UNIVERSITY OF BIRMINGHAM
University of Birmingham Research at Birmingham

Crypto quanto and inverse options

Alexander, Carol; Chen, Ding; Imeraj, Arben

DOI:
10.1111/mafi.12410

License:
Creative Commons: Attribution (CC BY)

Document Version
Publisher's PDF, also known as Version of record

Citation for published version (Harvard):
Alexander, C, Chen, D & Imeraj, A 2023, 'Crypto quanto and inverse options', *Mathematical Finance*, vol. 33, no. 4, pp. 1005-1043. https://doi.org/10.1111/mafi.12410

Link to publication on Research at Birmingham portal

General rights
Unless a licence is specified above, all rights (including copyright and moral rights) in this document are retained by the authors and/or the copyright holders. The express permission of the copyright holder must be obtained for any use of this material other than for purposes permitted by law.

•Users may freely distribute the URL that is used to identify this publication.
•Users may download and/or print one copy of the publication from the University of Birmingham research portal for the purpose of private study or non-commercial research.
•User may use extracts from the document in line with the concept of 'fair dealing' under the Copyright, Designs and Patents Act 1988 (?)
•Users may not further distribute the material nor use it for the purposes of commercial gain.

Where a licence is displayed above, please note the terms and conditions of the licence govern your use of this document.

When citing, please reference the published version.

Take down policy
While the University of Birmingham exercises care and attention in making items available there are rare occasions when an item has been uploaded in error or has been deemed to be commercially or otherwise sensitive.

If you believe that this is the case for this document, please contact UBIRA@lists.bham.ac.uk providing details and we will remove access to the work immediately and investigate.

Download date: 11. May. 2024

Received: 8 November 2022 | Accepted: 10 June 2023

DOI: 10.1111/mafi.12410

ORIGINAL ARTICLE

MATHEMATICAL FINANCE An International Journal of Mathematics, Statistics and Financial Economics

WILEY

Crypto quanto and inverse options

Carol Alexander[1] | Ding Chen[2] | Arben Imeraj[1]

[1]University of Sussex Business School, Brighton, UK

[2]School of Mathematics, University of Birmingham, Birmingham, UK

Correspondence
Carol Alexander, University of Sussex Business School, Brighton, UK.
Email: c.alexander@sussex.ac.uk

Abstract

Over 90% of exchange trading on crypto options has always been on the Deribit platform. This centralized crypto exchange only lists inverse products because they do not accept fiat currency. Likewise, other major crypto options platforms only list crypto–stablecoin trading pairs in so-called *direct* options, which are similar to the standard crypto options listed by the CME except the US dollar is replaced by a stablecoin version. Until now a clear mathematical exposition of these products has been lacking. We discuss the sources of market incompleteness in direct and inverse options and compare their pricing and hedging characteristics. Then we discuss the useful applications of currency protected "quanto" direct and inverse options for fiat-based traders and describe their pricing and hedging characteristics, all in the Black–Scholes setting.

KEYWORDS
cryptocurrency, foreign exchange, hedging, incomplete market, inverse option

JEL CLASSIFICATION
C02, G12, G23

1 | INTRODUCTION

Blockchain technology, on which every crypto is built, is renaître. The storage of information on a distributed ledger, which is visible to all participants, goes back to the 1990s at least (Haber

This is an open access article under the terms of the Creative Commons Attribution License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.
© 2023 The Authors. *Mathematical Finance* published by Wiley Periodicals LLC.

& Stornetta, 1991, 1997). But the idea of a cryptographically linked blockchain technology only grew in popularity after the introduction of a cryptocurrency called bitcoin as the native token for a blockchain called Bitcoin (Nakamoto, 2008). Bitcoin transactions are verified by a peer-to-peer network, stored on the blockchain, transparent to anyone with internet access and become immutable after a certain period of time. Today, blockchains of many different types have exceeded their initial purpose of recording simple peer-to-peer transactions, becoming the backbone of Web 3.0 by carrying fully autonomous and self-regulating smart contracts. As a result, there are tens of thousands of tokens which are transacted on blockchains and also off-chain on market places and centralized exchanges.

A token is a crypto asset that is transacted using a smart contract on a blockchain. Almost all crypto assets are tokens, and even those that were minted before the introduction of Ethereum Request for Comments (ERC) token standards are easily "wrapped" to become a token. Native tokens are generated through building the blocks in the chain. If the blockchain can carry smart contracts, the native token is also the unit of account for the "gas" required to fuel the smart-contract transactions.[1] Because a native token is used for payment, we can call it a cryptocurrency. Although the term "cryptocurrency" has slipped into popular use as a generic term for all crypto assets, the only other type of token, which is truly a cryptocurrency, that is, a fungible token that is commonly used for payment and settlements, is a stablecoin. A stablecoin is a token whose price is pegged to a fiat currency such as the US dollar.

Other tokens, which are not used for payment, can be divided into fungible and nonfungible classes. A nonfungible token (NFT) is a certificate of ownership of a physical or digital asset such as land or art. NFTs are frequently used as collateral for borrowing cryptocurrencies, but of course they themselves could not be a unit of account because they are nonfungible. An NFT is an asset, which is arguably more akin to a commodity than a security. How to classify the nonpayment but still fungible tokens that are issued by developers of projects in the digital economy is an even greater point of debate. Under the leadership of Gary Gensler and Dan Berkowitz, the U.S. Securities and Exchange Commission (SEC) has been arguing for years that such tokens are a type of security, and should, therefore, be regulated by the SEC. On the other hand, the Commodities and Futures Trading Commission (CFTC) produced a counter-argument that the CFTC should be regulating the crypto assets (such as native tokens) that have a utility value and are, therefore, more like a commodity than a security.

Crypto assets are exchanged in trading pairs via decentralized liquidity pools on-chain or on centralized exchanges off-chain. When both sides of the trading pair are cryptocurrencies, the trading pair could be regarded as an exchange rate. If neither side of the trading pair is a cryptocurrency, then the trading pair is more like an asset swap. And if only one side of the trading pair is a cryptocurrency, it is like a security or a commodity with the cryptocurrency being the unit of account. The most common type of platform for trading exchange-rate crypto pairs is the order book of an off-chain centralized exchange such as Binance or OKX. By contrast, most asset-swap crypto pairs are traded in liquidity pools of an on-chain decentralized exchange, such as Uniswap or Pancakeswap. Most crypto–crypto asset swaps are traded in on-chain liquidity pools. Trading volumes in all these markets have exploded over recent years. In particular, millions of retail traders, either truly believing in a projects' philosophy or for purely speculative purposes, have easily opened accounts on any number of so-called "self-regulated" centralized exchanges to trade bitcoin and other crypto assets and derivatives. Large proprietary trading firms have also been trading very actively.[2] All these traders, on entering this melting pot of traditional finance and modern computer science, are faced with an array of innovative crypto assets many of which

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

have very actively traded derivative products. However, in the crypto derivatives category, options are backed by very little academic research.

Almost every crypto derivative has a settlement price based on a nontraded asset. This is because of the extreme fragmentation of the crypto markets, where trading pairs are typically listed on several different centralized exchanges or traded in a number of competing liquidity pools. For instance, the bitcoin US dollar trading pair is listed on over 20 exchanges, and pairs of BTC against USD stablecoins are listed on even more (CryptoCompare, 2022a). For this reason, the settlement price for BTC/USD futures and options is derived from a *coin index* whose value is typically an average price taken across several different exchanges. The exact formula used for the settlement index depends on the exchange. Some also use the average of an index as the settlement price, for instance taken over the last hour or 30 min before settlement. So technically, these crypto derivatives markets are incomplete.

At the time of writing two-thirds of total trading volumes on all centralized crypto exchanges is on derivative products rather than spot trading pairs, see CryptoCompare (2022b). These products include standard, direct and inverse calendar futures and options, and perpetual futures, the latter presenting by far the greatest trading volume in the entire crypto asset ecosystem. Perpetual contracts are nonexpiring futures that mimic a margin-based spot account on the coin index. A regular funding payment assures the agent that the price of the perpetual stays close to the underlying coin index value, hence these products are often called perpetual swaps. While a perpetual contract is traded and never settled, its price is not the same as the settlement price for the corresponding futures and options because the basis risk can be high on these highly volatile instruments.

