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ABSTRACT

We prove that for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that every regular oriented graph on $n > n_0$ vertices and degree at least $(1/4 + \varepsilon)n$ has a Hamilton cycle. This establishes an approximate version of a conjecture of Jackson from 1981. We also establish a result related to a conjecture of Kühn and Osthus about the Hamiltonicity of regular directed graphs with suitable degree and connectivity conditions.

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1. Introduction

A Hamilton cycle in a (directed) graph is a (directed) cycle that visits every vertex of the (directed) graph. Hamilton cycles are one of the most intensely studied structures

in graph theory. There is a rich body of results that establish (best-possible) conditions guaranteeing existence of Hamilton cycles in (directed) graphs. Degree conditions that guarantee Hamiltonicity have been of particular interest, as well as the trade-off between degree conditions and other conditions (e.g. various types of connectivity conditions).

In this paper, we are concerned with directed graphs (or digraphs for short) and oriented graphs. Recall that a digraph can have up to two directed edges between any pair of vertices (one in each direction), while an oriented graph can have only one.

The seminal result in the area is Dirac's theorem [2], which states that every graph on $n \geq 3$ vertices with minimum degree at least $n/2$ contains a Hamilton cycle. The disjoint union of two cliques of equal size or the slightly imbalanced complete bipartite graph shows that the bound is best possible. Ghouila-Houri [3] proved the corresponding result for directed graphs, which states that every digraph on n vertices with minimum semi-degree (i.e. the smaller of the minimum in- and outdegree) at least $n/2$ contains a Hamilton cycle. The bound here is again tight by doubling the edges in the extremal examples for the graph setting. The proofs of both of these results are relatively short, while the corresponding result for oriented graphs, due to Keevash, Kühn, and Osthus [8] given below, is more difficult and uses the Regularity Lemma together with a stability method. Again the degree threshold is tight as demonstrated by examples given in [8].

Theorem 1.1. *There exists an integer n_0 such that any oriented graph G on $n \geq n_0$ vertices with minimum semi-degree $\delta^0(G) \geq \lceil (3n-4)/8 \rceil$ contains a Hamilton cycle.*

Here we consider the question of minimum degree thesholds for Hamiltonicity in *regular* (di)graphs possibly with some mild connectivity constraints. In this direction, for the undirected setting, Bollobás and Häggkvist (see [4]) independently conjectured that a t-connected regular graph with degree at least $n/(t+1)$ is Hamiltonian. That is, the threshold for Hamiltonicity is significantly reduced compared to Dirac's Theorem if we consider regular graphs (with some relatively mild connectivity conditions). Note that the connectivity conditions without regularity is not enough to guarantee Hamiltonicity due to the slightly imbalanced complete bipartite graph. Jackson [4] proved the conjecture for $t = 2$, while Jung [7] and Jackson, Li, and Zhu [6] gave an example showing the conjecture does not hold for $t \geq 4$. Finally, Kühn, Lo, Osthus, and Staden [9,10] resolved the conjecture by proving the case $t = 3$ for large regular graphs. Results in [17] which use ideas from [9,10] also show that the algorithmic Hamilton cycle problem behaves quite differently for dense regular graphs compared to dense graphs.

Jackson conjectured in 1981 that, for oriented graphs, regularity alone is enough to reduce the semi-degree threshold for Hamiltonicity from $\lceil (3n-4)/8 \rceil$ in Theorem 1.1 to $n/4$.

Conjecture 1.2 *([5]). For each $d > 2$, every d-regular oriented graph on $n \leq 4d + 1$ vertices has a Hamilton cycle.*

Note that the disjoint union of two regular tournaments shows that the degree bound above cannot be improved. Furthermore, one cannot reduce the degree bound even if the oriented graph is strongly 2-connected; see Proposition 1.6. Our main result is an approximate version of Jackson's conjecture.

Theorem 1.3. *For every $\varepsilon > 0$, there exists an integer $n_0(\varepsilon)$ such that every d-regular oriented graph on $n \geq n_0(\varepsilon)$ vertices with $d \geq (1/4 + \varepsilon)n$ is Hamiltonian.*

Recall that Jackson [4] proved the $t = 2$ case of the Bollobás–Häggkvist conjecture, namely that every 2-connected regular graph of degree at least $n/3$ has a Hamilton cycle. Kühn and Osthus gave a corresponding conjecture for digraphs.

Conjecture 1.4 *([11]). Every strongly 2-connected d-regular digraph on n vertices with $d \geq n/3$ contains a Hamilton cycle.*

We give a counterexample to this conjecture (see Proposition 1.6), but we show that a slight modification of the conjecture is true. In particular, 2-connectivity is replaced with the following slightly different condition. We call a digraph G on at least four vertices *strongly well-connected* if for any partition (A, B) of $V(G)$ with $|A|, |B| \geq 2$, there exist two vertex-disjoint edges ab and cd such that $a, d \in A$ and $b, c \in B$. Note that the property of being strongly well-connected and that of being strongly 2-connected are incomparable[4]; on the other hand being strongly well-connected is stronger than being strongly connected but weaker than being strongly 3-connected. Our second result is an approximate version of a slightly modified statement of Conjecture 1.4.

Theorem 1.5. *For every $\varepsilon > 0$, there exists an integer $n_0(\varepsilon)$ such that every strongly well-connected d-regular digraph on $n \geq n_0(\varepsilon)$ vertices with $d \geq (1/3 + \varepsilon)n$ is Hamiltonian.*

Note that Kühn and Osthus [11] give an example that shows the degree bound in Conjecture 1.4 cannot be reduced, i.e. an example of a strongly 2-connected regular digraph on n vertices and degree close to $n/3$. The same example is easily seen to be strongly well-connected, showing that we cannot take the degree to be smaller than $n/3$ in Theorem 1.5.

Our methods are based on the robust expanders technique of Kühn and Osthus which have been used to resolve a number of conjectures (see [12,13]). Any directed dense graph that is a robust expander automatically contains a Hamilton cycle. An important part of this paper is to gain an understanding of dense directed graphs that are not robust expanders. In particular, we are able to construct vertex partitions of such digraphs with useful expansion properties. Although we do not show it directly, such partitions almost immediately allow us to construct very long cycles in the required settings (that is cycles

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[4] A directed cycle (on at least 4 vertices) is strongly well-connected but not strongly 2-connected; see Proposition 1.6 for the converse example.
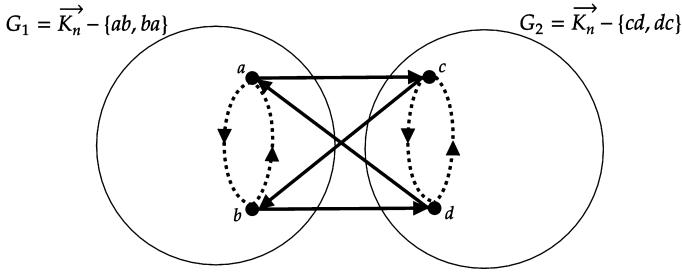
Fig. 1. A strongly 2-connected $(n-1)$-regular digraph G on $2n$ vertices.

that pass through all but a small proportion of the vertices). The remainder of the paper is devoted to giving delicate balancing arguments to obtain full Hamilton cycles.

The paper is organised as follows. In the next subsection, we give the counterexample to Conjecture 1.4. In Section 2 we give notation, preliminaries and a sketch proof. In Section 3 we develop the necessary language of partitions and establish some of their basic properties. Section 4 is devoted mainly to giving the balancing arguments that will allow us to construct full Hamilton cycles. Section 5 shows how to combine earlier results to show that dense directed and oriented graphs with certain vertex partitions contain Hamilton cycles. In Section 6 we prove Theorems 1.5 and 1.3. We pose some open problems in Section 7.

1.1. Counterexample to Conjecture 1.4

Proposition 1.6. *For $n \geq 3$, there exists a strongly 2-connected $(n-1)$-regular digraph on $2n$ vertices with no Hamilton cycle. For $n \geq 3$, there exists a strongly 2-connected $(n-1)$-regular oriented graph on $4n+2$ vertices with no Hamilton cycle.*

Proof. Let G' be the digraph that is the disjoint union of two complete digraphs G_1 and G_2 each of size n. Let $a, b \in V(G_1)$ and $c, d \in V(G_2)$. Let G be the digraph obtained from G' by deleting the edges ab, ba, cd, and dc, and adding the edges ac, cb, bd, da (see Fig. 1). It is clear that G is a strongly 2-connected $(n-1)$-regular digraph on $2n$ vertices.

It is easy to see that G has no Hamilton cycle. Indeed, any Hamilton cycle H of G must use at least one edge inside one of the cliques (since $n \geq 3$). Let P be a maximal path of H inside one of the cliques (say G_1) with at least one edge. Let e and e' be the edges of H that extend P into G_2. Then e and e' must be vertex-disjoint edges that cross between G_1 and G_2 in opposite directions. But G does not have such a pair of edges.

The oriented graph is constructed similarly. It is easy to construct a regular tournament of order $2n+1$ that contains two cycles that together span the tournament and which have exactly two vertices in common. Indeed, we start with the two directed cycles with common vertices say a and b. The (undirected) complement is Eulerian, that is, all

vertices have even degree, and so we orient these edges using an Euler tour. This gives the desired tournament.

Let G' be the disjoint union of two such regular tournaments G_1 and G_2 each of order $2n+1$. Let C_1 and C_1' be the two directed cycles in G_1 such that $V(C_1) \cup V(C_1') = V(G_1)$ and $V(C_1) \cap V(C_1') = \{a, b\}$. Similarly, let C_2 and C_2' be two directed cycles in G_2 such that $V(C_2) \cup V(C_2') = V(G_2)$ and $V(C_2) \cap V(C_2') = \{c, d\}$. Let G be obtained from G' by deleting the edges of $C_1 \cup C_1' \cup C_2 \cup C_2'$, and adding the edges ac, cb, bd, da. It is easy to check that G is a strongly 2-connected, $(n-1)$-regular, oriented graph on $4n+2$ vertices. Note that G is not Hamiltonian by a similar argument as above. $\quad\square$

2. Notation and preliminaries

Throughout the paper, we use standard graph theory notation and terminology. For a digraph G, we denote its vertex set by $V(G)$ and its edge set $E(G)$. For $a, b \in V(G)$, we write ab for the directed edge from a to b. We sometimes write $|G|$ for the number of vertices in G and $e(G)$ for the number of edges in G. We write $H \subseteq G$ to mean H is a subdigraph of G, i.e. $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We sometimes think of $F \subseteq E(G)$ as a subdigraph of G with vertex set consisting of those vertices incident to edges in F and edge set F. For $S \subseteq V(G)$, we write $G[S]$ for the subdigraph of G induced by S and $G - S$ for the digraph $G[V(G) \setminus S]$. For $A, B \subseteq V(G)$ not necessarily disjoint, we define $E_G(A, B) := \{ab \in E(G) : a \in A, \ b \in B\}$ and we write $G[A, B]$ for the graph with vertex set $A \cup B$ and edge set $E_G(A, B)$. We write $e_G(A, B) := |E_G(A, B)|$. We often drop subscripts if these are clear from context. For two digraphs H_1 and H_2, the union $H_1 \cup H_2$ is the digraph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. We say that an undirected graph G is bipartite with bipartition (A, B) if $V(G) = A \cup B$ and $E(G) \subseteq \{ab : a \in A, \ b \in B\}$.

For a digraph G and $v \in V(G)$, we denote the set of outneighbours and inneighbours of v by $N_G^+(v)$ and $N_G^-(v)$ respectively, and we write $d_G^+(v) = |N_G^+(v)|$ and $d_G^-(v) = |N_G^-(v)|$ for the out- and indegree of v respectively. For $S \subseteq V(G)$ we write $d_S^-(v) := |N_G^-(v) \cap S|$ and $d_S^+(v) := |N_G^+(v) \cap S|$. We write $\delta^+(G)$ and $\delta^-(G)$ respectively for the minimum out- and indegree of G, and $\delta^0(G) := \min\{\delta^+(G), \delta^-(G)\}$ for the minimum semi-degree. Similarly, the maximum semi-degree $\Delta^0(G)$ of G is defined by $\Delta^0(G) := \max\{\Delta^+(G), \Delta^-(G)\}$ where $\Delta^+(G)$ and $\Delta^-(G)$ denote the maximum out- and maximum indegree of G respectively. A digraph is called d-regular if each vertex has exactly d outneighbours and d inneighbours. For undirected graphs G, we write $\Delta(G)$ and $\delta(G)$ respectively for the maximum degree and the minimum degree. A graph is called d-regular if each vertex has exactly d neighbours.

A directed path Q in a digraph G is a subdigraph of G where $V(Q) = \{v_1, \ldots, v_k\}$ for some $k \in \mathbb{N}$ and where $E(Q) = \{v_1 v_2, v_2 v_3, \ldots, v_{k-1} v_k\}$. A directed cycle in G is exactly the same except that it also includes the edge $v_k v_1$. A set of vertex-disjoint directed paths $\mathcal{Q} = \{Q_1, Q_2, \ldots\}$ in G is called a *path system* in G. We interchangeably think of $\mathcal{Q}$ as a set of vertex-disjoint directed paths in G and as a subdigraph of G with vertex set

$V(\mathcal{Q}) = \cup_i V(Q_i)$ and edge set $E(\mathcal{Q}) = \cup_i E(Q_i)$. We sometimes call this subgraph the graph induced by $\mathcal{Q}$. A matching M in a digraph (or undirected graph) G is a set of edges $M \subseteq E(G)$ such that every vertex of G is incident to at most one edge in M. We say that a matching M *covers* $S \subseteq V(G)$ if every vertex in S is incident to some edge in M.

For two sets A and B, the *symmetric difference* of A and B is the set $A \triangle B := (A \setminus B) \cup (B \setminus A)$. For $k \in \mathbb{N}$, we sometimes denote the set $\{1, 2, \ldots, k\}$ by $[k]$. For $x, y \in (0, 1]$, we often use the notation $x \ll y$ to mean that x is sufficiently small as a function of y i.e. $x \leq f(y)$ for some implicitly given non-decreasing function $f : (0, 1] \to (0, 1]$.

2.1. Tools

We will require Vizing's theorem for multigraphs in the proof of Lemma 4.1. Let H be an (undirected) multigraph (without loops). The multiplicity $\mu(H)$ of H is maximum number of edges between two vertices of H, and, as usual, $\Delta(H)$ is the maximum degree of H. A proper k-edge-colouring of H is an assignment of k colours to the edges of H such that incident edges receive different colours.

Theorem 2.1 *([18]; see e.g. [1]). Any multigraph H has a proper k-edge colouring with $k = \Delta(H) + \mu(H)$ colours. In particular, by taking the largest colour class, there is a matching in H of size at least $e(H)/(\Delta(H) + \mu(H))$.*

In Lemma 4.2, we will require a Chernoff inequality for bounding the tail probabilities of binomial random variables. For a random variable X, write $\mathbb{E}[X]$ for the expectation of X. We write $X \sim \mathrm{Bin}(n, p)$ to mean that X is distributed as a binomial random variable with parameters n and p, that is a random variable that counts the number of heads in n independent coin flips where the probability of heads is p. In that case we have $\mathbb{E}[X] = np$ and the following bound.

Theorem 2.2 *(see [16]). Suppose $X_1, X_2, \ldots, X_n$ are independent random variables taking values in $\{0, 1\}$ and $X = X_1 + \cdots + X_n$. Then, for all $0 \leq \delta \leq 1$, we have*

$$\mathbb{P}(X \leq (1 - \delta)\mathbb{E}(X)) \leq \exp\left(-\delta^2 \mathbb{E}(X)/2\right).$$

In particular, this holds for $X \sim \mathrm{Bin}(n, p)$.

2.2. Robust expanders

In this subsection we define robust expanders and discuss some of their useful properties.

Definition 2.3. Fix a digraph G on n vertices and parameters $0 < \nu < \tau < 1$. For $S \subseteq V(G)$, the *robust ν-outneighbourhood of S* is the set $RN_\nu^+(S) := \{v \in V(G) :$

$|N_G^-(v) \cap S| \geq \nu n\}$. We say G is a *robust (ν, τ)-outexpander* if $|RN_\nu^+(S)| \geq |S| + \nu n$ for all subsets $S \subseteq V(G)$ satisfying $\tau n \leq |S| \leq (1 - \tau)n$.

If the constant ν used is clear from context, we write $\mathrm{RN}^+(S)$. The notion of robust expansion has been key to proving numerous conjectures about Hamilton cycles. One of the starting points is the following seminal result which states that robust expanders with certain minimum degree condition are Hamiltonian.

Theorem 2.4 *([14]; see also [15]). Let $1/n \ll \nu \leq \tau \ll \gamma < 1$. If G is an n-vertex digraph with $\delta^0(G) \geq \gamma n$ such that G is a robust (ν, τ)-outexpander, then G contains a Hamilton cycle.*

The following straightforward lemma shows that robust expansion is a "robust" property, i.e. if G is a robust (ν, τ)-outexpander, then adding or deleting a small number of vertices results in another robust outexpander with slightly worse parameters.

Lemma 2.5 *([9]). Let $0 < \nu \ll \tau \ll 1$. Suppose that G is a digraph and $U, U' \subseteq V(G)$ are such that $G[U]$ is a robust (ν, τ)-outexpander and $|U \triangle U'| \leq \nu|U|/2$. Then, $G[U']$ is a robust $(\nu/2, 2\tau)$-outexpander.*

By taking $(U, U') = (V(G) - S, V(G))$, Lemma 2.5 has the following corollary.

Corollary 2.6. *Let $1/n \ll \nu \ll \tau \ll 1$. If G is an n-vertex digraph and $S \subset V(G)$ such that $|S| \leq \nu|G|/2$ and $G - S$ is a robust (ν, τ)-outexpander then G is a robust $(\nu/2, 2\tau)$-outexpander.*

The next lemma shows that any digraph G with minimum semi-degree slightly higher than $|G|/2$ is a robust outexpander.

Lemma 2.7 *([12]). Let $0 < \nu \leq \tau \leq \varepsilon < 1$ be constants such that $\varepsilon \geq 2\nu/\tau$. Let G be a digraph on n vertices with $\delta^0(G) \geq (1/2 + \varepsilon)n$. Then, G is a robust (ν, τ)-outexpander.*

In fact we can relax the degree condition in Lemma 2.7 and allow a small number of vertices to violate the minimum degree condition.

Corollary 2.8. *Let $1/n < \nu, \rho \ll \tau \ll \varepsilon \ll \alpha < 1$ be constants. If G is an n-vertex digraph such that $d^+(v), d^-(v) \geq (1/2 + \varepsilon)n$ for all but at most ρn vertices $v \in V(G)$, then G is a robust (ν, τ)-outexpander. In particular, if additionally $\delta^0(G) \geq \alpha n$, then G contains a Hamilton cycle.*

Proof. Fix ν' and τ' such that $\nu, \rho \ll \nu' \ll \tau' \ll \tau$. Let W be the set of vertices v in G such that $\min\{d^+(v), d^-(v)\} < (1/2 + \varepsilon)n$. Then, observe that $G' = G - W$ satisfies

$$d_{G'}^+(v), d_{G'}^-(v) \geq (1/2 + \varepsilon - \rho)n \geq (1/2 + \varepsilon - \rho)|G'|$$

for all $v \in V(G')$. By our choice of parameters, we can conclude that G' is a robust (ν', τ')-outexpander by Lemma 2.7 since $\tau' \leq \varepsilon - \rho$ and $2\nu'/\tau' \leq \varepsilon - \rho$. Moreover, we have $|W| = \rho n \leq \nu' n/2$. Therefore, G is a robust (ν, τ)-outexpander by Corollary 2.6, and the result follows by Theorem 2.4. □

2.3. Sketch proof

Note that the sketch proof we give below only makes reference to Definition 2.3, Theorem 2.4, and Lemma 2.7. We will sketch the proof of Theorem 1.5 and then explain how these ideas are generalised and refined to prove Theorem 1.3.

