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2-generated axial algebras of Monster type $(2\beta, \beta)$

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A B S T R A C T

Axial algebras of Monster type (α, β) are a class of non-associative algebras that includes, besides associative algebras, other important examples such as the Jordan algebras and the Griess algebra. 2-generated primitive axial algebras of Monster type (α, β) naturally split into three cases: the case when $\alpha \notin \{2\beta, 4\beta\}$, the case $\alpha = 4\beta$ and $\alpha = 2\beta$. In this paper we give a complete classification all 2-generated primitive axial algebras of Monster type $(2\beta, \beta)$.

1. Introduction

This paper is part of a programme aimed at classifying all 2-generated axial algebras of Monster type (α, β) over a field of characteristic other than 2. Such algebras appear in different areas of mathematics and physics and are of particular interest for the study of several classes of finite simple groups, such as the 3-transposition groups, and many of

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the Sporadics, including the Monster (see [2,14] and the introductions of [20,13], and [5]). Just like for Lie algebras, a fundamental step towards the understanding of axial algebras of Monster type is the classification of the 2-generated ones. This project has its origins in work by S. Norton (see [18]), who classified the 2-generated subalgebras of the Griess algebra (which is a real axial algebra of Monster type $(\frac{1}{4}, \frac{1}{32})$). The general case was first attacked by F. Rehren [20,19], who proved that every 2-generated primitive axial algebra of Monster type (α, β) over a ring R in which 2, α, β, $\alpha-\beta$, $\alpha-2\beta$, and $\alpha-4\beta$ are invertible can be generated as R-module by 8 vectors and computed the structure constants with respect to these elements. In that paper the main case subdivision for this project is implicitly indicated, namely:

- *the* $\alpha \notin \{2\beta, 4\beta\}$ *case*;
- *the* $\alpha = 2\beta$ *case*;
- *the* $\alpha = 4\beta$ *case*.

Each of these cases is divided into two further subcases, depending whether the algebra admits an automorphism that swaps the two generating axes (the *symmetric case*) or not (the *non symmetric case*).

At present, results of T. Yabe [22], C. Franchi, M. Mainardis, S. Shpectorov, and J. McInroy [6,3,4], give a complete understanding of the symmetric case.

The non symmetric case is still wide open. A first result in this direction is the classification of the 2-generated primitive axial algebras over $\mathbb{Q}(\alpha, \beta)$, where α and β are algebraically independent indeterminates over $\mathbb{Q}$ obtained in [5]. In that case it turns out that every such algebra is symmetric. There are, however, non symmetric examples, e.g. the algebra $Q_2(\beta)$ constructed in [12,8], by V. Joshi. Another family of examples, pointed out by Michael Turner [21], comes from the Matsuo algebras $3C(\alpha)$. Such an algebra has basis b_0, b_1, b_2 and multiplication defined by $ab = \frac{\alpha}{2}(a+b-c)$, whenever $\{a, b, c\} = \{b_0, b_1, b_2\}$; each b_i is an axis of Jordan type α and in fact the algebra itself is a primitive 2-generated algebra of Jordan type α. Provided $\alpha \neq -1$, it has an identity $\mathbb{1} = \frac{1}{1+\alpha}(b_0 + b_1 + b_2)$. It is straightforward to see that, for $i \in \{1, 2, 3\}$, $a_i := \mathbb{1} - b_i$ is an axis of Jordan type $1-\alpha$. Hence, for $\alpha \neq \frac{1}{2}$, with respect to the generators a_i, b_j (with $i \neq j$ and $i, j \in \{1, 2, 3\}$), the algebra is a primitive 2-generated algebra of Monster type $(\alpha, 1-\alpha)$ and it is clearly non symmetric. We'll denote this algebra as $3C(\alpha, 1-\alpha)$.

In this paper we deal with the case $\alpha = 2\beta$ and give a complete classification of such algebras, namely we prove

Theorem 1.1. *Let V be a primitive axial algebra of Monster type $(2\beta, \beta)$ over a field $\mathbb{F}$ of characteristic other than 2. Then one of the following holds*

(1) *V is symmetric;*
(2) *V is isomorphic to $Q_2(\beta)$, or, when $\beta = -\frac{1}{2}$, to the 3-dimensional quotient of $Q_2(\beta)$;*
(3) *V is isomorphic to the algebra $3C(\frac{2}{3}, \frac{1}{3})$.*

The paper is organised as follows: in Section 2 we recall the basic definitions and properties of axial algebras which will be needed in the sequel. In Section 3 we construct a universal object for the category of 2-generated primitive axial algebras of Monster type $(2\beta, \beta)$, in a similar way as we did in [5, Section 4] for the case where $\alpha \notin \{2\beta, 4\beta\}$, and we prove that it is linearly spanned by 8 vectors. In Section 4 we apply the machinery developed in [5, Section 5] to the case $\alpha = 2\beta$. Precisely, we consider a ring R of characteristic other than 2, we denote by R_0 the prime subring of R and let $R_0[\frac{1}{2}, \beta, \frac{1}{\beta}][x, y, z, t]$ be the polynomial ring in 4 variables over $R_0[\frac{1}{2}, \beta, \frac{1}{\beta}]$. For a subset T of $R_0[\frac{1}{2}, \beta, \frac{1}{\beta}][x, y, z, t]$, let $\mathcal{V}(T)$ be the variety associated to (the ideal generated by) T. Finally we denote by $\mathcal{M}_2(2\beta, \beta, R)$ a set of representatives of the isomorphism classes of 2-generated primitive axial algebras of Monster type $(2\beta, \beta)$ over R. With such notation, we prove the following result.

Theorem 1.2. *Assume R is a ring of characteristic other than 2. Then there exists a subset $T \subseteq R_0(\alpha, \beta)[x, y, z, t]$ of size 5 and a map*

$$\xi \colon \mathcal{M}_2(2\beta, \beta, R) \to \mathcal{V}(T)$$

such that, for every P in the image of ξ, there exists an element V_P of $\mathcal{M}_2(2\beta, \beta, R)$ with the property that every element in $\xi^{-1}(P)$ is a quotient of V_P.

In Section 5 we show how Theorem 1.2 can be used to obtain an alternative and independent proof of Yabe's classification of the primitive symmetric axial algebras of Monster type $(2\beta, \beta)$ over a field (see Theorem 5.7). Having at hand the classification in the symmetric case, in Section 6 we deal with the non symmetric case and prove Theorem 1.1. The key observation is that, for a primitive 2-generated axial algebra of Monster type V with generating axes a_0 and a_1, the orbit under the Miyamoto group of each a_i ($i \in \{1, 2\}$) generates a primitive 2-generated axial subalgebra of Monster type which is symmetric.

2. Basics

We start by recalling the definition and basic features of axial algebras. Let R be a ring with identity and let $\mathcal{S}$ be a finite subset of R with $1 \in \mathcal{S}$. A *fusion law* on $\mathcal{S}$ is a map

$$\star \colon \mathcal{S} \times \mathcal{S} \to 2^{\mathcal{S}}.$$

An *axial algebra* over R with *spectrum* $\mathcal{S}$ and fusion law $\star$ is a commutative non-associative R-algebra V generated by a set $\mathcal{A}$ of nonzero idempotents (called *axes*) such that, for each $a \in \mathcal{A}$,

Table 1
Fusion law $\mathcal{M}(\alpha, \beta)$.

$\star$	1	0	α	β
1	1	$\emptyset$	α	β
0	$\emptyset$	0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

(Ax1) $ad(a) : v \mapsto av$ is a semisimple endomorphism of V with spectrum contained in $\mathcal{S}$;
(Ax2) for every $\lambda, \mu \in \mathcal{S}$, the product of a λ-eigenvector and a μ-eigenvector of ad_a is the sum of δ-eigenvectors, for $\delta \in \lambda \star \mu$.

Furthermore, V is called *primitive* if

(Ax3) $V_1 = \langle a \rangle$.

An axial algebra over R is said to be *of Monster type* (α, β) if it satisfies the fusion law $\mathcal{M}(\alpha, \beta)$ given in Table 1, with $\alpha, \beta \in R \setminus \{0, 1\}$, and $\alpha \neq \beta$.

Let V be a primitive axial algebra of Monster type (α, β) and let $a \in \mathcal{A}$. Let $\mathcal{S}^+ := \{1, 0, \alpha\}$ and $\mathcal{S}^- := \{\beta\}$. The partition $\{\mathcal{S}^+, \mathcal{S}^-\}$ of $\mathcal{S}$ induces a $\mathbb{Z}_2$-grading on $\mathcal{S}$ which, in turn, induces a $\mathbb{Z}_2$-grading $\{V_+^a, V_-^a\}$ on V, where $V_+^a := V_1^a + V_0^a + V_\alpha^a$ and $V_-^a = V_\beta^a$. It follows that, if τ_a is the map from the disjoint union $R \dot{\cup} V$ to $R \dot{\cup} V$ such that $\tau_{a|_V}$ is the multiplication by ε on V_ϵ^a for $\varepsilon \in \{\pm 1\}$ and $\tau_{a|_R}$ is the identity, then τ_a is an involutory automorphism of V (see [10, Proposition 3.4]). The map τ_a is called the *Miyamoto involution* associated to the axis a. By definition of τ_a, the element $av - \beta v$ of V is τ_a-invariant and, since a lies in $V_+^a \leq C_V(\tau_a)$, also $av - \beta(a + v)$ is τ_a-invariant. In particular, by symmetry,

Lemma 2.1. *Let a and b be axes of V. Then $ab - \beta(a + b)$ is fixed by the 2-generated group $\langle \tau_a, \tau_b \rangle$.*

If V is generated by the set of axes $\mathcal{A} := \{a_0, a_1\}$, for $i \in \{1, 2\}$. Set $\rho := \tau_{a_0}\tau_{a_1}$, and for $i \in \mathbb{Z}$, $a_{2i} := a_0^{\rho^i}$ and $a_{2i+1} := a_1^{\rho^i}$ (note that the orbits can be finite and even of different lengths, so there can be multiple labels for the same axis). Since ρ is an automorphism of V, for every $j \in \mathbb{Z}$, a_j is an axis. Denote by $\tau_j := \tau_{a_j}$ the corresponding Miyamoto involution.

Lemma 2.2. *For every $n \in \mathbb{N}$, and $i, j \in \mathbb{Z}$ such that $i \equiv j \mod n$ we have*

$$a_i a_{i+n} - \beta(a_i + a_{i+n}) = a_j a_{j+n} - \beta(a_j + a_{j+n}),$$

Proof. This follows immediately from Lemma 2.1. $\square$

For $n \in \mathbb{N}$ and $r \in \{0, \ldots, n-1\}$ set

$$s_{r,n} := a_r a_{r+n} - \beta(a_r + a_{r+n}). \tag{1}$$

If $\{0, 1, \alpha, \beta\}$ are pairwise distinguishable in R, i.e. α, β, $\alpha - 1$, $\beta - 1$, and $\alpha - \beta$ are invertible in R, by [5, Proposition 2.4], for every $a \in \mathcal{A}$, there is a linear function $\lambda_a : V \to R$, such that every $v \in V$ can be written as

$$v = \lambda_a(v)a + u \text{ with } u \in \bigoplus_{\delta \neq 1} V_\delta^a.$$

Remark 2.3. Note that, when R is a field, then $\{0, 1, \alpha, \beta\}$ are always pairwise distinguishable in R. Since we are interested in algebras over fields, from now on we assume $0, 1, \alpha, \beta$ are pairwise distinguishable in R. For the same reason, since the characteristic cannot be 2, we also assume that 2 is invertible in R.

For $i \in \mathbb{Z}$, let

$$a_i = \lambda_{a_0}(a_i)a_0 + u_i + v_i + w_i \tag{2}$$

be the decomposition of a_i into ad_{a_0}-eigenvectors, where u_i is a 0-eigenvector, v_i is an α-eigenvector and w_i is a β-eigenvector.

Lemma 2.4. *[5, Lemma 4.4] With the above notation,*

(1) $u_i = \frac{1}{\alpha}((\lambda_{a_0}(a_i) - \beta - \alpha\lambda_{a_0}(a_i))a_0 + \frac{1}{2}(\alpha - \beta)(a_i + a_{-i}) - s_{0,i})$;
(2) $v_i = \frac{1}{\alpha}((\beta - \lambda_{a_0}(a_i))a_0 + \frac{\beta}{2}(a_i + a_{-i}) + s_{0,i})$;
(3) $w_i = \frac{1}{2}(a_i - a_{-i})$.

Lemma 2.5. *Let I be an ideal of V, a an axis of V, $x \in V$ and let*

$$x = x_1 + x_0 + x_\alpha + x_\beta$$

be the decomposition of x as sum of ad_a-eigenvectors. If $x \in I$, then $x_1, x_0, x_\alpha, x_\beta \in I$. Moreover, I is τ_a-invariant.

Proof. Suppose $x \in I$. Then I contains the vectors

$$\begin{aligned} x - ax &= x_0 + (1-\alpha)x_\alpha + (1-\beta)x_\beta, \\ a(x - ax) &= \alpha(1-\alpha)x_\alpha + \beta(1-\beta)x_\beta, \\ a(a(x-ax)) - \beta a(x - ax) &= \alpha(\alpha - \beta)(1-\alpha)x_\alpha. \end{aligned}$$

Since $0, 1, \alpha, \beta$ are pairwise distinguishable in R, it follows that I contains x_1, x_0, x_α, x_β. Since $x^{\tau_a} = x_1 + x_0 + x_\alpha - x_\beta \in I$, the last assertion follows. $\square$

3. The universal object

From now on we assume that $\alpha = 2\beta$, $\{1, 0, 2\beta, \beta\}$ is a set of pairwise distinguishable elements in R, that is the difference of any two distinct elements of this set is a unit in R (note, in particular, that this implies that also 2 is a unit in R). We proceed by adapting to the case $\alpha = 2\beta$ the construction given in [5] of the universal object in the category of 2-generated primitive axial algebras of Monster type. Let

- D be the polynomial ring

$$\mathbb{Z}[x, y, w, t],$$

  where x, y, w, t are algebraically independent indeterminates over $\mathbb{Z}$;
- L be the ideal of D generated by the set

$$\Sigma := \{xy - 1,\ (1 - x)w - 1,\ 2t - 1\};$$

- $\hat{D} := D/L$. For $d \in D$, we denote the element $L + d$ by $\hat{d}$. Note that $\hat{1}$, $\hat{0}$, $\hat{x}$, and $2\hat{x}$ are pairwise distinguishable in $\hat{D}$;
- $\mathrm{A} := \{\mathrm{a}_0, \mathrm{a}_1\}$ be a set of cardinality 2 and W be the free commutative magma generated by the elements of $\mathcal{A}$ subject to the condition that every element of $\mathcal{A}$ is idempotent;
- $\hat{R} := \hat{D}[\Lambda]$ be the ring of polynomials with coefficients in $\hat{D}$ and indeterminates set $\Lambda := \{\lambda_{c,w} \mid c \in \mathcal{A}, w \in W, c \neq w\}$ where $\lambda_{c,w} = \lambda_{c',w'}$ if and only if $c = c'$ and $w = w'$.
- $\hat{V} := \hat{R}[W]$ be the set of all formal linear combinations $\sum_{w \in W} \gamma_w w$ of the elements of W with coefficients in $\hat{R}$, with only finitely many coefficients different from zero. Endow $\hat{V}$ with the usual structure of a commutative non-associative $\hat{R}$-algebra;
- $\hat{\mathcal{S}}$ be the set $\{1, 0, \hat{x}, 2\hat{x}\}$;
- for $\mu \in \hat{\mathcal{S}}$,

$$f_\mu := \prod_{\lambda \in \hat{\mathcal{S}} \setminus \{\mu\}} (x - \lambda),$$

- for every $c \in \mathcal{A}$, $\lambda_{c,c} := 1$;
- J be the ideal of $\hat{V}$ generated by all the elements

$$(f_1(\mathrm{ad}_c))(w - \lambda_{c,w} c), \tag{3}$$

  for all $c \in \mathcal{A}$ and $w \in W$, and

$$\left(\prod_{\eta \in \gamma \star \delta} (\mathrm{ad}_c - \eta\, \mathrm{id}_{\hat{V}}) \right) ((f_\gamma(ad_c))(v) \cdot (f_\delta(ad_c))(w)) \tag{4}$$

for all $v, w \in \hat{V}$, $\gamma, \delta \in \hat{\mathcal{S}}$, $c \in \mathcal{A}$;

- I_0 be the ideal of $\hat{R}$ generated by all the elements

$$\sum_{w \in W} \gamma_w \lambda_{c,w} \tag{5}$$

for all $c \in \mathcal{A}, w \in W$, and $\gamma_w \in \hat{R}$ such that $\sum_{w \in W} \gamma_w w \in J$.

Finally set

- $J_0 := J + I_0 \hat{V}$,
- $\overline{R} := \hat{R}/I_0$,
- $\overline{V} := \hat{V}/J_0$,
- $\beta := x + I_0$, and $\alpha := 2x + I_0$,
- $\mathbf{a}_i := \mathrm{a}_i + J_0$, for $i \in \{0, 1\}$ and $\mathcal{A} := \{\mathbf{a}_0, \mathbf{a}_1\}$.

From now on, we use bold letters to denote the elements of $\overline{V}$.

For $i \in \mathbb{Z}$, set

$$\lambda_i := \lambda_{\mathbf{a}_0}(\mathbf{a}_i).$$

By Corollary 3.8 in [5], the permutation that swaps $\mathbf{a}_0$ and $\mathbf{a}_1$ induces a semi-automorphism f of $\overline{V}$, i.e. f acts on $\overline{R}$ as a ring homomorphism, on $\overline{V}$ as a (non-associative) ring homomorphism and $(\gamma \mathbf{v})^f = \gamma^f \mathbf{v}^f$, for every $\gamma \in \overline{R}$ and $\mathbf{v} \in \overline{V}$. By [5, Lemma 3.1],

$$\lambda_{\mathbf{a}_1}(\mathbf{a}_0) = \lambda_1^f, \text{ and } \lambda_{\mathbf{a}_1}(\mathbf{a}_{-1}) = \lambda_2^f.$$

Set $G_0 := \langle \tau_0, \tau_1 \rangle$ and $G := \langle \tau_0, f \rangle$ (where τ_0 and τ_1 are the involutory automorphisms of $\overline{V}$ as defined in Section 2).

Lemma 3.1. *[5, Lemma 4.6] The groups G_0 and G are dihedral groups, G_0 is a normal subgroup of G such that $|G : G_0| \leq 2$. For every $n \in \mathbb{N}$, the set $\{\mathbf{s}_{0,n}, \ldots, \mathbf{s}_{n-1,n}\}$ is invariant under the action of G. In particular, if K_n is the kernel of this action, we have*

(1) *$K_1 = G$;*
(2) *$K_2 = G_0$, in particular $\mathbf{s}_{0,2}^{\tau_1} = \mathbf{s}_{0,2}$ and $\mathbf{s}_{0,2}^f = \mathbf{s}_{1,2}$;*
(3) *G/K_3 induces the full permutation group on the set $\{\mathbf{s}_{0,3}, \mathbf{s}_{1,3}, \mathbf{s}_{2,3}\}$ with point stabilisers generated by $\tau_0 K_3$, $\tau_1 K_3$ and $f K_3$, respectively. In particular $\mathbf{s}_{0,3}^f = \mathbf{s}_{1,3}$, $\mathbf{s}_{1,3}^{\tau_0} = \mathbf{s}_{2,3}$, and $\mathbf{s}_{0,3}^{\tau_1} = \mathbf{s}_{2,3}$.*

Proof. This follows immediately from the definitions. $\square$

For $i, j \in \{1, 2, 3\}$, with the notation fixed before Lemma 2.4 and $\mathbf{s}_{0,i} \in \overline{V}$ defined as in Equation (1), set

$$P_{ij} := \mathbf{u}_i\mathbf{u}_j + \mathbf{u}_i\mathbf{v}_j \quad \text{and} \quad Q_{ij} := \mathbf{u}_i\mathbf{v}_j + \frac{1}{\alpha^2}\mathbf{s}_{0,i}\mathbf{s}_{0,j}.$$

Lemma 3.2. *[5, Lemma 4.5] For $i, j \in \{1, 2, 3\}$ we have*

$$\mathbf{s}_{0,i} \cdot \mathbf{s}_{0,j} = -\alpha(\mathbf{a}_0 P_{ij} - \alpha Q_{ij}). \tag{6}$$

Proof. Since $\mathbf{u}_i$ and $\mathbf{v}_j$ are a 0-eigenvector and an α-eigenvector for $\mathrm{ad}_{\mathbf{a}_0}$, respectively, by the fusion rule, we have $\mathbf{a}_0 P_{ij} = \alpha(\mathbf{u}_i \cdot \mathbf{v}_j)$ and the result follows. □

The following polynomial will play a crucial rôle in the classification of the non symmetric algebras in Section 6:

$$Z(x, y) := \frac{2}{\beta}x + \frac{(2\beta - 1)}{\beta^2}y - \frac{(4\beta - 1)}{\beta}. \tag{7}$$

Lemma 3.3. *In $\overline{V}$ the following equalities hold:*

$$\begin{aligned}\mathbf{s}_{0,2} = &-\frac{\beta}{2}(\mathbf{a}_{-2} + \mathbf{a}_2) + \beta Z(\lambda_1, \lambda_1^f)(\mathbf{a}_1 + \mathbf{a}_{-1}) \\ &- \left[2Z(\lambda_1, \lambda_1^f)(\lambda_1 - \beta) - (\lambda_2 - \beta)\right]\mathbf{a}_0 + 2Z(\lambda_1, \lambda_1^f)\mathbf{s}_{0,1},\end{aligned}$$

and

$$\begin{aligned}\mathbf{s}_{1,2} = &-\frac{\beta}{2}(\mathbf{a}_{-1} + \mathbf{a}_3) + \beta Z(\lambda_1^f, \lambda_1)(\mathbf{a}_0 + \mathbf{a}_2) \\ &- \left[2Z(\lambda_1^f, \lambda_1)(\lambda_1^f - \beta) - (\lambda_2^f - \beta)\right]\mathbf{a}_1 + 2Z(\lambda_1^f, \lambda_1)\mathbf{s}_{0,1}.\end{aligned}$$

Proof. Since $\alpha = 2\beta$, from the first formula in [5, Lemma 4.7] we deduce the expression for $\mathbf{s}_{0,2}$. The expression for $\mathbf{s}_{1,2}$ follows by applying f. □