More recently, a great many centralized exchanges have also listed European-style options, mainly on BTC and ether (ETH) against USD. Some of these platforms are at least semi-regulated by the CFTC, including the Chicago Mercantile Exchange (CME) and IQ Option. Others, such as Deribit are registered in off-shore tax havens where there is no need to comply with know-your-customer protocols, market abuse directives or indeed any other form of regulation aimed at protecting the client's interests. Due to the rapid growth in popularity of crypto options, by May 2021, Deribit's monthly trading volume of bitcoin options was $26bn and on ether options, it was $15.5bn. At the time of writing, just after the merge of the Ethereum chain to proof-of-stake consensus, ETH options have a greater volume than BTC options on most exchanges.[3]

Apart from regulatory oversight, the main factor that differentiates the centralized crypto derivatives platforms is the type of products that they offer. For example, CME and Robinhood run order books in standard European put and calls with a contract size in bitcoin or ether, and all contracts are margined and settled in USD. But $\sim 90\%$ of open interest and trading volume on crypto options has always been on Deribit, which only runs order books in so-called *inverse* options. These have a contract size of one bitcoin, ether or solana, and they track the USD value of these coins even though they are margined and settled in BTC, ETH, or SOL. Such an inverse structure is necessary because clients cannot use fiat currency for on-boarding or off-boarding the Deribit exchange, even though they wish to trade an option on a cryptocurrency-USD pair. This means that any USD-denominated agent trading inverse options on Deribit must bear the expense of first transferring the profit in BTC or ETH to their wallet, and then sending it to another crypto exchange, where it can eventually be converted to USD. Other large but so-called "selfregulated" (but let us call them what they really are, i.e. unregulated) exchanges also prevent fiat off-boarding but they do allow two-way transfers of stablecoins such as USDT (tether). This allows them to list *direct* options, which are exactly like the CME and Robinhood products, except that margining and settlement is in a stablecoin. Either way, the absence of

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

on-exchange crypto-fiat brokerage services makes crypto options trading a cost-inefficient operation for USD-denominated traders.

This discussion has highlighted three important questions to answer before we can really understand crypto options should be priced and hedged:

1. What is the currency that the trader uses as a unit of account? This could be a cryptocurrency like BTC or a fiat currency such as USD. A BTC-based trader with a BTC-denominated trading account has a different perspective on profit and loss to a US trader whose account is measured in USD;
2. Is the underlying a security or a currency?[4] If a security, then the option should be priced like a stock or bond option with the unit of account specified in 1, and if a currency, then the option is equivalent to an FX option;
3. Is the settlement price that of a tradable instrument? The vast majority of crypto options are settled using an index, sometimes also averaged prior to settlement. So should pricing and hedging take account of the incompleteness of this market or not?

Clarifying the answers to those questions justifies proposing that *quanto* options be added to the array of crypto products. Quanto direct options are similar to traditional quantos, but the quanto inverse option is a completely new type of exotic option. We argue that both direct and inverse quantos are better products for risk-averse USD-denominated agents than their vanilla counterparts, which have no currency protection.

Given their novelty, a substantial number of research papers have been written on bitcoin options. Siu and Elliott (2021), Jalan et al. (2021), and Chen and Huang (2021) all study the empirical application of various pricing models and Hou et al. (2020) propose the use of a stochastic volatility correlated jump (SVCJ) model to price bitcoin options. Cao and Celik (2021) suppose that the bitcoin price is a function of domestic money supply and S&P500 returns and value bitcoin options within this equilibrium model. Alexander et al. (2022) examine the behavior of implied volatility smiles of bitcoin options to infer whether demand pressures on market makers are motivated by directional or volatility traders. Matic et al. (2023) use the daily implied volatilities quoted by the Deribit exchange to calibrate a stochastic-volatility (SV) inspired surface and simulate the underlying through the SV process introduced by Duffie et al. (2000) with the GARCH-filtered kernel density of McNeil and Frey (2000). Four other papers focus exclusively on inverse options, including the pricing of such options under the standard FX framework (Lucic, 2022); pricing within a SV jump model framework (Sepp & Rakhmonov, 2022; Teng & Härdle, 2022); and their empirical dynamic delta hedging performance relative to the BS delta (Alexander & Imeraj, 2023; Teng & Härdle, 2022).

By contrast with all previous research in this area, we do not focus on a particular class of model and neither do we include an empirical study. Instead, we discuss the properties of four different types of crypto options that are of practical and academic interest, with a specific focus on direct and inverse quanto products. In the following, Section 2 analyzes the current state of the crypto options markets; Section 3 clarifies the exact differences between standard, direct, and inverse options, discusses the applications of quanto direct options for crypto traders, and introduces quanto inverse options; Section 4 derives pricing formulae for inverse and quanto inverse options under the geometric Brownian motion assumption and examines the associated issue of market incompleteness; Section 5 derives the Black–Scholes type hedge ratios for quanto inverse options and describes their characteristics; and Section 6 concludes.

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

2 | CRYPTO OPTION MARKETS

We have highlighted the need for regulation in spot trading, but the "wild west" term that regulators use for crypto markets really stems from the trading on nonstandard derivatives, like perpetual futures and inverse products, especially on bitcoin and ether. While the prices and volatilities of different tokens can be very highly correlated, mainly led by bitcoin perpetuals, the market places for crypto derivatives are highly fragmented and split between regulated and unregulated trading platforms. Similar to spot trading, the vast majority of crypto derivatives exchanges employ electronic limit order book trading, on a variety of underlying assets and different contract sizes.[5] A distinct feature of the unregulated platforms is that they act as their own clearing houses, and they issue no margin calls. Instead, any long or short leveraged position in futures or perpetuals that trends adversely, drying up the margin account, is automatically and immediately liquidated. This involves closing the position at prevailing market prices, without allowing the client to deposit more collateral in the margin account. This type of forced selling has resulted in some vicious cycles of extremely rapid price changes leading to further liquidations, as happened on May 19, 2021 and on numerous other occasions since.[6] These daily liquidations potentially contribute to bitcoin's infamously high volatility, which in turn acts as a catalyst for growth in crypto options markets. Options on bitcoin and a few other crypto have been available since 2017, but their trading volume and open interest remained low until institutional interest in CME options started to grow during late 2021. By March 2023, daily volumes exceeded 35 billion USD notional and major commentators now view the crypto options market as too big to ignore.[7]

The fragmentation of crypto derivatives trading is also behind the two different settlement mechanisms, that is, standard (direct) and inverse, which we will discuss in greater depth in the next section. The CME launched its first European cash-settled option in January 2020. However, as the bitcoin price surged, the contract size of five bitcoins quickly became too large, and the anticipated influx of institutional investors did not occur. This is why, in March 2022, the CME launched micro bitcoin options with a contract size of a tenth of a bitcoin, aiming to compete with the unregulated platforms and target retail traders. Despite the CME introducing a wide spectrum of reference rates for other native tokens—such as Polkadot (DOT) and Solana (SOL)—currently they only offer bitcoin and ether options. Some unregulated exchanges like Binance or OKX, which now accept fiat currencies, offer direct European cash-settled options for bitcoin and ether, in addition to options on the Binance native token (BNB) and Ripple (XRP). Most unregulated exchanges started as nonfiat venues, even if some now accept USD and other currencies. For this reason, they first introduced "inverse" derivatives products that are margined and settled in crypto, despite the underlying being USD or USDT value of bitcoin or any proxy. By adopting this approach, Deribit and similar exchanges have facilitated a fully functional options market without the necessity of onboarding any fiat currency, thus circumventing potential regulatory or legal complications. While the CME reported a trading volume exceeding 1.6 billion USD in March 2023, unregulated exchanges recorded a volume surpassing 35 billion USD, with Deribit accounting for over 88% of this total. In terms of open interest, Deribit indisputably dominates the market, possessing over 85% of the total market share. At present, inverse options are available for bitcoin, ether, and solana but, given the trends over recent years, it may only be a matter of time before options on many other tokens are listed.

For the Deribit exchange, based on data from January 2020 to January 2023, Table 1 shows the total number of traded call and put contracts ("Trades"), the average daily number of available strikes ("Strikes") and the average end of day open interest ("OI") for different maturities

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

TABLE 1 Summary statistics on trades, strikes, and open interest.