Let $G = (V, E)$ be an n-vertex, d-regular digraph with $d \geq (1/3+\varepsilon)n$. If G is a robust (ν, τ)-outexpander (for suitable parameters ν and τ), then by Theorem 2.4, we know G has a Hamilton cycle. So assume G is not a robust (ν, τ)-outexpander. We describe a useful vertex partition of G.

Partitioning non-robust expanders - Since G is not a robust (ν, τ)-outexpander we know by Definition 2.3 that there exists $S \subseteq V(G)$ such that $\tau n \leq |S| \leq (1-\tau)n$ and $|\mathrm{RN}_\nu^+(S)| \leq |S|+\nu n$. This immediately gives us a partition of $V(G)$ into four parts given by

$$V_{11} = S \cap \mathrm{RN}^+(S), \quad V_{12} = S \setminus \mathrm{RN}^+(S),$$
$$V_{21} = \mathrm{RN}^+(S) \setminus S, \quad V_{22} = V \setminus (S \cup \mathrm{RN}^+(S)).$$

We see that most outedges from vertices in S go to $\mathrm{RN}^+(S)$ by the definition of $\mathrm{RN}^+(S)$. Moreover, S and $\mathrm{RN}^+(S)$ must be of similar size; indeed we already know $\mathrm{RN}^+(S)$ is not significantly bigger than S, and it cannot be significantly smaller because otherwise the degrees in $\mathrm{RN}^+(S)$ would be larger than degrees in S violating that G is regular. Also most outedges of vertices in $V \setminus S$ go to $V \setminus \mathrm{RN}^+(S)$ because if many of these edges went to $\mathrm{RN}^+(S)$, the degrees in $\mathrm{RN}^+(S)$ would again be too large violating that G is regular. All of this is straightforward to show and captured in Lemma 3.6. The structure we obtain is depicted in Fig. 2. To summarise, we have that

(a) $|S| \approx |\mathrm{RN}^+(S)|$ so $|V_{12}| \approx |V_{21}|$,
(b) most edges of G are from S to $\mathrm{RN}^+(S)$ and from $V \setminus S$ to $V \setminus \mathrm{RN}^+(S)$. We call these the good edges of G, and
(c) (b) implies that we must have $|S|, |V \setminus S| \gtrsim d$ so that in particular $n/3 \lesssim |S|, |V \setminus S| \lesssim 2n/3$

Next we describe the strategy to construct a Hamilton cycle in G using this partition.

Constructing the Hamilton cycle for balanced partitions - We first describe how to construct the Hamilton cycle in the special case $|V_{12}| = |V_{21}| > 0$. In that case, let $V_{12} = \{x_1, \ldots, x_k\}$ and $V_{21} = \{y_1, \ldots, y_k\}$. Consider the two edge-disjoint subgraphs G_1 and G_2 of G given by (see Fig. 3)
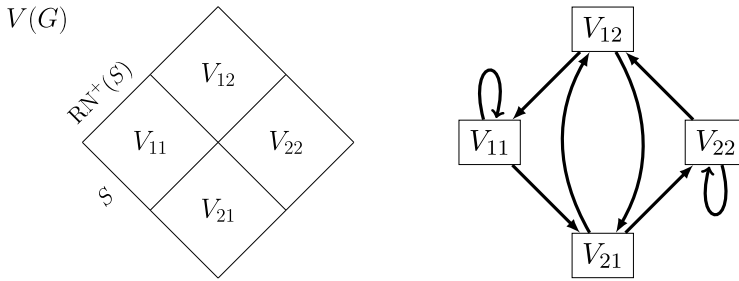
Fig. 2. The 4-partition of $V(G)$ with $|V_{12}| \approx |V_{21}|$, and directions of the good edges.
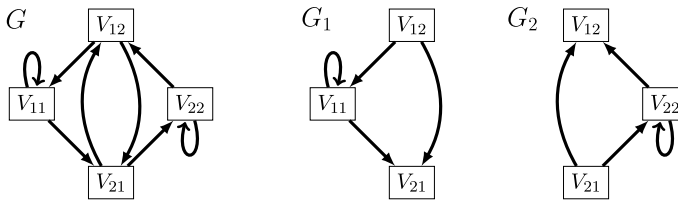
Fig. 3. The edge-disjoint subgraphs G_1 and G_2 of G.

$$G_1 = \left(S \cup \mathrm{RN}^+(S),\ E_G(S, \mathrm{RN}^+(S))\right)$$
$$= (V_{11} \cup V_{12} \cup V_{21},\ E(V_{12}, V_{11}) \cup E(V_{11}, V_{11}) \cup E(V_{11}, V_{21}) \cup E(V_{12}, V_{21})),$$

and

$$G_2 = \left((V \setminus S) \cup (V \setminus \mathrm{RN}^+(S)),\ E_G(V \setminus S, V \setminus \mathrm{RN}^+(S))\right)$$
$$= (V_{22} \cup V_{12} \cup V_{21},\ E(V_{21}, V_{22}) \cup E(V_{22}, V_{22}) \cup E(V_{22}, V_{12}) \cup E(V_{21}, V_{12})).$$

Suppose we can find

(i) vertex-disjoint paths $Q_1^1, \ldots, Q_k^1$ in G_1 that together span $V(G_1)$ and where Q_i^1 is from x_i to $y_{\sigma(i)}$ for some permutation σ on $[k]$,

(ii) vertex-disjoint paths $Q_1^2, \ldots, Q_k^2$ in G_2 that together span $V(G_2)$ and where Q_i^2 is from y_i to $x_{\pi(i)}$ for some permutation π on $[k]$,

(iii) and where the permutation $\pi\sigma$ is a cyclic permutation.

Then it is easy to see that the union of these paths forms a Hamilton cycle. We find these paths as follows.

Consider G_1 first. We construct the graph J_1 from G_1 by identifying x_i with y_i for every $i \in [k]$ and keeping all edges (except any self loops). The vertex which replaces x_i and y_i is called i. From the structure of G_1, it is not hard to see that most vertices in J_1 have degree roughly $d = (1/3 + \varepsilon)n$, while $|J_1| = |S| \lesssim 2n/3$ by (c). So most vertices in
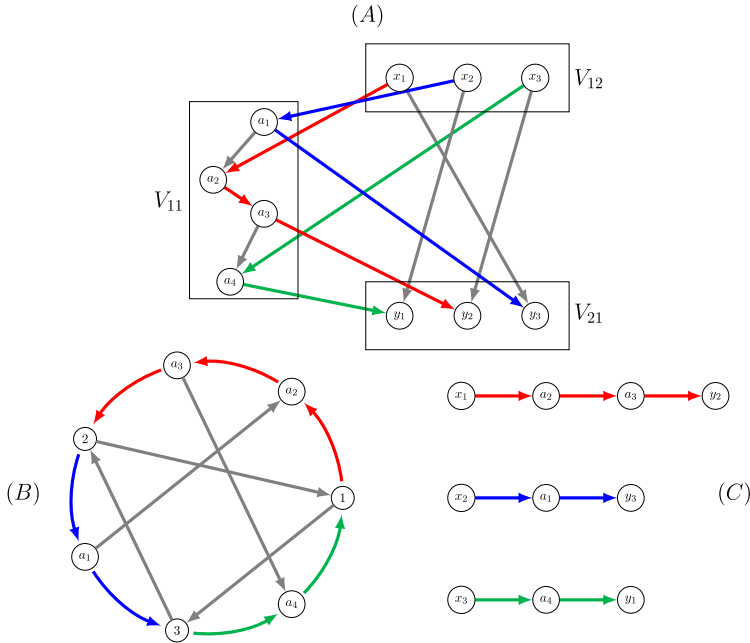
Fig. 4. An example illustration of (A) G_1, (B) the corresponding graph J_1 with a Hamilton cycle H_1, and (C) the vertex-disjoint paths $Q_1^1, \ldots, Q_k^1$ spanning G_1 (with $k = 3$ in this case) corresponding to H_1. In this case $\sigma = (231)$, i.e. the cyclic permutation that sends 1 to 2, 2 to 3, and 3 to 1.

J_1 have in- and outdegree at least $(1/2 + \varepsilon/2)|J_1|$, which implies J_1 is a robust expander by Lemma 2.7.[5] Therefore J_1 has a Hamilton cycle H_1 by Theorem 2.4.

Let σ be the permutation on $[k]$ where $\sigma(i)$ is the vertex in $[k]$ after i that is visited by H_1. Therefore H_1 is the union of paths $R_1, \ldots, R_k$ where R_i is from i to $\sigma(i)$, which corresponds in G_1 to the path Q_i^1 from x_i to $y_{\sigma(i)}$; these paths can easily be seen to satisfy (i) (see Fig. 4). Next, we obtain J_2 from G_2 by identifying the vertex x_i with $y_{\sigma(i)}$, and labelling the resulting vertex i, for every $i \in [k]$ similarly as for J_1. Again, we find that J_2 is a robust expander and so has a Hamilton cycle H_2. Let π be the permutation on $[k]$ such that $\pi(i)$ is the next vertex in $[k]$ after i visited by H_2. Using the same argument as before, we obtain paths $Q_1^2, \ldots, Q_k^2$ satisfying (ii). By our choice of identification in J_2, and since H_2 is a Hamilton cycle, it is easy to see that π and σ satisfy (iii).

Constructing the Hamilton cycle for unbalanced partitions - We have seen how to find the Hamilton cycle when $|V_{12}| = |V_{21}|$. If instead we only have (by (a)) that $|V_{12}| \approx |V_{21}|$, then we will find vertex-disjoint paths $S_1, \ldots, S_\ell$ that use only bad edges (and only a relatively small number of bad edges) such that "contracting" these paths results in a slightly modified graph G' with a slightly modified vertex partition $V_{11}', V_{12}', V_{21}', V_{22}'$, which has essentially the same properties as before but also that $|V_{12}'| = |V_{21}'|$. Here G' is

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[5] Any enumeration of the vertices in V_{12} and V_{21} would lead to J_1 being a robust expander.

not regular, but almost regular; this however is enough for us. So we can find a Hamilton cycle in G' using the previous argument, and "uncontracting" the paths $S_1, \ldots, S_\ell$ gives a Hamilton cycle in G.[6]

The case of regular oriented graphs - For Theorem 1.3, i.e. when G is an n-vertex regular oriented graph with degree $d \geq (1/4 + \varepsilon)n$, we start by applying the same argument as before. Recall that we construct digraphs J_1 and J_2 and wish to find Hamilton cycles in these digraphs. However, whereas before, we could guarantee that both J_1 and J_2 would be robust expanders, this time we find that (at most) one of them, say J_2 might not be. This is because G and J_i have lower degree, and so we cannot necessarily apply Lemma 2.7. It is not too hard to see that the J_i are almost regular and so we can iterate our partition argument on J_2. In particular we can partition $V(J_2)$ into four parts $Z_{11}, Z_{12}, Z_{21}, Z_{22}$ that satisfy slightly modified forms of (a) and (b). Again if $|Z_{12}| = |Z_{21}|$, then we can create digraphs K_1 and K_2 such that Hamilton cycles in K_1 and K_2 lift to a Hamilton cycle in J_2 (just as Hamilton cycles in J_1 and J_2 lift to a Hamilton cycle in G). This time the increase in density is enough to guarantee that both K_1 and K_2 are robust expanders, which gives the desired Hamilton cycle by Theorem 2.4. If $|Z_{12}| \neq |Z_{21}|$ then, as before, we need to construct paths whose contraction results in a modified graph with a modified partition that is balanced. In fact, we need to be able to find and contract paths in such a way that we simultaneously have $|V'_{12}| = |V'_{21}|$ and $|Z'_{12}| = |Z'_{21}|$. For this purpose, and generally for a cleaner and more transparent argument, rather than working with two iterations of the 4-partition described earlier, we work equivalently with a 9-partition of $V(G)$. The required paths are constructed in Lemma 4.6.

3. Partitions of regular digraphs and oriented graphs

We have seen that (essentially) any dense digraph that is a robust expander is Hamiltonian. If the digraph is not a robust expander, then we will see (Lemma 3.6) that the witness sets to this non-expansion naturally forms a partition of the vertices into 4 parts. Throughout the paper we will be working with such partitions and their iterations. In this section, we introduce the language of partitions and establish some of their basic properties.

Definition 3.1. For a given digraph G and $k \in \mathbb{N}$, a partition $\mathcal{P}_k = \{V_{ij} : i, j \in [k]\}$ of $V(G)$ is called a k^2-partition of $V(G)$ (we allow the sets V_{ij} to be empty). The set of *good edges with respect to* $\mathcal{P}_k$ is defined as

$$\mathcal{G}_k(\mathcal{P}_k, G) := \bigcup_i E(V_{i*}, V_{*i}),$$

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[6] For Theorem 1.5, these paths are constructed directly in the proof of the theorem in Section 6, but in the more complicated case of Theorem 1.3, they are constructed in Lemma 4.6.

where $V_{i*} := \bigcup_j V_{ij}$ and $V_{*j} := \bigcup_i V_{ij}$. The set of *bad edges with respect to* $\mathcal{P}_k$ is defined as

$$\mathcal{B}_k(\mathcal{P}_k, G) := E(G) - \mathcal{G}_k(\mathcal{P}_k, G) = \bigcup_{i \neq j} E(V_{i*}, V_{*j}).$$

We write $G_{ij} := G[V_{i*}, V_{*j}]$.

Note that while we define k^2-partitions and prove properties for general k, in fact we only require the cases $k = 2, 3$. For regular digraphs, we have a useful equality relating the sizes of different parts in a k^2-partition and the number of bad edges.

Proposition 3.2. *Let G be a d-regular digraph, $k \in \mathbb{N}$, and $\mathcal{P}_k = \{V_{ij} : i, j \in [k]\}$ be a k^2-partition of $V(G)$. Then, for all $i \in [k]$, we have*

$$d(|V_{i*}| - |V_{*i}|) = \sum_{j \neq i} \left(e(G_{ij}) - e(G_{ji}) \right).$$

Proof. By considering outneighbours of the vertices in V_{i*}, we can write

$$d|V_{i*}| = e(V_{i*}, V_{*i}) + \sum_{j \neq i} e(V_{i*}, V_{*j}).$$

Similarly, by considering the inneighbours of the vertices in V_{*i}, we have

$$d|V_{*i}| = e(V_{i*}, V_{*i}) + \sum_{j \neq i} e(V_{j*}, V_{*i}).$$

By subtracting the second equality from the first one, the result follows. $\quad\square$

If the number of bad edges is small compared to $E(G)$, then Proposition 3.2 implies that V_{i*} and V_{*i} are similar in size.

Corollary 3.3. *Let $k \in \mathbb{N}$ and γ be a positive constant. Let G be a d-regular digraph on n vertices, and $\mathcal{P}_k = \{V_{ij} : i, j \in [k]\}$ be a k^2-partition of $V(G)$. If $|\mathcal{B}_k(\mathcal{P}, G)| \leq \gamma n^2$, then we have $\big||V_{i*}| - |V_{*i}|\big| \leq \gamma n^2/d$ for all $i \in [k]$.*

Proof. Fix $i \in [k]$. We have

$$\left| \sum_{j \neq i} \left(e(V_{i*}, V_{*j}) - e(V_{j*}, V_{*i}) \right) \right| \leq \sum_{j \neq i} \left(e(V_{i*}, V_{*j}) + e(V_{j*}, V_{*i}) \right) \leq |\mathcal{B}_k(\mathcal{P}, G)| \leq \gamma n^2.$$

Hence, by Proposition 3.2, we know $d\big||V_{i*}| - |V_{*i}|\big| \leq \gamma n^2$, so the result follows. $\quad\square$

We will be especially interested in partitions with a small number of bad edges and where certain parts are not too small.

Definition 3.4. For a given digraph G on n vertices and positive constants γ, τ, and $k \in \mathbb{N}$, we say a k^2-partition $\mathcal{P}_k = \{V_{ij} : i, j \in [k]\}$ of $V(G)$ is a (k^2, τ, γ)-*partition* if the following hold:

$$|\mathcal{B}_k(\mathcal{P}_k, G)| \le \gamma n^2 \text{ and } |V_{i*}|, |V_{*j}| \ge \tau n \text{ for all } i, j \in [k].$$

Remark 3.5. In general, the constants γ and τ are taken to satisfy $1/n \ll \gamma \ll \tau \ll 1$. When working with regular graphs, we sometimes implicitly take the conclusion of Corollary 3.3 as a property of a (k^2, τ, γ)-partition.

Next, we show that every almost regular digraph which is dense and not a robust (ν, τ)-outexpander admits a $(4, \tau/2, 4\nu)$-partition.

Lemma 3.6. *Let* $1/n \ll \nu \ll \tau \ll \alpha \ll 1$, *and* G *be a digraph on n vertices such that* $e(G) \ge (\alpha - \nu)n^2$ *and* $\Delta^0(G) \le \alpha n$. *If* G *is not a robust* (ν, τ)-*outexpander, then* G *admits a* $(4, \tau, 4\nu)$-*partition.*

Proof. Assume G is not a robust (ν, τ)-outexpander. Then we can find a subset $S \subseteq V(G)$ such that $\tau n \le |S| \le (1-\tau)n$ and $|\text{RN}_\nu^+(S)| < |S| + \nu n$. Let us define $V_{11} = S \cap \text{RN}_\nu^+(S)$, $V_{12} = S - \text{RN}_\nu^+(S)$, $V_{21} = \text{RN}_\nu^+(S) - S$, and $V_{22} = V(G) - (S \cup \text{RN}_\nu^+(S))$. Therefore $V_{1*} = S$ and $V_{*1} = \text{RN}_\nu^+(S)$. Note that $\mathcal{P}_2 = \{V_{ij} : i, j \in [2]\}$ is a 4-partition of $V(G)$. Moreover, since $\tau n \le |S| \le (1-\tau)n$, we have $|V_{1*}|, |V_{2*}| \ge \tau n$.

We first show that $|\mathcal{B}_2(\mathcal{P}_2, G)| \le 4\nu n^2$. By the definition of $\text{RN}_\nu^+(S)$, we know that every vertex in V_{*2} has fewer than νn inneighbours from V_{1*}. Thus, we have

$$e(V_{1*}, V_{*2}) \le \nu n^2 \tag{3.1}$$

and

$$\begin{aligned}
e(V_{1*}, V_{*1}) &= e(V_{1*}, V(G)) - e(V_{1*}, V_{*2}) \ge e(V_{1*}, V(G)) - \nu n^2 \\
&= e(G) - e(V_{2*}, V(G)) - \nu n^2 \\
&\ge (\alpha - \nu)n^2 - \alpha n |V_{2*}| - \nu n^2 = \alpha n |V_{1*}| - 2\nu n^2. \tag{3.2}
\end{aligned}$$

Since $|V_{*1}| = |\text{RN}_\nu^+(S)| < |S| + \nu n = |V_{1*}| + \nu n$, we have

$$e(V(G), V_{*1}) \le \alpha n |V_{*1}| < (|V_{1*}| + \nu n)\alpha n \le \alpha n |V_{1*}| + \nu n^2.$$

Thus, together with (3.2), we have

$$e(V_{2*}, V_{*1}) = e(V(G), V_{*1}) - e(V_{1*}, V_{*1}) \le 3\nu n^2.$$

Therefore (3.1) implies that $|\mathcal{B}_2(\mathcal{P}_2, G)| = e(V_{1*}, V_{*2}) + e(V_{2*}, V_{*1}) \le 4\nu n^2$.

