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Abstract

We prove estimates for the optimal volume of thick embeddings of finite graphs into symmetric spaces, generalising results of Kolmagorov-Barzdin and Gromov-Guth for embeddings into Euclidean spaces. We distinguish two very different behaviours depending on the rank of the non-compact factor. For rank at least 2, we construct thick wirings of N-vertex graphs with volume $CN\ln(N)$ and prove that this is optimal. For rank at most 1 we prove lower bounds of the form cN^a for some (explicit) $a > 1$ which depends on the dimension of the Euclidean factor and the conformal dimension of the boundary of the non-compact factor. The key ingredient is a coarse geometric analogue of a thick embedding called a coarse wiring, with the key property that the minimal volume of a thick embedding is comparable to the minimal volume of a coarse wiring for symmetric spaces of dimension at least 3.

1 Introduction

The focus of this paper is on thick embeddings of graphs as considered by Kolmogorov-Barzdin and Gromov-Guth [KB93, GG12]. By a graph, we mean a pair $(V\Gamma, E\Gamma)$ where $V\Gamma$ is a set whose elements are called vertices, and $E\Gamma$ is a set of unordered pairs of distinct elements of $V\Gamma$. Elements of $E\Gamma$ are called edges. The topological realisation of a graph is the (metric) topological space obtained from a disjoint union of unit intervals indexed by $e \in E\Gamma$, whose end points we label using the two elements contained in e. We then identify the end points of two intervals whenever they are labelled by the same element of $V\Gamma$.

The idea behind thick embeddings of graphs is that they are the appropriate embeddings to consider in situations where the graph models a physical object (i.e. vertices and edges are "thick" and therefore need to remain a prescribed distance apart). Two key examples are: a brain, where neurons are represented by vertices and axons by edges; and an electronic network, where components are vertices and wires are edges. We briefly summarise the relevant results from [KB93, GG12] in the following two theorems.

Theorem 1.1. *Let Γ be a finite graph with maximal degree d. For each $k \geq 3$, there is a topological embedding $f_k : \Gamma \to \mathbb{R}^k$ and a constant $C = C(d,k)$ with the following properties:*

(i) $d_{\mathbb{R}^k}(f_k(X), f_k(Y)) \geq 1$ *whenever X, Y are: two distinct vertices; an edge and a vertex not contained in that edge; or two disjoint edges.*

(ii) $\mathrm{diam}(f_3) := \mathrm{diam}(\mathrm{im}(f_3)) \leq C|\Gamma|^{1/2}$.

(iii) $\mathrm{diam}(f_k) \leq C|\Gamma|^{1/(k-1)} \log(1+|\Gamma|)^4$.

We say a topological embedding $g : \Gamma \to Z$ is **ε-thick** if it satisfies the inequality $d_Z(g(X), g(Y)) \geq \varepsilon$ whenever X, Y are as in condition (i).

Theorem 1.2. *For every $\delta, \varepsilon > 0$ and $d \in \mathbb{N}$ there is a constant $c > 0$ such that given any finite graph Γ with maximal degree d and Cheeger constant (cf. Definition 5.2) $\geq \delta$ and any ε-thick topological embedding $g : \Gamma \to \mathbb{R}^k$, we have $\mathrm{diam}(g) \geq c^{-1}|\Gamma|^{1/(k-1)} - c$ and the 1-neighbourhood of the image of g has volume $\geq c^{-1}|\Gamma|^{k/(k-1)}$.*

We define the **volume** $\mathrm{vol}(g)$ of an ε-thick topological embedding g to be the measure of the ε-neighbourhood of its image. From Theorem 1.1 we get obvious upper bounds on the volume of 1-thick embeddings into $\mathbb{R}^k$. Namely, $\mathrm{vol}(f_3) \leq C'|\Gamma|^{3/2}$ and $\mathrm{vol}(f_k) \leq C'|\Gamma|^{k/(k-1)} \log(1+|\Gamma|)^{4k}$.

Our goal is to develop the theory of thick embeddings into spaces other than $\mathbb{R}^k$. In this paper, we prove versions of Theorems 1.1 and 1.2 for thick embeddings into symmetric spaces.

1.1 Thick embeddings into symmetric spaces

Our main results are analogues of Theorems 1.1 and 1.2 for more general symmetric spaces. We recall that each symmetric space X decomposes as a direct product of symmetric spaces $K \times \mathbb{R}^d \times N$ where K is compact and N has no non-trivial compact or Euclidean factor. N is called the non-compact factor. We begin with the case where the rank of N is at most 1. When the non-compact factor is a real hyperbolic space, we have the following upper bounds:

Theorem 1.3. *Let $X = K \times \mathbb{R}^r \times \mathbb{H}^q_{\mathbb{R}}$ where K is compact and $q + r \geq 3$. Let $d \in \mathbb{N}$. There is a constant $C = C(X,d)$ such that for any finite graph Γ with maximal degree at most d there is an 1-thick topological embedding of Γ into X with volume*

$$\leq \begin{cases} C|\Gamma|^{1+1/(q+r-2)} & \text{if} \quad q+r = 3,4, \\ C|\Gamma|^{1+1/(q+r-2)} \ln(1+|\Gamma|)^{4(q+r-1)} & \text{if} \quad q+r \geq 5. \end{cases}$$

For $q+r \geq 4$ this follows from Theorem 1.1 by composing the topological embedding with a suitable coarse embedding $\mathbb{R}^r \times \mathbb{R}^{q-1} \to K \times \mathbb{R}^r \times \mathbb{H}^q_{\mathbb{R}}$ where $\mathbb{R}^{q-1}$ embeds as a horosphere in $\mathbb{H}^q_{\mathbb{R}}$. The case $q+r=3$ is new and is treated separately (cf. Theorem 1.7).

Our next result gives lower bounds on the volume of wirings into all symmetric spaces whose non-compact factor has rank at most one. In particular, it states that the embeddings constructed in Theorem 1.3 are within a poly-logarithmic error of being optimal.

Theorem 1.4. *Let $X = K \times \mathbb{R}^r \times \mathbb{H}^q_{\mathbb{F}}$, where K is compact and $q \dim_{\mathbb{R}}(\mathbb{F}) + r \geq 3$. Let $d \in \mathbb{N}$. Set $Q = (q+1)\dim_{\mathbb{R}}(\mathbb{F}) - 2$, the conformal dimension of the boundary of $\mathbb{H}^q_{\mathbb{F}}$. For any $d, \varepsilon, \delta > 0$ there is a constant $c = c(d, \varepsilon, \delta) > 0$ with the following property. For any N-vertex graph Γ with maximal degree d and Cheeger constant $h(\Gamma) \geq \delta$ every ε-thick topological embedding $g : \Gamma \to X$ has volume*

$$\geq \begin{cases} cN^{1+1/r}\log(1+N)^{-1/r} & \text{if} \quad Q = 1, \\ cN^{1+1/(Q+r-1)} & \text{if} \quad Q \geq 2. \end{cases}$$

In particular, our results in Theorem 1.3 are sharp for $\mathbb{H}^3_{\mathbb{R}}$, $\mathbb{H}^3_{\mathbb{R}} \times \mathbb{R}$ and $\mathbb{H}^4_{\mathbb{R}}$. If the $k=1$ case of the conjecture in [GG12] is verified (giving optimal bounds for Euclidean spaces), then the lower bound in Theorem 1.4 would also be optimal for all products $\mathbb{R}^r \times \mathbb{H}^q_{\mathbb{R}}$ where $q \geq 3$.

When the rank of N is at least 2, we provide matching upper and lower bounds.

Theorem 1.5. *Let X be a symmetric space whose non-compact factor has rank ≥ 2 and let $d \in \mathbb{N}$. There are constants $\varepsilon, C > 0$ which depend on X and d such that for any finite graph Γ with maximal degree at most d, there is an ε-thick topological embedding of Γ into X with diameter $\leq C\ln(1+|\Gamma|)$ and volume $\leq C|\Gamma|\ln(1+|\Gamma|)$.*

Theorem 1.6. *Let X be a symmetric space whose non-compact factor has rank ≥ 2 and let $d \in \mathbb{N}$. For any $d, \varepsilon, \delta > 0$ there is a constant $c = c(d, \varepsilon, \delta) > 0$ with the following property. For any finite graph Γ with maximal degree d and Cheeger constant $h(\Gamma) \geq \delta$ every ε-thick[1] topological embedding $g : \Gamma \to X$ satisfies $\mathrm{vol}(g) \geq c|\Gamma|\ln(1+|\Gamma|)$.*

This "gap" between the rank at most 1 and the higher rank case is similar in flavour to the gap in the separation profiles of symmetric spaces found in [HMT20b]. This is no coincidence. The lower bounds on the volumes of topological embeddings found in Theorems 1.6 and 1.4 are inverse functions

[1] Unlike topological embeddings into Euclidean space, there does not seem to be an obvious way to relate the volumes of optimal topological embeddings with different thickness parameters.

of the separation profiles of the symmetric spaces[2], and our approach to prove both of these theorems utilises separation profiles in a crucial way. In order to use separation profiles, we will reformulate the above theorems in terms of carefully chosen continuous maps (called coarse wirings) between bounded degree graphs.

We present one further result in this section, which is an upper bound for thick embeddings into real hyperbolic 3-space which is asymptotically optimal (with the corresponding lower bound provided by 1.4) but which does not depend on the degree of the graph.

Theorem 1.7. *There is a* 1*-thick topological embedding of* K_N *(the complete graph on* N *vertices) into* $\mathbb{H}^3$ *with diameter* $\leq 2\ln(N) + 9$ *and volume* $\leq 2039N^2$.

1.2 Coarse k-wirings

Definition 1.8. Let Γ, Γ' be graphs. A **wiring** of Γ into Γ' is a continuous map $f : \Gamma \to \Gamma'$ which maps vertices to vertices and edges to unions of edges.

A wiring f is a **coarse k-wiring** if

1. the restriction of f to $V\Gamma$ is $\leq k$-to-1, i.e. $|\{v \in V\Gamma \mid f(v) = w\}| \leq k$ for all $w \in V\Gamma'$; and

2. each edge $e \in E\Gamma'$ is contained in at most k of the paths in $\mathcal{P}$.

We consider the **image** of a wiring $\mathrm{im}(f)$ to be the subgraph of Γ' consisting of all vertices and edges in the image of f. The **diameter** of a wiring $\mathrm{diam}(f)$ is the diameter of its image (measured in Γ'), the **volume** of a wiring $\mathrm{vol}(f)$ is the number of vertices in its image.