Lemma 3.4. *In $\overline{V}$ we have*

$$\begin{aligned}\mathbf{a}_4 = \mathbf{a}_{-2} &- 2Z(\lambda_1, \lambda_1^f)(\mathbf{a}_{-1} - \mathbf{a}_3) \\ &+ \frac{1}{\beta}\left[4Z(\lambda_1, \lambda_1^f)\,(\lambda_1 - \beta) - (2\lambda_2 - \beta)\right](\mathbf{a}_0 - \mathbf{a}_2)\end{aligned}$$

and

$$\begin{aligned}\mathbf{a}_{-3} = \mathbf{a}_3 &- 2Z(\lambda_1^f, \lambda_1)(\mathbf{a}_2 - \mathbf{a}_{-2}) \\ &+ \frac{1}{\beta}\left[4Z(\lambda_1^f, \lambda_1)\left(\lambda_1^f - \beta\right) - \left(2\lambda_2^f - \beta\right)\right](\mathbf{a}_1 - \mathbf{a}_{-1}).\end{aligned}$$

Proof. Since $\mathbf{s}_{0,2}$ is invariant under τ_1, we have $\mathbf{s}_{0,2} - \mathbf{s}_{0,2}^{\tau_1} = 0$. On the other hand, in the expression $\mathbf{s}_{0,2} - \mathbf{s}_{0,2}^{\tau_1}$ obtained from the first formula of Lemma 3.3, the coefficient of $\mathbf{a}_4$ is $-\beta/2$, which is invertible in $\overline{R}$. Hence we deduce the expression for $\mathbf{a}_4$. By applying the map f to the expression for $\mathbf{a}_4$ we get the expression for $\mathbf{a}_{-3}$. $\square$

Lemma 3.5. *In $\overline{V}$ we have*

$$
\begin{aligned}
&\mathbf{s}_{0,1}\mathbf{s}_{0,1} = \\
&+\frac{\beta^2}{4} Z(\lambda_1^f, \lambda_1)(\mathbf{a}_{-2} + \mathbf{a}_2) \\
&+\frac{1}{2}\Big[-2(2\beta-1)(\lambda_1^2 + {\lambda_1^f}^2) - \frac{(8\beta^2-4\beta+1)}{\beta}\lambda_1\lambda_1^f + (16\beta^2-7\beta+1)\lambda_1 \\
&\quad +(14\beta^2-8\beta+1)\lambda_1^f - \beta(14\beta^2-7\beta+1)\Big](\mathbf{a}_{-1}+\mathbf{a}_1) \\
&+\Big[\frac{2(2\beta-1)}{\beta}\lambda_1^3 + \frac{(8\beta^2-4\beta+1)}{\beta^2}\lambda_1^2\lambda_1^f + \frac{2(2\beta-1)}{\beta}\lambda_1{\lambda_1^f}^2 - (18\beta-6)\lambda_1^2 \\
&\quad -\frac{2(10\beta^2-5\beta+1)}{\beta}\lambda_1\lambda_1^f - 2(\beta-1){\lambda_1^f}^2 - \frac{(2\beta-1)}{2}\lambda_1\lambda_2 - \beta\lambda_1^f\lambda_2 \\
&\quad +\frac{(54\beta^2-17\beta+1)}{2}\lambda_1 + (9\beta^2-6\beta+1)\lambda_1^f + \frac{\beta(5\beta-1)}{2}\lambda_2 - \frac{\beta^2}{2}\lambda_2^f \\
&\quad -\frac{\beta(24\beta^2-9\beta+1)}{2}\Big]\mathbf{a}_0 \\
&+\Big[-\frac{2(2\beta-1)}{\beta}\lambda_1^2 - \frac{(6\beta^2-3\beta+1)}{\beta^2}\lambda_1\lambda_1^f - \frac{2(\beta-1)}{\beta}{\lambda_1^f}^2 + \frac{(16\beta^2-7\beta+1)}{\beta}\lambda_1 \\
&\quad +\frac{(10\beta^2-7\beta+1)}{\beta}\lambda_1^f - \frac{\beta}{2}\lambda_2^f - \frac{(57\beta^2+26\beta-4)}{4}\Big]\mathbf{s}_{0,1} \\
&+\frac{\beta^2}{4}\mathbf{s}_{0,3}.
\end{aligned}
$$

Proof. We apply Lemma 3.2 with $i = j = 1$. By Lemma 2.4 and by the definition of $\mathbf{s}_{0,1}$ and $\mathbf{s}_{1,2}$, the product $\mathbf{u}_1\mathbf{u}_1$ (resp. $\mathbf{u}_1\mathbf{v}_1$) can be written as a linear combination of $\mathbf{a}_{-1}$, $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{s}_{0,1}$, $\mathbf{s}_{1,2}$, and $\mathbf{s}_{0,1}\mathbf{s}_{0,1}$, with the coefficient of $\mathbf{s}_{0,1}\mathbf{s}_{0,1}$ equal to $\frac{1}{4\beta^2}$ (resp. $-\frac{1}{4\beta^2}$). Thus, $P_{1,1}$ and $Q_{1,1}$ are linear combinations of $\mathbf{a}_{-1}$, $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{s}_{0,1}$, and $\mathbf{s}_{1,2}$ and, by Lemma 3.3, we can replace $\mathbf{s}_{1,2}$ by a linear combination of $\mathbf{a}_{-1}$, $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{s}_{0,1}$. By Lemma [5, Lemma 4.3], it follows that $\mathbf{a}_0P_{1,1}$ is a linear combination of $\mathbf{a}_{-2}$, $\mathbf{a}_{-1}$, $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{s}_{0,1}$, $\mathbf{s}_{0,2}$, and $\mathbf{s}_{0,3}$. The result then follows by Lemma 3.2. $\square$

Lemma 3.6. *In $\overline{V}$ we have*

$$\mathbf{s}_{1,3} = \mathbf{s}_{0,3} + \beta Z(\lambda_1^f, \lambda_1)\mathbf{a}_{-2} - \beta Z(\lambda_1, \lambda_1^f)\mathbf{a}_3$$

$$
\begin{aligned}
&+\frac{1}{\beta^3}\Big[-4\beta(2\beta-1)(\lambda_1^2+\lambda_1^{f\,2})-2(8\beta^2-4\beta+1)\lambda_1\lambda_1^f+2\beta(15\beta^2-7\beta+1)\lambda_1\\
&\quad+\beta(26\beta^2-15\beta+2)\lambda_1^f-\beta^2(24\beta^2-13\beta+2)\Big]\,\mathbf{a}_{-1}\\
&+\frac{1}{\beta^4}\Big[8\beta(2\beta-1)\lambda_1^3+4(8\beta^2-4\beta+1)\lambda_1^2\lambda_1^f+8\beta(2\beta-1)\lambda_1\lambda_1^{f\,2}\\
&\quad-4\beta^2(15\beta-5)\lambda_1^2-2\beta(32\beta^2-16\beta+3)\lambda_1\lambda_1^f+4\beta^2\lambda_1^{f\,2}-2\beta^2(2\beta+1)\lambda_1\lambda_2\\
&\quad-4\beta^3\lambda_1^f\lambda_2+2\beta^3(40\beta-9)\lambda_1+2\beta^2(2\beta^2-5\beta+1)\lambda_1^f+2\beta^3(5\beta-1)\lambda_2\\
&\quad-2\beta^4\lambda_2^f-4\beta^4(5\beta-1)\Big]\,\mathbf{a}_0\\
&+\frac{1}{\beta^4}\Big[-8\beta(2\beta-1)\lambda_1^2\lambda_1^f-4(8\beta^2-2\beta+1)\lambda_1\lambda_1^{f\,2}-8\beta(2\beta-1)\lambda_1^{f\,3}-4\beta^2\lambda_1^2\\
&\quad+2\beta(32\beta^2-16\beta+3)\lambda_1\lambda_1^f+4\beta^2(16\beta-5)\lambda_1^{f\,2}+4\beta^3\lambda_1\lambda_2^f\\
&\quad+2\beta^2(2\beta-1)\lambda_1^f\lambda_2^f-2\beta^2(2\beta^2+5\beta+1)\lambda_1-2\beta^3(40\beta-9)\lambda_1^f+2\beta^4\lambda_2\\
&\quad-2\beta^3(5\beta-1)\lambda_2^f+4\beta^4(5\beta-1)\Big]\,\mathbf{a}_1\\
&+\frac{1}{\beta^3}\Big[4\beta(2\beta-1)\lambda_1^2+2(8\beta^2-4\beta+1)\lambda_1\lambda_1^f+\beta(8\beta-4)\lambda_1^{f\,2}\\
&\quad-\beta(26\beta^2-15\beta+2)\lambda_1-2\beta(15\beta^2-7\beta+1)\lambda_1^f+\beta^2(24\beta^2-13\beta+2)\Big]\,\mathbf{a}_2\\
&+\frac{1}{\beta^2}\Big[-8(\lambda_1^2-\lambda_1^{f\,2})+24\beta(\lambda_1-\lambda_1^f)+2\beta(\lambda_2-\lambda_2^f)\Big]\,\mathbf{s}_{0,1}.
\end{aligned}
$$

Similarly, $\mathbf{s}_{2,3}$ *belongs to the linear span of the elements* $\mathbf{a}_{-2}$, $\mathbf{a}_{-1}$, $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{s}_{0,1}$, *and* $\mathbf{s}_{0,3}$ *(since* $\mathbf{s}_{2,3}=\mathbf{s}_{1,3}^{\tau_0}$*, the exact expression for* $\mathbf{s}_{2,3}$ *is obtained by applying* τ_0 *to the above equation).*

Proof. Since, by Lemma 3.1, $\mathbf{s}_{0,1}$ is invariant under f, we have $\mathbf{s}_{0,1}\mathbf{s}_{0,1}-(\mathbf{s}_{0,1}\mathbf{s}_{0,1})^f=0$. Comparing the expressions for $\mathbf{s}_{0,1}\mathbf{s}_{0,1}$ and $(\mathbf{s}_{0,1}\mathbf{s}_{0,1})^f$ obtained from Lemma 3.5 and since by Lemma 3.1(3) $\mathbf{s}_{0,3}^f=\mathbf{s}_{1,3}$, we deduce the expression for $\mathbf{s}_{1,3}$. By applying the map τ_0 to the expression for $\mathbf{s}_{1,3}$ and since, by Lemma 3.1(3), $\mathbf{s}_{1,3}^{\tau_0}=\mathbf{s}_{2,3}$, we get an expression for $\mathbf{s}_{2,3}$ as a linear combination of the vectors $\mathbf{a}_{-3}$, $\mathbf{a}_{-2}$, $\mathbf{a}_{-1}$, $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{s}_{0,1}$, and $\mathbf{s}_{0,3}$. Finally, the last assertion follows since, by Lemma 3.4, $\mathbf{a}_{-3}\in\langle\mathbf{a}_{-2},\mathbf{a}_{-1},\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\rangle$. □

As a consequence of the *resurrection principle* [15, Lemma 1.7], we can now prove the following result. We use double angular brackets to denote algebra generation and singular angular brackets for linear span.

Proposition 3.7. $\overline{V}=\langle\mathbf{a}_{-2},\mathbf{a}_{-1},\mathbf{a}_0,\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{s}_{0,1},\mathbf{s}_{0,3}\rangle$. *In particular,* $\overline{V}$ *has dimension at most* 8.

Proof. Set $U := \langle \mathbf{a}_{-2}, \mathbf{a}_{-1}, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{s}_{0,1}, \mathbf{s}_{0,3} \rangle$. By Lemma 3.4, $\mathbf{a}_4, \mathbf{a}_{-3} \in U$, and by Lemma 3.6 also $\mathbf{s}_{1,3}$ and $\mathbf{s}_{2,3}$ belong to U. It follows that U is invariant under the maps τ_0, τ_1, and f. Hence, a_i belongs to U, for every $i \in \mathbb{Z}$. Now we show that U is closed under the algebra product. Since it is invariant under the maps τ_0, τ_1, and f, it is enough to show that it is invariant under the action of $\mathrm{ad}_{\mathbf{a}_0}$ and it contains $\mathbf{s}_{0,1}\mathbf{s}_{0,1}$, $\mathbf{s}_{0,3}\mathbf{s}_{0,3}$, and $\mathbf{s}_{0,1}\mathbf{s}_{0,3}$. The products $\mathbf{a}_0\mathbf{a}_i$, for $i \in \{-2,-1,0,1,2,3\}$ belong to U by the definition of U and by Lemma 3.3. By [5, Lemma 4.3], U contains $\mathbf{a}_0\mathbf{s}_{0,1}$ and $\mathbf{a}_0\mathbf{s}_{0,3}$. The product $\mathbf{s}_{0,1}\mathbf{s}_{0,1}$ belongs to U by Lemma 3.5 and similarly, by Lemma 2.4 and Lemma 3.2, the products $\mathbf{s}_{0,3}\mathbf{s}_{0,3}$, and $\mathbf{s}_{0,1}\mathbf{s}_{0,3}$ belong to U. Hence U is a subalgebra of $\overline{V}$ and, since it contains the generators $\mathbf{a}_0$ and $\mathbf{a}_1$, we get $U = \overline{V}$. □

Remark 3.8. Note that the above proof gives a constructive way to compute the structure constants of the algebra $\overline{V}$ relative to the generating set B. This has been done with the use of GAP [9] in [7, 2btnon-Symmetric.s]. The explicit expressions however are far too long to be written here.

Corollary 3.9. *In $\overline{R}$ the following identities hold:*

$$\begin{aligned}
\lambda_{\mathbf{a}_0}(\mathbf{a}_{-1}) &= \lambda_1,\\
\lambda_{\mathbf{a}_0}(\mathbf{a}_{-2}) &= \lambda_2,\\
\lambda_{\mathbf{a}_0}(\mathbf{s}_{0,1}) &= \lambda_1 - \beta - \beta\lambda_1,\\
\lambda_{\mathbf{a}_0}(\mathbf{s}_{0,3}) &= \lambda_{\mathbf{a}_0}(\mathbf{a}_3) - \beta - \beta\lambda_{\mathbf{a}_0}(\mathbf{a}_3),\\
\lambda_{\mathbf{a}_0}(\mathbf{a}_3) &= \tfrac{8(\beta-1)}{\beta^3}\lambda_1^3 - \tfrac{4(2\beta^2+\beta-1)}{\beta^4}\lambda_1^2\lambda_1^f - \tfrac{4(2\beta-1)^2}{\beta^4}\lambda_1{\lambda_1^f}^2\\
&\quad - \tfrac{4(4\beta^2-7\beta+1)}{\beta^3}\lambda_1^2 + \tfrac{16(2\beta-1)}{\beta^2}\lambda_1\lambda_1^f + \tfrac{6}{\beta}\lambda_1\lambda_2 + \tfrac{2(2\beta-1)}{\beta^2}\lambda_1^f\lambda_2\\
&\quad + \tfrac{(\beta^2-22\beta+4)}{\beta^2}\lambda_1 - \tfrac{2(2\beta-1)}{\beta^2}\lambda_1^f - \tfrac{2(5\beta+1)}{\beta}\lambda_2 + \tfrac{2(5\beta-1)}{\beta}\\
\lambda_{\mathbf{a}_1}(\mathbf{s}_{0,1}) &= \lambda_1^f - \beta - \beta\lambda_1^f,\\
\lambda_{\mathbf{a}_1}(\mathbf{a}_{-2}) &= \lambda_{\mathbf{a}_0}(\mathbf{a}_3)^f,\\
\lambda_{\mathbf{a}_1}(\mathbf{a}_2) &= \lambda_1^f,
\end{aligned}$$

In particular, $\overline{R}$ is generated as a $\hat{D}$-algebra by λ_1, λ_2, λ_1^f, and λ_2^f.

Proof. By Proposition 3.7, $\overline{V}$ is generated as $\overline{R}$-module by the set $B := \{\mathbf{a}_{-2}, \mathbf{a}_{-1}, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{s}_{0,1}, \mathbf{s}_{0,3}\}$. Since $\lambda_{\mathbf{a}_1}(v) = (\lambda_{\mathbf{a}_0}(\mathbf{v}^f))^f$, $\lambda_{\mathbf{a}_0}$ is a linear function, and $\overline{R} = \overline{R}^f$, we just need to show that $\lambda_{\mathbf{a}_0}(\mathbf{v}) \in \hat{D}[\lambda_1, \lambda_1^f, \lambda_2, \lambda_2^f]$ for every $\mathbf{v} \in B$. By definition we have

$$\lambda_{\mathbf{a}_0}(\mathbf{a}_0) = 1,\ \ \lambda_{\mathbf{a}_0}(\mathbf{a}_1) = \lambda_1,\ \ \lambda_{\mathbf{a}_0}(\mathbf{a}_2) = \lambda_2,\ \text{and}\ \ \lambda_{\mathbf{a}_0}(\mathbf{a}_3) = \lambda_3.$$

Since τ_0 fixes $\mathbf{a}_0$ and is an $\overline{R}$-automorphism of $\overline{V}$, we get

$$\lambda_{\mathbf{a}_0}(\mathbf{a}_{-1}) = \lambda_{\mathbf{a}_0}((\mathbf{a}_1)^{\tau_0}) = \lambda_1,$$
$$\lambda_{\mathbf{a}_0}(\mathbf{a}_{-2}) = \lambda_{\mathbf{a}_0}((\mathbf{a}_2)^{\tau_0}) = \lambda_2,$$

and

$$\lambda_{\mathbf{a}_0}(\mathbf{s}_{0,1}) = \lambda_{\mathbf{a}_0}(\mathbf{a}_0\mathbf{a}_1 - \beta\mathbf{a}_0 - \beta\mathbf{a}_1) = \lambda_1 - \beta - \beta\lambda_1,$$

and

$$\lambda_{\mathbf{a}_0}(\mathbf{s}_{0,3}) = \lambda_{\mathbf{a}_0}(\mathbf{a}_0\mathbf{a}_3 - \beta\mathbf{a}_0 - \beta\mathbf{a}_3) = \lambda_3 - \beta - \beta\lambda_3.$$

We conclude the proof by showing that $\lambda_3 \in \hat{D}[\lambda_1, \lambda_1^f, \lambda_2, \lambda_2^f]$. Set

$$\phi := \mathbf{u}_1\mathbf{u}_1 - \mathbf{v}_1\mathbf{v}_1 - \lambda_{\mathbf{a}_0}(\mathbf{u}_1\mathbf{u}_1 - \mathbf{v}_1\mathbf{v}_1)\mathbf{a}_0$$

and

$$\mathbf{z} := \phi - 2(2\beta - \lambda_1)\mathbf{u}_1.$$

Then, by the fusion law, ϕ is a 0-eigenvector for $\mathrm{ad}_{\mathbf{a}_0}$ and so $\mathbf{z}$ is a 0-eigenvector for $\mathrm{ad}_{\mathbf{a}_0}$ as well. Since $\mathbf{s}_{0,1}$ is τ_0-invariant, it lies in $\overline{V}_+^{\mathbf{a}_0}$ and the fusion law implies that the product $\mathbf{z}\mathbf{s}_{0,1}$ is a sum of a 0- and an α-eigenvector for $\mathrm{ad}_{\mathbf{a}_0}$. In particular, $\lambda_{\mathbf{a}_0}(\mathbf{z}\mathbf{s}_{0,1}) = 0$. By Remark 3.8 we can compute explicitly the product $\mathbf{z}\mathbf{s}_{0,1}$ (see [7, 2btnon-Symmetric.s]):

$$\begin{aligned}
\mathbf{z}\mathbf{s}_{0,1} = &-\frac{\beta^3}{4}\mathbf{a}_3 \\
&+\frac{\beta}{4}\left[2\beta\lambda_1 - \lambda_1^f - \beta(\beta-1)\right]\mathbf{a}_{-2} \\
&+\Bigg[-\beta^2\lambda_1^2 - \frac{(2\beta^2+\beta-1)}{2}\lambda_1\lambda_1^f - \frac{(2\beta^2-4\beta+1)}{2\beta}\lambda_1^{f\,2} + \frac{(4\beta^3-\beta^2-\beta)}{2}\lambda_1 \\
&+(2\beta^2-4\beta+1)\lambda_1^f + \frac{\beta^3}{4}\lambda_2 - \frac{\beta^2}{4}\lambda_2^f + \frac{\beta(4\beta-1}{2}\Bigg]\mathbf{a}_{-1} \\
&+\Bigg[2(2\beta-1)\lambda^3 + \frac{(2\beta-1)^2}{\beta}\lambda_1^2\lambda_1^f - (10\beta^2-8\beta+1)\lambda_1^2 + (-2\beta^2+3\beta-1)\lambda_1\lambda_1^f \\
&-\frac{\beta(2\beta-1)}{2}\lambda_1\lambda_2 + \beta(2\beta-1)^2\lambda_1 + \frac{\beta(2\beta-1)}{2}\lambda_1^f + \frac{\beta^2(\beta-1)}{2}\lambda_2 - \frac{\beta^2(4\beta-1)}{2}\Bigg]\mathbf{a}_0 \\
&+\Bigg[-\beta^2)\lambda_1^2 - \frac{(\beta+3)(2\beta-1)}{2}\lambda_1\lambda_1^f - \frac{(6\beta^2-4\beta+1)}{2\beta}\lambda_1^{f\,2} + \frac{\beta(4\beta^2+3\beta-3)}{2}\lambda_1 \\
&+(8\beta^2-5\beta+1)\lambda_1^f + \frac{bt^3}{4}\lambda_2 + \frac{\beta^2}{4}\lambda_2^f - \frac{\beta(17\beta^2-12\beta+2)}{4}\Bigg]\mathbf{a}_1 \\
&+\left[\frac{\beta(3\beta-1)}{2}\lambda_1 + \frac{\beta(4\beta-1)}{4}\lambda_1^f - \frac{3\beta^2(3\beta-1)}{4}\right]\mathbf{a}_2
\end{aligned}$$

$$+\left[-2\beta\lambda_1^2-(2\beta-1)\lambda_1\lambda_1^f+\beta(4\beta+1)\lambda_1+(2\beta-1)\lambda_1^f+\frac{\beta^2}{2}\lambda_2-\frac{\beta(9\beta+2)}{2}\right]\mathbf{s}_{0,1}.$$

Since $\lambda_{a_0}(\mathbf{z}\mathbf{s}_{0,1})=0$, taking the image under $\lambda_{\mathbf{a}_0}$ of both sides, we get the expression for $\lambda_3=\lambda_{\mathbf{a}_0}(\mathbf{a}_3)$. □

We conclude this section giving some relations in $\overline{V}$ and $\overline{R}$ which will be useful in the sequel for the classification of the algebras. The explicit expressions for these relations can be found in [7, 2btnon-Symmetric.s].

