Robustness, Scott continuity, and Computability

Farjudian, Amin; Moggi, Eugenio

DOI:
10.1017/S0960129523000233

Mathematical Structures in Computer Science (2023), **33**, pp. 536–572
doi:10.1017/S0960129523000233

CAMBRIDGE UNIVERSITY PRESS

PAPER

Robustness, Scott continuity, and computability

Amin Farjudian[1] and Eugenio Moggi[2]

[1]School of Computer Science, University of Nottingham Ningbo China, Ningbo, China, 315100 and [2]DIBRIS, Genova Univ., Genova, Italy

Corresponding author: Eugenio Moggi; Email: moggi@unige.it

(Received 26 August 2022; revised 06 April 2023; accepted 21 June 2023; first published online 4 August 2023)

Abstract

Robustness is a property of system analyses, namely monotonic maps from the complete lattice of subsets of a (system's state) space to the two-point lattice. The definition of robustness requires the space to be a metric space. Robust analyses cannot discriminate between a subset of the metric space and its closure; therefore, one can restrict to the complete lattice of closed subsets. When the metric space is compact, the complete lattice of closed subsets ordered by reverse inclusion is ω-continuous, and robust analyses are exactly the Scott-continuous maps. Thus, one can also ask whether a robust analysis is computable (with respect to a countable base). The main result of this paper establishes a relation between robustness and Scott continuity when the metric space is not compact. The key idea is to replace the metric space with a compact Hausdorff space, and relate robustness and Scott continuity by an adjunction between the complete lattice of closed subsets of the metric space and the ω-continuous lattice of closed subsets of the compact Hausdorff space. We demonstrate the applicability of this result with several examples involving Banach spaces.

Keywords: Robustness; continuous lattices; category theory; topology

1. Introduction

The main contribution of this paper is relating *robust analyses* and *Scott-continuous maps* (between ω-continuous lattices). This contribution is relevant to the broader endeavor of developing software tools for system analysis based on mathematical models. Typically, the behavior of a controlled system is given a priori, but for most systems the open system approach is insufficient as the correctness of the controlling system depends on properties of the environment. This requires modeling the environment as well. Software tools (for system analysis) manipulate formal descriptions. The key point of formal descriptions is their mathematical exactness. However, exactness should not be confused with precision. In particular, mathematical descriptions should make explicit *known unknowns* and the amount of imprecision. There are two unavoidable sources of imprecision: errors in measurements (on physical systems) and representations of continuous quantities in software tools.

The key feature of robust analyses is the ability to cope with small amounts of imprecision. On the other hand, analyses can be implemented in software tools only if they are *computable*. A definition of computability for *effectively given domains* has been proposed in Smyth (1977, Definition 3.1), where the domains considered include those of interest for us, namely ω-continuous lattices. The key point of this, and similar proposals, is that computable maps (between effectively given domains) are necessarily Scott continuous.

For the benefit of readers, before outlining the main result of the paper, we review the context in which it is placed and recall related results, while keeping technicalities to a minimum.

Systems. First, one has to decide how to model systems. The simplest systems, i.e., *discrete* systems, can be modeled by a set $\mathbb{S}$ (of states) and a transition map $t:\mathbb{S} \to \mathbb{S}$ describing the *deterministic* state change of the system in one step. The systems we consider are *closed*, i.e., they do not interact with the *environment*, or to put it differently, a model should account also for the environment. In this respect, it is important to model also *known unknowns* (for discrete systems, imprecision is not an issue). The simplest way to do this is by *non-determinism*, namely a state in $\mathbb{S}$ is replaced by a set of states and the transition map is replaced by a transition relation $T \subseteq \mathbb{S} \times \mathbb{S}$. In theoretical computer science, the pair $(\mathbb{S}, T)$ is called a *transition system*.

Analyses. We move from the category of sets and relations to the category of complete lattices and monotonic maps. More precisely, we replace $\mathbb{S}$ with the complete lattice $\mathbb{P}(\mathbb{S})$ of subsets of $\mathbb{S}$, and a relation $T \subseteq \mathbb{S} \times \mathbb{S}'$ with the monotonic map $T_*:\mathbb{P}(\mathbb{S}) \to \mathbb{P}(\mathbb{S}')$ such that:

$$T_*(S) \stackrel{\triangle}{=} \{s' \in \mathbb{S}' \mid \exists s \in S.T(s, s')\}.$$

We take reverse inclusion as the partial order $\leq$ on $\mathbb{P}(\mathbb{S})$, i.e., $S' \leq S \iff S' \supseteq S$. The rationale for this choice is that a smaller set (of states) provides more information on the actual state of the system, because it constitutes a more accurate approximation of the state. The transition relations on $\mathbb{S}$ form a complete lattice $\mathbb{P}(\mathbb{S} \times \mathbb{S})$, where $T' \leq T$ means that T is more deterministic than T'.

Several *analyses* correspond to monotonic maps between complete lattices. For instance, reachability analysis for transition systems on $\mathbb{S}$ corresponds to the map $R:\mathbb{P}(\mathbb{S} \times \mathbb{S}) \times \mathbb{P}(\mathbb{S}) \to \mathbb{P}(\mathbb{S})$ given by $R(T, I) \stackrel{\triangle}{=} T^*(I)$, i.e., the set of states reachable in a finite number of steps from (a state in) I, while safety analysis corresponds to the map $S:\mathbb{P}(\mathbb{S} \times \mathbb{S}) \times \mathbb{P}(\mathbb{S}) \times \mathbb{P}(\mathbb{S}) \to \Sigma$ in which Σ is the two-point lattice $\bot < \top$ and $S(T, I, E) = \top \stackrel{\triangle}{\iff} T^*(I) \cap E = \emptyset$, i.e., no bad state in E is reachable from I.

Approximation. The partial order on a complete lattice X allows (qualitative) comparisons, in particular, we say that x' is an *over-approximation* of x when $x' \leq x$. The category of complete lattices and monotonic maps is also the natural setting for *abstract interpretation* (Cousot, 1996; Cousot and Cousot, 1977). More precisely, given an interpretation $[\![-]\!]$ of a (programming) language in a complete lattice X, one can choose another complete lattice X_a related to X by an *adjunction* $X_a \underset{\gamma}{\overset{\alpha}{\underset{\top}{\leftrightarrows}}} X$, i.e., a pair of monotonic maps α (called abstraction) and γ (called concretization) such that $x' \leq_a \alpha(x) \iff \gamma(x') \leq x$. In general, X_a is *simpler* than X (e.g., X_a can be finite) and allows interpretations $[\![-]\!]_a$ of the language for computing over-approximations of $[\![-]\!]$, i.e., $\gamma([\![p]\!]_a) \leq [\![p]\!]$ for every (program) p in the language.

The adjunction between X_a and X gives a systematic way of defining a $[\![-]\!]_a$ from $[\![-]\!]$. For instance, if the language is given by the BNF $p ::= c \mid f(p)$, then an interpretation $[\![-]\!]$ in X is uniquely determined by $[\![c]\!] \in X$ and a monotonic map $[\![f]\!]:X \to X$, and the *abstract* interpretation (computing over-approximations) in X_a is determined by taking $[\![c]\!]_a \stackrel{\triangle}{=} \alpha([\![c]\!]) \in X_a$ and $[\![f]\!]_a \stackrel{\triangle}{=} \alpha \circ [\![f]\!] \circ \gamma:X_a \to X_a$. In static analysis, the choice of X_a and $[\![-]\!]_a$ is a matter of trade-offs between the cost of computing $[\![p]\!]_a$ and the information provided by $\gamma([\![p]\!]_a)$.

Spaces. To model more complex systems, e.g., continuous or hybrid (Goebel et al., 2009), one may have to replace sets with more complex spaces. For instance, in Moggi et al. (2018), hybrid systems were modeled by triples $(\mathbb{S}, F, G)$, where $\mathbb{S}$ is a Banach space, and F and G are binary relations on $\mathbb{S}$—called flow and jump relation, respectively—which constrain how the system may evolve continuously (in time) and discontinuously (instantaneously). As in transition systems, the relations F and G allow modeling known unknowns. Despite the increased complexity of models, it is still possible and useful to move to the category of complete lattices and monotonic maps and use it for analyses and abstract interpretations of these more complex systems.

Imprecision. In defining reachability for hybrid systems, we realized the need to cope with *imprecision*, see also Fränzle (1999). Thus, in Moggi et al. (2018), we introduced *safe and robust* reachability analysis. In Moggi et al. (2019), we used metric spaces to formalize the notions of imprecision and robust analysis. In a metric space $\mathbb{S}$, a level of imprecision $\delta > 0$ means that one cannot distinguish two points s and s' when their distance $d(s, s')$ is less than δ. If one considers subsets instead of points, and allows δ to become arbitrarily small, then one cannot distinguish two subsets that have the same closure, namely $\forall \delta > 0.B(S, \delta) = B(\overline{S}, \delta)$, where $B(S, \delta)$ is the open subset $\{s' \mid \exists s \in S.d(s, s') < \delta\}$ and the closure $\overline{S}$ is the smallest closed subset containing S. Therefore, one can replace $\mathbb{P}(\mathbb{S})$ with the complete lattice $\mathbb{C}(\mathbb{S})$ of closed subsets, which is related to the former by the adjunction $\mathbb{C}(\mathbb{S}) \underset{\gamma}{\overset{\alpha}{\underset{\hookrightarrow}{\overset{\twoheadleftarrow}{\top}}}} \mathbb{P}(\mathbb{S})$, with γ an inclusion map and α a surjective map. This replacement is convenient, since the cardinality of $\mathbb{C}(\mathbb{S})$ can be smaller than that of $\mathbb{P}(\mathbb{S})$, e.g., when $\mathbb{S}$ is the real line $\mathbb{R}$.

Robustness. A monotonic map $A:\mathbb{P}(\mathbb{S}) \to \mathbb{P}(\mathbb{S}')$, with $\mathbb{S}$ and $\mathbb{S}'$ metric spaces, is *robust* when *small* input changes cause small output changes, i.e.,

$$\forall S \in \mathbb{P}(\mathbb{S}).\forall \varepsilon > 0.\exists \delta > 0.B(A(S), \varepsilon) \leq A(B(S, \delta)).$$

When A is robust, there is no *loss of information* in restricting to closed subsets, namely there exists a unique monotonic map $A_c:\mathbb{C}(\mathbb{S}) \to \mathbb{C}(\mathbb{S}')$ such that $A_c \circ \alpha = \alpha \circ A$. Thus, the focus of Moggi et al. (2018, 2019) was on analyses between complete lattices of closed subsets. In Moggi et al. (2019), sufficient (and almost necessary) conditions were identified to ensure that every monotonic map $A : \mathbb{C}(\mathbb{S}) \to \mathbb{C}(\mathbb{S}')$ has a *best robust approximation*, i.e., the biggest robust map $\Box_R(A):\mathbb{C}(\mathbb{S}) \to \mathbb{C}(\mathbb{S}')$ such that $\Box_R(A) \leq A$ in the lattice of monotonic maps with the pointwise order.

Scott continuity. We refer to Gierz et al. (2003) for the definitions of Scott-continuous map, way-below relation $\ll$, and continuous lattice. Restricting to compact metric spaces is mathematically appealing, since in this case a monotonic map $A : \mathbb{C}(\mathbb{S}) \to \mathbb{C}(\mathbb{S}')$ is robust exactly when it is Scott continuous, and the complete lattices $\mathbb{C}(\mathbb{S})$ and $\mathbb{C}(\mathbb{S}')$ are ω-continuous (Moggi et al., 2018).

A complete lattice X is *ω-continuous* when it has a countable *base* B, i.e., a countable subset of X such for every $x \in X$, the subset $B_x \triangleq \{b \in B \mid b \ll x\}$ is directed and $x = \sup B_x$. Moreover, by fixing an enumeration e of the base, one can define when an element $x \in X$ is *computable*, namely, when the set $\{n \in \mathbb{N} \mid e(n) \in B_x\}$ is a recursively enumerable subset of $\mathbb{N}$. The notion of computable can be extended to Scott-continuous maps between ω-continuous lattices because these maps, under the pointwise order, form an ω-continuous lattice.

Ideally, we would like to focus on computable analyses, but we settle for the broader class of Scott-continuous analyses, since they are better behaved. For instance, every monotonic map $A:X \to X'$ between complete lattices has a *best (i.e., tightest) Scott-continuous approximation* $\Box_S(A):X \to X'$, while there is no *best computable approximation* of a monotonic map between (*effectively given*) ω-continuous lattices.

Related results. In Moggi et al. (2018), we defined *robustness* as a property of *analyses*, i.e., monotonic maps $A : \mathbb{C}(\mathbb{S}_1) \to \mathbb{C}(\mathbb{S}_2)$, where $\mathbb{S}_i$ are metric spaces, and $\mathbb{C}(\mathbb{S}_i)$ are the complete lattices of closed subsets of $\mathbb{S}_i$, ordered by reverse inclusion. In the same paper, we proved that:

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

- Robustness of A amounts to continuity with respect to suitable T_0-topologies $\tau_R(\mathbb{S}_i)$ on the carrier sets of $\mathbb{C}(\mathbb{S}_i)$, called *robust topologies* (see Definition 2.25). In general, the topology $\tau_R(\mathbb{S}_i)$ depends on the metric of $\mathbb{S}_i$.
- When $\mathbb{S}_i$ is compact, the topology $\tau_R(\mathbb{S}_i)$ coincides with the *Scott topology* $\tau_S(\mathbb{S}_i)$ on $\mathbb{C}(\mathbb{S}_i)$. Thus, in this case, $\tau_R(\mathbb{S}_i)$ depends only on the *topology* induced by the metric on $\mathbb{S}_i$.

In particular, when both $\mathbb{S}_1$ and $\mathbb{S}_2$ are compact, robustness and Scott continuity are equivalent properties of A, and the complete lattices $\mathbb{C}(\mathbb{S}_i)$ are ω-continuous. In Moggi et al. (2019), we prove that every analysis $A : \mathbb{C}(\mathbb{S}_1) \to \mathbb{C}(\mathbb{S}_2)$ has a *best robust approximation* $\Box_R(A)$, when $\mathbb{S}_2$ is compact, with $\Box_R(A)(C)$ given by $\bigcap\{A(C_\delta) \mid \delta > 0\}$, where the closed subset $C_\delta \stackrel{\triangle}{=} \overline{B(C, \delta)}$ is called *δ-fattening* of C. When $\mathbb{S}_1$ is not compact, however, $\Box_R(A)$ may fail to be Scott continuous, and $\mathbb{C}(\mathbb{S}_1)$ may fail to be ω-continuous.

Motivating examples. Examples of metric spaces that are not compact are Banach Spaces. In applications, one usually considers closed bounded subsets of Banach spaces. In finite-dimensional Banach spaces, all closed bounded subsets are compact, but this fails in the infinite-dimensional case. To motivate the need to go beyond compact subsets, we present some examples of closed bounded subsets of infinite-dimensional Banach spaces that are not compact:

- Probability distributions for a system with a countable set of states form a closed bounded subset of ℓ_1, i.e., the Banach space of sequences $(x_n \mid n \in \omega)$ in $\mathbb{R}^\omega$ such that $\sum_{n\in\omega} | x_n |$ is bounded. More generally, probability distributions on a measurable space (X, Σ) form a closed bounded subset of $ca(\Sigma)$, i.e., the Banach space of countably additive bounded signed measures on Σ. This subset is not compact when the cardinality of Σ is infinite.
- Continuous maps from a compact Hausdorff space X to a compact interval $[a, b]$ in $\mathbb{R}$ form a closed bounded subset of $C(X)$, i.e., the Banach space of continuous maps from X to $\mathbb{R}$. For instance, these maps could represent the height as a function of the position.
- Closed bounded subsets of feature spaces arising from kernel methods in machine learning (Hofmann et al., 2008). Usually, feature spaces are Hilbert spaces, whose carrier sets consist of real-valued maps. All Hilbert spaces with a countable base are isomorphic to ℓ_2, i.e., the Hilbert space of sequences $(x_n \mid n \in \omega)$ in $\mathbb{R}^\omega$ such that $\sum_{n\in\omega} | x_n^2 |$ is bounded.
- Closed bounded subsets of Sobolev spaces $W^{m,p}(\Omega)$, in which $\Omega \subseteq \mathbb{R}^n$ is an open set. These sets commonly appear in solution of partial differential equations (Brezis, 2011).

Contribution. For simplicity, we consider analyses of the form $A : \mathbb{C}(\mathbb{S}) \to \mathbb{C}(1)$, with 1 denoting the one-point metric space, although the results hold also when 1 is replaced by a compact metric space, e.g., a compact interval $[a, b]$ of the real line. The lattice $\mathbb{C}(1)$ is (isomorphic to) the two-point lattice Σ and the complete lattice $\mathbb{C}(\mathbb{S}) \to \Sigma$ is isomorphic to that of upward closed subsets of $\mathbb{C}(\mathbb{S})$, ordered by inclusion.

This paper proposes a way to reconcile robustness and Scott continuity when $\mathbb{S}$ is not compact. The general idea is to *construct* an ω-continuous lattice D related to $\mathbb{C}(\mathbb{S})$ by an adjunction:

$$\begin{tikzcd} \mathbb{C}(\mathbb{S}) \arrow[r, shift left, "\iota_*"] \arrow[r, phantom, "\top"] & D, \arrow[l, shift left, dashed, "\iota^*"] \end{tikzcd}$$

such that the composite map $A' \circ \iota_*: \mathbb{C}(\mathbb{S}) \to \Sigma$ is robust whenever the map $A' : D \to \Sigma$ is Scott continuous. Therefore, given an analysis $A : \mathbb{C}(\mathbb{S}) \to \Sigma$, we can take the *best Scott-continuous approximation* A' of $A \circ \iota^* : D \to \Sigma$—in fact, any Scott-continuous approximation will do—and the composite map $A' \circ \iota_*$ is guaranteed to be a robust approximation of A.

The ω-continuous lattice D that we construct is of the form $\mathbb{C}(\overline{\mathbb{S}})$, where $\overline{\mathbb{S}}$ is a compact Hausdorff space given by the limit of an ω^{op}-chain of compact metric spaces related to $\mathbb{S}$ (see Theorem 3.10), and the adjunction $\iota_* \vdash \iota^*$ is determined by a continuous map $\iota : \mathbb{S} \to \overline{\mathbb{S}}$. Thus, by moving from $\mathbb{S}$ to $\overline{\mathbb{S}}$, we gain compactness by giving up the metric structure.

In general, $\overline{\mathbb{S}}$ is not uniquely determined by $\mathbb{S}$, although Theorem 3.11 provides some criteria for choosing the ω^{op}-chain of compact metric spaces which determines $\overline{\mathbb{S}}$.

Summary

The rest of the paper is organized as follows:

- Section 2 contains the mathematical preliminaries, where we fix notation and definitions. We will also present some basic results, usually without proofs, unless the results are not available in textbooks, in which case we provide proofs or pointers to other papers which include the relevant proofs. Most definitions are standard or taken from other papers. The only exception is the category $\mathbf{Top}_A$ of topological analyses (Definition 2.9).
- The main theoretical results are in Section 3. These include properties of idempotents and their splittings in a generic category $\mathbb{A}$ (e.g., Theorem 3.9) and the construction (Theorem 3.10) of a continuous map $\iota : \mathbb{S} \to \overline{\mathbb{S}}$ relating a metric space $\mathbb{S}$ to a compact Hausdorff space $\overline{\mathbb{S}}$.
- In Section 4, we apply the results of Section 3 to several examples of $\mathbb{S}$, which include finite-dimensional Banach spaces $\ell_{m,p}$, infinite -dimensional Banach spaces ℓ_p (i.e., sequence spaces), and closed unit balls B_p in sequence spaces.
- In Section 5, we investigate loss of precision when moving from the complete lattice $\mathbb{C}(\mathbb{S})$ to the ω-continuous lattice $\mathbb{C}(\overline{\mathbb{S}})$, when $\mathbb{S}$ is a closed unit ball B_p. In Theorem 5.4, we characterize the closed subsets of ℓ_p (with $1 < p < \infty$) for which there is no loss of precision as those that can be expressed as a non-empty intersection of finite unions of closed balls.
- We conclude the paper with some remarks and suggestions for future work in Section 6.

2. Mathematical Preliminaries

In this section, we present the basic technical background—including the notation—that will be used throughout the paper. We assume some familiarity with Category Theory, see Borceux (1994). We use standard terminology for topological and metric spaces. At times, we may refer to a structure by its carrier set. For instance, for a metric space (X, d), we may simply write "the metric space X." For brevity, in the text, we will henceforth use the word "carrier" instead of "carrier set."

