conjugacy class of involutions by Lemma 6.8. Let $t \in H$ be an involution, then we may suppose that $t \in Q$. Hence, as Q is large,

$$C_{G_1}(t) \leq N_G(Q) \cap G_1 = N_{G_1}(Q) = P_1 N_G(QQ_M) \leq H.$$

It follows from [6, Proposition 17.11] that H is strongly p-embedded in G_1. Now Bender's Theorem [4] implies that $H = G_1$. Hence, as $F^*(H) = F^*(P)$,

$$F^*(G) = F^*(G_1) = F^*(H) = F^*(P) \cong \mathrm{PSL}_3(q)$$

by Lemma 6.17. This completes the proof of Proposition 6.6. □

Finally Proposition 6.1 follows by combining Proposition 6.3 and Proposition 6.6.

7. The proof of Theorem 1.2 when $Q_M \neq V_M$

In this section we complete the proof Theorem 1.2. To do this we first prove

Proposition 7.1. *Assume that Hypothesis 5.1 holds and that $Q_M \neq V_M$. Then one of the following holds*

(i) $F^*(\langle \mathcal{L}_G(S)\rangle) \cong \mathrm{PSL}_n(q)$, $n \geq 4$, *and, if q is even,* $F^*(G) \cong \mathrm{PSL}_n(q)$; *or*
(ii) $G \cong \mathrm{Alt}(9)$ *or* $\mathrm{Mat}(24)$.

Throughout this section we assume

$$V_M \neq Q_M.$$

We begin with a closer inspection of $N_G(Q)$.

Lemma 7.2. *We have* $\langle \widetilde{V_M^{N_G(Q)}}\rangle \cong \mathrm{SL}_n(q)$*, Q is a natural* $\mathrm{SL}_n(q)$*-module and $\widetilde{V_M}$ is a long root group of* $\langle \widetilde{V_M^{N_G(Q)}}\rangle$.

Proof. By Lemma 5.4,

$$[\widetilde{V_M}, O_{p'}(\widetilde{N_G(Q)})] = 1.$$

By Lemma 5.5, $\widetilde{V_M}$ acts faithfully on some component $\widetilde{L}$ of $\widetilde{N_G(Q)}$ and $\widetilde{L}$ is unique with this property. Furthermore, L is normalized by $M \cap N_G(Q)$ and Q is an irreducible L-module. Since, by Lemma 5.2 (vi), Q is a strong dual F-module with offender $\widetilde{V_M}$, Lemma 2.2 applies to give the candidates for $\widetilde{L}$. As $Q_M > V_M$, we have $V_M \leq Q_M'$ by Lemma 3.4 and so, as Q_M normalizes L, the structure of $\mathrm{Aut}(\widetilde{L})$ and the description of the possibilities for $\widetilde{V_M}$ given in Lemma 2.2 shows that $\widetilde{L} \cong \mathrm{SL}_n(p^f)$ or $\mathrm{Sp}_{2n}(p^f)$. The assertion in the lemma now follows from Lemma 5.6. $\square$

Because of Lemma 7.2, we set $D = \langle V_M^{N_G(Q)}\rangle$ and note that as Q is irreducible $Q \leq D$. Thus $D = L$ and

$$\widetilde{D} \cong \mathrm{SL}_n(q).$$

This means that $\widetilde{N_G(Q)}$ is a subgroup of $\Gamma\mathrm{L}_n(q)$ which contains $\mathrm{SL}_n(q)$. If $\widetilde{D} \cong \mathrm{SL}_2(q)$, then $|QQ_M| = q^3$, $V_M = Q_M$, a contradiction. We therefore assume

$$n \geq 3 \text{ and } Q_M > V_M.$$

We also introduce the following notation

$$H = \langle D, M \rangle,$$
$$L_1 = O^{p'}(N_D(Q \cap Q_M)),$$
$$\widehat{L}_1 = \langle Q_M^{L_1} \rangle,$$
$$U = \langle M^\circ Q_M, L_1 \rangle,$$
$$\widehat{B} = N_G(QQ_M),\ B_D = \widehat{B} \cap D,\ B_M = M \cap \widehat{B}.$$