We now bound $|V_{*1}|$ and $|V_{*2}|$ from below. Let T be the set of vertices with outdegree at most $(\alpha - \sqrt{\nu})n$. Then as $\Delta^0(G) \le \alpha n$

$$(\alpha - \nu)n^2 \le e(G) \le (\alpha - \sqrt{\nu})n|T| + \alpha n(n - |T|) = \alpha n^2 - \sqrt{\nu}n|T|,$$

which implies that $|T| \le \sqrt{\nu}n$. For $\{i, j\} = [2]$, recall that $|V_{i*}| \ge \tau n$ and so we have

$$\begin{aligned}
(|V_{*i}| + 4\nu/\tau)|V_{i*}| \ge |V_{i*}||V_{*i}| + 4\nu n^2 &\ge e(V_{i*}, V_{*i}) + |\mathcal{B}_2(\mathcal{P}_2, G)| \\
&\ge e(V_{i*}, V_{*i}) + e(V_{i*}, V_{*j}) \\
&= e(V_{i*}, V(G)) \ge (\alpha - \sqrt{\nu})n|V_{i*} \setminus T| \ge (\alpha - \sqrt{\nu})n|V_{i*}|/2
\end{aligned}$$

As a result, we obtain $|V_{*i}| \ge (\alpha - \sqrt{\nu})n/2 - 4\nu/\tau \ge \tau n$, so the result follows. □

One can construct an (ℓ^2, τ, γ)-partition of G from a (k^2, τ, γ)-partition of G for $\ell \le k$.

Proposition 3.7. *Let G be a digraph with a (k^2, τ, γ)-partition $\mathcal{P}_k = \{V_{ij} : i, j \in [k]\}$. Let $\{I_1, I_2, \ldots, I_\ell\}$ be a partition of $[k]$ with $I_t \ne \emptyset$ for all $t \in [\ell]$. For $i', j' \in [\ell]$, let $W_{i'j'} = \bigcup_{i \in I_{i'}, j \in I_{j'}} V_{ij}$. Then, $\mathcal{P}_\ell = \{W_{i'j'} : i', j' \in [\ell]\}$ is an (ℓ^2, τ, γ)-partition of G.*

Proof. Let $n = |G|$. For $i' \in [\ell]$, note that

$$W_{i'*} = \bigcup_{j' \in [\ell]} W_{i'j'} = \bigcup_{i \in I_{i'}, j \in [k]} V_{ij} = \bigcup_{i \in I_{i'}} V_{i*}$$

and so $|W_{i'*}| \ge \tau n$. Similarly, we have $|W_{*j'}| \ge \tau n$ for all $j' \in [\ell]$. Moreover, note that

$$\begin{aligned}
|\mathcal{B}_\ell(\mathcal{P}_\ell, G)| = \sum_{i',j' \in [\ell]:\, i' \ne j'} e(W_{i'*}, W_{*j'}) &= \sum_{i',j' \in [\ell]:\, i' \ne j'} \sum_{i \in I_{i'}, j \in I_{j'}} e(V_{i*}, V_{*j}) \\
&\le \sum_{i,j \in [k]:\, i \ne j} e(V_{i*}, V_{*j}) = |\mathcal{B}_k(\mathcal{P}_k, G)|,
\end{aligned}$$

so the result follows. □

Next, we show that if a regular digraph is dense and admits a (k^2, τ, γ)-partition, then certain unions of parts have size at least roughly the degree of the digraph.

Proposition 3.8. *Let $1/n \ll \gamma \ll \tau \ll \varepsilon \ll \alpha \ll 1$, $k \in \mathbb{N}$, and G be a d-regular digraph on n vertices where $d \ge (\alpha + \varepsilon)n$. Suppose that G has a (k^2, τ, γ)-partition $\mathcal{P}_k = \{V_{ij} : i, j \in [k]\}$. Then we have $|V_{i*}|, |V_{*i}| \ge d - \varepsilon n/2$ for all $i \in [k]$. In particular, $\mathcal{P}_k$ is a $(k^2, \alpha + \varepsilon/2, \gamma)$-partition for G.*

Proof. Let $i \in [k]$. By looking at the outneighbours of V_{i*}, we have

$$|V_{*i}| \geq \frac{e(V_{i*}, V_{*i})}{|V_{i*}|} = \frac{d|V_{i*}| - \sum_{j \neq i, \, j \in [k]} e(V_{i*}, V_{*j})}{|V_{i*}|}$$

$$\geq d - \frac{|\mathcal{B}_k(\mathcal{P}_k, G)|}{|V_{i*}|} \geq d - \frac{\gamma n^2}{|V_{i*}|} \geq d - \frac{\gamma n}{\tau} \geq d - \varepsilon n/2$$

since $|\mathcal{B}_k(\mathcal{P}_k, G)| \leq \gamma n^2$, $|V_{i*}| \geq \tau n$ and $\gamma \ll \tau \ll \varepsilon$. Similarly, we have $|V_{i*}| \geq d - \varepsilon n/2$. $\square$

If a (k^2, τ, γ)-partition has the minimum possible number of bad edges among all (k^2, τ, γ)-partitions of a digraph, then we give it a special name.

Definition 3.9. Let $1/n \ll \gamma \ll \tau \ll 1$, $k \in \mathbb{N}$, and G be a digraph on n vertices. A (k^2, τ, γ)-partition $\mathcal{P}_k = \{V_{ij} : i, j \in [k]\}$ of $V(G)$ is called an *extremal (k^2, τ, γ)-partition* if $\mathcal{B}_k(\mathcal{P}_k, G) \leq \mathcal{B}_k(\mathcal{P}'_k, G)$ for all (k^2, τ, γ)-partitions $\mathcal{P}'_k$ of $V(G)$.

We establish some useful degree conditions for extremal (k^2, τ, γ)-partitions of dense regular digraphs.

Proposition 3.10. *Let $1/n \ll \gamma \ll \tau \ll \alpha \ll 1$, $k \in \mathbb{N}$, and G be a d-regular digraph on n vertices with $d \geq \alpha n$ and an extremal (k^2, τ, γ)-partition $\mathcal{P}_k = \{V_{ij} : i, j \in [k]\}$. Then, for all $i, j \in [k]$ and $w \in V_{ij}$, we have $d^+_{V_{*i'}}(w) \leq d^+_{V_{*i}}(w)$ and $d^-_{V_{j'*}}(w) \leq d^-_{V_{j*}}(w)$ for all $i', j' \in [k]$. In particular, we have $d^+_{V_{j'*}}(w), d^-_{V_{*i'}}(w) \leq d/2$ for all $i' \neq i$ and $j' \neq j$, and $d^+_{\mathcal{G}_k(\mathcal{P}_k, G)}(v), d^-_{\mathcal{G}_k(\mathcal{P}_k, G)}(v) \geq d/k$ for all $v \in V(G)$.*

Proof. Let ε be a constant such that $\tau \ll \varepsilon \ll \alpha$. Let $\alpha' = \alpha - \varepsilon$. Suppose the contrary and without loss of generality that there exists $w \in V_{ij}$ and $a \in [k]$ such that $d^+_{V_{*a}}(w) > d^+_{V_{*i}}(w)$. Let $V'_{ij} = V_{ij} \backslash \{w\}$, $V'_{aj} = V_{aj} \cup \{w\}$, and $V'_{i'j'} = V_{i'j'}$ for all $(i', j') \in [k] \times [k] \backslash \{(i, j), (a, j)\}$. Let $\mathcal{P}'_k = \{V'_{i'j'} : i', j' \in [k]\}$. By Proposition 3.8,

$$|V'_{i*}| = |V_{i*}| - 1 \geq d - \varepsilon n/2 - 1 \geq \tau n$$

since $\tau \ll \varepsilon \ll \alpha$. Similarly, we have $|V'_{*j}| \geq \tau n$. Moreover, for all $i' \neq i$ and $j' \neq j$, we know $|V'_{i'*}| \geq |V_{i'*}| \geq \tau n$ and $|V'_{*j'}| \geq |V_{*j'}| \geq \tau n$. On the other hand, we obtain

$$|\mathcal{B}_k(\mathcal{P}'_k, G)| = |\mathcal{B}_k(\mathcal{P}_k, G)| - d^+_{V_{*a}}(w) + d^+_{V_{*i}}(w) < |\mathcal{B}_k(\mathcal{P}_k, G)|.$$

Hence $\mathcal{P}'_k$ is a (k^2, τ, γ)-partition of G having fewer bad edges than the extremal (k^2, τ, γ)-partition $\mathcal{P}_k$, which is a contradiction. As a result, for all $1 \leq i', j' \leq k$, we have $d^+_{V_{*i'}}(w) \leq d^+_{V_{*i}}(w)$ and $d^-_{V_{j'*}}(w) \leq d^-_{V_{j*}}(w)$. The rest of the proof is immediate. $\square$

For any dense regular oriented graph, we show that certain unions of sets in a (k^2, τ, γ)-partition have strictly positive size.

Proposition 3.11. *Let $1/n < \gamma \ll \tau \ll \varepsilon < 1$ be constants, $k \in \mathbb{N}$, and G be a d-regular oriented graph on n vertices with $d \geq (1/4+\varepsilon)n$. Suppose that G has a (k^2, τ, γ)-partition $\mathcal{P}_k = \{V_{ij} : i, j \in [k]\}$. Then, for $i \in [k]$, we have*

$$|\bigcup_{j \neq i} V_{ij}|, |\bigcup_{j \neq i} V_{ji}| \geq \tau n.$$

Proof. First suppose that $k = 2$. Without loss of generality, assume $|V_{11}| \leq |V_{22}|$, which gives $d - |V_{11}|/2 \geq d - n/4 \geq \varepsilon n$. By Corollary 3.3, we know that $|V_{12}| - |V_{21}| \leq \gamma n^2/d \leq \tau n$. Hence, it suffices to show that $|V_{12}| \geq 2\tau n$. By Proposition 3.8, we have $|V_{11}| + |V_{12}| = |V_{1*}| \geq (1/4 + \varepsilon/2)n$. Since $\tau \ll \varepsilon$, we may assume that $|V_{11}| \geq n/4$. Then, since G is oriented and d-regular, we can write

$$\begin{aligned} d|V_{11}| = e(V(G), V_{11}) &= e(V_{11}, V_{11}) + e(V_{12}, V_{11}) + e(V_{2*}, V_{11}) \\ &\leq |V_{11}|^2/2 + |V_{12}||V_{11}| + \gamma n^2 \\ &\leq |V_{11}| \cdot (|V_{11}|/2 + |V_{12}| + 4\gamma n). \end{aligned}$$

This implies $|V_{12}| \geq d - |V_{11}|/2 - 4\gamma n \geq (\varepsilon - 4\gamma)n \geq 2\tau n$ as required.

Now, fix $k \geq 3$ and define $\mathcal{W}_i = \{W^i_{ab} : a, b \in [2]\}$ for all $i \in [k]$ where

$$W^i_{11} = V_{ii}, \ W^i_{12} = \bigcup_{j \neq i} V_{ij}, \ W^i_{21} = \bigcup_{j \neq i} V_{ji}, \ W^i_{22} = \bigcup_{a, b \neq i} V_{ab}.$$

Notice that $\mathcal{W}_i$ is a $(4, \tau, \gamma)$-partition by Proposition 3.7, so we get $|W^i_{12}|, |W^i_{21}| \geq \tau n$ from the case $k = 2$. Then, we obtain

$$\left|\bigcup_{j \neq i} V_{ij}\right| = |W^i_{12}| \geq \tau n \ \text{ and } \ \left|\bigcup_{j \neq i} V_{ji}\right| = |W^i_{21}| \geq \tau n,$$

so the result follows for any k. □

4. Balancing partitions

Let G be a regular digraph or oriented graph and suppose $\mathcal{P}_k$ is a (k^2, τ, γ)-partition of G that is "not balanced", in the sense that $|V_{i*}| \neq |V_{*i}|$ for some $i \in [k]$. Then, Proposition 3.2 implies that any Hamilton cycle C must contain a number of bad edges (i.e. edges from $\mathcal{B}_k(\mathcal{P}_k, G)$) that depends on the extent of the "imbalance" of $\mathcal{P}_k$. Since $\mathcal{B}_k(\mathcal{P}_k, G)$ is small (at most γn^2 edges), when constructing a Hamilton cycle of G, it is necessary to first pick the edges of $\mathcal{B}_k(\mathcal{P}_k, G)$ that will be in C. Let us write $\mathcal{Q}$ for the bad edges in our target Hamilton cycle, and note that $\mathcal{Q}$ is a path system.

By Proposition 3.2 (applied with $d = 1$ and $G = C$), we must ensure that $\mathcal{Q}$ satisfies that for all $i \in [k]$,

$$\sum_{j \neq i} |E(\mathcal{Q}) \cap E(V_{i*}, V_{*j})| - \sum_{j \neq i} |E(\mathcal{Q}) \cap E(V_{j*}, V_{*i})| = |V_{i*}| - |V_{*i}|.$$

A naive approach to construct $\mathcal{Q}$ is to take a suitable size matching in each of G_{ij} for $i \neq j$, where as before $G_{ij} = G[V_{i*}, V_{*j}]$. However, the union of these matchings may not be a path system since it might contain cycles or might satisfy $\Delta^0(\mathcal{Q}) \geq 2$. The main purpose of this section is to adapt the naive approach to construct $\mathcal{Q}$; see Lemma 4.10.

Our first goal is to show that given several edge-disjoint subdigraphs of some given digraph, we are able to pick a relatively large path system from each subdigraph such that the union of these path systems does not contain a directed cycle; this is Lemma 4.3. The first two lemmas below are technical results needed to prove this.

Lemma 4.1. *Let G be a digraph with $\Delta^0(G) \leq d$. Let $0 < \theta < 1$, and define the sets $W^+ = \{w \in G : d^+(w) \geq \theta d\}$ and $W^- = \{w \in G : d^-(w) \geq \theta d\}$. Then, there exists a matching M satisfying*

  (i) $4\theta e(M) + |W^+| + |W^-| \geq e(G)/d$,
  (ii) $x \notin W^+$ *and* $y \notin W^-$ *for all* $xy \in E(M)$,
  (iii) $e(M) \leq e(G)/\theta d$.

Proof. If $\theta d < 1$, then we obtain $W^+ = \{w \in G : d^+(w) \geq 1\}$. Then, we have $x \in W^+$ for any $xy \in E(G)$, which, in particular, implies $d|W^+| \geq e(G)$. Therefore, we can set M to be empty in that case. Hence, we may assume $\theta d \geq 1$. Let H be the multigraph obtained from G by deleting all the edges ab with either $a \in W^+$ or $b \in W^-$, and by making all the edges undirected. Note that we have $\Delta(H) + \mu(H) \leq 2\theta d + 2$ and

$$e(H) \geq e(G) - d(|W^+| + |W^-|). \tag{4.1}$$

Then, by Theorem 2.1 (Vizing's theorem for multigraphs), there exists a matching M_1 in H of size at least $e(H)/(2\theta d + 2)$. Moreover, we can assume that $e(M_1) \leq e(H)/\theta d$ because otherwise we can remove some edges from M_1. Let M be the corresponding matching in G. Clearly (ii) holds. By using $\theta d \geq 1$, we obtain

$$\frac{e(G)}{\theta d} \geq \frac{e(H)}{\theta d} \geq e(M) \geq \frac{e(H)}{2\theta d + 2} \geq \frac{e(H)}{4\theta d},$$

so (iii) holds. Hence, together with (4.1), we have $4\theta e(M) + |W^+| + |W^-| \geq e(G)/d$, proving (i). □

Now, given some matchings in a graph, we show that one can pick a significant number of edges from each matching such that all the chosen edges form a matching.

Lemma 4.2. *Let $k, r \in \mathbb{N}$ and $M_1, M_2, \ldots, M_k$ be matchings with $\Delta \left(\bigcup_{i \in [k]} M_i \right) \leq r$. Suppose $e(M_i) > 2(r^3 + r)^2 \ln k$ for all $i \in [k]$. Then, there exists a matching $H \subseteq \bigcup_{i \in [k]} M_i$ with $|E(H) \cap M_i| \geq e(M_i)/(r^2 + 1)$ for all $i \in [k]$.*

Proof. Letting $G = \bigcup_{i \in [k]} M_i$, we have $\Delta(G) \leq r$. We mark edges of G randomly as follows. For each vertex $v \in G$, pick an edge incident to v uniformly at random and mark all other edges incident to v. Do this independently for every vertex v (so some edges may be marked twice). Then, let H be the graph where all the marked edges are deleted. Note that H is a matching. We now show that H satisfies the desired property with positive probability. Observe that the probability of an edge uv surviving into H is at least $1/r^2$ because the probability of uv being marked due to u is at least $1/r$, and independently the probability of uv being marked due to v is at least $1/r$. Moreover, these events are independent for vertex-disjoint edges. Now, for any $i \in [k]$, let $X_i = \mathrm{Bin}(e(M_i), 1/r^2)$. Since M_i is a matching, we have

$$\mathbb{P} \left(|E(H) \cap M_i| \leq \frac{e(M_i)}{r^2 + 1} \right) \leq \mathbb{P} \left(X_i \leq \frac{e(M_i)}{r^2 + 1} \right).$$

Note that $\mathbb{E}[X_i] = e(M_i)/r^2$. Hence, by Theorem 2.2 (Chernoff bound), we obtain

$$\mathbb{P} \left(X_i \leq \frac{e(M_i)}{r^2 + 1} \right) = \mathbb{P} \left(X_i \leq \frac{r^2}{r^2 + 1} \cdot \mathbb{E}[X_i] \right)$$

$$\leq \exp \left(-\frac{\mathbb{E}[X_i]}{2(r^2 + 1)^2} \right) = \exp \left(\frac{-e(M_i)}{2(r^3 + r)^2} \right).$$

Then, by using $e(M_i) > 2(r^3 + r)^2 \ln k$, we obtain $\mathbb{P} \left(|E(H) \cap M_i| \leq \frac{e(M_i)}{r^2 + 1} \right) < \frac{1}{k}$ for each $i \in [k]$. Hence, by the union bound, we have

$$\mathbb{P} \left(|E(H) \cap M_i| \geq \frac{e(M_i)}{r^2 + 1} \text{ for all } i \in [k] \right) > 0.$$

Therefore, there exists a matching $H \subseteq \bigcup_{i \in [k]} M_i$ with $|E(H) \cap M_i| \geq \frac{e(M_i)}{r^2 + 1}$ for all $i \in [k]$. $\square$

By using Lemmas 4.1 and 4.2, we will prove an edge selection lemma which will be used in the proof of Lemma 4.6.