Lemma 3.10. *Set* $\mathbf{d_0}:=\mathbf{a}_4^2-\mathbf{a}_4,\ \ \mathbf{d}_1:=\mathbf{s}_{2,3}^f-\mathbf{s}_{2,3},\ \ \mathbf{d}_2:=\mathbf{d}_1{}^{\tau_1}$*, and, for* $i\in\{1,2\}$, $\mathbf{D}_i:=\mathbf{d}_i{}^{\tau_0}-\mathbf{d}_i$*. Further, let* $\mathbf{e}:=\mathbf{u}_1^{\tau_1}\mathbf{v}_3^{\tau_1}$ *and* $\mathbf{E}:=\mathbf{a}_2\mathbf{e}-2\beta\mathbf{e}$*.*

Then, for $i\in\{1,2\}$*, the following identities hold:*

(1) $\mathbf{d}_i=0,\ \ \mathbf{D}_i=0,\ \ \mathbf{E}=0$*;*
(2) *there exists an element* $t(\lambda_1,\lambda_1^f,\lambda_2,\lambda_2^f)\in\overline{R}$ *such that*

$$\begin{aligned}\mathbf{g}\ &:=\ t(\lambda_1,\lambda_1^f,\lambda_2,\lambda_2^f)\mathbf{a}_0\\&\quad+\left\{4(\lambda_1-\lambda_1^f)\left[\beta\lambda_1+(\beta-1)(\lambda_1^f-\beta)\right]+\beta^2(\lambda_2-\lambda_2^f)\right\}(\mathbf{a}_{-1}+\mathbf{a}_1+\frac{2}{\beta}\mathbf{s}_{0,1})=0.\end{aligned}$$

Proof. Identities involving the $\mathbf{d}_i$'s and $\mathbf{D}_i$'s follow from Lemma 3.1 and the fact that $\mathbf{a}_4$ is idempotent. By the fusion law, the product $\mathbf{u}_1\mathbf{u}_2$ is a 0-eigenvector for $\mathrm{ad}_{\mathbf{a}_0}$ and the product $\mathbf{u}_1^{\tau_1}\mathbf{v}_3^{\tau_1}$ is a 2β-eigenvector for $\mathrm{ad}_{\mathbf{a}_2}$. The last claim follows by an explicit computation of the product $\mathbf{a}_0(\mathbf{u}_1\mathbf{u}_2)$, which gives the left hand side of the equation. □

Lemma 3.11. *In the ring* $\overline{R}$ *the following holds:*

(1) $\lambda_{\mathbf{a}_0}(\mathbf{d}_0)=0$,
(2) $\lambda_{\mathbf{a}_0}(\mathbf{d}_1)=0$,
(3) $\lambda_{\mathbf{a}_0}(\mathbf{d}_2)=0$,
(4) $\lambda_{\mathbf{a}_1}(\mathbf{d}_1)=0$,
(5) $\lambda_{\mathbf{a}_1}(\mathbf{g}-\mathbf{g}^f)=0$.

Proof. This follows immediately from Lemma 3.10. □

4. Proof of Theorem 1.2

The five identities in Lemma 3.11 produce five polynomials $p_i(x,y,z,t)$ for $i\in\{1,\dots,5\}$ in $\hat{D}[x,y,z,t]$ (with x,y,z,t indeterminates over $\hat{D}$), that simultaneously annihilate on the quadruple $(\lambda_1,\lambda_1^f,\lambda_2,\lambda_2^f)$. Precisely, by Lemmas 3.4 and 3.6 we can write explicitly $\mathbf{a}_4$, $\mathbf{s}_{2,3}$, and $\mathbf{s}_{2,3}^f$, whence the $\mathbf{d}_i$'s, as linear combinations of the vectors $\mathbf{a}_{-2}$, $\mathbf{a}_{-1}$, $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{s}_{0,1}$, and $\mathbf{s}_{0,3}$ with coefficients in $\overline{R}$. Using linearity of $\lambda_{\mathbf{a}_0}$ and $\lambda_{\mathbf{a}_1}$

and Corollary 3.9, we get the desired polynomials. These are too long to be displayed here but can be found in [7, 2btnon-Symmetric.s].

Let further, for $i \in \{1,2,3\}$, $q_i(x,z) := p_i(x,x,z,z)$. Then

$$
\begin{aligned}
q_1(x,z) = {} & \frac{128}{\beta^{10}}(-384\beta^5 + 608\beta^4 - 376\beta^3 + 114\beta^2 - 17\beta + 1)x^7 \\
& + \frac{64}{\beta^{10}}(4352\beta^6 - 6080\beta^5 + 2992\beta^4 - 516\beta^3 - 40\beta^2 + 23\beta - 2)x^6 \\
& + \frac{64}{\beta^8}(64\beta^4 - 96\beta^3 + 52\beta^2 - 12\beta + 1)x^5 z \\
& + \frac{16}{\beta^9}(-38720\beta^6 + 42912\beta^5 - 9252\beta^4 - 5928\beta^3 + 3477\beta^2 - 660\beta + 44)x^5 \\
& + \frac{16}{\beta^8}(-3168\beta^5 + 4832\beta^4 - 2782\beta^3 + 747\beta^2 - 92\beta + 4)x^4 z \\
& + \frac{32}{\beta^5}(8\beta^2 - 6\beta + 1)x^3 z^2 \\
& + \frac{8}{\beta^8}(84832\beta^6 - 48224\beta^5 - 50482\beta^4 + 55573\beta^3 - 20164\beta^2 + 3262\beta - 200)x^4 \\
& + \frac{8}{\beta^7}(19792\beta^5 - 30292\beta^4 + 17700\beta^3 - 4917\beta^2 + 647\beta - 32)x^3 z \\
& + \frac{16}{\beta^5}(-72\beta^3 + 62\beta^2 - 15\beta + 1)x^2 z^2 \\
& + \frac{8}{\beta^7}(-45888\beta^6 - 33584\beta^5 + 119184\beta^4 - 85132\beta^3 + 27054\beta^2 - 4089\beta + 240)x^3 \\
& + \frac{4}{\beta^6}(-52880\beta^5 + 81156\beta^4 - 47828\beta^3 + 13527\beta^2 - 1838\beta + 96)x^2 z \\
& + \frac{32}{\beta^4}(48\beta^3 - 44\beta^2 + 12\beta - 1)xz^2 + \frac{4}{\beta^2}(2\beta - 1)z^3 \\
& + \frac{4}{\beta^6}(19648\beta^6 + 114384\beta^5 - 204648\beta^4 + 128262\beta^3 - 38411\beta^2 + 5598\beta - 320)x^2 \\
& + \frac{8}{\beta^5}(16288\beta^5 - 25096\beta^4 + 14904\beta^3 - 4272\beta^2 + 593\beta - 32)xz \\
& + \frac{2}{\beta^3}(-322\beta^3 + 301\beta^2 - 86\beta + 8)z^2 \\
& + \frac{8}{\beta^5}(-26112\beta^5 + 40040\beta^4 - 23878\beta^3 + 6959\beta^2 - 995\beta + 56)x \\
& + \frac{2}{\beta^4}(-15264\beta^5 + 23658\beta^4 - 14169\beta^3 + 4110\beta^2 - 580\beta + 32)z \\
& + \frac{4}{\beta^4}(7632\beta^5 - 11668\beta^4 + 6932\beta^3 - 2011\beta^2 + 286\beta - 16),
\end{aligned}
$$

$$
\begin{aligned}
q_2(x,z) = &-\frac{8}{\beta^4}(8\beta^2-6\beta+1)x^4+\frac{4}{\beta^4}(40\beta^3-14\beta^2-7\beta+2)x^3+\frac{4}{\beta^2}(2\beta-1)x^2z\\
&-\frac{4}{\beta^3}(24\beta^3+24\beta^2-28\beta+5)x^2-\frac{2}{\beta^2}(22\beta^2-15\beta+2)xz\\
&+\frac{2}{\beta^2}(70\beta^2-51\beta+8)x+\frac{2}{\beta}(18\beta^2-13\beta+2)z\\
&-\frac{2}{\beta}(18\beta^2-13\beta+2),\\
q_3(x,z) = &-\frac{8}{\beta^5}(16\beta^3-20\beta^2+8\beta-1)x^4+\frac{8}{\beta^4}(64\beta^2-48\beta+8)x^3z\\
&+\frac{4}{\beta^5}(72\beta^4-70\beta^3+5\beta^2+10\beta-2)x^3-\frac{4}{\beta^4}(28\beta^3-12\beta^2-3\beta+1)x^2z\\
&-\frac{4}{\beta^2}(2\beta-1)xz^2-\frac{4}{\beta^4}(40\beta^4-2\beta^3-57\beta^2+34\beta-5)x^2\\
&+\frac{2}{\beta^3}(6\beta^3+35\beta^2-27\beta+4)xz+\frac{4}{\beta}(2\beta-1)z^2+\frac{2}{\beta^3}(74\beta^3-123\beta^2+59\beta-8)x\\
&+\frac{2}{\beta^2}(18\beta^3-35\beta^2+17\beta-2)z-\frac{2}{\beta^2}(18\beta^3-31\beta^2+15\beta-2).
\end{aligned}
$$

Now let V be a primitive axial algebra of Monster type $(2\beta,\beta)$ over a ring R of characteristic other than 2, generated by the two axes a_0 and a_1. By [5, Corollary 3.8], V is a homomorphic image of $\overline{V}\otimes_{\hat{D}} R$ and R is a homomorphic image of $\overline{R}\otimes_{\hat{D}} R$. We identify the elements of $\hat{D}$ with their images in R so that the polynomials p_i and q_i defined above are considered as polynomials in $R[x,y,z,t]$ and $R[x,z]$, respectively. For each $i\in\mathbb{Z}$, let a_i be the image of the axis $\mathbf{a}_i$ and let

$$P := (\lambda_{a_0}(a_1), \lambda_{a_1}(a_0), \lambda_{a_0}(a_2), \lambda_{a_1}(a_{-1})).$$

By Corollary 3.9, $\overline{R}\otimes_{\hat{D}} R$ is isomorphic to $R[\lambda_1,\lambda_1^f,\lambda_2,\lambda_2^f]$. Let

$$v_P : R[x,y,z,t]\to R$$

be the R-algebra homomorphism that associates to each polynomial $f\in R[x,y,z,t]$ its value on the quadruple P (of course we are assuming x,y,z,t to be algebraically independent indeterminates over R too). Let I_P be the kernel of v_P and set

$$U_P := \frac{\overline{V}\otimes_{\hat{D}} R}{(\overline{V}\otimes_{\hat{D}} R)I_P.}$$

Then, U_P is a primitive axial R-algebra of Monster type $(2\beta,\beta)$. We denote the images of an element δ of $\overline{R}\otimes_{\hat{D}} R$ in R via v_P by $\bar{\delta}$ and by $\overline{p}_i$ and $\overline{q}_i$ the polynomials in $R[x,y,z,t]$ and $R[x,z]$ corresponding to p_i and q_i, respectively. Set

$$T := \{\overline{p}_1, \overline{p}_2, \overline{p}_3, \overline{p}_4, \overline{p}_5\}. \tag{8}$$

By definition, the $\overline{p}_i$'s have the coefficients in the ring $R_0(\beta)$, where R_0 is the prime subring of R (see Section 1).

Proof of Theorem 1.2. With the above notation, by [5, Corollary 3.8], P is the homomorphic image in R^4 of the quadruple $(\lambda_1, \lambda_1^f, \lambda_2, \lambda_2^f)$ and so it is a common zero of the set T and V is isomorphic to a quotient of U_P. Hence the claim follows by setting ξ to be the map

$$\begin{array}{rccc} \xi\colon & \mathcal{M}_2(2\beta, \beta, R) & \to & \mathcal{V}(T) \\ & V & \mapsto & P \end{array} \quad \square$$

In the symmetric case we have

Corollary 4.1. *If V is a symmetric primitive axial algebra of Monster type $(2\beta, \beta)$ generated by two axes a_0 and a_1, then*

$$\lambda_{a_0}(a_1) = \lambda_{a_1}(a_0), \;\; \lambda_{a_0}(a_2) = \lambda_{a_1}(a_{-1})$$

and the pair $(\lambda_{a_0}(a_1), \lambda_{a_0}(a_2))$ is a solution of the system

$$\begin{cases} \bar{q}_1(x, z) = 0 \\ \bar{q}_2(x, z) = 0 \\ \bar{q}_3(x, z) = 0 \end{cases} \tag{9}$$

Proof. Since V is symmetric, $\lambda_{a_0}(a_1) = \lambda_{a_1}(a_0)$ and $\lambda_{a_0}(a_2) = \lambda_{a_1}(a_{-1})$. Hence the claim follows from Theorem 1.2 and the definitions of the polynomials $\bar{q}_i$'s. $\square$

Lemma 4.2. *For any field $\mathbb{F}$, the resultant $r_{2,3}(x)$ of the polynomials $\bar{q}_2(x, z)$ and $\bar{q}_3(x, z)$ with respect to z is*

$$\gamma x(x-1)(2x-\beta)(x-\beta)^3[(16\beta-6)x + (-18\beta^2+\beta+2)][(8\beta-2)x + (-9\beta^2+2\beta)],$$

where

$$\gamma := \frac{-16(2\beta-1)^3(4\beta-1)}{\beta^{10}}.$$

Proof. The resultant has been computed in the ring $\mathbb{Z}[\beta, \beta^{-1}][x]$ using [1] (see [7, SolutionsSystem.s]. $\square$

Set

$$\begin{aligned}
c_1(x) &:= 16x^4 - 48x^3 - 51x^2 + 46x - 8;\\
c_2(x) &:= 4x^2 + 2x - 1;\\
c_3(x) &:= 5x^2 + x - 1;\\
c_4(x) &:= 18x^2 - x - 2;\\
c_5(x) &:= 3x - 1;\\
c_6(x) &:= 7x - 2;\\
c_7(x) &:= 4x - 1;
\end{aligned}$$

and $\mathcal{C} := \{\varepsilon \in \mathbb{F} \mid \prod_{i=1}^{7} c_i(\varepsilon) = 0\}$. Further set

$$\begin{aligned}
\mathcal{S}_0 &:= \left\{ \left(\frac{\beta}{2}, \frac{\beta}{2}\right),\ (\beta, 0)\,,\ \left(\beta, \frac{\beta}{2}\right) \right\};\\
\mathcal{S}_1 &:= \{(1,1),\ (0,1),\ (\beta,1)\}\,;\\
\mathcal{S}_2 &:= \left\{ \left(\frac{9\beta^2 - 2\beta}{2(4\beta-1)}, \frac{9\beta^2 - 2\beta}{2(4\beta-1)} \right) \right\},\ \text{if } \beta \neq \frac{1}{4}\\
\mathcal{S}_3 &:= \emptyset,\ \text{if } \beta \notin \mathcal{C} \setminus \left\{\frac{3}{8}\right\} \text{ and}\\
\mathcal{S}_3 &:= \left\{ \left(\frac{18\beta^2 - \beta - 2}{2(8\beta-3)}, \frac{(48\beta^4 - 28\beta^3 + 7\beta - 2)(3\beta-1)}{2\beta^2(8\beta-3)^2} \right) \right\},\ \text{if } \beta \in \mathcal{C} \setminus \left\{\frac{3}{8}\right\}
\end{aligned}$$

Lemma 4.3. *Let $\mathbb{F}$ be a field of characteristic other than 2, $\beta \in \mathbb{F} \setminus \{0, 1, \frac{1}{2}\}$ and let $\mathcal{S}$ be the set of solutions of the system of equations (9) in Corollary 4.1. Then*

(1) *if $\beta = \frac{1}{4}$, $\mathcal{S} = \mathcal{S}_0 \cup \{(\mu, 1) \mid \mu \in \mathbb{F}\}$;*
(2) *if $\beta \neq \frac{1}{4}$, $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$.*

Proof. If $\beta = \frac{1}{4}$, then

$$\begin{aligned}
\bar{q}_1(x,z) &= -\,4z(z-1)(8z-1),\\
\bar{q}_2(x,z) &= -\,(8x-1)(4x-1)(z-1),\\
\bar{q}_3(x,z) &= (z-1)(4x-1)(16x+8z-3)
\end{aligned}$$

and claim (1) follows. Let $\beta \neq \frac{1}{4}$. A direct check (see [7, SolutionsSystem.s]) shows that $\mathcal{S}_i \subseteq \mathcal{S}$ for every $i \in \{0,1,2,3\}$. To prove the reverse inclusions, let (x_0, z_0) be a solution of the system (9). Then, by the properties of the resultant of two polynomials (see e.g. [16, Proposition 8.1]), x_0 is a root of the polynomial $r_{2,3}(x)$ given in Lemma 4.2. As $\beta \neq \frac{1}{4}$, $r_{2,3}(x)$ is a non zero polynomial and so we get

$$x_0 \in \left\{0, 1, \frac{\beta}{2}, \beta, \frac{(9\beta^2 - 2\beta)}{2(4\beta - 1)}, \frac{(18\beta^2 - \beta - 2)}{2(8\beta - 3)}\right\}.$$

Assume $x_0 = 0$. Then $\bar{q}_2(0, z) = \frac{2}{\beta}(9\beta - 2)(2\beta - 1)(z - 1)$ and so either $z_0 = 1$ or $\beta = \frac{2}{9}$. In the former case we get the pair $(0, 1)$ which belongs to $\mathcal{S}_1$. In the latter case, we get $\bar{q}_1(0, z) = 5z(z - 1)(9z + 1)$ and $\bar{q}_3(0, z) = -10z(z - 1)$. Since $ch\,\mathbb{F} = 5$ implies $\beta = \frac{2}{9} = -2$ and $\alpha = 2\beta = 1$ which is not allowed, we get $z_0 \in \{0, 1\}$ and the solutions $(0, 0) = \left(\frac{9\beta^2-2\beta}{2(4\beta-1)}, \frac{9\beta^2-2\beta}{2(4\beta-1)}\right) \in \mathcal{S}_2$ and $(0, 1) \in \mathcal{S}_1$.

Assume $x_0 = 1$. Then $\bar{q}_2(1, z) = \frac{2}{\beta^2}(\beta - 1)(9\beta - 4)(2\beta - 1)(z - 1)$ and so either $z_0 = 1$ or $\beta = \frac{4}{9}$. In the former case we get the pair $(1, 1)$ which belongs to $\mathcal{S}_1$. Suppose $\beta = \frac{4}{9}$ and $z_0 \neq 1$. Then note that $\beta \neq \frac{2}{9}$ and, since also $\beta \neq \frac{1}{4}$ by assumption, $ch\,\mathbb{F} \neq 7$. We have $\bar{q}_1(1, z) = -\frac{1}{64}(z - 1)(144z^2 + 1193z - 4900)$ and $\bar{q}_3(1, z) = \frac{5}{64}(z - 1)(16z - 23)$. Hence we must have $z_0 = \frac{23}{16}$. Thus $\bar{q}_1(1, \frac{23}{16}) = \frac{3\cdot5^2\cdot7^2\cdot11}{2^{11}}$. Since $ch\,\mathbb{F} \neq 7$ and we get non admissible values of β ($\frac{1}{2}$ and 1) when $ch\ \mathbb{F} \in \{3, 5\}$, we have $ch\,\mathbb{F} = 11$. Then $z = \frac{1}{5}$ and we see that $(1, \frac{1}{5}) = \left(\frac{18\beta^2-\beta-2}{2(8\beta-3)}, \frac{(48\beta^4-28\beta^3+7\beta-2)(3\beta-1)}{2\beta^2(8\beta-3)^2}\right) \in \mathcal{S}_3$.

Assume $x_0 = \frac{\beta}{2}$. Then $\bar{q}_2(\frac{\beta}{2}, z) = -\frac{1}{\beta}(4\beta - 1)(2\beta - 1)(\beta - 2z)$, whence $z_0 = \frac{\beta}{2}$ and we get the pair $\left(\frac{\beta}{2}, \frac{\beta}{2}\right)$ which is in $\mathcal{S}_0$.

Assume $x_0 = \beta$, then $\bar{q}_2(\beta, z) = 0$ and $\bar{q}_1(\beta, z) = \frac{2}{\beta^2}(2\beta - 1)z(z - 1)(2z - \beta)$. Since $\beta \notin \{0, \frac{1}{2}\}$, we get $z_0 \in \{0, 1, \frac{\beta}{2}\}$ and the pairs (x_0, z_0) belong to $\mathcal{S}_0 \cup \mathcal{S}_1$.

Assume $x_0 = \frac{9\beta^2-2\beta}{2(4\beta-1)}$. Then

$$\bar{q}_2(x_0, z) = -\frac{(2\beta - 1)(3\beta - 1)(9\beta - 2)}{2(4\beta - 1)^3}[(8\beta - 2)z - \beta(9\beta - 2)]$$

and thus either $z_0 = \frac{9\beta^2-2\beta)}{2(4\beta-1)}$ and so the pair (x_0, z_0) belongs to $\mathcal{S}_2$, or $\beta \in \{\frac{1}{3}, \frac{2}{9}\}$. If $\beta = \frac{1}{3}$, then $x_0 = \frac{1}{2} = \frac{18\beta^2-\beta-2}{2(8\beta-3)} = \frac{9\beta^2-2\beta}{2(4\beta-1)}$, $\bar{q}_1(x_0, z) = -2z(2z - 1)(3z - 1)$ and $\bar{q}_2(x_0, z) = z(2z - 1)$. Hence $z_0 \in \{0, \frac{1}{2}\}$ and we see that

$$\left(\frac{1}{2}, 0\right) = \left(\frac{18\beta^2 - \beta - 2}{2(8\beta - 3)}, \frac{(48\beta^4 - 28\beta^3 + 7\beta - 2)(3\beta - 1)}{2\beta^2(8\beta - 3)^2}\right) \in \mathcal{S}_3$$

and $(\frac{1}{2}, \frac{1}{2}) = \left(\frac{9\beta^2-2\beta)}{2(4\beta-1)}, \frac{9\beta^2-2\beta)}{2(4\beta-1)}\right) \in \mathcal{S}_2$. If $\beta = \frac{2}{9}$, then $x_0 = 0$ and we are in one of the cases considered above.

Finally assume $x_0 = \frac{18\beta^2-\beta-2}{2(8\beta-3)}$. Then

$$\bar{q}_2(x_0, z) = -\omega\left[2\beta^2(8\beta - 3)^2 z - (48\beta^4 - 28\beta^3 + 7\beta - 2)(3\beta - 1)\right]$$

where $\omega := \frac{1}{\beta^4(8\beta-3)^4}(2\beta^2 + 5\beta - 2)(2\beta - 1)(3\beta - 4)(3\beta - 1)$. Thus either $z_0 = \frac{(48\beta^4-28\beta^3+7\beta-2)(3\beta-1)}{2\beta^2(8\beta-3)^2}$ or $\omega = 0$. In the former case we see that the pair

$$\left(\frac{(18\beta^2 - \beta - 2)}{2(8\beta - 3)}, \frac{(48\beta^4 - 28\beta^3 + 7\beta - 2)(3\beta - 1)}{2\beta^2(8\beta - 3)^2}\right)$$

Table 2
Multiplication table for the algebra $3A(2\beta, \beta)$, $\beta \neq \frac{1}{4}$.