We use the "$\in$" symbol to denote set membership (e.g., $x \in X$) and ":" symbol to denote function types (e.g., $f{:}X \to Y$) and also to denote objects and arrows in categories (e.g., $A{:}\mathbf{Top}_0$ and $f{:}\mathbf{Top}_0(A, B)$). A natural number is identified with the set of its predecessors, i.e., $0 = \emptyset$ and $n = \{0, \ldots, n-1\}$, for any $n \geq 1$. We write $\mathbb{N}$ or ω for the set of natural numbers. When the order matters ω denotes the set of natural numbers ordered by inclusion, while $\mathbb{N}$ denotes the set of natural numbers with the discrete order. We write $(x_n \mid n \in \omega)$ to denote a countable sequence, and when the indexing set is clear from the context, we just write $(x_n \mid n)$.

The powerset of a set X is denoted by $\mathsf{P}(X)$, $\subseteq$ denotes subset inclusion, and $\subset$ denotes strict subset inclusion, i.e., $A \subset B \iff A \subseteq B \wedge A \neq B$. Similarly, the finite powerset (i.e., the set of finite subsets) of X is denoted by $\mathsf{P}_f(X)$, and $A \subseteq_f B$ denotes that A is a finite subset of B. When X is a topological space, we write $\mathsf{O}(X)$ and $\mathsf{C}(X)$ for the subsets of $\mathsf{P}(X)$ consisting of the open subsets and the closed subsets of X, respectively.

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

$$
\begin{tikzcd}
 & \mathbf{Ban} \arrow[r, hook, shift right] \arrow[d] & \mathbf{NVS} \arrow[l, dashed, shift right] \arrow[l, phantom, "\perp"] \arrow[d] & & \text{normed vector space} \\
\mathbf{KMS} \arrow[r, hook] \arrow[d] & \mathbf{CMS} \arrow[r, hook, shift right] & \mathbf{Met} \arrow[l, dashed, shift right] \arrow[l, phantom, "\perp"] \arrow[d] & & \text{(extended) metric space} \\
\mathbf{KH} \arrow[rr, hook, shift right] & & \mathbf{Haus} \arrow[ll, dashed, shift right] \arrow[ll, phantom, "\perp"] \arrow[r, hook, shift right] & \mathbf{Top}_0 \arrow[l, dashed, shift right] \arrow[l, phantom, "\perp"] \arrow[d, shift left] & \text{topological space} \\
 & & & \mathbf{Set} \arrow[u, dashed, hook, shift left, "D"] \arrow[u, phantom, "\dashv"] &
\end{tikzcd}
$$

The notation $\longrightarrow$ denotes a faithful (*forgetful*) functor, $\hookrightarrow$ denotes a full&faithful (*inclusion*) functor, and $\vdash$ indicates the existence of a left adjoint to a functor. The left adjoints from top to bottom and left to right are: the Cauchy completion $\overline{X}$ (for X in **NVS** or **Met**), the Stone-Čech compactification βX (for X in **Haus**), the Hausdorff reflection HX (for X in $\mathbf{Top}_0$), the Discrete topology DX (for X in **Set**).

Figure 1. Categories of spaces.

2.1 Categories of spaces

The spaces of interest for this paper are (extended) metric spaces[1] and Hausdorff spaces. However, in examples we restrict to Banach spaces, and some constructions extend to arbitrary topological spaces. Figure 1 summarizes the relations among the following categories of spaces.

Definition 2.1 (Categories of Spaces).

- $\mathbf{Top}_0$ *is the category of* T_0*-topological spaces* (X, τ) *and continuous maps.* **Haus** *and* **KH** *are the full sub-categories consisting of Hausdorff spaces (aka* T_2*-spaces) and compact Hausdorff spaces, respectively.*
- **Met** *is the category of* ***extended metric spaces*** (X, d)*, i.e., the metric* d *can be* ∞*, and* ***short maps****, i.e., maps* $f{:}X_1 \to X_2$ *such that* $d_2(f(x), f(x')) \leq d_1(x, x')$ *for* $x, x' \in X_1$*. There are other maps one can consider between (extended) metric spaces, in particular* ***isometries****, i.e., metric preserving maps. The forgetful functor* $U{:}\mathbf{Met} \to \mathbf{Haus}$ *maps a metric* d *on* X *to the* T_2*-topology* τ_d *on* X *generated by the open balls.* **CMS** *and* **KMS** *are the full sub-categories of* **Met** *consisting of Cauchy complete extended metric spaces and compact extended metric spaces, respectively. The objects in* **KMS** *are exactly the extended metric spaces whose underlying topological spaces are compact.*
- **NVS** *is the category of normed vector spaces* $(X, \cdot, +, \| - \|)$ *and short linear maps. The forgetful functor* $U{:}\mathbf{NVS} \to \mathbf{Met}$ *maps a normed vector space to the metric space with (the same carrier and) metric* $d(x', x) \stackrel{\triangle}{=} \| x' - x \|$*.* **Ban** *is the full sub-category of* **NVS** *consisting of Banach spaces. The objects in* **Ban** *are exactly the normed vector spaces whose underlying metric spaces are complete.*

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

Example 2.2. As a running example, we consider the one-dimensional Banach space $\mathbb{R}$:*Ban*. As a (complete) metric space $\mathbb{R}$:**CMS**, its metric is $d(x,y) \triangleq |x-y|$, while as a Hausdorff space $\mathbb{R}$:**Haus**, its topology $\mathsf{O}(\mathbb{R})$ is the set of (Euclidean) open subsets of $\mathbb{R}$.

Recall that, K:**KH** is said to be a compactification of X:**Haus**, if X can be regarded as a dense subspace (via a topological embedding) of K. There are several compactifications of $\mathbb{R}$, for example:

- The one-point (aka Alexandroff) compactification $\mathbb{R}_\infty$ with the carrier $\mathbb{R} \cup \{\infty\}$, whose open subsets are either in $\mathsf{O}(\mathbb{R})$ or subsets in $\mathsf{P}(\mathbb{R}_\infty)$ of the form $A \cup \{\infty\}$, where $A \in \mathsf{O}(\mathbb{R})$ and its complement A^c in $\mathbb{R}$ is compact.
- The two-point compactification $\overline{\mathbb{R}}$, which is topologically homeomorphic to the interval $[0,1]$ with the Euclidean topology. The two-point compactification of $\mathbb{R}$ has $\mathbb{R} \cup \{-\infty, +\infty\}$ as its carrier, with the order on $\mathbb{R}$ extended so that $\forall x \in \mathbb{R}.\ -\infty < x < +\infty$. The collection of sets of the form $[-\infty, x)$, (x,y), $(y, +\infty]$, with $x, y \in \mathbb{R}$, forms a base for the two-point compactification.
- The Stone-Čech compactification $\beta\mathbb{R}$, which is characterized by the universal property that any continuous function from $\mathbb{R}$ to a compact Hausdorff space K can be extended to a continuous function from $\beta\mathbb{R}$ to K in a unique way. For more details see, e.g., Munkres (2000, Chapter 5).

All three methods of compactification exemplified above have limitations. The two-point compactification may work only for topological spaces *induced* by a linear order. The one-point compactification works only for *locally compact* spaces, which do not include infinite-dimensional Banach spaces. The Stone-Čech compactification works for all *Tychonoff* spaces. The problem is that, even for simple spaces such as $\mathbb{R}$, there is no concrete description of $\beta\mathbb{R}$, which makes it impossible to be used in an effective framework as we intend to do.

The following theorems recall some properties of categories and functors in Figure 1.

Theorem 2.3. *The categories in the following diagram have finite limits and finite sums, and the functors preserve them:*

$$\mathbf{KMS} \hookrightarrow \mathbf{CMS} \hookrightarrow \mathbf{Met} \longrightarrow \mathbf{Haus}$$

The categories **CMS**, **Met**, *and* **Haus** *have also small limits and small colimits.*

For the existence of sums and infinitary products, it is essential to use extended metric spaces.

Theorem 2.4. *The categories in the following diagram have small limits and finite sums, and the functors preserve them:*

$$\mathbf{KH} \hookrightarrow \mathbf{Haus} \hookrightarrow \mathbf{Top}_0$$

The categories have also small colimits.

Definition 2.5. (Imprecision and Fattening). *Given a metric space $\mathbb{S}$, with metric d, we define:*

(1) $B(S,\delta) \triangleq \{y \mid \exists x \in S. d(x,y) < \delta\} \in \mathsf{O}(\mathbb{S})$, where $S \in \mathsf{P}(\mathbb{S})$ and $\delta > 0$. The subset $B(S,\delta)$ is open, because it is the union of the open balls $B(x,\delta) \triangleq \{y \mid d(x,y) < \delta\}$ with $x \in S$.
*(2) the **closure** $\overline{S} \in \mathsf{C}(\mathbb{S})$ of $S \in \mathsf{P}(\mathbb{S})$, i.e., the smallest $C \in \mathsf{C}(\mathbb{S})$ such that $S \subseteq C$.*
*(3) $S_\delta \triangleq \overline{B(S,\delta)}$ is the δ-**fattening** of $S \in \mathsf{P}(\mathbb{S})$.*

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

While S_δ is always closed, the subset $C(S,\delta) \stackrel{\triangle}{=} \{y \mid \exists x \in S.d(x,y) \leq \delta\}$ may fail to be closed, e.g., in the metric space $\mathbb{R}$ we have $B(S,1) = C(S,1) = (-2,2) \subset [-2,2] = S_1$, when S is the open interval $(-1,1)$. Moreover, when $C(S,\delta)$ is closed it may fail to coincide with S_δ, e.g., in a 1-*discrete* metric space $\mathbb{S}$ (i.e., the metric is either 0 or 1), we have $B(S,1) = S_1 = S \subset \mathbb{S} = C(S,1)$, for any proper non-empty subset S of $\mathbb{S}$. We summarize some basic properties of $B(S,\delta)$ and S_δ, see also Moggi et al. (2019).

Proposition 2.6. *In a metric space $\mathbb{S}$, the following claims hold for any $S \in \mathsf{P}(\mathbb{S})$ and $\delta, \delta' > 0$.*

(1) $B(B(S,\delta),\delta') \subseteq B(S,\delta+\delta')$.
(2) $S \subseteq \overline{S} \subseteq B(S,\delta) = B(\overline{S},\delta)$.
(3) $B(S,\delta) \subseteq S_\delta \subseteq B(S,\delta+\delta')$.
(4) $\overline{S} = \bigcap_{\delta>0} B(S,\delta) = \bigcap_{\delta>0} S_\delta$.

We now prove two properties of short maps (between metric spaces) and compact metric spaces expressed in terms of δ-fattening. These properties are relevant to the proof of Theorem 3.10.

Proposition 2.7. *If f:**Met**$(\mathbb{S},\mathbb{S}')$, then the following claims hold for any $S \in \mathsf{P}(\mathbb{S})$ and $\delta > 0$.*

(1) $f(B(S,\delta)) \subseteq B(f(S),\delta)$.
(2) $\overline{f(S_\delta)} \subseteq f(S)_\delta$.

Proof. For (1) observe that $y' \in f(B(S,\delta))$ means $\exists y \in B(S,\delta).y' = f(y)$, i.e., $\exists y.\exists x \in S.y' = f(y) \wedge d(y,x) < \delta$. Thus, $\exists x \in S.d'(y',f(x)) < \delta$, because $d'(y',f(x)) = d'(f(y),f(x)) \leq d(y,x) < \delta$ (by f short), i.e., $y' \in B(f(S),\delta)$.

Claim (2) follows from (1) and the chain of equivalences $f(A) \subseteq C' \iff A \subseteq f^{-1}(C') \iff \overline{f(\overline{A})} \subseteq C'$, which holds for any f:**Haus**$(\mathbb{S},\mathbb{S}')$, $A \in \mathsf{P}(\mathbb{S})$ and $C' \in \mathsf{C}(\mathbb{S}')$. Namely, take $A = B(S,\delta)$ and $C' = f(S)_\delta$. □

Proposition 2.8. *If $\mathbb{S}$:**KMS**, then $\forall K \in \mathsf{C}(\mathbb{S}).\forall O \in \mathsf{O}(\mathbb{S}).K \subseteq O \implies \exists \delta > 0.K_\delta \subseteq O$.*

Proof. First, in a compact metric space closed subsets are compact, thus $K \in \mathsf{C}(\mathbb{S})$ and K_δ are compact. If the claim were false, then there exist $K \in \mathsf{C}(\mathbb{S})$ and $O \in \mathsf{O}(\mathbb{S})$ such that $K \subseteq O$ and $\forall n.\exists x_n \in K_{\delta_n}.x_n \notin O$, where $\delta_n \stackrel{\triangle}{=} 2^{-n}$. Since the sequence $(x_n \mid n)$ is in the compact subset K_{δ_0}, it must have an accumulation point x.

For simplicity, we assume that $(x_n \mid n)$ is a Cauchy sequence and x is its limit. Since the Cauchy sequence $(x_n \mid n)$ is eventually in K_δ for any $\delta > 0$, we conclude that $x \in K$, and also $x \in O$. Since O is open, there exists an open ball $B(x,\delta) \subseteq O$, which contradicts the assumption $\forall n.x_n \notin O$. □

2.2 Categories of analyses

We define an analysis as a monotonic map between complete lattices. However, we need to consider further properties of analyses, that (with the exception of computability) can be defined as continuity with respect to suitable topologies on (the carrier of) complete lattices. For this reason, we introduce the category $\mathbf{Top}_A$ of *topological analyses*, which *refines* the category $\mathbf{Po}_A$ of *analyses* (see Figure 2).

Definition 2.9. (Category of Analyses).

- **Po** *is the category of posets and monotonic maps.*
- *The forgetful functor U:$\mathbf{Top}_0 \to \mathbf{Po}$ maps a T_0-topology τ on X to the* ***specialization order*** *$\leq_\tau$ on X, i.e., $x \leq_\tau y \stackrel{\triangle}{\iff} \forall O \in \tau.(x \in O \implies y \in O)$.*

$$
\begin{tikzcd}
\mathbf{Top}_A \arrow[r, hook] \arrow[d, shift right] \arrow[d, phantom, "\vdash"] & \mathbf{Top}_0 \arrow[d, shift right, "U"'] \arrow[d, phantom, "\vdash"] & \text{topologies} \\
\mathbf{Po}_A \arrow[r, hook] \arrow[u, shift right] & \mathbf{Po} \arrow[u, shift right, "A"'] & \text{posets}
\end{tikzcd}
$$

The notation $\longrightarrow$ denotes a faithful enriched (*forgetful*) functor, $\hookrightarrow$ denotes a full&faithful enriched (*inclusion*) functor, and $\vdash$ indicates an adjunction in the enriched sense.

Figure 2. Poset-enriched categories.

- *The inclusion functor* A:$\mathbf{Po} \hookrightarrow \mathbf{Top}_0$ *maps a poset* $\leq$ *on* X *to the* ***Alexandroff topology*** $\tau_{\leq}$ *of the upward closed subsets of* X*, i.e.,* $O \in \tau_{\leq} \overset{\triangle}{\iff} \forall x, y \in X.x \in O \wedge x \leq y \implies y \in O$.
- $\mathbf{Po}_A$*, the category of* ***analyses****, is the full sub-category of* $\mathbf{Po}$ *consisting of complete lattices.*
- $\mathbf{Top}_A$*, the category of* ***topological analyses****, is the full sub-category of* $\mathbf{Top}_0$ *consisting of* T_0*-spaces whose specialization order is a complete lattice.*

Theorem 2.10. *The following statements hold:*

(1) The categories $\mathbf{Po}$ *and* $\mathbf{Top}_0$ *are* $\mathbf{Po}$*-enriched and have small limits and small colimits.*
(2) The functors U *and* A *are* $\mathbf{Po}$*-enriched, and* A *is left adjoint to* U*.*
(3) The functor U *preserves small limits and small sums.*
(4) The functor A *preserves finite limits and small colimits.*

Proof. The $\mathbf{Po}$-enrichment of $\mathbf{Po}$ is given by its cartesian closed structure. Since U is faithful, $\mathbf{Top}_0(X, Y)$ is a subset of $\mathbf{Po}(UX, UY)$ and inherits the $\mathbf{Po}$-enrichment. It is easy to prove that $\mathbf{Top}_0(AX, Y) = \mathbf{Po}(X, UY)$. Thus, A is left adjoint to U also as $\mathbf{Po}$-enriched functors. $\square$

Corollary 2.11. *The following statements hold:*

(1) The category $\mathbf{Po}_A$ *is* $\mathbf{Po}_A$*-enriched and has small products.*
(2) The category $\mathbf{Top}_A$ *is* $\mathbf{Po}$*-enriched and has small products.*
(3) The $\mathbf{Po}$*-enriched functors* U *and* A *restrict to functors between* $\mathbf{Top}_A$ *and* $\mathbf{Po}_A$*.*

Proof. The $\mathbf{Po}_A$-enrichment of $\mathbf{Po}_A$ is given by its cartesian closed structure. The other claims are easy consequences of Theorem 2.10 and the definition of $\mathbf{Top}_A$. $\square$

Theorem 2.12. (Topologies on a poset). *Given a partial order* $\leq$ *on* X*, the set of* T_0*-topologies on* X *with specialization order* $\leq$ *ordered by reverse inclusion is a complete lattice* $\mathbf{Top}(\leq)$*, where:*

- *the least element* $\tau_\bot$ *is the Alexandroff topology* $\tau_{\leq}$
- *the top element* $\tau_\top$ *is the topology generated by the set* $\{\not\downarrow y \mid y \in X\}$*, where* $\not\downarrow y \overset{\triangle}{=} \{x \in X \mid x \not\leq y\}$
- *the non-empty sups are given by intersection.*

Moreover, when X *is finite* $\mathbf{Top}(\leq)$ *is trivial, i.e.,* $\tau_\bot = \tau_\top$.

Proof. It is easy to show that $(X, \tau_\top)$ and $(X, \tau_\bot)$ are T_0-spaces with specialization order $\leq$. Given a T_0-topology τ on X with specialization order $\leq$ we have $\tau_\top \subseteq \tau \subseteq \tau_\bot$, because:

- each $O \in \tau$ is upward closed, by definition of $\leq_\tau$,
- each $\not\downarrow y$ is in τ, because $\not\downarrow y = \bigcup\{O \in \tau \mid y \notin O\}$.

Therefore, $\tau_\top$ and $\tau_\bot$ are respectively the top and bottom element in $\mathbf{Top}(\leq)$. Since the topologies on a set X (ordered by reverse inclusion) form a complete lattice, with (non-empty) sups given by intersections, so do the topologies τ on X such that $\tau_\top \subseteq \tau \subseteq \tau_\bot$. Moreover, such topologies are T_0, because $\tau_\top$ is.

For every $x \in X$, we have $\uparrow x = \bigcap\{\not\downarrow y \mid y \in X \wedge x \not\leq y\}$. When X is finite, the right-hand side of the equality is a finite intersection of open sets in $\tau_\top$, thus $\uparrow x \in \tau_\top$, and therefore, $\tau_\bot \subseteq \tau_\top$. □

When $\leq$ is an object in $\mathbf{Po}_A$, the topologies in $\mathbf{Top}(\leq)$ are objects in $\mathbf{Top}_A$.

2.3 Adjunctions and best approximations

A key property of categories of analyses is poset-enrichment, which provides a qualitative criterion for comparing analyses, and allows the definition of *adjunctions* between two complete lattices.

Definition 2.13. (Adjunction). *An* ***adjunction*** *in a* **Po***-enriched category* $\mathbb{A}$*, notation* $f \dashv g$*, is a pair of maps* $X \underset{-f\rightarrow}{\overset{\leftarrow g-}{}} Y$ *in* $\mathbb{A}$ *such that* $f \circ g \leq id_Y$ *and* $g \circ f \geq id_X$*. The maps* f *and* g *are called left and right adjoint, respectively, and any one of these two maps uniquely determines the other.*

Remark 2.14. A characterization of adjunctions in **Po** is $f \dashv g$ iff $\forall x \in X.\forall y \in Y.(x \leq_X g(y) \iff f(x) \leq_Y y)$. This characterization implies that in **Po** left adjoints preserve sups and (dually) right adjoints preserve infs.

Theorem 2.15. *The* **Po***-enriched categories* $\mathbf{Po}_A$ *and* $\mathbf{Top}_A$ *have limits of* ω^{op}*-chains of right adjoints.*

Proof. First, we prove the property for $\mathbf{Po}_A$. Given an ω^{op}-chain $(p_n{:}D_{n+1} \to D_n \mid n)$ of right adjoints in $\mathbf{Po}_A$, its limit in **Po** is the subset of $\prod_n \mid D_n \mid$ given by $\mid D \mid \stackrel{\triangle}{=} \{d \mid \forall n.d_n = p_n(d_{n+1})\}$ with the pointwise order $\leq_D$. Right adjoints preserve infs; thus, infs in D exist and are computed pointwise. Therefore, $D{:}\mathbf{Po}_A$ and the maps $\pi_n{:}D \to D_n$ with $\pi_n(d) = d_n$ form a limit cone (and preserve infs).