Lemma 4.3. *Let $k \in \mathbb{N}$ with $k \leq 10$, let $0 < \gamma \ll \alpha < 1$ be constants, and let G be a digraph on n vertices. Let $G_1, G_2, \ldots, G_k$ be pairwise edge-disjoint subgraphs of G with $\sum_{i \in [k]} e(G_i) \leq \gamma n^2$ and $\Delta^0(G_i) \leq \alpha n$ for each $i \in [k]$. Then, each G_i contains a path system $\mathcal{Q}_i$ such that $\bigcup_{i \in [k]} \mathcal{Q}_i$ is cycle-free and $e(\mathcal{Q}_i) \geq \lfloor e(G_i)/\alpha n \rfloor$ for all $i \in [k]$.*

Proof. Since $\gamma \ll \alpha$, we can choose a constant θ with $\sqrt{8\gamma}/\alpha < \theta < 1/\left(8 \ln k(k^3+k)^2\right)$. Then, let us define the sets

$$W_i^+ = \{w \in V(G_i) : d_{G_i}^+(w) \geq \alpha \theta n\} \text{ and } W_i^- = \{w \in V(G_i) : d_{G_i}^-(w) \geq \alpha \theta n\}.$$

By Lemma 4.1, for each $i \in [k]$, we can find a matching M_i in G_i with

$$4\theta e(M_i) + |W_i^+| + |W_i^-| \geq e(G_i)/\alpha n,$$
$$M_i \subseteq G_i[V - W_i^+, V - W_i^-], \ e(M_i) \leq e(G_i)/\alpha \theta n.$$

For each i, we have either $|W_i^+| + |W_i^-| > (e(G_i)/\alpha n) - 1$ or $4\theta e(M_i) \geq 1$. In the latter case, we have $e(M_i) \geq 2(k^3+k)^2 \ln k$ due to the definition of θ. Let R be the set of indices $i \in [k]$ satisfying $4\theta e(M_i) \geq 1$. By applying Lemma 4.2 for the matchings M_i with $i \in R$, we find a matching $M \subseteq \bigcup_{i \in R} M_i$ such that $e(M \cap M_i) \geq e(M_i)/(k^2+1)$ for all $i \in R$. Therefore, we have

$$e(M \cap M_i) + |W_i^+| + |W_i^-| \geq e(M_i)/(k^2+1) + |W_i^+| + |W_i^-|$$
$$\geq 4\theta e(M_i) + |W_i^+| + |W_i^-| \geq e(G_i)/\alpha n$$

for all $i \in R$. On the other hand, if $i \notin R$, we know $|W_i^+| + |W_i^-| > (e(G_i)/\alpha n) - 1$, which, in particular implies $|W_i^+| + |W_i^-| \geq \lfloor e(G_i)/\alpha n \rfloor$. Write $N_i = M \cap M_i$ if $i \in R$, write $N = \bigcup_{i \in R} N_i$, and set $N_i = \emptyset$ if $i \notin R$. Thus, we obtain $e(N_i) + |W_i^+| + |W_i^-| \geq \lfloor e(G_i)/\alpha n \rfloor$ for all $i \in [k]$. By deleting edges in N_i or removing vertices from $W_i^+ \cup W_i^-$, we may assume

$$e(N_i) + |W_i^+| + |W_i^-| = \lfloor e(G_i)/\alpha n \rfloor \text{ for all } i \in [k].$$

Let us write $W = \bigcup_{i \in [k]} (W_i^+ \cup W_i^-)$. Note that

$$|V(N) \cup W| \leq \sum_{i \in [k]} \left(2e(N_i) + |W_i^+| + |W_i^-|\right) \leq 2 \cdot \sum_{i \in [k]} (e(G_i)/\alpha \theta n) \leq 2\gamma n/\alpha \theta. \ (4.2)$$

We now construct the desired path systems $\mathcal{Q}_1, \ldots \mathcal{Q}_k$ by induction. Suppose we have found path systems $\mathcal{Q}_1, \ldots, \mathcal{Q}_j$ for some $0 \leq j \leq k$ such that $N \cup \left(\bigcup_{i \in [j]} \mathcal{Q}_i\right)$ is cycle-free, and the following hold for all $i \in [j]$:

(i) $N_i \subseteq \mathcal{Q}_i \subseteq G_i$,
(ii) $e(\mathcal{Q}_i) = \lfloor e(G_i)/\alpha n \rfloor$.

If $j = k$, then we are done. If $j < k$, then we construct $\mathcal{Q}_{j+1}$ as follows. First, we define $U = (V(N) \cup W) \cup V\left(\bigcup_{i \in [j]} \mathcal{Q}_i\right)$. By using (4.2), we have

$$|U| \leq 2\gamma n / \alpha\theta + 2 \sum_{i \in [j]} e(\mathcal{Q}_i) \leq (2\gamma n / \alpha\theta) + (2\gamma n / \alpha).$$

We construct the undirected bipartite graph $\mathcal{B}$ with bipartition (A, B) as follows. Let $B = V(G) - U$, and let A be the disjoint union of W_{j+1}^+ and W_{j+1}^-. We add the edge ab for each $a \in W_{j+1}^+$ and $b \in B$ if $b \in N_G^+(a)$, and add the edge ab for each $a \in W_{j+1}^-$ and $b \in B$ if $a \in N_G^-(b)$. Due to the choice of θ, we have

$$d_{\mathcal{B}}(a) \geq \alpha\theta n - (2\gamma n / \alpha\theta) - (2\gamma n / \alpha)$$

$$\geq \gamma n / \alpha \geq e(G_{j+1}) / \alpha n \geq |W_{j+1}^+| + |W_{j+1}^-| \geq |A|.$$

Therefore, we can greedily pick a matching in $\mathcal{B}$ that covers A. Note that the corresponding edges in G with respect to this matching give a path system $\mathcal{Q}_{j+1}'$ in G_{j+1} containing paths of length one or two with $e(\mathcal{Q}_{j+1}') = |W_{j+1}^+| + |W_{j+1}^-|$ and $E(\mathcal{Q}_{j+1}') \cap E\left(N \cup \bigcup_{i \in [j]} \mathcal{Q}_i\right) = \emptyset$. Moreover, each edge in $\mathcal{Q}_{j+1}'$ will contain a vertex in $W_{j+1}^+ \cup W_{j+1}^-$ and one unique vertex not in U. Since $x \notin W_{j+1}^+$ and $y \notin W_{j+1}^-$ for all $xy \in N_{j+1}$, we can add N_{j+1} into $\mathcal{Q}_{j+1}'$ to obtain another path system $\mathcal{Q}_{j+1}$ in G_{j+1} with $e(\mathcal{Q}_{j+1}) = \lfloor e(G_{j+1}) / \alpha n \rfloor$.

Finally, suppose $N \cup \left(\bigcup_{i \in [j+1]} \mathcal{Q}_i\right) = N \cup \left(\bigcup_{i \in [j]} \mathcal{Q}_i\right) \cup \mathcal{Q}_{j+1}$ has a cycle C. Since $N \cup \left(\bigcup_{i \in [j]} \mathcal{Q}_i\right)$ has no cycle, C contains an edge e in $\mathcal{Q}_{j+1}'$. However, e contains a unique vertex x not in U, that is, x has (total) degree 1 in $N \cup \left(\bigcup_{i \in [j+1]} \mathcal{Q}_i\right)$, a contradiction. This completes the inductive construction of the $\mathcal{Q}_i$ and the proof of the lemma. As a result, $N \cup \left(\bigcup_{i \in [j+1]} \mathcal{Q}_i\right)$ is cycle-free, and we are done. $\quad\square$

Suppose G is an oriented graph and consider a 9-partition $\{V_{ij} : i, j \in [3]\}$ of $V(G)$. For $i, j \in [3]$, $i \neq j$, we say a path system $\mathcal{Q}$ is *type-ij* if $E(\mathcal{Q}) \subseteq E(V_{i*}, V_{*j})$. Our next lemma describes the structure of the graph which is the union of several path systems that are of different types. First some further notation.

We denote the set of all type-ij path systems by $\mathcal{Q}(i, j)$. Let $\mathcal{S} \subset \bigcup_{i \neq j} \mathcal{Q}(i, j)$ be a set consisting of three path systems of different types. We say $\mathcal{S}$ is a *symmetric 3-set* if either $|\mathcal{S} \cap \mathcal{Q}(1, 2)| = |\mathcal{S} \cap \mathcal{Q}(2, 3)| = |\mathcal{S} \cap \mathcal{Q}(3, 1)| = 1$ or $|\mathcal{S} \cap \mathcal{Q}(2, 1)| = |\mathcal{S} \cap \mathcal{Q}(3, 2)| = |\mathcal{S} \cap \mathcal{Q}(1, 3)| = 1$. Otherwise, we say $\mathcal{S}$ is an *anti-symmetric 3-set*. For an anti-symmetric 3-set $\mathcal{S}$, if $|\mathcal{S} \cap (\mathcal{Q}(1, 2) \cup \mathcal{Q}(2, 3) \cup \mathcal{Q}(3, 1))| = 2$ and $|\mathcal{S} \cap (\mathcal{Q}(2, 1) \cup \mathcal{Q}(3, 2) \cup \mathcal{Q}(1, 3))| = 1$, then we call the unique path system in $\mathcal{S} \cap (\mathcal{Q}(2, 1) \cup \mathcal{Q}(3, 2) \cup \mathcal{Q}(1, 3))$ a *special element* of $\mathcal{S}$. Similarly, if $|\mathcal{S} \cap (\mathcal{Q}(1, 2) \cup \mathcal{Q}(2, 3) \cup \mathcal{Q}(3, 1))| = 1$ and $|\mathcal{S} \cap (\mathcal{Q}(2, 1) \cup \mathcal{Q}(3, 2) \cup \mathcal{Q}(1, 3))| = 2$, then we call the unique path system in $\mathcal{S} \cap (\mathcal{Q}(1, 2) \cup \mathcal{Q}(2, 3) \cup \mathcal{Q}(3, 1))$ as a *special element* of $\mathcal{S}$. We will show that the graph induced by $\mathcal{S}$ has some structural properties if $\mathcal{S}$ is a symmetric or anti-symmetric 3-set. First, we need the definition of an anti-directed path.

Let G be a digraph. A subgraph P of G is called an *anti-directed path* in G if its edges can be ordered as $E(P) = \{e_1, e_2, \ldots, e_k\}$ for some $k \in \mathbb{N}$ such that

(i) $(e_1, e_2, \ldots, e_k)$ induces an (undirected) path when we forgot the directions of the edges, and
(ii) $(e_1, e_2, \ldots, e_k)$ does not contain a directed path of length at least two.

An anti-directed path P in G is said to be *maximal* if it is not entirely contained in any other anti-directed path.

Lemma 4.4. *Let $\mathcal{P}_3 = \{V_{ij} : i, j \in [3]\}$ be a 9-partition of an oriented graph G. Let $\mathcal{S}$ be a set consisting of three path systems in G of different types; thus either $\mathcal{S}$ is a symmetric 3-set or an anti-symmetric 3-set. Let H be the graph induced by all the paths in $\mathcal{S}$. If $\mathcal{S}$ is a symmetric 3-set, then H is the disjoint union of paths and cycles. If $\mathcal{S}$ is an anti-symmetric 3-set with special element S, then $E(H)$ can be partitioned into maximal anti-directed paths Q of length at most three with the following properties:*

(i) *If Q is a maximal anti-directed path of length two, then Q has a unique edge belonging to S.*
(ii) *If Q is a maximal anti-directed path of length three, then each edge of Q belongs to a distinct path system in $\mathcal{S}$ where the middle edge belongs to S.*

Proof. If $\mathcal{S}$ is a symmetric 3-set, without loss of generality, assume $\mathcal{S} = \{\mathcal{Q}_{23}, \mathcal{Q}_{31}, \mathcal{Q}_{12}\}$ where the path system $\mathcal{Q}_{ij}$ is type-ij. Then, for any $x_{23}y_{23} \in E(\mathcal{Q}_{23})$, $x_{31}y_{31} \in E(\mathcal{Q}_{31})$, $x_{12}y_{12} \in E(\mathcal{Q}_{12})$, we obtain x_{23}, x_{31}, x_{12} are all distinct since $x_{23} \in V_{2*}$, $x_{31} \in V_{3*}$, and $x_{12} \in V_{1*}$. Similarly, we have y_{23}, y_{31}, y_{12} are all distinct. Therefore, for any vertex $v \in H$, we have $d^+(v), d^-(v) \leq 1$, which implies H is the disjoint union of paths and cycles.

Let $\mathcal{S}$ be an anti-symmetric 3-set. Without loss of generality, it is enough to examine the cases $\mathcal{S} = \{\mathcal{Q}_{23}, \mathcal{Q}_{12}, \mathcal{Q}_{13}\}$ and $\mathcal{S} = \{\mathcal{Q}_{23}, \mathcal{Q}_{12}, \mathcal{Q}_{21}\}$ where $\mathcal{Q}_{ij}$ is type-ij. Let us first examine the case $\mathcal{S} = \{\mathcal{Q}_{23}, \mathcal{Q}_{12}, \mathcal{Q}_{13}\}$. Note that $\mathcal{Q}_{13}$ is the special element of $\mathcal{S}$. For any $x_{23}y_{23} \in E(\mathcal{Q}_{23})$, $x_{13}y_{13} \in E(\mathcal{Q}_{13})$, $x_{12}y_{12} \in E(\mathcal{Q}_{12})$, we have $x_{23} \in V_{2*}$, $x_{13}, x_{12} \in V_{1*}$, $y_{12} \in V_{*2}$, $y_{23}, y_{13} \in V_{*3}$, which shows that $\Delta^0(H) \leq 2$. Thus, we can conclude that two different maximal anti-directed paths in H are edge-disjoint, which implies every edge of H lies in a unique maximal anti-directed path. Let Q be a maximal anti-directed path in H of length at least two. Let e_i, e_{i+1} be two consecutive edges in Q. It is easy to check that

$$(e_i, e_{i+1}) \in (E(\mathcal{Q}_{13}) \times E(\mathcal{Q}_{12})) \cup (E(\mathcal{Q}_{12}) \times E(\mathcal{Q}_{13}))$$
$$\cup (E(\mathcal{Q}_{13}) \times E(\mathcal{Q}_{23})) \cup (E(\mathcal{Q}_{23}) \times E(\mathcal{Q}_{13})).$$

Therefore, if Q has two edges, property (i) follows. If Q has three consecutive edges e_i, e_{i+1}, e_{i+2}, then

$$(e_i, e_{i+1}, e_{i+2}) \in E(\mathcal{Q}_{12}) \times E(\mathcal{Q}_{13}) \times E(\mathcal{Q}_{23}) \text{ or}$$

$$(e_i, e_{i+1}, e_{i+2}) \in E(\mathcal{Q}_{23}) \times E(\mathcal{Q}_{13}) \times E(\mathcal{Q}_{12}).$$

This shows property (ii), and in particular that the middle of the three edges is in the special element $\mathcal{Q}_{13}$. Finally, if Q has at least four edges, take any four consecutive edges. These four edges contain two anti-directed paths of length three and the middle edge of each of these paths lies in $\mathcal{Q}_{13}$ from the argument above. Therefore we obtain two consecutive edges in Q both in $\mathcal{Q}_{13}$, which is impossible since $\mathcal{Q}_{13}$ is a path system.

If $\mathcal{S} = \{\mathcal{Q}_{23}, \mathcal{Q}_{12}, \mathcal{Q}_{21}\}$, it is easy to check that we have $d_H^+(v) \leq 2$ and $d_H^-(v) \leq 1$ for all $v \in V(H)$. Note that $\mathcal{Q}_{21}$ is the special element of $\mathcal{S}$. As before, we see that $E(H)$ can be partitioned into maximal anti-directed paths since $\Delta^0(H) \leq 2$. Also, since each anti-directed path of length at least three has at least one vertex of indegree two, we have that all the maximal anti-directed paths in H have at most two edges. Moreover, if Q is a maximal anti-directed path of length two, say e and f are the edges of Q, then we have either $(e, f) \in E(Q_{23}) \times E(Q_{21})$ or $(e, f) \in E(Q_{21}) \times E(Q_{23})$, which completes the proof. $\square$

We need one more technical proposition before we prove the lemma that shows how to select the bad edges that will be part of our final Hamilton cycle.

Proposition 4.5. *Let $t, x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}$ be such that*

$$x_1 + x_2 + x_3 \equiv t \equiv x_1 + x_4 + x_5 \pmod{2}. \tag{4.3}$$

Then, one can find $m_i \in \{-1, 1\}$ for $i \in [5]$ with

$$m_1 x_1 + m_2 x_2 + m_3 x_3 = t = m_1 x_1 + m_4 x_4 + m_5 x_5.$$

Proof. Without loss of generality, we can assume $x_2 \leq x_3$, $x_4 \leq x_5$, and $x_2 + x_3 \leq x_4 + x_5$. By (4.3), we must have $(x_2 + x_3, x_4 + x_5) \in \{(0, 0), (1, 1), (2, 2), (0, 2)\}$.

1. If $x_2 + x_3 = 0 = x_4 + x_5$, then we have $t = x_1$ and $x_2 = x_3 = x_4 = x_5 = 0$. Hence, we only need $m_1 x_1 = x_1$, which can be done by choosing $m_1 = 1$.
2. If $x_2 + x_3 = 1 = x_4 + x_5$, then we have $t = 1 - x_1$, $x_2 = x_4 = 0$ and $x_3 = x_5 = 1$. Hence, we need $m_1 x_1 + m_3 = 1 - x_1 = m_1 x_1 + m_5$, which can be done by choosing $m_3 = m_5 = 1$, and $m_1 = -1$.
3. If $x_2 + x_3 = 2 = x_4 + x_5$, then we have $t = x_1$ and $x_2 = x_3 = x_4 = x_5 = 1$. Hence, we need $m_1 x_1 + m_2 + m_3 = x_1 = m_1 x_1 + m_4 + m_5$, which can be done by choosing $m_1 = 1$, $m_2 = m_4 = 1$, and $m_3 = m_5 = -1$.
4. If $x_2 + x_3 = 0$ and $x_4 + x_5 = 2$, then we have $t = x_1$, $x_2 = x_3 = 0$ and $x_4 = x_5 = 1$. Hence, we need $m_1 x_1 = x_1 = m_1 x_1 + m_4 + m_5$, which can be done by choosing $m_4 = 1$, $m_5 = -1$, and $m_1 = 1$. $\square$

We are now ready to prove the main result of this section.

Lemma 4.6. *Let $1/n \ll \gamma \ll \tau \ll \alpha \ll 1$ be some constants, let G be a d-regular oriented graph on n vertices with $d \geq \alpha n$ and an extremal $(9, \tau, \gamma)$-partition $\mathcal{P}_3 = \{V_{ij} : i, j \in [3]\}$. Then, there exists a path system $\mathcal{Q}$ in $\mathcal{B}_3(\mathcal{P}_3, G)$ such that, writing $a_{ij} = |E(\mathcal{Q}) \cap E(V_{i*}, V_{*j})|$ for all $i \neq j$, we have*

(i) $e(\mathcal{Q}) \leq 2\gamma n / \alpha$, and
(ii) $a_{i*} - a_{*i} = |V_{i*}| - |V_{*i}|$ for all $i \in [3]$, where $a_{i*} = \sum_{j \neq i} a_{ij}$ and $a_{*i} = \sum_{j \neq i} a_{ji}$.

Proof. We first give the main idea of the proof. Note that by Proposition 3.2, it suffices to find a path system $\mathcal{Q}$ satisfying

$$a_{i*} - a_{*i} = \sum_{j \neq i} e(G_{ij})/d - \sum_{j \neq i} e(G_{ji})/d \tag{4.4}$$

for each $i \in [3]$. By Proposition 3.10, we know $\Delta(G_{ij}) \leq d/2$, so by using Lemma 4.3, we can find path systems $\mathcal{Q}_{ij}$ in G_{ij} such that $\bigcup_{i \neq j} \mathcal{Q}_{ij}$ is cycle-free and $e(\mathcal{Q}_{ij})$ has roughly $2e(G_{ij})/d$ edges. Therefore, we need only (roughly) half of the edges from each $\mathcal{Q}_{ij}$ to satisfy (4.4). Moreover, for each $i \neq j$, it makes sense to include edges from only one of G_{ij} and G_{ji}. We will choose $\mathcal{Q}_{ij}^0 \subseteq \mathcal{Q}_{ij}$, where $\mathcal{Q}_{ij}^0$ has size (roughly) $e(\mathcal{Q}_{ij})/2$ for three different pairs (i, j) and is empty for the remaining three pairs, by using the structural properties of $\bigcup_{i \neq j} \mathcal{Q}_{ij}$ (ensured by Lemma 4.4) so that $\Delta^0(\bigcup_{i \neq j} \mathcal{Q}_{ij}^0) = 1$. Since $\bigcup_{i \neq j} \mathcal{Q}_{ij}^0 \subseteq \bigcup_{i \neq j} \mathcal{Q}_{ij}$ is cycle-free, $\Delta^0(\bigcup_{i \neq j} \mathcal{Q}_{ij}^0) = 1$ guarantees that $\bigcup_{i \neq j} \mathcal{Q}_{ij}^0$ is a path system. Also, $e(\bigcup_{i \neq j} \mathcal{Q}_{ij}^0)$ is small enough by the construction since $e(\bigcup_{i \neq j} \mathcal{Q}_{ij}) \leq 2|\mathcal{B}_3(\mathcal{P}_3, G)|/d \leq 2\gamma n/\alpha$.