By Lemma 7.2, $\widetilde{V_M}$ is a central root subgroup of a Sylow p-subgroup of $\widetilde{D}$ and so, as $Q_M Q$ centralizes $[V_M, Q] = C_{V_M}(Q)$ and $C_S(C_{V_M}(Q)) = S \cap M^\circ Q_M = QQ_M$, we deduce $QQ_M \in \mathrm{Syl}_p(D)$. We record

Lemma 7.3. *We have $QQ_M \in \mathrm{Syl}_p(M^\circ Q_M) \cap \mathrm{Syl}_p(D)$ and $Q \cap Q_M = [Q, QQ_M]$ is a $\mathrm{GF}(q)$-hyperplane of Q.* $\square$

Lemma 7.4. *We have $U = M^\circ \widehat{L}_1$,*

$$U/O_p(U) \cong \mathrm{SL}_2(q) \circ \mathrm{SL}_{n-1}(q)$$

and $O_p(U) = J(O_p(L_1)) = J(U_M Q) = U_M$. Furthermore, $[M^\circ, \widehat{L}_1] \leq U_M$ and U_M is the tensor product module for $U/O_p(U)$. In particular, $QQ_M \in \mathrm{Syl}_p(U)$.

Proof. We have that $Q \cap Q_M$ is a $\mathrm{GF}(q)$-hyperplane in Q by Lemma 7.3. Fix $x \in M^\circ \setminus N_G(Q)$. Then, by Lemma 5.3, $Q^x \cap Q_M \cap Q = 1$,

$$U_M = (Q \cap Q_M)(Q \cap Q_M)^x \leq M^\circ$$

and U_M is elementary abelian. Hence, as $|Q^x \cap Q_M| = q^{n-1} = |O_p(\widetilde{L_1})|$ and $Q^x \cap Q$ centralizes $Q_M \cap Q$, $\widetilde{U_M} = O_p(\widetilde{L_1})$ and $|U_M Q : U_M| = q$. Assume $A \leq U_M Q$ is a maximal order abelian subgroup of $U_M Q$ with $A \neq U_M$. Then $A \geq Z(U_M Q) \geq Q \cap U_M = Q \cap Q_M$, $q^{2n-2} > |A \cap U_M| \geq q^{2n-3}$ and $AU_M \cap Q > Q \cap Q_M$. Since $A \cap U_M$ centralizes $AU_M \cap Q$ and Q is a $\mathrm{GF}(q)$-space, we deduce that $A \cap U_M \leq Q$ and this implies that $q^{2n-3} \leq q^{n-1}$ which contradicts $n \geq 3$. Hence U_M is the unique abelian subgroup of maximal order in $U_M Q$. Therefore $U_M = J(U_M Q)$ and so U_M is normalized by L_1. Since M normalizes U_M, so does U. As $\widetilde{O_p(U)} \leq O_p(\widetilde{L_1})$ and $Q \not\leq O_p(U)$, we deduce $O_p(U) = U_M$. In particular, we have demonstrated

$$O_p(U) = J(O_p(L_1)) = J(U_M Q) = U_M.$$

By Theorem A (iii) we have that $US \in \mathcal{VL}(S)$ as $M = M^\circ S \leq US$. Using $\overline{M^\circ} \cong \mathrm{SL}_2(q)$ and V_M is the natural module and reading Table 1 right to left, yields $\overline{U^\circ} \cong \mathrm{SL}_m(q)$ for some $m \geq 2$ and V_U is the natural module or $q = p = 2$, $\overline{U^\circ} \cong 3{\cdot}\mathrm{Alt}(6)$ and V_U is the 2^6 module. In particular, $V_M \leq V_U \leq U_M$ and so U_M centralizes V_U. By Lemma 5.3,