Basis	Products	Form
a_{-1}, a_0, a_1, s	$a_i^2 = a_i$ $a_i a_{i+1} = s + \beta(a_i + a_{i+1})$ where $a_i := a_{i+3}$ $a_i s = \beta s + \frac{\beta^2}{2}(a_{i-1} + a_{i+1}) - \frac{\beta^2(10\beta-3)}{2(4\beta-1)} a_i$ $s^2 = -\frac{\beta(6\beta^2+2\beta-1)}{2(4\beta-1)} s - \frac{\beta^3(2\beta-1)}{4(4\beta-1)}(a_{-1} + a_0 + a_1)$	$(a_i, a_i) = 1$ $(a_i, a_{i+1}) = \frac{\beta(9\beta-2)}{2(4\beta-1)}$ $(s, a_i) = -\frac{3\beta^2(3\beta-1)}{2(4\beta-1)}$

is a root of $\bar{q}_1(x, z)$ only if $\beta \in \mathcal{C} \setminus \{\frac{3}{8}\}$. In the latter case, if $\beta = \frac{4}{9}$ (and hence $ch\,\mathbb{F} \neq 5$ since $\beta \neq 1$), we get $x_0 = 1$ and the above discussion holds. Similarly, if $\beta = \frac{1}{3}$, then $x_0 = \frac{1}{2} = \frac{9\beta^2-2\beta}{2(4\beta-1)}$ and we are in one of the cases already considered. Finally if $2\beta^2+5\beta-2 = 0$, then $x_0 = \beta$ and we are again in one of the cases considered above. □

In order to classify primitive axial algebras of Monster type $(2\beta, \beta)$ over $\mathbb{F}$ generated by two axes a_0 and a_1 we can proceed, similarly as we did in [5], in the following way. We first solve the system (9) and classify all symmetric algebras. Then we observe that, the even subalgebra $\langle\langle a_0, a_2\rangle\rangle$ and the odd subalgebra $\langle\langle a_{-1}, a_1\rangle\rangle$ are symmetric, since the automorphisms τ_1 and τ_0 respectively, swap their generating axes. Hence, from the classification of the symmetric case, we know all possible configurations for the subalgebras $\langle\langle a_0, a_2\rangle\rangle$ and $\langle\langle a_{-1}, a_1\rangle\rangle$ and from the relations found in Section 3, we derive the structure of the entire algebra.

5. The symmetric case

Let, in this and in the following section, V be a primitive axial algebra of Monster type $(2\beta, \beta)$ over a field $\mathbb{F}$ of characteristic other than 2, generated by the two axes a_0 and a_1. By [5, Corollary 3.8], V is a homomorphic image of $\overline{V} \otimes_{\hat{D}} \mathbb{F}$. For every element $\mathbf{v} \in \overline{V}$, we denote by v its image in V. In particular a_0 and a_1 are the images of $\mathbf{a}_0$ and $\mathbf{a}_1$ and all the formulas obtained from the ones in Lemmas 3.2, 3.3, 3.4, 3.10 with a_i and $s_{r,j}$ in the place of $\mathbf{a}_i$ and $\mathbf{s}_{r,j}$, respectively, hold in V. With an abuse of notation we identify the elements of $\overline{R}$ with their images in $\mathbb{F}$, so that in particular $\lambda_1 = \lambda_{a_0}(a_1)$, $\lambda_1^f = \lambda_{a_1}(a_0)$, $\lambda_2 = \lambda_{a_0}(a_2)$, and $\lambda_2^f = \lambda_{a_1}(a_2)$.

In this section we shall classify the 2-generated primitive symmetric algebras of Monster type $(2\beta, \beta)$. This classification could be obtained by a careful analysis of Yabe's list [22]. We give here, however, an alternative proof independent of Yabe's result.

We begin with an overview of the 2-generated primitive symmetric algebras of Monster type $(2\beta, \beta)$. We shall show that these are precisely the quotients of the following algebras:

(1) the algebras of Jordan type $1A$, $2B$, $3C(\beta)$, $3C(2\beta)$, and $\widehat{S}(\frac{1}{2})^\circ$ (see [11]);
(2) the algebra $3A(2\beta, \beta)$, where $\beta \neq \frac{1}{4}$ (see [20] and Table 2);

Table 3
Multiplication table for the algebra $4Y(2\beta,\beta)$, where $4\beta^2+2\beta-1=0$.

Basis	Products	Form
a_0,a_1,a_2,a_3,s	$a_i^2=a_i$ $a_ia_{i+1}=s+\beta(a_i+a_{i+1})$ where $a_4:=a_0$ $a_ia_{i+2}=4\beta s-\frac{2\beta-1}{2}(a_i+a_{i+2})$ where $a_5:=a_1$ $a_is=\beta s+\frac{\beta^2}{2}(a_{i-1}+a_{i+1})$ $s^2=\frac{3\beta-1}{8}(4s-a_3-a_0-a_1-a_2)$	$(a_i,a_i)=1$ $(a_i,a_{i+1})=\beta+\frac{1}{4}$ $(a_i,a_{i+2})=\beta$ $(a_i,s)=\frac{1}{4}\beta$ $(s,s)=\frac{1}{8}\beta$

(3) the algebras $4J(2\beta,\beta)$ and $6J(2\beta,\beta)$ (see [8][1]);
(4) the 5-dimensional algebra $4Y(2\beta,\beta)$ where $4\beta^2+2\beta-1=0$, with basis (a_3,a_0,a_1,a_2,s) and multiplication table as in Table 3. Note that it coincides with the algebra $IV_2(\xi,\beta,\mu)$ defined by Yabe [22], when $\beta=\frac{1-\xi^2}{2}$, $\xi=2\beta$, and $\mu=\frac{-1}{\xi+1}$.
(5) the 5-dimensional algebra $6Y(\frac{1}{2},2)$ where $ch\ \mathbb{F}=7$ and $\beta=2$, with basis (a_0,a_2,a_4,d,z), where $d:=a_i-a_{i+3}$, $s:=a_id-\frac{1}{2}d$, and multiplication table as in [17, Table 1]. It coincides with the algebra $IV_3(\frac{1}{2},2)$ defined by Yabe [22] and it is also a quotient of the Highwater algebra (see [6, §11]).

Note that some of the algebras listed above may have different names, namely one can prove that $4J(\frac{1}{4},\frac{1}{8})\cong 4A(\frac{1}{4},\frac{1}{8})$, $4J(\frac{1}{2},\frac{1}{4})\cong 4Y(\frac{1}{2},\frac{1}{4})$ and $6J(\frac{2}{5},\frac{1}{5})\cong 6A(\frac{2}{5},\frac{1}{5})$.

It is immediate that the values of (λ_1,λ_2) corresponding to the trivial algebra $1A$ and to the algebra $2B$ are $(1,1)$ and $(0,1)$, respectively. In the following lemmas we list the key features of the remaining algebras.

Lemma 5.1. *Let $\mathbb{F}$ be a field of characteristic other than 2 and $\beta\in\mathbb{F}\setminus\{0,1,\frac{1}{2},\frac{1}{4}\}$. The algebra $3A(2\beta,\beta)$ is a 2-generated symmetric Frobenius axial algebra of Monster type $(2\beta,\beta)$ with $\lambda_1=\lambda_2=\frac{\beta(9\beta-2)}{2(4\beta-1)}$. It is simple except when $(18\beta^2-\beta-1)(9\beta^2-10\beta+2)(5\beta-1)=0$, in which case one of the following holds*

(1) *$\beta=\frac{1}{5}$, $ch\ \mathbb{F}\neq 3$, and there is a unique quotient of maximal dimension which is isomorphic to $3C(\beta)$;*
(2) *$18\beta^2-\beta-1=0$, $ch\ \mathbb{F}\neq 3$, and there is a unique non trivial quotient denoted by $3A(2\beta,\beta)^{\times}$;*
(3) *$9\beta^2-10\beta+2=0$, $ch\ \mathbb{F}\neq 3$, and there is a unique non trivial quotient which is isomorphic to $1A$;*

Proof. Let V be the algebra $3A(2\beta,\beta)$. Then V has a Frobenius form such that $(a,a)\neq 0$ for every axis a. Since the Miyamoto group of V is transitive on the set of axes, by [13, Theorem 4.11] every proper ideal of V is contained in the radical of the form. Now, the

[1] We adopt here the notation used in [17].

Gram matrix of the Frobenius form with respect to the basis $(a_0, a_1, a_2, s_{1,0})$ can be computed easily and has determinant

$$\Delta := -\frac{\beta^2(9\beta^2 - 10\beta + 2)^3(18\beta^2 - \beta - 1)(5\beta - 1)}{16(4\beta - 1)^5}.$$

Thus, if $\Delta \neq 0$, V is simple. In particular, this happens if $ch\,\mathbb{F} = 3$, since Δ annihilates only for $\beta \in \{0, -1\}$, whence $\alpha \in \{0, 1\}$, but these values are not allowed. So assume $ch\,\mathbb{F} \neq 3$. When $\beta = \frac{1}{5}$ we see that the radical is generated by the vector $\frac{\beta}{2}(a_0 + a_1 + a_2) + s_{0,1}$ and hence the quotient over the radical is isomorphic to the algebra $3C(\beta)$. If $(18\beta^2 - \beta - 1) = 0$, then the radical is one-dimensional generated by the vector $-\frac{\beta}{2}(a_0 + a_1 + a_2) + s_{0,1}$ and so the quotient over the radical is the unique non trivial quotient. Finally, if $(9\beta^2 - 10\beta + 2) = 0$, then the radical is three dimensional, with generators

$$a_0 - a_2,\ \ a_0 - a_1,\ \ (2\beta - 1)a_0 + s_{0,1}.$$

It is immediate to see that the quotient over the radical is the trivial algebra $1A$. Using Lemma 2.5, it is straightforward to prove that the radical is a minimal ideal. □

Lemma 5.2. *Let $\mathbb{F}$ be a field of characteristic other than 2 and $\beta \in \mathbb{F} \setminus \{0, 1, \frac{1}{2}\}$.*

(1) *The algebra $3C(2\beta)$ has $(\lambda_1, \lambda_2) = (\beta, \beta)$. It is simple except when $\beta = -\frac{1}{2}$, in which case it has a non trivial quotient of dimension 2, denoted by $3C(-1)^\times$.*
(2) *The algebra $3C(\beta)$ has $\lambda_1 = \lambda_2 = \frac{\beta}{2}$. It is simple except when $\beta \in \{-1, 2\}$. The algebra $3C(-1)$ has a non trivial quotient of dimension 2, denoted by $3C(-1)^\times$, and the algebra $3C(2)$ has a one dimensional quotient isomorphic to the algebra $1A$.*
(3) *The algebra $4J(2\beta, \beta)$ has $(\lambda_1, \lambda_2) = (\beta, 0)$. It is simple except when $\beta = -\frac{1}{4}$, in which case it has a unique non trivial quotient, denoted by $4J(2\beta, \beta)^\times$, over the ideal generated by $a_{-1} + a_0 + a_1 + a_2 - \frac{2}{\beta}s_{0,1}$, which is a simple algebra of dimension 4.*
(4) *The algebra $6J(2\beta, \beta)$ has $(\lambda_1, \lambda_2) = \left(\beta, \frac{\beta}{2}\right)$. It is simple provided $\beta \notin \{2, -\frac{1}{7}\}$.*
    If $\beta = -\frac{1}{7}$, the algebra has a unique non trivial quotient, denoted by $6J(\frac{2}{7}, \frac{1}{7})^\times$, over the ideal $\mathbb{F}(a_{-2} + a_{-1} + a_0 + a_1 + a_2 + a_3 - \frac{1}{\beta}s_{0,3} - \frac{2}{\beta}s_{0,1})$, which is a simple algebra of dimension 7.
    If $\beta = 2$, then the algebra has a unique non trivial quotient which is isomorphic to $3C(2\beta)$.

Proof. (1) and (2) are proved in [11, (3.4)]. Let $V \in \{4J(2\beta, \beta), 6J(2\beta, \beta)\}$. Then, V is a subalgebra of a Matsuo algebra and so it is endowed of a Frobenius form. As in the proof of Lemma 5.1, every proper ideal of V is contained in the radical of the form. When $V = 4J(2\beta, \beta)$, the Gram matrix, with respect to the basis $a_{-1}, a_0, a_1, a_2, -\frac{2}{\beta}s_{0,1}$, is

$$2\begin{pmatrix} 1 & \beta & 0 & \beta & 2\beta \\ \beta & 1 & \beta & 0 & 2\beta \\ 0 & \beta & 1 & \beta & 2\beta \\ \beta & 0 & \beta & 1 & 2\beta \\ 2\beta & 2\beta & 2\beta & 2\beta & 2 \end{pmatrix}.$$

The determinant of this matrix is $2(2\beta-1)^2(4\beta+1)$ and so, if $\beta \neq -\frac{1}{4}$ we get the thesis. If $\beta = -\frac{1}{4}$, the radical of the form is the 1-dimensional ideal $\mathbb{F}(a_{-1}+a_0+a_1+a_2-\frac{2}{\beta}s_{0,1})$. When $V = 6J(2\beta,\beta)$, the Gram matrix, with respect to the basis $a_0, a_2, a_{-2}, a_1, a_{-1}, a_3, -\frac{1}{\beta}s_{0,3}, -\frac{2}{\beta}s_{0,1}$ given in [8, Table 8], is

$$\begin{pmatrix} 2 & \beta & \beta & 2\beta & 2\beta & 2\beta & 2\beta & 4\beta \\ \beta & 2 & \beta & 2\beta & 2\beta & 2\beta & 2\beta & 4\beta \\ \beta & \beta & 2 & 2\beta & 2\beta & 2\beta & 2\beta & 4\beta \\ 2\beta & 2\beta & 2\beta & 2 & \beta & \beta & 2\beta & 4\beta \\ 2\beta & 2\beta & 2\beta & \beta & 2 & \beta & 2\beta & 4\beta \\ 2\beta & 2\beta & 2\beta & \beta & \beta & 2 & 2\beta & 4\beta \\ 2\beta & 2\beta & 2\beta & 2\beta & 2\beta & 2\beta & 2 & 2\beta \\ 4\beta & 4\beta & 4\beta & 4\beta & 4\beta & 4\beta & 2\beta & 4+2\beta \end{pmatrix}.$$

The determinant of this matrix is $-16(2\beta-1)^2(\beta-2)^5(7\beta+1)$ and so, if $\beta \notin \{2, -\frac{1}{7}\}$, the algebra is simple. If $\beta = -\frac{1}{7}$, then the radical of the form is the 1-dimensional ideal $\mathbb{F}(a_{-2}+a_{-1}+a_0+a_1+a_2+a_3-\frac{1}{\beta}s_{0,3}-\frac{2}{\beta}s_{0,1})$ and the result follows. Finally suppose $\beta = 2$ (and hence $ch\,\mathbb{F} \neq 3$). Then the radical I of the form is 5-dimensional with basis

$$a_0 - a_2,\ a_0 - a_{-2},\ a_1 - a_{-1},\ a_1 - a_3,\ s_{0,1} - s_{0,3}$$

and a direct computation shows that V/I is an algebra of type $3C(2\beta)$. It remains to prove that I is a minimal ideal. Suppose J is a non zero ideal of V. Then $J \subseteq I$. Since I is invariant under the action of the Miyamoto group and

$$\begin{array}{rcccl} s_{0,1}-s_{0,3} & = & a_0(a_1-a_3)-2(a_1-a_3) & = & a_1(a_0-a_{-2})-2(a_0-a_{-2}) \\ a_0-a_{-2} & = & \frac{1}{6}(a_3-a_1)(s_{0,1}-s_{0,3}) & = & (a_0-a_2)^{\tau_0} \\ a_1-a_3 & = & \frac{1}{6}(a_{-2}-a_0)(s_{0,1}-s_{0,3}) & = & (a_1-a_{-1})^{\tau_1} \end{array}$$

in order to prove that $J = I$ it is enough to show that J contains one of the above five basis vectors.

Set

$$v_0 := -2a_0 + a_{-2} + a_2,$$

$$w_0 := a_{-1} + a_1 - 2a_3 - s_{0,1} + s_{0,3},$$

$$w_\alpha := a_1 - a_3 + a_{-1} - a_3 + s_{0,1} - s_{0,3}.$$

Then, with respect to ad_{a_0}, (v_0, w_0) is a basis for the 0-eigenspace of I, w_α is a generator of the α-eigenspace and (a_1-a_{-1}, a_2-a_{-2}) is a basis for the β-eigenspace. By Lemma 2.5, J contains a non zero eigenvector v for ad_{a_0}. If v is an α-eigenvector, then it is a multiple of w_α and hence J contains $\frac{1}{3}(w_\alpha - w_\alpha^{\tau_1})^{\tau_0} = a_1 - a_3$. Suppose that v is a β-eigenvector, that is $v \in \langle a_1 - a_{-1}, a_2 - a_{-2}\rangle$. If v is a multiple of either $a_2 - a_{-2} = (a_0 - a_{-2})^{\tau_1}$ or $a_1 - a_{-1}$, we are done. So assume v is neither a multiple of $a_2 - a_{-2}$ nor of $a_1 - a_{-1}$. Then we may assume that $v = a_1 - a_{-1} + \eta(a_2 - a_{-2})$, for some $\eta \in \mathbb{F}$, $\eta \neq 0$. Then $v - v^{\tau_2} = a_1 - a_3 + \eta(a_0 - a_{-2})$ and J contains $a_0 - a_2 = \frac{1}{3\eta}[2(v - v^{\tau_2}) - a_0(v - v^{\tau_2}) - 2(v - v^{\tau_2})^{\tau_1} + a_2(v - v^{\tau_2})^{\tau_1}]$. Finally suppose v is a 0-eigenvector, that is $v \in \langle v_0, w_0\rangle$. If v is a multiple of w_0, then $a_{-1} - a_3$ is a multiple of $v - v^{\tau_1}$ and so it lies in J. Otherwise we may assume $v = v_0 + \eta w_0$ with $\eta \in \mathbb{F}$. Then J contains $\frac{1}{3}(v - v^{\tau_1}) = \eta(a_{-1} - a_3) - a_0 + a_2$ and we may conclude that $a_0 - a_2 \in J$ as in the previous case. □

Lemma 5.3. *Let $\mathbb{F}$ be a field of characteristic other than 2 and $\beta \in \mathbb{F}$ such that $4\beta^2 + 2\beta - 1 = 0$. The algebra $4Y(2\beta, \beta)$ is a simple 2-generated primitive symmetric Frobenius axial algebra of Monster type $(2\beta, \beta)$, with $\lambda_1 = \beta + \frac{1}{4}$ and $\lambda_2 = \beta$.*

Proof. All the properties are easily verified. Note that the Frobenius form is defined by $(a_i, a_i) = 1$, $(a_i, a_j) = \lambda_1$, for $i, j \in \{0, 1, 2, 3\}$ such that $i - j \equiv_2 1$, $(a_0, a_2) = (a_1, a_3) = \lambda_2$, and $(a_i, s) = \frac{1}{4}\beta$ for $i \in \{0, 1, 2, 3\}$. Then the Frobenius is the one specified in Table 3, and its Gram matrix has determinant $\frac{1}{32}\beta(\beta-1)^2(2\beta-1)(2\beta+3)$. Note that, for $\beta = -\frac{3}{2}$, condition $4\beta^2 + 2\beta - 1 = 0$ implies $ch\, \mathbb{F} = 5$ and $\beta = 1$. Hence the Frobenius form is always non degenerate. Note that $\beta \neq -\frac{1}{4}$, for otherwise the condition $4\beta^2 + 2\beta - 1 = 0$ implies $ch\, \mathbb{F} = 5$, whence $\beta = 1$, a contradiction. Thus the projection graph is connected and the result follows with the argument already used to prove Lemma 5.2. □

Lemma 5.4. *Let $\mathbb{F}$ be a field of characteristic 7. The algebra $6Y(\frac{1}{2}, 2)$ has $(\lambda_1, \lambda_2) = (1, 1)$, it is a baric algebra with a unique maximal ideal $R = \langle a_0 - a_2, a_0 - a_4, d, z\rangle$ of codimension 1. The algebra has 4 non trivial quotients*

(1) *$V/R \cong 1A$;*
(2) *$V/\langle d, z\rangle \cong 3C(2)$;*
(3) *$V/\langle a_0 - a_2, a_2 - a_4\rangle \cong \widehat{S}(2)^\circ$, the 3-dimensional axial algebra of Jordan type $\frac{1}{2}$;*
(4) *$V/\mathbb{F}q$, a 4-dimensional algebra denoted by $6Y(\frac{1}{2}, 2)^\times$.*

Proof. The algebra V is endowed with the Frobenius form given in [17, Table 1]: $(a_i, a_j) = 1$ for every $i, j \in \mathbb{Z}$ and $(d, x) = (z, x) = 0$ for all $x \in V$. It is immediate to see that the radical is $R = \langle a_0 - a_2, a_0 - a_4, d, z\rangle$ and $V/R \cong 1A$. Moreover, by [17, Corollary 4.15], every proper ideal of V is contained in R. Note that a basis of ad_{a_0}-eigenvectors for R is given by $(z, u := -a_0 - 3(a_2 + a_4) - 2z, -3d + z, a_2 - a_4)$, where z and u are 0-eigenvectors, $-3d + z$ is an α-eigenvector and $a_2 - a_4$ is a β-eigenvector. Let I be a non zero ideal of V. By Lemma 2.5, I contains an eigenvector v for ad_{a_0}.

Routine computations show that $Vz = \langle z\rangle$, $I_1 := V(a_4 - a_2) = \langle a_0 - a_2, a_2 - a_4\rangle$, and $I_2 := V(-3d + z) = \langle d, z\rangle$. Finally, suppose v is a 0-eigenvector which is not a multiple of z. Then we may assume $v = u + \mu z$, with $\mu \in \mathbb{F}$. If $\mu = 2$ we see that $v \in I_1$; if $\mu \neq 2$, then $u \in I$ and so I contains $Vu = \langle a_0 - a_2, a_2 - a_4, z\rangle$. A direct check shows that $z \in I_2$, $I_1 \cap I_2 = \{0\}$ and $I_1 + I_2 = R$. Therefore, Vz, I_1, I_2, and R are the proper non trivial ideals of V and the result follows. □

The next lemma will be useful to deal the case of algebras of dimension at most 3.