Given an ω^{op}-chain $(p_n{:}D_{n+1} \to D_n \mid n)$ of right adjoints in $\mathbf{Top}_A$, its limit in $\mathbf{Top}_0$ is the subspace of $\prod_n D_n$ corresponding to the subset $\mid D \mid \stackrel{\triangle}{=} \{d \mid \forall n.d_n = p_n(d_{n+1})\}$, and the maps $\pi_n{:}D \to D_n$ with $\pi_n(d) = d_n$ form a limit cone. Since $U{:}\mathbf{Top}_0 \to \mathbf{Po}$ is **Po**-enriched and preserves limits, we have that $(Up_n \mid n)$ is an ω^{op}-chain of right adjoints in $\mathbf{Po}_A$ and $(U\pi_n \mid n)$ is a limit cone in **Po**. By the result for $\mathbf{Po}_A$, we have $UD{:}\mathbf{Po}_A$. Hence, $D{:}\mathbf{Top}_A$. □

A similar result holds if right adjoints are replaced by left adjoints (i.e., $\mathbf{Po}_A$ and $\mathbf{Top}_A$ have limits of ω^{op}-chains of left adjoints). We recall further properties of adjunctions in **Po**-enriched categories (and in **Po**). Each of these properties has a dual, which we do not state explicitly.

Proposition 2.16. *If* $f \dashv g$ *in a* **Po***-enriched category* $\mathbb{A}$ *and* $f{:}X \to Y$ *is monic, then* $g \circ f = id_X$*.*

Proof. Since $f \dashv g$ one has $f \circ g \circ f = f$. When f is monic, this implies $g \circ f = \mathrm{id}_X$. □

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

In other words, if a left adjoint is monic, then it is split monic and its right adjoint is split epic.

Proposition 2.17. *If X is a complete lattice, then $f{:}\mathbf{Po}(X,Y)$ is a left adjoint iff f preserves sups.*

Proof. By the dual of the adjoint functor theorem Borceux (1994, Theorem 3.3.3) for posets. □

Proposition 2.18. *If Y is a complete lattice and X is a sub-poset of Y, then the inclusion $f{:}X\to Y$ is a left adjoint in* **Po** *iff X is a complete lattice and sups in X are computed as in Y (i.e., f preserves sups).*

Proof. The right-to-left implication follows from Proposition 2.17. For the other implication, consider the right adjoint g to f. By Remark 2.14, f preserves sups and g preserves infs. Moreover, X has all infs (i.e., is a complete lattice), because $g\circ f=\mathrm{id}_X$ (by Proposition 2.16) and $\inf D=g(\inf f(D))$ for any subset D of X. □

Definition 2.19. (Best Approximation). *Given a subset X of a poset Y, $x\in X$ is the* ***best*** *X-**approximation** of $y\in Y$ iff $\forall x'\in X.x'\le x\iff x'\le y$.*

A subset X of a poset Y can be identified with the sub-poset of Y with carrier X and the partial order inherited from Y. Then, the inclusion $f{:}X\to Y$ is a left adjoint in **Po** exactly when every $y\in Y$ has a best X-approximation, and the right adjoint to f maps y to its best X-approximation. When Y is a complete lattice, Proposition 2.18 characterizes the subsets X of Y for which every $y\in Y$ has a best X-approximation.

We are mainly interested in the cases where Y is a complete lattice of analyses $\mathbf{Po}_A(\le_1,\le_2)$ and X is a sub-poset of the form $\mathbf{Top}_A(\tau_1,\tau_2)$, where $\le_i$ is the specialization order of the topology τ_i. In these cases, the best X-approximations exist whenever the poset $\mathbf{Top}_A(\tau_1,\tau_2)$ is a complete lattice with sups computed as in $\mathbf{Po}_A(\le_1,\le_2)$.

Example 2.20. Given a complete lattice $\le$ (on a set X), we can define two topologies in $\mathbf{Top}(\le)$:

(1) the **Scott topology** $\tau_S(\le)$ of upward closed subsets O of X such that $\sup D\in O\implies\exists d\in D.d\in O$ for any directed subset D of X, equivalently, $\sup S\in O\implies\exists S_0\subseteq_f S.\sup S_0\in O$ for any subset S of X.

(2) the ω-**topology** $\tau_\omega(\le)$ of upward closed subsets O of X such that $\sup D\in O\implies\exists d\in D.d\in O$ for any ω-chain D in X, equivalently, $\sup S\in O\implies\exists S_0\subseteq_f S.\sup S_0\in O$ for any countable subset S of X.

Clearly, $\tau_S(\le)\subseteq\tau_\omega(\le)$. Given a map $f{:}\mathbf{Po}_A(\le_1,\le_2)$, there is an order-theoretic characterization of continuity with respect to these two topologies, namely:

- $f{:}\mathbf{Top}_A(\tau_S(\le_1),\tau_S(\le_2))\iff f$ preserves sups of directed sets, i.e., f is **Scott continuous**.
- $f{:}\mathbf{Top}_A(\tau_\omega(\le_1),\tau_\omega(\le_2))\iff f$ preserves sups of ω-chains, i.e., f is ω-**continuous**.

These characterizations imply $\mathbf{Top}_A(\tau_S(\le_1),\tau_S(\le_2))\subseteq\mathbf{Top}_A(\tau_\omega(\le_1),\tau_\omega(\le_2))\subseteq\mathbf{Po}_A(\le_1,\le_2)$ and that these subsets are closed with respect to sups computed in $\mathbf{Po}_A(\le_1,\le_2)$. Therefore, every analysis $f{:}\mathbf{Po}_A(\le_1,\le_2)$ has a best Scott-continuous approximation $\Box_S(f)$, and a best ω-continuous approximation $\Box_\omega(f)$, with $\Box_S(f)\le\Box_\omega(f)\le f$.

We introduce two sub-categories of $\mathbf{Po}_A$, related to the example above.

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

Definition 2.21. (Category of Continuous Lattices (Gierz et al., 1980)).

- **CL** *is the category of continuous lattices, i.e., every element x in the lattice is the sup of the directed set formed by the elements way-below x, and Scott-continuous maps.*
- ω**CL** *is the full sub-category of* **CL** *whose objects are* ω*-continuous lattices, i.e., continuous lattices with a countable subset B (called a* base*) such that every element x in the lattice is the sup of the directed set formed by the elements in the base way-below x.*

Proposition 2.22. *The* **Po***-enriched categories in the following diagram have finite products and limits of* ω^{op}*-chains of right adjoints and the functors preserve them:*

$$\omega\mathbf{CL} \hookrightarrow \mathbf{CL} \hookrightarrow \mathbf{Top}_A \longrightarrow \mathbf{Po}_A$$

where the full&faithful functor from **CL** *to* $\mathbf{Top}_A$ *maps a continuous lattice* $\leq$ *to the Scott topology* $\tau_S(\leq)$*. Moreover, the categories* ω**CL** *and* **CL** *have exponentials, the functor* $\omega\mathbf{CL} \hookrightarrow \mathbf{CL}$ *preserves them, and every* ω*-continuous map between* ω*-continuous lattices is necessarily Scott continuous.*

Proof. According to Abramsky and Jung (1994, Proposition 3.2.4), if D_1 and D_2 are two directed-complete partial orders with bases B_1 and B_2, respectively, then $B_1 \times B_2$ forms a basis for $D_1 \times D_2$. In particular, if B_1 and B_2 are both countable, then so is $B_1 \times B_2$. This implies that both ω**CL** and **CL** have finite products. Exponentials in **CL** are discussed in Gierz et al. (1980, Section II-4). Scott continuity of ω-continuous maps between ω-continuous lattices is proven in Abramsky and Jung (1994, Proposition 2.2.14). $\square$

2.4 From spaces to complete lattices

Given a topological space $\mathbb{S}$, the set of closed subsets of $\mathbb{S}$, ordered by reverse inclusion, forms a complete lattice $\mathbb{C}(\mathbb{S})$, with sups given by intersection. We introduce several topologies on these complete lattices, but first we give the main properties of $\mathbb{C}$ as a functor from **Haus** to $\mathbf{Po}_A$.

Definition 2.23. *The functor* $\mathbb{C}$:**Haus** $\to \mathbf{Po}_A$ *is defined as follows:*

- $\mathbb{C}(\mathbb{S})$ *is the complete lattice of closed subsets of* $\mathbb{S}$ *under reverse inclusion.*
- *If* f:**Haus**$(\mathbb{S}, \mathbb{S}')$*, then the map* $\mathbb{C}(f) \stackrel{\triangle}{=} f_*$:$\mathbf{Po}_A(\mathbb{C}(\mathbb{S}), \mathbb{C}(\mathbb{S}'))$ *is given by* $f_*(C) = \overline{f(C)}$*, i.e., it maps C to the closure of the image of C along f.*
  There is also a contravariant version, whose action on maps f^*:$\mathbf{Po}_A(\mathbb{C}(\mathbb{S}'), \mathbb{C}(\mathbb{S}))$ *is given by* $f^*(C') = f^{-1}(C')$*, i.e., it maps* C' *to the inverse image of* C' *along f.*

Theorem 2.24. *If* $\mathbb{S} \ {-f\to}\ \mathbb{S}'$ *in* **Haus***, then* $f^* \dashv f_*$ *in* $\mathbf{Po}_A$.

Proof. See Moggi et al. (2018, Example 5.5). $\square$

Definition 2.25. (Topologies on $\mathbb{C}(\mathbb{S})$ (Moggi et al., 2018)).

(1) Given a Hausdorff space $\mathbb{S}$*, the* ***upper Vietoris topology*** $\tau_U(\mathbb{S})$ *is the topology on* $\mathsf{C}(\mathbb{S})$ *for which* $U \in \tau_U(\mathbb{S}) \stackrel{\triangle}{\iff} \forall C \in U.\exists O \in \mathsf{O}(\mathbb{S}).C \in \uparrow O \subseteq U$*, where* $\uparrow S \stackrel{\triangle}{=} \{C \in \mathsf{C}(\mathbb{S}) \mid C \subseteq S\}$*, i.e., the set of closed subsets of* $\mathbb{S}$ *included in S.*

(2) Given an extended metric space $\mathbb{S}$*, the* ***Robust topology*** $\tau_R(\mathbb{S})$ *is the topology on* $\mathsf{C}(\mathbb{S})$ *for which* $U \in \tau_R(\mathbb{S}) \stackrel{\triangle}{\iff} \forall C \in U.\exists \delta > 0.C \in \uparrow B(C, \delta) \subseteq U$*, where* $B(S, \delta) \in \mathsf{O}(\mathbb{S})$ *is defined in Definition 2.5.*

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

(3) *For uniformity, given a Hausdorff space $\mathbb{S}$, we write $\tau_S(\mathbb{S})$ and $\tau_A(\mathbb{S})$ for the Scott and Alexandroff topologies on $\mathsf{C}(\mathbb{S})$ induced by the partial order on $\mathbb{C}(\mathbb{S})$, and for conciseness we write $\mathbb{A}_{\alpha\beta}(\mathbb{S}_1, \mathbb{S}_2)$ for the poset $\mathbf{Top}_A(\tau_\alpha(\mathbb{S}_1), \tau_\beta(\mathbb{S}_2))$, where α and β range over $\{A, R, S, U\}$.*

Remark 2.26. The Robust topology $\tau_R(\mathbb{S})$ on $\mathsf{C}(\mathbb{S})$ induced by the metric d on $\mathbb{S}$ coincides with the topology induced by the *Hausdorff-Smyth hemi-metric* d' on $\mathsf{C}(\mathbb{S})$ induced by d, see Goubault-Larrecq (2008, Proposition 1).

Goubault-Larrecq defines $d'(C, C') \triangleq \sup_{x' \in C'} \inf_{x \in C} d(x, x')$ under the assumption that d is a hemi-metric, i.e., d satisfies only the properties $d(x, x) = 0$ and $d(x, z) \leq d(x, y) + d(y, z)$. However, unlike Proposition 1, we do not restrict d' to the subset of $\mathsf{C}(\mathbb{S})$ of the non-empty compact subsets, thus what is claimed in the proposition may fail for closed subsets.

The topology induced by a hemi-metric d is the smallest topology containing all open balls $B(x, \delta) = \{x' \mid d(x, x') < \delta\}$. Therefore, the topology induced by d' on $\mathsf{C}(\mathbb{S})$ coincides with the Robust topology, because the following inclusions hold for every $C \in \mathsf{C}(\mathbb{S})$ and $0 < \delta < \delta'$

$$B'(C, \delta) \triangleq \{C' \mid d'(C, C') < \delta\} \subseteq \uparrow B(C, \delta) \triangleq \uparrow \{x' \mid \exists x \in C.d(x, x') < \delta\} \subseteq B'(C, \delta').$$

Theorem 2.27. *The following statements hold:*

(1) *If $\mathbb{S}$:**Haus**, then $\tau_U(\mathbb{S})$ is in $\mathbf{Top}(\leq)$, where $\leq$ is the partial order on $\mathbb{C}(\mathbb{S})$.*
(2) *If $\mathbb{S}$:**KH**, then $\mathbb{C}(\mathbb{S})$:**CL** and $\tau_U(\mathbb{S}) = \tau_S(\mathbb{S})$.*
(3) *If $\mathbb{S}$:**Met**, then $\tau_S(\mathbb{S}) \subseteq \tau_R(\mathbb{S}) \subseteq \tau_U(\mathbb{S})$. Therefore, $\tau_S(\mathbb{S}) = \tau_R(\mathbb{S}) = \tau_U(\mathbb{S})$ when $\mathbb{S}$:**KMS**.*
(4) *If $\mathbb{S}$:**KMS** is finite, then $\mathbb{C}(\mathbb{S})$ is finite and $\tau_S(\mathbb{S}) = \tau_A(\mathbb{S})$.*

Proof.

(1) We prove that $\tau_\top \subseteq \tau_U(\mathbb{S}) \subseteq \tau_\perp = \tau_A(\mathbb{S})$ (see Theorem 2.12). $\tau_U(\mathbb{S}) \subseteq \tau_A(\mathbb{S})$, because the open subsets of the upper topology are upward closed with respect to reverse inclusion.

To prove $\tau_\top \subseteq \tau_U(\mathbb{S})$ we show that $\not\downarrow C \triangleq \{C' \in \mathsf{C}(\mathbb{S}) \mid C \not\subseteq C'\}$ is in $\tau_U(\mathbb{S})$ when $C \in \mathsf{C}(\mathbb{S})$. If $C \not\subseteq C'$, then there exists x in $C - C'$. But every singleton is closed in $\mathbb{S}$ (because $\mathbb{S}$:**Haus**), thus the complement O of the singleton $\{x\}$ is open and $C' \in \uparrow O \subseteq \not\downarrow C$.
(2) Follows from Edalat (1995, Proposition 3.3).
(3) The inclusions are proved in Moggi et al. (2018, Lemma A.3).
(4) If $\mathbb{S}$ is finite, then $\mathbb{C}(\mathbb{S})$ is also finite. Hence, $\mathbf{Top}(\leq)$ contains only one topology. □

In item (3) of Theorem 2.27, when $\mathbb{S}$ is not compact, the inclusions may be strict. One such counter-example is the metric space $\mathbb{R}$ of real numbers.

Example 2.28. Let $\mathbb{R}$ be the metric space defined in Example 2.2, which is not compact. Consider the closed subset $\mathbb{N}$ of natural numbers and the open subset $O \triangleq \bigcup_{n \in \mathbb{N}} B(n, 2^{-(n+1)})$, then:

- $\uparrow O$ is in $\tau_U(\mathbb{R})$, but it is not in $\tau_R(\mathbb{R})$, because $\mathbb{N} \in \uparrow O$, but there is no $\delta > 0$ such that $B(\mathbb{N}, \delta) \subseteq O$. The reason is that, for any given $\delta > 0$, if we take $n_0 \in \mathbb{N}$ to be large enough to satisfy $2^{-(n_0+1)} < \delta$, then $B(n_0, \delta) \subseteq B(\mathbb{N}, \delta)$, but $B(n_0, \delta) \not\subseteq O$.
- $\uparrow \emptyset = \{\emptyset\}$ is in $\tau_R(\mathbb{R})$, because $\forall \delta > 0.\ B(\emptyset, \delta) = \emptyset$, but $\uparrow \emptyset$ is not in $\tau_S(\mathbb{R})$. The reason is that, if we define $C_n \triangleq \{x \in \mathbb{R} \mid x \geq 2^n\}$ for every $n \in \omega$, then $(C_n \mid n \in \omega)$ is a chain of closed subsets of $\mathbb{R}$ satisfying $\bigcap_{n \in \omega} C_n = \emptyset$, but $\forall n \in \omega.\ \emptyset \neq C_n$.

Theorem 2.29. *If* f:**Haus**$(\mathbb{S}, \mathbb{S}')$*, then* f^*:$\mathbb{A}_{SS}(\mathbb{S}', \mathbb{S})$ *and* f_*:$\mathbb{A}_{UU}(\mathbb{S}, \mathbb{S}')$.

Proof. Since the map f^* is a left adjoint in $\mathbf{Po}_A$, it preserves all sups. Thus, it is Scott continuous. The map f_* is upper Vietoris continuous, because for any $C \in \mathsf{C}(\mathbb{S})$ and $O' \in \mathsf{O}(\mathbb{S}')$ we have $\overline{f(C)} \subseteq O' \implies f(C) \subseteq O' \implies C \subseteq f^{-1}(O')$ and $f^{-1}(O') \in \mathsf{O}(\mathbb{S})$ by continuity of f. □

Theorem 2.30. *The functor* $\mathbb{C}$:**Haus** $\to$ $\mathbf{Po}_A$ *restricted to* **KH** *factors through* **CL***, and when restricted to* **KMS***, it factors through* ω**CL**.

Proof. From Theorem 2.27, we know that for any X:**KH**, the lattice $\mathbb{C}(X)$ is continuous. The fact that for any X:**KMS**, the lattice $\mathbb{C}(X)$ is ω-continuous is a straightforward consequence. It remains to show that, if f:**KH**(X, Y), then f_* is Scott continuous. But this also follows from item (2) of Theorem 2.27 and Theorem 2.29. □

Example 2.31. Consider the compact Hausdorff spaces $\mathbb{R}_\infty$ and $\overline{\mathbb{R}}$ defined in Example 2.2, i.e., the one-point and two-point compactifications of $\mathbb{R}$. These spaces are related by the maps $\mathbb{R} \overset{\bar{\iota}}{\hookrightarrow} \overline{\mathbb{R}} \xrightarrow{f} \mathbb{R}_\infty$ in **Haus**, where $\bar{\iota}$ is the obvious sub-space inclusion and f maps $-\infty$ and $+\infty$ to ∞ and is the identity on the other points. Moreover, the sub-space inclusion $\mathbb{R} \overset{\iota_\infty}{\hookrightarrow} \mathbb{R}_\infty$ is given by the composition $f \circ \bar{\iota}$. By Theorem 2.24, we get the adjunctions $\mathbb{C}(\mathbb{R}) \underset{\bar{\iota}_*}{\overset{\bar{\iota}^*}{\rightleftarrows}} \mathbb{C}(\overline{\mathbb{R}}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbb{C}(\mathbb{R}_\infty)$ in $\mathbf{Po}_A$. Since $\bar{\iota}^* \circ \bar{\iota}_*$ is the identity on $\mathbb{C}(\mathbb{R})$ and $f_* \circ f^*$ is the identity on $\mathbb{C}(\mathbb{R}_\infty)$, we conclude that $(\bar{\iota}_*, \bar{\iota}^*)$ is an insertion-closure pair and (f^*, f_*) is an embedding-projection pair in $\mathbf{Po}_A$. In fact, (f^*, f_*) is an embedding-projection pair in **CL**, by Theorem 2.30. On the other hand, $\bar{\iota}_*$ is not Scott continuous, e.g., the sup of the ω-chain $([n, +\infty) \mid n \in \omega)$ in $\mathbb{C}(\mathbb{R})$ is $\emptyset$, while the sup of its image in $\mathbb{C}(\overline{\mathbb{R}})$, namely $([n, +\infty] \mid n \in \omega)$, is the singleton $\{+\infty\}$. However, by Theorem 2.29 and $\tau_U(\overline{\mathbb{R}}) = \tau_S(\overline{\mathbb{R}})$, we have that $\bar{\iota}_*$ is in $\mathbb{A}_{US}(\mathbb{R}, \overline{\mathbb{R}})$. Furthermore, by the results in Section 3, we have that $\bar{\iota}_*$ is in $\mathbb{A}_{RS}(\mathbb{R}, \overline{\mathbb{R}})$.