$M^\circ S/U_M$ has no non-central M°-chief factors. Hence $M^\circ(S\cap U^\circ)C_{U^\circ}(V_U)/C_{U^\circ}(V_U)$ is not characteristic p, it follows that $\overline{U^\circ}\cong \mathrm{SL}_2(q)$ and so $U^\circ = M^\circ$. Therefore L_1 normalizes M°. In particular, QQ_M is a Sylow p-subgroup of $U = M^\circ L_1$. As $[Q_M, M^\circ]\le U_M$ and L_1 normalizes M°, $[\widehat{L}_1, M^\circ]\le U_M$ and also $M^\circ\widehat{L}_1 = U$. As $\widehat{L}_1$ does not normalize V_M, we find, by Theorem A and Table 1 for example, that U_M is the tensor product module of V_M for M° with $Q\cap Q_M$ for $\widehat{L}_1$. Since US has characteristic p, we have $C_U(U_M) = U_M$ and the remaining details of the lemma easily follow. $\square$

Lemma 7.5. *The group $\widehat{B}$ normalizes D, M° and U. In particular $F^*(\langle D, M^\circ\rangle) = F^*(H)$, $B = B_D B_M \le \widehat{B}$.*

Proof. By Lemma 7.2, we have that $\widetilde{D}\cong \mathrm{SL}_n(q)$. Furthermore Q is the natural $\widetilde{D}$-module and $\widetilde{V_M}$ induces $\mathrm{GF}(q)$-transvections on Q with $Q_M Q\in \mathrm{Syl}_p(D)$ by Lemma 7.3. Furthermore, Lemma 7.4 shows that QQ_M is a Sylow p-subgroup of U.
As Q is weakly closed in QQ_M by Lemma 3.1 (i), $\widehat{B}$ normalizes Q and so $\widehat{B}\le N_G(Q)$ and normalizes D. Furthermore, as Q is the natural $\widetilde{D}$-module, $\widehat{B}$ normalizes all the preimages of parabolic subgroups of $\widetilde{D}$, which contain $\widetilde{Q_M}$. In particular it normalizes both L_1 and $QU_M = O_p(L_1)$. By 7.4, $U_M = J(U_M Q)$ and so $\widehat{B}$ normalizes U_M.

Suppose that L_1/QU_M is quasisimple and let D_1 be the preimage of $E(L_1/(Q\cap Q_M))$. Then by Lemma 7.4 $D_1 = \langle Q_M^{L_1}\rangle$ and $QQ_M\cap D_1 = Q_M$. As $\widehat{B}$ normalizes D_1, we see that $\widehat{B}$ normalizes Q_M as well.

If L_1/QU_M is not quasisimple, then $\widetilde{D}\cong \mathrm{SL}_3(2)$, or $\mathrm{SL}_3(3)$. In the first case, $N_G(QQ_M) = QQ_M = S$ and there is nothing is to prove. Thus assume that $L\cong \mathrm{SL}_3(3)$. Then $|U_M| = 3^4$ and this group is normalized by $\widehat{B}$. Furthermore by Lemma 7.4 QU_M/U_M and Q_M/U_M act quadratically on U_M and no other subgroup of order three in QQ_M/U_M has this property. This again shows that $\widehat{B}$ normalizes Q_M. Hence in any case

$$\widehat{B}\text{ normalizes } Q_M. \tag{1}$$

By (1) $\widehat{B}$ normalizes V_M and then M° and by Lemma 7.4 it also normalizes U. This proves the first part of the assertion. In particular, as $N_G(Q) = \widehat{B}D$, and $M\le M^\circ\widehat{B}$, we have $F^*(\langle D, M^\circ\rangle) = F^*(H)$.

Finally, as $B_D, B_M\le \widehat{B}$ and $\widehat{B}$ normalizes D and $M^\circ Q_M$, we get that B_D and B_M normalize each other and so $B = B_D B_M$ is a group. $\square$

The next two propositions provide the major step in the proof of Proposition 7.1. As in Section 6 we use geometric arguments to identify $F^*(H)$.
We continue with our standard notation. By Lemma 7.2 we have that $\widetilde{D}\cong \mathrm{SL}_n(q)$ and, as $Q_M\ne V_M$, $n\ge 3$.