Under mild hypotheses on the target space, we can convert a thick topological embedding into a coarse k-wiring.

Proposition 1.9. *Let* M *be a Riemannian manifold and let* Y *be a graph quasi-isometric to* M*, let* $d \in \mathbb{N}$ *and let* $T > 0$*. There exist constants* C *and* k *such for every finite graph* Γ *with maximal degree* d *the following holds:*

If there is a T*-thick topological embedding* $\Gamma \to M$ *with diameter* D *and volume* V *then there is a coarse* k*-wiring of* Γ *into* Y *with diameter at most* CD *and volume at most* CV.

With stronger hypotheses we are able to convert coarse k-wirings into thick topological embeddings.

[2]By the separation profile of a symmetric space we mean either the 1-Poincaré profile of the symmetric space as defined in [HMT20a] or equivalently, the separation profile as defined in [BST12] of any graph quasi-isometric to the symmetric space.

Theorem 1.10. *Let M be a compact Riemannian manifold of dimension $n \geq 3$, let Y be a graph quasi-isometric to the universal cover $\widetilde{M}$ of M and let $k, d \in \mathbb{N}$. There exist constants C and $\varepsilon > 0$ such that the following holds:*

If there is a coarse k-wiring of a finite graph Γ with maximal degree d into Y with diameter D and volume V then there is a ε-thick embedding of Γ into $\widetilde{M}$ with diameter at most CD and volume at most CV.

Using Proposition 1.9 and Theorem 1.10 we can prove Theorems 1.5, 1.6, 1.3 and 1.4 purely in terms of coarse wirings. We introduce wiring profiles in order to discuss coarse wirings between infinite graphs.

Definition 1.11. Let Γ be a finite graph and let Y be a graph. We denote by $\mathrm{wir}^k(\Gamma \to Y)$ the minimal volume of a coarse k-wiring of Γ into Y. If no such coarse k-wiring exists, we say $\mathrm{wir}^k(\Gamma \to Y) = +\infty$.

Let X and Y be graphs. The k-**wiring profile** of X into Y is the function

$$\mathrm{wir}^k_{X\to Y}(n) = \max\left\{\mathrm{wir}^k(\Gamma \to Y) \ \mid \Gamma \leq X,\ |\Gamma| \leq n\right\}.$$

A simple example of a situation where $\mathrm{wir}^k(\Gamma \to Y) = +\infty$ is when Γ has a vertex whose degree is greater than k times the maximal degree of Y.

The reason for working with wiring profiles is that they have three very useful properties. Firstly, wirings between graphs can be composed and there is a natural inequality which controls the volume of the composition.

Proposition 1.12. *Let X, Y, Z be graphs. Suppose $\mathrm{wir}^k_{X\to Y}$ and $\mathrm{wir}^l_{Y\to Z}$ take finite values. Then*

$$\mathrm{wir}^{kl}_{X\to Z}(n) \leq \mathrm{wir}^l_{Y\to Z}\left(\mathrm{wir}^k_{X\to Y}(n)\right).$$

Secondly, for bounded degree graphs, the wiring profile of X into Y grows linearly whenever there is a regular map from X to Y.

Definition 1.13. Let X, Y be metric spaces and let $\kappa > 0$. A map $r : X \to Y$ is κ-regular if

1. $d_Y(r(x), r(x')) \leq \kappa(1 + d_X(x, x'))$, and
2. the preimage of every ball of radius 1 in Y is contained in a union of at most κ balls of radius 1 in X.

Quasi-isometric and coarse embeddings between bounded degree graphs are examples of regular maps.

Proposition 1.14. *Let X and Y be graphs with maximal degree $\Delta > 0$ and let $r : X \to Y$ be a κ-regular map. Then there exists $k = k(\kappa, \Delta)$ such that*

$$\mathrm{wir}^k_{X\to Y}(n) \leq \left(\kappa + \frac{1}{2}\right)\Delta n.$$

These two propositions naturally combine to show that wiring profiles are well-behaved with respect to regular maps.

Corollary 1.15. *Let X, X', Y and Y' be graphs with maximal degree Δ and let $r_X : X' \to X$ and $r_Y : Y \to Y'$ be κ-regular maps. Then for every k such that $\mathrm{wir}^k_{X\to Y}$ takes finite values there is some l such that*

$$\mathrm{wir}^l_{X\to Y'}(n) \leq \left(\kappa + \frac{1}{2}\right) \Delta \dot{\mathrm{wir}}^k_{X\to Y}(n). \tag{1}$$

$$\mathrm{wir}^l_{X'\to Y'}(n) \leq \left(\kappa + \frac{1}{2}\right) \Delta \dot{\mathrm{wir}}^k_{X\to Y}\left(\left(\kappa + \frac{1}{2}\right)\Delta n\right). \tag{2}$$

The third benefit of coarse wirings is that we can find lower bounds on the wiring profile of two bounded degree graphs in terms of their separation profiles: a measure of the combinatorial connectivity of their finite subgraphs introduced in [BST12].

Theorem 1.16. *Let X and Y be graphs of bounded degree where $\mathrm{sep}_X \gtrsim n^r \log(n)^s$ and $\mathrm{sep}_Y \simeq n^p \log(n)^q$. Then, for any k,*

$$wir^k_{X\to Y}(n) \gtrsim \begin{cases} n^{r/p}\log(n)^{(s-q)/p} & \text{if } p>0, \\ \exp(n^{r/(q+1)}\log(n)^{s/(q+1)}) & \text{if } p=0. \end{cases}$$

The separation profiles of (graphs quasi-isometric to) symmetric spaces have all been calculated [BST12, HMT20a, HMT20b] and are all of the form $n^p \log(n)^q$. Combining these calculations with Theorem 1.16 and Theorem 1.10 is sufficient to prove Theorems 1.6 and 1.4.

The coarse geometric approach also has great benefits when computing upper bounds. For instance, we can deduce the upper bound on volumes of thick embeddings in Theorem 1.5 from the following theorem.

Theorem 1.17. *There is a Cayley graph Y of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ with the following property. For any N-vertex graph Γ with maximal degree d, we have*

$$\mathrm{wir}^{2d}(\Gamma \to Y) \leq 6dN\ln(1+N).$$

The deduction works as follows. The graph Y is quasi-isometric to the Diestel-Leader graph $\mathrm{DL}(2,2)$ [Woe05]. Next, $\mathrm{DL}(2,2)$ quasi-isometrically embeds into any symmetric space M whose non-compact factor has rank ≥ 2 [HMT20b, Proposition 2.8 and Theorem 3.1]. Choose a graph X which is quasi-isometric to M. By Corollary 1.15, there are constants l, C' which depend on Y and d but not N such that $\mathrm{wir}^l(\Gamma \to X) \leq C'N\ln(1+N)$. Theorem 1.5 then follows from Theorem 1.10.

It is important to stress that the analogy between thick embeddings and coarse wirings only holds when there is a bound on the degree of the graphs and the dimension of the symmetric space is at least 3. This is evidenced by Theorem 1.7 which holds independent of the degree of the graph, where no such result for coarse wirings is possible. On the other hand, we can consider coarse wirings into spaces where not all graphs admit topological embeddings, such as $\mathbb{R}^2$ and $\mathbb{H}^2$.

Theorem 1.18. *Let $d \geq 3$ and let $X(d)$ be the disjoint union of all finite graphs with maximal degree $\leq d$. Let Y and Z be graphs which are quasi-isometric to $\mathbb{R}^2$ and $\mathbb{H}^2$ respectively. For all sufficiently large k, we have*

$$\mathrm{wir}^k_{X(d)\to Y}(n) \simeq n^2 \quad \text{and} \quad \exp(n^{1/2}) \lesssim \mathrm{wir}^k_{X(d)\to Z}(n) \lesssim \exp(n).$$

The lower bounds both follow from Theorem 1.16, since $\mathrm{sep}_{X(d)}(n) \simeq n$ as it contains a family of expanders of at most exponentially growing size [Hum17]. For the upper bound we will make direct constructions. We believe that it is possible to improve the bound in the $p = 0$ case of Theorem 1.16 to $\exp(n^{r/q}\log(n)^{s/q})$. This would have the consequence that $\mathrm{wir}^k_{X(d)\to Z}(n) \simeq \exp(n)$ in Theorem 1.18.

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2 Thick topological embeddings into hyperbolic 3-space

Our goal in this section is to prove Theorems 1.3 and 1.7, which we do by directly constructing thick topological embeddings. We start with the proof of Theorem 1.3 in the case $q + r \geq 4$.

Proof. Define $h_0 = (2(\cosh(1) - 1)^{-1/2}$. Consider the map

$$\phi_{q,r} : \mathbb{R}^r \times \mathbb{R}^{q-1} \to \mathbb{R}^r \times \mathbb{H}^q_{\mathbb{R}} \quad \text{given by} \quad \phi_{q,r}(\underline{x}, \underline{y}) = (\underline{x}, (\underline{y}; h_0)).$$

Claim: $d(\phi_{q,r}(\underline{x}, \underline{y}), \phi_{q,r}(\underline{x}', \underline{y}')) \geq 1$ whenever $\|\underline{x} - \underline{x}'\|_2 \geq 1$ or $\|\underline{y} - \underline{y}'\|_2 \geq 1$.

Proof of Claim. If $\|\underline{x} - \underline{x}'\|_2 \geq 1$ then this is obvious. If $\|\underline{y} - \underline{y}'\|_2 \geq 1$, then

$$\begin{aligned} d(\phi_{q,r}(\underline{x}, \underline{y}), \phi_{q,r}(\underline{x}', \underline{y}')) &\geq d_{\mathbb{H}^q_{\mathbb{R}}}((\underline{y}; h_0), (\underline{y}'; h_0)) \\ &= \cosh^{-1}\left(1 + \frac{\|\underline{y} - \underline{y}'\|_2^2}{2h_0^2}\right) \\ &\geq \cosh^{-1}\left(1 + \frac{1}{2h_0^2}\right) \\ &= \cosh^{-1}(1 + (\cosh(1) - 1)) = 1. \end{aligned}$$

$\square$

Let Γ be a finite graph with maximal degree d and let $\psi = \sqrt{2}.f_{q+r-1}$ where f_{q+r-1} is the 1-thick topological embedding of Γ into $\mathbb{R}^{q+r-1}$ defined in Theorem 1.1. Let us first show that $\psi \circ \phi$ is a 1-thick embedding of Γ into $\mathbb{R}^r \times \mathbb{H}^q_{\mathbb{R}}$.