Theorem 2.32. *The functor* $\mathbb{C}$:**KH** $\to$ **CL** *preserves limits of* ω^{op}*-chains.*

Proof. Given an ω^{op}-chain $(p_n:X_{n+1} \to X_n \mid n)$ in **KH**, let $(\pi_n:X \to X_n \mid n)$ be its limit in **KH** (and **Haus**), where X is the sub-space of $\prod_n X_n$ such that $|X| = \{x \mid \forall n.x_n = p_n(x_{n+1})\}$. By Theorem 2.15, the limit of $(\mathbb{C}(p_n) \mid n)$ in $\mathbf{Po}_A$ is given by $(\pi'_n:D \to D_n \mid n)$, where D_n is $\mathbb{C}(X_n)$ and D is the sub-poset of $\prod_n D_n$ such that $|D| = \{d \mid \forall n.d_n = \mathbb{C}(p_n)(d_{n+1})\}$. But for spaces in **KH** compact and closed subsets coincide, thus $\mathbb{C}(p_n)(C) = p_n(C)$, i.e., it is the image of C along p_n.

By the universal property of D, there exists a unique map ϕ:$\mathbb{C}(X) \to D$ in $\mathbf{Po}_A$ such that:

$$\mathbb{C}(X \xrightarrow{\pi_n} X_n) = \mathbb{C}(X) \xrightarrow{\phi} D \xrightarrow{\pi'_n} D_n,$$

namely $\phi(C) = (\pi_n(C) \mid n)$ for every $C \in \mathbb{C}(X)$. Moreover, ϕ preserves infs, since the $\mathbb{C}(\pi_n)$ are right adjoints and preserve infs. Therefore, ϕ has a left adjoint ϕ':$D \to \mathbb{C}(X)$, namely $\phi'(d) = \bigcap_n \pi^*_n(d_n)$. We prove that ϕ' is the inverse of ϕ, and the limit in $\mathbf{Po}_A$ is actually in **CL**, because **CL** is *replete* in $\mathbf{Po}_A$ (i.e., if A:**CL** and f:$A \to B$ is an iso in $\mathbf{Po}_A$, then f is in **CL**). Because of the adjunction $\phi' \dashv \phi$ we have:

(1) $\forall n.\forall d \in D.\phi(\phi'(d))_n = \pi_n(\bigcap_n \pi^*_n(d_n)) \subseteq d_n$, and
(2) $\forall C \in \mathbb{C}(X).C \subseteq \phi'(\phi(C)) = \bigcap_n \pi^*_n(\pi_n(C))$.

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

We prove that the inclusions in (1) and (2) are indeed equalities. In the case of (1), given $d \in D$, we have (by the Axiom of Choice) $\forall n.\forall x \in d_n.\exists y \in d_{n+1}.x = p_n(y)$, which leads to the construction of a sequence $\hat{y} \in X$, from which we obtain: $\forall n.\forall x \in d_n.\exists \hat{y} \in X.x = \hat{y}_n \wedge (\forall i.\hat{y}_i \in d_i)$. But $\{\hat{y} \in X \mid \forall n.\hat{y}_n \in d_n\}$ is another way of denoting $\bigcap_n \pi_n^*(d_n) \in \mathbb{C}(X)$. Thus, the first inclusion is an equality.

In the case of (2), given an $x \in \phi'(\phi(C))$, i.e., $\forall n.x_n \in \pi_n'(C)$, we prove that $x \in C$. Since C is closed and a base for the topology on X is given by the subsets $[O]_n = \{y \in X \mid y_n \in O\}$ with $O \in \mathsf{O}(X_n)$, it suffices to prove that $\forall n.\forall O \in \mathsf{O}(X_n).x \in [O]_n \implies \exists y \in C.y \in [O]_n$. But, $x_n \in \pi_n(C)$ means that $x_n = y_n$ for some $y \in C$. Hence, $x \in [O]_n \iff x_n \in O \iff y_n \in O \iff y \in [O]_n$. $\square$

3. Main Results

Ideally, given a metric space $\mathbb{S}$ (or more generally, an extended metric space), we would like to find a *compactification* $\overline{\mathbb{S}}$ of $\mathbb{S}$ such that the complete lattice $\mathbb{C}(\overline{\mathbb{S}})$ is ω-continuous. For this, it suffices for the topology on $\overline{\mathbb{S}}$ to be *second-countable*, i.e., to have a countable base. Compactification, however, may not always give us the desired result. For instance, when $\mathbb{S}$ is the sequence space ℓ_∞ (see Section 4.4)—which is not second-countable—it is impossible to obtain an ω-continuous lattice $\mathbb{C}(\overline{\mathbb{S}})$ if $\overline{\mathbb{S}}$ is taken to be any compactification of ℓ_∞.

We establish a weaker result, namely given an ω-chain $(g_n \mid n)$ of short idempotents (with certain additional properties) on a metric space $\mathbb{S}$, we define a compact Hausdorff space $\overline{\mathbb{S}}$ with a countable base and a continuous map $\iota : \mathbb{S} \to \overline{\mathbb{S}}$ such that the monotonic map $\mathbb{C}(\iota){:}\mathbb{C}(\mathbb{S}) \to \mathbb{C}(\overline{\mathbb{S}})$ is continuous when $\mathbb{C}(\mathbb{S})$ is equipped with the Robust topology and $\mathbb{C}(\overline{\mathbb{S}})$ is equipped with the Scott topology. In general, $\overline{\mathbb{S}}$ is not a compactification of $\mathbb{S}$, nor is it *uniquely* determined (up to iso) by $\mathbb{S}$, as it depends on the choice of $(g_n \mid n)$.

The result above follows from Theorem 3.10, which is applicable under more general assumptions than having an ω-chain $(g_n \mid n)$ of short idempotents. Another result, Theorem 3.11, gives sufficient conditions to ensure that $\iota : \mathbb{S} \to \overline{\mathbb{S}}$ is both monic and epic, which is as close as we can get to having that $\overline{\mathbb{S}}$ is a compactification of $\mathbb{S}$.

3.1 Idempotents, sections, and retractions

In this section, we establish some general properties of idempotents, sections (aka split monos), and retractions (aka split epis). All these notions are preserved by functors and reflected by full&faithful functors. The properties of section-retraction pairs that we consider are weaker variants of those considered in Domain Theory for embedding-projection pairs, such as the limit-colimit coincidence, see e.g., Abramsky and Jung (1994, Sec 3.3.2). The notion of embedding-projection pair (and insertion-closure pair) is available only in **Po**-enriched categories, see e.g., Abramsky and Jung (1994, Sec 3.1). Therefore, in the categories (of spaces) that we consider, one can rely only on the more general notion of section-retraction pair.

Definition 3.1. (idempotents & co). *In a category $\mathbb{A}$:*

*(1) g is an **idempotent** on X $\overset{\triangle}{\iff}$ $X \xrightarrow{g} X$ and $g \circ g = g$.*

(2) given two idempotents g_1 and g_2 on X we write $g_1 \leq g_2$ $\overset{\triangle}{\iff}$ $g_1 \circ g_2 = g_1 = g_2 \circ g_1$.

*(3) (e, p) is an **sr-pair** (sr for section-retraction) from X to Y $\overset{\triangle}{\iff}$ $X \underset{e}{\overset{p}{\leftrightarrows}} Y$ and $p \circ e = id_X$.*

*(4) (e, p) is a **splitting** of g $\overset{\triangle}{\iff}$ (e, p) is an sr-pair such that $g = e \circ p$.*

*(5) **idempotents split** $\overset{\triangle}{\iff}$ every idempotent has a splitting.*

We write $\mathbb{I}(X)$ for the poset of idempotents on X and $\mathbb{A}_{sr}$ for the category with sr-pairs as arrows.

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

When $\mathbb{A}$ is **Po**-enriched, there is a partial order $\sqsubseteq$ on the hom-set $\mathbb{A}(X, X)$ of endomorphisms on X, but its restriction to the subset of idempotents on X is unrelated to the partial order $\leq$. However, the two partial orders coincide when restricted to the idempotents below id_X, i.e., the idempotents g such that $g \sqsubseteq \mathrm{id}_X$.

In Category Theory, a definition or result given for a generic category $\mathbb{A}$ can be recast in the dual category $\mathbb{A}^{op}$. In particular, sections are the dual of retractions and idempotents are self-dual.

Proposition 3.2. (Duality). *The following statements hold:*

(1) The definition of $g_1 \leq g_2$ is self-dual, i.e., $g_1 \leq g_2$ in $\mathbb{A} \iff g_1 \leq g_2$ in $\mathbb{A}^{op}$.
(2) (e, p) is an sr-pair from X to Y in $\mathbb{A} \iff (p, e)$ is an sr-pair from X to Y in $\mathbb{A}^{op}$.

The definition of epic (and monic) generalizes from maps (aka arrows) to families of maps.

Definition 3.3. *We say that the family of maps $(h_i{:}X_i \to X \mid i{:}I)$ into X is **jointly epic** in $\mathbb{A} \overset{\triangle}{\iff}$*

$$\forall Y{:}\mathbb{A}.\forall f, f'{:}\mathbb{A}(X, Y), (\forall i.f \circ h_i = f' \circ h_i) \implies f = f'.$$

*Dually, we say that the family of maps $(h_i{:}X \to X_i \mid i \in I)$ from X is **jointly monic** in $\mathbb{A} \overset{\triangle}{\iff}$*

$$\forall Y{:}\mathbb{A}.\forall f, f'{:}\mathbb{A}(Y, X), (\forall i.h_i \circ f = h_i \circ f') \implies f = f'.$$

The family $(h_i \mid i \in I)$ determines X, provided I is not empty. As expected, a family consisting of one map h (i.e., I is a singleton) is jointly epic exactly when h is epic. The following proposition extends well-known properties of epis (namely, $h \circ g$ epic implies h epic, and compositions of epis are epic) to families of maps.

Proposition 3.4. *Given a family of maps $(h_i{:}X_i \to X \mid i \in I)$ into X and for each $i \in I$ a family of maps $(g_{i,j}{:}X_{i,j} \to X_i \mid j \in J_i)$ into X_i, let $(f_{i,j}{:}X_{i,j} \to X \mid i \in I, j \in J_i)$ be the family of maps into X such that $f_{i,j} = h_i \circ g_{i,j}$. The following properties hold:*

(1) if $(h_i \mid i \in I)$ is jointly epic and $(g_{i,j} \mid j \in J_i)$ is jointly epic for each $i \in I$, then $(f_{i,j} \mid i \in I, j \in J_i)$ is jointly epic.
(2) if $(f_{i,j} \mid i \in I, j \in J_i)$ is jointly epic, then $(h_i \mid i \in I)$ is jointly epic.

Moreover, if $(h_i{:}X_i \to X \mid i \in I)$ is a colimit cone from some diagram D in $\mathbb{A}$ to X, then the family of maps $(h_i \mid i \in I)$ into X is jointly epic.

Proposition 3.5. (Basic facts). *In any category $\mathbb{A}$, the following hold:*

(1) the relation $\leq$ is a partial order on $\mathbb{I}(X)$, i.e., $g_1 \leq g_2 \leq g_1 \implies g_1 = g_2$ and $g_1 \leq g_2 \leq g_3 \implies g_1 \leq g_3$, and id_X is the top element in $\mathbb{I}(X)$, i.e., $g \leq id_X$ for g idempotent on X.
(2) if (e, p) is an sr-pair from X to Y, then $g = e \circ p$ is an idempotent on Y.
(3) if $(g_i \mid i \in I)$ is a family in $\mathbb{I}(X)$ which is jointly monic (or jointly epic) in $\mathbb{A}$, then its sup in $\mathbb{I}(X)$ is id_X.
(4) all arrows in $\mathbb{A}_{sr}$ are monic, and (e, p) is an iso in $\mathbb{A}_{sr} \iff p$ is the inverse of e in $\mathbb{A}$.
(5) if $(e_i, p_i){:}X_i \to Y$ is a splitting of the idempotent $g_i{:}\mathbb{I}(Y)$ (for $i = 1, 2$), then $g_1 \leq g_2 \iff$ there exists, necessarily unique, $(e, p){:}X_1 \to X_2$ such that $(e_1, p_1) = (e_2, p_2) \circ (e, p)$.

Proof. The proofs for Items (1) and (2) are straightforward. For the other items, we have:

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

$$\begin{tikzcd} & \underline{X} & \\ X_n \arrow[ur, "\underline{f}_n"] \arrow[rr, "e_n"'] & & X_{n+1} \arrow[ul, "\underline{f}_{n+1}"'] \end{tikzcd} \quad \Longrightarrow \exists! \underline{q} \quad \begin{tikzcd} & \underline{X} & \\ X_n \arrow[ur, "(\underline{f}_n, \underline{q}_n)"] \arrow[rr, "(e_n, p_n)"'] & & X_{n+1} \arrow[ul, "(\underline{f}_{n+1}, \underline{q}_{n+1})"'] \end{tikzcd}$$

colimit in $\mathbb{A}$ cone in $\mathbb{A}_{sr}$

Figure 3. Partial diagrammatic recast of Proposition 3.6.

Item (3). Consider an idempotent g of X which is an upper-bound of $(g_i \mid i)$, i.e., $\forall i.g_i \leq g$. Then, $\forall i.g_i \circ g = g_i$, which implies $g = \mathrm{id}_X$, because $(g_i \mid i)$ is jointly monic.

Item (4). To prove that any arrow (e, p) in $\mathbb{A}_{sr}$ is a mono, it suffices to observe that in $\mathbb{A}$ every section e is monic and every retraction p is epic. If (e', p') is the inverse of (e, p) in $\mathbb{A}_{sr}$, then e' is the inverse of e in $\mathbb{A}$, thus $p = p \circ e \circ e' = e'$, because $p \circ e = \mathrm{id}_X$.

Item (5). Uniqueness of (e, p) is immediate, because (e_2, p_2) is a mono in $\mathbb{A}_{sr}$. For existence, define $e = p_2 \circ e_1$ and $p = p_1 \circ e_2$. Then, from the assumption $e_1 \circ p_1 = g_1 \leq g_2 = e_2 \circ p_2$ we derive:

(i) $p \circ e = \mathrm{id}_{X_1}$ (i.e., (e, p) is an sr-pair), because
$(p_1 \circ e_2) \circ (p_2 \circ e_1) =$ by definition of g_2
$p_1 \circ g_2 \circ e_1 =$ by (e_1, p_1) sr-pair
$p_1 \circ g_2 \circ e_1 \circ (p_1 \circ e_1) =$ by definition of g_1
$p_1 \circ g_2 \circ g_1 \circ e_1 =$ by $g_1 \leq g_2$
$p_1 \circ g_1 \circ e_1 =$ by definition of g_1
$p_1 \circ (e_1 \circ p_1) \circ e_1 =$ by (e_1, p_1) sr-pair
$\mathrm{id}_{X_1} \circ \mathrm{id}_{X_1} = \mathrm{id}_{X_1}$

(ii) $e_2 \circ e = e_1$, because
$e_2 \circ (p_2 \circ e_1) =$ by definition of g_2
$g_2 \circ e_1 =$ by (e_1, p_1) sr-pair
$g_2 \circ e_1 \circ (p_1 \circ e_1) =$ by definition of g_1
$g_2 \circ g_1 \circ e_1 =$ by $g_1 \leq g_2$
$g_1 \circ e_1 =$ by definition of g_1
$(e_1 \circ p_1) \circ e_1 = e_1$ by (e_1, p_1) sr-pair

(iii) $p \circ p_2 = p_1$, dual of the previous proof. □

Proposition 3.6. (ω-colimits of sections). *Given an ω-chain $((e_n, p_n){:}X_n \to X_{n+1} \mid n \in \omega)$ in $\mathbb{A}_{sr}$ and a colimit cone $(\underline{f}_n{:}X_n \to \underline{X} \mid n)$ in $\mathbb{A}$ from the ω-chain $(e_n{:}X_n \to X_{n+1} \mid n)$ to $\underline{X}$, there exists a unique cone $((\underline{f}_n, \underline{q}_n){:}X_n \to \underline{X} \mid n)$ in $\mathbb{A}_{sr}$ from the ω-chain $((e_n, p_n) \mid n)$ to $\underline{X}$ (see Figure 3). If $\underline{g}_n$ is the idempotent $\underline{f}_n \circ \underline{q}_n$ on $\underline{X}$, then $(\underline{g}_n \mid n)$ is an ω-chain in $\mathbb{I}(\underline{X})$ and is jointly epic in $\mathbb{A}$.*

Proof. First, we define the family $(h_{i,j}{:}X_i \to X_j \mid i, j \in \omega)$ of maps in $\mathbb{A}$ (by induction on $|j - i|$):

- $h_{i,i}$ is the identity on X_i
- $h_{i,j} = h_{i+1,j} \circ e_i$ when $i < j$, thus $h_{i,i+1} = e_i$
- $h_{i,j} = p_j \circ h_{i,j+1}$ when $i > j$, thus $h_{i+1,i} = p_i$

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

Second, we prove (by induction on $|j-i|$) the property:

$$\forall i, j. h_{i+1,j} \circ e_i = h_{i,j} = p_j \circ h_{i,j+1}. \tag{1}$$

- case $i=j$: immediate, since $h_{i+1,i} = p_i$ and $h_{i,i+1} = e_i$.
- case $i<j$: $h_{i+1,j} \circ e_i = h_{i,j}$ by definition of $h_{i,j}$. For the other equality:
  (1) $h_{i,j} =$ by definition of $h_{i,j}$
  (2) $h_{i+1,j} \circ e_i =$ by IH on $(i+1, j)$
  (3) $p_j \circ h_{i+1,j+1} \circ e_i =$ by definition of $h_{i,j+1}$
  (4) $p_j \circ h_{i,j+1}$.
- case $i>j$: $h_{i,j} = p_j \circ h_{i,j+1}$ by definition of $h_{i,j}$. For the other equality:
  (1) $h_{i+1,j} \circ e_i =$ by definition of $h_{i+1,j}$
  (2) $p_j \circ h_{i+1,j+1} \circ e_i =$ by IH on $(i, j+1)$
  (3) $p_j \circ h_{i,j+1} =$ by definition of $h_{i,j}$
  (4) $h_{i,j}$.

Property (1) implies that $(h_{i,n} \mid i)$ is a cone in $\mathbb{A}$ from $(e_i \mid i)$ to X_n. Thus, there exists a unique $\underline{q}_n{:}X_n \to \underline{X}$ such that $\forall i. \underline{q}_n \circ \underline{f}_i = h_{i,n}$. In particular, $\underline{q}_n \circ \underline{f}_n = h_{n,n} = \mathrm{id}_{X_n}$, i.e., $(\underline{f}_n, \underline{q}_n){:}X_n \to \underline{X}$ in $\mathbb{A}_{sr}$. Since the colimit cone $(\underline{f}_n \mid n)$ is jointly epic (by Proposition 3.4), property (1) implies also that $\underline{q}_n = p_n \circ \underline{q}_{n+1}$ Thus, $((\underline{f}_n, \underline{q}_n) \mid n)$ is a cone in $\mathbb{A}_{sr}$ from $((e_n, p_n) \mid n)$ to $\underline{X}$.

For uniqueness of $(\underline{q}_n \mid n)$, we use again that $(\underline{f}_n \mid n)$ is jointly epic and prove (by induction on $|j-i|$) that $\forall i, j. q'_j \circ \underline{f}_i = h_{i,j}$ when $(q'_n \mid n)$ is a cone from $\underline{X}$ to $(p_n \mid n)$ such that $\forall n. q'_n \circ \underline{f}_n = \mathrm{id}_{X_n}$.

- case $i=j$: immediate, by $q'_i \circ \underline{f}_i = \mathrm{id}_{X_i} = h_{i,i}$, assumption on $(q'_n \mid n)$ and definition of $h_{i,i}$.
- case $i<j$:
  (1) $q'_j \circ \underline{f}_i =$ by definition of $(\underline{f}_n \mid n)$
  (2) $q'_j \circ \underline{f}_{i+1} \circ e_i =$ by IH on $(i+1, j)$
  (3) $h_{i+1,j} \circ e_i = h_{i,j}$ by definition of $h_{i,j}$.
- case $i>j$:
  (1) $q'_j \circ \underline{f}_i =$ by assumption on $(q'_n \mid n)$
  (2) $p_j \circ q'_{j+1} \circ \underline{f}_i =$ by IH on $(i, j+1)$
  (3) $p_j \circ h_{i,j+1} = h_{i,j}$ by definition of $h_{i,j}$.