Let P be the preimage of the minimal parabolic subgroup of $\widetilde{D}$, which contains QQ_M, but which is not contained in L_1. This means that P normalizes $V_M \cap Q$. Set

$$U_1 = \langle M^\circ Q_M, P\rangle.$$

Proposition 7.6. *Suppose $V_M \neq Q_M$ and $\widetilde{D} \cong \mathrm{SL}_3(q)$. Then $F^*(H) \cong \mathrm{PSL}_4(q)$.*

Proof. As $\widetilde{D} \cong \mathrm{SL}_3(q)$, we have $|Q_M Q| = q^6$, $|Q_M| = q^5$, $|U_M| = q^4$ and $\Phi(\widetilde{Q_M}) = \widetilde{V_M}$. Especially, we have

$$\Phi(Q_M) \leq U_M \cap V_M Q = V_M(Q \cap U_M)$$

and so, by Lemma 7.4, Q_M/V_M is elementary abelian. Also by Lemma 7.4, $U_M = [Q_M, M^\circ]$ and U_M/V_M is a natural module for M°. If q is odd, then the central involution of $M^\circ Q_M/Q_M$ and coprime action, can be used to show Q_M/V_M is a direct sum of a natural $M^\circ Q_M/Q_M$-module and trivial modules.

Assume that q is even. Then there is an element $\nu \in M^\circ$ of order $q+1$, which acts fixed-point-freely on U_M and satisfies

$$Q_M = U_M C_{Q_M}(\nu) \text{ and } U_M \cap C_{Q_M}(\nu) = 1.$$

By Lemma 7.4, $C_{Q_M}(U_M) = U_M$. Since Q_M/V_M is elementary abelian, $C_{Q_M}(\nu)$ is elementary abelian, as $[U_M, x] = V_M$ for all $x \in C_{Q_M}(\nu)^\#$, all the involutions in xU_M are contained in xV_M. It follows that $V_M C_{Q_M}(\nu) \cup U_M$ is the set of elements of order at most two in Q_M. In particular, $V_M C_{Q_M}(\nu)$ is normalized by $M^\circ Q_M$.

We have shown

$$\begin{aligned} &Q_M/V_M = U_M/V_M \times W/V_M, \text{ is elementary abelian,} \\ &|W/V_M| = q, \ [W/V_M, M^\circ] = 1 \text{ and } U_M/V_M = [Q_M/V_M, M^\circ] \end{aligned} \tag{1}$$

and further

$$\text{if } p = 2, \text{ then } U_M \cup W \text{ contains every involution in } Q_M. \tag{2}$$

Set $W_P = \langle V_M^P\rangle$ and select $g \in P$ such that $O^{p'}(P) = \langle Q_M, Q_M^g\rangle$. By Lemma 7.5, B_D normalizes V_M. Thus $|\{V_M^x \mid x \in P\}| = q+1$ and Q_M acts transitively on $\{V_M^x \mid x \in P\} \setminus \{V_M\}$ by conjugation. Hence, using (1), we obtain the first equality in

$$W_P = V_M V_M^g[V_M^g, Q_M] = V_M V_M^g[V_M^g, U_M] = V_M V_M^g(Q \cap U_M)$$

which has order at most q^4. Since $V_M \cap Q$ is normalized by P, and $O_p(P)/(V_M \cap Q)$ has order q^4 and involves 2 natural $\mathrm{SL}_2(q)$-modules, we have $|W_P| = q^3$. If $W_P \not\leq Q_M$, then we obtain $q^3 = |V_M[U_M, W_P]| \leq W_P \cap U_M < W_P$, which is nonsense. Hence $W_P \leq Q_M$