	Moneyness	Short-term Trades	Short-term Strikes	Short-term OI	Mid-term Trades	Mid-term Strikes	Mid-term OI	Long-term Trades	Long-term Strikes	Long-term OI	Total Trades	Total Strikes	Total OI
	$m < 0.5$	8983	0.13	846	21,311	0.79	884	145,384	8.03	638	175,579	8.96	663
Put	$0.5 \leq m < 0.6$	34,072	0.54	821	51,928	1.14	726	114,605	4.08	662	200,606	5.76	690
	$0.6 \leq m < 0.7$	120,575	1.55	711	85,863	1.51	857	187,473	4.63	652	393,876	7.96	704
	$0.7 \leq m < 0.8$	322,031	3.88	585	190,503	2.13	820	254,433	5.22	665	766,967	11,23	667
	$0.8 \leq m < 0.9$	884,592	7.07	591	345,633	2.77	816	326,138	5.74	605	1,556,363	15.58	636
	$0.9 \leq m < 1$	1,826,346	12.3	401	326,814	3.05	573	248,470	5.29	461	2,401,631	20.63	442
Call	$1 \leq m < 1.1$	1,880,291	11.8	456	344,449	2.95	679	250,958	5.28	588	2,475,698	20.03	523
	$1.1 \leq m < 1.2$	752,401	6.32	590	357,160	2.61	841	256,538	4.96	652	1,366,099	13.88	659
	$1.2 \leq m < 1.3$	241,781	3.48	567	214,005	1.99	939	262,422	4.45	737	718,208	9.92	718
	$1.3 \leq m < 1.4$	94,026	1.73	591	116,778	1.46	1052	224,446	3.65	846	435,251	6.84	825
	$1.4 \leq m < 1.5$	43,488	0.85	792	72,178	1.46	961	188,001	3.37	889	303,668	5.35	889
	$1.5 \leq m < 1.6$	23,828	0.45	769	37,893	0.9	996	146,096	2.96	947	207,818	4.31	938
	$1.6 \leq m < 1.7$	12,709	0.21	1040	33,242	0.62	1016	115,117	2.62	911	161,069	3.44	938
	$1.7 \leq m < 2$	18,290	0.22	1106	43,213	0.82	1362	223,228	6.23	996	284,731	7.26	1041
	$m \geq 2$	6415	0.07	1515	21,909	10.55	1309	373,696	14.7	966	402,013	15.32	981
Total		6,269,833	50.59	528	2,262,876	24.42	844	3,316,972	81.21	758	11,849,681	156	697

Note: Summary statistics on bitcoin option contracts traded on Deribit. We consider out-of-the-money (OTM) calls and puts with varying moneyness levels and divide into three maturity buckets: short-term options (up to 2 weeks); mid-term options (between 2 weeks and 1 month); and long-term options (longer than 1 month). "Total" aggregates all maturities at each moneyness level. The sample covers trades between January 2020 and January 2023 but does not consider in-the-money (ITM) options, which are less traded overall. The column "Trades" presents the number of traded contracts in the respective moneyness and maturity category. The "Strikes" column provides the average number of tradable strikes in the respective moneyness and maturity category. The "OI" column shows the average number of contracts open at end-of-day in each moneyness and maturity category.

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and moneyness subcategories.[8] Cumulatively, more than 11.8 million OTM contracts have been traded across all maturities. Every day, traders had access on average to over 156 distinct strike levels, but preferred to trade more call contracts (6.3 million) than put options (5.5 million). The majority of trading focuses on short-term options and at-the-money (ATM) call and put options (up to twelve options either side of the current underlying price) account for over 1.8 million contracts. Indeed, even when all other moneyness ranges for this maturity class are aggregated, they do not match the volume of ATM trading. Simultaneously, ATM calls and puts exhibit the smallest average daily open interest, suggesting the presence of much high-frequency and speculative day trading, presumably dominated by algorithms. As we move further out of the money within the short-term maturity bracket, we observe a decline in trading activity and available option strikes. However, there is a concurrent increase in open interest. A plausible explanation for this trend is that traders might be speculating with short-term OTM options, willing to hold onto them until maturity given their appealing risk-return profile. This hypothesis is supported by the enhanced trading volume and open interest for deep OTM calls ($m > 1.7$). Recall that the short-term maturity category considers options expiring within the next 2 weeks. Thus, it seems unlikely that institutional traders would retain options with such short maturity if they have strikes situated 70% above the current underlying level. Rather, this trend might indicate retail trading activities, specifically those speculating on price jumps while holding options expiring in 0, 1, or 2 trading days. An alternative explanation could be related to liquidity constraints resulting from the very wide spreads, which might induce traders to hold option contracts until expiry rather than closing before the option matures.

The findings for mid-term maturities in Table 1 are different from those for short-term maturities. As expected, the bulk of the trading volume still remains concentrated around the ATM level, reaching its peak within a range of 10%–20% above and below the current underlying price. We continue to observe the decreasing trend in trades and available strikes as we progress further out of the money, albeit at a less pronounced rate, with the sole exception being the deep OTM call options. The trading volume on all mid-term options together is approximately one quarter of the trading volume on short-term options; at the ATM level alone, this ratio diminishes to one fifth. Yet at strike levels 30% above or 40% below the current underlying price, these mid-term options have a higher trading volume than short-term options of the same moneyness categories. As expected, there are fewer tradable strikes in the ATM range as maturity increases. Across the entire options chain, mid-term options have an average of 25 tradable strike levels, which is about half of that for short-term options. The relatively balanced number of mid-term contracts traded around the underlying price level, that is, ±20%, may serve as an indicator for potential options trading strategies, such as strangles or butterfly spreads, pointing towards the presence of professional traders exploiting arbitrage opportunities.

Recall that the "long-term" category includes options with maturities from 1 month to a year, yet these options together have only half the trading volume on short-term options, which have maturities from 1 day to 2 weeks. Nevertheless, large block trades are more common on longer-term options, often comprising 25 contracts within a single transaction. Interestingly, long-term deep OTM calls have an unusually large open interest, and a great number of trades. Indeed, roughly one in every five long-term call options traded is at a strike price, which double that of the current underlying! In fact, the far deep OTM call options ($m > 2$) display a higher trading volume than any near-the-money subclass and almost three times more available strike prices. This shows that there are many highly speculative trades based on extremely bullish long-term projections for the bitcoin price.

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

3 | CRYPTO OPTION TYPES

We now define the payoffs and settlements for direct and inverse options precisely using USD as the fiat side and either bitcoin (BTC) or ether (ETH) as the crypto side of the trading pair. Both bitcoin and ether are cryptocurrencies, but bitcoin is only a cryptocurrency while ether can also be regarded as a security or commodity because Ethereum is a smart contract blockchain whereas Bitcoin is not. Indeed, the Securities and Exchange Commission (SEC) have argued that *every* token apart from bitcoin is a security and should, therefore, fall under the jurisdiction of the SEC rather than the CFTC.[9] We also consider two types of traders, one USD-denominated whose trading book is denominated in USD and the other crypto-based whose trading book is denominated in BTC.

Both settlement mechanisms for crypto options, which differ between exchanges as well as the different types of products, are not yet widely understood. For instance, on Deribit, the underlying is a nontradable index of spot prices and on the CME, it is a futures contract on a similarly nontradable reference rate. The settlement price on Deribit is the average value of the underlying over the 30 min prior to settlement and on the CME, it is the spot value of the reference rate.[10] These settlement differences, combined with a widespread lack of proper documentation from the unregulated exchanges, may lead to confusion about a seemingly trivial European-style product.

We begin by comparing standard options with direct options and inverse (sometimes also called indirect) options, then we discuss the uses of quanto direct options, as well as a new type of quanto inverse option. It is quite possible that quanto direct and/or inverse options are already being traded on-chain, just as traditional quanto options are traded over-the-counter, and so our first through specification of details in this section could be very useful to their users.

3.1 | Standard versus direct options

Either standard or direct options are widely traded on all the regulated and some unregulated centralized exchanges. A bitcoin (ether) option trader can choose whether to trade the pair BTC–USD (ETH–USD) or BTC–USDT (ETH–USDT) or the other side of the pair could be some other stablecoin such as USDC. Because of the risks surrounding stablecoins,[11] a USD-denominated trader may prefer the standard option, which is a plain vanilla European product. The call has payoff in USD given by

$$V_T^{\$} = \left(S_T^{\$} - K^{\$}\right)^+, \tag{1}$$

where $K^{\$}$ denotes the strike price and $S_T^{\$}$ denotes the underlying price at maturity T. On the CME (at the time of writing), the underlying is either the BTC–USD pair—which from now on, we regard as the BTC/USD, exchange rate, that is, the number of USD for 1 unit of bitcoin[12]—or the pair ETH–USD, which we regard as either the ETH/USD exchange rate or the USD price of the ETH security token.

The margining and settlement of standard bitcoin or ether options depends on the exchange. On the CME, they are margined and settled in USD and the settlement price is that of the CME bitcoin or ether futures, so both the option premium and the payoff are denominated in USD. They are also cleared by the exchange, thus omitting counterparty risk. This procedure does not differ in any way from trading other commodities on the CME exchange. Binance lists *direct* BTC–USDT and ETH–USDT (and many other crypto) options, which are margined and settled in USDT.