The topological embedding ψ is $\sqrt{2}$-thick. If $\|(\underline{x}, \underline{y}) - (\underline{x}', \underline{y}')\|_2 \geq \sqrt{2}$, then either $\|\underline{x} - \underline{x}'\|_2 \geq 1$ or $\|\underline{y} - \underline{y}'\|_2 \geq 1$. Applying the claim, we see that $\psi \circ \phi$ is 1-thick.

Finally we bound $\mathrm{vol}(\psi \circ \phi)$. Firstly note that if $\|(\underline{x}, \underline{y}) - (\underline{x}', \underline{y}')\|_2 \leq 1$, then

$$\begin{aligned} d(\phi_{q,r}(\underline{x}, \underline{y}), \phi_{q,r}(\underline{x}', \underline{y}')) &= \left(\|\underline{x} - \underline{x}'\|_2 + d_{\mathbb{H}^q_{\mathbb{R}}}((\underline{y}; h_0), (\underline{y}'; h_0)) \right)^{1/2} \\ &\leq \left(1 + \cosh^{-1}\left(1 + \frac{\|\underline{y} - \underline{y}'\|_2^2}{2h_0^2} \right) \right)^{1/2} \\ &\leq \left(1 + \cosh^{-1}\left(1 + \frac{1}{2h_0^2} \right) \right)^{1/2} = \sqrt{2}. \end{aligned}$$

Now let Y be a $\frac{1}{2}$-separated 1-net in $\mathrm{im}(\psi)$. It follows from the above equation that $\phi(Y)$ is a $\sqrt{2}$-net in $\mathrm{im}(\psi \circ \phi)$. Denote by α, β the volumes of the balls of radius $\frac{1}{4}$ and $\sqrt{2} + 1$ in $\mathbb{R}^{q+r-1}$ and $\mathbb{R}^r \times \mathbb{H}^q_{\mathbb{R}}$ respectively. We have

$$\mathrm{vol}(\psi \circ \phi) \leq \beta |Y| \quad \text{and} \quad \alpha |Y| \leq \mathrm{vol}(\psi).$$

Hence, using the volume bounds from Theorem 1.1, there is a constant C which depends on q, r, d but not Γ such that

$$\begin{aligned} \mathrm{vol}(\psi \circ \phi) &\leq \beta |Y| \\ &\leq \beta \alpha^{-1} \mathrm{vol}(\psi) \\ &\leq \begin{cases} \beta \alpha^{-1} C' |\Gamma|^{3/2} & \text{if} \quad q + r = 4, \\ \beta \alpha^{-1} C' |\Gamma|^{1+1/(q+r-2)} \log(1 + \Gamma)^{4(q+r-1)} & \text{if} \quad q + r \geq 5. \end{cases} \end{aligned}$$

$\square$

It remains to tackle the case $q + r = 3$.

Theorem 2.1. *There is a* 1*-thick topological embedding* $g : K_N \to \mathbb{H}^3$ *with* $\mathrm{diam}(g) \leq 2\ln(N) + 9$ *and* $\mathrm{vol}(g) \leq 2039 N^2$.

We split the proof into two parts. Firstly, we build a 1-thick topological embedding of the complete graph on N vertices into $[0, N-1]^2 \times [0, 1]$. Then we use an embedding of $\mathbb{R}^2$ as a horosphere in $\mathbb{H}^3$ to construct a 1-thick topological embedding into $\mathbb{H}^3$.

Lemma 2.2. *Let K_N denote the complete graph on N vertices. There is a 1-thick topological embedding $f : K_N \to [0, N-1]^2 \times [0,1] \subset (\mathbb{R}^3, \|\cdot\|_\infty)$.*

Proof. Enumerate the vertices of K_N as $v_0, \ldots, v_{N-1}$. Now we map v_k to $(k,k,0)$. We connect $(k,k,0)$ to $(l,l,0)$ using the following piecewise linear path P_{kl}:

$$(k,k,0) \to (l,k,0) \to (l,k,1) \to (l,l,1) \to (l,l,0). \tag{3}$$

Let us verify that this embedding is 1-thick. Any two distinct vertices v_k and v_l are mapped at distance $|k-l| \geq 1$. Next, consider a path P_{kl} and the image $(i,i,0)$ of a vertex v_i with $i \neq k,l$. Since one of the first two coordinates of the path P_{kl} is always either k or l, we have

$$d_\infty(P_{kl}, (i,i,0)) \geq \min\{|i-k|, |i-l|\} \geq 1.$$

Finally, consider paths P_{ij}, P_{kl}. Let $(w,x,a) \in P_{ij}$ and $(y,z,b) \in P_{kl}$ and suppose $d((w,x,a),(y,z,b)) < 1$.

If $a = 1$, then $b > 0$, so $w = j$ and $y = k$. Since $d_\infty((w,x,a),(y,z,b)) \geq |w-y|$, we have $|j-k| < 1$. Thus $j = k$ and the two paths come from edges which share a vertex.

If $a \in (0,1)$ then $w = x \in \{i,j\}$. Since $d_\infty((w,x,a),(y,z,b)) \geq \max\{|w-y|, |x-z|\}$ and at least one of y,z is equal to either k or l, one of i,j must be equal to one of k,l. Thus the two paths come from edges which share a vertex.

If $a = 0$ then either $x = i$ or $w = x = j$. Also $b < 1$ so either $z = k$ or $y = z = l$. If $x = i$ and $z = k$ then the argument from the $a = 1$ case holds. Next, suppose $w = x = j$. Since $z \in \{k,l\}$ and $d_\infty((w,x,a),(y,z,b)) \geq |x-z|$, we have $j = k$ or $j = l$. If $x = i$ and $y = z = l$, then $i = l$ following the same reasoning. $\square$

Next, we embed $[0,N-1]^2 \times [0,1]$ into $\mathbb{H}^3$. We work in the upper-half space model of $\mathbb{H}^3 = \{(x,y;z) \mid z > 0\}$.

Consider the map $\phi : \mathbb{R}^2 \times [0,1] \to \mathbb{H}^3$ defined by

$$(x,y,a) \mapsto (x,y; h_0 e^{-a}).$$

Lemma 2.3. *Let $f : K_N \to [0,N-1]^2 \times [0,1]$ be the 1-thick topological embedding from Lemma 2.2. The map $g = \phi \circ f$ is a 1-thick embedding of K_N into $\mathbb{H}^3$ with diameter $\leq 2\ln N + 9$ and volume $\leq 2039N^2$.*

Proof. Firstly, recall that $d_{\mathbb{H}^3}((x_1,y_1;z_1),(x_2,y_2;z_2))$ is given by the formula

$$\cosh^{-1}\left(1 + \frac{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}{2z_1 z_2}\right)$$

We first prove that g is 1-thick. Since f is 1-thick with respect to the L^∞ metric, it suffices to prove that $d_{\mathbb{H}^3}(\phi(a_1,b_1,c_1),\phi(a_2,b_2;c_2)) \geq 1$ whenever $(a_1,b_1,c_1),(a_2,b_2,c_2) \in [0,n-1]^2 \times [0,1]$ are at L^∞ distance ≥ 1.

Suppose $\max\{|a_2-a_1|,|b_2-b_1|,|c_2-c_1|\} \geq 1$. If $\max\{|a_2-a_1|,|b_2-b_1|\} \geq 1$, then

$$d_{\mathbb{H}^3}(\phi(a_1,b_1,c_1),\phi(a_2,b_2,c_2)) \geq \cosh^{-1}\left(1+\frac{1}{2h_0^2}\right) = 1.$$

If $|c_2-c_1| \geq 1$, then

$$\begin{aligned} d_{\mathbb{H}^3}(\phi(a_1,b_1,c_1),\phi(a_2,b_2,c_2)) &\geq \cosh^{-1}\left(1+\frac{h_0^2(1-e^{-1})^2}{2h_0^2e^{-1}}\right) \\ &= \cosh^{-1}(\cosh(1)) = 1. \end{aligned}$$

Next we bound the diameter and the volume. For every point $(x,y;z)$ in the image of g, we have $|x|,|y| \leq N-1$ and $h_1 = h_0e^{-1} \leq z \leq h_0$. Thus

$$\begin{aligned} d_{\mathbb{H}^3}((0,0;h_0),(x,y;z)) &\leq \cosh^{-1}\left(1+\frac{2(N-1)^2+h_0^2(1-e^{-1})^2}{2h_0^2e^{-2}}\right) \\ &\leq \cosh^{-1}\left(1+\frac{2e^2N^2+e^2h_0^2}{2h_0^2}\right) \\ &\leq \cosh^{-1}\left(1+\frac{e^2}{2}+2e^2(\cosh(1)-1)N^2\right) \\ &\leq \cosh^{-1}\left(2e^2\cosh(1)N^2\right) \\ &\leq \ln\left(4e^2\cosh(1)N^2\right) \\ &= 2\ln(N)+\ln(4e^2\cosh(1)) \leq 2\ln(N)+9. \end{aligned}$$

Next, we bound the volume. For each point $(x,y;z)$ in the image of g there is a point $(a,b;h_0)$ with $a,b \in \{0,\ldots,N-1\}$ such that $|x-a| \leq \frac{1}{2}$, $|y-b| \leq \frac{1}{2}$ and $z \in [h_0e^{-1},h_0]$. We have

$$\begin{aligned} d_{\mathbb{H}^3}((a,b;h_0),(x,y;z)) &\leq \cosh^{-1}\left(1+\frac{\frac{1}{2}^2+\frac{1}{2}^2+h_0^2(1-e^{-1})^2}{2h_0^2e^{-2}}\right) \\ &\leq \cosh^{-1}\left(1+\frac{1}{4h_0^2e^{-2}}+\frac{1}{2e^{-2}}\right) \\ &= \cosh^{-1}\left(1+\frac{e^2\cosh(1)}{2}\right) =: \lambda. \end{aligned}$$

Thus, the volume of the 1-neighbourhood of the image of g is at most CN^2 where C is the volume of the ball of radius $\lambda+1$ in $\mathbb{H}^3$. We have

$$C = \pi(\sinh(2(\lambda+1)) - 2(\lambda+1)) \leq 2039$$

as required. $\square$

Using the same strategy, we can also prove the following.