Consider the idempotents $\underline{g}_n = \underline{f}_n \circ \underline{q}_n$ on $\underline{X}$. From item (5) of Proposition 3.5, it follows that $\forall n. \underline{g}_n \leq \underline{g}_{n+1}$. Moreover, by Proposition 3.4, the family $(\underline{g}_n \mid n)$ is jointly epic, because it is the composition of jointly epic families, namely the colimit cone $(\underline{f}_n \mid n)$ and the singleton families consisting of the (split) epic $\underline{q}_n$. □

The following is the dual of Proposition 3.6:

Corollary 3.7. (ω^{op}-colimits of retractions). *Given an ω-chain $((e_n, p_n){:}X_n \to X_{n+1} \mid n \in \omega)$, in $\mathbb{A}_{sr}$ and a limit cone $(\overline{q}_n{:}\overline{X} \to X_n \mid n)$ in $\mathbb{A}$ from $\overline{X}$ to the ω^{op}-chain $(p_n{:}X_{n+1} \to X_n \mid n)$, there exists a unique cone $((\overline{f}_n, \overline{q}_n){:}X_n \to \overline{X} \mid n)$ in $\mathbb{A}_{sr}$ from the ω-chain $((e_n, p_n) \mid n)$ to $\overline{X}$. If $\overline{g}_n$ is the idempotent $\overline{f}_n \circ \overline{q}_n$ on $\overline{X}$, then $(\overline{g}_n \mid n)$ is an ω-chain in $\mathbb{I}(\overline{X})$ and is jointly monic in $\mathbb{A}$.*

Theorem 3.9 on the next page implies existence and uniqueness of $\iota{:}\underline{X} \to \overline{X}$ in $\mathbb{A}$ such that $\forall n. \iota \circ \underline{f}_n = \overline{f}_n$ and $\forall n. \underline{q}_n = \overline{q}_n \circ \iota$, where $((\underline{f}_n, \underline{q}_n){:}X_n \to \underline{X} \mid n)$ and $((\overline{f}_n, \overline{q}_n){:}X_n \to \overline{X} \mid n)$ are the cones in

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

$\mathbb{A}_{sr}$ given by Proposition 3.6 and Corollary 3.7. In general, there is no reason for ι to be monic, epic, or iso, as demonstrated in the examples below.

Example 3.8. We consider some examples of ω-chains $((e_n, p_n){:}X_n \to X_{n+1} \mid n \in \omega)$ in $\mathbb{A}_{sr}$ for several categories $\mathbb{A}$, in which the colimit cone of Proposition 3.6 and the limit cone of Corollary 3.7 exist. For each example, we give an explicit description of the colimit object $\underline{X}$ and the limit object $\overline{X}$, and the properties of the unique map $\iota{:}\underline{X} \to \overline{X}$ in $\mathbb{A}$.

- We start with the category **CL**. In fact, the example is within the full sub-category ω**CL**, whose objects are the retracts of $\mathbb{P}(\omega)$. Since **CL** is poset-enriched, one can ask whether a section is an embedding. Let X_n be the finite ordinal $n+1$ and e_n be the inclusion of X_n into X_{n+1}. There is only one possible retraction p_n for e_n (because of monotonicity), which maps the largest element of $n+2$—i.e., $n+1$—to the largest element of $n+1$—i.e., n—and is the identity otherwise. Moreover, (e_n, p_n) is an embedding-projection pair. Since in **CL** the *limit-colimit coincidence* for ω-chains of embedding-projection pairs holds, we have $\underline{X} = \overline{X} = \omega + 1$ and ι is an iso.
- Since the inclusion functor from **CL** to **Po** is poset-enriched, the ω-chain of embedding-projection pairs defined in the previous example is also an ω-chain of embedding-projection pairs in **Po**. However, the *limit-colimit coincidence* does not hold in **Po**. In particular, in **Po** we have $\underline{X} = \omega$, $\overline{X} = \omega + 1$, and ι is the inclusion of ω into $\omega + 1$. Therefore, ι is monic, but it is not epic (nor a section).
- Since the forgetful functor from **Po** to **Set** is not poset-enriched, an ω-chain of embedding-projection pairs in **Po** becomes an ω-chain of section-retraction pairs. However, the forgetful functor preserves limits and colimits. Thus, in **Set**, we have $\underline{X} = \omega$, $\overline{X} = \omega + 1$, and ι is the inclusion of ω into $\omega + 1$. In **Set**, the map ι is monic (actually a section), but it is not epic.
- The ω-chain of section-retraction pairs defined in **Set** can be viewed in the dual category **Set**^{op}, by swapping the components of a pair. Therefore, ι in **Set**^{op} is epic (actually a retraction), but it is not monic.
- The previous two examples can be combined in the product category **Set** $\times$ **Set**^{op}. In this case, $\underline{X}$ is the pair $(\omega, \omega + 1)$, $\overline{X}$ is the pair $(\omega + 1, \omega)$, and ι is neither epic nor monic.
- The full&faithful functor D from **Set** to **Haus** (see Figure 1) preserve colimits. Thus, for the image of the ω-chain of section-retraction pairs in **Set**, we have $\underline{X} = D\omega$, $\overline{X}$ is the one-point (aka Alexandroff) compactification of $D\omega$, whose carrier is $\omega + 1$, and ι is the inclusion of ω into $\omega + 1$. In **Haus**, the map ι is both monic and epic, since $\underline{X}$ is a dense sub-space of $\overline{X}$.

If we start from an ω-chain $(g_n \mid n)$ in $\mathbb{I}(X)$, where X is an object of interest in $\mathbb{A}$, then we can get, by splitting the idempotents, an ω-chain in $\mathbb{A}_{sr}$ which is unique up to iso. From this ω-chain, there are three cones in $\mathbb{A}_{sr}$, whose vertices are $\underline{X}$, X and $\overline{X}$, respectively. The following theorem states that these cones are related by unique maps $\underline{\iota}{:}\underline{X} \to X$ and $\overline{\iota}{:}X \to \overline{X}$ in $\mathbb{A}$. There is also an *inverse* (which we do not prove), where we start from an ω-chain $((e_n, p_n) \mid n)$ in $\mathbb{A}_{sr}$, and consider the unique map $\iota{:}\underline{X} \to \overline{X}$ in $\mathbb{A}$, relating the cones in $\mathbb{A}_{sr}$ from the ω-chain to $\underline{X}$ and $\overline{X}$. In this case, any factorization $(\underline{\iota}, \overline{\iota})$ of ι through an object X, induces a ω-chain $(g_n \mid n)$ of idempotents in $\mathbb{I}(X)$, where $g_n = \underline{\iota} \circ \underline{f}_n \circ \overline{q}_n \circ \overline{\iota}$.

Theorem 3.9. *If $(g_n \mid n)$ is an ω-chain in $\mathbb{I}(X)$ and every g_n has a splitting $(f_n, q_n){:}X_n \to X$, then there exists a unique ω-chain $((e_n, p_n){:}X_n \to X_{n+1} \mid n)$ in $\mathbb{A}_{sr}$ such that $((f_n, q_n){:}X_n \to X \mid n)$ is a cone in $\mathbb{A}_{sr}$ from $((e_n, p_n){:}X_n \to X_{n+1} \mid n)$ to X. Moreover:*

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

(1) *If* $(\underline{f}_n{:}X_n \to \underline{X} \mid n)$ *is a colimit cone in* $\mathbb{A}$ *from the* ω*-chain* $(e_n{:}X_n \to X_{n+1} \mid n)$ *to* $\underline{X}$ *and* $\underline{\iota}{:}\underline{X} \to X$ *is the unique map such that* $\forall n.f_n = \underline{\iota} \circ \underline{f}_n$, *then* $\forall n.q_n \circ \underline{\iota} = \underline{q}_n$ *(see Proposition 3.6 for* $\underline{q}_n$*).*

(2) *If* $(\overline{q}_n{:}\overline{X} \to X_n \mid n)$ *is a limit cone in* $\mathbb{A}$ *from* $\overline{X}$ *to the* ω^{op}*-chain* $(p_n{:}X_{n+1} \to X_n \mid n)$ *and* $\overline{\iota}{:}X \to \overline{X}$ *is the unique map such that* $\forall n.q_n = \overline{q}_n \circ \overline{\iota}$, *then* $\forall n.\overline{\iota} \circ f_n = \overline{f}_n$ *(see Corollary 3.7 for* $\overline{f}_n$*).*

Proof. By Item (5) of Proposition 3.5, we get that for each $n \in \omega$ there exists a unique sr-pair (e_n, p_n) such that $(f_{n+1}, q_{n+1}) \circ (e_n, p_n) = (f_n, q_n)$, which implies that $((f_n, q_n){:}X_n \to X \mid n)$ is a cone in $\mathbb{A}_{sr}$ from the ω-chain $((e_n, p_n){:}X_n \to X_{n+1} \mid n)$ to X.

Item (1). We rely on the proof of Proposition 3.6, where $\underline{q}_n$ is defined. Since $(\underline{f}_i \mid i)$ is jointly epic, $\forall j.q_j \circ \underline{\iota} = \underline{q}_j$ follows from $\forall i, j.q_j \circ \underline{\iota} \circ \underline{f}_i = \underline{q}_j \circ \underline{f}_i$, or equivalently, from $\forall i, j.q_j \circ f_i = h_{i,j}$, which we prove by induction on $|j - i|$:

- case $i = j$: immediate, since $q_i \circ f_i = \mathrm{id}_{X_i} = h_{i,i}$, because (f_i, q_i) is a map in $\mathbb{A}_{sr}$ and by definition of $h_{i,i}$.
- case $i < j$:
  (a) $q_j \circ f_i =$ because $(f_{i+1}, q_{i+1}) \circ (e_i, p_i) = (f_i, q_i)$ in $\mathbb{A}_{sr}$
  (b) $q_j \circ f_{i+1} \circ e_i =$ by IH on $(i+1, j)$
  (c) $h_{i+1,j} \circ e_i = h_{i,j}$ by definition of $h_{i,j}$.
- case $i > j$:
  (a) $q_j \circ f_i =$ because $(f_{i+1}, q_{i+1}) \circ (e_i, p_i) = (f_i, q_i)$ in $\mathbb{A}_{sr}$
  (b) $p_j \circ q_{j+1} \circ f_i =$ by IH on $(i, j+1)$
  (c) $p_j \circ h_{i,j+1} = h_{i,j}$ by definition of $h_{i,j}$.

Item (2). By duality, since it is the dual of Item (1). □

3.2 Extended metric spaces versus compact Hausdorff spaces

The following result requires moving between four categories (and we have added also **Set**) using four functors (where $\longrightarrow$ denotes a faithful functor and $\hookrightarrow$ a full&faithful functor):[2]

$$\begin{tikzcd}
\mathbf{KMS} \arrow[r, hook] \arrow[d, dashed, "U"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Met} \arrow[d, "U"] \arrow[dr, dashed, "U"] & \\
\mathbf{KH} \arrow[r, hook] & \mathbf{Haus} \arrow[r, "U"] & \mathbf{Set}
\end{tikzcd}$$

All the categories in the above diagram have:

- finite limits and finite sums (Theorem 2.3);
- enough points, more precisely, the faithful forgetful functor U from $\mathbb{A}$ into **Set** is (isomorphic) to the global section functor $\mathbb{A}(1, -)$, where $\mathbb{A}$ is any of the other four categories (**KMS**, **Met**, **KH** or **Haus**), and 1 is the terminal object in $\mathbb{A}$;
- splittings of idempotents, because each of the five categories has equalizers.

Moreover, all functors in the diagrams preserve finite limits and finite sums; **Haus** has all small limits and small colimits (Theorem 2.4), where limits are computed as in **Top**, and limits (computed in **Haus**) of diagrams in **KH** are in **KH**.

In applications, we start from a metric space $\mathbb{S}$, then we identify an ω-chain $(g_n \mid n)$ of idempotents on $\mathbb{S}$ in **Met**, and by applying Theorem 3.9 in **Haus**, we get a map $\overline{\iota}{:}\mathbb{S} \to \overline{\mathbb{S}}$ in **Haus**. The theorems below provide sufficient conditions to ensure that $\overline{\mathbb{S}}$ is compact, $\overline{\iota}$ is monic and

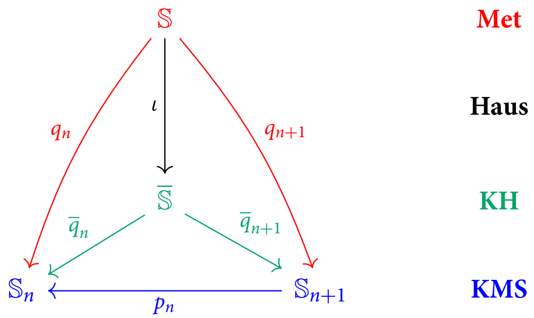

Figure 4. Partial diagrammatic recast of Theorem 3.10.

epic, and, above all, that the complete lattice $\mathbb{C}(\overline{\mathbb{S}})$ is ω-continuous and the monotonic map $\mathbb{C}(\bar{\iota})$ is in $\mathbb{A}_{RS}(\mathbb{S}, \overline{\mathbb{S}})$. These properties can be proved for maps $\iota : \mathbb{S} \to \overline{\mathbb{S}}$ that are not necessarily obtained through Theorem 3.9, and the theorems below capture this greater generality. If we start from an ω-chain $(g_n \mid n)$ of idempotents on $\mathbb{S}$ in **Met**, then Theorem 3.9 provides candidates for $(p_n \mid n)$ and $(q_n \mid n)$ in the following theorem, because **Met** has splittings of idempotents.

Theorem 3.10. *If $(p_n{:}\mathbb{S}_{n+1} \to \mathbb{S}_n \mid n)$ is an ω^{op}-chain in* **KMS**, *$(q_n \mid n)$ is a cone from $\mathbb{S}$ to $(p_n \mid n)$ in* **Met**, *$(\overline{q}_n{:}\overline{\mathbb{S}} \to \mathbb{S}_n \mid n)$ is a limit cone from $\overline{\mathbb{S}}$ to $(p_n \mid n)$ in* **KH** *(and* **Haus***), and $\iota : \mathbb{S} \to \overline{\mathbb{S}}$ is the unique map in* **Haus** *such that $q_n = \overline{q}_n \circ \iota$ (see Figure 4), then:*

(1) $\overline{\mathbb{S}}$ is compact and has a countable base. Thus, $\mathbb{C}(\overline{\mathbb{S}})$ is ω-continuous.
(2) The monotonic map $\mathbb{C}(\iota)$ is in $\mathbb{A}_{RS}(\mathbb{S}, \overline{\mathbb{S}})$.

Proof. The Hausdorff space $\overline{\mathbb{S}}$ can be identified with the set $\{s \in \prod_n |\mathbb{S}_n| \mid \forall n.s_n = p_n(s_{n+1})\}$, equipped with the coarsest topology $\mathsf{O}(\overline{\mathbb{S}})$ making the maps $\overline{q}_n(s) = s_n$ continuous, i.e., the topology generated by the sub-base $[O]_n = \{s \in |\overline{\mathbb{S}}| \mid \overline{q}_n(s) \in O\}$, where O is an open set in $\mathbb{S}_n$.

By Theorem 2.4, the limit $\overline{\mathbb{S}}$ is in **KH**, because it is the limit (in **Haus**) of a diagram in **KH**. The topology on $\mathbb{S}_n$ has a countable base τ_n^b, because $\mathbb{S}_n$ is in **KMS**. Thus, the topology on $\overline{\mathbb{S}}$ has a countable sub-base too, namely the set of open subsets of the form $[B]_n$ with $B \in \tau_n^b$.

By Theorem 2.27, the complete lattice $\mathbb{C}(X)$ is in **CL** and the topologies $\tau_S(X)$ and $\tau_U(X)$ coincide, when X:**KH**, as in the case of $\mathbb{S}_n$ and $\overline{\mathbb{S}}$. Therefore, $\tau_S(X)$ is generated by $\uparrow O \stackrel{\Delta}{=} \{K \in \mathsf{C}(X) \mid K \subseteq O\}$ with $O \in \mathsf{O}(X)$, and the way-below relation is given by $K_1 \ll K_2 \iff K_2 \subseteq K_1{}^\circ$, where $K_1{}^\circ$ is the interior of K_1.

By Theorem 2.32, the continuous lattice $\mathbb{C}(\overline{\mathbb{S}})$ is (isomorphic to) the limit of the ω^{op}-chain of right adjoints $(\mathbb{C}(p_n) \mid n)$ in **CL** (and in $\mathbf{Po}_A$). To be more precise, the iso is $K \mapsto (\overline{q}_n(K) \mid n)$ with inverse $(K_n \mid n) \mapsto \bigcap_n \overline{q}_n^*(K_n)$.

The sub-base of $\mathsf{O}(\overline{\mathbb{S}})$ given above, i.e., the set of $[O]_n = \{s \in |\overline{\mathbb{S}}| \mid \overline{q}_n(s) \in O\}$, where O is an open set in $\mathbb{S}_n$, is actually a base, because $[O]_n = [p_n^*(O)]_{n+1}$. Therefore, every $O \in \mathsf{O}(\overline{\mathbb{S}})$ is of the form $\bigcup_{i \in I} [O_i]_{n_i}$ with $O_i \in \mathsf{O}(\mathbb{S}_{n_i})$ for $i \in I$. Since $\overline{\mathbb{S}}$ is compact, also $K \in \mathsf{C}(\overline{\mathbb{S}})$ is compact, and $K \subseteq O$ implies $K \subseteq \bigcup_{i \in J} [O_i]_{n_i}$ for some $J \subseteq_f I$. In particular, $O \supseteq \bigcup_{i \in J} [O_i]_{n_i} = [O_J]_{n_J}$, where $n_J = \sup_{i \in J} n_i$ and $O_J \in \mathsf{O}(\mathbb{S}_{n_J})$ is the union for $i \in J$ of the O_i moved from $\mathbb{S}_{n_i}$ to $\mathbb{S}_{n_J}$. Therefore:

$$\forall K \in \mathsf{C}(\overline{\mathbb{S}}).\forall O \in \mathsf{O}(\overline{\mathbb{S}}).K \subseteq O \iff \exists n.\exists O_n \in \mathsf{O}(\mathbb{S}_n).\overline{q}_n(K) \subseteq O_n \wedge [O_n]_n \subseteq O. \quad (2)$$

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

Using the above property, and δ-fattening (see Definition 2.5), we have:

- $\mathbb{C}(\iota){:}\mathbb{A}_{RS}(\mathbb{S},\overline{\mathbb{S}})$ means $\forall C \in \mathsf{C}(\mathbb{S}).\forall O \in \mathsf{O}(\overline{\mathbb{S}}).\overline{\iota(C)} \subseteq O \implies \exists \delta > 0.\overline{\iota(C_\delta)} \subseteq O$, in which C_δ is as defined in item (3) of Definition 2.5.
- By property (2), this is implied by

$$\forall C \in \mathsf{C}(\mathbb{S}).\forall n.\forall O \in \mathsf{O}(\mathbb{S}_n).\overline{\iota(C)} \subseteq [O]_n \implies \exists \delta > 0.\overline{\iota(C_\delta)} \subseteq [O]_n$$

- which is equivalent to

$$\forall C \in \mathsf{C}(\mathbb{S}).\forall n.\forall O \in \mathsf{O}(\mathbb{S}_n).\overline{q_n(C)} \subseteq O \implies \exists \delta > 0.\overline{q_n(C_\delta)} \subseteq O$$

  because $\overline{\iota(C)} \subseteq [O]_n \iff \overline{q}_n(\overline{\iota(C)}) \subseteq O$, by definition of $[O]_n$, and $\overline{q}_n(\overline{\iota(C)}) = \overline{\overline{q}_n(\iota(C))} = \overline{q_n(C)}$, by $q_n = \overline{q}_n \circ \iota$ and compactness of $\overline{\iota(C)}$.

Since $q_n{:}\mathbf{Met}(\mathbb{S},\mathbb{S}_n)$ and $\mathbb{S}_n{:}\mathbf{KMS}$, by applying Proposition 2.7 to q_n and Proposition 2.8 to $\mathbb{S}_n$, we get the chain of implications $\overline{q_n(C)} \subseteq O \implies \exists \delta > 0.\overline{q_n(C)}_\delta \subseteq O \implies \exists \delta > 0.\overline{q_n(C_\delta)} \subseteq O$ for $C \in \mathsf{C}(\mathbb{S})$ and $O \in \mathsf{O}(\mathbb{S}_n)$. □

If we start from an ω-chain $(g_n \mid n)$ of idempotents on $\mathbb{S}$ in **Met**, rather than in **Haus**, then $((f_n, q_n) \mid n)$ and $((e_n, p_n) \mid n)$ in the following theorem consist of short maps. Moreover, families of maps that are jointly monic in **Met** are also jointly monic in **Haus**, because these categories have enough points. Finally, by Proposition 3.5, if an ω-chain $(g_n \mid n)$ of idempotents on $\mathbb{S}$ is jointly monic, then the identity on $\mathbb{S}$ is the sup of the ω-chain in the poset of idempotents on $\mathbb{S}$.