Lemma 5.5. *Let V be a symmetric primitive axial algebra of Monster type (α, β) over a field $\mathbb{F}$, generated by two axes a_0 and a_1. Suppose there exists $\eta \in \mathbb{F}$ such that*

$$a_2 = a_{-1} + \eta(a_0 - a_1).$$

Then, one of the following holds

(1) $\eta = 0$ *and* $a_2 = a_{-1}$*;*
(2) $\eta = 1$*,* $a_1 = a_{-1}$*,* $a_2 = a_0$*, and V is spanned by* $a_0, a_1, s_{0,1}$*.*

Proof. If $\eta = 0$, the claim is trivial. Suppose $\eta \neq 0$. By substituting the expression for a_2 in the definition of $s_{0,2}$ we get

$$\begin{aligned} s_{0,2} &= a_0a_2 - \beta(a_0 + a_2) \\ &= a_0[a_{-1} + \eta(a_0 - a_1)] - \beta[a_0 + a_{-1} + \eta(a_0 - a_1)] \\ &= (1 - \eta)s_{0,1} + \beta(a_{-1} - a_2) + \eta a_0 - \eta\beta(a_0 + a_1). \end{aligned}$$

Thus $s_{0,2}^{\tau_1} = (1-\eta)s_{0,1} + \beta(a_3 - a_0) + \eta a_2 - \eta\beta(a_2 + a_1)$. Since $a_{-1} = a_2 - \eta(a_0 - a_1)$ and $a_3 = (a_2)^{\tau_0 f} = a_0 + \eta(a_1 - a_2)$, by Lemma 3.1(2) and substituting the above expressions for $s_{0,2}$ and $s_{0,2}^{\tau_1}$, we get

$$\begin{aligned} 0 &= s_{0,2} - s_{0,2}^{\tau_1} \\ &= \beta(a_{-1} - a_3) - \beta(a_2 - a_0) + \eta(a_0 - a_2) - \eta\beta(a_0 - a_2) \\ &= \beta[a_2 - \eta(a_0 - a_1) - a_0 - \eta(a_1 - a_2)] + (\beta + \eta - \eta\beta)(a_0 - a_2) \\ &= \eta(1 - 2\beta)(a_0 - a_2). \end{aligned}$$

Since $\beta \neq 1/2$, we have $a_2 = a_0$ and, by the symmetry, $a_{-1} = a_1$. The second assertion follows by [5, Lemma 4.3]. □

Proposition 5.6. *Let V be a symmetric primitive axial algebra of Monster type $(2\beta, \beta)$ over a field $\mathbb{F}$ of characteristic other than 2, generated by two axes a_0 and a_1. If V has dimension at most 3, then either V is an algebra of Jordan type β or 2β, or $18\beta^2 - \beta - 1 = 0$ in $\mathbb{F}$ and V is isomorphic to the algebra $3A(2\beta, \beta)^\times$.*

Proof. Since V is symmetric, ad_{a_0} and ad_{a_1} have the same eigenvalues. Since 1 is an eigenvalue for ad_{a_0}, it follows from the fusion law that if 0 is an eigenvalue for ad_{a_0}, or V has dimension at most 2, then V is of Jordan type β or 2β. Let us assume that 0 is not an eigenvalue for ad_{a_0}. Then $u_1 = 0$ (recall the definition of u_1 in Section 2) and, by Lemma 2.4, we get

$$s_{0,1} = [\lambda_1(1-2\beta) - \beta]a_0 + \frac{\beta}{2}(a_1 + a_{-1}). \tag{10}$$

Since we have also $u_1^f = 0$ we deduce

$$a_2 = a_{-1} + \left[\frac{2}{\beta}(\lambda_1(1-2\beta) - \beta) - 1\right](a_0 - a_1).$$

Thus we can apply Lemma 5.5: if claim (2) holds, then $a_1 = a_{-1}$, V is spanned by a_0, a_1, and $s_{0,1}$, so by Equation (10) has dimension at most 2 and we are done. Suppose claim (1) holds, that is $\frac{2}{\beta}(\lambda_1(1-2\beta) - \beta) - 1 = 0$. Then

$$s_{0,1} = \frac{\beta}{2}(a_0 + a_1 + a_{-1}) \text{ and } a_0a_1 = \frac{3}{2}\beta(a_0 + a_1) + \frac{\beta}{2}a_{-1},$$

whence we get that V satisfies the multiplication given in Table 2. The vector $v := 3\beta a_0 + (2\beta - 1)(a_{-1} + a_1)$ is a 2β-eigenvector for ad_{a_0} and, in order to satisfy the fusion law (in particular $v \cdot v$ must be a 1-eigenvector for ad_{a_0}), β must be such that $18\beta^2 - \beta - 1 = 0$. Hence V is isomorphic to a quotient of $3A(2\beta, \beta)^\times$. Since by hypothesis $\beta \notin \{1, \frac{1}{2}\}$, by Lemma 5.1, $3A(2\beta, \beta)^\times$ is simple and $V \cong 3A(2\beta, \beta)^\times$. □

Theorem 5.7. *Let V be a primitive symmetric axial algebra of Monster type $(2\beta, \beta)$ over a field $\mathbb{F}$ of characteristic other than 2, generated by two axes a_0 and a_1. Then, one of the following holds:*

(1) *V is an algebra of Jordan type β or 2β;*
(2) *V is isomorphic to $3A(2\beta, \beta)$;*
(3) *V is isomorphic to $4J(2\beta, \beta)$;*
(4) *V is isomorphic to $6J(2\beta, \beta)$;*
(5) *$4\beta^2 + 2\beta - 1 = 0$ in $\mathbb{F}$ and V is isomorphic to $4Y(2\beta, \beta)$;*
(6) *$ch\,\mathbb{F} = 7$, $\beta = 2$ and V is isomorphic to $6Y(\frac{1}{2}, 2)$;*
(7) *$18\beta^2 - \beta - 1 = 0$ in $\mathbb{F}$ and V is isomorphic to $3A(2\beta, \beta)^\times$;*
(8) *$\beta = -\frac{1}{4}$ and V is isomorphic to $4J(-\frac{1}{2}, -\frac{1}{4})^\times$;*
(9) *$ch\,\mathbb{F} = 7$, $\beta = 2$ and V is isomorphic to $6Y(\frac{1}{2}, 2)^\times$;*
(10) *$\beta = -\frac{1}{7}$ and V is isomorphic to $6J(-\frac{2}{7}, -\frac{1}{7})^\times$.*

Proof. By Theorem 1.2, V is determined, up to homomorphic images, by the pair (λ_1, λ_2), which must be a solution of the system (9) in Corollary 4.1. The solutions of that system are given by Lemma 4.3 and the proof proceeds by considering each case. Recall that, by [5, Corollary 3.8] and Proposition 3.7, V is spanned on $\mathbb{F}$ by the set a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, $s_{0,1}$, and $s_{0,3}$.

Claim 1. *If $\lambda_1 = \beta$, then V falls into one of the cases (1), (3), (4), (8), (10).*

Assume $\lambda_1 = \beta$. Then, by Lemma 4.3,

$$(\lambda_1, \lambda_2) \in \left\{ \left(\beta, \frac{\beta}{2}\right), (\beta, 0), (\beta, 1) \right\}.$$

Evaluating the formula in Lemma 3.6 for $\lambda_1 = \beta$, we get $s_{1,3} = s_{2,3} = s_{0,3}$ and, by Lemma 3.1,

$$s_{0,3}^{\tau_1} = s_{0,3}.$$

If $(\lambda_1, \lambda_2) = (\beta, \frac{\beta}{2})$, the algebra satisfies the multiplication table of the algebra $6J(2\beta, \beta)$. Hence V is isomorphic to a quotient of $6J(2\beta, \beta)$ and, by Lemma 5.2, we get that either (4) or (10) holds.

Suppose $(\lambda_1, \lambda_2) \in \{(\beta, 0), (\beta, 1)\}$. Let E be as in Lemma 3.10. Using the explicit expression for E computed in [7, 2btsymmetric_lm=bt.g] we get

$$0 = E = \frac{(2\beta - 1)(2\lambda_2 - \beta)}{4} \left[s_{0,3} - s_{0,1} + \beta(a_3 - a_{-1})\right],$$

hence, since $(2\beta-1)(2\lambda_2-\beta) \neq 0$, we get $s_{0,3} = s_{0,1} + \beta(a_{-1} - a_3)$. Then, since $s_{0,3}^{\tau_1} = s_{0,3}$ we get

$$\begin{aligned} 0 =& s_{0,3} - s_{0,3}^{\tau_1} \\ =& s_{0,1} + \beta(a_{-1} - a_3) - s_{0,1} - \beta(a_3 - a_{-1}) \\ =& 2\beta(a_{-1} - a_3) \end{aligned}$$

whence $a_3 = a_{-1}$, $s_{0,1} = s_{0,3}$, and $a_{-2} = a_2$. It follows that the dimension of V is at most 5.

Now, if $(\lambda_1, \lambda_2) = (\beta, 0)$ we see that V satisfies the multiplication table of $4J(2\beta, \beta)$ and either (3) or (8) holds. Finally, if $(\lambda_1, \lambda_2) = (\beta, 1)$, then $Z(\beta, \beta) = 0$ and so by Lemma 3.3 and Equation (1) we get $a_0 a_2 = a_0$, that is a_0 is a 1-eigenvector for ad_{a_2}. By primitivity, this implies $a_2 = a_0$. Consequently, $a_{-1} = a_2^f = a_0^f = a_1$ and V satisfies the multiplication table of $3C(2\beta)$ and so it is a quotient of that algebra. In particular V an axial algebra of Jordan type 2β (case (1)).

Claim 2. *If $\lambda_1 \neq \beta$, then $V = \langle a_{-1}, a_0, a_1, a_2, s_{0,1} \rangle$ and there exist $b, c \in \mathbb{F}$ such that*

$$b(a_{-1} - a_2) + c(a_0 - a_1) = 0. \tag{11}$$

The precise expressions for b and c are too long to be displayed here, but can be found in [7, 2btsymmetric_lm<>bt.g].

Assume $\lambda_1 \neq \beta$. Let D_1 be as in Lemma 3.10. By Lemma 3.6 or using the explicit expression for D_1 computed in [7, 2btsymmetric_lm<>bt.g], we get

$$\begin{aligned} 0 = D_1 = \frac{-2(2\beta-1)(\lambda_1-\beta)}{\beta^2}\Big[(\beta-1)(a_{-2}-a_2) \\ + \left(\frac{2\lambda_1(4\beta-1)(2\lambda_1-3\beta)}{\beta^2} + 10\beta - 3 - 2\lambda_2\right)(a_{-1}-a_1)\Big]. \end{aligned}$$

Since $\lambda_1 \neq \beta$ and $\beta \notin \{1, \frac{1}{2}\}$, the coefficient of a_{-2} in D_1 is non zero, so

$$a_{-2} = a_2 + \frac{1}{(\beta-1)}\left[\frac{2\lambda_1(4\beta-1)(2\lambda_1-3\beta)}{\beta^2} + 10\beta - 3 - 2\lambda_2\right](a_1 - a_{-1}). \tag{12}$$

Since V is symmetric, the map f swapping a_0 and a_1 is an algebra automorphism, whence

$$a_3 = a_{-1} + \frac{1}{(\beta-1)}\left[\frac{2\lambda_1(4\beta-1)(2\lambda_1-3\beta)}{\beta^2} + 10\beta - 3 - 2\lambda_2\right](a_0 - a_2). \tag{13}$$

It follows that

$$s_{0,3} = a_0a_3 - \beta(a_0 + a_3) \in \langle a_0a_{-1}, a_0, a_0a_2, a_3\rangle.$$

Since, by Lemma 3.3 and Equation (13), $\langle a_0a_{-1}, a_0, a_0a_2, a_3\rangle \leq \langle a_{-1}, a_0, a_1, a_2, s_{0,1}\rangle$, we get $V = \langle a_{-1}, a_0, a_1, a_2, s_{0,1}\rangle$, as claimed. Moreover, equation $d_1 = 0$ of Lemma 3.10 becomes Equation (11).

Claim 3. *If $\beta \neq \frac{1}{4}$ and $(\lambda_1, \lambda_2) = \left(\frac{(9\beta^2-2\beta)}{2(4\beta-1)}, \frac{(9\beta^2-2\beta)}{2(4\beta-1)}\right)$, then V falls into either case (2) or case (7).*

Note that under the above condition, since $\beta \neq 0$, $\lambda_1 \neq \beta$ and Claim 2 holds. Since both $a_{-2}^2 - a_{-2}$ and $a_3^2 - a_3$ are the zero vector, by Equations (12) and (13), we get

$$\frac{2\beta(18\beta-5)}{(4\beta-1)}(a_2 - a_{-1}) = 0 \text{ and } \frac{2(18\beta^2-9\beta+1)}{(4\beta-1)}(a_2 - a_{-1}) = 0.$$

Now note that, in any field $\mathbb{F}$, there is no element β which simultaneously satisfies $(18\beta - 5) = 0$ and $(18\beta^2 - 9\beta + 1) = 0$. Hence $a_2 = a_{-1}$, $a_{-2} = a_1$, and V satisfies the multiplication table of the algebra $3A(2\beta, \beta)$. The result then follows by Lemma 5.1.

Claim 4. *If $(\lambda_1, \lambda_2) = \left(\frac{\beta}{2}, \frac{\beta}{2}\right)$, then V falls in one of the cases (1), (2), (6), (7) or (9).*

Since $\beta \neq 0$, $\frac{\beta}{2} \neq \beta$, so Equation (11) in Claim 2 holds and a direct computation gives

$$\frac{(\beta-1)(4\beta-1)}{2\beta}(a_{-1}-a_2)=0. \tag{14}$$

Moreover, Equation (12) and the identity $a_{-2}^2 = a_{-2}$ give

$$\frac{(\beta^2+2\beta-1)}{\beta}(a_{-1}-a_2)=0. \tag{15}$$

Suppose $\beta \neq \frac{1}{4}$ (resp. $\beta^2+2\beta-1 \neq 0$). From Equation (14) (resp. Equation (15)) we get $a_{-1}=a_2$, $a_{-2}=a_1$, whence $s_{0,2}=s_{0,1}$. Thus, from Lemma 3.3, we get

$$\frac{(5\beta-1)}{\beta}\left[s_{0,1}+\frac{\beta}{2}(a_0+a_1+a_{-1})\right]=0.$$

If $\beta \neq \frac{1}{5}$, it follows that $s_{0,1} = -\frac{\beta}{2}(a_0+a_1+a_{-1})$, V satisfies the multiplication table of the algebra $3C(\beta)$ and so it is an algebra of Jordan type β (case (1)).

If $\beta = \frac{1}{5}$, we have $\frac{\beta}{2} = \frac{(9\beta^2-2\beta)}{2(4\beta-1)}$ and, by Claim 3, V is a quotient of the algebra $3A(2\beta,\beta)$ (cases (2) and (7)).

Finally, assume $\beta = \frac{1}{4}$ and $\beta^2+2\beta-1=0$. Then $-\frac{7}{16}=0$, whence $ch\,\mathbb{F}=7$, $\beta=2$, and $\alpha = 4 = \frac{1}{2}$. In this case, Equation (12) becomes

$$a_{-2}=a_2+a_1-a_{-1} \tag{16}$$

and V satisfies the multiplication table of the algebra $6Y(\frac{1}{2},2)$. Hence V is isomorphic to a quotient of this algebra and, by Lemma 5.4, one of the cases (1), (6), or (9) holds.

Claim 5. *If* $(\lambda_1,\lambda_2)=(0,1)$, *then*

$$\frac{-(9\beta-4)(4\beta^2+2\beta-1)}{\beta(\beta-1)}(a_{-1}+a_0-a_1-a_2)=0 \tag{17}$$

$$\frac{5(2\beta-1)}{\beta(\beta-1)^2}\left[2b\beta(\beta-1)(a_1-a_{-1})-(18\beta^3+27\beta^2-24\beta+4)(a_0-a_2)\right]=0 \tag{18}$$

$$\frac{-2(2\beta-1)}{\beta}b(a_{-1}-a_1)=0. \tag{19}$$

Since $(\lambda_1,\lambda_2)=(0,1)$ and $\beta \neq 0$, $\lambda_1 \neq \beta$ and Claim 2 holds. In this case, the expressions for the coefficients b and c in Equation (11) reduce to

$$b=c=\frac{-(9\beta-4)(4\beta^2+2\beta-1)}{\beta(\beta-1)}, \tag{20}$$

giving Equation (17). Equation (13) and the property $a_3^2=a_3$ give Equation (18) and the equation $D_2=0$ of Lemma 3.10 becomes Equation (19).

Claim 6. *If* $(\lambda_1, \lambda_2) = (0, 1)$ *and* $a_{-1} = a_1$*, then* V *is isomorphic to the algebra* $2B$*.*

Since $a_{-1} = a_1$, we get $a_2 = a_{-1}^f = a_1^f = a_0$, whence, by Claim 2, $V = \langle a_0, a_1, s_{0,1}\rangle$. Further, $s_{0,2} = a_0a_2 - \beta(a_0 + a_2) = (1 - 2\beta)a_0$. Substituting this expression in the first equation of Lemma 3.3, we get $s_{0,1} = -\beta(a_0 + a_1)$ (note that the coefficient of $s_{0,1}$ in that equation is $2Z(\lambda_1, \lambda_1) = 2Z(0, 0) = -\frac{2(4\beta-1)}{\beta} \neq 0$), in particular $V = \langle a_0, a_1\rangle$. By the definition of $s_{0,1}$, $a_0a_1 = 0$, whence $V \cong 2B$.

Claim 7. *If* $(\lambda_1, \lambda_2) = (0, 1)$*, then* V *is isomorphic to the algebra* $2B$*.*

If $b \neq 0$, since $\beta \neq \frac{1}{2}$, by Equation (19), $a_{-1} = a_1$ and we conclude by Claim 6. Assume $b = 0$. Since $\beta \notin \{0, 1, \frac{1}{2}\}$, Equation (18) is equivalent to

$$5(18\beta^3 + 27\beta^2 - 24\beta + 4)(a_0 - a_2) = 0.$$

If $5(18\beta^3 + 27\beta^2 - 24\beta + 4) \neq 0$, we get $a_0 = a_2$, so $a_1 = a_0^f = a_2^f = a_{-1}$ and the result follows by Claim 6.

Suppose $5(18\beta^3 + 27\beta^2 - 24\beta + 4) = 0$. Since $b = 0$ (and $\beta \neq 1$), Equation (20) implies $ch\, \mathbb{F} = 31$ and $\beta = 9$. In this case, the vector E defined in Lemma 3.10 is equal to $-4a_{-1} + 6a_0 + 6a_1 - 4a_2 + 14s_{0,1}$. Since, by Lemma 3.10, $E = 0 = E^{\tau_0}$, we have

$$6(a_1 - a_{-1}) = E - E^{\tau_0} = 0,$$

whence $a_1 = a_{-1}$ and again the result follows by Claim 6.

Claim 8. *If* $(\lambda_1, \lambda_2) = (1, 1)$*, then* V *falls into cases (1), (2), (6), (7), or (9).*

Evaluating equation $D_2 = 0$ of Lemma 3.10 on the pair $(\lambda_1, \lambda_2) = (1, 1)$, we get

$$\frac{2(4\beta^3 - 14\beta^2 + 11\beta - 2)(9\beta^2 - 10\beta + 2)(\beta - 2)(2\beta - 1)}{\beta^6}(a_{-1} - a_1) = 0.$$

Hence, if $(4\beta^3 - 14\beta^2 + 11\beta - 2)(9\beta^2 - 10\beta + 2)(\beta - 2) \neq 0$, we get $a_{-1} = a_1$ and $a_2 = a_{-1}^f = a_1^f = a_0$. Thus, by Claim 2, V has dimension at most 3 and we conclude by Proposition 5.6 that V is of Jordan type or isomorphic to $3A(2\beta, \beta)^\times$. So let us assume

$$(4\beta^3 - 14\beta^2 + 11\beta - 2)(9\beta^2 - 10\beta + 2)(\beta - 2) = 0. \tag{21}$$

Further, the coefficients b and c of Equation (11) reduce to

$$b = -\frac{1}{\beta^3}\left(36\beta^4 - 126\beta^3 + 127\beta^2 - 48\beta + 6\right)$$

and

$$c = -\frac{1}{\beta^4}(4\beta^2 + 2\beta - 1)(9\beta^2 - 10\beta + 2)(\beta - 2).$$

Assume first $\beta - 2 = 0$. Then $(\lambda_1, \lambda_2) = (\frac{\beta}{2}, \frac{\beta}{2})$ and we reduce to Claim 4.

Suppose now $4\beta^3 - 14\beta^2 + 11\beta - 2 = 0$. Then routine computations show that, in every characteristic other than 2, b and c are not zero. Thus, by Lemma 5.5, V has dimension at most 3 and we conclude by Proposition 5.6, as in the previous case.

Finally assume $9\beta^2 - 10\beta + 2 = 0$. Then $c = 0$. If $b \neq 0$, from Equation (11) we get $a_2 = a_{-1}$ and $a_1 = a_{-1}^{\tau_0} = a_2^{\tau_0} = a_{-2}$. Thus, Equation (12) becomes

$$a_1 - a_{-1} = \frac{1}{(\beta - 1)}\left[\frac{2(4\beta - 1)(2 - 3\beta)}{\beta^2} + 10\beta - 5\right](a_{-1} - a_1),$$

whence, either $a_1 = a_{-1}$ and again V has dimension at most 3 and we conclude by Proposition 5.6, or

$$\frac{1}{(\beta - 1)}\left[\frac{2(4\beta - 1)(2 - 3\beta)}{\beta^2} + 10\beta - 5\right] = -1$$

that is

$$\frac{11\beta^3 - 30\beta^2 + 22\beta - 4}{\beta^2(\beta - 1)} = 0.$$

It is now straightforward to check that the two polynomials $9\beta^2 - 10\beta + 2$ and $11\beta^3 - 30\beta^2 + 22\beta - 4$ have no common roots in any field of characteristic other than 2 and we get a contradiction. We are therefore left with the case $b = 0$. Then the two polynomials $9\beta^2 - 10\beta + 2$ and b have a common root if and only if $ch\,\mathbb{F} = 7$ and the common root if $\beta = -1$. Thus we get $(\lambda_1, \lambda_2) = (1, 1) = \frac{(9\beta^2 - 2\beta)}{2(4\beta - 1)}$ and we conclude by Claim 3.