Theorem 2.4. *There is a constant C such that for every $N \in \mathbb{N}$, there is a 1-thick topological embedding $g : K_N \to \mathbb{R} \times \mathbb{H}^2_{\mathbb{R}}$ with $\mathrm{vol}(g) \leq CN^2$.*

Proof. Repeat the proof of Theorem 2.1 but replace the map ϕ by

$$\phi : \mathbb{R}^2 \times [0,1] \to \mathbb{R} \times \mathbb{H}^2_{\mathbb{R}} \quad \text{given by} \quad \phi(x,y,z) = (x; y, h_0 e^{-z}).$$

$\square$

3 Coarse wiring

In this section, we present some elementary properties of coarse wirings and construct coarse wirings of finite graphs into a Cayley graph of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$.

Recall that a map $r : X \to Y$ between metric spaces is **κ-regular** if $d_Y(r(x), r(y)) \leq \kappa(1 + d_X(x,y))$ for all $x, y \in X$ and the preimage of every ball of radius 1 in Y is contained in a union of at most κ balls of radius 1 in X.

Lemma 3.1. *Let X and Y be graphs with maximal degree Δ and let $r : VX \to VY$ be a κ-regular map. Then for all sufficiently large k we have*

$$\mathrm{wir}^k_{X\to Y}(n) \leq \left(\kappa + \frac{1}{2}\right)\Delta n.$$

Proof. Let $\Gamma \subset X$ be a subgraph with $|V\Gamma| \leq n$. For $xx' \in E\Gamma$ let $P_{xx'}$ be any minimal length path from $r(x)$ to $r(x')$ and let $\Gamma' = \bigcup_{E\Gamma} P_{xx'}$. We construct a wiring $f : \Gamma \to \Gamma'$ as follows. For each vertex $v \in V\Gamma$ we define $f(v) = r(v)$. We then map each edge xx' continuously to the path $P_{xx'}$.

Since each path $P_{xx'}$ contains at most $2\kappa + 1$ vertices and $|E\Gamma| \leq \frac{1}{2}\Delta n$, we have $|V\Gamma'| \leq n\Delta(\kappa + \frac{1}{2})$.

If $P_{xx'}$ contains an edge e then the distance from $r(x)$ to the initial vertex of e is at most 2κ, so there are at most $1 + \Delta^{2\kappa+1}$ possibilities for $r(x)$; r is at most $\kappa(1+\Delta)$-to-one so there are at most $k := \kappa(1+\Delta)(1+\Delta^{2\kappa+1})$ possibilities for x. Therefore there are at most k edges $xx' \in E\Gamma$ such that $f(xx') = P_{xx'}$ contains a given edge e of $E\Gamma'$. It follows that $\mathrm{wir}^k_Y(\Gamma) \leq (\kappa + \frac{1}{2})\Delta n$. $\square$

Lemma 3.2. *Suppose* $\mathrm{wir}^k_{X\to Y}(N) < \infty$. *Then*

$$\mathrm{wir}^{kl}_{X\to Z}(N) \leq \mathrm{wir}^l_{Y\to Z}\left(\mathrm{wir}^k_{X\to Y}(N)\right).$$

Proof. If $\mathrm{wir}^l_{Y\to Z}\left(\mathrm{wir}^k_{X\to Y}(N)\right) = +\infty$ then there is nothing to prove, so assume it is finite. Let $\Gamma \subset X$ with $|V\Gamma| \leq N$. Then there exists a coarse

k-wiring ψ of Γ into Y with $\mathrm{vol}(W) \leq \mathrm{wir}^k_{X\to Y}(N)$ and a coarse l-wiring ψ' of $\mathrm{im}(W)$ into Z with $\mathrm{vol}(W') \leq \mathrm{wir}^l_{Y\to Z}\left(\mathrm{wir}^k_{X\to Y}(N)\right)$.

We now construct a coarse kl-wiring ψ'' of Γ into Z. For each $v \in V\Gamma$, define $\psi''(v) = \psi'(\psi(v))$. For each $e \in E\Gamma$, let $e_1, \ldots, e_m$ be the edge path P_e. We define P''_e to be the concatenation of paths $P'_{e_1}P'_{e_2}\ldots P'_{e_m}$. We extend ψ'' continuously so that the image of e is P''_e. It is clear that $\psi''|_{V\Gamma}$ is $\leq kl$-to-1 and $\mathrm{im}(\psi'') \subseteq \mathrm{im}(\psi')$, so $\mathrm{vol}(\psi'') \leq \mathrm{vol}(\psi')$. Since each edge in $\mathrm{im}(\psi'')$ is contained in at most l of the paths $P'_{e'}$ and each $P'_{e'}$ is used in at most k of the paths P_e, we have that each edge in $\mathrm{im}(\psi'')$ is contained in the image of at most kl of the edges in $E\Gamma$, as required. □

Proof of Corollary 1.15. This follows immediately from Lemmas 3.1 and 3.2. □

Finally in this section we construct coarse wirings into a Cayley graph of the lamplighter group. This construction is crucial for Theorem 1.10. We identify $\mathbb{Z}_2 \wr \mathbb{Z}$ with the semidirect product $\bigoplus_{\mathbb{Z}} \mathbb{Z}_2 \rtimes \mathbb{Z}$ and define Y to be the Cayley graph of $\mathbb{Z}_2 \wr \mathbb{Z}$ using the generating set $\{(\delta_0, 0), (0,1), (0,-1)\}$ where $\delta_0(i) = 1$ if $i = 0$ and 0 otherwise.

Proposition 3.3. *Let Γ be an n-vertex graph with maximal degree d. There is a coarse $2d$-wiring of Γ into Y with diameter at most $6\lceil\log_2(n)\rceil$ and volume at most $dn\left(3\lceil\log_2(n)\rceil + \frac{1}{2}\right)$.*

Proof. Set $k = \lceil\log_2(n)\rceil$. For each $0 \leq i \leq n-1$ and $0 \leq j \leq k-1$ fix $i_j \in \{0,1\}$ such that $\sum_{j=0}^{k-1} 2^j i_j = i$.

Enumerate the vertices of Γ as $v_0, \ldots, v_{n-1}$. All the points in the image of the wiring will have their lamplighter position and lamp functions supported on the set $\{0, \ldots, 2k-1\}$, so we represent elements of $\mathbb{Z}_2 \wr \mathbb{Z}$ by a binary string of length exactly $2k$ (for the element of $\bigoplus_{\mathbb{Z}} \mathbb{Z}_2$) with one entry marked by a hat ($\hat{}$) to indicate the position of the lamplighter (for the element of $\mathbb{Z}$). Note that this set has diameter at most $6k = 6\lceil\log_2(n)\rceil$.

Now we map each v_i to $\hat{i_0} i_1 \ldots i_{k-1} i_0 i_1 \ldots i_{k-1}$ and for each edge $v_i v_j$ we assign the path P_{ij} which travels from left to right correcting the binary string as it goes, then returns to the leftmost position:

$$\begin{aligned}
\hat{i_0} i_1 \ldots i_{k-1} i_0 i_1 \ldots i_{k-1} &\to \widehat{j_0} i_1 \ldots i_{k-1} i_0 i_1 \ldots i_{k-1} &(4)\\
&\to j_0 \widehat{i_1} \ldots i_{k-1} i_0 i_1 \ldots i_{k-1} &(5)\\
\ldots &\to j_0 j_1 \ldots \widehat{j_{k-1}} i_0 i_1 \ldots i_{k-1} &(6)\\
\ldots &\to j_0 j_1 \ldots j_{k-1} j_0 j_1 \ldots \widehat{j_{k-1}} &(7)\\
&\to j_0 j_1 \ldots j_{k-1} j_0 j_1 \ldots \widehat{j_{k-2}} j_{k-1} &(8)\\
\ldots &\to \widehat{j_0} j_1 \ldots j_{k-1} j_0 j_1 \ldots j_{k-1}. &(9)
\end{aligned}$$

Now suppose an edge e lies on one of the paths P_{ij}. Choose one of the end vertices and denote the binary string associated to this vertex by $a_0 \ldots a_{2k-1}$.

We claim that at least one of the following holds:

$$i = \sum_{l=0}^{k-1} 2^l a_{k+l} \; (\dagger) \qquad j = \sum_{l=0}^{k-1} 2^l a_l \; (\ddagger)$$

In particular, as the graph Γ has maximal degree at most d, this means that there are at most $2d$ paths containing the edge e.

If e appears on P_{ij} during stages (4), (5) or (6), then $a_{k+l} = i_l$ for $0 \leq l \leq k-1$. Thus ($\dagger$) holds. Otherwise, e appears on P_{ij} during stages (7), (8) or (9), then $a_l = j_l$ for $0 \leq l \leq k-1$. Thus ($\ddagger$) holds.

For the volume estimate, each path P_{ij} meets at most $6k+1$ vertices and there are $|E\Gamma| \leq \frac{1}{2}nd$ paths. $\square$

Proof of Theorem 1.17. This follows immediately from Proposition 3.3 and Corollary 1.15, since $\mathbb{Z}_2 \wr \mathbb{Z}$ is quasi-isometric to $\mathrm{DL}(2,2)$ [Woe05]. $\square$

4 From fine wirings to coarse wirings and back

In this section we prove Proposition 1.9 and Theorem 1.10, which describe circumstances in which one can translate between thick embeddings of a graph into a metric space and coarse wirings of that graph into a graph quasi-isometric to the metric space.

4.1 Fine to coarse

Proposition 4.1. *Let M be a Riemannian manifold and let Y be a graph quasi-isometric to M. For any $d \in \mathbb{N}$ and $T > 0$, there exists a constant k depending only on d, M, T and Y such that if Γ is a finite graph with maximal degree d and there is a T-thick embedding $\phi : \Gamma \to M$ with diameter D and volume V then there is a coarse k-wiring of Γ into Y with diameter at most kD and volume at most kV.*

Proof. Let $f \colon M \to Y$ be a (possibly discontinuous) quasi-isometry. Let $\lambda \geq 1$ be such that

1. $\frac{1}{\lambda} d_Y(f(x_1), f(x_2)) - \lambda \leq d_M(x_1, x_2) \leq \lambda d_Y(f(x_1), f(x_2)) + \lambda$ for x_1 and x_2 in M, and

2. for any $y \in Y$, there exists $x \in M$ with $d_Y(y, f(x)) \leq \lambda$.

We show that $f\phi$ can be perturbed to obtain a coarse wiring ψ.

For $v \in V\Gamma$, let $\psi(v)$ be any vertex of Y within distance $\frac{1}{2}$ of $f\phi(v)$. If w is another vertex of Γ with $\psi(w) = \psi(v)$ then $d_M(\phi(v), \phi(w)) \leq 3\lambda$. But, for any distinct pair of vertices v, w, $d_M(\phi(v), \phi(w)) \geq T$, so it follows that at most $C_{3\lambda+T/2}/c_{T/2}$ vertices of Γ map under ψ to $\psi(v)$.