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

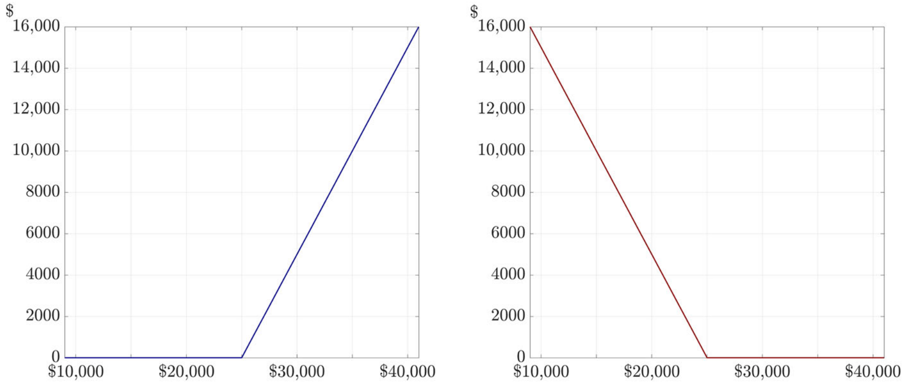

FIGURE 1 Standard and direct option payoff. The payoff to a long call (left: blue graph) and put (right: red graph) as a function of the settlement price. These payoffs present a standard option but could also display the direct payoff using USDT ($\mathbb{T}$) instead of USD (\$) for both the payoff (vertical axis) as well as the underlying (vertical axis). The strike level is set at $K^{\$} = \$25{,}000$. [Color figure can be viewed at wileyonlinelibrary.com]

The settlement price is that of the stablecoin-margined product BTCUSDT (or ETHUSDT) on their spot platform at the time of settlement, and the payoff is in USDT; but in other ways, the Binance settlement procedure mimics the CME's. Hence, using the symbol $\mathbb{T}$ to denote the price of the stablecoin USDT, the call pay-off may be written:

$$V_T^{\mathbb{T}} = \left(S_T^{\mathbb{T}} - K^{\mathbb{T}}\right)^+, \tag{2}$$

which is identical to Equation (1) except the currency is USD not but the stablecoin USDT.

We have selected these two particular centralized exchanges because they are the largest to list standard and direct crypto options, and they illustrate how two exchanges can offer the same basic product, with but have different underlyings and settlement procedures.[13] We also note that despite being the only crypto options exchange not to run 24/7 on every day of the year, currently the CME has the greatest trading volume and open interest of the three. This could be because of the dominance of USD-denominated traders, who prefer to use USD rather than a stablecoin as the unit of account. Figure 1 illustrates the well-known payoffs of standard and direct call and puts as a function of the underlying. For the sake of clarity, we omit the graphs for USDT as these would not add any further information.

Determining the pricing approach for standard or direct options is fundamentally linked to the trader's interpretation of the underlying asset, that is, the USD price of bitcoin (or ether, or another token). Is it a currency pair, a security, or a commodity? Or is crypto an entirely new asset class of its own? When the trader classifies the underlying as a currency pair, the payoff structure allows a direct pricing approach using the Garman and Kohlhagen (1983), or any other FX pricing model. However, a question about the risk-free rate arises here. There is no governmental body issuing treasury notes for bitcoin (or any crypto) and crypto lending platforms that guarantee fixed annual returns also bear substantial risks, as we shall discuss in the next section. On the other hand, a trader might classify the underlying as a security (following the SEC's announcements referred to above) or as a commodity when the crypto is a native token for a smart-contract blockchain. Either of these interpretations enables the use of standard option pricing models like the Black and Scholes (1973), or another SV-jump model (Matic et al., 2023). The challenge here then lies in identifying elements comparable to dividend payouts, or cost-of-carry. The proof-of-work architecture of bitcoin does not pay any reward to investors (only the miners receive rewards) but the proof-of-stake structure of Ethereum allows investors to deposit, or stake, their ETH and collect a weekly return like a dividend yield. Alternatively, since ETH is a native token, investors possessing a sizable amount of ETH might pay the costs of a secure custody service. In short, irrespective of the trader's interpretation of the nature of the underlying, there exists a plethora of available option pricing models due to the conventional, straightforward payoffs of crypto options.

3.2 | Inverse options

Inverse options are the only type of option that Deribit lists. Therefore, over 90% of the trading volume on centralized crypto options exchanges is on inverse options. The reason Deribit only lists inverse options is that it is a nonfiat exchange so there is no USD transacted anywhere on the platform. The inverse structure allows Deribit to list options on cryptocurrency–USD trading pairs, currently BTC–USD, ETH–USD, and SOL–USD. Each option is margined and settled in the cryptocurrency of the underlying option.

Many other nonfiat exchanges list inverse options, precisely because they can trade against the USD without using it as the unit of account or indeed allowing any fiat currency onto the platform. And most inverse options track the USD value of a coin *index* not a single spot price. Importantly, margining and settlement is always in a cryptocurrency, not in fiat. If held to maturity, the settlement price S_T is either the coin index value exactly at the settlement time, or its average value over a time period immediately prior to settlement. For example, Deribit bitcoin inverse options use the "Deribit Bitcoin Index" for settlement. This is currently an average of bitcoin spot prices on eleven major centralized exchanges. Any constituent value that falls outside the +/−0.5% range of the median price is adjusted to the closest bandwidth price limit, and then the index is calculated as the equally weighted average of these values. The exchange also reserves the right to manually exclude exchanges from this index, if it sees fit.[14]

Due to its frequent rebalancing, physical replication of this index is an immensely difficult and expensive task. An agent would be required to hold bitcoin positions on multiple exchanges and rebalance these constantly. Complicating replication, even more, the final option settlement value is the average of the index during the last 30 min before expiry. This important feature about Deribit inverse options is often left out. The underlying is not directly tradable. For this, we must consider pricing within an incomplete market.

In general, an inverse contract specifies a notional number N of coins, which is multiplied by a point value to obtain a payoff expressed as a number of coins, that is, in the units of the coin. The terminal payoff (and indeed all trading profits) are transferred to the trader in the cryptocurrency, not in USD. For example, the payoff $V_T^{\mathbb{B}}$ to an inverse call on BTC/USD is denominated in BTC and may be written:

$$V_T^{\mathbb{B}} = N \frac{\left(S_T^{\$} - K^{\$}\right)^+}{S_T^{\$}}, \tag{3}$$

where the second term is a dimensionless quantity called the point value. There is actually no need for our notation to specify the units for the settlement price and strike (even though we have

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

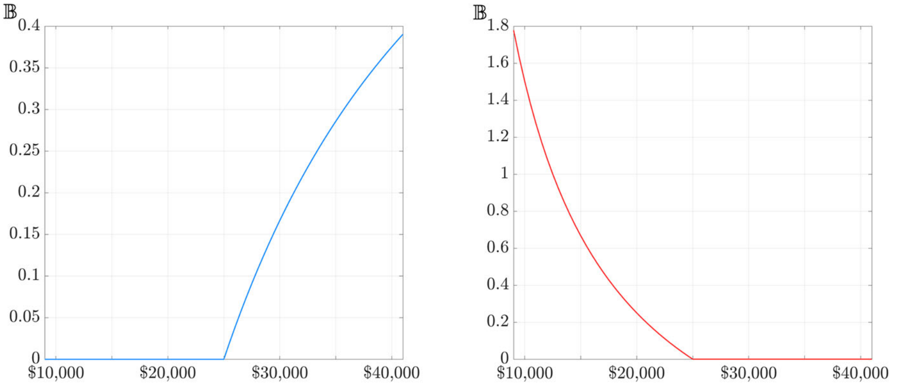

FIGURE 2 Inverse option payoff. The payoff in bitcoin to a long inverse call (left: blue graph) and put (right: red graph) as a function of the settlement price. The strike is \$25,000. Note that the call payoff is capped at maximum 𝔹1 whereas the put can theoretically pay out an infinite amount of bitcoin. [Color figure can be viewed at wileyonlinelibrary.com]

done so above) because their difference becomes dimensionless when divided by $S_T^\$$. But we do need to know what the underlying is and in this example, it is the BTC/USD exchange rate.