Let us write $n_i = |V_{i*}| - |V_{*i}|$ for $i \in [3]$. Since $n_1 + n_2 + n_3 = 0$, without loss of generality, we can assume $n_1, n_2 \geq 0$. Recall $G_{ij} = G[V_{i*}, V_{*j}]$, and write $m_{ij} = e(G_{ij}) - e(G_{ji})$ for $i, j \in [3]$, $i \neq j$. Note that $m_{ij} = -m_{ji}$. Without loss of generality, we can assume $m_{12} \geq 0$. By Proposition 3.2, we have

$$dn_1 = m_{12} + m_{13} = m_{12} - m_{31}, \tag{4.5}$$

$$dn_2 = m_{21} + m_{23} = m_{23} - m_{12}. \tag{4.6}$$

Since $n_1, n_2 \geq 0$, (4.5) and (4.6) imply that $m_{23} \geq m_{12} \geq m_{31}$. So, it suffices to consider the cases

$$m_{23} \geq m_{12} \geq m_{31} \geq 0 \quad \text{and} \quad m_{23} \geq m_{12} \geq 0 \geq m_{31}.$$

Let $m_{12} = dx$ for some $x \geq 0$, and write $x = s + t$ where $s = \lfloor x \rfloor$ and $0 \leq t < 1$.

Case 1: Suppose we have $m_{23} \geq m_{12} \geq m_{31} \geq 0$. Then, we can write $m_{31} = d(x - n_1)$ and $m_{23} = d(x + n_2)$ by using (4.5) and (4.6). Let $H = G_{23} \cup G_{31} \cup G_{12}$. Notice that $e(H) \leq |\mathcal{B}_3(\mathcal{P}_3, G)| \leq \gamma n^2$. Also, by Proposition 3.10, we know $\Delta^0(G_{23}), \Delta^0(G_{31}), \Delta^0(G_{12}) \leq d/2$. By Lemma 4.3, we can find path systems $\mathcal{Q}_{23} \subseteq G_{23}$, $\mathcal{Q}_{31} \subseteq G_{31}$, $\mathcal{Q}_{12} \subseteq G_{12}$ such that $\mathcal{Q}_{23} \cup \mathcal{Q}_{31} \cup \mathcal{Q}_{12}$ has no cycle and

$$e(\mathcal{Q}_{23}) = \left\lfloor \frac{e(G_{23})}{d/2} \right\rfloor \geq \left\lfloor \frac{m_{23}}{d/2} \right\rfloor = \lfloor 2x + 2n_2 \rfloor \geq s + n_2,$$

$$e(\mathcal{Q}_{31}) = \left\lfloor \frac{e(G_{31})}{d/2} \right\rfloor \geq \left\lfloor \frac{m_{31}}{d/2} \right\rfloor = \lfloor 2x - 2n_1 \rfloor \geq s - n_1,$$

$$e(\mathcal{Q}_{12}) = \left\lfloor \frac{e(G_{23})}{d/2} \right\rfloor \geq \left\lfloor \frac{m_{12}}{d/2} \right\rfloor = \lfloor 2x \rfloor \geq s.$$

Moreover, by Lemma 4.4, we have $\mathcal{Q}_{23} \cup \mathcal{Q}_{31} \cup \mathcal{Q}_{12}$ is a disjoint union of paths and cycles since $\{\mathcal{Q}_{23}, \mathcal{Q}_{31}, \mathcal{Q}_{12}\}$ is a symmetric 3-set. However, we know it is cycle-free, which implies it is a path system. Note that (4.5) implies that $d(x - n_1) = m_{31} \geq 0$ by assumption, so we have $s - n_1 \geq 0$. We now define $\mathcal{Q}$ by choosing $s + n_2$ edges from $\mathcal{Q}_{23}$, $s - n_1$ edges from $\mathcal{Q}_{31}$, and s edges from $\mathcal{Q}_{12}$. Note that

$$e(\mathcal{Q}) \leq \frac{e(G_{12}) + e(G_{23}) + e(G_{31})}{d/2} \leq \frac{|\mathcal{B}_3(\mathcal{P}_3, G)|}{d/2} \leq \frac{2\gamma n}{\alpha}.$$

Since $a_{23} = s + n_2$, $a_{31} = s - n_1$, $a_{12} = s$ and $a_{21} = a_{32} = a_{13} = 0$, the result follows.

Case 2: Suppose we have $m_{23} \geq m_{12} \geq 0 \geq m_{31}$. Recall $m_{12} = dx$. Then, we can write $m_{13} = d(n_1 - x)$ and $m_{23} = d(n_2 + x)$ by using (4.5) and (4.6). As with the previous case, we can find path systems $\mathcal{Q}_{23} \subseteq G_{23}$, $\mathcal{Q}_{13} \subseteq G_{13}$, $\mathcal{Q}_{12} \subseteq G_{12}$ such that $\mathcal{Q}_{23} \cup \mathcal{Q}_{13} \cup \mathcal{Q}_{12}$ is cycle-free and

$$e(\mathcal{Q}_{13}) = 2n_1 + \lfloor -2x \rfloor, \ e(\mathcal{Q}_{23}) = \lfloor 2x \rfloor + 2n_2, \ e(\mathcal{Q}_{12}) = \lfloor 2x \rfloor, \tag{4.7}$$

$$e(\mathcal{Q}_{13} \cup \mathcal{Q}_{23} \cup \mathcal{Q}_{12}) \leq 2\gamma n/\alpha. \tag{4.8}$$

Let H be the graph induced by $\mathcal{Q}_{23} \cup \mathcal{Q}_{13} \cup \mathcal{Q}_{12}$. Note that $\mathcal{Q}_{13}$ is the special element of the anti-symmetric 3-set $\{\mathcal{Q}_{23}, \mathcal{Q}_{13}, \mathcal{Q}_{12}\}$. For simplicity, we write $A = 13$, $B = 23$ and $C = 12$ (so e.g. $m_A = m_{13}$ and $G_A = G_{13}$). By Lemma 4.4, we can decompose $E(H)$ into six sets $\mathcal{S}_T$ with $T \in \{ABC, AB, AC, A, B, C\}$ such that $\mathcal{S}_T$ is the set of maximal anti-directed paths of length $|T|$ containing an edge in each $\mathcal{Q}_S$ for $S \in T$, e.g. $\mathcal{S}_{ABC}$ is the set of anti-directed paths of length three with one edge in each of $\mathcal{Q}_{13}, \mathcal{Q}_{23}, \mathcal{Q}_{12}$.

From the definition of the decomposition, clearly we have

$$e(\mathcal{Q}_{13}) = e(\mathcal{Q}_A) = |\mathcal{S}_{ABC}| + |\mathcal{S}_{AB}| + |\mathcal{S}_{AC}| + |\mathcal{S}_A|,$$

$$e(\mathcal{Q}_{23}) = e(\mathcal{Q}_B) = |\mathcal{S}_{ABC}| + |\mathcal{S}_{AB}| + |\mathcal{S}_B|,$$

$$e(\mathcal{Q}_{12}) = e(\mathcal{Q}_C) = |\mathcal{S}_{ABC}| + |\mathcal{S}_{AC}| + |\mathcal{S}_C|.$$

By (4.7), we obtain

$$2(|\mathcal{S}_{ABC}| + |\mathcal{S}_{AC}|) + |\mathcal{S}_{AB}| + |\mathcal{S}_A| + |\mathcal{S}_C|$$

$$= e(\mathcal{Q}_A) + e(\mathcal{Q}_C) = 2n_1 + \lfloor -2t \rfloor + \lfloor 2t \rfloor \tag{4.9}$$

$$|\mathcal{S}_{AB}| + |\mathcal{S}_B| - |\mathcal{S}_{AC}| - |\mathcal{S}_C| = e(\mathcal{Q}_B) - e(\mathcal{Q}_C) = 2n_2. \tag{4.10}$$

Hence, letting $|\mathcal{S}_T| \equiv r_T \pmod 2$ for $T \in \{ABC, AB, AC, A, B, C\}$ where $r_T \in \{0,1\}$ (so $|\mathcal{S}_T| \pm r_T$ is even), we have the following equivalence by summing (4.9) and (4.10):

$$r_{AB} + r_A + r_C \equiv -\lfloor 2t \rfloor - \lfloor -2t \rfloor \equiv r_{AC} + r_A + r_B \pmod 2.$$

Since $0 \le t < 1$, we have $-\lfloor 2t \rfloor - \lfloor -2t \rfloor \in \{0,1\}$. Then, by Proposition 4.5, we can find $i_{AB}, i_{AC}, i_A, i_B, i_C \in \{-1, 1\}$ such that

$$i_{AB} r_{AB} + i_A r_A + i_C r_C = -\lfloor 2t \rfloor - \lfloor -2t \rfloor = i_{AC} r_{AC} + i_A r_A + i_B r_B. \qquad (4.11)$$

We now construct $\mathcal{Q} \subseteq H$ as follows. Initializing $\mathcal{Q} = \emptyset$, we will add some edges into $\mathcal{Q}$ as follows:

1. Choose $(|\mathcal{S}_{ABC}| + r_{ABC})/2$ many paths from $\mathcal{S}_{ABC}$, $(|\mathcal{S}_{AB}| + i_{AB} r_{AB})/2$ many paths from $\mathcal{S}_{AB}$, and $(|\mathcal{S}_{AC}| + i_{AC} r_{AC})/2$ many paths from $\mathcal{S}_{AC}$. For each such path, we add the unique edge from $\mathcal{Q}_A \subseteq G_{13}$ to $\mathcal{Q}$.
2. Take the remaining $(|\mathcal{S}_{ABC}| - r_{ABC})/2$ many paths from $\mathcal{S}_{ABC}$. For each such path, we add the unique edge from $\mathcal{Q}_B \subseteq G_{23}$ and the unique edge from $\mathcal{Q}_C \subseteq G_{12}$ to $\mathcal{Q}$.
3. Take the remaining $(|\mathcal{S}_{AB}| - i_{AB} r_{AB})/2$ many paths from $\mathcal{S}_{AB}$. For each such path, we add the unique edge from $\mathcal{Q}_B \subseteq G_{23}$ to $\mathcal{Q}$.
4. Take the remaining $(|\mathcal{S}_{AC}| - i_{AC} r_{AC})/2$ many paths from $\mathcal{S}_{AC}$. For each such path, we add the unique edge from $\mathcal{Q}_C \subseteq G_{12}$ to $\mathcal{Q}$.
5. For each $T \in \{A, B, C\}$, take $(|\mathcal{S}_T| + i_T r_T)/2$ many paths from $\mathcal{S}_T$. Add them to $\mathcal{Q}$.

If $\Delta^0(\mathcal{Q}) \ge 2$, then there exists an anti-directed path Q' of length 2 in $\mathcal{Q}$. This path Q' must be contained in some maximal anti-directed path Q^* in $\mathcal{S}_{ABC} \cup \mathcal{S}_{AB} \cup \mathcal{S}_{AC}$. Only in Step 2 do we add more than one edge from a maximal anti-directed path to $\mathcal{Q}$. However, the two edges added in that case are not incident by Lemma 4.4(ii) as $\mathcal{Q}_{13} = \mathcal{Q}_A$ is the special element. Therefore no such Q' exists, and so $\Delta^0(\mathcal{Q}) \le 1$. Recall that $\bigcup \mathcal{Q} \subseteq H$ is cycle-free and so $\mathcal{Q}$ is a path system. By (4.8)

$$e(\mathcal{Q}) \le e(H) \le 2\gamma n/\alpha.$$

Note that

$$
\begin{aligned}
2(a_{1*} - a_{*1}) &= 2\left(e(\mathcal{Q} \cap G_{12}) + e(\mathcal{Q} \cap G_{13})\right) = 2\left(e(\mathcal{Q} \cap \mathcal{Q}_A) + e(\mathcal{Q} \cap \mathcal{Q}_C)\right) \\
&= 2(|\mathcal{S}_{ABC}| + |\mathcal{S}_{AC}|) + |\mathcal{S}_{AB}| + |\mathcal{S}_A| + |\mathcal{S}_C| + i_{AB} r_{AB} + i_A r_A + i_C r_C \\
&= 2n_1,
\end{aligned}
$$

where the last equality is due to (4.9) and (4.11). Similarly,

$$2(a_{2*} - a_{*2}) = 2(e(\mathcal{Q} \cap G_{23}) - e(\mathcal{Q} \cap G_{12})) = 2(e(\mathcal{Q} \cap G_B) - e(\mathcal{Q} \cap G_C))$$

$$= (|\mathcal{S}_{AB}| + |\mathcal{S}_B| - |\mathcal{S}_{AC}| - |\mathcal{S}_C|) + (i_{AC}r_{AC} + i_B r_B - i_{AB}r_{AB} - i_C r_C)$$
$$= 2n_2,$$

where the last equality is due to (4.10) and (4.11). So we have $a_{1*} - a_{*1} = n_1$ and $a_{2*} - a_{*2} = n_2$. Since $n_1 + n_2 + n_3 = 0$, we deduce that $a_{3*} - a_{*3} = n_3$ as required. $\square$

The previous lemma shows how to obtain the (path system of) bad edges that will be part of our final Hamilton cycle. It will be convenient to suitably contract this path system because the resulting contracted graph will have a "balanced" partition and finding a Hamilton cycle in the contracted graph will give us a Hamilton cycle in the original graph by "uncontracting" the path system. We now define the right notion of contraction and establish some of its properties.

Definition 4.7. Let G be a digraph, $k \in \mathbb{N}$, and $\mathcal{P}_k = \{V_{ij} : i, j \in [k]\}$ be a k^2-partition of $V(G)$. Let $\mathcal{Q}$ be a path system in G. We define the *contraction of $\mathcal{Q}$ in G with respect to $\mathcal{P}_k$* as follows: for each $Q \in \mathcal{Q}$, create a new vertex x associated to Q such that $N^-(x) = N_G^-(u)$ and $N^+(x) = N_G^+(v)$ where Q goes from u to v. If $u \in V_{ij}$ and $v \in V_{i'j'}$, put x into $V_{i'j}$. Then, we delete all the vertices in $\mathcal{Q}$. We call $\mathcal{P}_k' = \{V_{ij}' : i, j \in [k]\}$ the *resulting partition* where V_{ij}' is the updated version of V_{ij} for all $i, j \in [k]$, and we denote the *resulting* graph by G'.

Since we often use the following fact, we state it as a proposition.

Proposition 4.8. *Let G be a digraph, $\mathcal{Q}$ be a path system in G, and $\mathcal{P}_k = \{V_{ij} : i, j \in [k]\}$ be a k^2-partition of $V(G)$. If G' is the graph obtained from G by contracting $\mathcal{Q}$ with respect to $\mathcal{P}_k$, and G' is Hamiltonian, then so is G.*

Next we see that the number of bad edges cannot increase from contracting a path system with respect to the given partition.

Proposition 4.9. *Let $1/n \ll \theta, \gamma \ll \tau \le 1$ and $k \in \mathbb{N}$ be constants. Let G be a digraph on n vertices, $\mathcal{Q}$ be a path system in G, and $\mathcal{P}_k = \{V_{ij} : i, j \in [k]\}$ be a k^2-partition of $V(G)$. Let us contract $\mathcal{Q}$ with respect to the partition $\mathcal{P}_k$. Then, we have $|\mathcal{B}_k(\mathcal{P}_k', G')| \le |\mathcal{B}_k(\mathcal{P}_k, G)|$. Moreover, if $\mathcal{P}_k$ is a (k^2, τ, γ)-partition of G and $e(\mathcal{Q}) \le \theta n$, then $\mathcal{P}_k'$ is a $(k^2, \tau/2, 2\gamma)$-partition of G'.*

Proof. Consider a path $P \in \mathcal{Q}$ that goes from u to v, let x be the created vertex corresponding to P during the contraction process with $x \in V_{cb}'$. In particular, we have $v \in V_{c*}$. If $xy \in \mathcal{B}_k(\mathcal{P}_k', G')$, then $y \notin V_{*c}$ and $y \in N^+(v)$, which shows $vy \in \mathcal{B}_k(\mathcal{P}_k, G)$. Similarly, for any bad edge in G' with respect to $\mathcal{P}_k'$, we can find a different bad edge in G with respect to $\mathcal{P}_k$, which shows $|\mathcal{B}_k(\mathcal{P}_k', G')| \le |\mathcal{B}_k(\mathcal{P}_k, G)|$.

Notice that we have $|G'| \geq (1 - \theta)n$. Also, since we deleted at most $2\theta n$ vertices and $\theta \ll \tau$, we have $|V'_{i*}| \geq \tau n - 2\theta n \geq \tau |G'|/2$ for all $i \in [k]$. Moreover, we get $2(1-\theta)^2 > 1$ since $\theta \ll 1$, which implies $|\mathcal{B}_k(\mathcal{P}'_k, G')| \leq \gamma n^2 \leq 2\gamma(1-\theta)^2 n^2 \leq 2\gamma|G'|^2$. $\quad \square$

We end this section with a lemma which states that if a path system $\mathcal{Q}$ in $\mathcal{B}(\mathcal{P}_k, G)$ satisfies condition (ii) of Lemma 4.6, then the contraction of $\mathcal{Q}$ with respect to $\mathcal{P}_k$ balances the partition.

Lemma 4.10. *Let $k \in \mathbb{N}$, and let $\mathcal{P}_k = \{V_{ij} : i, j \in [k]\}$ be a k^2-partition for a digraph G. Let $\mathcal{Q}$ be a path system in $\mathcal{B}_k(\mathcal{P}_k, G)$ such that, for all $i \in [k]$,*

$$\sum_{j \neq i} a_{ij} - \sum_{j \neq i} a_{ji} = |V_{i*}| - |V_{*i}|,$$

where a_{ij} denotes the number of edges in $E(\mathcal{Q}) \cap E(V_{i*}, V_{*j})$ for all $i \neq j$. Then, the contraction of $\mathcal{Q}$ with respect to $\mathcal{P}_k$ results in a digraph G' with a k^2-partition $\mathcal{P}'_k = \{V'_{ij} : i, j \in [k]\}$ such that $|V'_{i*}| = |V'_{*i}|$ for all $i \in [k]$.