and W_P is elementary abelian. Since $|\widetilde{W_P}|$ is normalized by $\widetilde{P}$ and $\widetilde{U_M}$ is normalized by $\widetilde{L_1}$, $W_P \not\leq U_M$. If $p = 2$, (2) yields $W_P = W$. If q is odd, then, as B_M normalizes P and V_M, it also normalizes W_P. Since the involution in B_D/QQ_M centralizes Q_M/U_M, we deduce that $W_P = W$ is this case as well. In particular

$$W = W_P = O_p(U_1).$$

Now $C_{U_1}(W_P) = W_P$, and $[W_P, O_p(P)] = [V_M, y] = V_M \cap Q$ for all $y \in Q_P \setminus W$. Hence W_P is a strong dual F-module with offender $O_p(P)/W$ of order q^2 and so Lemma 2.2 implies $U_1/W_P \cong \mathrm{SL}_3(q)$ and $W_P = V_{U_1}$ is a natural module. We apply Proposition 2.5 with $X_1 = U_1$, $X_2 = D$, $T = QQ_M$, $B_2 = B_D$. Then $B_1 = \langle B_M, B_D \cap X_1\rangle$ and $R = N_D(Q \cap Q_M)$ and $M^* = MB_1$. Then $(X_1 \cap X_2)B_i/O_p(X_i)$ is a parabolic subgroup of $X_i/O_p(X_i)$, $[O^p(O^{p'}(M^*), O^p(O^{p'}(R)] \leq T$ by Lemma 7.4 and $B_i \leq N_G(X_{3-i})$ for $i = 1, 2$ by Lemma 7.5. Thus Proposition 2.5 implies that $F^*(H) = \langle X_1, X_2\rangle \cong \mathrm{PSL}_4(q)$. □

Proposition 7.7. *Suppose $V_M \neq Q_M$ and $\widetilde{D} \cong \mathrm{SL}_n(q)$ with $n > 3$. Then $F^*(H) \cong \mathrm{PSL}_{n+1}(q)$ or $G \cong \mathrm{Mat}(24)$.*

Proof. Select L_2^* such that $QQ_M \leq L_2^* \leq L_1$ to be the maximal subgroup which satisfies $\langle (V_M \cap Q)^{L_2^*}\rangle$ has order q^{n-2} and put $L_2 = O^{p'}(L_2^*)$. Then $L_2/O_p(L_2) \cong \mathrm{SL}_{n-2}(q)$. Set $\widehat{L}_2 = O^p(L_2)$. Notice that, as $n > 3$, $L_2 \not\leq B_D$.

By Lemma 7.4, we have that $\langle M^\circ, L_1\rangle = M^\circ \widehat{L}_1$, and, as $O^p(L_2) \leq O^p(L_1) \leq \widehat{L}_1$, $\langle M^\circ, L_2\rangle = M^\circ \widehat{L}_2$. In L we see that P normalizes $\widehat{L}_2$. Thus

$$\langle U_1, L_2\rangle = U_1 \widehat{L}_2. \tag{3}$$

Since $O_p(\widehat{L}_2)$ is non-trivial and is normalized by $\langle M^\circ, O^p(P)\rangle B$, we obtain

$$O_p(U_1) \neq 1.$$

Hence $U_1 S \in \mathcal{L}_G(S)$ and Theorem A implies $U_1 S \in \mathcal{VL}(S)$. Using $\overline{M^\circ} \cong \mathrm{SL}_2(q)$ and V_M is the natural module and Table 1 reading right to left, we obtain $\overline{U_1^\circ} \cong \mathrm{SL}_m(q)$ for some $m \geq 2$ and V_{U_1} is the natural module or $q = p = 2$, $\overline{U_1^\circ} \cong 3{}^{\cdot}\mathrm{Alt}(6)$ and V_{U_1} is the 2^6 module.