Currently, all exchanges that list inverse options use a notional of exactly one coin.[15] Moreover, they all quote the option prices in USD, as well as in the cryptocurrency of the trading pair, but settlement is always in the cryptocurrency. Now, for a USD-denominated trader, the true payoff (which is in cryptocurrency) may just as well be translated into USD, in which case, $V_T^{\mathbb{B}}$ should be multiplied by the price of the underling at the time of settlement $\bar{S}_T^\$$. Note that this price is not the same as the settlement price, the latter being the average price over the 30 min before the settlement time. Given how volatile crypto markets are, there can actually be a large difference. Anyway, we can write the payoff to a USD-denominated trader as

$$V_T^\$ = \bar{S}_T^\$ V_T^{\mathbb{B}} = \bar{S}_T^\$ \frac{\left(S_T^\$ - K^\$\right)^+}{S_T^\$} \approx \left(S_T^\$ - K^\$\right)^+. \tag{4}$$

This shows that one can think of the inverse option payoff as equivalent to a standard FX option, except that the payoff is denominated in the foreign currency as remarked by Lucic (2022). There is a large body of academic research on FX options, their pricing, hedging, volatility dynamics, and so forth, see Levy (1992), Carr and Wu (2007), Demeterfi (1998), and many others. However, FX options are usually denominated in the same currency as the underlying. Inverse options are denominated in the foreign rather than the domestic currency and this rather unusual denomination of the payoff is a potential source of confusion. Similar characteristics in FX options, when one uses the foreign-domestic symmetry relationship to convert a domestic call to a foreign put, has been documented previously (Grabbe, 1983; Reiswich & Wystup, 2010). We also provide more details on the foreign–domestic relationship in Section 4.1.

Figure 2 illustrates the true terminal payoff an agent would receive when trading a long inverse option. It is a piece-wise concave (call) or entirely convex (put) structure. Note that the call payoff is capped at maximum 𝔹1 whereas the put can theoretically pay out an infinite amount of bitcoin. For USD-denominated traders, but only for these traders, the payoff to an inverse option can be approximated by a standard piece-wise linear form,[16] and hence their pricing is almost identical to

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

that of the direct option (Garman & Kohlhagen, 1983). But very much depends only on the trader's base currency. USD-denominated traders might only consider the USD value of their option position on the balance sheet, but crypto option traders are international, and so may prefer to use a cryptocurrency—or a different fiat currency—as their unit of account on Deribit. And even for USD-denominated traders, cryptocurrency-denominated profits stemming from large positions in inverse options cannot be instantly exchanged for USD. This is a particular issue for puts which can, in theory, pay out an infinite amount of bitcoin. Deribit also offers block trades of at least 25 bitcoin or 250 ether for which it does not require very large negative price movements for the position holder to receive a high number of bitcoin or ether. Thus, while the *paper* value of profits in BTC or ETH may be very high, converting such an amount into fiat may well result in liquidity problems. Then profits would be reduced through spillage on the currency conversion trades. The absence of adequate brokerage services, and of exchange requirements to hold large margin reserves, is yet another expense factor for institutional bitcoin option trading. Professional traders transacting large amounts might refrain from denominating their profit and loss in USD and use cryptocurrency their balance sheets instead, but still the currency risk faced traders in inverse options cannot be ignored.

3.3 | Standard quanto and direct options

A quantity-adjusted option, or quanto option for short, allows traders to gain exposure to a foreign market without taking any currency risk. In traditional markets, the underlying is often a single security, or a security index, or another asset like a commodity. The settlement price S_T of this asset and the option strike K are denominated in the foreign currency but the standard option payoff is converted into domestic currency using a predetermined exchange rate $\bar{X}$, which is agreed upon entering the contract (quanto futures and options are usually traded OTC). For instance, a standard quanto call has payoff:

$$V_T = \bar{X}\left(S_T - K\right)^+. \tag{5}$$

Because they are well-known products in traditional markets, the pricing and hedging for quanto options have been researched very extensively (Clark, 2011; Demeterfi, 1998; Jamshidian, 1993; Jeanblanc et al., 2009). So it is somewhat surprising that there has been no previous research documenting quanto products in crypto markets. In fact, the first futures ever traded on crypto were quanto products.[17]

To illustrate the usefulness of a quanto option to crypto traders, first suppose that a BTC-based option trader is interested to gain exposure to ETH. She could use existing exchange-traded products to convert BTC to USD and then trade ETH–USD options. But this way she is exposed to the risk of BTC fluctuating against USD. To remove the currency risk, the trader can obtain a payoff denominated in BTC by agreeing a fixed USD/BTC rate $\bar{X}^{\$/\mathbb{B}}$ with the quanto option issuer before entering the contract.[18] We express the payoff to a quanto call, for a BTC-based trader as

$$V_T^{\mathbb{B}} = \bar{X}^{\$/\mathbb{B}}\left(S_T^{\$} - K^{\$}\right)^+, \tag{6}$$

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

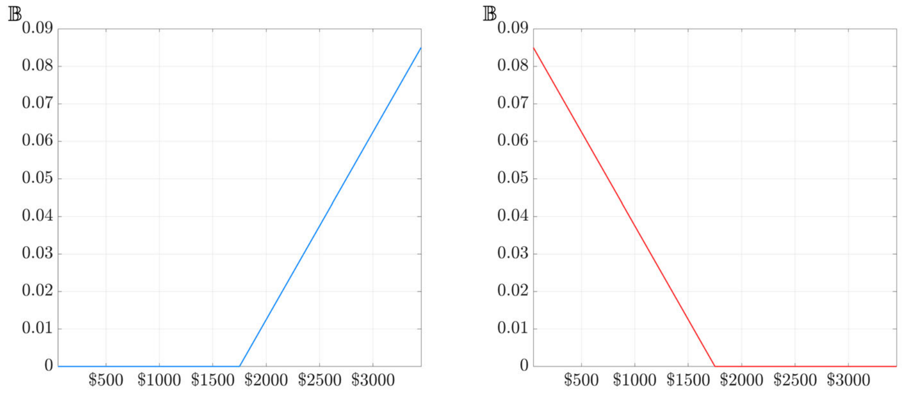

FIGURE 3 Standard quanto option payoffs. The payoffs to a long standard quanto call (left: blue graph) and put (right: red graph). The security (in this case) is ether, the foreign currency is USD and the domestic currency is BTC, that is, the buyer is a BTC-based trader. Thus, the payoff is in BTC and the underlying on the horizontal axis is the USD price of ETH. We set $K^{\mathbb{E}}$ = \$1750. [Color figure can be viewed at wileyonlinelibrary.com]

where $S_T^{\$}$ is the price of ETH in USD at maturity T and the option strike $K^{\$}$ is also denominated in USD. These types of options would enable traders to participate in ETH without physically owning or depositing ether, thus avoiding gas or other blockchain fees.

Figure 3 depicts the payoffs to a quanto call and put given by Equation (6). Note the difference between inverse and $\$/\mathbb{B}$ quanto option, that is, the call quanto payoff is not capped upwards, whereas the inverse pays out a maximum of $\mathbb{B}1$. It is the opposite for the put, the inverse is uncapped and could theoretically pay out an infinite amount.

The main cryptocurrencies like bitcoin and ether have very active options markets but there are thousands of minor tokens that are *only* paired with stablecoins on the major nonfiat exchanges like Binance, never with USD. Therefore, a USD-denominated trader wishing to trade such a token—let us call it XYZ—must take on the (very real) risk of the stablecoin de-pegging from USD while the trade on XYZ is in place. To remove this currency risk, at the same time as leveraging their exposure to XYZ though an option trade, a USD-denominated agent might agree a fixed stablecoin exchange rate with the quanto option issuer. For instance, using a fix of the tether rate $\bar{X}^{\mathbb{T}/\$}$, the quanto *direct* call USD-denominated payoff becomes

$$V_T^{\$} = \bar{X}^{\mathbb{T}/\$}\left(S_T^{\mathbb{T}} - K^{\mathbb{T}}\right)^+, \tag{7}$$

where $S_T^{\mathbb{T}}$ is the tether price of the XYZ token at the time of the option maturity T and the option strike $K^{\mathbb{T}}$ of the quanto is also denominated in USDT.

3.4 | Quanto inverse options

A quanto inverse option is a natural extension of both inverse and quanto direct options, which converts the inverse option payoff to another currency using an exchange rate that is fixed upfront and agreed between both parties when the contract is issued. This allows traders to mitigate the currency risk that is unavoidable when trading inverse options.