Theorem 3.11. *Given an ω-chain $(g_n \mid n)$ of idempotents on $\mathbb{S}$ in* **Haus**, *which is jointly monic, consider:*

- *a splitting* $\mathbb{S} \xrightarrow{q_n} \mathbb{S}_n \xhookrightarrow{f_n} \mathbb{S}$ *of* g_n;
- *the unique ω-chain $((e_n, p_n){:}\mathbb{S}_n \to \mathbb{S}_{n+1} \mid n)$ in* $\mathbf{Haus}_{sr}$ *such that $((f_n, q_n){:}\mathbb{S}_n \to \mathbb{S} \mid n)$ is a cone in* $\mathbf{Haus}_{sr}$ *from $((e_n, p_n){:}\mathbb{S}_n \to \mathbb{S}_{n+1} \mid n)$ to $\mathbb{S}$ (see Theorem 3.9);*
- *the limit cone $(\overline{q}_n{:}\overline{\mathbb{S}} \to \mathbb{S}_n \mid n)$ in* **Haus** *from $\overline{\mathbb{S}}$ to the ω^{op}-chain $(p_n{:}\mathbb{S}_{n+1} \to \mathbb{S}_n \mid n)$;*
- *the unique map $\iota : \mathbb{S} \to \overline{\mathbb{S}}$ in* **Haus** *such that $\forall n.q_n = \overline{q}_n \circ \iota$.*

Then, ι is both monic and epic in **Haus**.

Proof. If $(g_n \mid n)$ is jointly monic, then also $(q_n \mid n)$ is jointly monic, by (the dual of) Proposition 3.4, and ι is monic, for the same reason.

If $((\overline{f}_n, \overline{q}_n){:}\mathbb{S}_n \to \overline{\mathbb{S}} \mid n)$ is the unique cone in $\mathbf{Haus}_{sr}$ given by Corollary 3.7, then $\forall n.\iota \circ f_n = \overline{f}_n$, by Theorem 3.9. Therefore, to prove that ι is epic it suffices, by Proposition 3.4, to prove that $(\overline{f}_n \mid n)$ is jointly epic in **Haus**. This amounts to proving that the union of the images of the maps in $(\overline{f}_n \mid n)$ is dense in $\overline{\mathbb{S}}$. To do this we use the base of $\mathsf{O}(\overline{\mathbb{S}})$ as in the proof of Theorem 3.10. For every $s \in \overline{\mathbb{S}}$ and $[O]_n$ in the base (i.e., $O \in \mathsf{O}(\mathbb{S}_n)$) such that $s \in [O]_n$ (i.e., $\overline{q}_n(s) \in O$), we have to give an $s_n \in \mathbb{S}_n$ such that $\overline{f}_n(s_n) \in [O]_n$. It suffice to take $s_n = \overline{q}_n(s)$, since $s_n = \overline{q}_n(\overline{f}_n(s_n))$. □

4. Examples

In this section, we consider examples of Banach spaces $\mathbb{S}$, demonstrating how to apply the results of Section 3 to define a specific $\overline{\mathbb{S}}$ in **KH**, and in which cases $\overline{\mathbb{S}}$ is a compactification of $\mathbb{S}$ (a summary is given at the end of this section, see Figure 5). In Section 5, we will study loss of precision when going from $\mathbb{S}$ to $\overline{\mathbb{S}}$. All examples considered in this section are *sequence spaces*. Hence, we recall some general definitions and fix notation.

Definition 4.1. (Uniform notation). *We write $\mathbb{R}$ for the standard Banach space on the reals, and also for the underlying vector space, metric space, topological space, and set.*

- *Given a set I, we write $\mathbb{R}^I$ for the product of I copies of the set $\mathbb{R}$, which is also the carrier of the product of I copies of $\mathbb{R}$ in the categories of the vector spaces, extended metric spaces and Hausdorff spaces.*
- *Given a real number p in the interval $[1,\infty)$, we write $\| - \|_{I,p}$ for the map from $\mathbb{R}^I$ to $[0,\infty]$ given by $\| x \|_{I,p} \stackrel{\triangle}{=} (\sum_{i\in I} | x_i |^p)^{1/p}$, and the notation is extended to $p=\infty$ by defining $\| x \|_{I,\infty} \stackrel{\triangle}{=} \sup_{i\in I} | x_i |$. Since I is determined by x, we drop the subscript I and write $\| x \|_p$.*
- *We write $\ell_{I,p}$ for the Banach space with carrier the sub-space $\{x \in \mathbb{R}^I \mid \| x \|_p < \infty\}$ of (the vector space) $\mathbb{R}^I$ and norm $\| - \|_{I,p}$. We write $B_{I,p}$ for the closed unit ball in $\ell_{I,p}$, whose elements are those x such that $\| x \|_p \leq 1$. The subset $B_{I,p}$ inherits from $\ell_{I,p}$ the metric.*
- *If $I \subseteq J$, then $\ell_{I,p}$ is isomorphic (in the category of Banach spaces and short linear maps) to the sub-space of $\ell_{J,p}$ with carrier $\{x \mid \forall j \in J \setminus I. x_j = 0\}$, and $B_{I,p}$ is a sub-space of $B_{J,p}$ (modulo this iso).*

We consider only countable I, specifically, either ω or a natural number m. We write ℓ_p for $\ell_{\omega,p}$ and $\ell_{*,p}$ for the (normed vector) sub-space of ℓ_p with carrier $\{x \mid \exists n.\forall i > n. x_i = 0\}$. We write B_p for $B_{\omega,p}$, and $B_{*,p}$ for $B_p \cap \ell_{*,p}$. Note that $\ell_{0,p}$ is trivial and $\ell_{1,p} = \mathbb{R}$ for every p.

In the sequel, we use the following characterization of limits in **Top** and general properties of limits and colimits (valid in any category).

Proposition 4.2. *Given a small diagram $D{:}I \to$ **Top**, a limit cone $(\pi_i{:}(X,\tau) \to D_i \mid i \in I)$ in* **Top** *is obtained by taking a limit cone $(\pi_i{:}X \to U(D_i) \mid i \in I)$ of $U \circ D{:}I \to$ **Set** in **Set**, and by defining τ as the coarsest topology on X making the maps $\pi_i{:}(X,\tau) \to D_i$ continuous.*

Proposition 4.3. (Limits commute with Limits). *Given an $I \times J$-diagram $D{:}I \times J \to \mathbb{A}$ in a category $\mathbb{A}$ (with the relevant limits), if for each $i \in I$, $(p^i_j{:}X_i \to D_{i,j} \mid j \in J)$ is a limit cone for the J-diagram $D(i,-){:}J \to \mathbb{A}$, then the family $(X_i \mid i \in I)$ extends* canonically *to an I-diagram $X{:}I \to \mathbb{A}$, namely, for $f{:}i \to i'$ in I, the map $X_f{:}X_i \to X_{i'}$ is the unique map in $\mathbb{A}$ such that for all $j \in J$, the following diagram commutes:*

$$\begin{tikzcd} X_i \arrow[r, dashed, "X_f"] \arrow[d, "p^i_j"'] & X_{i'} \arrow[d, "p^{i'}_j"] \\ D_{i,j} \arrow[r, "D_{f,j}"] & D_{i',j} \end{tikzcd}$$

Moreover, if $(p_i{:}x \to X_i \mid i \in I)$ is a limit cone for $X{:}I \to \mathbb{A}$, then $(p^i_j \circ p_i{:}x \to D_{i,j} \mid i \in I, j \in J)$ is a limit cone for D. Since one can exchange the role of I and J, there are two alternative ways of computing limits of $I \times J$-diagrams, which necessarily produce canonically isomorphic results.

Proposition 4.4. (Colimits of cofinal diagrams). *Given an ω-diagram $D{:}\omega \to \mathbb{A}$ in a category $\mathbb{A}$ (with the relevant colimits), if $(f_n{:}D_n \to X \mid n \in \omega)$ is a colimit cone for D and $h{:}\omega \to \omega$ is a strictly increasing map, then $(f_{h(n)}{:}D_{h(n)} \to X \mid n \in \omega)$ is a colimit cone for the ω-diagram $D \circ h{:}\omega \to \mathbb{A}$.*

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

4.1 Banach space $\mathbb{R}$

Consider the metric space $\mathbb{R}$ with metric $d(x, y) = |\, x - y\,|$ and the ω-chain $(r_n \mid n \in \omega)$ such that

$$r_n(x) \triangleq \begin{cases} n, & \text{if } n < x, \\ x, & \text{if } |\, x\,| \leq n, \\ -n, & \text{if } x < -n. \end{cases} \tag{3}$$

Each r_n is idempotent and short, because:

$$d(r_n(x), r_n(y)) = \left\{ \begin{array}{c|c|c||c} x < -n & -n \leq x \leq n & n < x & \\ \hline\hline 0 & n + x & 2n & y < -n \\ \hline n + y & d(x, y) & n - y & -n \leq y \leq n \\ \hline 2n & n - x & 0 & n < y \end{array} \right\} \leq d(x, y).$$

The image of r_n is the compact sub-space $\mathbb{R}_n \triangleq [-n, n]$, and the union $\mathbb{S}_*$ of the $\mathbb{R}_n$ is $\mathbb{R}$.

Let (f_n, q_n) be the splitting of r_n through $\mathbb{R}_n$ in **Met** and $p_n = q_n \circ f_{n+1}{:}\mathbb{R}_{n+1} \to \mathbb{R}_n$. Let $(\overline{q}_n \mid n)$ be the limit cone from $\overline{\mathbb{S}}$ to the ω^{op}-chain $(p_n \mid n)$ in **Haus**. Then, by Theorem 3.10, $\overline{\mathbb{S}}$ is compact, and by Theorem 3.11, the map $\iota : \mathbb{R} \to \overline{\mathbb{S}}$ is both epic and monic in **Haus**.

We show that $(\overline{q}_n \mid n)$ is isomorphic to the cone $(\hat{q}_n \mid n)$ from $\overline{\mathbb{R}} = [-\infty, +\infty]$ (the two-point compactification of $\mathbb{R}$ defined in Example 2.2) to $(p_n \mid n)$, where $\hat{q}_n$ is the extension of q_n to $\overline{\mathbb{R}}$ mapping $-\infty$ to $-n$ and $+\infty$ to $+n$. Let $\phi{:}\overline{\mathbb{R}} \to \overline{\mathbb{S}}$ be the unique map such that $\forall n.\hat{q}_n = \overline{q}_n \circ \phi$, namely $\phi(x) \triangleq (\hat{q}_n(x) \mid n)$. The map ϕ is a bijection (in **Set**), since the elements $(s_n \mid n)$ in $\overline{\mathbb{S}}$ satisfy one of the following disjoint properties:

- $\forall n.s_n = -n$, i.e., $(s_n \mid n) = \phi(-\infty)$;
- $(s_n \mid n)$ is eventually constant. This happens when $|\, s_m\,| < m$ for some $m \in \omega$. In this case $(s_n \mid n) = \phi(s_m)$;
- $\forall n.s_n = +n$, i.e., $(s_n \mid n) = \phi(+\infty)$.

Therefore, $(\hat{q}_n \mid n)$ is a limit cone from $\overline{\mathbb{R}}$ to $(p_n \mid n)$ in **Set**. To prove that ϕ is an iso in **Top**, it suffices to show that the topology on $\overline{\mathbb{R}}$ is the coarsest topology making the maps $\hat{q}_n$ continuous (Proposition 4.2). Since a base for the topology on $\overline{\mathbb{R}}$ consists of the subsets of the form $[-\infty, x)$, (x, y) and $(y, +\infty]$ for $x, y \in \mathbb{R}$, it suffices to show that every element in the base is of the form $\hat{q}_n^{-1}(O)$ for some $n \in \omega$ and open subset $O \in \mathsf{O}(\mathbb{R}_n)$. This is immediate by taking n such that $|\, x\,|, |\, y\,| < n$, and taking O of the form $[-n, x)$, (x, y) and $(y, +n]$, respectively.

4.2 Banach spaces $\ell_{m,\infty}$ for $1 < m$

Fix a natural number $m > 1$, consider the metric space $\mathbb{S} = \ell_{m,\infty}$ with metric $d_\infty(x, y) = \max_{i \in m} d(x_i, y_i)$, which coincides with the product $\mathbb{R}^m$ in **Met**, and the ω-chain $(g_n \mid n)$, where

$$\forall x \in \mathbb{R}^m.\ g_n(x) \triangleq (r_n(x_i) \mid i \in m), \tag{4}$$

where r_n is as defined in (3). Since g_n is defined pointwise, it is idempotent and short by *inheritance*, since $d_\infty(g_n(x), g_n(y)) = \max_{i \in m} d(r_n(x_i), r_n(y_i)) \leq \max_{i \in m} d(x_i, y_i) = d_\infty(x, y)$. The image of g_n is the compact sub-space $\mathbb{S}_n \triangleq \mathbb{R}_n^m$ and, once again, the union $\mathbb{S}_*$ of the $\mathbb{S}_n$ is $\mathbb{S}$.

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

From Section 4.1 and Proposition 4.3, we have that $\overline{\mathbb{S}}$ is isomorphic to $\overline{\mathbb{R}}^m$ in **KH**. In fact, **KH** has all small limits. Thus, we can take $I = m$ and $J = \omega^{op}$, and consider the $I \times J$-diagram $D{:}I \times J \to \mathbf{KH}$ such that $D(i, n) = \mathbb{R}_n$ and $D(i, n+1 \to n)$ is the map $p_n{:}\mathbb{R}_{n+1} \to \mathbb{R}_n$ defined in Section 4.1. The limit $\overline{\mathbb{S}}$ is obtained by first computing the limits $\mathbb{R}_n^m$ of I-diagrams $D(-, n)$ and then the J-limit, while $\overline{\mathbb{R}}^m$ is obtained by first computing the limits $\overline{\mathbb{R}}$ of the J-diagrams $D(i, -)$ and then the I-limit.

4.3 Banach spaces $\ell_{m,p}$ for $1 < m$ and $1 \le p < \infty$

This is a modification of Section 4.2, where we consider the metric space $\mathbb{S} = \ell_{m,p}$ that has the carrier of $\ell_{m,\infty} = \mathbb{R}^m$, but with metric

$$d_p(x, y) \triangleq \left(\sum_{i\in m} d(x_i, y_i)^p\right)^{1/p} \ge d_\infty(x, y).$$

We take the same g_n used for $\ell_{m,\infty}$, as defined in (4). Clearly g_n is idempotent, since this property does not depend on the metric, and is short also with respect to d_p (again by inheritance), since

$$d_p(g_n(x), g_n(y)) = \left(\sum_{i\in m} d(r_n(x_i), r_n(y_i))^p\right)^{1/p} \le \left(\sum_{i\in m} d(x_i, y_i)^p\right)^{1/p} = d_p(x, y).$$

Thus, the metric space $\mathbb{S}_n$ has the same carrier of $\mathbb{R}_n^m$ and metric d_p. Since d_p and d_∞ induce the same topology on $\mathbb{R}_n^m$, the spaces $\overline{\mathbb{S}}$ for $\ell_{m,p}$ and for $\ell_{m,\infty}$ are equal and isomorphic to $\overline{\mathbb{R}}^m$ in **KH**.

4.4 Banach spaces ℓ_∞ and c_0

Consider the metric space $\mathbb{S} = \ell_\infty$ with metric $d_\infty(x, y) \triangleq \sup_{i\in\omega} d(x_i, y_i)$ and the ω-chain $(g_n \mid n)$ of idempotents on ℓ_∞ defined by

$$\forall x \in \ell_\infty.\, g_n(x) \triangleq (r_n(x_i) \mid i \in n) \cdot 0^\omega. \tag{5}$$

Since g_n is defined pointwise, it is idempotent and short by *inheritance*, e.g.,

$$d_\infty(g_n(x), g_n(y)) = \sup_{i\in n} d(r_n(x_i), r_n(y_i)) \le \sup_{i\in n} d(x_i, y_i) \le d_\infty(x, y).$$

The image of g_n is the compact sub-space $\mathbb{S}_n = \ell_{\infty n} \triangleq \mathbb{R}_n^n \times \{0\}^\omega$, which is isomorphic to the finite product $\mathbb{R}_n^n$ in **Met** (since $\{0\}$ is a terminal object in **Met**), and the union $\mathbb{S}_*$ of the $\mathbb{S}_n$ is the sub-space $\ell_{*,\infty}$ of ω-sequences eventually equal to 0, which is not dense in ℓ_∞. For instance, consider $1^\omega \in \ell_\infty$, then $\ell_{*,\infty} \cap B(1^\omega, 0.5) = \emptyset$. The closure of $\ell_{*,\infty}$ in ℓ_∞ is the sub-space c_0 of ω-sequences converging to 0. In functional analysis, the elements of c_0 are sometimes called *null sequences*, e.g., see Narici and Beckenstein (2011).

From Section 4.1, Proposition 4.3, and the dual of Proposition 4.4, we have that $\overline{\mathbb{S}}$ for ℓ_∞, which we denote as $\overline{\ell}_\infty$, is isomorphic to $\overline{\mathbb{R}}^\omega$ in **KH**. In fact, take $I = \mathbb{N}$, $J = \omega^{op}$, and consider the $I \times J$-diagram $D{:}I \times J \to \mathbf{KH}$ such that $D(i, n) = \mathbb{R}_n$ if $i < n$, else $\{0\}$, and $D(i, n+1 \to n) = p_n{:}\mathbb{R}_{n+1} \to \mathbb{R}_n$ if $i < n$, else the unique map from $D(i, n+1)$ to $\{0\}$. The limit $\overline{\mathbb{S}}$ is obtained by first computing the limits of I-diagrams $D(-, n)$, which are isomorphic to $\mathbb{R}_n^n$, and then the J-limit, while $\overline{\mathbb{R}}^\omega$ is obtained by first computing the limits $\overline{\mathbb{R}}$ of the J-diagrams $D(i, -)$ and then the I-limit.

Note that the map $\iota{:}\ell_\infty \to \overline{\ell}_\infty$ given by Theorem 3.11 is monic and epic in **Haus**. Moreover, let $\prod_{i\in\omega} O_i$ be a non-empty basic open set in the topology of $\overline{\ell}_\infty$. This means that, for some

$J \subseteq_f \mathbb{N}$, we have $\forall i \in \mathbb{N} \setminus J{:}O_i = \overline{\mathbb{R}}$. For each $j \in J$, choose a point $y_j \in O_j$. Then, the sequence $x \in \ell_{*,\infty}$ defined by:

$$\forall i \in \omega: \quad x_i \triangleq \begin{cases} y_i, & \text{if } i \in J, \\ 0, & \text{if } i \notin J, \end{cases}$$

is in $\prod_{i\in\omega} O_i$. Thus, although $\ell_{*,\infty}$ is a subset of both $\overline{\ell}_\infty$ and ℓ_∞, and, not dense in ℓ_∞, it is dense in $\overline{\ell}_\infty$. As such, ℓ_∞ is not a sub-space of $\overline{\ell}_\infty$. In particular, $\overline{\ell}_\infty$ is not a compactification of ℓ_∞.

Remark 4.5. Since ℓ_∞ is not second-countable, no compactification K of ℓ_∞ can be second-countable either, and the lattice $\mathbb{C}(K)$ cannot be ω-continuous. The case of c_0 or ℓ_p with $1 \le p < \infty$ is more subtle. These spaces are second-countable and normal, and in theory, it is possible to obtain second-countable compactifications using a variant of the Stone-Čech construction. The problem with the Stone-Čech compactification (and its variants) is that concrete descriptions are either non-existent, or unwieldy, even for relatively simple spaces. As such, these compactification methods are not suitable for an effective framework.

4.5 Banach spaces ℓ_p for $1 \le p < \infty$

Consider the metric spaces $\mathbb{S} = \ell_p$ with metric $d_p(x, y) \triangleq \left(\sum_{i\in\omega} d(x_i, y_i)^p\right)^{1/p} \ge d_\infty(x, y)$. The carrier $|\,\ell_p\,|$ of ℓ_p satisfies the following strict inclusions

$$|\,\ell_{*,\infty}\,| \subset |\,\ell_p\,| \subset |\,c_0\,| \subset \ell_\infty.$$

We can consider the restrictions to ℓ_p of the idempotents g_n defined in (5). It is straightforward to prove that g_n is short and idempotent on ℓ_p, its image $\mathbb{S}_n = \ell_{p_n}$ is isomorphic to the metric space with carrier $\mathbb{R}^n_n$ and metric d_p, and the union $\mathbb{S}_*$ of the $\mathbb{S}_n$ is the dense sub-space $\ell_{*,p}$ of ℓ_p.

By analogy with Section 4.4, we have that the map $\iota : \mathbb{S} \to \overline{\mathbb{S}}$ is monic and epic in **Haus**, and $\overline{\mathbb{S}}$ for ℓ_p is isomorphic to $\overline{\mathbb{R}}^\omega$ in **KH**. In particular, $\overline{\mathbb{S}}$ is independent of p. In summary, the relations between ℓ_p and ℓ_q, for $1 \le p < q \le \infty$, are:

- The carrier of ℓ_p is a proper subset of the carrier of ℓ_q, and the inclusion of ℓ_p into ℓ_q is a short map in **Met**, since $\forall x, y \in \ell_p.d_q(x, y) \le d_p(x, y)$.
- The compact metric spaces ℓ_{p_n} and ℓ_{q_n} have the same carrier, but different metrics, and the inclusion of ℓ_{p_n} into ℓ_{q_n} is a short map in **KMS**.
- As topological spaces, ℓ_{p_n} and ℓ_{q_n} are equal. Thus, $\overline{\ell}_p = \overline{\ell}_q$.