Proof. Let $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_t\}$. Let a^p_{ij} denote the number of edges in $E(Q_p) \cap E(V_{i*}, V_{*j})$ for all $1 \leq p \leq t$ and $i \neq j$. Consider a path Q_p, say from $u \in V_{xy}$ to $v \in V_{zt}$. Recall that we delete all the vertices in Q_p and add a new vertex into V_{zy} (see Definition 4.7). By applying Proposition 3.2 with $d = 1$ and $G = Q_p \cup \{vu\}$, we obtain

$$|V_{i*} \cap V(Q_p)| - |V_{*i} \cap V(Q_p)| = \sum_{j \neq i} a^p_{ij} - \sum_{j \neq i} a^p_{ji} + \mathbb{1}\{i = z\} - \mathbb{1}\{i = y\}$$

for each $i \in [k]$ since $v \in V_{z*}$ and $u \in V_{*y}$. By considering the new vertex added into V_{zy}, we see that the contraction of the path Q_p leads to a decrease in $|V_{i*}| - |V_{*i}|$ by $\sum_{j \neq i} a^p_{ij} - \sum_{j \neq i} a^p_{ji}$. Since all the paths in $\mathcal{Q}$ can be contracted independently, we have

$$|V'_{i*}| - |V'_{*i}| = (|V_{i*}| - |V_{*i}|) - \sum_{p \in [t]} \left(\sum_{j \neq i} a^p_{ij} - \sum_{j \neq i} a^p_{ji} \right)$$

$$= (|V_{i*}| - |V_{*i}|) - \left(\sum_{j \neq i} a_{ij} - \sum_{j \neq i} a_{ji} \right).$$

Since we have $\sum_{j \neq i} a_{ij} - \sum_{j \neq i} a_{ji} = |V_{i*}| - |V_{*i}|$, the result follows. $\quad \square$

5. Hamilton cycles from partitions

The main goal of this section is to prove that regular directed or oriented graphs of suitably high degree that admit a (k^2, τ, γ)-partition for suitable k, τ, γ have a Hamilton cycle. We begin by formally defining certain contracted graphs associated with
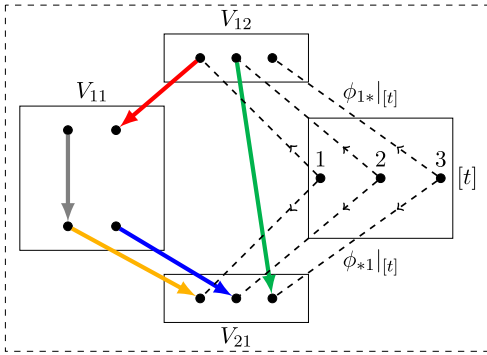
Partition $\mathcal{P}$ with $|V_{12}| = |V_{21}| = t > 0$, and a proper 1-pair $\phi^1 = (\phi_{1*}, \phi_{*1})$ with respect to $\mathcal{P}$
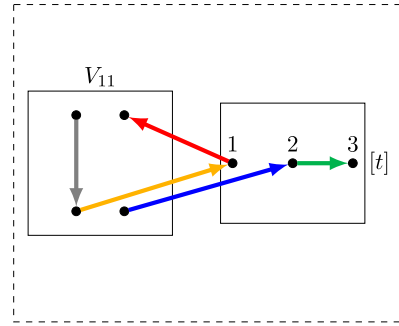
The digraph $\mathcal{J}^1(\mathcal{P}, G, \phi^1)$ on the vertex set $[t] \cup V_{11}$

Fig. 5. An illustration for how $\mathcal{J}^1(\mathcal{P}, G, \phi^1)$ is constructed.

4-partitions (i.e. the graphs J_i discussed in the sketch proof). These will be used in this and the next section.

Let H be a (undirected) bipartite graph with bipartition (A, B) and $|A| = |B| = n$. Given a set K of size n and bijections $\phi_A : K \to A$ and $\phi_B : K \to B$, the *identification of H with respect to (K, ϕ_A, ϕ_B)* is defined to be the digraph G, where $V(G) = K$ and for each $a, b \in K$, we have $ab \in E(G)$ if and only if $\phi_A(a)\phi_B(b) \in E(H)$.[7]

Let G be a digraph and $\mathcal{P} = \{V_{ij} : i, j \in [2]\}$ be a 4-partition of $V(G)$. For each $i \in [2]$, we define $\mathcal{B}^i(\mathcal{P}, G)$ to be the (undirected) bipartite graph with bipartition (V_{i*}, V_{*i}), where, for each $a \in V_{i*}$ and $b \in V_{*i}$, we have $ab \in E(\mathcal{B}^i(\mathcal{P}, G))$ if and only if $ab \in E(G)$. (Although V_{i*} and V_{*i} are not disjoint as subsets of $V(G)$, namely $V_{i*} \cap V_{*i} = V_{ii}$, we duplicate any vertices in V_{ii}, so $\mathcal{B}^i(\mathcal{P}, G)$ has $|V_{i*}| + |V_{*i}|$ vertices.)

Let G be a digraph and $\mathcal{P} = \{V_{ij} : i, j \in [2]\}$ be a 4-partition of $V(G)$ such that $|V_{12}| = |V_{21}| = t > 0$. For $i \in [2]$, we call $\phi^i = (\phi_{i*}, \phi_{*i})$ a *proper i-pair with respect to $\mathcal{P}$* if $\phi_{i*} : [t] \cup V_{ii} \to V_{i*}$ and $\phi_{*i} : [t] \cup V_{ii} \to V_{*i}$ are bijections satisfying $\phi_{i*}(x) = \phi_{*i}(x) = x$ for all $x \in V_{ii}$. In this case we define $\mathcal{J}^i(\mathcal{P}, G, \phi^i)$ to be the identification of $\mathcal{B}^i(\mathcal{P}, G)$ with respect to $([t] \cup V_{ii}, \phi_{i*}, \phi_{*i})$. Formally, $V(\mathcal{J}^i(\mathcal{P}, G, \phi^i)) = [t] \cup V_{ii}$ and $xy \in E(\mathcal{J}^i(\mathcal{P}, G, \phi^i))$ if and only if $\phi_{i*}(x)\phi_{*i}(y) \in E(\mathcal{B}^i(\mathcal{P}, G))$. One can think of $\mathcal{J}^i(\mathcal{P}, G, \phi^i)$ as the digraph obtained from $G[V_{i*} \cup V_{*i}]$ by pairing vertices in $V_{i*} \setminus V_{ii}$ with vertices in $V_{*i} \setminus V_{ii}$ and identifying them, where the pairing is determined by ϕ_{i*} and ϕ_{*i}; if we pair $x \in V_{i*} \setminus V_{ii}$ with $y \in V_{*i} \setminus V_{ii}$, the identified vertex has the same outneighbours as x and the same inneighbours as y. Note that there is a one-to-one correspondence between the edges in $\mathcal{J}^i(\mathcal{P}, G, \phi^i)$ and those in $G[V_{i*}, V_{*i}]$. Fig. 5 illustrates this construction by a small example.

---

[7] If $\phi_A(a)\phi_B(a) \in E(H)$ for some $a \in K$, then we will have a loop $aa \in E(G)$. The small number of loops in G play no role in our arguments, but we keep them for convenience so that H and G have the same number of edges.

The first proposition shows how Hamiltonicity of $\mathcal{J}^i$ translates into Hamiltonicity for G.

Proposition 5.1. *Let G be a digraph on n vertices, and let $\mathcal{P} = \{V_{ij} : i, j \in [2]\}$ be a 4-partition of $V(G)$ with $|V_{12}| = |V_{21}| > 0$. Suppose that for every $i \in [2]$ and every proper i-pair ϕ^i with respect to $\mathcal{P}$, we have that $\mathcal{J}^i(\mathcal{P}, G, \phi^i)$ is Hamiltonian. Then, G is Hamiltonian.*

Proof. Let $|V_{12}| = |V_{21}| = t$ and ϕ^1 be a proper 1-pair with respect to $\mathcal{P}$. Consider a Hamilton cycle C in $\mathcal{J}^1(\mathcal{P}, G, \phi^1)$. Recall that the vertex set of $\mathcal{J}^1(\mathcal{P}, G, \phi^1)$ is $[t] \cup V_{11}$. Let $p_1, \ldots, p_t$ be the order in which the vertices in $[t]$ are visited by C so that C can be partitioned into paths $P_1, \ldots P_t$ where P_r is a path from p_r to p_{r+1} (with the convention that $p_{t+1} = p_1$). Each P_r corresponds to a path P_r^1 in $G[V_{11} \cup V_{12} \cup V_{21}]$ from $\phi_{1*}(p_r) \in V_{12}$ to $\phi_{*1}(p_{r+1}) \in V_{21}$, and moreover the paths $P_1^1, \ldots, P_t^1$ are vertex-disjoint and span $V_{11} \cup V_{12} \cup V_{21}$.

Let ϕ^2 be the proper 2-pair with respect to $\mathcal{P}$ satisfying $\phi_{2*}(p_r) = \phi_{*1}(p_{r+1}) \in V_{21}$ and $\phi_{*2}(p_r) = \phi_{1*}(p_r) \in V_{12}$ for all $r \in [t]$. Note that $\mathcal{J}^2(\mathcal{P}, G, \phi^2)$ can be obtained from $G[V_{2*} \cup V_{*2}]$ by identifying the start and end points of P_r^1 for each r and calling the resulting vertex p_r (here we keep only the inedges of the start point $\phi_{1*}(p_r)$ and the outedges of the end point $\phi_{*1}(p_{r+1})$). Since $\mathcal{J}^2(\mathcal{P}, G, \phi^2)$ has some Hamilton cycle H, we see that G also has a Hamilton cycle, obtained by replacing each vertex p_r in H with the path P_r^1. $\square$

Next, we will prove that digraphs admitting a $(4, 1/3, \gamma)$-partition with additional degree conditions are Hamiltonian. Recall that for a k^2-partition $\mathcal{P}_k = \{V_{ij} : i, j \in [k]\}$ of $V(G)$, the set of good edges was defined as $\mathcal{G}_k(\mathcal{P}_k, G) = \bigcup_i E(V_{i*}, V_{*i})$ (see Definition 3.1), and we also think of $\mathcal{G}_k(\mathcal{P}_k, G)$ as the subdigraph of G with the vertex set consisting of those vertices incident to edges in $\mathcal{G}_k(\mathcal{P}_k, G)$.

Lemma 5.2. *Let $1/n \ll \gamma, \rho \ll \varepsilon \ll 1$ be constants. Let G be a digraph on n vertices with a $(4, 1/3, \gamma)$-partition $\mathcal{P} = \{V_{ij} : i, j \in [2]\}$. Suppose that*

(i) $d^+_{\mathcal{G}_2(\mathcal{P}, G)}(v), d^-_{\mathcal{G}_2(\mathcal{P}, G)}(v) \geq (1/3 + \varepsilon)n$ *holds for all but at most ρn vertices $v \in V(G)$,*
(ii) $\delta^0(\mathcal{G}_2(\mathcal{P}, G)) \geq n/20$,
(iii) $|V_{12}| = |V_{21}| > 0$.

Then G is Hamiltonian.

Proof. Let $|V_{12}| = |V_{21}| = t$. For $i \in [2]$, let ϕ^i be a proper i-pair with respect to $\mathcal{P}$. Let $J_i := \mathcal{J}^i(\mathcal{P}, G, \phi^i)$. Since $|V(J_i)| = |V_{i*}| \geq n/3$ (the inequality holds because $\mathcal{P}$ is a $(4, 1/3, \gamma)$-partition), we obtain $n/3 \leq |J_i| \leq 2n/3$. On the other hand, for any $v \in V_{ii}$, we have $d^+_{J_i}(v) = d^+_{\mathcal{G}_2(\mathcal{P}, G)}(v)$ and $d^-_{J_i}(v) = d^-_{\mathcal{G}_2(\mathcal{P}, G)}(v)$. Similarly, for

any $r \in [t]$, we have $d^+_{J_i}(r) = d^+_{\mathcal{G}_2(\mathcal{P},G)}(\phi_{i*}(r))$ and $d^-_{J_i}(r) = d^-_{\mathcal{G}_2(\mathcal{P},G)}(\phi_{*i}(r))$. Then, $d^+_{J_i}(x), d^-_{J_i}(x) \geq (1/2 + \varepsilon)|J_i|$ holds for all but at most $3\rho|J_i|$ vertices x in J_i by (i). Moreover, (ii) implies $\delta^0(J_i) \geq |J_i|/20$. Therefore, J_i is Hamiltonian for $i \in [2]$ by Corollary 2.8. Hence, the result follows from Proposition 5.1. □

We end this section by showing that every regular oriented graph of sufficiently high degree that admits a $(9, \tau, \gamma)$-partition is Hamiltonian.

Lemma 5.3. *Let $1/n < \gamma \ll \tau \ll \varepsilon < 1$ be constants. Then every d-regular oriented graph G on n vertices with $d \geq (1/4 + \varepsilon)n$ and that admits a $(9, \tau, \gamma)$-partition is Hamiltonian.*

Proof. Let $\mathcal{P} = \{V_{ij} : i, j \in [3]\}$ be an extremal $(9, \tau, \gamma)$-partition of G. Firstly, we claim at least two of the following are true:

(a) $\tau n + |V_{11}| \leq |V_{22}| + |V_{33}| + |V_{23}| + |V_{32}|$,
(b) $\tau n + |V_{22}| \leq |V_{33}| + |V_{11}| + |V_{31}| + |V_{13}|$,
(c) $\tau n + |V_{33}| \leq |V_{11}| + |V_{22}| + |V_{12}| + |V_{21}|$.

If not, then without loss of generality, say (a) and (b) are false. By adding up those inequalities, we obtain $2\tau n > 2|V_{33}| + |V_{31} \cup V_{32}| + |V_{13} \cup V_{23}|$. However, by Proposition 3.11, we know $|V_{31} \cup V_{32}|, |V_{13} \cup V_{23}| \geq \tau n$, so we have a contradiction. Similarly, it can be easily shown that at least two of the following are true:

$\quad$ (a′) $\tau n \leq |V_{12}| + |V_{21}|$, $\qquad$ (b′) $\tau n \leq |V_{13}| + |V_{31}|$, $\qquad$ (c′) $\tau n \leq |V_{23}| + |V_{32}|$.

Thus, without loss of generality, we can assume that (c) and (a′) hold, that is,

$$\tau n + |V_{33}| \leq |V_{11}| + |V_{22}| + |V_{12}| + |V_{21}| \text{ and } \tau n \leq |V_{12}| + |V_{21}|. \tag{5.1}$$

By Lemma 4.6, there exists a path system $\mathcal{Q}$ in $\mathcal{B}_3(\mathcal{P}, G)$ containing at most $8\gamma n$ edges such that $\sum_{j \neq i} a_{ij} - \sum_{j \neq i} a_{ji} = |V_{i*}| - |V_{*i}|$ for all $i \in [3]$, where $a_{ij} = |E(V_{i*}, V_{*j}) \cap \mathcal{Q}|$. We contract $\mathcal{Q}$ with respect to $\mathcal{P}$ and write G' for the resulting graph and $\mathcal{P}' = \{V'_{ij} : i, j \in [3]\}$ for the resulting partition. By Proposition 3.8, $\mathcal{P}$ is actually a $(9, 1/4 + \varepsilon/2, \gamma)$-partition of G, so Proposition 4.9 implies that $\mathcal{P}'$ is a $(9, 1/8, 2\gamma)$-partition for G'. By Lemma 4.10, we have

$$|V'_{i*}| = |V'_{*i}| \geq |V_{*i}| - |V(\mathcal{Q})| \geq n/4 \text{ for all } i \in [3]. \tag{5.2}$$

Moreover, by using Proposition 3.11, we have

$$\sum_{j \neq i} |V'_{ij}| = \sum_{j \neq i} |V'_{ji}| \geq \tau n \text{ for all } i \in [3]. \tag{5.3}$$

Also, using (5.1) and the facts that $\gamma \ll \tau$ and $e(\mathcal{Q}) \leq 8\gamma n$, we have

$$|V'_{33}| \leq |V'_{11}| + |V'_{22}| + |V'_{12}| + |V'_{21}| \text{ and } |V'_{12}| + |V'_{21}| \geq \tau n/2. \tag{5.4}$$

Since $|V(\mathcal{Q})| \leq 16\gamma n$, we have

$$\delta^0(G') \geq d - 16\gamma n \geq (1/4 + \varepsilon/2)\, n. \tag{5.5}$$

Similarly, by Proposition 3.10,[8]

$$\begin{aligned} &\text{for any } v \in V(G'), \text{ if } v \in V'_{ab} \text{ for some } a, b \in [3], \\ &\text{then } d^+_{V'_{*a}}(v), d^-_{V'_{b*}}(v) \geq d/3 - 16\gamma n. \end{aligned} \tag{5.6}$$

In other words, we have

$$d^+_{\mathcal{G}_3(\mathcal{P}', G')}(v), d^-_{\mathcal{G}_3(\mathcal{P}', G')}(v) \geq d/3 - 16\gamma n. \tag{5.7}$$

Let

$$\begin{aligned} W_{11} &= V'_{33}, & W_{12} &= V'_{32} \cup V'_{31}, \\ W_{21} &= V'_{23} \cup V'_{13}, & W_{22} &= V'_{11} \cup V'_{22} \cup V'_{12} \cup V'_{21}. \end{aligned}$$

By Proposition 3.7, we have $\mathcal{W} = \{W_{ij} : i, j \in [2]\}$ is a $(4, 1/8, 2\gamma)$-partition for G'. Furthermore, (5.4) and (5.3) imply that

$$|W_{11}| \leq |W_{22}| \text{ and } |W_{12}| = |W_{21}| \geq \tau n/2.$$

By Proposition 4.8, if G' is Hamiltonian then so is G. For $i \in [2]$, let ϕ^i be a proper i-pair with respect to $\mathcal{W}$. In order to prove the lemma, it is enough to show that $J_i := \mathcal{J}^i(\mathcal{W}, G', \phi^i)$ is Hamiltonian for $i \in [2]$ by Proposition 5.1.

First, for J_1, (5.2) and the fact that $|W_{11}| \leq |W_{22}|$ imply that

$$n/4 \leq |V'_{3*}| = |J_1| \leq |G'|/2 \leq n/2. \tag{5.8}$$

Let $B^+(J_1)$ be the set of vertices in J_1 satisfying $d^+_{J_1}(x) < (1/2+\varepsilon/2)|J_1|$. Similarly, define $B^-(J_1)$. For any vertex $x \in V(J_1)$, we have $\phi_{1*}(x) \in V'_{3*}$ and $d^+_{J_1}(x) = d^+_{V'_{*3}}(\phi_{1*}(x))$. Together with (5.5) and (5.8), we deduce that

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[8] It is clear that all the vertices of G' inherited from G satisfy these degree conditions; for the new vertices in G' (created from contracting paths), one can easily check in the definition of contraction that the vertices are placed in such a way that the degree conditions hold.

$$\begin{aligned}
2\gamma n^2 \geq |\mathcal{B}_2(\mathcal{W}, G')| &\geq e(\phi_{1*}(B^+(J_1)), V(G') \setminus V'_{*3}) \\
&\geq \sum_{x \in B^+(J_1)} \left(d^+_{G'}(\phi_{1*}(x)) - d^+_{V'_{*3}}(\phi_{1*}(x)) \right) \\
&\geq \sum_{x \in B^+(J_1)} \left(\delta^0(G') - d^+_{J_1}(x) \right) \geq \sum_{x \in B^+(J_1)} ((1/4 + \varepsilon/2)n - (1/2 + \varepsilon/2)|J_1|) \\
&\geq |B^+(J_1)|\varepsilon n/4.
\end{aligned}$$

So $|B^+(J_1)| \leq 8\gamma n/\varepsilon$ and, similarly, $|B^-(J_1)| \leq 8\gamma n/\varepsilon$. By (5.8),

$$|B^+(J_1)| + |B^-(J_1)| \leq 16\gamma n/\varepsilon \leq 64\gamma|J_1|/\varepsilon \leq \sqrt{\gamma}|J_1|.$$

Thus $d^+_{J_1}(x), d^-_{J_1}(x) \geq (1/2 + \varepsilon/2)|J_1|$ holds for all but at most $\sqrt{\gamma}|J_1|$ vertices. Also, by (5.6), we have $\delta^0(J_1) \geq d/3 - 16\gamma n \geq |J_1|/10$. Therefore, by Corollary 2.8, J_1 is Hamiltonian.