Assume that $\overline{U_1^\circ} \cong \mathrm{SL}_m(q)$ for some $m \geq 2$. Then as $M^\circ \leq U_1^\circ$, and P does not normalize M°, $m \geq 3$. Assume $m \geq 4$. Then $N_{\overline{U_1}}(V_M \cap Q)$ acts irreducibly on $V_{U_1}/V_M \cap Q$ and as $V_{U_1} \not\leq Q$, we deduce that $V_{U_1} \cap Q = V_M \cap Q$ and $|\widetilde{V_{U_1}}| = q^{m-1}$. Now $V_{U_1} \cap U_M = V_M$, and so $[V_{U_1}, M^\circ] = V_M$. Hence M° normalizes $\langle V_M^P\rangle$ and so we conclude that $V_{U_1} = \langle V_M^P\rangle$ and $m = 3$. Set $X_1 = U_1$, $X_2 = D$. Then Proposition 2.5 yields $F^*(H) \cong \mathrm{PSL}_{n+1}(q)$.

Suppose that $\overline{U_1^\circ} \cong 3{}^{\cdot}\mathrm{Alt}(6)$. Then $p = q = 2$ and $|V_{U_1}| = 2^6$. Then $N_{U_1S}(Q)/C_{U_1S}(V_{U_1}) \cong \mathrm{Sym}(3) \times \mathrm{Sym}(4)$. Assume that $n > 4$. Then there exists $P^* \leq L_1$ such that $\widetilde{P^*}$ is a parabolic subgroup of $\widetilde{D}$ which permutes with $\widetilde{P}$ but not with $N_{U_1S}(QQ_M)$.

Then P^* permutes with P and M but not with $\langle P, M\rangle$, a contradiction. Hence $n = 4$, $\widetilde{N_G(Q)} \cong \mathrm{SL}_4(2)$ and $|Q| = 2^4$. We claim that this extension splits. By Lemma 7.4, U has shape $2^6.(\mathrm{SL}_2(2)\times\mathrm{SL}_3(2))$ and $O_2(U)$ is a direct sum of two natural $\mathrm{SL}_3(2)$ modules. By considering the normalizer of a Sylow 3-subgroup of M° in U, we see that U splits over $O_2(U)$. In particular, there is a subgroup of U isomorphic to $2^3 : \mathrm{SL}_3(2)$ and so S splits over Q. Now [9, (I.17.4)] implies that $N_G(Q)$ is a split extension of Q by $\mathrm{SL}_4(2)$. Finally [1, (41.5)] yields $G \cong \mathrm{Mat}(24)$. This completes the proof of the proposition. □

To prove Proposition 7.1 we have to deal with the case that $F^*(H) \cong \mathrm{PSL}_n(q)$, $n \geq 4$. In the next lemma we prove the first assertion in Proposition 7.1(i).

Lemma 7.8. *Assume that $F^*(H) \cong \mathrm{PSL}_n(q)$, $n \geq 4$. Then $F^*(H) = F^*(\langle \mathcal{L}_G(S)\rangle)$.*

Proof. Suppose that the statement is false and select $P \in \mathcal{L}_G(S)$ minimal so that $P \nleq H$. Then Q is not normal in P as $N_G(Q) \leq H$. Since Q is large and abelian, we have that $Y_P \nleq Q$. In particular, $P \in \mathcal{VL}(S)$. By Theorem A(iii), $\langle P, N_G(Q)\rangle = \langle M_P, N_G(Q)\rangle$ and so $M_P \nleq H$ and therefore

$$P = M_P \in \mathcal{VL}^{min}(S) = \mathcal{VL}^{min}_{lin} \cup \mathcal{VL}^{min}_{orthsymp}(S) \cup \mathcal{VL}^{min}_{wreath}(S).$$