Claim 9. *If $\beta = \frac{1}{4}$ and $\lambda_2 = 1$, then V falls into cases (1), (2), (3), (4), (6), (7), (8), (9), or (10).*

Since $1 \neq \beta = \frac{1}{4}$, $ch\,\mathbb{F} \neq 3$. If $\lambda_1 \in \{0, 1, \beta\}$ we conclude using Claims 7, 8, and 1, respectively. Assume $\lambda_1 \in \mathbb{F} \setminus \{0, 1, \beta\}$. Then the identity $D_2 = 0$ in Lemma 3.10 gives the relation

$$\frac{28}{3}(4\lambda_1 - 1)(a_1 - a_{-1}) = 0.$$

Since $(4\lambda_1 - 1) \neq 0$, either $a_1 = a_{-1}$ or $ch\,\mathbb{F} = 7$. In the former case $a_2 = a_{-1}^f = a_1^f = a_0$ and, by Claim 2, V ha dimension at most 3, so we conclude by Proposition 5.6. Assume $a_1 - a_{-1} \neq 0$. So $ch\,\mathbb{F} = 7$ and $\beta = \frac{1}{4} = 2$. Then, Equation (12) becomes

$$a_{-2} = a_2 + (a_1 - a_{-1}), \tag{22}$$

the identity $E = 0$ in Lemma 3.10 reduces to

$$(\lambda_1 - 1)(\lambda_1 - 2)\left[2(a_{-2} - a_2) + (\lambda_1^2 + 1)(a_0 - a_2)\right] = 0, \tag{23}$$

and condition $E - E^{\tau_0} = 0$ gives

$$(\lambda_1^2 - 2)(a_{-2} - a_2) = 0.$$

Since, by assumption $a_{-2} - a_2 = a_1 - a_{-1} \neq 0$, it follows $\lambda_1^2 = 2$. Then by Equations (22) and (23), we get

$$2(a_1 - a_{-1}) + 3(a_0 - a_2) = 0$$

and applying the map f,

$$3(a_1 - a_{-1}) + 2(a_0 - a_2) = 0.$$

Taking the difference we get

$$a_0 - a_2 - a_1 + a_{-1} = 0,$$

whence, applying the map τ_0,

$$a_0 - a_{-2} - a_{-1} + a_1 = 0.$$

Now, taking the difference of these two and using Equation (22), we get

$$0 = a_2 - a_{-2} + 2(a_1 - a_{-1}) = a_1 - a_{-1},$$

against the assumption $a_1 - a_{-1} \neq 0$.

Claim 10. *Let $\mathcal{C}$ be the set defined before Lemma 4.3. If $\beta \in \mathcal{C} \setminus \{\frac{3}{8}\}$ and $(\lambda_1, \lambda_2) = \left(\frac{18\beta^2-\beta-2}{2(8\beta-3)}, \frac{(48\beta^4-28\beta^3+7\beta-2)(3\beta-1)}{2\beta^2(8\beta-3)^2}\right)$, then either $2\beta^2 + 5\beta - 2 = 0$, in which case $(\lambda_1, \lambda_2) = (\beta, 1)$ and Claim 1 holds, or $\lambda_1 \neq \beta$ and the following relation holds*

$$\frac{(2\beta^2 + 5\beta - 2)(4\beta^2 + 2\beta - 1)(5\beta^2 + \beta - 1)(7\beta - 2)(2\beta - 1)}{\beta^5(8\beta - 3)^2(\beta - 1)}(a_1 - a_{-1}) = 0. \tag{24}$$

Since we are assuming $\lambda_1 = \frac{18\beta^2-\beta-2}{2(8\beta-3)}$, we have $\lambda_1 = \beta$ if and only if

$$2\beta^2 + 5\beta - 2 = 0.$$

Since $\beta \in \mathcal{C}$, β is a root of at least one of the polynomials c_i, $i \in \{1, \ldots, 7\}$ defined before Lemma 4.3. Routine calculations show that this is possible if and only if $ch\,\mathbb{F} = 5$ and either $\beta = 1$, which is not allowed, or $\beta = -1$. In the latter case, $\lambda_2 = \frac{(48\beta^4-28\beta^3+7\beta-2)(3\beta-1)}{2\beta^2(8\beta-3)^2} = 1$. If $\lambda_1 \neq \beta$, Claim 2 holds and the identity $D_2 = 0$ in Lemma 3.10 becomes Equation (24).

Claim 11. *Assume (λ_1, λ_2) are as in Claim 10 and $2\beta^2 + 5\beta - 2 \neq 0$. Then V falls in one of the cases (1), (2), (5), (6), (7), or (9).*

If $(4\beta^4+2\beta-1)(5\beta^2+\beta-1)(7\beta-2) \neq 0$ in $\mathbb{F}$, from Equation (24) we deduce $a_1 = a_{-1}$ and $a_0 = a_2$. Thus V has dimension at most 3 and we conclude by Proposition 5.6.

If $(5\beta^2 + \beta - 1) = 0$, then $\lambda_1 = \lambda_2 = \frac{\beta}{2}$ and we are in the case of Claim 4.

If $(4\beta^4 + 2\beta - 1) = 0$, we get $\lambda_1 = \beta + \frac{1}{4}$ and $\lambda_2 = \beta$. Then Equation (13) becomes $a_3 = a_{-1}$, whence, applying f, $a_{-2} = a_2$. It follows that V satisfies the multiplication table of the algebra $4Y(2\beta, \beta)$. Hence, by Lemma 5.3, (6) holds.

Finally assume $7\beta - 2 = 0$, that is $ch\,\mathbb{F} \neq 7$ and $\beta = \frac{2}{7}$. In this case $\lambda_1 = \lambda_2 = \frac{4}{7} = \frac{9\beta^2-2\beta}{2(4\beta-1)}$ and we conclude by Claim 3. □

6. The non symmetric case

In this section we deal with the nonsymmetric case and prove Theorem 1.1. Let V be generated by the two axes a_0 and a_1. Recall that, by Proposition 3.7, V has dimension at most 8. Set $V_e := \langle\langle a_0, a_2\rangle\rangle$ and $V_o := \langle\langle a_{-1}, a_1\rangle\rangle$. As noticed at the end of Section 4, V_e and V_o are symmetric, since the automorphisms τ_1 and τ_0 respectively, swap their generating axes. Hence, from Theorem 5.7 we get the possible values for the pair (λ_2, λ_2^f) and the structure of those subalgebras. Note that V is symmetric if and only if $\lambda_1 = \lambda_1^f$ and $\lambda_2 = \lambda_2^f$ in $\mathbb{F}$.

Lemma 6.1. *If V has dimension 8, then $V \cong 6J(2\beta, \beta)$.*

Proof. Suppose V has dimension 8. Then the generators $a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, s_{0,1}$, and $s_{0,3}$ are linearly independent. We express d_1 defined in Lemma 3.10 as a linear combination of the basis vectors (see [7, 2btnon-Symmetric.s]). Since $d_1 = 0$ in V, every coefficient must be zero. In particular, considering the coefficients of a_{-2}, a_3, and $s_{0,1}$ we get, respectively, the equations

$$\frac{(6\beta-1)}{\beta}\lambda_1 + \frac{(2\beta-2)}{\beta}\lambda_1^f - (8\beta-3) = 0$$
$$\frac{(2\beta-2)}{\beta}\lambda_1 + \frac{(6\beta-1)}{\beta}\lambda_1^f - (8\beta-3) = 0$$
$$\frac{8}{\beta^2}(\lambda_1^2 - {\lambda_1^f}^2) - \frac{24}{\beta}(\lambda_1 - \lambda_1^f) - \frac{2}{\beta}(\lambda_2 - \lambda_2^f) = 0$$

whose common solutions have $\lambda_1 = \lambda_1^f$ and $\lambda_2 = \lambda_2^f$. Hence V is symmetric and the result follows from Theorem 5.7. □

Lemma 6.2. *If V is non symmetric, then V_e and V_o are not isomorphic to the seven dimensional quotient of* $6J(-\frac{2}{7}, -\frac{1}{7})$.

Proof. It is enough to show that the claim holds for V_o. Let us assume by contradiction that V_o is isomorphic to the seven dimensional quotient $6J(-\frac{2}{7}, -\frac{1}{7})^{\times}$ of $6J(-\frac{2}{7}, -\frac{1}{7})$. By Lemma 6.1, V has dimension smaller than 8, hence $V = V_o$ and so V has basis $a_{-5}, a_{-3}, a_{-1}, a_1, a_3, a_5, s_{1,2}$ and the product with respect to this basis is known (see [8, Table 8]). Let $a_0 = x_{-5}a_{-5} + x_{-3}a_{-3} + x_{-1}a_{-1} + x_1a_1 + x_3a_3 + x_5a_5 + zs_{1,2}$ be the decomposition of a_0 with respect to this basis. Since $a_0^{\tau_0} = a_0$, and τ_0 swaps a_{-i} and a_i, for $i \in \{1, 3, 5\}$ and fixes $s_{1,2}$, we must have $x_{-i} = x_i$, for $i \in \{1, 3, 5\}$. Moreover, $a_1 - a_{-1}$ and $a_3 - a_{-3}$ are β-eigenvectors for a_0, and so we have

$$\begin{aligned} 0 =& 14[a_0(a_1 - a_{-1}) - \beta(a_1 - a_{-1})] \\ =& (x_3 + x_5 - 2z)(a_5 - a_{-5}) + (x_3 + x_5)(a_{-3} - a_3) \\ &+ (14x_1 - 3x_3 - 3x_5 - 2z + 2)(a_1 - a_{-1}) \end{aligned}$$

and, since all the coefficients must be zero, we get $x_5 = -x_3$, $z = 0$, and $x_1 = -\frac{1}{7}$. Then

$$\begin{aligned} 0 =& 14[a_0(a_3 - a_{-3}) - \beta(a_3 - a_{-3})] \\ =& (x_3 + \tfrac{1}{7})(a_1 - a_{-1} + a_5 - a_{-5}) - 17(x_3 + \tfrac{1}{7})(a_3 - a_{-3}) \end{aligned}$$

whence $x_3 = -\frac{1}{7}$ and $a_0 = \frac{1}{7}(a_{-5} - a_{-3} - a_{-1} - a_1 - a_3 + a_5)$. But then $a_0^2 \neq a_0$, a contradiction. □

Remark 6.3. Lemma 6.1, Lemma 6.2 and Theorem 5.7 imply that if V is non symmetric, then the dimensions of the even subalgebra and of the odd subalgebra are both at most 5.

As a consequence, from Lemma 3.4 we derive some relations between the odd and even subalgebras.

Lemma 6.4. *If $a_{-3} = a_5$, then*

$$\begin{aligned} &Z(\lambda_1^f, \lambda_1)(a_0 - a_2 + a_{-2} - a_4) \\ &= \frac{1}{\beta}\left[2Z(\lambda_1^f, \lambda_1)\left(\lambda_1^f - \beta\right) - \left(\lambda_2^f - \beta\right)\right](a_{-1} - a_3). \end{aligned} \tag{25}$$

If $a_{-4} = a_4$, then

$$\begin{aligned} &Z(\lambda_1, \lambda_1^f)(a_{-1} - a_3 + a_{-3} - a_1) \\ &= \frac{1}{\beta}\left[2Z(\lambda_1, \lambda_1^f)(\lambda_1 - \beta) - (\lambda_2 - \beta)\right](a_{-2} - a_2). \end{aligned} \tag{26}$$

If $a_{-3} = a_3$, then

$$
\begin{aligned}
&2Z(\lambda_1^f, \lambda_1)(a_2 - a_{-2}) \\
&= \frac{1}{\beta}\left[4Z(\lambda_1^f, \lambda_1)\left(\lambda_1^f - \beta\right) - \left(2\lambda_2^f - \beta\right)\right](a_1 - a_{-1}).
\end{aligned} \tag{27}
$$

If $a_{-2} = a_4$, then

$$
\begin{aligned}
&2Z(\lambda_1, \lambda_1^f)(a_{-1} - a_3) \\
&= \frac{1}{\beta}\left[4Z(\lambda_1, \lambda_1^f)(\lambda_1 - \beta) - (2\lambda_2 - \beta)\right](a_0 - a_2).
\end{aligned} \tag{28}
$$

If $a_{-2} = a_4$ and $a_3 = a_{-1}$, then

$$
\begin{aligned}
&2\left[\beta Z(\lambda_1, \lambda_1^f) - 2(\lambda_1^f - \beta)\right](a_2 - a_{-2}) \\
&= \frac{1}{\beta^2}\left[4(2\beta - 1)\beta Z(\lambda_1, \lambda_1^f)(\lambda_1^f - \beta) - 8\beta(\lambda_1 - \lambda_1^f)\left(2\beta - \lambda_1 - \lambda_1^f\right)\right. \\
&\quad \left. + 2(2\beta - 1)^2(\lambda_1^f - \beta) - 2\beta^2(\lambda_2 - \lambda_2^f)\right](a_1 - a_{-1}).
\end{aligned} \tag{29}
$$

Proof. By applying the maps τ_0 and τ_1 to the formulas of Lemma 3.4 we find similar formulas for a_{-4} and a_5. Equations (25), (26), (27), and (28) follow. To prove Equation (29), note that if $a_{-2} = a_4$ and $a_3 = a_{-1}$, then $s_{0,3} = s_{0,1}$. Thus $s_{1,3} - s_{2,3} = 0$ and the claim follows from Lemma 3.6. □

In view of the above relations, it is important to investigate some subalgebras of the symmetric algebras.

Lemma 6.5.

(1) *If U is one of the algebras $4Y(2\beta, \beta)$, $4J(2\beta, \beta)$, or $4J(2\beta, \beta)^\times$, then $U = \langle\langle a_{-1} - a_1, a_0 - a_2\rangle\rangle$.*
(2) *If U is one of the algebras $3C(\beta)$, $3C(-1)^\times$, $3A(2\beta, \beta)$, or $3A(2\beta, \beta)^\times$, then $U = \langle\langle a_{-1} - a_1, a_0 - a_1, \rangle\rangle$, unless $U = 3C(2)$, or $3A(2\beta, \beta)$, where $9\beta^2 - 10\beta + 2 = 0$, or $\beta = \frac{2}{7}$.*

Proof. To prove (1), set $W := \langle\langle a_{-1} - a_1, a_0 - a_2\rangle\rangle$. Let U be equal to $4J(2\beta, \beta)$ or its four dimensional quotient when $\beta = -\frac{1}{4}$. Then $a_{-1}a_1 = 0 = a_0a_2$. Thus $(a_{-1} - a_1)^2 = a_{-1} + a_1$ and $(a_0 - a_2)^2 = a_0 + a_2$, whence we get that a_0, a_1 belong to W and the claim follows.

Let U be equal to $4Y(2\beta, \beta)$. Then, W contains the vectors

$$
\begin{aligned}
&(a_{-1} - a_1)^2 = a_{-1} + a_1 + (2\beta - 1)(a_0 + a_2) - 8\beta s_{0,1}, \\
&(a_0 - a_2)^2 = a_0 + a_2 + (2\beta - 1)(a_{-1} + a_1) - 8\beta s_{0,1},
\end{aligned}
$$

$$a_2 - a_1 = \frac{1}{4(\beta-1)}\left[(a_{-1}-a_1)^2 - (a_0-a_2)^2 - 2(\beta-1)(a_{-1}-a_1+a_0-a_2)\right],$$

and

$$(a_2-a_1)^2 = a_2 + a_1 - 2\beta(a_2+a_1) - 2s_{0,1}.$$

The matrix of the coefficients of the five vectors $a_{-1}-a_1$, a_0-a_2, $(a_{-1}-a_1)^2$, a_2-a_1, and $(a_2-a_1)^2$ (with respect to the basis $(a_{-2}, a_{-1}, a_0, a_1, s_{0,1})$) has determinant $8\beta(1-4\beta)$ which annihilates only for $\beta \in \{0, \frac{1}{4}\}$. However, β cannot be neither 0 nor $\frac{1}{4}$ since the algebra $4Y(2\beta,\beta)$ is defined only when $4\beta^2+2\beta-1=0$. Therefore the five vectors are linearly independent and so generate the entire algebra in this case.

To prove (2), set $W := \langle\langle a_{-1}-a_1, a_0-a_1\rangle\rangle$. Let U be equal to $3C(\beta)$. Then W contains also $(a_0-a_1)^2 = (1-\beta)(a_0+a_1)+\beta a_{-1}$ and the three vectors generate U, unless $\beta = 2$. Since $3C(-1)^\times$ is a quotient of $3C(-1)$, we also obtain that this algebra is an exception only when $-1 = 2$, that is $ch\,\mathbb{F} = 3$. But then $2\beta = 1$, which is not allowed. If $U = 3A(2\beta,\beta)$, then W contains also the vectors $c := (a_0-a_1)(a_{-1}-a_1) = (1-2\beta)a_1 - s_{0,1}$ and

$$\begin{aligned} c^2 = &\frac{1}{(16\beta-4)}[\beta^2(30\beta^2-23\beta+4)(a_{-1}+a_0) - (82\beta^4-129\beta^3+92\beta^2-32\beta+4)a_1 \\ &+ 2\beta(26\beta^2-26\beta+5)s_{0,1}]. \end{aligned}$$

The matrix of the coefficients of the four vectors $a_{-1}-a_1$, a_0-a_1, c, and c^2 is non degenerate, whence the four vectors generate U, unless $9\beta^2-10\beta+2=0$, or $\beta=\frac{2}{7}$. Since $3A(2\beta,\beta)^\times$ is a quotient of $3A(2\beta,\beta)$ provided $18\beta^2-\beta-1=0$, if $U = 3A(2\beta,\beta)^\times$, we get that $W = U$ unless $9\beta^2-10\beta+2=0$, or $\beta=\frac{2}{7}$. In the former case we get no solutions for β; in the latter case we see that $\beta=\frac{2}{7}$ satisfies $18\beta^2-\beta-1=0$ only if $ch\,\mathbb{F}=3$ and $\beta=2$, whence $2\beta=1$, which is not allowed. □

We start now to consider the possible configurations. Many of them are forbidden by the following result which is an immediate consequence of Theorem 1.1 in [17].

Proposition 6.6. *Let V be a primitive 2-generated axial algebra of Monster type with generators a_0 and a_1. Let $D_e := |\{a_i \mid i \in 2\mathbb{Z}\}|$ and $D_o := |\{a_i \mid i \in 1+2\mathbb{Z}\}|$. Then one of the following occurs: $D_e = D_o$, $D_e = 2D_o$, or $2D_e = D_o$.*

Lemma 6.7. *If V is non symmetric and $D_e = 6$, then $V = V_e$.*

Proof. Let us assume that $D_e = 6$. Then, by Theorem 5.7, V_e is isomorphic to either a quotient of $6J(\beta, 2\beta)$ or to $6Y(\frac{1}{2}, 2)$, with $ch\,\mathbb{F} = 7$. By Proposition 3.7, Lemma 6.1 and Lemma 6.2, we get $V_e \cong 6Y(\frac{1}{2}, 2)$, $ch\,\mathbb{F} = 7$ and $\lambda_2 = 1$.

Assume that $\lambda_2^f = 1$. By Theorem 1.2, $(\lambda_1, \lambda_1^f, 1, 1) \in \mathcal{V}(T)$, where T is the set of the five polynomials p_i, $i \in \{1,2,3,4,5\}$, defined in Section 4. Now, a direct computation (see [7, 2btnon-Symmetric.s]) gives

$$\begin{aligned} p_1(x,y,1,1) &= 3x(x-y)^2(3x^4+2x^3y+2x^2y^2+xy^3-y^4+2x^3-3x^2y+xy^2 \\ &\quad -x^2+3xy-2y^2+3x-3) \\ p_2(x,y,1,1) &= x(x-y)(3x^2-3xy+y^2+2x-2y+3) \\ p_3(x,y,1,1) &= x(x-y)(3x^3+3xy^2+y^3+x^2+3xy+y^2-3x-2y-3) \\ p_4(x,y,1,1) &= -y(x-y)(2x^2-2xy-y^2+2x-2y-3) \\ p_5(x,y,1,1) &= 3y(x-y)(3x^3+3x^2y+xy^2+x^2+3y^2-x-3y+2). \end{aligned}$$

One can check that a pair $(\lambda_1, \lambda_1^f) \in \mathbb{F}^2$ annihilates the above five polynomials if and only if $\lambda_1 = \lambda_1^f$, which implies that V is symmetric, a contradiction. Thus

$$\lambda_2^f \neq 1. \tag{30}$$

By Proposition 6.6, $D_o \in \{3,6\}$. If $D_o = 6$, then, as above, $V_o \cong 6Y(\frac{1}{2}, 2)$, whence $\lambda_2^f = 1$, against Equation (30). Thus $D_o = 3$. By Theorem 5.7 and [17, Paragraph 5.2 and Section 7], V_o is isomorphic to one of $3A(2\beta,\beta)$, $3C(2)$, or $3C(4)$. Since $3A(2\beta,\beta)$ is not defined when $\beta = \frac{1}{4}$, we are left with the latter two cases. If $V_o \cong 3C(2)$, then $\lambda_2^f = \frac{\beta}{2} = 1$, against Equation (30). Thus $V_o \cong 3C(4)$ and $\lambda_2^f = \beta = 2$. Since in this algebra $a_{-3} = a_3$, Equation (27) holds and, as $Z(\lambda_1^f, \lambda_1) = \lambda_1^f - \lambda_1 \neq 0$, it is equivalent to

$$a_2 - a_{-2} = (\lambda_1^f - \beta)(a_1 - a_{-1}).$$

Since $a_2 \neq a_{-2}$, $\lambda_1^f - \beta \neq 0$ and so $a_1 - a_{-1} \in V_e$. Then $a_1 - a_3 = (a_1 - a_{-1})^{\tau_1} \in V_e$, whence, by Lemma 6.5, $V_o \subseteq V_e = V$. □

Lemma 6.8. *If V is non symmetric and $D_e = 4$, then $V = V_o$.*

Proof. Let us assume that $D_e = 4$. Then, by Lemma 6.6, $D_o \in \{2,4\}$. Moreover, the vectors a_{-2}, a_0, a_2, a_4 are linearly independent and, from the first formula in Lemma 3.4, we get that $Z(\lambda_1, \lambda_1^f) \neq 0$ and $a_3 \neq a_{-1}$. Hence $D_o = 4$ and a_1, a_{-1}, a_3, a_{-3} are linearly independent. Since $a_{-4} = a_4$, Equation (26) in Lemma 6.4 holds.

Suppose $2Z(\lambda_1, \lambda_1^f)(\lambda_1 - \beta) - (\lambda_2 - \beta) = 0$. Then, by Equation (26), we have $a_1 - a_{-1} + a_3 - a_{-3} = 0$, a contradiction. Hence $2Z(\lambda_1, \lambda_1^f)(\lambda_1 - \beta) - (\lambda_2 - \beta) \neq 0$, and so, by Equation (26), $(a_{-2} - a_2) \in \langle\langle a_{-1}, a_1 \rangle\rangle$. Since V_o is invariant under τ_1, it contains also $a_0 - a_4$. Thus by Lemma 6.5, $V_e \subseteq V_o = V$. □

Lemma 6.9. *For every field $\mathbb{F}$ and every $\beta, \lambda, \mu \in \mathbb{F}$, with $\beta \neq 0$, $Z(\lambda, \mu) = Z(\mu, \lambda)$ implies $\lambda = \mu$.*

Proof. By definition of the polynomial $Z(x, y)$ in Equation (7), we get

$$\begin{aligned}0 =& Z(\lambda, \mu) - Z(\mu, \lambda)\\ =& \frac{2}{\beta}\lambda + \frac{2\beta-1}{\beta^2}\mu - \frac{4\beta-1}{\beta} - \left(\frac{2}{\beta}\mu + \frac{2\beta-1}{\beta^2}\lambda - \frac{4\beta-1}{\beta}\right)\\ =& \frac{1}{\beta^2}(\lambda-\mu). \quad \square\end{aligned}$$

Lemma 6.10. *If V is non symmetric and $D_e = 3$, then either $V = V_e$ or $V = V_o$.*

Proof. Let us assume that $D_e = 3$, that is $a_4 = a_{-2}$. Then, by Proposition 6.6, $D_o \in \{3, 6\}$. If $D_o = 6$, then by Remark 6.3, $V_o \cong 6Y(\frac{1}{2}, 2)$ with $ch\,\mathbb{F} = 7$ and thus Lemma 6.7 yields $V = V_o$. From now on assume $D_o = 3$. In particular, $a_{-3} = a_3$ and Equations (27) and (28) in Lemma 6.4 hold.