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

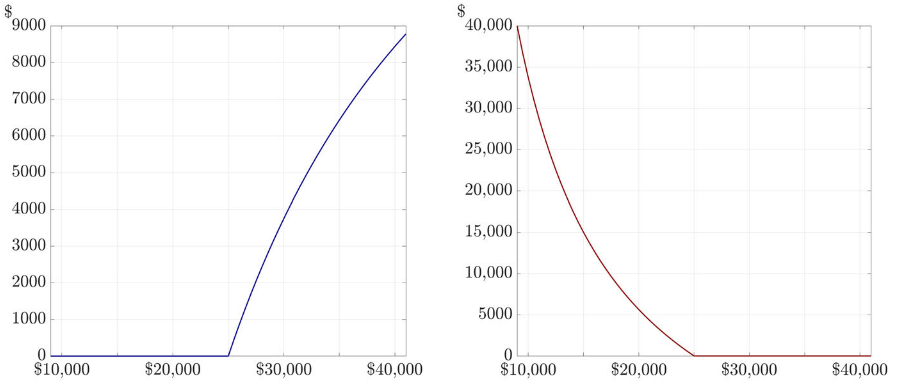

FIGURE 4 Quanto inverse options. The payoff to a long quanto inverse call (left: blue graph) and put (right: red graph) as a function of the settlement price. In Equation (8), we set $\mathbb{Y} = \mathbb{B}$, that is, the notional is 1 bitcoin, and translate the inverse option payoff to USD by setting $\mathbb{Z} = \$$. Both the quanto factor and the option strike are set to \$25,000. [Color figure can be viewed at wileyonlinelibrary.com]

The payoff to a quanto inverse call with a notional of $N = 1$ coin in cryptocurrency $\mathbb{Y}$ can be converted to a currency $\mathbb{Z}$ (either crypto or fiat) at a fixed exchange rate $\bar{X}^{\mathbb{Y}/\mathbb{Z}}$, yielding the payoff:

$$V_T^{\mathbb{Z}} = \bar{X}^{\mathbb{Y}/\mathbb{Z}} \frac{\left(S_T^{\mathbb{Z}} - K^{\mathbb{Z}}\right)^+}{S_T^{\mathbb{Z}}}, \tag{8}$$

where $S_T^{\mathbb{Z}}$ and $K^{\mathbb{Z}}$ are the settlement and strike prices of an option on a token XYZ, both denominated the cryptocurrency $\mathbb{Z}$ These options have payoffs that mimic the convex put and concave call payoffs of inverse options, but they are denominated in a different currency.[19] The quanto factor $\bar{X}^{\mathbb{Y}/\mathbb{Z}}$ changes the slope of the terminal payoff and consequently also the option price prior to expiry. Depending on the choice of $\bar{X}^{\mathbb{Y}/\mathbb{Z}}$, a quanto inverse call (or put) could have higher or lower prices than a direct call (or put), as shown later—see Figure 6. Intuitively, the buyer of such a call may seek to fix the exchange rate slightly higher or lower to her expected future exchange rate, depending on her position. Compared with their direct counterpart, a quanto inverse call loses more value as the option moves further in the money. As such, a quanto inverse call provides an affordable alternative to gain exposure to a crypto options market by paying a lower price than one would for an in-the-money (ITM) direct call of the same strike.

Figure 4 displays the payoffs to quanto inverse calls and puts on the BTC–USD trading pair. In this example, we also use a fixed BTC/USD exchange rate (but in general the exchange-rate fix could be in BTC relative to any other crypto or fiat currency), so the shapes of these payoffs are exactly similar to the nonquanto inverse option payoff in Figure 2. The only difference is that the vertical axis is now in USD units. In particular, the quanto inverse put payoff again increases rapidly as the underlying depreciates. This feature makes quanto inverse puts an excellent insurance against a black swan crypto event. For instance, suppose that in February 2020, a trader bought a quanto inverse put, with a notional of 1 bitcoin, expiring on March 13, 2020. Also, suppose both the strike and the quanto factor of this put were fixed at \$9000, which is reasonable since this was about the BTC/USD rate in February 2020. On March 12, 2020, the BTC/USD rate fell to about \$3500, so suppose this was the settlement price. Thus, a standard put would have paid \$5500 if held to expiry. But the quanto inverse put would have paid $\$9000 \times \$5500/3500 =$ \$14,143, which is more than double that of the standard put.

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

TABLE 2 Payoffs to standard and inverse options ± currency protection.

Call	Pay-off function	$S_T^\$$ $10,000	$20,000	$30,000	$40,000	$50,000
Standard	$(S_T^\$ - K^\$)^+$	$0	$0	$5000	$15,000	$25,000
Inverse	$(S_T^\$ - K^\$)^+/S_T^\$$	₿0	₿0	₿0.17	₿0.38	₿0.5
Standard quanto	$\bar{X}^{\$/₿}(S_T^\$ - K^\$)^+$	₿0	₿0	₿0.22	₿0.67	₿1.11
Quanto inverse	$\bar{X}^{₿/\$}(S_T^\$ - K^\$)^+/S_T^\$$	$0	$0	$3750	$8438	$11,250
Put						
Standard	$(K^\$ - S_T^\$)^+$	$15,000	$5000	$0	$0	$0
Inverse	$(K^\$ - S_T^\$)^+/S_T^\$$	₿1.5	₿0.25	₿0	₿0	₿0
Standard quanto	$\bar{X}^{\$/₿}(K^\$ - S_T^\$)^+$	₿0.67	₿0.22	₿0	₿0	₿0
Quanto inverse	$\bar{X}^{₿/\$}(K^\$ - S_T^\$)^+/S_T^\$$	$33,750	$5625	$0	$0	$0
Call	**Pay-off function**	$S_T^\$$ **$500**	**$1000**	**$1500**	**$2000**	**$2500**
Standard	$(S_T^\$ - K^\$)^+$	$0	$0	$0	$250	$750
Inverse	$(S_T^\$ - K^\$)^+/S_T^\$$	𝔼0	𝔼0	𝔼0	𝔼0.125	𝔼0.3
Standard quanto	$\bar{X}^{\$/₿}(S_T^\$ - K^\$)^+$	₿0	₿0	₿0	₿0.01	₿0.03
Quanto inverse	$\bar{X}^{𝔼/\$}(S_T^\$ - K^\$)^+/S_T^\$$	$0	$0	$0	$250	$600
Put						
Standard	$(K^\$ - S_T^\$)^+$	$1250	$750	$250	$0	$0
Inverse	$(K^\$ - S_T^\$)^+/S_T^\$$	𝔼2.5	𝔼0.75	𝔼0.17	𝔼0	𝔼0
Standard quanto	$\bar{X}^{\$/₿}(K^\$ - S_T^\$)^+$	₿0.06	₿0.03	₿0.01	₿0	₿0
Quanto inverse	$\bar{X}^{𝔼/\$}(K^\$ - S_T^\$)^+/S_T^\$$	$5000	$1500	$333	$0	$0

Note: We compare the payoffs to standard and inverse calls and puts with and without the currency protection provided by a fixed quanto factor. The currency of each payoff is recorded and numbers are rounded. The upper part is for options on BTC/USD, that is, with $S^\$$ = BTC/USD, with strike $K^\$$ = $25,000 and different settlement prices from $10,000 to $50,000. The quanto factor for converting the standard option payoff from USD to BTC is $\bar{X}^{\$/₿}$ = $22,500, and for converting the inverse option payoff from BTC to USD, it is $\bar{X}^{₿/\$}$ = $22,500^{-1} = ₿0.0000444. The lower part is for options on ETH–USD, that is, where $S^\$$ = ETH–USD, with strike $K^\$$ = $1750 and different settlement prices from $500 to $2500. The quanto factor for converting the standard option payoff from USD to BTC is again $\bar{X}^{\$/₿}$ = $22,500, but now for converting the inverse option payoff from ETH to USD, we use $\bar{X}^{𝔼/\$}$ = $2000^{-1} = 𝔼0.0005.

Finally, consider how a quanto inverse option might be constructed for a USD-denominated trader who seeks exposure to a crypto asset XYZ, but instead of trading a direct option on XYZ with currency protection against decoupling of stablecoin price from USD, as in Section 3.1, the trader prefers to gain exposure to an inverse option pay-off. Reasons for this could be (1) for the same terminal value of the underling, the quanto inverse option profit is greater than that from direct option; and (2) a quanto inverse put option payoff is not only convex but uncapped. In this case, a USD-denominated trader could fix the USD price of XYZ upfront, but still retain an optional exposure to XYZ in the form of an inverse option payoff, which has payoff in XYZ.

3.5 | Summary and comparison of payoffs

Table 2 summarizes the payoffs to different types of calls and puts with and without currency protection. As before, we use $, ₿, and 𝔼 to denote payoffs in USD, bitcoin, and ether, for example, ₿1 is 1 bitcoin or $10 is ten dollars. Note that option strikes and underlyings are always quoted in

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

USD. The upper panel considers standard and inverse options on the exchange rate BTC/USD with strike $K^{\$} = \$25,000$ and we report the payoffs for different settlement prices. The quanto versions suppose a prefixed exchange rate of $\bar{X}^{\$/\mathbb{B}} = \$22,500$. The lower panel is for options on the ETH–USD trading pair, where ETH is regarded as a security and as such could be replaced by any other crypto asset, as discussed above. In the table, we let the option strike be $K^{\$} = \$1750$ and again consider different scenarios for the settlement price.