By Theorem 1.1, $P \in \mathcal{VL}^{min}_{lin} \cup \mathcal{VL}^{min}_{orthsymp}(S)$. Since Q is abelian, Q acts quadratically on W_P and so Corollary 4.6 implies that $\overline{P^\circ} \cong \mathrm{SL}_2(r)$ for some $r = p^b$. Assume first that $Q_P = W_P$. Then $N_G(Q)$ has no section isomorphic to $\mathrm{PSL}_n(q)$ for $n \geq 3$ by Proposition 6.1. Hence $Q_P > W_P$. Therefore we may apply Lemma 7.4. The group L_1 there is defined with respect to $Q \cap Q_M$. As $Q \cap Q_M$ and $Q \cap Q_P$ both are hyperplanes in Q which are S-invariant, we get $Q \cap Q_M = Q \cap Q_P$. Hence $O^{p'}(N_D(Q \cap Q_M)) = O^{p'}(N_D(Q \cap Q_P))$. Now Lemma 7.4 yields

$$U_P = J(U_P Q) = J(O_p(L_1)) = J(U_M Q) = U_M.$$

Hence $\langle P, M\rangle \leq N_G(U_M) \in \mathcal{L}_G(S)$. By Theorem A (i), $N_G(U_M)$ contains a unique element of $\mathcal{VL}^{min}(S)$. Therefore $P = M \leq H$, a contradiction. We conclude that $H = \langle \mathcal{L}_G(S)\rangle$ and the lemma is proved. □

Proof of Proposition 7.1. By Propositions 7.6, 7.7 and Lemma 7.8,

$$F^*(H) = F^*(\langle \mathcal{L}_G(S)\rangle) \cong \mathrm{PSL}_n(q)$$

with $n \geq 4$.

To complete the proof of the proposition, we assume that $p = 2$ and demonstrate that $G = H$. Let $K = F^*(H)$ and $S_0 = S \cap K$. So we know $K \cong \mathrm{PSL}_n(q)$. By Lemma 3.6, G has a component E. Since $E \cap S \in \mathrm{Syl}_2(E)$, $E \cap H \neq 1$ and so $K \leq E$. It follows that

$F^*(G) = E$. Since $F^*(K) \leq E$, E satisfies the hypothesis of Theorem 1.2 and so we may suppose that G is simple.

As M is normalized by S, S normalizes all the parabolic subgroups of K which contain S_0. Hence H is isomorphic to a subgroup of $\mathrm{P\Gamma L}_n(q)$.

Let $z \in Z(S)^\#$. Assume $g \in G$ and $1 \neq Q^g \cap H \leq S$. Then, as Q is large, z normalizes Q^g and $|C_{Q^g}(z)| \geq 4$ as $|Q| \geq q^3$. Hence $|Q^g \cap S| \geq 4$ and, as S/S_0 is cyclic, we then have $Q^g \cap K \neq 1$. By Proposition 2.6 (i) and (ii), there exists $h \in H$, such that $Q^{gh} \cap Z(J(S)) \neq 1$. Thus, as Q is large, $J(S) = J(Q^{gh}J(S))$ and Lemma 7.8 implies $Q^{gh} \leq N_G(J(S)) \leq H$. Hence there exists $h_1 \in H$ such that $Q^{ghh_1} = Q$ as Q is weakly closed. Since $N_G(Q) \leq H$, we deduce $g \in H$. In particular,

$$z^G \cap H = z^H.$$

Now, as H is not soluble, application of [8] (see also [23, Lemma 2.6]) shows $G = F^*(H)$ or $H \cong \mathrm{PSL}_4(2)$ and $G \cong \mathrm{Alt}(9)$. This proves the proposition. □

Proof of Theorem 1.2. This follows from Proposition 6.1 and Proposition 7.1. □

Data availability

No data was used for the research described in the article.