4.6 Unit ball B_∞

Let us now consider the unit ball B_∞ in the metric space ℓ_∞. As a metric space, B_∞ coincides with the infinite product $\mathbb{R}^\omega_1$ in **Met**, and the idempotents g_n defined is Section 4.4 restrict to idempotents on B_∞. The image of g_n (restricted to B_∞) is the compact sub-space $\mathbb{S}_n = B_{\infty n} \triangleq \mathbb{R}^n_1 \times \{0\}^\omega$, which is isomorphic to the finite product $\mathbb{R}^n_1$ in **Met**, and coincides with the closed unit ball $B_{n,\infty}$ in $\ell_{n,\infty} = \mathbb{R}^n$. The union $\mathbb{S}_*$ of the $\mathbb{S}_n$ is the sub-space $B_{*,\infty}$ (as defined in Definition 4.1) and the closure of $\mathbb{S}_*$ is the sub-space $B_\infty \bigcap c_0$.

Similar to Section 4.4, we can prove that $\overline{\mathbb{S}}$ for B_∞, which we denote with $\overline{B}_\infty$, is isomorphic to the product $\mathbb{R}^\omega_1$ in **KH**. More precisely, take $I = \mathbb{N}$, $J = \omega^{op}$, and consider the $I \times J$-diagram $D{:}I \times J \to$ **KH** such that $D(i, n) = \mathbb{R}_1$ if $i < n$ else $\{0\}$, and $D(i, n+1 \to n)$ is the identity on $\mathbb{R}_1$ if $i < n$, else the unique map from $D(i, n+1)$ to $\{0\}$.

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

As in the case of ℓ_∞, the map $\iota{:}B_\infty \to \overline{B}_\infty$ given by Theorem 3.11 is monic and epic in **Haus**. Moreover, as a set-theoretic map, ι is a bijection. In fact, the metric space B_∞ is (equal to) the product $\mathbb{R}_1^\omega$ in **Met** and the topological space $\overline{B}_\infty$ is (isomorphic to) the product $\mathbb{R}_1^\omega$ in **KH**. As such, without loss of generality, we assume that the map ι is an identity map.

4.7 Unit ball B_p for $1 \leq p < \infty$

Let us now consider the unit ball B_p in the metric space ℓ_p. One can proceed in analogy with Section 4.6. In particular, $B_{p_n} \triangleq B_{n,p} \times \{0\}^\omega$ is isomorphic to the closed unit ball $B_{n,p}$ in $\ell_{n,p}$, the map $\iota{:}B_p \to \overline{B}_p$ given by Theorem 3.11 is monic and epic in **Haus**.

Moreover, as a set-theoretic map, ι is a bijection. The map ι is clearly injective. So, it suffices to prove that it is surjective. Let us consider an element $(x_n \mid n)$ in $\overline{B}_p$. Each $x_n{:}B_{p_n}$ may differ from x_{n+1} only in the nth component, namely $x_{n,n} = 0$, while $x_{n+1,n} \in \mathbb{R}_1$. Consider $y \in \mathbb{R}_1^\omega$ defined by $y_n \triangleq x_{n+1,n}$ for $n \in \omega$. We have $\| y \|_p = \lim_{n\to\infty} \| x_n \|_p \leq 1$, which entails that $y \in B_p$. Furthermore,

$$\forall i, n \in \omega. \quad x_{n,i} = \begin{cases} y_i, & \text{if } i < n, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $\iota(y) = (x_n \mid n)$, and ι is surjective. The relations among the ℓ_p spaces established in Section 4.5 imply the following relations among unit balls, for $1 \leq p < q \leq \infty$:

- The carrier of B_p is a proper subset of the carrier of B_q, and the inclusion of B_p into B_q is a short map in **Met**.
- The carrier of B_{p_n} is a proper subset of the carrier of B_{q_n}, and the inclusion of B_{p_n} into B_{q_n} is a short map in **KMS**.
- As a topological space, B_{p_n} is a closed sub-space of B_{q_n} in **KH**, since the metrics d_p and d_q induce the same topology on $\mathbb{R}^n$. Thus, $\overline{B}_p$ is a closed sub-space of $\overline{B}_q$ in **KH**.

The last point implies that the carrier of $\overline{B}_p$ depends on p, but its topology is that on $\overline{B}_\infty$.

5. Precision

Figure 5 gives a summary of the examples in Section 4. We observe the following:

(1) In the finite-dimensional cases, $\mathbb{S}$ is a dense sub-space of $\overline{\mathbb{S}}$ in **Haus**, and $\overline{\mathbb{S}}$ is a compactification of $\mathbb{S}$. Therefore, every closed subset C of $\mathbb{S}$ is the intersection $C' \bigcap \mathbb{S}$ for some closed subset C' of $\overline{\mathbb{S}}$. Since the metrics d_p and d_∞ induce the same topology on $\mathbb{R}^m$—the carrier of both $\ell_{m,p}$ and $\ell_{m,\infty}$—we have $\mathbb{C}(\ell_{m,p}) = \mathbb{C}(\ell_{m,\infty})$ for each $p \in [1, \infty]$. Furthermore, $\overline{\mathbb{S}}$ does not depend on p. Hence, it suffices to consider the cases $\mathbb{S} = \ell_{m,\infty}$. The map $\iota{:}\ell_{m,\infty} \to \overline{\ell}_{m,\infty}$ is a sub-space inclusion and $\iota^* \circ \iota_* = \mathrm{id}_{\mathbb{C}(\ell_{m,\infty})}$.

(2) In the infinite-dimensional cases—as we will demonstrate—$\mathbb{S}$ is not a sub-space of $\overline{\mathbb{S}}$. More precisely, $\iota{:}\mathbb{S} \to \overline{\mathbb{S}}$ is monic and epic; thus, the image of ι is dense in $\overline{\mathbb{S}}$. However, ι is not a sub-space inclusion; thus, there are closed subsets C of $\mathbb{S}$ which cannot be written as $C' \bigcap \mathbb{S}$ for some closed subset C' of $\overline{\mathbb{S}}$, and $\iota^* \circ \iota_*$ is not the identity on $\mathbb{C}(\mathbb{S})$.

In what follows, we focus mainly on the case of the unit ball B_p. As discussed in Sections 4.6 and 4.7, without loss of generality, we assume that the bijective map $\iota{:}B_p \to \overline{B}_p$ is an identity. As

Notation: $\hookrightarrow$ (sub-space), $\rightarrowtail$ (sub-object), X^I product of I copies of X.

$$
\begin{tikzcd}
B_\infty \cong \mathbb{R}_1^\omega \arrow[r, hook] & \ell_\infty & \ell_{m,\infty} \cong \mathbb{R}^m \arrow[l, hook'] \\
B_p \arrow[u, hook] \arrow[r, hook] & \ell_p \arrow[u, hook] & \ell_{m,p} \arrow[u, hook] \arrow[l, hook']
\end{tikzcd}
\quad \text{in } \mathbf{Met}
$$

$$
\begin{tikzcd}
\overline{B}_p \arrow[r] & \overline{B}_\infty \cong \mathbb{R}_1^\omega \arrow[r, hook] & \overline{\ell}_\infty \cong \overline{\mathbb{R}}^\omega & \overline{\ell}_{m,\infty} \cong \overline{\mathbb{R}}^m \arrow[l, hook'] \\
& B_\infty \arrow[u, hook] \arrow[r, hook] & \ell_\infty \arrow[u, hook] & \ell_{m,\infty} \cong \mathbb{R}^m \arrow[u, hook] \arrow[l, hook'] \\
& B_p \arrow[u, hook] \arrow[uul, tail] \arrow[r, hook] & \ell_p \arrow[u, hook] & \ell_{m,p} \arrow[u, equal] \arrow[l, hook']
\end{tikzcd}
\quad \text{in } \mathbf{Haus}
$$

In the following table, for each metric space $\mathbb{S}$ considered in Section 4 (first column), we give:

- the compact metric sub-space $\mathbb{S}_n$ (second column), i.e., its nth-approximant;
- the metric sub-space $\mathbb{S}_*$ (third column), i.e., the union of its approximants;
- the Hausdorff space corresponding to $\mathbb{S}$ (fourth column);
- the compact Hausdorff sub-space corresponding to $\mathbb{S}_n$ (fifth column);
- the compact Hausdorff space $\overline{\mathbb{S}}$ given by our construction (sixth column);
- the property of the map $\iota{:}\mathbb{S} \to \overline{\mathbb{S}}$ in **Haus** (seventh column);
- the section where the example is explained in details (eighth column).

Met			**Haus**				see
$\mathbb{S}$	$\mathbb{S}_n$	$\mathbb{S}_*$	$\mathbb{S}$	$\mathbb{S}_n$	$\overline{\mathbb{S}}$	$\iota : \mathbb{S} \to \overline{\mathbb{S}}$	Section
finite-dimensional Banach spaces							
$\mathbb{R}$	$\mathbb{R}_n$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}_n$	$\overline{\mathbb{R}}$	sub-space	4.1
$\ell_{m,\infty} = \mathbb{R}^m$	$\mathbb{R}_n^m$	$\mathbb{R}^m$	$\mathbb{R}^m$	$\mathbb{R}_n^m$	$\overline{\mathbb{R}}^m$	sub-space	4.2
$\ell_{m,p} \rightarrowtail \mathbb{R}^m$	$(\mid \mathbb{R}_n^m \mid, d_p)$	$\ell_{m,p}$					4.3
infinite-dimensional Banach spaces							
ℓ_∞	$\mathbb{R}_n^n$	$\ell_{*,\infty}$	ℓ_∞	$\mathbb{R}_n^n$	$\overline{\mathbb{R}}^\omega$	sub-object	4.4
$\ell_p \rightarrowtail \ell_\infty$	$(\mid \mathbb{R}_n^n \mid, d_p)$	$\ell_{*,p}$	ℓ_p				4.5
closed bounded convex subsets of infinite-dimensional Banach spaces							
$B_\infty = \mathbb{R}_1^\omega$	$B_{n,\infty} = \mathbb{R}_1^n$	$B_{*,\infty}$	B_∞	$\mathbb{R}_1^n$	$\mathbb{R}_1^\omega$	sub-object	4.6
$B_p \rightarrowtail \mathbb{R}_1^\omega$	$B_{n,p}$	$B_{*,p}$	B_p	$B_{n,p}$	$\overline{B}_p$		4.7

Figure 5. Summary of Examples of Section 4.

a result, by going from B_p to $\overline{B}_p$, the carrier does not change, and we have to compare only the topologies, or equivalently $\mathbb{C}(\overline{B}_p) \subset \mathbb{C}(B_p)$. Since the left adjoint ι^* is the inclusion map of $\mathbb{C}(\overline{B}_p)$ into $\mathbb{C}(B_p)$, we have $\iota^*(\iota_*(C)) = \iota_*(C)$. Thus, the loss of precision is measured by how bigger is $\iota_*(C)$ in comparison to the closed subset C in $\mathbb{C}(B_p)$.

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

Remark 5.1. In applications, it is reasonable to restrict to closed bounded subsets of Banach spaces, i.e., closed subsets included in a ball of finite radius. The claims in this section are true for closed balls in ℓ_p with center x and radius $r > 0$, but we state them for the paradigmatic case $x = 0$ and $r = 1$, to avoid extra parameters in notation. Note that, compact subsets of a Banach space are always closed and bounded, and in the finite-dimensional case the converse also holds.

To start, we present a positive result where there is no loss of precision.

Proposition 5.2. (Compact sets). *If $\mathbb{S}_1 \rightarrowtail \mathbb{S}_2$ is a mono in* **Haus**, *then compact subsets of $\mathbb{S}_1$ are compact in $\mathbb{S}_2$.*

Proof. Write ι for the mono $\mathbb{S}_1 \rightarrowtail \mathbb{S}_2$ which we can assume to be an inclusion between the carriers. Since ι is continuous, the claim now follows from the general fact that the image of a compact set under a continuous function is always compact. □

Corollary 5.3. *Assume that $\mathbb{S}$:***Haus**, *and $\iota : \mathbb{S} \to \overline{\mathbb{S}}$ (in* **Haus***) is as in Theorem 3.11. If $C \in \mathbb{C}(\mathbb{S})$ is compact, then $C \in \mathbb{C}(\overline{\mathbb{S}})$, and therefore $\iota_*(C) = C$.*

Proof. As $\mathbb{S} \rightarrowtail \overline{\mathbb{S}}$ is mono, by Proposition 5.2, every compact $C \in \mathbb{C}(\mathbb{S})$ is also compact in $\overline{\mathbb{S}}$. The result now follows from the fact that in Hausdorff spaces, compact subsets are closed. □

Compact subsets of infinite-dimensional Banach spaces are not so relevant in applications (e.g., they always have empty interior). For $1 < p < \infty$, however, we have a characterization of the closed bounded subsets of ℓ_p for which there is no loss of precision, i.e., $C = \iota^*(\iota_*(C))$.

Theorem 5.4. (**No loss of Precision**). *For any $1 < p < \infty$ and $C \in \mathbb{C}(\ell_p)$, the following are equivalent:*

(1) C is bounded and $C = \iota^*(\iota_*(C))$, i.e., there is no loss of precision over C.
(2) C is a non-empty intersection of finite unions of closed balls in ℓ_p.
(3) C is a non-empty intersection of finite unions of bounded-closed-convex subsets of ℓ_p.

Proof. See the proof on page 566. □

To prove Theorem 5.4, we relate the topology on $\overline{B}_p$ (indeed on any closed ball in ℓ_p) to the weak-* topology, one of the fundamental topologies studied in functional analysis.

5.1 Weak-* topology on | B_p | for $1 \leq p \leq \infty$

Let us first present a quick reminder of weak and weak-* topologies for the case of normed vector spaces over the field of real numbers. The reader may refer to any standard book on functional analysis, e.g., Conway (1990); Rudin (1991), for the more general treatment of these topologies.

Definition 5.5. (dual). *Given X:***NVS** *with norm $\| . \|_X$, its continuous* ***dual*** *X' is the Banach space of linear continuous functions from X to $\mathbb{R}$ with norm $\| f \|_{X'} \triangleq \sup \{ | f(x) | \mid x \in X \wedge \| x \|_X \leq 1\}$.*

Proposition 5.6. *If Y:***Ban** *is the Cauchy completion of X:***NVS**, *then Y' and X' are isomorphic.*

Proof. See, e.g., Narici and Beckenstein (2011, Page 270). □

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

Table 1. Some duals and double duals (up to iso)

X	X'	X''	**Note**
c_0	ℓ_1	ℓ_∞	
ℓ_p	$\ell_{p'}$	ℓ_p	$1 < p < \infty$

Definition 5.7. (reflexive, weak, weak-*). *For X:**NVS**, the map $\eta_X{:}X \to X''$ (a linear isometry) is defined as $\eta_X(x)(f) \triangleq f(x)$ for $x \in X$ and $f \in X'$. When the map η_X is an iso, X is called **reflexive**.*

*(1) The **weak** topology W_X on X is the coarsest topology making each $f \in X'$ continuous.*
*(2) The **weak-*** topology W^*_X on X' is the coarsest topology on X' making $\eta_X(x)$ continuous for each $x \in X$.*

Proposition 5.8. *Let the **conjugate** p' of p be the unique $q \in [1, \infty]$ such that $1/p + 1/q = 1$.*

(1) For every $p \in [1, \infty)$, the Cauchy completion of $\ell_{*,p}$ is the Banach space ℓ_p.
(2) The Cauchy completion of $\ell_{*,\infty}$ is the Banach sub-space c_0 of ℓ_∞.
(3) For every $p \in [1, \infty]$, the map $\xi{:}\ell_{p'} \to (\ell_{*,p})'$, given by $\xi(x')(x) \triangleq \sum_{i\in\omega} x'_i * x_i$, is an iso.

Proof. Proofs of (1) and (2) are straightforward. To prove (3), by Proposition 5.6, and by (1) and (2), we may regard ξ as a function from $\ell_{p'}$ to $(\ell_p)'$, when $p \in [1, \infty)$, or to $(c_0)'$, when $p = \infty$. The proof that ξ is an iso may now be found in:

- Conway (1990, Appendix B), for $p \in [1, \infty)$.
- Albiac and Kalton (2006, Page 50), for $p = \infty$. □

The duals and double duals of relevance in this section are summarized in Table 1.

Definition 5.9. (Topologies τ_p, τ^*_p, $\overline{\tau}_p$). *We define the following topologies on the carrier of ℓ_p:*

(1) τ_p is the original (or, norm) topology on ℓ_p, i.e., the topology induced by the norm $\|\,.\,\|_p$.
(2) τ^*_p denotes the weak- topology on ℓ_p as the continuous dual of $\ell_{*,p'}$.*
(3) $\overline{\tau}_p$ is the topology on ℓ_p as a subset of the compact Hausdorff space $\overline{\mathbb{R}}^\omega$.

We use the same notation for the topologies when restricted to a subset of ℓ_p, such as B_p.

According to Figure 5, by using the notation of Definition 5.9, we have that $\overline{\mathbb{S}} = (\mid B_p \mid, \overline{\tau}_p)$ when $\mathbb{S} = (\mid B_p \mid, \tau_p)$. Therefore, we obtain:

Proposition 5.10. *For every $C \in \mathbb{C}(\ell_p)$, the set $\iota^*(\iota_*(C))$ is the $\overline{\tau}_p$ closure of C.*

In Theorem 5.12, we show that the topological spaces $(\mid B_p \mid, \overline{\tau}_p)$ and $(\mid B_p \mid, \tau^*_p)$ coincide for any $1 \leq p \leq \infty$. First, we recall the lemma on the "rigidity" of compact Hausdorff topologies in Rudin (1991, Section 3.8).

Lemma 5.11. *If $\tau_1 \subseteq \tau_2$ are topologies on a set X, with τ_1 Hausdorff and τ_2 compact, then $\tau_1 = \tau_2$.*

Proof. Let $F \subseteq X$ be τ_2-closed. Since X is τ_2-compact, so is F. Since $\tau_1 \subseteq \tau_2$, it follows that F is τ_1-compact. Since τ_1 is a Hausdorff topology, it follows that F is τ_1-closed. □

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

Theorem 5.12. *For each $1 \leq p \leq \infty$, $\overline{B}_p$ is (isomorphic to) the topological space $(| B_p |, \tau_p^*)$.*

Proof. The topology $\overline{\tau}_p$ is Hausdorff. On the other hand, by the Banach-Alaoglu theorem (see, e.g., Rudin (1991, Section 3.8)) the closed unit ball $| B_p |$ is weak-* compact, i.e., τ_p^*-compact. Therefore, by Lemma 5.11, it suffices to prove that $\overline{\tau}_p \subseteq \tau_p^*$.

For $i \in \omega$, consider the retractions $\pi_i : \ell_p \to \mathbb{R}$ defined by

$$\forall x \in \ell_p. \quad \pi_i(x) \triangleq x_i. \tag{6}$$

The set $\mathscr{Y} = \{| B_p | \bigcap \pi_i^{-1}(O) \mid i \in \omega, O \subseteq \mathbb{R} \text{ Euclidean open}\}$ is a sub-base for $\overline{\tau}_p$. For each $i \in \omega$, consider the sequence e_i defined by

$$\forall n \in \omega. \quad e_i(n) \triangleq \begin{cases} 0, & \text{if } i \neq n, \\ 1, & \text{if } i = n. \end{cases} \tag{7}$$

For all $i \in \omega$, the sequence e_i is in all the relevant pre-duals as specified in Definition 5.9 above. Furthermore

$$\forall i \in \omega . \forall x \in \ell_p. \quad \pi_i(x) = x(e_i).$$

Thus, each π_i is weak-* continuous and $\mathscr{Y} \subseteq \tau_p^*$, which entails that $\overline{\tau}_p \subseteq \tau_p^*$. □

Remark 5.13. As pointed out in Remark 5.1, although we present results for closed unit balls, they hold for arbitrary closed balls. In particular, Theorem 5.12 holds for closed balls in ℓ_p, because they are all τ_p^*-compact.