We now describe a collection of paths $P_{vv'}$ in Y as v and v' range over pairs of adjacent vertices in Γ. The restriction of ϕ to the edge vv' is a continuous path in M; choose a sequence $\phi(v) = w'_0, \dots, w'_n = \phi(v')$ of points on this path with n minimal such that $d(w'_i, w'_{i+1}) \leq 2T$ for each i. Denote this minimal n by $n_{vv'}$. Choose $w_0 = \psi(v)$, $w_n = \psi(v')$ and for each $1 \leq i \leq n-1$ let w_i is a vertex of Y within distance $\frac{1}{2}$ of $f(w'_i)$. For each i we have

$$\begin{aligned} d_Y(w_i, w_{i+1}) &\leq 1 + d_Y(f(w'_i), f(w'_{i+1})) \\ &\leq 1 + \lambda d_M(w'_i, w'_{i+1}) + \lambda^2 \\ &\leq 1 + 2\lambda T + \lambda^2 := L, \end{aligned}$$

so can be joined by an edge path comprising at most L edges. We define the path $P_{vv'}$ to be the concatenation of these $n_{vv'}$ paths of length at most L.

We extend ψ to a continuous map which sends each edge vv' to the path $P_{vv'}$. We claim that ψ is a coarse wiring with the appropriate bounds on diameter and volume.

Firstly, we bound the diameter. Note that every point in $\mathrm{im}(\psi)$ is within distance $(L+1)/2$ of some $f(w')$ with $w' \in \mathrm{im}(\phi)$. Let $x, y \in \mathrm{im}(\psi)$ and let $v, w \in \Gamma$ satisfy $d_Y(x, f\phi(v)), d_Y(y, f\phi(w)) \leq (L+1)/2$. We have

$$\begin{aligned} d_Y(x,y) &\leq d_Y(x, f\phi(v)) + \lambda\left(d_M(\phi(v), \phi(w)) + \lambda\right) + d_Y(f\phi(w), y) \\ &\leq L + 1 + \lambda.\mathrm{diam}(\psi) + \lambda^2 \\ &\leq C(T,\lambda).\mathrm{diam}(\psi). \end{aligned}$$

The final inequality fails if Γ is a single vertex, but the proposition obviously holds in this situation. Otherwise $\mathrm{diam}(\phi) \geq T$ and the inequality holds for a suitable C.

Next we bound the volume of the wiring. The bound follows from the two inequalities

$$\mathrm{vol}(\phi) \geq \frac{c_{T/2}}{2d+1}\left(|V\Gamma| + \sum_{vv' \in E\Gamma} n_{vv'}\right) \quad \text{and} \quad \mathrm{vol}(\psi) \leq |V\Gamma| + L \sum_{vv' \in E\Gamma} n_{vv'}.$$

For the second bound, each vertex in $V\Gamma$ contributes at most 1 vertex to $\mathrm{vol}(\psi)$ and each path $P_{vv'}$ contributes at most $Ln_{vv'}$ vertices to $\mathrm{vol}(\psi)$. For the first bound, notice that the (open) balls of radius $T/2$ around the image of each vertex are necessarily disjoint. Similarly, the balls of radius $T/2$ centred at any two points in one of the sequences $\phi(v) = w'_0, \dots, w'_n = \phi(v')$ defined above are necessarily disjoint: if this were not the case for w'_j and $w'_{j'}$, we must have $|j - j'| \geq 2$ since $d(w'_i, w'_{i+1}) \geq T$ for all i, but then we can remove $w'_{j+1}, \dots, w'_{j'-1}$ from the above sequence, contradicting the minimality assumption. Moreover, if two balls of radius $T/2$ centred at points on sequences corresponding to different edges have non-trivial intersection, then these edges must have a common vertex since ϕ is a T-thick embedding. Thus, the T-neighbourhood of the image of ϕ contains a family of $\big(|V\Gamma| + \sum_{vv'\in E\Gamma} n_{vv'}\big)$ balls of radius $T/2$, such that no point is contained in more than $2d+1$ of these balls (d for each end vertex, and an extra 1 if the point is within distance $T/2$ of the image of a vertex). As a result

$$\mathrm{vol}(\phi) \geq \frac{c_{T/2}}{2d+1}\left(|V\Gamma| + \sum_{vv'\in E\Gamma} n_{vv'}\right).$$

It remains to prove that we have defined a coarse k-wiring. It is sufficient to show that there is a constant k depending only on λ and the growth rates c and C of volumes in M such that any edge of Y is contained in $P_{vv'}$ for at most k edges $vv' \in E\Gamma$.

Let uu' be an edge of Y contained in at least one path in the collection P. Let A be the subset of $E\Gamma$ comprising edges e such that P_e contains uu'. As noted during the proof of the diameter bound every point in P_e is contained in the $(L+1)/2$-neighbourhood of $f(\phi(e))$ so there is a point $x_e \in \phi(e)$ such that $d_Y(u, f(x_e)) \leq (L+1)/2$, and so for any other edge $e' \in A$,

$$d_M(x_e, x_{e'}) \leq \lambda\left(d_Y(f(x_e), u) + d_Y(u, f(x_{e'}))\right) + \lambda \leq \lambda(L+2).$$

For any edge $e' \in A$, $x_{e'}$ is within distance T of at most $2d$ of the points $x_{e''}$ for $e'' \in A$. It follows that the size of A is at most $2dc_{T/2}^{-1}C_{\lambda(L+2)+T/2}$. □

4.2 Coarse to fine

The return direction is more sensitive and we are not able to obtain 1-thick embeddings in all cases. When the target space is Euclidean this is easily resolved by rescaling, but in other spaces changing thickness potentially has a more drastic effect on the volume.

Theorem 4.2. *Let M be a compact Riemannian manifold of dimension $n \geq 3$, let Y be a graph quasi-isometric to the universal cover $\widetilde{M}$ of M and let $k, d \in \mathbb{N}$. There exist constants C and $\varepsilon > 0$ such that the following holds:*

If $\mathrm{wir}^k_Y(\Gamma) = V < \infty$ then there is a ε-wiring of Γ into $\widetilde{M}$ with volume at most CV.

Using Lemma 3.1 and the fact that quasi-isometries of bounded degree graphs are regular, it suffices to prove Proposition 4.2 for a specific bounded-degree graph quasi-isometric to $\widetilde{M}$.

We require a standard Švarc-Milnor lemma.

Lemma 4.3. *Let $x \in M$. Then, for sufficiently large L, the graph $\mathcal{G}_x^L$ with vertex set equal to the preimage of x in $\overline{M}$, with vertices connected by an edge if and only if they are separated by a distance of at most L in $\widetilde{M}$, is quasi-isometric to $\widetilde{M}$.*

Now we assume that $Y = \mathcal{G}_x^L$ for a suitably chosen L. The next step is to "thicken" Y to a graph Y' to obtain injective wirings.

Definition 4.4. A wiring f of a finite graph Γ into a graph Y' is called an **injective wiring** if $f : V\Gamma \to VY'$ is injective, each $f(v)$ lies in only those $f(e)$ with $e \in E\Gamma$ when v is an end vertex of e, and each $w \in VY' \setminus V\Gamma$ is contained in the interior of at most one $f(e)$.

Definition 4.5. Given a graph Y and $T \in \mathbb{N}$ we define the **T-thickening** of T to be the graph $K_T(Y)$ with vertex set $VY \times \{1, \ldots, T\}$ and edges $\{(v, i), (w, j)\}$ whenever either $v = w$ and $1 \leq i < j \leq T$, or $\{v, w\} \in EY$ and $1 \leq i \leq j \leq T$.

Lemma 4.6. *For all $d, k \in \mathbb{N}$ there exists some T with the following property. If there is a coarse k-wiring ψ of a finite graph Γ with maximal degree d into Y then there is an injective wiring ψ' of Γ into $K_T(Y)$. Moreover,* $\operatorname{diam}(\psi') = \operatorname{diam}(\psi) + 2$ *and* $\operatorname{vol}(\psi') \leq T\operatorname{vol}(\psi)$.

Proof. For each $v^i \in V\Gamma'$ enumerate $\psi^{-1}(v^i) = \left\{v_1^i, \ldots, v_{l_i}^i\right\}$ for some $l_i \leq k$. Define $\psi'(v_j^i) = (\psi(v_j^i), j)$. We now define new paths $P'_{xx'}$.

If $\psi(x) = \psi(x')$ then we define $\psi'(xx')$ to be the edge $\psi'(x)\psi'(x')$. We then define each $\psi'(xx')$ where $\psi(x) \neq \psi(x')$ in turn.

Note that $\psi(xx')$ is a path $\psi(x) = x^0, \ldots, x^m = \psi(x')$ in Γ. We have $\psi'(x) = (x^0, j_0)$ for some j_0. For the second vertex in the path we choose the minimal j_1 such that (x^1, j_1) is not in $\psi'(V\Gamma)$ and has not previously appeared in any $\psi(vv')$. Repeat this process to construct the remaining vertices in the path $(x^2, j_2), \ldots, (x^m, j_m) = \psi'(x^m)$, and extend ψ' so that $\psi'(xx')$ is this path. Since ψ is a coarse k-wiring, each vertex lies in the interior of at most $\frac{1}{2}kd$ of the $\psi(e)$, so we can always complete this process, provided $T \geq k + \frac{1}{2}kd$. It is immediate from the construction that ψ' is an injective wiring.

Note that if $(x, j), (y, j')$ are contained in $\operatorname{im}(\psi')$ then $(x, 1), (y, 1) \in \operatorname{im}(\psi')$ and there is a path of length at most $\operatorname{diam}(\psi)$ connecting $(x, 1)$ to $(y, 1)$ in $K_T(Y)$. Hence $\operatorname{diam}(\psi') \leq \operatorname{diam}(\psi) + 2$. If $(x, j) \in \operatorname{im}(\psi')$ for some j then $x \in \operatorname{im}(\psi)$. Therefore $\operatorname{vol}(\psi') \leq T\operatorname{vol}(\psi)$. $\square$

Lemma 4.7. *Suppose that M is a compact manifold of dimension $n \geq 3$ with fundamental group G and let $\widetilde{M}$ be the universal cover of M. Let $x \in M$ and denote by Gx the orbit of x in $\overline{M}$ under G. Then for any L, T there is an embedding of $Y' = K_T(\mathcal{G}_X^L(Gx))$ into $\widetilde{M}$ that is equivariant with respect to the obvious action of G on Y'.*

This embedding is ε-thick for some $\varepsilon > 0$, and there is a uniform upper bound on the length of the paths obtained as the images of edges of Y' under the embedding.