The lower part of Table 2 displays the payoff to standard and inverse calls and puts on the ETH–USD trading pair. Below this, the standard quanto option payoff is for a *BTC-based* trader in ETH–USD, seeking currency protection against a fall in the price of BTC relative to the dollar. The fourth payoff is for a *USD-based* trader entering a quanto inverse option on ETH–USD, seeking currency protection against a fall in the price of ETH relative to the US dollar. Notice the use of notation ETH–USD means we assume that ETH is security here—and indeed could be replaced by any other crypto XYZ that is regarded as a security. We are particular to use notation ETH–USD in this case rather than ETH/USD, the latter specifically denoting an exchange rate in this paper.

As already remarked, for every settlement price yielding a nonzero payoff, the standard call has a greater payoff than the quanto inverse call, but the standard put has a much smaller payoff than the quanto inverse put. This ordering holds for both BTC/USD and ETH–USD options, as shown in the table. Furthermore, the table compares the payoffs to other types of options. For instance, in the BTC/USD case (upper panel), the inverse and standard quanto options are both denominated in BTC, and again the calls and puts have opposite ordering. That is, the inverse call has a smaller payoff than the standard quanto but the inverse put has a greater payoff than the standard quanto. Finally, we note that Table 2 only considers standard and inverse options because the direct options are similar to the standard ones, except the USD payoff is in a stablecoin.

4 | PRICING CONSIDERATIONS

The technical issues concerning inverse option pricing focus on the debate about bitcoin—is it a currency, commodity, or a security? While the fundamental idea behind bitcoin was to facilitate a distributed payment system, and its use today reflects a store of value to a certain extent, its high volatility suggests that it acts more like a commodity or security than a currency. Other coins like ether were never created as a medium of exchange or a store of value, but rather as consumption goods. We can picture ether as the gas for the smart contract cars that run on the highway called the Ethereum blockchain. The debate continues at the highest level of regulation and we cannot hope to bring closure to it here. Nevertheless, the question whether a crypto asset is a security, a commodity, or a currency is actually central to the arguments presented here.

In the following, we skip the pricing for direct and quanto options because the already voluminous extant literature still applies to these. Instead, we hope to shed some light on the rather confusing properties of inverse options and present an in-depth analysis of pricing quanto inverse options.

4.1 | Inverse option valuation

For this case, we consider bitcoin as a currency, and the underlying of a bitcoin option contract is a *tradable* BTC/USD exchange rate, denoted $S_t^{\$}$. Assuming well-functioning money markets exist for both currencies, we can define two cash bond accounts $B_t^{\$} = e^{r^{\$} t}$ and $B_t^{\mathbb{B}} = e^{r^{\mathbb{B}} t}$ as the

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

respective numéraires, where $r^{\$}$ and $r^{\mathbb{B}}$ are the risk-free interest rates in the corresponding currencies. Furthermore, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space with filtration $\mathcal{F}$ and probability measure $\mathbb{P}$. We assume log-normal asset dynamics under the physical measure $\mathbb{P}$:

$$\frac{dS_t^{\$}}{S_t^{\$}} = \mu^{\$}\, dt + \sigma^{\$}\, dW_t^{\$}, \tag{9}$$

where $\mu^{\$}$ is the drift, $\sigma^{\$}$ is the volatility, and $W^{\$}$ denotes a standard Brownian motion. In this framework, we can rewrite the payoff $V_T^{\mathbb{B}}$ to a T-maturity vanilla call on $S_T^{\$}$ from Equation (3) to

$$V_T^{\mathbb{B}} = K^{\$}\left(K^{\mathbb{B}} - S_T^{\mathbb{B}}\right)^+, \tag{10}$$

where $S^{\mathbb{B}} = \frac{1}{S^{\$}}$ the opposite exchange rate, that is, USD/BTC. Note that a USD-denominated call becomes a BTC-based put with strike $K^{\mathbb{B}}$ being the inverse of the USD-strike price. Note that the option can be priced trivially via standard FX option pricing formulae, and satisfies the put–call duality conditions as articulated, for example, in Garman and Kohlhagen (1983). The price of the put in BTC is given by

$$\begin{aligned}
P_t^{\mathbb{B}} &= e^{-r^{\mathbb{B}}\tau} K^{\mathbb{B}} \Phi\left(-d_2^{\mathbb{B}}\right) - e^{-r^{\$}\tau} S_t^{\mathbb{B}} \Phi\left(-d_1^{\mathbb{B}}\right), \\
d_1^{\mathbb{B}} &= \frac{1}{\sigma^{\mathbb{B}}\sqrt{\tau}}\left[\ln\left(\frac{S_t^{\mathbb{B}}}{K^{\mathbb{B}}}\right) + \left(r^{\mathbb{B}} - r^{\$} + \frac{(\sigma^{\mathbb{B}})^2}{2}\right)\tau\right], \\
d_2^{\mathbb{B}} &= d_1^{\mathbb{B}} - \sigma^{\mathbb{B}}\sqrt{\tau}.
\end{aligned} \tag{11}$$

Similarly, we can express the variables in a USD-denominated framework:

$$\begin{aligned}
d_1^{\mathbb{B}} &= \frac{1}{\sigma^{\$}\sqrt{\tau}}\left[-\ln\left(\frac{S_t^{\$}}{K^{\$}}\right) - \left(r^{\$} - r^{\mathbb{B}} - \frac{(\sigma^{\$})^2}{2}\right)\tau\right] = -d_2^{\$}, \\
d_2^{\mathbb{B}} &= -d_1^{\$}.
\end{aligned}$$

Clearly, $K^{\$}$ units of BTC-denominated puts are equivalent to $\frac{1}{S_t^{\$}}$ units of USD-denominated calls:

$$K^{\$} P_t^{\mathbb{B}} = e^{-r^{\mathbb{B}}\tau}\Phi\left(-d_2^{\mathbb{B}}\right) - e^{-r^{\$}\tau} K^{\$} S_t^{\mathbb{B}} \Phi\left(-d_1^{\mathbb{B}}\right) = \frac{1}{S^{\$}} C_t^{\$}. \tag{12}$$

This pricing approach can be readily extended beyond Black–Scholes world. Solutions to option pricing problems are attainable, at least in the Fourier transform sense, for any tractable Lévy processes governing the evolution of the USD/BTC exchange rate. In particular, Sepp and Rakhmonov (2022) show that put–call parity and duality conditions similar to Equation (12) still hold under a log-normal SV model with quadratic drift. They also derive the corresponding pricing formulae for direct and inverse options and the necessary and sufficient conditions under which price and inverse price processes are martingales with finite moments, for the log-normal SV models with quadratic and linear drifts, the Heston model, and the exponential Ornstein–

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Uhlenbeck model. Teng and Härdle (2022) calibrate a stochastic volatility with correlated jumps (SVCJ) model to Deribit BTC inverse options and perform a dynamic delta hedging study using an array of nested model-dependent deltas, measuring out-of-sample performance using the hedging error relative to the BS delta. Like Alexander and Imeraj (2023), they find that it is difficult to identify one single model that consistently outperforms the BS delta. They also conclude that the broader SVCJ model outperforms its nested models in terms of in-sample and out-of-sample pricing.

Crucially, this simple pricing approach relies on the assumptions that both markets denominated in USD and in BTC are complete, and there are no restrictions on exchanging wealth from one to the other. However, considering the current state of bitcoin options, we must note that this assumption is not satisfied. Not only is there no well-functioning money market for bitcoin, it does not exist at all.[20] Second, the exchanges on which inverse options are traded (which accept no fiat and are unregulated) use nontraded underlyings. For example, the Deribit bitcoin inverse options use their own "Deribit Bitcoin Index" as the underlying, which is an average of bitcoin spot prices from (currently) 11 different centralized exchanges. Due to its frequent rebalancing, the physical replication of this index is an immensely difficult and expensive task. An agent would be required to hold bitcoin positions on multiple exchanges and rebalance these constantly. Moreover, the final option settlement price is the average of the index during the last 30 min before expiry. This important feature about Deribit inverse options is often ignored. But the price of bitcoin can change considerably during 30 min—much more than we see for traditional financial instruments. Hence, the underlying is not tradable and the market is incomplete. Therefore, Equation (11) provides only an approximate option price, even in a Black–Scholes world.