Corollary 5.14. *For each $1 \leq p \leq \infty$ and $C \in \mathbb{C}(B_p)$, $\iota_*(C) = C$ iff C is a τ_p^*-closed subset.*

Proof. Follows from Proposition 5.10 and Theorem 5.12. □

Corollary 5.15. *For each $1 < p < \infty$ and $C \in \mathbb{C}(B_p)$, $\iota_*(C) = C$ iff C is a weakly closed subset.*

Proof. Since the space ℓ_p for $1 < p < \infty$ is reflexive, i.e., the weak and weak-* topologies on ℓ_p coincide. The result follows from Corollary 5.14. □

We have established all the preliminaries for presenting the proof of Theorem 5.4:

Proof. **(Theorem 5.4)**

- **(1)** $\Rightarrow$ **(2)** As C is assumed to be bounded, then it must be a subset of a closed ball B of finite radius. If $\iota^*(\iota_*(C)) = C$, then, by Corollary 5.15, C must be weakly closed. Thus, C is a weakly closed subset of (the weakly compact set) B. As a result, it is weakly compact.
  According to Corson and Lindenstrauss (1966, Theorem 3), a subset of a separable reflexive space is weakly compact if and only if it is the non-empty intersection of finite unions of closed balls. Each space ℓ_p for $1 < p < \infty$ is separable and reflexive. Hence, the result follows.
- **(2)** $\Rightarrow$ **(3)** This is straightforward as every closed ball is a bounded-closed-convex subset.
- **(3)** $\Rightarrow$ **(1)** Assume that C is a non-empty intersection of finite unions of bounded-closed-convex subsets of ℓ_p. To be precise, $C = \bigcap_{i \in I} \bigcup_{j \in k_i} C_{i,j}$ with $k \in \omega^I$ and $C_{i,j}$ bounded-closed-convex subset of ℓ_p.

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

As a consequence of Hahn-Banach separation theorem, every closed and convex subset of a Banach space is weakly closed (see, e.g., Rudin (1991, Theorem 3.12)). This entails that each $C_{i,j}$ is weakly closed. As the set of closed subsets (under any topology) are closed under finite unions and arbitrary intersections, then C itself is also weakly closed. Clearly, C is also bounded. The result now follows from Corollary 5.15. □

Item (3) of Theorem 5.4 provides examples where no loss of precision is incurred.

Example 5.16. (Sequence intervals). For a pair $s, t \in \ell_p$, the sequence interval $[s, t]$ is given by:

$$[s, t] \triangleq \{u \in \ell_p \mid \forall i \in \omega.\ s_i \leq u_i \leq t_i\}.$$

In general, sequence intervals are not norm-compact in ℓ_p, but they are bounded, *norm-closed, and convex*. Hence, when $1 < p < \infty$, there is no loss of precision over sequence intervals.

5.1.1 Metrizability of the $\overline{\tau}_p$-topologies over bounded closed balls

Consider the map $d_*:\ell_\infty \times \ell_\infty \to \mathbb{R}$ defined as follows:

$$\forall x, y \in \ell_\infty. \quad d_*(x, y) \triangleq \sum_{n\in\omega} \frac{d(x_n, y_n)}{2^{n+1}}. \tag{8}$$

It is straightforward to prove that d_* is a metric on the carrier of ℓ_∞.

Proposition 5.17. *The metric d_* induces the $\overline{\tau}_\infty$-topology on closed balls of finite radius in ℓ_∞.*

Proof. According to Rudin (1991, Section 3.8(c), page 63), if X is a compact topological space and if some uniformly bounded sequence $(f_n \mid n \in \omega)$ of continuous real-valued functions separates points on X, then X is metrizable, with the metric $\rho(x, y) = \sum_{n\in\omega} 2^{-n} d(f_n(x), f_n(y))$.

By the Banach-Alaoglu theorem, a closed ball of finite radius B is weak-* compact. The countable set $\{\pi_i \mid i \in \omega\}$ of retractions from (6) is separating on B, in the sense that:

$$\forall x, y \in B. \quad x \neq y \implies \exists i \in \omega. \pi_i(x) \neq \pi_i(y).$$

Furthermore, each π_i is weak-* continuous. Thus, it suffices to take $f_n = \pi_n/2$, and the claim follows from Rudin (1991, Section 3.8(c), page 63) and Theorem 5.12. □

The metric d_* on the carrier of ℓ_∞ satisfies the following property

$$\forall p \in [1, \infty]. \forall x, y \in \ell_p. \quad d_*(x, y) = \sum_{n\in\omega} \frac{d(x_n, y_n)}{2^{n+1}} \leq \sum_{n\in\omega} \frac{d_\infty(x, y)}{2^{n+1}} = d_\infty(x, y) \leq d_p(x, y). \tag{9}$$

Corollary 5.18. *For each $p \in [1, \infty]$, the metric d_* induces the $\overline{\tau}_p$-topology on closed balls of finite radius in ℓ_p.*

Proof. Let B be the closed ball in ℓ_p centered at x_0 with radius $r > 0$. Let B' denote the closed ball in ℓ_∞ centered at x_0 with radius r. The $\overline{\tau}_p$-topology on B is the restriction of the $\overline{\tau}_\infty$-topology on B'. The claim now follows from Proposition 5.17. □

By going from B_p to $\overline{B}_p$, robustness with respect to the metric d_p is replaced by robustness with respect to the metric d_*. Indeed, inequality (9) shows that for any subset S of $\mid B_p \mid$ its δ-neighborhood $B(S, \delta)$ under d_p is included its δ-neighborhood under d_*.

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

Proposition 5.19. *For every $C \in \mathbb{C}(B_p)$:*

(1) *The set $\iota_*(C)$ is the closure of C under the d_* metric.*
(2) *For every $x \in B_p$, $x \in \iota_*(C)$ if and only if:*

$$\forall n \in \omega. \forall \delta > 0. \exists y \in C. \forall i \in n. \quad d(x_i, y_i) < \delta. \tag{10}$$

Proof. Claim (1) follows from Corollary 5.18. From it, we deduce that $x \in \iota_*(C)$ if and only if:

$$\forall \varepsilon > 0. \exists y \in C. \quad d_*(x, y) < \varepsilon. \tag{11}$$

Thus, for the second claim, it suffices to prove that (10) and (11) are equivalent.

(10) $\Rightarrow$ (11)**:** For any given $\varepsilon > 0$, choose n large enough such that $\varepsilon 2^{n-2} > 1$ and $\delta = \varepsilon/2$. By (10), there exists a $y \in C$ such that $\forall i \in n$. $d(x_i, y_i) < \delta$. For this y we have:

$$\begin{aligned} d_*(x, y) &= \sum_{i \in n} \frac{d(x_i, y_i)}{2^{i+1}} + \sum_{i \geq n} \frac{d(x_i, y_i)}{2^{i+1}} \\ &\leq \sum_{i \in n} \frac{\delta}{2^{i+1}} + \sum_{i \geq n} \frac{2}{2^{i+1}} \\ &\leq \delta + \frac{2}{2^n} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

(11) $\Rightarrow$ (10)**:** For any given $n \in \omega$ and $\delta > 0$, choose $\varepsilon = \delta/2^{n+1}$. By (11), there exists a $y \in C$ such that $d_*(x, y) < \varepsilon$, which implies that:

$$\sum_{i \in \omega} \frac{d(x_i, y_i)}{2^{i+1}} < \varepsilon = \frac{\delta}{2^{n+1}}. \tag{12}$$

From (12), we obtain $\forall i \in \omega . d(x_i, y_i) < \delta * 2^{i-n}$. In particular, for every $i \in n$, we have $d(x_i, y_i) < \delta$. This completes the proof of the second claim. $\square$

5.2 Loss of precision

In this subsection, we discuss loss of precision through a few examples.

Example 5.20. (bounded, closed, non-convex, discrete). Take the sequences $(e_i \mid i \in \omega)$ as defined in (7), that is:

$$\forall n \in \omega. \quad e_i(n) \triangleq \begin{cases} 0, & \text{if } i \neq n, \\ 1, & \text{if } i = n, \end{cases}$$

and consider the set $C \triangleq \{e_i \mid i \in \omega\}$, which is a discrete, bounded, and closed subset of B_p for $p \in [1, \infty]$. Using the metric d_* of (8), we show that $\iota_*(C) = C \bigcup \{0\}$.

It is indeed clear that 0 is in the weak-* closure of C as:

$$\lim_{i \to \infty} d_*(e_i, 0) = \lim_{i \to \infty} 2^{-(i+2)} = 0.$$

Furthermore, $\forall i, j \in \omega.\ d_*(e_i, e_j) \leq 2^{-(i+2)} + 2^{-(j+2)}$. This entails that $\lim_{i,j\to\infty} d_*(e_i, e_j) = 0$. Hence, the sequence $(e_i \mid i \in \omega)$ is a Cauchy sequence under d_*, and it can have only one limit point, i.e., zero.

Example 5.21. (bounded, closed, non-convex, connected). Consider the unit sphere $S_p \triangleq \{s \in B_p \mid \| s \|_p = 1\}$, which is norm-closed and non-convex. The weak-* closure of the unit sphere in $\mid B_p \mid$ is the entire unit ball $\mid B_p \mid$. In this case, the loss of precision is quite noticeable.

In Theorem 5.4, the assumption $p \notin \{1, \infty\}$ is crucial, as demonstrated in Example 5.22 and Proposition 5.25.

Example 5.22. (bounded, closed, convex, $p = \infty$). Consider the set $C \triangleq c_0 \bigcap B_\infty$, which is a bounded, closed, and convex subset of B_∞. In Section 4.6, we proved that $\iota_*(C) = B_\infty$.

Theorem 5.4 relies on the fact that norm-closed and convex subsets of reflexive Banach spaces are weakly closed. On the other hand, norm-closed and convex subsets of non-reflexive Banach spaces are not, in general, weak-* closed. Recall that the map η_X in Definition 5.7 is a linear isometry which embeds X into X''. When X is not reflexive, we have $X'' \setminus X \neq \emptyset$.

Proposition 5.23. *If X is a non-reflexive Banach space and $\theta \in X'' \setminus X$, then the kernel $\theta^{-1}(0)$ of θ is a closed linear sub-space of X' which is not weak-* closed.*

Proof. This is a consequence of Conway (1990, Theorem 3.1, page 108) and Rudin (1991, Theorem 3.10, page 64). □

Using the following result, for any given non-reflexive Banach space, one may construct many examples of sets for which there is a loss of precision, provided the space is the dual of another Banach space, i.e., has a pre-dual. This rules out spaces such as c_0 which have no pre-duals (Albiac and Kalton, 2006, Theorem 6.3.7).

Corollary 5.24. (bounded, closed, convex, non-reflexive with pre-dual). *If X is a non-reflexive Banach space, B is the closed unit ball in X' and $\theta \in X'' \setminus X$, then $C \triangleq B \bigcap \theta^{-1}(0)$ is closed and convex in X'—hence, weakly closed—but not weak-* closed.*

Proof. As a consequence of Proposition 5.23, the set C is closed and convex, hence, weakly closed. By Conway (1990, Corollary 12.6, page 160), however, C cannot be weak-* closed. □

In general, an exact description of the weak-* closures of sets C in Corollary 5.24 is not known. There are, however, inner and outer approximations available in the literature. For instance, according to Jameson (1982, Proposition 2), the weak-* closure of C must contain a closed ball centered at the origin. Proposition 5.25 gives an instance of this result, which we prove directly.

Proposition 5.25. (bounded, closed, convex, $p = 1$). *If $C \triangleq \{x \in B_1 \mid \sum_{n\in\omega} x_n = 0\}$, D is the closed ball in ℓ_1 centered at 0 with radius $1/2$, and $\overline{C}$ is the τ_1^*-closure of C in B_1, then:*

(1) C is closed and convex—hence, also weakly closed—in B_1. But, C is not weak- closed.*
(2) $D \setminus C \neq \emptyset$.
(3) $D \subset \overline{C} \subset B_1$, where both inclusions are strict.

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

Proof. Take $X = c_0$ (which implies that $X' = \ell_1$ and $X'' = \ell_\infty$) and let $\theta = 1^\omega \in \ell_\infty \setminus c_o$, i.e., the constant sequence of ones. We have $C = B_1 \bigcap \theta^{-1}(0)$.

(1) Follows from Corollary 5.24.
(2) $\left(\frac{1}{2^{n+2}} \mid n \in \omega\right) \in D \setminus C$.
(3) Assume that $x = (x_n \mid n \in \omega) \in D$. For any $n \in \omega$ and $\delta > 0$, consider $\hat{x} = (\hat{x}_i \mid i \in \omega)$ defined as follows:

$$\forall i \in \omega.\ \hat{x}_i \triangleq \begin{cases} x_i, & \text{if } i < n, \\ -x_{i-n}, & \text{if } n \leq i < 2n, \\ 0, & \text{otherwise.} \end{cases}$$

We have $\theta(\hat{x}) = \sum_{i\in\omega} \hat{x}_i = 0$. Furthermore, $\| \hat{x} \|_1 \leq 2\| x \|_1 \leq 1$. Hence, $\hat{x} \in \theta^{-1}(0) \bigcap B_1 = C$. Clearly, for all $i \in n$, we have $d(x_i, \hat{x}_i) = 0 \leq \delta$. Therefore, equation (10) of Proposition 5.19 holds, and we have $D \subseteq \overline{C}$.
Take the sequence $y = (y_n \mid n \in \omega)$ defined as follows:

$$\forall n \in \omega.\ y_n \triangleq \begin{cases} 0.5, & \text{if } n = 0, \\ -0.5, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $y \in C \setminus D \subseteq \overline{C} \setminus D$. Thus, we have proved that $D \subset \overline{C}$.
It remains to prove that $B_1 \setminus \overline{C} \neq \emptyset$. Take the sequence $z = (z_n \mid n \in \omega)$ defined as follows:

$$\forall n \in \omega.\ z_n \triangleq \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $z \in B_1$. We use Proposition 5.19 to prove that $z \notin \overline{C}$. In (10), choose $n = 1$ and $\delta = 1/2$. For any given $\hat{z} \in C$, we have $\sum_{n\in\omega} \hat{z}_n = 0$. Thus, $\sum_{n=1}^{\infty} \hat{z}_n = -\hat{z}_0$, which implies that:

$$\left| \sum_{n=1}^{\infty} \hat{z}_n \right| = | \hat{z}_0 |. \tag{13}$$

On the other hand, as $\hat{z} \in C$, we must have $\| \hat{z} \|_1 \leq 1$. As a result:

$$1 \geq \| \hat{z} \|_1 = \sum_{n\in\omega} | \hat{z}_n | \geq | \hat{z}_0 | + \left| \sum_{n=1}^{\infty} \hat{z}_n \right|. \tag{14}$$

By combining (13) and (14), we obtain $| \hat{z}_0 | \leq 1/2$, which implies $1 - \hat{z}_0 \geq 1/2$, equivalently $d(z_0, \hat{z}_0) \geq 1/2 = \delta$.

□

6. Concluding Remarks

The results in this paper are part of an overall study of robust maps. We have chosen the theory of ω-continuous lattices, within which computability can be studied using the framework of effectively given domains (Smyth, 1977), and robustness can be analyzed using the Robust topology (Definition 2.25) over the lattice of closed subsets of the state space.

https://doi.org/10.1017/S0960129523000233 Published online by Cambridge University Press

In a related work, Edalat (1995) has considered locally compact Hausdorff spaces instead of metric spaces and has worked with the domain of compact subsets (ordered by reverse inclusion) instead of the complete lattice of closed subsets. Furthermore, he has investigated the relationship between the Scott topology and the upper Vietoris topology, but has not studied robustness. The Robust topology lies in between the Scott and the upper Vietoris topologies (Theorem 2.27).

The case of compact metric spaces has been studied in Moggi et al. (2018). This suffices to deal with the input space of typical machine learning systems and the state space of common hybrid systems. In this paper, the focus has been non-compact metric spaces, for which, a novel approach has been presented based on approximation of the space via a (growing) sequence of compact metric sub-spaces. Non-compact spaces are relevant when dealing with perturbations of the *model* parameters of a system, e.g., perturbations of the activation function(s) of a neural network, or the flow F and jump G relations of a hybrid system $(\mathbb{S}, F, G)$.

We presented a detailed account of some examples, including (closed bounded subsets of) infinite-dimensional Banach spaces, and analyzed the important issue of precision, when it is retained, and when precision is lost. In particular, we have obtained a complete characterization of the closed subsets of reflexive spaces ℓ_p (i.e., those with $1 < p < \infty$), for which there is no loss of precision (Theorem 5.4).

All examples studied in this paper are sequence spaces. As such, studying other relevant spaces provides an immediate direction for future work. For instance, let $\Omega \subseteq \mathbb{R}^n$ be an open set. Lebesgue spaces $L^p(\Omega)$ are examples for future work, which are relevant in the study of partial differential equations (Brezis, 2011).

Other cases for future study include infinite-dimensional feature spaces arising in machine learning and spaces of bounded measures. In particular, by applying our results to (closed subsets of) probability measures, we obtain a framework for computation of probability measures using finitary approximations. It will be interesting to compare the finitary approximations obtained in this way, with those obtained by Edalat (1997) for computation of probability measures over separable metric spaces.

Competing interests. The authors declare none.

Notes

1 We use extended metric spaces because they have better category-theoretic properties.

2 The reason for using different colors in the diagram is that the same colors will also be used in Figure 4 to indicate from which of the four categories an object or arrow comes.

References

Abramsky, S. and Jung, A. (1994). Domain theory. In Abramsky, S., Gabbay, D. M. and Maibaum, T. S. E., editors, *Handbook of Logic in Computer Science*, vol. 3, pp. 1–168. Clarendon Press.

Albiac, F. and Kalton, N. J. (2006). *Topics in Banach Space Theory*. Springer.

Borceux, F. (1994). *Handbook of Categorical Algebra: Volume 1, Basic Category Theory*, volume 1. Cambridge University Press.

Brezis, H. (2011). *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, New York.

Conway, J. B. (1990). *A Course in Functional Analysis*, 2nd edition. Springer, New York.

Corson, H. H. and Lindenstrauss, J. (1966). On weakly compact subsets of Banach spaces. *Proceedings of the American Mathematical Society* **17** (2) 407–412.

Cousot, P. (1996). Abstract interpretation. *ACM Computing Surveys* **28** (2) 324–328.

Cousot, P. and Cousot, R. (1977). Abstract interpretation: A unified lattice model for static analysis of programs by construction or approximation of fixpoints. In *Proceedings of the 4th ACM SIGACT-SIGPLAN Symposium on Principles of Programming Languages*, 238–252.

Edalat, A. (1995). Dynamical systems, measures and fractals via domain theory. *Information & Computation* **120** (1) 32–48.

Edalat, A. (1997). When Scott is weak on the top. *Mathematical Structures in Computer Science* **7** (5) 401–417.

Fränzle, M. (1999). Analysis of hybrid systems: An ounce of realism can save an infinity of states. In *Computer Science Logic: 13th International Workshop, CSL'99 8th Annual Conference of the EACSL Madrid, Spain, September 20–25, 1999 Proceedings 13*. Springer, 126–139.

Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. W. and Scott, D. S. (1980). *A Compendium of Continuous Lattices*. Springer, Berlin Heidelberg.

Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. W. and Scott, D. S. (2003). *Continuous Lattices and Domains*, vol. 93. *Encycloedia of Mathematics and its Applications*. Cambridge University Press.

Goebel, R., Sanfelice, R. G. and Teel, A. (2009). Hybrid dynamical systems. *IEEE Control Syst. Mag.*, 29(2):28–93.

Goubault-Larrecq, J. (2008). *Simulation Hemi-Metrics between Infinite-State Stochastic Games*. Lecture Notes in Computer Science, vol. 4962, 50–65.

Hofmann, T., Schölkopf, B. and Smola, A. J. (2008). Kernel methods in machine learning. *Annals of Statistics* **36** (3) 1171–1220.

Jameson, G. J. O. (1982). The weak-star closure of the unit ball in a subspace. *Proceedings of the Edinburgh Mathematical Society* **25** (1) 87–95.

Moggi, E., Farjudian, A., Duracz, A. and Taha, W. (2018). Safe & robust reachability analysis of hybrid systems. *Theoretical Computer Science* **747** 75–99.

Moggi, E., Farjudian, A. and Taha, W. (2019). System analysis and robustness. In Cherubini, A., Sabadini, N. and Tini, S., editors, *Proceedings of the 20th Italian Conference on Theoretical Computer Science, ICTCS 2019, Como, Italy, September 9-11, 2019*, vol. 2504. *CEUR Workshop Proceedings*. CEUR-WS.org, 1–7.

Munkres, J. R. (2000). *Topology*, 2nd edition. Prentice Hall.

Narici, L. and Beckenstein, E. (2011). *Topological Vector Spaces*, 2nd edition. Chapman and Hall/CRC, New York.

Rudin, W. (1991). *Functional Analysis*. McGraw-Hill, Singapore, 2nd edition.

Smyth, M. B. (1977). Effectively given domains. *Theoretical Computer Science*, 5(3):257–274.

Cite this article: Farjudian A and Moggi E (2023). Robustness, Scott continuity, and computability. *Mathematical Structures in Computer Science* **33**, 536–572. https://doi.org/10.1017/S0960129523000233