Proof. Let B be a ball in M centred at x which is homeomorphic to $\mathbb{R}^n$. Fix a topological embedding f of the complete graph on T vertices into B. For each pair $y, z \in VK_T$, and each homotopy class $[\ell]$ in $\pi_1(X, x)$ which has a representative of length at most L, choose an arc $\gamma_{y,z,[\ell]}$ connecting $f(y)$ to $f(z)$ such that the loop obtained from concatenating $f(yz)$ and $\gamma_{y,z,[\ell]}$ is in $[\ell]$ and such that $\gamma_{y,z,[\ell]}$ intersects the union of $f(K_T)$ and all arcs previously added only at the points $f(y)$ and $f(z)$. This is always possible using a general position argument.

Lifting this embedding to $\overline{M}$, we obtain a G-equivariant embedding of $K_T(\mathcal{G}^L_X(Gx))$ into $\widetilde{M}$. As only finitely many arcs were added during the above procedure, the embedding is ε-thick, where $\varepsilon = \min\{d_M(X, Y)\}$ as X, Y range over the following:

- $X = \{f(v)\}$, $Y = \{f(w)\}$ for distinct vertices $v, w \in VK_T$; or
- $X = \{f(v)\}$ and Y is either $f(yz)$ or $\gamma_{y,z,[\ell]}$ with v, y, z all distinct; or
- X is either $f(vw)$ or $\gamma_{v,w,[\ell]}$ and Y is either $f(yz)$ or $\gamma_{y,z,[\ell']}$ with v, w, y, z all distinct.

Similarly, since there are only finitely many G-orbits of edges, there is a uniform upper bound on the lengths of images □

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. Let M be a compact manifold of dimension $n \geq 3$, let $\widetilde{M}$ be the universal cover of M and let Y be any graph quasi-isometric to $\widetilde{M}$. Fix $d, k \in \mathbb{N}$ and assume that there is a coarse k-wiring of Γ into Y with diameter D and volume V. We may assume $D \geq 1$ as the $D = 0$ case is obvious.

By Lemma 4.3 there is some L such that $\mathcal{G}^L_x$ is quasi-isometric to $\widetilde{M}$, so by Corollary 1.15(1), there exists some $l = l(k, d)$ so that there is a coarse l-wiring of Γ into $\mathcal{G}^L_x$ with diameter $\leq lD + l \leq 2lD$ and volume $\leq lV + l \leq 2lV$.

Now we apply Lemma 4.6: for some $T = T(l, d)$ there is an injective wiring ψ of Γ into $T_T(\mathcal{G}^L_x)$ with diameter $\leq 2lD + 2 \leq 4lD$ and volume $\leq 2TlV$. Composing this injective wiring with the ε-thick topological embedding ϕ of $T_T(\mathcal{G}^L_x)$ into $\widetilde{M}$ gives an ε-thick embedding $f : \Gamma \to \widetilde{M}$. The diameter of the image of f is bounded from above by a constant multiple of $\operatorname{diam}(\psi)$. For the volume, note that the sum of the lengths of all paths in the wiring is at most a constant times $\operatorname{vol}(\psi)$, and the volume of the thick embedding is at most this sum of lengths multiplied by the maximal volume of a ball of radius ε in M. Hence the volume of this thick embedding is at most a constant multiple of V. $\square$

5 Lower bounds on coarse wiring

5.1 Background on the separation profile

Recall that $f \lesssim g$ if there is a constant C such that

$$f(x) \leq Cg(Cx) + C \quad \text{for all } x \in X.$$

We write $f \simeq g$ if $f \lesssim g$ and $g \lesssim f$.

Definition 5.1. [BST12] Let Γ be a finite graph. We denote by $\mathrm{cut}(\Gamma)$ the minimal cardinality of a set $S \subset V\Gamma$ such that no component of $\Gamma - S$ contains more than $\frac{1}{2}|V\Gamma|$ vertices. A set S satisfying this property is called a **cut set** of Γ.

Let X be a (possibly infinite) graph. We define the **separation profile** of X to be the function $\mathrm{sep}_X : [0, \infty) \to [0, \infty)$ given by

$$\mathrm{sep}_X(n) = \max\{\mathrm{cut}(\Gamma) \mid \Gamma \leq X,\ |V\Gamma| \leq n\}.$$

For convenience, we will define $\mathrm{sep}_X(r) = 0$ whenever $r < 1$.

Definition 5.2. The Cheeger constant of a finite graph Γ is

$$h(\Gamma) = \min\{\frac{|\partial A|}{|A|} \mid A \subseteq V\Gamma,\ |A| \leq \frac{1}{2}|V\Gamma|\}$$

where $\partial A = \{v \in V\Gamma \mid d_\Gamma(v, A) = 1\}$.

The main result of [Hum17] states that for any bounded degree graph X

$$\mathrm{sep}_X(n) \simeq \max\left\{|\Gamma| h(\Gamma) \ \mid \Gamma \leq X,\ |\Gamma| \leq n\right\}.$$

The only part of this result we need here is the following.

Proposition 5.3. [Hum17, Proposition 2.1] *Let Γ be a finite graph. There is a subgraph $\Gamma' \leq \Gamma$ such that $|V\Gamma'| \geq \frac{1}{2}|V\Gamma|$ and $h(\Gamma') \geq \frac{\mathrm{cut}^{1/2}(\Gamma)}{4|\Gamma|}$.*

5.2 Lower bounds on wiring profiles

Proposition 5.4. *Let X, Y be graphs of bounded degree where $\mathrm{wir}^k_{X\to Y}(n) < \infty$. Then, for all $n \geq 3$,*

$$\sum_{s\geq 0} \mathrm{sep}_Y(2^{-s}\mathrm{wir}^k_{X\to Y}(n)) \geq \frac{\mathrm{sep}_X(n)}{40k^2}.$$

Proof. Let $n \geq 3$ and choose $\Gamma \leq X$ which satisfies $|\Gamma| \leq n$ and $\mathrm{cut}(\Gamma) = \mathrm{sep}_X(n) = l \leq 2|\Gamma|/3$. Let $\Gamma'' \leq \Gamma$ satisfy $|\Gamma''| \geq \frac{1}{2}|\Gamma|$ and $h(\Gamma'') \geq \frac{l}{2|\Gamma'|} \geq \frac{l}{4|\Gamma|}$ (using [Hum17]). Let ψ be a coarse k-wiring of Γ'' into Y such that $\mathrm{vol}(\psi) = m = wir^k_Y(\Gamma'')$. Set $\Gamma' = \mathrm{im}(\psi)$.

Let C_1 be a cut set of Γ' of minimal size. Our goal is to find an upper bound on $|C_1|$ in terms of $l = \mathrm{sep}_X(n)$. If $|C_1| \geq l/40k^2$, then

$$\mathrm{sep}_Y(wir^k_{X\to Y}(n)) \geq |C_1| \geq \frac{\mathrm{sep}_X(n)}{40k^2}.$$

Now suppose $|C_1| < l/40k^2$. It follows that $C_1 \cap \psi(V\Gamma'')$ contains less than 1/10th of the elements of $\psi(V\Gamma'')$.

Claim: There is a connected component of $\Gamma' \setminus C_1$ containing more than $\frac{4}{5}$ths of the elements of $\psi(V\Gamma'')$.

Proof of Claim. Suppose not and denote the connected components of $\Gamma' \setminus C_1$ by $A_1, \ldots, A_m$ where $A_i \cap \psi(V\Gamma'') \geq A_j \cap \psi(V\Gamma'')$ whenever $i \leq j$. Choose t minimal so that

$$\left| \psi(V\Gamma'') \cap \left(\bigcup_{i=1}^{t} A_i \right) \right| \geq \frac{1}{10} \left|\psi(V\Gamma'')\right|,$$

set $A = \bigcup_{i=1}^t A_i$ and $B = \bigcup_{i=t+1}^m A_i$. Then the subsets A, B of $\psi(V\Gamma'')$ are unions of connected components of $\psi(V\Gamma'') \setminus C_1$ and contain at least 1/10th of the vertices in $\psi(V\Gamma'')$. Using the bound on Cheeger constant we see that $\psi^{-1}A$ (which has at least $|V\Gamma''|/10k$ elements) has at least $l/40k$ neighbours in Γ'', none of which are mapped to A by ψ. Therefore, at least $l/40k$ of the paths P_{ab} must intersect C_1. It follows that $|C_1| \geq l/40k^2$. ☐

Now we repeat the argument. Let D_1 be the connected component of $\Gamma' \setminus C_1$ which contains more than $\frac{4}{5}$ths of the elements of $\psi(V\Gamma'')$. Note that $|D_1| \leq \frac{1}{2}|V\Gamma'|$.

Let C_2 be a cut set for D_1 and run the same analysis. Either $|C_1 \cup C_2| \geq l/40k^2$ or there is a component D_2 of $\Gamma'' \setminus (C_1 \cup C_2)$ which contains at least $\frac{4}{5}$ths of elements of $\psi(V\Gamma'')$. It is impossible for D_s to contain at least $\frac{4}{5}$ths of the elements of $\psi(V\Gamma'')$ as soon as

$$2^{-s}m \leq \frac{4}{5k}\left|\Gamma''\right|$$

and we have removed at most $\sum_{i=0}^{s-1}|C_i| \leq \sum_{i=0}^{s-1}\mathrm{sep}_Y(2^{-s}|\Gamma'|)$ vertices. Hence

$$\sum_{i=0}^{s-1} \mathrm{sep}_Y(2^{-s}\left|\Gamma'\right|) \geq l/40k^2 = \frac{\mathrm{sep}_X(n)}{40k^2} \tag{10}$$

holds for $s = 1 + \lceil \log_2(km/|\Gamma''|) \rceil$. ☐

In practice, the separation profiles of graphs we are interested in here are of the form $n^r \log(n)^s$ with $r \geq 0$ and $s \in \mathbb{R}$. Restricted to these functions, Proposition 5.4 says the following.