If $Z(\lambda_1, \lambda_1^f) = 0 = Z(\lambda_1^f, \lambda_1)$, then, by Lemma 6.9, $\lambda_1 = \lambda_1^f$ and, by Equations (27) and (28), $\lambda_2 = \lambda_2^f = \beta$. Thus V is symmetric: a contradiction. Therefore, without loss of generality, we may assume $Z(\lambda_1^f, \lambda_1) \neq 0$. Then, Equation (27) implies

$$a_2 - a_{-2},\ \ a_0 - a_2 \in V_o.$$

If $V_e = \langle\langle a_2 - a_{-2},\ \ a_0 - a_2\rangle\rangle$, then $V_e \subseteq V_o = V$ and we are done. Let us assume that $V_e \neq \langle\langle a_2 - a_{-2},\ \ a_0 - a_2\rangle\rangle$. Then Lemma 6.5(2) yields that one of the following holds

(*i*) $\beta = 2$ and $V_e \cong 3C(2)$.
(*ii*) $V_e \cong 3A(2\beta, \beta)$, where $9\beta^2 - 10\beta + 2 = 0$, or $\beta = \frac{2}{7}$.

In particular $a_{-2} \neq a_2$ and so, by Equation (27),

$$4Z(\lambda_1^f, \lambda_1)\left(\lambda_1^f - \beta\right) - \left(2\lambda_2^f - \beta\right) \neq 0. \tag{31}$$

From Equation (27) we get that $a_{-1} - a_1$ and $a_{-1} - a_3$ are contained in V_e. As above, if $V_o = \langle\langle a_{-1} - a_1, a_{-1} - a_3\rangle\rangle$, then $V_o = V_e = V$ and we are done. Let us assume that $V_o \neq \langle\langle a_{-1} - a_1, a_{-1} - a_3\rangle\rangle$. As above, by Lemma 6.5(2), either $V_o \cong 3C(2)$, or $V_o \cong 3A(2\beta, \beta)$, where $9\beta^2 - 10\beta + 2 = 0$, or $\beta = \frac{2}{7}$. Note that 2 and $\frac{2}{7}$ are roots of the polynomial $9x^2 - 10x + 2$, or $2 = \frac{2}{7}$, only if $ch\,\mathbb{F} = 3$, in which case $\beta = 2$ and $\alpha = 1$ and this is not allowed. Therefore, in any case, $V_e \cong V_o$.

Suppose $V_e \cong V_o \cong 3C(2)$. Then $s_{0,2} = -\frac{\beta}{2}(a_{-2} + a_0 + a_2) = -a_{-2} - a_0 - a_2$ and, similarly, $s_{1,2} = -a_{-1} - a_1 - a_3$. The second formula in Lemma 3.3 becomes

$$\begin{aligned}-a_{-1} - a_1 - a_3 = &-a_1 - a_3 + 2Z(\lambda_1^f, \lambda_1)(a_0 + a_2)\\ &-[2Z(\lambda_1^f, \lambda_1)(\lambda_1^f - 2) + 1]a_1 + 2Z(\lambda_1^f, \lambda_1)s_{0,1}\end{aligned}$$

whence (since $Z(\lambda_1^f, \lambda_1) \neq 0$)

$$s_{0,1} = (\lambda_1^f - \beta)a_1 - (a_0 + a_2). \tag{32}$$

If also $Z(\lambda_1, \lambda_1^f) \neq 0$, a similar computation with the first formula in Lemma 3.3 yields

$$s_{0,1} = (\lambda_1 - \beta)a_0 - (a_{-1} + a_1). \tag{33}$$

Comparing Equations (32) and (33) and using the invariance of $s_{0,1}$ under τ_0 and τ_1, we get

$$\begin{aligned}(\lambda_1 - 1)a_0 - a_{-1} &= (\lambda_1^f - 1)a_1 - a_2\\ (\lambda_1 - 2)(a_0 - a_2) &= a_{-1} - a_3\\ (\lambda_1^f - 2)(a_1 - a_{-1}) &= a_2 - a_{-2}.\end{aligned}$$

From the above identities we can express a_{-1}, a_{-2}, and a_3 as linear combinations of a_0, a_1, a_2. So V has dimension at most 3 and hence $V = V_o = V_e$. Suppose $Z(\lambda_1, \lambda_1^f) = 0$. Then $\lambda = -\frac{3}{4}\lambda_1^f + \frac{7}{2}$ and, since $\lambda \neq \lambda_1^f$ as V is non symmetric, we have $\lambda_1^f \neq 2$ and $ch\, \mathbb{F} \neq 7$. Further (see [7, 2btnon-Symmetric.s]

$$\begin{aligned}p_2\left(-\frac{3}{4}\lambda_1^f + \frac{7}{2}, \lambda_1^f, 1, 1\right) &= \frac{7}{32}(\lambda_1^f - 2)(3\lambda_1^f - 14)(\lambda_1^f - 7)\\ p_4\left(-\frac{3}{4}\lambda_1^f + \frac{7}{2}, \lambda_1^f, 1, 1\right) &= \frac{7}{512}\lambda_1^f(325{\lambda_1^f}^2 - 1420\lambda_1^f + 1668)(\lambda_1^f - 2)\\ p_5\left(-\frac{3}{4}\lambda_1^f + \frac{7}{2}, \lambda_1^f, 1, 1\right) &= \frac{7}{2048}\lambda_1^f(849{\lambda_1^f}^2 - 8668\lambda_1^f + 18292)(\lambda_1^f - 2).\end{aligned}$$

It is straightforward to check that the only value of λ_1^f which annihilates the above three polynomials is $\lambda_1^f = 2$, which leads to the contradiction $\lambda_1 = \lambda_1^f$.

Suppose $V_e \cong V_o \cong 3A(2\beta, \beta)$ where $9\beta^2 - 10\beta + 2 = 0$, or $\beta = \frac{2}{7}$ and suppose also by contradiction that $V \neq V_e$ and $V \neq V_o$. Thus $\lambda_2 = \lambda_2^f = \frac{\beta(9\beta-2)}{2(4\beta-1)}$ and V has dimension at least 5. Further, being V non symmetric, $\lambda_1 \neq \lambda_1^f$. Set

$$\eta = \frac{2\beta Z(\lambda_1^f, \lambda_1)}{4Z(\lambda_1^f, \lambda_1)(\lambda_1^f - \beta) - (2\lambda_2^f - \beta)}$$

so that, by Equation (27), $a_{-1} = a_1 + \eta(a_{-2} - a_2)$ and, applying τ_2, we get $a_3 = a_{-1} - \eta(a_0 - a_2) = a_1 + \eta(a_{-2} - a_0)$. Thus

$$\begin{aligned}s_{0,3} =& a_0 a_3 - \beta(a_0 + a_3)\\ =& a_0[a_1 + \eta(a_{-2} - a_0)] - \beta[a_0 + a_1 + \eta(a_{-2} - a_0)]\\ =& (2\beta - 1)\eta a_0 + s_{0,1} + \eta s_{0,2}\end{aligned}$$

Then, by Lemma 3.3 and Proposition 3.7, $V = \langle a_{-2}, a_0, a_1, a_2, s_{0,1}\rangle$ and hence the five vectors a_{-2}, a_0, a_1, a_2, $s_{0,1}$ are linearly independent. Hence from Lemma 3.10(2), since $\lambda_1 \neq \lambda_1^f$, we get

$$\beta\lambda_1 + (\beta - 1)(\lambda_1^f - \beta) = 0. \tag{34}$$

Recall the definition of vector d_1 in Lemma 3.10. By Lemma 3.10(1), $d_1 = 0$ in V. Therefore every coefficient of d_1 with respect to the basis $(a_{-2}, a_0, a_1, a_2, s_{0,1})$ must be zero. From the coefficient of $s_{0,1}$ we get

$$-\frac{8}{\beta^2}(\lambda_1 - \lambda_1^f)(\lambda_1 + \lambda_1^f - 3\beta) = 0.$$

Since $\lambda_1 \neq \lambda_1^f$, we get $\lambda_1^f + \lambda_1 - 3\beta = 0$. From Equation (34) it follows $\lambda_1 = 2\beta(1-\beta)$ and $\lambda_1^f = \beta(2\beta+1)$. Now, the difference between the coefficient of a_0 and a_{-2} (as computed in [7, 2btnon-Symmetric.s]) is $5\beta - 1$. Hence $\beta = \frac{1}{5}$. Finally we see that $\beta = \frac{1}{5}$ satisfies the conditions $9\beta^2 - 10\beta + 2 = 0$ or $\beta = \frac{2}{7}$ if and only if $ch\,\mathbb{F} = 3$ and $\beta = 2$, hence $\alpha = 1$, which is not allowed. □

Lemma 6.11. *Assume V is non symmetric and $D_e = 1$. Then $V = V_o$.*

Proof. Let us assume that $D_e = 1$. Then $V_e \cong 1A$, $\lambda_2 = 1$, for every $i \in \mathbb{Z}$, $a_0 = a_{2i}$, and $s_{0,2} = (1 - 2\beta)a_0$. It follows that the Miyamoto involution τ_1 fixes a_0 and a_1, so it is the identity on V, whence the β-eigenspace for ad_{a_1} is trivial. In particular, $a_3 = a_{-1}^{\tau_1} = a_{-1}$ and, by Theorem 5.7, V_o is isomorphic to one of $1A$, $3C(2\beta)$, $3C(-1)^\times$, $2B$. In any case we have $s_{0,3} = s_{0,1}$ and, by Proposition 3.7, V is spanned by a_{-1}, a_0, a_1, and $s_{0,1}$. Suppose $V_o \cong 1A$. Then, V is spanned by a_0, a_1, and $s_{0,1}$ and τ_0 is the identity on V, in particular the β-eigenspace for ad_{a_0} is trivial too. This implies that V is an axial algebra of Jordan type, and hence it is symmetric by [11], a contradiction.

Now suppose $V_o \cong 3C(2\beta)$, $3C(-1)^\times$, or $2B$. If $Z(\lambda_1, \lambda_1^f) = 0$, then

$$Z(\lambda_1^f, \lambda_1) = \frac{(4\beta - 1)}{2\beta^3}(\lambda_1^f - \beta)$$

and, since $a_3 = a_{-1}$ and $a_{-3} = a_1$, from the formula for a_{-3} in Lemma 3.4, we get

$$\frac{1}{\beta}[4Z(\lambda_1^f, \lambda_1)(\lambda_1^f - \beta) - 2\lambda_2^f] = 0.$$

Therefore, from Equation (29) and Theorem 1.2, the quadruple $(\lambda_1, \lambda_1^f, 1, \lambda_2^f)$ must be a solution of the system

$$\begin{cases} Z(\lambda_1, \lambda_1^f) = 0 \\ (4\beta - 1)(\lambda_1^f - \beta)^2 = \beta^3 \lambda_2^f \\ -\frac{8}{\beta}(\lambda_1 - \lambda_1^f)\left(2\beta - \lambda_1 - \lambda_1^f\right) + \frac{2(2\beta-1)^2}{\beta^2}(\lambda_1^f - \beta) - 2(1 - \lambda_2^f) = 0 \\ p_2(\lambda_1, \lambda_1^f, 1, \lambda_2^f) = 0 \\ p_4(\lambda_1, \lambda_1^f, 1, \lambda_2^f) = 0. \end{cases} \tag{35}$$

When $V_o \cong 2B$, then $\lambda_2^f = 0$ and the second equation yields that either $\lambda_1^f = \beta$ or $\beta = \frac{1}{4}$. In the former case, we obtain $\lambda = \lambda_1^f = \beta$ from the first equation and the third gives $\lambda_2^f = 1$: a contradiction. In the latter case $\beta = \frac{1}{4}$, from the first equation we get $\lambda = \lambda_1^f$; then the third equation gives $2\lambda_1^f - 1 = 0$, whence $\lambda_1^f = \frac{1}{2}$ and $p_2(\frac{1}{2}, \frac{1}{2}, 1, 0) = -4$, a contradiction (see [7, 2btnon-Symmetric.s]).

Finally, assume $V_o \cong 3C(2\beta)$ or to its quotient $3C(-1)^\times$, when $\beta = -\frac{1}{2}$, whence $\lambda_2^f = \beta$. By adding the second and the third equations of system (35) we get

$$\frac{2(2\beta-1)^2}{\beta^2}\lambda_1^f - \frac{2(4\beta^2 - 3\beta + 1)}{\beta} = 0$$

whence

$$\lambda_1^f = \frac{\beta(4\beta^2 - 3\beta + 1)}{(2\beta - 1)^2}.$$

Thus, since $\beta \notin \{0, 1, \frac{1}{2}\}$, the second, the fourth and the sixth equations of system (35) reduce to

$$\begin{cases} 8\beta^4 - 16\beta^3 + 12\beta^2 - 6\beta + 1 = 0 \\ (4\beta - 3)(16\beta^5 - 16\beta^3 + 20\beta^2 - 8\beta + 1) = 0 \\ p_4(\lambda_1, \lambda_1^f, 1, \beta) = 0. \end{cases} \tag{36}$$

If $\beta = \frac{3}{4}$, then by the first equation $ch\,\mathbb{F} \neq 3$ and the first and the third equations give respectively $-\frac{31}{32} = 0$ and $-\frac{2639}{24} = 0$ (see [7, 2btnon-Symmetric.s]), which is a contradiction. Hence β must annihilate the two polynomials

$$16x^5 - 16x^3 + 20x^2 - 8x + 1 \text{ and } 8x^4 - 16x^3 + 12x^2 - 6x + 1. \tag{37}$$

The remainder of the division of these two is $24x^3 - 16x^2 + 12x - 3$ and it also must be annihilated by β. Thus, if $ch\,\mathbb{F} = 3$, this last polynomial reduces to $-x^2$, giving $\beta = 0$: a contradiction since 0 does not annihilate the polynomials in Equation (37). Hence $ch\,\mathbb{F} \neq 3$. Using the Euclidean algorithm we get that the two polynomials in Equation (37) have a common root if and only if $-\frac{328}{729} = 0$, in which case the common root is $\beta = \frac{99}{243}$. Then $ch\,\mathbb{F} = 41$, $\beta = 8$ and $p_4(\lambda_1, \lambda_1^f, 1, \beta) = 12$, a contradiction.

Hence $Z(\lambda_1, \lambda_1^f) \neq 0$. Then, from the formula for $s_{0,2}$ in Lemma 3.3 we get $s_{0,1} = (\lambda_1 - \beta)a_0 - \frac{\beta}{2}(a_1 + a_{-1})$, whence V is spanned by a_{-1}, a_0, and a_1 and $a_0 a_1 = \lambda_1 a_0 + \frac{\beta}{2}(a_1 - a_{-1})$. In particular, we see that ad_{a_0} has eigenvalues 0, 1 and β. If $V_o \cong 3C(2\beta)$, then, having the same dimension, $V = V_o$ as claimed. Assume $V_o \cong 2B$. Then $a_{-1}a_1 = 0$ and we see that ad_{a_1} has eigenvalues $0, 1$, and λ_1. Since 2-generated axial algebras of Jordan type are symmetric, we must have $\lambda_1 = 2\beta$. Then it is straightforward to see that ad_{a_0} has 0-eigenvector $4\beta a_0 - (a_1 + a_{-1})$ and, in order to satisfy the fusion law $\alpha \star \alpha = \{0, 1\}$, β must be equal to $\frac{1}{4}$. Then $v := a_0 - \frac{1}{2}(a_1 + a_{-1})$ is an α-eigenvector for ad_{a_1}, but $v^2 = \frac{1}{2}a_0 + \frac{1}{16}(a_1 + a_{-1}) \notin \langle a_1, a_{-1}\rangle$. Therefore, the fusion law $\alpha \star \alpha = \{0, 1\}$ is not satisfied, a contradiction. Therefore, we are left with the case $\beta = -\frac{1}{2}$ ($ch\mathbb{F} \neq 3$ since $\beta \neq 1$) and $V_o \cong 3C(-1)^\times$. In this case, by [11, §3.4], V_o has basis (a_{-1}, a_1) and $a_1 a_{-1} = -a_{-1} - a_1$. Since ad_{a_0} has three distinct eigenvalues in the 3-dimensional space V, the 0-eigenspace is 1-dimensional, generated by $v := a_1 + a_{-1} - 2\lambda_1 a_0$. In order to satisfy the fusion law $0 \star 0 = 0$, we have

$$v^2 = (a_1 + a_{-1} - 2\lambda_1 a_0)(a_1 + a_{-1} - 2\lambda_1 a_0) = -a_{-1} - a_1 - 4\lambda_1^2 a_0 \in \langle v\rangle,$$

whence $\lambda_1 \in \{0, -\frac{1}{2}\}$. Further, ad_{a_1} has eigenvalues $1, -1$, and λ_1. Since, as we observed at the beginning of this proof, the β-eigenspace for ad_{a_1} is trivial, we must have $\lambda_1 = 0$. Then we get a contradiction since the 0-eigenspace for ad_{a_1} is 1-dimensional, generated by $w := a_{-1} + 4a_0 + 2a_1$, and $w^2 = -a_{-1} + 16a_0 - 2a_1 \notin \langle w\rangle$, against the fusion law $0 \star 0 = 0$. $\square$

Lemma 6.12. *Let V be non symmetric and suppose that $V = V_o$. Then $\beta = \frac{1}{3}$, $V_o \cong 3C(2\beta)$. Moreover, if a_{-1}, a_1, u is a basis of V_o such that $ab = \beta(a + b - c)$ whenever $\{a, b, c\} = \{a_{-1}, a_1, u\}$, then $a_0 = \frac{3}{5}(a_{-1} + a_1) - \frac{2}{5}u$.*

Proof. Clearly $V = V_o$ has dimension greater than 1 and so, by Lemmas 6.1 and 6.2, V_o is isomorphic to one of the following: $2B$, $3C(2\beta)$, $3C(\beta)$, $3C(-1)^\times$, $3A(2\beta, \beta)$, $3A(2\beta, \beta)^\times$, $4J(2\beta, \beta)$, $4J(2\beta, \beta)^\times$, $4Y(2\beta, \beta)$, or $6Y(\frac{1}{2}, 2)$ with $ch\,\mathbb{F} = 7$.

Suppose $V_o \cong 2B$, so that $V = V_o = \langle a_{-1}, a_1\rangle$ and $a_{-1}a_1 = 0$. Since a_0 is an idempotent distinct from a_1 and a_{-1}, we get $a_0 = a_{-1} + a_1$. Then both a_{-1} and a_1 are 1-eigenvectors for ad_{a_0}, a contradiction.

Suppose $V_o \cong 3C(2\beta)$, so that $V = V_o$ has basis (a_{-1}, a_1, u) where $u^2 = u$, and $ab = \beta(a + b - c)$ whenever $\{a, b, c\} = \{a_{-1}, a_1, u\}$. Let $a_0 = x_{-1}a_{-1} + x_1 a_1 + zu$ be the decomposition of a_0 with respect to the basis a_{-1}, a_1, u. Since $a_0^{\tau_0} = a_0$, and τ_0 swaps a_{-1} and a_1 and fixes u, we have immediately that $x_{-1} = x_1$. Then we have

$$0 = a_0(a_1 - a_{-1}) - \beta(a_1 - a_{-1}) = (2\beta z + x_1 - \beta)(a_1 - a_{-1})$$

whence $x_1 = \beta - 2\beta z$. Then,

$$0 = a_0^2 - a_0$$

$$= x_1(2\beta x_1 + x_1 - 1)(a_{-1} + a_1) - (2\beta x_1^2 - 4\beta x_1 z - z^2 + z)u.$$

If $x_1 = 0$, we get $z = \frac{1}{2}$ but also $z^2 - z = 0$, a contradiction. Hence $x_1 \neq 0$ and so $2\beta x_1 + x_1 - 1 = 0$, whence $2\beta + 1 \neq 0$ and $x_1 = \frac{1}{2\beta+1}$. Then, since $x_1 = \beta - 2\beta z$, we get $z = \frac{2\beta^2+\beta-1}{2\beta(2\beta+1)}$ and $a_0^2 - a_0 = \frac{(3\beta-1)(\beta-1)}{4\beta^2}u$. Since $\beta \neq 1$, we get $\beta = \frac{1}{3}$ and $a_0 = \frac{3}{5}(a_{-1} + a_1) - \frac{2}{5}u = \mathbb{1} - u$, where $\mathbb{1} = \frac{3}{5}(a_{-1} + a_1 + u)$ is the identity of the algebra $3C(\frac{2}{3})$. A straightforward computation shows that a_0 is an axis with eigenvalues 1, 0 and β.

Suppose $V_o \cong 3C(\beta)$, so that $V = V_o$ has basis (a_{-1}, a_1, u) where $u^2 = u$, and $ab = \frac{\beta}{2}(a + b - c)$ whenever $\{a, b, c\} = \{a_{-1}, a_1, u\}$. Let $a_0 = x_{-1}a_{-1} + x_1a_1 + zu$ be the decomposition of a_0 with respect to the basis a_{-1}, a_1, u. As in the previous case, $x_{-1} = x_1$. Then we have

$$0 = a_0(a_1 - a_{-1}) - \beta(a_1 - a_{-1}) = (\beta z + x_1 - \beta)(a_1 - a_{-1})$$

whence $x_1 = \beta - \beta z$. Then,

$$\begin{aligned} 0 &= a_0^2 - a_0 \\ &= x_1(\beta x_1 + x_1 - 1)(a_{-1} + a_1) - (\beta x_1^2 - 2\beta x_1 z - z^2 + z)u. \end{aligned}$$

Now, if $x_1 = 0$, then $z = 1$ and $a_0 = u$. This means that $V = \langle\langle a_0, a_1\rangle\rangle$ is symmetric, a contradiction. Hence $x_1 \neq 0$, whence $\beta + 1 \neq 0$, $x_1 = \frac{1}{\beta+1}$ and $z = \frac{\beta^2+\beta-1}{\beta(\beta+1)}$. Then we get $a_0^2 - a_0 = \frac{(2\beta-1)(\beta-1)}{\beta^2}u \neq 0$, a contradiction.