For J_2, we first show that J_2 has a $(4, 1/3, 8\gamma)$-partition. By (5.8)

$$n/2 \leq |J_2| = |G'| - |V'_{3*}| \leq 3n/4, \tag{5.9}$$

so (5.2) implies that

$$|V'_{*i}| = |V'_{i*}| \geq n/4 \geq |J_2|/3 \text{ for } i \in [2].$$

Let $t := |W_{12}| = |W_{21}|$. Recall that $\phi_{2*} : [t] \cup W_{22} \to W_{2*}$ and $\phi_{*2} : [t] \cup W_{22} \to W_{*2}$ are bijections satisfying $\phi_{2*}(x) = \phi_{*2}(x) = x$ for all $x \in W_{22}$, so we have $\phi_{2*}(q) \in V'_{32} \cup V'_{31}$ and $\phi_{*2}(q) \in V'_{23} \cup V'_{13}$ for any $q \in [t]$ since $W_{12} = V'_{32} \cup V'_{31}$ and $W_{21} = V'_{23} \cup V'_{13}$. Then, we partition $[t]$ into parts $\{T_{ij} : i, j \in [2]\}$ as follows:

$$\begin{aligned}
T_{11} &= \{q \in [t] : \phi_{2*}(q) \in V'_{13}, \ \phi_{*2}(q) \in V'_{31}\}, \\
T_{12} &= \{q \in [t] : \phi_{2*}(q) \in V'_{13}, \ \phi_{*2}(q) \in V'_{32}\}, \\
T_{21} &= \{q \in [t] : \phi_{2*}(q) \in V'_{23}, \ \phi_{*2}(q) \in V'_{31}\}, \\
T_{22} &= \{q \in [t] : \phi_{2*}(q) \in V'_{23}, \ \phi_{*2}(q) \in V'_{32}\}.
\end{aligned}$$

Then, let us write $Z_{ij} = V'_{ij} \cup T_{ij}$ for $i, j \in [2]$, and $\mathcal{Z} = \{Z_{ij} : i, j \in [2]\}$. Notice that $\mathcal{Z}$ is a partition of $V(J_2)$. By using $|T_{11}| + |T_{12}| = |V'_{13}|$, we deduce that $|Z_{1*}| = |V'_{1*}| \geq |J_2|/3$. More generally, for $i \in \{1, 2\}$

$$|Z_{i*}| = |Z_{*i}| = |V'_{i*}| \geq |J_2|/3.$$

Note that $Z_{12} \cup Z_{21} \supseteq V'_{12} \cup V'_{21} \neq \emptyset$ by (5.4). We deduce that $|Z_{12}| = |Z_{21}| > 0$. On the other hand, for $i \in \{1, 2\}$, we have $Z_{i*} = V'_{i1} \cup V'_{i2} \cup \{q \in [t] : \phi_{2*}(q) \in V'_{i3}\}$ and $Z_{*i} = V'_{1i} \cup V'_{2i} \cup \{q \in [t] : \phi_{*2}(q) \in V'_{3i}\}$. Since $\phi_{2*}(x) = \phi_{*2}(x) = x$ for all

$x \in V'_{11} \cup V'_{12} \cup V'_{21} \cup V'_{22}$, we see that $\phi_{2*}(V'_{i1} \cup V'_{i2}) = V'_{i1} \cup V'_{i2}$ and $\phi_{*2}(V'_{1i} \cup V'_{2i}) = V'_{1i} \cup V'_{2i}$. Therefore,

$$xy \in E(Z_{i*}, Z_{*j}) \quad \text{if and only if}$$
$$\phi_{2*}(x)\phi_{*2}(y) \in E(V'_{i*}, V'_{*j}) \quad \text{for all } i, j \in [2]. \tag{5.10}$$

Then, we have

$$e(Z_{i*}, Z_{*j}) = e(V'_{i*}, V'_{*j}) \text{ for } i, j \in [2]. \tag{5.11}$$

Hence, we obtain

$$|\mathcal{B}_2(\mathcal{Z}, J_2)| = e(V'_{1*}, V'_{*2}) + e(V'_{2*}, V'_{*1}) \leq |\mathcal{B}_3(\mathcal{P}', G')| \leq 2\gamma|G'|^2 \leq 8\gamma|J_2|^2.$$

As a result, $\mathcal{Z}$ is a $(4, 1/3, 8\gamma)$-partition for J_2 with $|Z_{12}| = |Z_{21}| > 0$.

Let $B^+(J_2)$ be the set of vertices in J_2 satisfying $d^+_{\mathcal{G}_2(\mathcal{Z}, J_2)}(x) < (1/3 + \varepsilon/3)|J_2|$. Similarly, define $B^-(J_2)$. Note that $d^+_{\mathcal{G}_2(\mathcal{Z}, J_2)}(x) = d^+_{\mathcal{G}_3(\mathcal{P}', G')}(\phi_{2*}(x))$ for any vertex $x \in V(J_2)$ by (5.10). Moreover, by (5.5) and (5.9), we have $\delta^0(G') \geq (1/3 + \varepsilon/2)|J_2|$. Hence, by (5.9), for any vertex $x \in B^+(J_2)$, we obtain

$$d^+_{\mathcal{B}_3(\mathcal{P}', G')}(\phi_{2*}(x)) > (1/3 + \varepsilon/2)|J_2| - (1/3 + \varepsilon/3)|J_2| \geq \varepsilon|J_2|/6 \geq \varepsilon n/12.$$

Since $|\mathcal{B}_3(\mathcal{P}', G')| \leq |\mathcal{B}_3(\mathcal{P}, G)| \leq \gamma n^2$, we find $\gamma n^2 \geq |B^+(J_2)|\varepsilon n/12$. So $|B^+(J_2)| \leq 12\gamma n/\varepsilon$ and, similarly, $|B^-(J_2)| \leq 12\gamma n/\varepsilon$. As a result, by (5.9), we have

$$|B^+(J_2)| + |B^-(J_2)| \leq 24\gamma n/\varepsilon \leq 48\gamma|J_2|/\varepsilon \leq \sqrt{\gamma}|J_2|.$$

On the other hand, by (5.7) and (5.9), for any vertex $x \in V(J_2)$, we obtain

$$d^+_{\mathcal{G}_2(\mathcal{Z}, J_2)}(x) = d^+_{\mathcal{G}_3(\mathcal{P}', G')}(\phi_{2*}(x)) \geq d/3 - 16\gamma n \geq |J_2|/20.$$

Similarly, we have $d^-_{\mathcal{G}_2(\mathcal{Z}, J_2)}(x) \geq |J_2|/20$. As a result, the partition $\mathcal{Z}$ for the digraph J_2 satisfies all the conditions of Lemma 5.2, so we are done. $\square$

6. Proofs of main results

In this section, we give the proofs of Theorems 1.5 and 1.3.

Proof of Theorem 1.5. Let $\varepsilon > 0$ be a constant. Let G be a strongly well-connected d-regular digraph on n (sufficiently large) vertices with $d \geq (1/3 + \varepsilon)n$. We will show that G is Hamiltonian. Let ν and τ be constants satisfying $1/n \ll \nu \ll \tau \ll \varepsilon$. If G is a robust (ν, τ)-outexpander, then we are done by Theorem 2.4. Assume not. Then, G admits a

$(4, \tau, 4\nu)$-partition by Lemma 3.6. Let $\mathcal{P} = \{V_{ij} : i, j \in [2]\}$ be an extremal $(4, \tau, 4\nu)$-partition for G. Notice that $|V_{1*}|, |V_{2*}| \geq (1/3 + \varepsilon/2)n$ by Proposition 3.8. Without loss of generality, assume $|V_{12}| \geq |V_{21}|$. We will choose a path system $\mathcal{Q}$ in $\mathcal{B}_2(\mathcal{P}, G)$ satisfying

$$|E(\mathcal{Q}) \cap E(V_{1*}, V_{*2})| - |E(\mathcal{Q}) \cap E(V_{2*}, V_{*1})| = |V_{12}| - |V_{21}|$$

as follows.

(i) If $V_{12} = V_{21} = \emptyset$, then $|V_{11}|, |V_{22}| \geq (1/3 + \varepsilon/2)n$. Since G is strongly well-connected, we can find disjoint edges $ab \in E(V_{11}, V_{22})$ and $cd \in E(V_{22}, V_{11})$. Then, set $\mathcal{Q} = \{ab, cd\}$.

(ii) If $|V_{12}| \geq |V_{21}| > 0$, then $d(|V_{12}| - |V_{21}|) = e(V_{1*}, V_{*2}) - e(V_{2*}, V_{*1})$ by Proposition 3.2. Hence, we have $e(V_{1*}, V_{*2}) \geq d(|V_{12}| - |V_{21}|)$. By Proposition 3.10, $E(V_{1*}, V_{*2})$ induces a subgraph H in G with $\Delta^0(H) \leq d/2$. Since $e(V_{1*}, V_{*2}) \leq 4\nu n^2$, by Lemma 4.3, we can find a path system $\mathcal{Q}'$ in H with $e(\mathcal{Q}') \geq 2e(V_{1*}, V_{*2})/d \geq 2(|V_{12}| - |V_{21}|)$. Then, we remove all but exactly $|V_{12}| - |V_{21}|$ edges in $\mathcal{Q}'$ to obtain $\mathcal{Q}$.

(iii) If $|V_{12}| \geq 2$ and $|V_{21}| = 0$, then as with the previous case, $E(V_{1*}, V_{*2})$ has a path system $\mathcal{Q}'$ containing $2|V_{12}|$ edges. We claim $\mathcal{Q}'$ has at least one path that starts in V_{11} and ends in V_{22}. If not, then any path in $\mathcal{Q}'$, with s edges say, is incident to at least s vertices in V_{12}, but since $\mathcal{Q}'$ contains more than $|V_{12}|$ edges, we have a contradiction. Next we claim that any path in $\mathcal{Q}'$ from V_{11} to V_{22} has at most $|V_{12}|$ edges. Indeed, if not, then $\mathcal{Q}'$ has a unique path which has $|V_{12}| + 1$ edges. But then $\mathcal{Q}'$ has $2|V_{12}| = |V_{12}| + 1$ edges, contradicting $|V_{12}| \geq 2$. Using the claims, we can remove all but exactly $|V_{12}|$ edges in $\mathcal{Q}'$ to obtain a path system $\mathcal{Q}$ with exactly $|V_{12}| = |V_{12}| - |V_{21}|$ edges and where at least one path starts in $|V_{11}|$ and ends in $|V_{22}|$.

(iv) If $|V_{12}| = 1$ and $|V_{21}| = 0$, let x be the unique vertex in V_{12}. By Proposition 3.2, we have $d = d^-_{V_{11}}(x) + d^+_{V_{22}}(x) + e(V_{11}, V_{22}) - e(V_{22}, V_{11})$. Note that, by Proposition 3.10, we know $d^-_{V_{11}}(x), d^+_{V_{22}}(x) \leq d/2$. If $d^-_{V_{11}}(x) = d^+_{V_{22}}(x) = d/2$, then we obtain another extremal $(4, \tau, 4\nu)$-partition by moving x into V_{11}, which results in case (i). If we have either $d^-_{V_{11}}(x) < d/2$ or $d^+_{V_{22}}(x) < d/2$, then we have $e(V_{11}, V_{22}) \geq 1$. We can take an arbitrary edge $ab \in E(V_{11}, V_{22})$, and set $\mathcal{Q} = \{ab\}$.

Now we contract this path system $\mathcal{Q}$ in G with respect to partition $\mathcal{P}$ to obtain a graph G' with resulting partition $\mathcal{P}' = \{V'_{ij} : i, j \in [2]\}$. By Lemma 4.10, we have $|V'_{12}| = |V'_{21}|$. Moreover, the choice of $\mathcal{Q}$ ensures that both V'_{12} and V'_{21} are nonempty as follows: In cases (i), (iii), and (iv) we include at least one path from V_{11} to V_{22} so that the vertex created when contracting this path is placed in V'_{21}; see Definition 4.7. In case (ii), V_{21} is nonempty and we do not use any vertices from V_{21} in the path system. Therefore, V'_{12} is nonempty after the contraction, which also means that V'_{12} is nonempty as $|V'_{12}| = |V'_{21}|$. We note that $\mathcal{Q}$ has at most $12\nu n$ edges since, by construction, $\mathcal{Q}$ has at most $||V_{12}| - |V_{21}||$

edges (except in case (i) where $\mathcal{Q}$ has two edges) and $|V_{12}| - |V_{21}| \leq 12\nu n$ by Corollary 3.3. Therefore, we delete at most $24\nu n$ vertices, which implies $\delta^0(G') \geq d - 24\nu n$. On the other hand, by Proposition 4.9, we have that $\mathcal{P}'$ is a $(4, \tau/2, 8\nu)$-partition. Also, by Proposition 3.8, we have

$$|V'_{i*}| \geq |V_{i*}| - 24\nu n \geq (1/3 + \varepsilon - 24\nu)n \geq |G'|/3$$

for $i \in [2]$. Similarly, we obtain $|V'_{*i}| \geq |G'|/3$, so $\mathcal{P}'$ is a $(4, 1/3, 8\nu)$-partition. Let $B^+(G')$ be the set of vertices in G' satisfying $d^+_{\mathcal{G}_2(\mathcal{P}', G')}(x) < (1/3 + \varepsilon/3)|G'|$. Similarly, define $B^-(G')$. Note that $\delta^0(G') \geq d - 24\nu n \geq (1/3 + \varepsilon/2)|G'|$. Hence, for any vertex $x \in B^+(G')$, we obtain

$$d^+_{\mathcal{B}_2(\mathcal{P}', G')}(x) > (1/3 + \varepsilon/2)|G'| - (1/3 + \varepsilon/3)|G'| \geq \varepsilon|G'|/6.$$

Since $|\mathcal{B}_2(\mathcal{P}', G')| \leq 8\nu|G'|^2$, we have $8\gamma|G'|^2 \geq |B^+(G')| \cdot \varepsilon|G'|/6$. So, $|B^+(G')| \leq 48\nu|G'|/\varepsilon$ and, similarly, $|B^-(G')| \leq 48\nu|G'|/\varepsilon$. As a result, we obtain

$$|B^+(G')| + |B^-(G')| \leq 96\nu|G'|/\varepsilon \leq \sqrt{\nu}|G'|.$$

Moreover, since $\mathcal{P}$ is an extremal $(4, \tau, 4\nu)$-partition and we deleted at most $24\nu n$ vertices, Proposition 3.10 implies that $d^+_{\mathcal{G}_2(\mathcal{P}', G')}(v), d^-_{\mathcal{G}_2(\mathcal{P}', G')}(v) \geq d/2 - 24\nu n \geq |G'|/20$ for all $v \in V(G')$. As a result, $\mathcal{P}'$ satisfies the properties in Lemma 5.2, so G' is Hamiltonian. Hence, the result follows by Proposition 4.8. $\quad\square$

Proof of Theorem 1.3. Let $\varepsilon > 0$ be a constant. Let G be a d-regular oriented graph on n (sufficiently large) vertices with $d \geq (1/4 + \varepsilon)n$. We will show that G is Hamiltonian. Fix constants ν and τ satisfying $1/n \ll \nu \ll \tau \ll \varepsilon$. By Theorem 2.4, we are done if G is a robust (ν, τ)-outexpander. Assume not. Then, by Lemma 3.6, G admits an extremal $(4, \tau, 4\nu)$-partition $\mathcal{P} = \{V_{ij} : i, j \in [2]\}$. Then, for each $i \in [2]$,

$$|V_{i*}|, |V_{*i}| \geq (1/4 + \varepsilon/2)n \tag{6.1}$$

by Proposition 3.8. Also, we have $|V_{12}|, |V_{21}| \geq \tau n$ by Proposition 3.11. Without loss of generality, assume $|V_{11}| \leq |V_{22}|$. Furthermore, by reversing the edges if necessary, we may assume that $|V_{12}| \geq |V_{21}|$. Let $r = |V_{12}| - |V_{21}|$. By Corollary 3.3 and the fact that $|\mathcal{B}_2(\mathcal{P}, G)| \leq 4\nu n^2$, we obtain

$$r \leq 4\nu n^2/d \leq 16\nu n. \tag{6.2}$$

Fix a subset R of V_{12} of size r. Let $\mathcal{W} = \{W_{ij} : i, j \in [2]\}$ where $W_{ij} = V_{ij} \setminus R$ for $i, j \in [2]$. Note that $|V_{11}| \leq |V_{22}|$ and $|V_{12}| \geq |V_{21}|$ imply that $|V_{*2}| \geq n/2$. Hence

$$|W_{2*}| = |W_{*2}| \geq (n - r)/2 \tag{6.3}$$

$$\text{and } |W_{1*}| = |W_{*1}| \geq |V_{21}| \geq \tau n. \tag{6.4}$$

We now split into cases depending on whether, for all proper 2-pairs ϕ^2 with respect to $\mathcal{W}$, the digraph $\mathcal{J}^2(\mathcal{W}, G - R, \phi^2)$ is a robust $(\nu^{1/2}, \tau)$-outexpander or not.

$\underline{\text{Case 1:}}$ Suppose that, for all 2-pairs ϕ^2 with respect to $\mathcal{W}$, $\mathcal{J}^2(\mathcal{W}, G - R, \phi^2)$ is a robust $(\nu^{1/2}, \tau)$-outexpander. Recall that $G_{ij} = G[V_{i*}, V_{*j}]$. We have by Proposition 3.2 that

$$e(G_{12}) \geq e(G_{12}) - e(G_{21}) = d(|V_{12}| - |V_{21}|) = dr.$$

By Proposition 3.10, $\Delta^0(G_{12}) \leq d/2$. Moreover, we have $e(G_{12}) \leq |\mathcal{B}_2(\mathcal{P}, G)| \leq 4\nu n^2$. Hence, by Lemma 4.3, G_{12} has a path system $\mathcal{Q}$ with r edges.

We contract $\mathcal{Q}$ in G with respect to $\mathcal{P}$ to obtain G' with resulting partition $\mathcal{P}' = \{V'_{ij} : i, j \in [2]\}$. Since $|E(\mathcal{Q}) \cap E(G_{12})| - |E(\mathcal{Q}) \cap E(G_{21})| = |V_{12}| - |V_{21}|$, Lemma 4.10 implies that $|V'_{12}| = |V'_{21}|$. Moreover, by Proposition 3.11, we have $|V_{12}|, |V_{21}| \geq \tau n$. Since $r \leq 16\nu n \ll \tau n$, we conclude that $|V'_{12}| = |V'_{21}| > 0$. By Proposition 4.8, it is enough to show that G' is Hamiltonian.

Consider a proper i-pair ψ^i with respect to $\mathcal{P}'$ for $i \in [2]$. Let $\mathcal{J}_i = \mathcal{J}^i(\mathcal{P}', G', \psi^i)$. To show that G' is Hamiltonian, by Proposition 5.1, it suffices to show that $\mathcal{J}_1$ and $\mathcal{J}_2$ are Hamiltonian.

We first prove that $\mathcal{J}_2$ is a robust $(\nu^{1/2}/2, 2\tau)$-outexpander by showing it is a small perturbation of $\mathcal{J}^2(\mathcal{W}, G - R, \phi^2)$ for a suitable proper 2-pair ϕ^2 with respect to $\mathcal{W}$, chosen as follows. Let $t = |W_{12}|$ and $t' = |V'_{12}|$. Recall that ψ^2 is a function from $[t'] \cup V'_{22}$ to $V'_{2*} \times V'_{*2}$. Pick ϕ^2 among all proper 2-pairs with respect to $\mathcal{W}$ such that $|X|$ is as large as possible where X is the set of $x \in ([t] \cup W_{22}) \cap ([t'] \cup V'_{22})$ satisfying $\phi^2(x) = \psi^2(x)$. We define $\mathcal{J}^2 = \mathcal{J}^2(\mathcal{W}, G - R, \phi^2)$.

We have that $V(\mathcal{J}_2) = [t'] \cup V'_{22}$ and $V(\mathcal{J}^2) = [t] \cup W_{22}$ and $X \subseteq V(\mathcal{J}_2) \cap V(\mathcal{J}^2)$. Moreover $\mathcal{J}_2[X] = \mathcal{J}^2[X]$; to see this, note that for $Y := \phi^2(X) = \psi^2(X) \subseteq V(G)$, the partitions $\mathcal{P}'$ and $\mathcal{W}$ are the same on $G'[Y] = (G - R)[Y]$.