Corollary 5.5. *Suppose* $\mathrm{sep}_X \gtrsim n^r \log(n)^s$ *and* $\mathrm{sep}_Y \simeq n^p \log(n)^q$*. then*

$$wir^k_{X\to Y}(n) \gtrsim \begin{cases} n^{r/p}\log(n)^{(s-q)/p} & \text{if} \quad p>0, \\ \exp(n^{r/(q+1)}\log(n)^{s/(q+1)}) & \text{if} \quad p=0. \end{cases}$$

Proof. Applying our hypotheses to Proposition 5.4, we have

$$n^r \log(n)^s \lesssim \sum_{i\geq 0}\left(2^{-i}\mathrm{wir}^k_{X\to Y}(n)\right)^p \log\left(2^{-i}\mathrm{wir}^k_{X\to Y}(n)\right)^q.$$

If $p>0$, then the sequence $\left(2^{-i}\mathrm{wir}^k_{X\to Y}(n)\right)^p$ decays exponentially as a function of i, so

$$\begin{aligned} n^r \log(n)^s &\lesssim \log\left(\mathrm{wir}^k_{X\to Y}(n)\right)^q \sum_{i\geq 0}\left(2^{-i}\mathrm{wir}^k_{X\to Y}(n)\right)^p \\ &\lesssim \mathrm{wir}^k_{X\to Y}(n)^p \log\left(\mathrm{wir}^k_{X\to Y}(n)\right)^q. \end{aligned}$$

Hence, there is some constant $C>0$ such that

$$w := \mathrm{wir}^k_{X\to Y}(n)^p \log\left(\mathrm{wir}^k_{X\to Y}(n)\right)^q \geq C^{-1}(C^{-1}n)^r\log(C^{-1}n)^s - C. \quad (11)$$

Now suppose $\mathrm{wir}^k_{X\to Y}(n) \leq dn^{r/p}\log(n)^{(s-q)/p}$. Then

$$\begin{aligned} w &\leq d^p n^r \log(n)^{s-q}\left(\log(d) + \frac{r}{p}\log(n) + \frac{s-q}{p}\log\log(n)\right)^q \\ &\leq \frac{(2r)^s d^p}{p^s} n^r \log(n)^{s-q}\log(n)^q = \frac{(2r)^s d^p}{p^s} n^r \log(n)^s \end{aligned}$$

for sufficiently large n. This contradicts (11) if d is small enough and n is large enough. Hence,

$$\mathrm{wir}^k_{X\to Y}(n) \gtrsim n^{r/p}\log(n)^{(s-q)/p}.$$

If $p=0$, then there is some $C>0$ such that

$$\begin{aligned} \log_2\left(\mathrm{wir}^k_{X\to Y}(n)\right)^{q+1} &\geq \sum_{i=0}^{\log_2(\mathrm{wir}^k_{X\to Y}(n))} \log_2\left(2^{-i}\mathrm{wir}^k_{X\to Y}(n)\right)^q \\ &\geq C^{-1-r}n^r\log_2(C^{-1}n)^s - C. \end{aligned}$$

Hence $\mathrm{wir}^k_{X\to Y}(n) \gtrsim \exp(n^{r/(q+1)}\log_2(n)^{s/(q+1)})$. $\square$

6 Completing Theorems 1.4, 1.5 and 1.6

In this section we give complete proofs of the main results of the paper.

Proof of Theorem 1.4. Let Γ be an N-vertex graph with maximal degree d and Cheeger constant $\geq \delta > 0$. Let g be an ε-thick embedding of Γ into $\mathbb{H}^3$ with volume V. Let Y be a graph which is quasi-isometric to $\mathbb{H}^q \times \mathbb{R}^r$. By Proposition 1.9 there is a coarse k-wiring from Γ to Y with volume at most kV.

We have $\mathrm{sep}_Y(n) \simeq n^{1-1/(r+1)} \log(n)^{1/(r+1)}$ if $q = 2$ and $\mathrm{sep}_Y(n) \simeq n^{1-1/(q+r-1)}$ if $q \geq 3$ [HMT20b, Theorem 1.7(ii)]. By [Hum17, Proposition 2.1], $\mathrm{sep}_\Gamma(N) \geq \frac{\delta}{2} N$. Using Corollary 5.5 we see that for $q \geq 3$

$$kV \geq \mathrm{wir}^k_{\Gamma \to Y}(N) \geq c^{-1} N^{1/(1-1/(q+r-1))} - c = c^{-1} N^{1+1/(q+r-2)} - c$$

while for $q = 2$

$$\begin{aligned} kV \geq \mathrm{wir}^k_{\Gamma \to Y}(N) &\geq c^{-1} N^{1/(1-1/(r+1))} \log(N)^{-(1/(r+1))/(1-1/(r+1))} - c \\ &= c^{-1} N^{1+1/r} \log(N)^{-1/r} - c. \end{aligned}$$

□

Proof of Theorem 1.5. This proof follows exactly the strategy of the result above. Let Γ be an N-vertex graph with maximal degree d. By Proposition 3.3 there is a $2d$-coarse wiring of Γ into a Cayley graph of $\mathbb{Z}_2 \wr \mathbb{Z}$ with volume $\leq 4dN \lceil \log_2(N) \rceil$. This Cayley graph is quasi-isometric to $\mathrm{DL}(2,2)$ [Woe05], and $\mathrm{DL}(2,2)$ quasi-isometrically embeds into any graph X quasi-isometric to a symmetric space whose non-compact factor has rank ≥ 2 [HMT20b, Proposition 2.8 and Theorem 3.1]. Thus, for some l we have

$$\mathrm{wir}^l_{\Gamma \to X} \leq C' N \ln(1+N).$$

□

Proof of Theorem 1.6. Let Γ be an N-vertex graph with maximal degree d and Cheeger constant $\geq \delta > 0$. Let g be an ε-thick embedding of Γ into a symmetric space M whose non-compact factor has rank ≥ 2 with volume V. Let Y be a graph which is quasi-isometric to M. By Proposition 1.9 there is a coarse k-wiring with volume at most kV.

We have $\mathrm{sep}_Y(n) \simeq n/\log(n)$ [HMT20b, Theorem 1.5, Proposition 2.8 and Theorem 3.1]. Using Corollary 5.5 we see that

$$kV \geq \mathrm{wir}^k_{\Gamma \to Y}(N) \geq c^{-1} N/\log(N) - c.$$

□

6.1 More examples of coarse wirings

Proposition 6.1. *Every N-vertex graph with maximal degree d admits a coarse $2d$-wiring into the standard 2-dimensional integer lattice $\mathbb{Z}^2$ with volume at most N^2.*

Let X be the disjoint union of all finite graphs with maximal degree 3. For any k there is some C such that $\mathrm{wir}^k_{X\to\mathbb{Z}^2}(n) \geq C^{-1}n^2 - C$.

Proof. The second claim follows immediately from Corollary 5.5 and the fact that $\mathrm{sep}_X(n) \simeq n$ [Hum17] and $\mathrm{sep}_{\mathbb{Z}^2}(n) \simeq n^{1/2}$ [BST12].

Let Γ be an n-vertex graph with maximal degree d. Enumerate the vertices of Γ by $v_0, \ldots, v_{n-1}$. We construct a d-wiring of Γ into $\{0, \ldots, n-1\}^2$ as follows:

Map the vertex v_k to the point (k,k). For each edge v_iv_j (with $i < j$) we define a path P_{ij} which travels horizontally from (i,i) to (j,i), then vertically from (j,i) to (j,j).

To see that this is a 1-wiring, note that if a horizontal edge $(a,b)(a+1,b)$ is in P_{ij} then $b = i$. Similarly, if a vertical edge $(a,b)(a,b+1)$ appears in P_{ij}, then $a = j$. Hence, any two paths containing a common edge have a common end vertex. Since, by assumption, there are at most d edges with a given end vertex, we have defined a coarse $2d$-wiring. The volume estimate is obvious. □

Proposition 6.2. *Let Y be a graph which is quasi-isometric to $\mathbb{H}^2$ and let $d \in \mathbb{N}$. There are constants $k = k(X,d)$ and $C = C(X,d)$ such that any N-vertex graph Γ with maximal degree d admits a coarse k-wiring into X with volume $\leq CN^2\exp(N)$.*

Let X be the disjoint union of all finite graphs with maximal degree 3. For any k there is some C such that $\mathrm{wir}^k_{X\to Y}(n) \geq C^{-1}\exp(C^{-1}n^{1/2})) - C$.

Proof. The second claim follows immediately from Corollary 5.5 and the fact that $\mathrm{sep}_X(n) \simeq n$ [Hum17] and $\mathrm{sep}_{\mathbb{Z}^2}(n) \simeq \log(n)$ [BST12].

We will construct 1-thick embeddings $K_N \to \mathbb{H}^2 \times [0,1]$ with volume $\leq C'N^2\exp(N)$. Since X is quasi-isometric to $\mathbb{H}^2 \times [0,1]$, the result will then follow from Proposition 1.9.

Firstly, recall the definition of the metric in the upper halfspace model of $\mathbb{H}^2$:

$$d_{\mathbb{H}^2}((x_1,y_1),(x_2,y_2)) = \cosh^{-1}\left(1 + \frac{(x_2-x_1)^2 + (y_2-y_1)^2}{2y_1y_2}\right).$$

We equip $\mathbb{H}^2 \times [0,1]$ with the metric

$$d((w,x;a),(y,z;b)) = \max\{d_{\mathbb{H}^2}((w,x),(y,z)), |a-b|\}.$$

Claim: If $d((w,x;a),(y,z;b)) < 1$ and $x, z \leq h_0 := (2(\cosh(1)-1)^{-1/2}$, then $|a-b| < 1$, $|w-y| < 1$ and $|\ln(x/h_0) - \ln(z/h_0)| < 1$.