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References

[1] M. Aschbacher, Sporadic Groups, Cambridge University Press, 1994.
[2] M. Aschbacher, Finite Group Theory, Cambridge Stud. Adv. Math., vol. 10, Cambridge University Press, New York, 2000.
[3] M.J.J. Barry, Large abelian subgroups of Chevalley groups, J. Aust. Math. Soc. Ser. A 27 (1) (1979) 59–87.
[4] H. Bender, Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festläßt, J. Algebra 17 (1971) 527–554.
[5] A. Delgado, D. Goldschmidt, B. Stellmacher, Groups and Graphs: New Results and Methods, DMV Seminar, vol. 6, Birkhäuser Verlag, Basel, 1985.
[6] D. Gorenstein, R. Lyons, R. Solomon, The classification of the finite simple groups, Am. Math. Soc. Surv. Monogr. 40 (2) (1996).
[7] D. Gorenstein, R. Lyons, R. Solomon, The classification of the finite simple groups, Am. Math. Soc. Surv. Monogr. 40 (3) (1998).
[8] D. Holt, Transitive permutation groups in which an involution central in a Sylow 2-subgroup fixes a unique point, Proc. Lond. Math. Soc. 37 (1978) 165–192.
[9] B. Huppert, Endliche Gruppen I, Springer, 1967.
[10] Z. Janko, A characterization of the Mathieu simple groups, I, J. Algebra 9 (1968) 1–19.
[11] Z. Janko, A characterization of the Mathieu simple groups, II, J. Algebra 9 (1968) 2–41.
[12] U. Meierfrankenfeld, B. Stellmacher, The general FF-module theorem, J. Algebra 351 (2012) 1–63.

[13] U. Meierfrankenfeld, B. Stellmacher, Applications of the FF-module theorem and related results, J. Algebra 351 (2012) 64–106.
[14] U. Meierfrankenfeld, B. Stellmacher, G. Stroth, Finite groups of local characteristic p: an overview, in: A. Ivanov, M. Liebeck, J. Saxl (Eds.), Groups, Combinatorics and Geometry, Durham, Cambridge Univ. Press, 2001, pp. 155–191.
[15] U. Meierfrankenfeld, B. Stellmacher, G. Stroth, The local structure theorem for finite groups with a large p-subgroup, Mem. Am. Math. Soc. 242 (2016) 1147.
[16] U. Meierfrankenfeld, G. Stroth, Quadratic $GF(2)$ - modules for sporadic groups and alternating groups, Commun. Algebra 18 (1990) 2099–2140.
[17] U. Meierfrankenfeld, G. Stroth, R. Weiss, Local identification of spherical buildings and finite simple groups of Lie type, Math. Proc. Camb. Philos. Soc. 154 (2013) 527–547.
[18] Chr Parker, P. Rowley, Symplectic Amalgams, Springer, 2002.
[19] Chr Parker, P. Rowley, Local characteristic p completions of weak BN-pairs, Proc. Lond. Math. Soc. (3) 93 (2) (2006) 325–394.
[20] Chr. Parker, G. Pientka, A. Seidel, G. Stroth, Finite groups which are almost groups of Lie type in characteristic p, Mem. Am. Math. Soc. (2024), in press, 2021 accepted.
[21] Chr Parker, G. Stroth, Strongly p-embedded subgroups, Pure Appl. Math. Q. 7 (3) (2011) 797–858 (Special Issue: in honor of Jacques Tits).
[22] Chr Parker, G. Stroth, On strongly p-embedded subgroups of Lie rank 2, Arch. Math. (Basel) 93 (5) (2009) 405–413.
[23] Chr Parker, G. Stroth, $\mathrm{F}_4(2)$ and its automorphism group, J. Pure Appl. Algebra 218 (5) (2014) 852–878.
[24] Chr Parker, G. Stroth, The local structure theorem: the wreath product case, J. Algebra 561 (2020) 374–401.
[25] R. Salarian, G. Stroth, Existence of strongly p-embedded subgroups, Commun. Algebra 43 (2015) 983–1024.
[26] M. Suzuki, Group Theory II, Springer, 1986.
[27] J. Tits, R. Weiss, Moufang Polygons, Springer, 2002.
[28] A. Wagner, On perspectivities of finite projective planes, Math. Z. 71 (1959) 113–123.
[29] A.E. Zalesskii, Eigenvalues of matrices of complex representations of finite groups of Lie type, Springer Lect. Notes 1352 (1986) 206–218.