First note that

$$|\mathcal{J}^2| = |W_{*2}| \geq (n - r)/2.$$

Since

$$|W_{2*} \triangle V'_{2*}|, |W_{*2} \triangle V'_{*2}| \leq 3e(\mathcal{Q}) + |R| = 4r,$$

we deduce that

$$|([t] \cup W_{22}) \setminus X|, |([t'] \cup V'_{22}) \setminus X| \leq 16r.$$

Therefore

$$|V(\mathcal{J}_2) \triangle V(\mathcal{J}^2)| \leq |([t] \cup W_{22}) \setminus X| + |([t'] \cup V'_{22}) \setminus X|$$
$$\leq 32r \leq 512\nu n \leq \nu^{1/2}|\mathcal{J}^2|/2.$$

Since $\mathcal{J}^2$ is a robust $(\nu^{1/2}, \tau)$-outexpander by assumption, we conclude that $\mathcal{J}_2$ is a robust $(\nu^{1/2}/2, 2\tau)$-outexpander by Lemma 2.5 (where $\mathcal{J}_2 \cup \mathcal{J}^2$ plays the role of G). Also, by Proposition 3.10, we know that $d^+_{\mathcal{G}_2(\mathcal{P}, G)}, d^-_{\mathcal{G}_2(\mathcal{P}, G)} \geq d/2$. Then, since $e(\mathcal{Q}) = r$, we have $d^+_{\mathcal{G}_2(\mathcal{P}', G')}(v), d^-_{\mathcal{G}_2(\mathcal{P}', G')}(v) \geq d/2 - 2r$ for all $v \in V(G')$, which shows

$$\delta^0(\mathcal{J}_2) \geq d/2 - 2r \geq n/10 \geq |\mathcal{J}_2|/10$$

by using $d \geq (1/4 + \varepsilon)n$, $r \leq 16\nu n$ and $\nu \ll \varepsilon$. Hence, $\mathcal{J}_2$ is Hamiltonian by Theorem 2.4.

We now show that $\mathcal{J}_1$ is Hamiltonian. By (6.1) and (6.2), we have $n/4 \leq |V_{1*}| - 2r \leq |V'_{1*}| = |\mathcal{J}_1|$. Also, since $|V_{11}| \leq |V_{22}|$ and $|V_{12}| - |V_{21}| = r$, we have $|V'_{1*}| \leq |V_{1*}| + r \leq (n+r)/2 + r \leq (1/2 + \tau)n$ as $\nu \ll \tau$. Then, we obtain

$$n/4 \leq |V'_{1*}| = |\mathcal{J}_1| \leq (1/2 + \tau)n.$$

Similarly as above, by Proposition 3.10, we have $\delta^0(\mathcal{J}_1) \geq d/2 - 2r \geq |\mathcal{J}_1|/10$. By Proposition 4.9, $\mathcal{P}'$ is a $(4, \tau/2, 8\nu)$-partition of G'. Also

$$e(\mathcal{J}_1) = e(V'_{1*}, V'_{*1}) \geq \delta^0(G')|V'_{1*}| - |\mathcal{B}_2(\mathcal{P}', G')|$$
$$\geq (d - 2r)|V'_{1*}| - 8\nu n^2 \geq (d - 64\nu n)|\mathcal{J}_1|$$

as $|V'_{1*}| = |\mathcal{J}_1| \geq n/4$. Let $B^+(\mathcal{J}_1)$ be the set of vertices in $\mathcal{J}_1$ satisfying $d^+_{\mathcal{J}_1}(v) < d - \varepsilon n/4$. Similarly define $B^-(\mathcal{J}_1)$. Since $\Delta^0(\mathcal{J}_1) \leq d$, we obtain

$$\left(|\mathcal{J}_1| - |B^+(\mathcal{J}_1)|\right) d + |B^+(\mathcal{J}_1)| (d - \varepsilon n/4) \geq e(\mathcal{J}_1) \geq (d - 64\nu n)|\mathcal{J}_1|,$$

which implies $64\nu n|\mathcal{J}_1| \geq |B^+(\mathcal{J}_1)|\varepsilon n/4$. Hence, we have $|B^+(\mathcal{J}_1)| \leq 256\nu|\mathcal{J}_1|/\varepsilon$, and similarly, $|B^-(\mathcal{J}_1)| \leq 256\nu|\mathcal{J}_1|/\varepsilon$. As a result, we obtain

$$|B^+(\mathcal{J}_1)| + |B^-(\mathcal{J}_1)| \leq 512\nu|\mathcal{J}_1|/\varepsilon \leq \sqrt{\nu}|\mathcal{J}_1|$$

as $\nu \ll \varepsilon$. Hence, for all but at most $\sqrt{\nu}|\mathcal{J}_1|$ vertices $v \in V(\mathcal{J}_1)$, we have

$$d^+_{\mathcal{J}_1}(v), d^-_{\mathcal{J}_1}(v) \geq (d - \varepsilon n/4) \geq (1/2 + \varepsilon/3)|\mathcal{J}_1|.$$

Therefore, $\mathcal{J}_1$ satisfies the conditions of Corollary 2.8, so it is Hamiltonian.

<u>Case 2:</u> Suppose that there exists a 2-pair $\phi^2 = (\phi_{2*}, \phi_{*2})$ with respect to $\mathcal{W}$ such that $\mathcal{J}^2(\mathcal{W}, G - R, \phi^2)$ is not a robust $(\nu^{1/2}, \tau)$-outexpander. Let $\mathcal{J}^2 = \mathcal{J}^2(\mathcal{W}, G - R, \phi^2)$. We now show that there is a $(9, \tau/6, 20\nu^{1/2})$-partition for G (so that we can apply Lemma 5.3).

Note that it suffices to show that $G - R$ admits a $(9, \tau/3, 10\nu^{1/2})$-partition since $|R| = r \leq 16\nu n$ (so we can arbitrarily add the vertices of R into those 9 parts, which would cause a small amount of increase in the number of bad edges). Recall that $\mathcal{J}^2$ is a digraph on $[t] \cup W_{22}$ where $t = |W_{12}| = |W_{21}|$, and $\phi_{2*} : [t] \cup W_{22} \to W_{2*}$, $\phi_{*2} : [t] \cup W_{22} \to W_{*2}$

are bijections satisfying $\phi_{2*}(x) = \phi_{*2}(x) = x$ for all $x \in W_{22}$. First we show $\mathcal{J}^2$ is almost regular, so it admits a $(4, \tau, 4\gamma^{1/2})$-partition by using Lemma 3.6 since we assumed it is not a robust $(\nu^{1/2}, \tau)$-outexpander. Note that any partition $\{U_{ij} : i, j \in [2]\}$ of $V(\mathcal{J}^2)$ also gives a 4-partition for W_{22}. Similarly, $\{U_{ij} : i, j \in [2]\}$ partitions W_{12} (resp. W_{21}) into 2 parts depending on $y \in U_{1*}$ or $y \in U_{2*}$ (resp. $y \in U_{*1}$ or $y \in U_{*2}$) for each $y \in [t]$, so we obtain a 9-partition of $G - R$. Then we show bad edges in this 9-partition (almost) correspond to $\mathcal{B}_2(\mathcal{W}, G - R)$, so we can find an upper bound for the number of them.

Let $\theta = d/|W_{2*}|$. Remove any loops in $\mathcal{J}^2$. Notice that we have $|\mathcal{J}^2| = |W_{2*}|$ and $\Delta^0(\mathcal{J}^2) \leq d = \theta|W_{2*}|$. Since $W_{2*} = V_{2*}$ and $W_{*2} = V_{*2} - R$, we have by (6.3)

$$e(\mathcal{J}^2) \geq e(W_{2*}, W_{*2}) - n = e(V_{2*}, W_{*2}) - n \geq d|W_{*2}| - e(\mathcal{B}_2(\mathcal{P}, G)) - n$$
$$\geq d|W_{2*}| - \nu n^2 - n \geq (\theta - \nu^{1/2})|W_{2*}|^2.$$

By Lemma 3.6, $\mathcal{J}^2$ admits a $(4, \tau, 4\nu^{1/2})$-partition $\mathcal{P}_2^* = \{U_{ij} : i, j \in [2]\}$. Let $X_i = \phi_{2*}(U_{i*})$ and $Y_i = \phi_{*2}(U_{*i})$ for $i \in [2]$. Hence we have

$$|X_1|, |X_2|, |Y_1|, |Y_2| \geq \tau|W_{2*}| \geq \tau(n - r)/2 \geq \tau n/3, \tag{6.5}$$
$$e_G(X_1, Y_2) + e_G(X_2, Y_1) \leq \mathcal{B}_2(\mathcal{P}_2^*, \mathcal{J}^2) + |W_{2*}| \leq 5\nu^{1/2}|W_{2*}|^2, \tag{6.6}$$

where we have used (6.3) and (6.2) for the first line. Then, let us define the partition $\mathcal{Z} = \{Z_{ij} : i, j \in [3]\}$ for $G - R$ as follows:

$$
\begin{array}{lll}
Z_{11} = W_{22} \cap X_1 \cap Y_1, & Z_{12} = W_{22} \cap X_1 \cap Y_2, & Z_{13} = W_{21} \cap X_1, \\
Z_{21} = W_{22} \cap X_2 \cap Y_1, & Z_{22} = W_{22} \cap X_2 \cap Y_2, & Z_{23} = W_{21} \cap X_2, \\
Z_{31} = W_{12} \cap Y_1, & Z_{32} = W_{12} \cap Y_2, & Z_{33} = W_{11}.
\end{array}
$$

Notice that, for $i \in [2]$

$$|Z_{i*}| = |X_i| \geq \tau n/3 \text{ and } |Z_{*i}| = |Y_i| \geq \tau n/3$$

by (6.5). Also, by (6.4), we have $|Z_{3*}| = |W_{1*}| \geq \tau n/3$ and $|Z_{*3}| = |W_{*1}| \geq \tau n/3$. Note that

$$\mathcal{B}_3(\mathcal{Z}, G - R) \subseteq E_G(X_1, Y_2) \cup E_G(X_2, Y_1) \cup \bigcup_{i,j \neq 3} (E_G(Z_{i*}, Z_{*3}) \cup E_G(Z_{3*}, Z_{*j}))$$
$$= E_G(X_1, Y_2) \cup E_G(X_2, Y_1) \cup \mathcal{B}_2(\mathcal{W}, G - R)$$
$$\subseteq E_G(X_1, Y_2) \cup E_G(X_2, Y_1) \cup \mathcal{B}_2(\mathcal{P}, G),$$

so (6.6) implies that $|\mathcal{B}_3(\mathcal{Z}, G - R)| \leq 5\nu^{1/2}|W_{2*}|^2 + 4\nu n^2 \leq 10\nu^{1/2}|G - R|^2$. Therefore, $\mathcal{Z}$ is a $(9, \tau/3, 10\nu^{1/2})$-partition for $G - R$. Let us distribute the vertices of R into elements of $\mathcal{Z}$ arbitrarily. Since $r \leq 16\nu n \ll \tau n$, the modified version of $\mathcal{Z}$ becomes a $(9, \tau/6, 20\nu^{1/2})$-partition for G. Hence, by Lemma 5.3, G is Hamiltonian, as required. $\quad \square$

7. Conclusion

The main result of this paper is a proof of the approximate version of Jackson's conjecture, namely Conjecture 1.2. It remains an open problem to prove this conjecture exactly. Similarly, it would be interesting (and probably easier) to obtain an exact version of Theorem 1.5, namely to show that every strongly well-connected n-vertex d-regular digraph with $d \geq n/3$ is Hamiltonian.

Another natural question is to ask for the analogue of Theorem 1.5 for oriented graphs. By suitably orienting the edges in a non-Hamiltonian 2-connected regular graph on n vertices with degree close to $n/3$ (see e.g. [10]), there exist non-Hamiltonian strongly well-connected regular oriented graphs on n vertices with d close to $n/6$.

Proposition 7.1. *For $n \in \mathbb{N}$, there exists a strongly well-connected $3n$-regular oriented graph on $18n + 5$ vertices with no Hamilton cycle (Fig. 6).*

Proof. Let G_1, G_2 and G_3 be vertex-disjoint regular tournaments each on $(6n + 1)$ vertices. For $i \in [3]$, let $M_i = \{x_j^i y_j^i : j \in [2n]\}$ be a matching of size $2n$ in G_i. Define G to be the oriented graph obtained from $\bigcup_{i \in [3]}(G_i - M_i)$ by adding two new vertices z and z' and edge set $\{x_j^i z, z y_j^i, x_{j+n}^i z', z' y_{j+n}^i : i \in [3], j \in [n]\}$. Note that G is a $3n$-regular oriented graph on $18n + 5$ vertices. We claim that G is strongly well-connected.
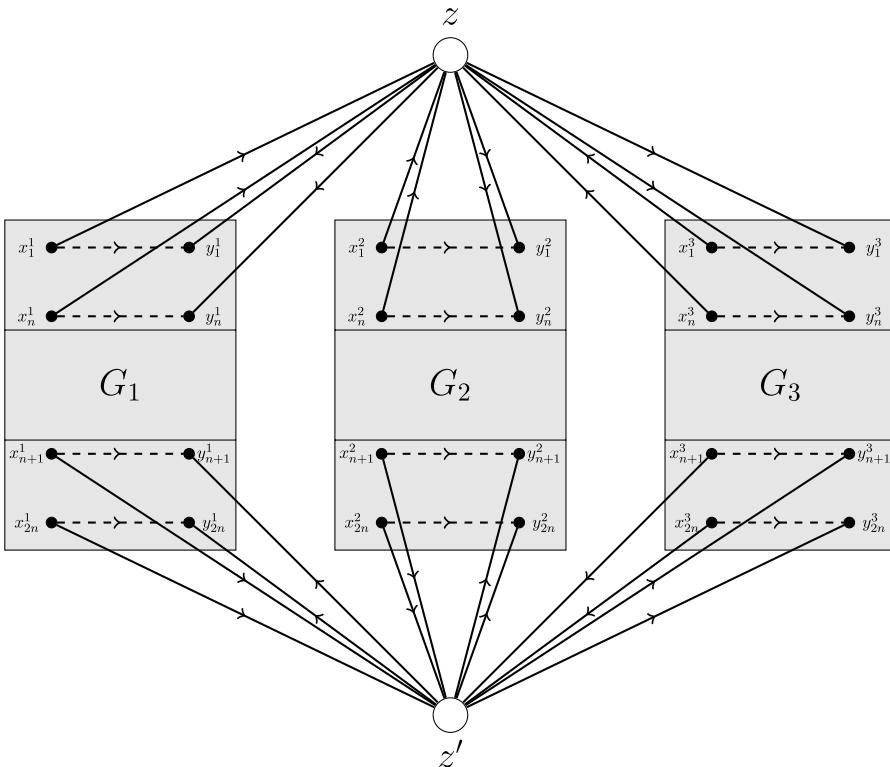

Fig. 6. A strongly well-connected $3n$-regular oriented graph G on $18n + 5$ vertices.

Indeed, G has a cycle with vertex set $V(G_1) \cup V(G_2) \cup \{z, z'\}$ and another cycle with vertex set $V(G_3) \cup \{z, z'\}$. The union of these two cycles (which is a subdigraph of G) is already strongly well-connected; hence G is strongly well-connected. However G is not Hamiltonian because deleting the two vertices z and z' from G disconnects it into 3 components (whereas deleting any 2 vertices from a Hamilton cycle disconnects it into at most 2 components). □

Are all strongly well-connected d-regular oriented graphs on n vertices with $d \geq n/6$ Hamiltonian? We note that a version of this question with "strongly 2-connected" in place of "strongly well-connected" was asked in [11], but Proposition 1.6 provides a counterexample for that.

Another interesting direction is to obtain an analogue of the Bollobás–Häggkvist Conjecture (which is discussed in the introduction) for oriented graphs. That is, given $t \geq 3$, determine the minimum value for d such that any strongly t-connected d-regular n-vertex oriented graph is Hamiltonian. For any choice of t, we must have $d \geq n/8$ by considering a suitable orientation of the example of Jung and of Jackson, Li, and Zhu (mentioned in the Section 1), as shown below.

Proposition 7.2. *For $n \in \mathbb{N}$, there exists a strongly n-connected $2n$-regular oriented graph on $16n + 1$ vertices with no Hamilton cycle.*

Proof. Consider a $2n$-regular oriented bipartite graph H with vertex classes A and B each of size $4n$. Fix $b \in B$ and let $N_H^+(b) = \{a_1^+, \dots, a_{2n}^+\}$ and $N_H^-(b) = \{a_1^-, \dots, a_{2n}^-\}$. Let G_1 and G_2 be regular tournaments each on $(4n+1)$ vertices. Suppose that $V(H)$, $V(G_1)$ and $V(G_2)$ are pairwise disjoint. For $i \in [2]$, let $M_i = \{x_j^i y_j^i : j \in [n]\}$ be a matching of size n in G_i. Define G to be the oriented graph obtained from $(H - \{b\}) \cup G_1 \cup G_2$ by removing the edges from $M_1 \cup M_2$ and adding the edges $\{x_j^1 a_j^+, a_j^- y_j^1, x_j^2 a_{j+n}^+, a_{j+n}^- y_j^2 : j \in [n]\}$. Note that G is a strongly n-connected $2n$-regular oriented graph on $16n + 1$ vertices (Fig. 7). However G is not Hamiltonian as removing A will create $2 + (|B| - 1) > |A|$ components. □

Are all strongly 3-connected d-regular oriented graphs on n vertices with $d \geq n/8$ Hamiltonian?

For digraphs, one can similarly ask whether all strongly well connected (or 3-connected) d-regular digraphs on n vertices with $d \geq n/3$ (or $d \geq n/4$, respectively) Hamiltonian? If the answer is yes, then the value of d is best possible by considering the digraph analogues of the examples given by Propositions 7.1 and 7.2.

Data availability

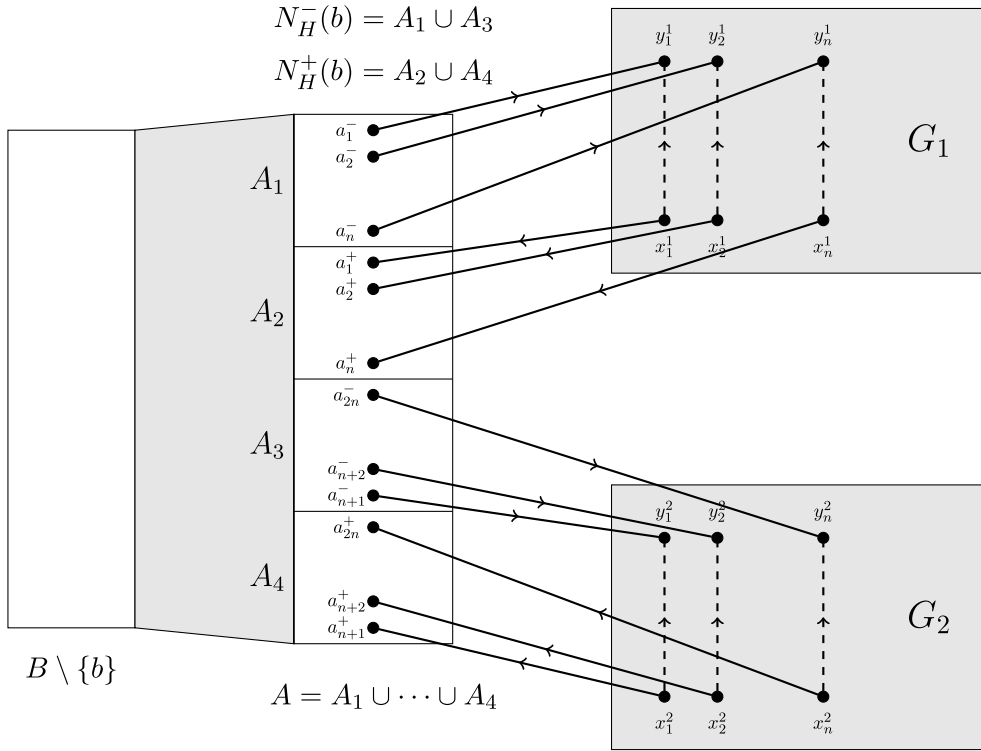No data was used for the research described in the article.

Fig. 7. A strongly n-connected $2n$-regular oriented graph G on $16n + 1$ vertices.

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