Proof of Claim. It is immediate from the definition that $|a-b|<1$. Since $x, z \leq h_0$,

$$\begin{aligned} 1 &> d((w,x;a),(y,z;b)) \\ &\geq d_{\mathbb{H}^2}((w,x),(y,z)) \\ &\geq \cosh^{-1}\left(1+\frac{(w-y)^2}{2h_0^2}\right) \\ &\geq \cosh^{-1}(1+(\cosh(1)-1)(w-y)^2). \end{aligned}$$

Hence $(w-y)^2 < 1$, so $|w-y|<1$. Finally, write $x = h_0 e^p$ and $z = h_0 e^q$ with $p, q \in \mathbb{R}$. We have

$$\begin{aligned} 1 &> d((w,x;a),(y,z;b)) \\ &\geq d_{\mathbb{H}^2}((w,x),(y,z)) \\ &\geq \cosh^{-1}\left(1+\frac{h_0^2(e^p-e^q)^2}{2h_0^2 e^{p+q}}\right) \\ &= \cosh^{-1}\left(\frac{1}{2}(e^{p-q}+e^{q-p})\right) = |p-q|. \end{aligned}$$

Hence $|\ln(x/h_0) - \ln(z/h_0)| = |p-q| < 1$. $\square$

Enumerate the vertices of K_N by $v_0, \ldots, v_{N-1}$. We map v_i to $(i, h_0 e^{-i}; 0)$ where $h_0 = (2(\cosh(1)-1))^{-1/2}$. For $i<j$, we connect $(i, h_0e^{-i}; 0)$ to $(j, h_0 e^{-j}; 0)$ using the path P_{ij} defined as follows:

$$\begin{aligned} (i, h_0 e^{-i}; 0) &\to (j, h_0 e^{-i}; 0) && (12) \\ &\to (j, h_0 e^{-i}; 1) && (13) \\ &\to (j, h_0 e^{-j}; 1) && (14) \\ &\to (j, h_0 e^{-j}; 0) && (15) \end{aligned}$$

where the first segment lies in the horocircle $y = h_0 e^i$ and the others are geodesics.

We first prove that this embedding is 1-thick. Let $(w,x;a) \in P_{ij}$ and $(y,z;b) \in P_{kl}$ with $d((w,x;a),(y,z;b)) < 1$. Set $p = \ln(x/h_0)$ and $q = \ln(z/h_0)$. From the claim we have $\max\{|w-y|, |p-q|, |a-b|\} < 1$.

If $a=1$, then $b>0$, so $w=j$ and $y=l$. Since w, y are both integers they must be equal. Thus $j=l$ and the two paths come from edges which share a vertex.

If $a \in (0,1)$ then $w=j$ and $p \in \{-i,-j\}$. Note that one of the four equalities $y=k$, $y=l$, $q=-k$, $q=-l$ holds at every point on P_{kl}. If it one of the first two, then $\min\{|j-k|, |j-l|\} < 1$ and $j \in \{k,l\}$, or if it is one of the last two, then one of $-i, -j$ is equal to one of $-k, -l$. In any case the two paths share an end vertex.

If $a = 0$ then either $p = -i$ or $w = j$ and $p = -j$. Also $b < 1$ so either $q = -k$ or $y = l$ and $q = -l$. If $p = -i$, then either $q = -k$ in which case $|-i-(-k| < 1$ by the claim, thus $i = k$; or $q = -l$ in which case $i = l$ by the same reasoning. Next, suppose $w = j$ and $p = -j$. Since $q \in \{-k, -l\}$ we have $j = k$ or $j = l$. If $p = -i$, $y = l$ and $q = -l$, then $i = l$ following the same reasoning.

Every point in the image of the embedding is of the form $(x, h_0 e^{-y}; z)$ where $|x|, |y| \leq n-1$ and $z \in [0,1]$. Set $p = \left(\frac{n-1}{2}, h_0, \frac{1}{2}\right)$. We have

$$\begin{aligned}
d\left((x, h_0 e^{-y}; z), p\right) &\leq \cosh^{-1}\left(1 + \frac{\frac{n-1}{2}^2 + h_0^2(1-e^{n-1})^2}{2h_0^2 e^{-(n-1)}}\right) + \frac{1}{2} \\
&\leq \cosh^{-1}\left(1 + \frac{\frac{n^2}{4} + h_0^2)}{2h_0^2 e^{-(n-1)}}\right) + \frac{1}{2} \\
&\leq \cosh^{-1}\left(1 + (\frac{n^2}{8} + 1)e^{n-1}\right) + \frac{1}{2} \\
&\leq \cosh^{-1}\left(\frac{\frac{17n^2}{8} e^{n-1}}{2}\right) + \frac{1}{2} \\
&\leq \cosh^{-1}\left(\cosh(n - 1 + 2\ln(n) + \ln(17) - \ln(8))\right) + \frac{1}{2} \\
&= n - 1 + 2\ln(n) + \ln(17) - \ln(8) + \frac{1}{2} \leq n + 2\ln(n)
\end{aligned}$$

Thus, the volume of the wiring is at most $4\pi \sinh^2((n + 2\ln(n) + 1)/2)$: the volume of the ball of radius $n + 2\ln(n) + 1$ in $\mathbb{H}^2$. We have

$$\begin{aligned}
4\pi \sinh^2((n + 2\ln(n) + 1)/2) &\leq 4\pi \left(\frac{\exp((n + 2\ln(n) + 1)/2)}{2}\right)^2 \\
&\leq \pi \exp(n + 2\ln(n) + 1) \\
&= e\pi n^2 e^n \simeq e^n
\end{aligned}$$

as required. $\square$

7 Questions

We first recall a conjecture from [GG12].

Conjecture 7.1. *For each $k \geq 3, d \in \mathbb{N}$ there is some constant $C = C(k,d)$ such that every N-vertex graph Γ with maximal degree d admits a 1-thick topological embedding into $\mathbb{R}^k$ with diameter $\leq CN^{1/(k-1)}$ and volume $\leq CN^{1+1/(k-1)}$.*

These bounds on diameter and volume are optimal. A positive resolution to this question would also give optimal volume bounds for 1-thick embeddings into all spaces $\mathbb{H}^k \times \mathbb{R}^l$ where $k \geq 3$ and $k + l \geq 4$.

Conjecture 7.2. *For each $k \geq 3$, $l \geq 4-k$, $d \in \mathbb{N}$ there is some constant $C = C(k,l,d)$ such that every N-vertex graph Γ with maximal degree d admits a 1-thick topological embedding into $\mathbb{H}^k \times \mathbb{R}^l$ with volume $\leq CN^{1+1/(k+l-2)}$.*

To improve our bounds on thick embeddings of graphs into other symmetric spaces whose non-compact factor has rank one requires constructions of thick embeddings into nilpotent Lie groups.

Question 7.3. Let P be a connected nilpotent Lie group with polynomial growth of degree $p \geq 3$ and let $d \in \mathbb{N}$. Do there exist constants $C, \varepsilon > 0$ which depend on p, d such that for any N-vertex graph Γ with maximal degree d there is a ε-thick embedding of Γ into P with diameter $\leq CN^{1/(p-1)}$.

Another important example worthy of consideration is a semidirect product of the Heisenberg group with $\mathbb{R}$, $H \rtimes_\psi \mathbb{R}$ where the action is given by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \psi(t) = \begin{pmatrix} 1 & e^t x & z \\ 0 & 1 & e^{-t} y \\ 0 & 0 & 1 \end{pmatrix}.$$

Conjecture 7.4. *For every d there exist constants $C = C(d)$ and $\varepsilon = \varepsilon(d)$ such that every N-vertex graph Γ with maximal degree d admits a ε-thick embedding into $H \rtimes_\psi \mathbb{R}$ with volume $\leq CN\ln(N)$.*

An immediate consequence of this conjecture is that the dichotomy at the heart of [HMT20b] is also detected by wiring profiles. Specifically, let G be a connected unimodular Lie group, let Y be a graph quasi-isometric to G and let X be the disjoint union of all finite graphs with degree ≤ 3. Either G is quasi-isometric to a product of a hyperbolic group and a nilpotent Lie group, in which case there is some $p > 1$ such that for all k sufficiently large $\mathrm{wir}^k_{X\to Y}(N) \gtrsim N^p$; or G contains a quasi-isometrically embedded copy of either $\mathrm{DL}(2,2)$ or $H \rtimes_\psi \mathbb{R}$, in which case for all k sufficiently large $\mathrm{wir}^k_{X\to Y}(N) \simeq N\ln(N)$.

The lower bound from separation profiles is incredibly useful, and our best results are all in situations where we can prove that the lower bound in Theorem 1.16 is optimal. As a result it is natural to record the following:

Question 7.5. For which bounded degree graphs Y does the following hold:
Let X be the disjoint union of all finite graphs with maximal degree ≤ 3. For all k sufficiently large

$$\mathrm{sep}_Y(\mathrm{wir}^k_{X\to Y}(N)) \simeq N.$$

A starting point would be to determine when the following strengthening of Proposition 5.4 holds:

Question 7.6. Let X, Y be graphs of bounded degree where $\mathrm{wir}^k_{X\to Y}(n) < \infty$. Does there exist a constant $C > 0$ such that for all n

$$\mathrm{sep}_Y(\mathrm{wir}^k_{X\to Y}(n)) \geq \frac{\mathrm{sep}_X(n)}{C} - C?$$

We certainly should not expect Theorem 1.16 give the correct lower bound in all cases. A natural example to consider would be a coarse wiring of an infinite binary tree B into $\mathbb{Z}^2$. The depth k binary tree B_k (with vertices considered as binary strings $v = (v_1, v_2, \dots v_m)$ of length $\leq k$) admits a 1-wiring into $\mathbb{Z}^2$ with volume $\lesssim |B_k| \log |B_k|$ as follows

$$\psi(v_1, v_2, \dots v_l) = \left(\sum_{i\in\{l|v_l=0\}} 2^{k-i}, \sum_{j\in\{l|v_l=1\}} 2^{k-i} \right)$$

where the path connecting $\psi(v_1, v_2, \dots v_l)$ to $\psi(v_1, v_2, \dots v_l, 0)$ (respectively $\psi(v_1, v_2, \dots v_l, 1)$) is a horizontal (resp. vertical) line.

Question 7.7. Is it true that for all sufficiently large k, $\mathrm{wir}^k_{B\to\mathbb{Z}^2}(N) \simeq N \ln(N)$? Does the lower bound hold for all coarse wirings $X \to Y$ where X has exponential growth and Y has (at most) polynomial growth?

It is also natural to ask whether other invariants which behave monotonically with respect to coarse embedding (and regular maps) provide lower bounds on wiring profiles.

We finish with another, arguably more fundamental question. How does wir^k depend on k? In the cases we compute, we show that wiring profiles stabilise, meaning there exists some k such that $\mathrm{wir}^k_{X\to Y}(n) \simeq \mathrm{wir}^k_{X\to Y}(n)$ whenever $l \geq k$.

Question 7.8. Do there exist two bounded degree graphs X and Y such that for every k there is some $l \geq k$ with

$$\mathrm{wir}^k_{X\to Y}(N) \not\lesssim \mathrm{wir}^l_{X\to Y}(N)?$$

Since every coarse k-wiring is a coarse l-wiring when $l \geq k$, we always have $\mathrm{wir}^k_{X\to Y}(N) \geq \mathrm{wir}^l_{X\to Y}(N)$.

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