Suppose $V_o \cong 3A(2\beta, \beta)$. Then $V = V_o$ has basis $(a_{-1}, a_1, a_3, s_{1,2})$. Let

$$a_0 = x_{-1}a_{-1} + x_1a_1 + x_3a_3 + zs_{1,2}$$

be the decomposition of a_0 with respect to this basis. As in the previous cases, $x_{-1} = x_1$. Then,

$$a_2 = a_0^{\tau_1} = x_3a_{-1} + x_1(a_1 + a_3) + zs_{1,2}$$

and

$$a_{-2} = a_0^{\tau_{-1}} = x_1(a_{-1} + a_3) + x_3a_1 + zs_{1,2}.$$

Now we compute

$$\begin{aligned} 0 &= a_0(a_1 - a_{-1}) - \beta(a_1 - a_{-1}) \\ &= \frac{1}{(4\beta - 1)}\left[(4\beta - 1)(x_1 + \beta x_3) - \beta^2(7\beta - 2)z - \beta(4\beta - 1)\right](a_1 - a_{-1}) \end{aligned}$$

and

$$= a_0^2 - a_0 - a_2^2 + a_2$$
$$= \frac{(x_1 - x_3)}{(4\beta - 1)}[(8\beta^2 + 2\beta - 1)x_1 - 2\beta^2(7\beta - 2)z + (4\beta - 1)(x_3 - 1)](a_{-1} - a_3).$$

Since, by Lemma 6.11, $V_e \not\cong 1A$, we have $x_3 \neq x_1$ and thus, the above two equations reduce respectively to Equations (38) and (39)

$$(4\beta - 1)(x_1 + \beta x_3) - \beta^2(7\beta - 2)z - \beta(4\beta - 1) = 0 \tag{38}$$
$$(8\beta^2 + 2\beta - 1)x_1 + (4\beta - 1)x_3 - 2\beta^2(7\beta - 2)z - (4\beta - 1) = 0. \tag{39}$$

By subtracting Equation (38) twice from Equation (39) we get

$$(4\beta - 1)(2\beta - 1)(x_1 - x_3 + 1) = 0 \tag{40}$$

whence (recall $\beta \notin \{\frac{1}{2}, \frac{1}{4}\}$ in this case) $x_3 = x_1 + 1$. Thus, Equations (38) and (39) become

$$(4\beta - 1)(\beta + 1)x_1 - \beta^2(7\beta - 2)z = 0. \tag{41}$$

Suppose $\beta = -1$. Equation (41) gives $9z = 0$. If $\mathbb{F}$ has characteristic other than 3, then $z = 0$ and

$$0 = a_0^2 - a_0 = -x_1(3x_1 + 3)(a_{-1} + a_1 + a_3) + 2x_1(3x_1 + 2)s_{1,2}$$

whence it follows $x_1 = 0$. Therefore $a_0 = a_3$, and this is a contradiction since V is non symmetric. If $\mathbb{F}$ has characteristic 3, then we get

$$0 = a_0^2 - a_0 = z(a_{-1} + a_1 + a_3) + x_1 s_{1,2},$$

whence $z = x_1 = 0$ and again the contradiction $a_0 = a_3$.

Hence $\beta \neq -1$ and from Equation (41) we get

$$x_1 = \frac{\beta^2(7\beta - 2)}{(4\beta - 1)(\beta + 1)}z.$$

In this case we get

$$\begin{aligned}
0 &= 4(4\beta - 1)^2(\beta + 1)^2(a_0^2 - a_0) \\
&= \beta^2 z \left[\beta(18\beta^2 - \beta - 1)(5\beta - 1)(8\beta - 1)z \right. \\
&\qquad \left. +4(18\beta^2 - 8\beta + 1)(\beta + 1)(4\beta - 1)\right](a_{-1} + a_1 + a_3) \\
&\quad +2z\left[\beta(18\beta^2 - \beta - 1)(5\beta - 1)(10\beta^2 - 1)z \right. \\
&\qquad \left. +2(12\beta^2 + 2\beta - 1)(\beta + 1)(4\beta - 1)(3\beta - 1)\right] s_{1,2}
\end{aligned}$$

and so either $z = 0$, whence $x_1 = 0$ and $a_0 = a_3$, a contradiction as above, or

$$\beta(18\beta^2 - \beta - 1)(5\beta - 1)(8\beta - 1)z + 4(18\beta^2 - 8\beta + 1)(\beta + 1)(4\beta - 1) = 0 \tag{42}$$

and

$$\beta(18\beta^2 - \beta - 1)(5\beta - 1)(10\beta^2 - 1)z + 2(12\beta^2 + 2\beta - 1)(\beta + 1)(4\beta - 1)(3\beta - 1) = 0. \tag{43}$$

We now multiply Equation (42) by $(10\beta^2 - 1)$ and subtract Equation (43) multiplied by $(8\beta - 1)$ to get

$$2(2\beta - 1)^2(18\beta^2 - \beta - 1)(\beta + 1)(4\beta - 1) = 0.$$

Therefore, $18\beta^2 - \beta - 1 = 0$. Then Equation (42) gives $18\beta^2 - 8\beta + 1 = 0$. Summing and subtracting these two equations we get $9\beta(4\beta - 1) = 0$ and $7\beta - 2 = 0$, whence $\beta = \frac{1}{4} = \frac{2}{7}$, which gives $7 = 8$, a contradiction in all characteristics.

Suppose $V_o \cong 3A(2\beta, \beta)^\times$, with $18\beta^2 - \beta - 1 = 0$ and $\mathbb{F}$ of characteristic other than 3. Then $V = V_o$ has basis (a_{-1}, a_1, a_3). Let $a_0 = x_{-1}a_{-1} + x_1a_1 + x_3a_3$ be the decomposition of a_0 with respect to this basis. As in the previous cases, $x_{-1} = x_1$. Then

$$0 = a_0(a_1 - a_{-1}) - \beta(a_1 - a_{-1}) = (x_3\beta + x_1 - \beta)(a_1 - a_{-1})$$

whence $x_1 = \beta(1 - x_3)$, and

$$\begin{aligned} 0 &= a_0^2 - a_0 \\ &= (x_3 - 1)\beta[3\beta(\beta - 1)x_3 - 3\beta^2 - \beta + 1](a_{-1} + a_1) \\ &\quad + (x_3 - 1)[(\beta^3 - 6\beta^2 + 1)x_3 - \beta^3]a_3. \end{aligned}$$

If $x_3 = 1$, then $x_1 = 0$ and $a_0 = a_3$, a contradiction since V is not symmetric. Hence $x_3 \neq 1$ and so $3\beta(\beta - 1)x_3 - 3\beta^2 - \beta + 1 = 0$ and $(\beta^3 - 6\beta^2 + 1)x_3 - \beta^3 = 0$. Since $\beta \neq 1$, from the first equation we see that $\mathbb{F}$ has characteristic other than 3 and $x_3 = \frac{3\beta^2 + \beta - 1}{3\beta(\beta - 1)}$, $x_1 = \frac{1 - 4\beta}{3(\beta - 1)}$. Then, from the second equation we get

$$(7\beta^2 - 1)(2\beta - 1)(\beta + 1) = 0.$$

Since $ch\mathbb{F} \neq 3$, we must have $7\beta^2 - 1 = 0$. Then, since by hypothesis $18\beta^2 - \beta - 1 = 0$, we get that $\mathbb{F}$ has characteristic 19, $\beta = 7$ and $a_0 = 8(a_{-1} + a_1 + a_3)$. It is straightforward to check that the β-eigenspace for ad_{a_0} is 2-dimensional, spanned by $v_1 := 3(a_{-1} + a_1) + 13a_3$ and $v_2 := 3(a_{-1} + a_3) + 13a_1$, while the 0-eigenspace and the α-eigenspace are trivial. Since $v_1v_2 = 6a_{-1} - 4(a_1 + a_3) \notin \langle a_0 \rangle$, the fusion law $\beta \star \beta = \{0, 1, \alpha\}$ is not satisfied, a contradiction.

Suppose $V_o \cong 3C(-1)^\times$ (and $\mathbb{F}$ has characteristic other than 3 since $\alpha = 2\beta \neq 1$). Then $V = V_o$ has basis (a_{-1}, a_1) and $a_{-1}a_1 = -a_{-1} - a_1$. Let $a_0 = x_{-1}a_{-1} + x_1a_1$ be the decomposition of a_0 with respect to this basis. We have

$$0 = a_0(a_1 - a_{-1}) - \beta(a_1 - a_{-1}) = -(2x_{-1} - x_1 + 1)a_{-1} - (x_{-1} - 2x_1 - 1)a_1$$

whence $x_{-1} = x_1 = -1$. Thus $a_0 = -a_{-1} - a_1$. Then τ_{-1} swaps a_1 and a_0 and hence V is symmetric, a contradiction.

Suppose $V_o \cong 4J(2\beta, \beta)$. Then a basis for V is given by $(a_{-3}, a_{-1}, a_1, a_3, u)$, where $u := -\frac{2}{\beta}s_{1,2}$. Let $a_0 = x_{-3}a_{-3} + x_{-1}a_{-1} + x_1a_1 + x_3a_3 + zu$ be the decomposition of a_0 with respect to this basis. As in the previous cases, since $a_0^{\tau_0} = a_0$, we have $x_{-3} = x_3$ and $x_{-1} = x_1$. Then, using the multiplication for the chosen basis given in [8, Table 7], we get

$$\begin{aligned}0 &= a_0(a_1 - a_{-1}) - \beta(a_1 - a_{-1})\\ &= \beta(x_3 - z)(a_3 - a_{-3}) + (x_1 + \beta x_3 + 3\beta z - \beta)(a_1 - a_{-1})\end{aligned}$$

and

$$\begin{aligned}0 &= a_0(a_3 - a_{-3}) - \beta(a_3 - a_{-3})\\ &= (\beta x_1 + 3\beta z + x_3 - \beta)(a_3 - a_{-3}) + \beta(x_1 - z)(a_1 - a_{-1})\end{aligned}$$

whence, $x_1 = x_3 = z$ and $(4\beta + 1)z - \beta = 0$. Thus, if $\beta = -\frac{1}{4}$ we get immediately $\beta = 0$, a contradiction. Hence $\beta \neq -\frac{1}{4}$, $z = \frac{\beta}{4\beta+1}$ and $a_0 = \frac{\beta}{4\beta+1}(a_{-3} + a_{-1} + a_1 + a_3 + u)$. Then, $a_0^2 = \beta a_0 \neq a_0$, a contradiction.

Suppose now that $\beta = -\frac{1}{4}$ and $V = V_o$ is isomorphic to $4J(2\beta, \beta)^\times = 4J(2\beta, \beta)/I$ where $I = \langle a_{-3} + a_{-1} + a_1 + a_3 + u\rangle$. Then the calculations used above for the case of $4J(2\beta, \beta)$ are easily adapted and produce a contradiction.

Suppose $V = V_o \cong 4Y(2\beta, \beta)$, with $4\beta^2 + 2\beta - 1 = 0$. Then a basis for V is given by $(a_{-3}, a_{-1}, a_1, a_3, s_{1,2})$. Let $a_0 = x_{-3}a_{-3} + x_{-1}a_{-1} + x_1a_1 + x_3a_3 + zs_{1,2}$ be the decomposition of a_0 with respect to this basis. As above, $x_{-3} = x_3$ and $x_{-1} = x_1$. Then, using the multiplication for the chosen basis given in Table 3, we get

$$\begin{aligned}0 &= a_0(a_1 - a_{-1}) - \beta(a_1 - a_{-1}) =\\ &= -[\beta x_1 + \beta x_3 + \frac{1}{2}x_3 - \frac{1}{2}\beta^2 z - \beta]a_{-3} + [\beta x_1 + \beta x_3 + \frac{1}{2}x_1 - \frac{1}{2}\beta^2 z - \beta]a_{-1}\\ &\quad +[(\beta(x_1 - x_3) + \frac{1}{2}x_3 + \frac{1}{2}\beta^2 z]a_1 + [(\beta(x_1 - x_3) - \frac{1}{2}x_1 - \frac{1}{2}\beta^2 z]a_3\\ &\quad -4\beta(x_1 - x_3)s\end{aligned}$$

whence $x_1 = x_3$, and

$$
\begin{aligned}
0 &= 2a_0(a_3 - a_{-3}) - \beta(a_3 - a_{-3}) = \\
&= (x_1 + \beta^2 z)(a_{-1} - a_{-3}) + \left[(4\beta+1)x_1 - \beta^2 z - 2\beta\right](a_1 - a_3)
\end{aligned}
$$

whence $x_1 = -\beta^2 z$ and $\beta(2\beta+1)z + 1 = 0$. Thus, if $\beta = -\frac{1}{2}$ we get immediately a contradiction. If $\beta \neq -\frac{1}{2}$, then $z = -\frac{1}{\beta(2\beta+1)}$ and

$$a_0 = \frac{\beta}{2\beta+1}(a_{-3} + a_{-1} + a_1 + a_3) - \frac{1}{\beta(2\beta+1)}s.$$

From the condition $a_0^2 = a_0$ we get that

$$32\beta^5 + 16\beta^4 - 16\beta^3 + 4\beta^2 + 5\beta - 1 = 0.$$

Since also $4\beta^2 + 2\beta - 1 = 0$, this leads to a contradiction.

Assume finally that $ch\,\mathbb{F} = 7$ and $V = V_o \cong 6Y(\frac{1}{2}, 2)$. Then a basis for V is given by (a_{-1}, a_3, a_7, d, z), where $d = a_1 - a_5 = a_{-5} - a_1$ and $z = -\frac{1}{2}d^2$. Note that τ_0 acts on the basis in the following way

$$a_{-1}^{\tau_0} = a_7 + d,\;\; a_3^{\tau_0} = a_3 + d,\;\; a_7^{\tau_0} = a_{-1} + d,\;\; d^{\tau_0} = d,\;\; z^{\tau_0} = z.$$

Let $a_0 = x_{-1}a_{-1} + x_3 a_3 + x_7 a_7 + xd + tz$ be the decomposition of a_0 with respect to this basis. Since a_0 is fixed by τ_0, we have $x_{-1} = x_7$ and $r = x_{-1} + x_3 + x_7 + r$, whence $x_3 = -2x_7$. Then, using the multiplication for the chosen basis given in [17, Table 1], since $a_1 = a_7 + d$, we get

$$
\begin{aligned}
0 &= a_0(a_1 - a_{-1}) - \beta(a_1 - a_{-1}) \\
&= [x_7(a_{-1} + a_7) - 2x_7 a_3 + rd + tz](a_7 - a_{-1} + d) \\
&= (3x_7 + 2)(a_{-1} - a_7) - 2d - 2rz,
\end{aligned}
$$

which is a contradiction. □

Proof of Theorem 1.1. Let V be a non symmetric 2-generated primitive axial algebra of Monster type $(2\beta, \beta)$ with generators a_0 and a_1. If $V = V_o$ or $V = V_e$, then Lemma 6.12 yields that claim (3) holds. So, let us assume that $V_e \neq V \neq V_o$. By Lemmas 6.1, 6.2, 6.7, 6.8, 6.10, and 6.11 the even and the odd subalgebras V_e and V_o are isomorphic to either $2B$, or to a quotient of $3C(2\beta)$ of dimension greater than 1 and by Lemma 5.2, $(\lambda_2, \lambda_2^f) \in \{(0,0), (\beta,\beta), (0,\beta), (\beta,0)\}$. Moreover, $a_{-1} = a_3$, $a_1 = a_{-3}$, $a_4 = a_0$, $a_{-2} = a_2$, and so, from Lemma 3.4, we get that $(\lambda_1, \lambda_1^f, \lambda_2, \lambda_2^f)$ satisfies the following equations

$$Z(\lambda_1, \lambda_1^f)(\lambda_1 - \beta) = \frac{\lambda_2}{2} \tag{44}$$

and

$$Z(\lambda_1^f, \lambda_1)(\lambda_1^f - \beta) = \frac{\lambda_2^f}{2}. \tag{45}$$

Suppose first that $\lambda_2 = \lambda_2^f$. Then we get $Z(\lambda_1, \lambda_1^f)(\lambda_1 - \beta) = Z(\lambda_1^f, \lambda_1)(\lambda_1^f - \beta)$, which is equivalent to $(\lambda_1 - \lambda_1^f)(\lambda_1 + \lambda_1^f - 2\beta) = 0$ and so $\lambda_1 + \lambda_1^f - 2\beta = 0$, since $\lambda_1 \neq \lambda_1^f$, as the algebra is non symmetric. Then, the system of Equations (44) and (45) is equivalent to

$$\begin{cases} \frac{1}{\beta^2}(\lambda_1 - \beta)^2 = \frac{\lambda_2}{2} \\ \lambda_1^f = 2\beta - \lambda_1 \end{cases} \tag{46}$$

If $\lambda_2 = 0$, we get the solution $(\beta, \beta, 0, 0)$ which corresponds to a symmetric algebra, a contradiction. Suppose $\lambda_2 = \beta$. It is long but straightforward to check that there is no quadruple $(\lambda_1, 2\beta - \lambda_1, \beta, \beta)$, with $\beta \notin \{0, \frac{1}{2}\}$, which is a common solution of (46) and of the set of polynomials T defined in (8) (see [7, 2btnon-Symmetric.s]). A contradiction to Theorem 1.2.

Finally assume that $\lambda_2 = \beta$ and $\lambda_2^f = 0$. Then, by Equation (45), either $Z(\lambda_1^f, \lambda_1) = 0$ or $\lambda_1^f = \beta$. If $Z(\lambda_1^f, \lambda_1) = 0$, then $\lambda_1^f = \frac{4\beta-1}{2} - \frac{2\beta-1}{2\beta}\lambda_1$ and we check that no quadruple $(\lambda_1, \lambda_1^f, \beta, 0)$, with $\beta \notin \{0, \frac{1}{2}\}$, is a common solution of Equation (44) and of the set of polynomials T defined in (8) (see [7, 2btnon-Symmetric.s]). So $\lambda_1^f = \beta$. Then Equation (44) becomes

$$(\lambda_1 - \beta)^2 = \frac{\beta^2}{4},$$

whence $\lambda_1 - \beta = \pm\frac{\beta}{2}$, and we get the two quadruples $(\frac{3}{2}\beta, \beta, \beta, 0)$ and $(\frac{\beta}{2}, \beta, \beta, 0)$. In the former case, we get $p_2(\frac{3}{2}\beta, \beta, \beta, 0) = 2(2\beta - 1)(4\beta - 1)$, whence $\beta = \frac{1}{4}$, and $p_1(\frac{3}{8}, \frac{1}{4}, \frac{1}{4}, 0) = 3$ which is equal zero only if $ch\mathbb{F} = 3$ and $\beta = 1$, a contradiction (see [7, 2btnon-Symmetric.s]). If $(\lambda_1, \lambda_1^f, \lambda_2, \lambda_2^f) = (\frac{\beta}{2}, \beta, \beta, 0)$, then from Lemma 3.3 we get $s_{0,1} = -\beta(a_0 + a_2)$ and so V has dimension at most 4. Moreover, V satisfies the same multiplication table as the algebra $Q_2(\beta)$. By Theorem 8.6 in [8], for $\beta \neq -\frac{1}{2}$ the algebra $Q_2(\beta)$ is simple, while it has a 3-dimensional quotient over the radical $\mathbb{F}(a_0+a_1+a_2+a_{-1})$ when $\beta = -\frac{1}{2}$. Thus claim (2) follows. □

Data availability

No data was used for the research described in the article.

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References

[1] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann, Singular 4-1-1 — a computer algebra system for polynomial computations, http://www.singular.uni-kl.de, 2018.
[2] C. Franchi, M. Mainardis, A.A. Ivanov, Majorana representations of the finite groups, Algebra Colloq. 27 (2020) 31–50.
[3] C. Franchi, M. Mainardis, Classifying 2-generated symmetric axial algebras of Monster type, J. Algebra 596 (2022) 200–218.
[4] C. Franchi, M. Mainardis, J. M^{c}Inroy, Quotients of the Highwater algebra and its cover, arXiv: 2205.02200, May 2022, 46 pages.
[5] C. Franchi, M. Mainardis, S. Shpectorov, 2-generated axial algebras of Monster type, arXiv:2101. 10315, Jan 2021, 22 pages.
[6] C. Franchi, M. Mainardis, S. Shpectorov, An infinite-dimensional 2-generated primitive axial algebra of Monster type, Ann. Mat. Pura Appl. 201 (2022) 1279–1293.
[7] C. Franchi, M. Mainardis, S. Shpectorov, https://github.com/shpectorov/generic_Sakuma, 2023.
[8] A. Galt, V. Joshi, A. Mamontov, S. Shpectorov, A. Staroletov, Double axes and subalgebras of Monster type in Matsuo algebras, Commun. Algebra 49 (10) (2021) 4208–4248.
[9] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.10.0, https://www.gap-system.org, 2019.
[10] J. Hall, F. Rehren, S. Shpectorov, Universal axial algebras and a theorem of Sakuma, J. Algebra 421 (2015) 394–424.
[11] J. Hall, F. Rehren, S. Shpectorov, Primitive axial algebras of Jordan type, J. Algebra 437 (2015) 79–115.
[12] V. Joshi, Axial algebras of Monster type $(2\eta, \eta)$, PhD thesis, University of Birmingham, 2020.
[13] S.M.S. Khasraw, J. McInroy, S. Shpectorov, On the structure of axial algebras, Trans. Am. Math. Soc. 373 (2020) 2135–2156.
[14] A.A. Ivanov, The future of Majorana theory, in: Group Theory and Computation, in: Indian Statistical Institute Series, Springer, Singapore, 2018, pp. 107–118.
[15] A.A. Ivanov, D.V. Pasechnik, À. Seress, S. Shpectorov, Majorana representations of the symmetric group of degree 4, J. Algebra 324 (2010) 2432–2463.
[16] S. Lang, Algebra, Graduate Texts in Mathematics, vol. 211, Springer, 2002.
[17] J. M^{c}Inroy, S. Shpectorov, From forbidden configurations to a classification of some axial algebras of Monster type, J. Algebra 627 (2023) 58–105.
[18] S.P. Norton, The Monster algebra: some new formulae, in: Moonshine, the Monster and Related Topics, South Hadley, Ma., 1994, in: Contemp. Math., vol. 193, AMS, Providence, RI, 1996, pp. 297–306.
[19] F. Rehren, Axial algebras, PhD thesis, University of Birmingham, 2015.
[20] F. Rehren, Generalized dihedral subalgebras from the Monster, Trans. Am. Math. Soc. 369 (2017) 6953–6986.
[21] M. Turner, Skew axial algebras of Monster type, arXiv:2303.09345, March 2023, 26 pages.
[22] T. Yabe, On the classification of 2-generated axial algebras of Majorana type, J. Algebra 619 (2023) 347–382.