To fully understand the valuation and hedging of bitcoin options on Deribit, we need to consider the actual state of the cryptocurrency option market. To circumvent the difficulties in reconstructing and trading the underlying index of the Deribit bitcoin options, we consider instead the Deribit perpetual futures because this tracks the index closely through the funding mechanism. However, this introduces an unhedgeable basis risk that can be sizable in a volatile market. Now, option pricing in an incomplete market in the presence of basis risk is typically solved through indifference pricing by formulating a stochastic control problem in the mean of the Hamilton–Jacobi–Bellman (HJB) partial differential equation (Monoyios, 2004), or by solving the corresponding forward–backward stochastic differential equation (Rouge & El Karoui, 2000). Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as in the standard inverse pricing case. The nontradable underlying asset (in this case the Deribit bitcoin index, $S_t^{\mathbb{B}}$) and the hedging instrument (the perpetual futures, $Y_t^{\mathbb{B}}$) evolve according to the geometric Brownian motions:

$$\frac{dS_t^{\mathbb{B}}}{S_t^{\mathbb{B}}} = \mu_{\mathbb{B}} dt + \sigma_{\mathbb{B}} dW_t^{\mathbb{B}}, \tag{13}$$

$$\frac{dY_t^{\mathbb{B}}}{Y_t^{\mathbb{B}}} = \mu dt + \sigma dW_t, \tag{14}$$

where $W^{\mathbb{B}}$ and W correlated standard Brownian motions with $\langle dW^{\mathbb{B}}, dW \rangle = \rho\, dt$. Note that S and Y are not perfectly correlated, that is, $|\, \rho \,| < 1$ and hence we cannot find a unique equivalent martingale measure (EMM) for which the discounted value of Y is a martingale under risk-neutral

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

measure. We rewrite

$$W_t^{\mathbb{B}} = \rho W_t + \epsilon \widetilde{W}_t^{\mathbb{B}},$$

where $\epsilon = \sqrt{1-\rho^2}$ with independent $\widetilde{W}_t^{\mathbb{B}}$ and W_t. We further denote by $\{\mathcal{G}_t\}_{0\le t\le T}$ the filtration generated by $W_t^{\mathbb{B}}$. Note that this Brownian motion drives the nontraded inverse index asset.

Assume that there exists an equivalent measure $\mathbb{Q}$ to $\mathbb{P}$ on $\mathcal{F}$. Then there exists adapted processes m_T and g_T where the Radon–Nykodym derivative is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = M_t, \tag{15}$$

where $\{M_t\}_{0\le t\le T}$ is a $\mathbb{P}$-martingale with M_t is given by

$$M_t = \exp\left[\int_0^t m_u dW_u + \int_0^t g_u d\widetilde{W}_u^{\mathbb{B}} - \frac{1}{2}\int_0^t m_u^2 du - \frac{1}{2}\int_0^t g_u^2 du\right]. \tag{16}$$

Using the multidimensional Girsanov theorem, we see that the processes $\{\widetilde{W}_t, \hat{W}{}_t^{\mathbb{B}}\}_{0\le t\le T}$ defined by

$$\begin{pmatrix} \widetilde{W}_t \\ \hat{W}{}_t^{\mathbb{B}} \end{pmatrix} = \begin{pmatrix} W_t + \int_0^t m_u du \\ \widetilde{W}_t^{\mathbb{B}} + \int_0^t g_u du \end{pmatrix}, \tag{17}$$

is an independent Brownian motion under Q. Further, the dynamics under Q are given by

$$\frac{dS_t^{\mathbb{B}}}{S_t^{\mathbb{B}}} = (\mu_{\mathbb{B}} - \sigma_{\mathbb{B}}(\rho m_t + \epsilon g_t))dt + \sigma_{\mathbb{B}} dW_t, \tag{18}$$

$$\frac{dY_t^{\mathbb{B}}}{Y_t^{\mathbb{B}}} = (\mu - \sigma m_t)dt + \sigma d\underset{t}{\hat{W}}, \tag{19}$$

where $\underset{t}{\hat{W}} = \rho \widetilde{W}_t + \epsilon \hat{W}{}_t^{\mathbb{B}}$ is a Brownian motion such that $\langle d\widetilde{W}_t, d\underset{t}{\hat{W}}\rangle = \rho\, dt$. For $\mathbb{Q}$ be a local EMM, Y_t needs to be a $\mathbb{Q}$-local martingale, that is, iff:

$$\mu - m_t\sigma = r \quad \Rightarrow \quad m_t = m := \frac{\mu - r}{\sigma}.$$

Note that the EMM is uniquely defined. On the other hand, X is nontradable, which lets g_T be of any arbitrary form, which results in an infinite set of possible EMM. We define this set as $\mathcal{M}$, which is in correspondence with the set of g_t. We further want to link the local martingale with an equivalent probability measure. Denote an equivalent measure $\widetilde{\mathbb{P}}$ to $\mathbb{P}$ on $\mathcal{G}$, the risk-neutral density given by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \widetilde{m}_t, \tag{20}$$

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$\widetilde{m}_t = \exp\left\{ -\int_0^t \theta_u dW_u^{\mathbb{B}} - \frac{1}{2}\int_0^t \theta_u^2 du \right\}, \tag{21}$$

where θ_t is a $\mathcal{G}_t$-adapted process. Under $\widetilde{\mathbb{P}}$ we define $\hat{W}$ as

$$\underset{t}{\hat{W}} = W_t^{\mathbb{B}} + \int_0^t \theta_u du \tag{22}$$

and underlying dynamics under $\widetilde{\mathbb{P}}$ given by

$$\frac{dS_t^{\mathbb{B}}}{S_t^{\mathbb{B}}} = \left(\mu_{\mathbb{B}} - \sigma_{\mathbb{B}}\theta_t\right)dt + \sigma_{\mathbb{B}} d\underset{t}{\hat{W}}. \tag{23}$$

Note that the dynamics of the nontradable inverse index are the same under $\mathbb{Q}$ and $\widetilde{\mathbb{P}}$ when the integrands m_t, g_t, and θ_t are given as

$$\rho m_t + \epsilon g_t = \theta_t.$$

Many arbitrage valuation models such as the BSM model rely on the assumption that any option's claim can be replicated by a portfolio and hence hedged perfectly. The absence of such a hedging portfolio leads to major difficulties when evaluating options in this fashion. A preference-dependent approach is, therefore, necessary for pricing these options. By making assumptions about the shape of the trader's utility function and their initial wealth endowments, the option pricing problem can be transformed into an optimal trading strategy problem where the trader seeks to maximize her expected utility with and without trading in options contracts. Many papers already address this topic in much detail—see Rouge and El Karoui (2000), Monoyios (2004), Ankirchner et al. (2010), Davis (2000), and many others. This way, it can be presented as the numerical solution to the corresponding HJB partial, or the forward–backward stochastic, differential equation. However, there are significant limitations to using this type of model for pricing and hedging purposes. The option pricing formula and the associated Greeks depend on whether the option position is long or short, and the type of utility functions used can often be restrictive—in some cases, it would even yield option prices that depend on the agent's initial wealth.

This discussion suggests that a preference-dependent approach is essential for pricing Deribit options. However, there exists a trade-off between the complete and the incomplete market model, that is, facing potential pricing error due to basis risk versus the necessity of preference-dependent pricing. On the one hand, Alexander and Imeraj (2023) show that the basis risk between index and perpetuals oscillates mostly around ±10 bps, particularly since 2022. On the other hand, it could be challenging to identify the correct preferences due to prominent sentimental and behavioral features in crypto markets, specifically, due to the diversity of traders, that is, (semi-professional) retail and institutional participants. Formulating correct assumptions of the agent's preference is likely unrealistic, as this space is continuously evolving. Specifically, the open interest for deep OTM options ($m > 2$) seen in Table 1 leads us to question the rationality of some bitcoin option traders. Thus, the risk of mis-specifying investor preference outweighs the risk of mis-pricing with

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

basis. Further analysis may be warranted to discern the sentimental and behavioral features, but will not be addressed in this paper.

4.2 | Quanto inverse option valuation

It is worth noting upfront that the quanto inverse option tabulated in Table 2 shares a payoff structure similar to the inverse option payoff, but is denominated differently. This subtle difference has important consequences in terms of how these options should be priced. Consider first a standard FX call on/USD that matures at time T, with either a USD-denominated payoff:

$$V_T^{\$} = \left(S_T^{\$} - K^{\$}\right)^+, \tag{24}$$

or a BTC-denominated payoff:

$$V_T^{\mathbb{B}} = K^{\$}\left(K^{\mathbb{B}} - S_T^{\mathbb{B}}\right)^+. \tag{25}$$

We have already shown that this option can be priced in a standard FX option pricing framework. Now consider an exotic option that pays

$$V_T^{\$} = \bar{X}^{\mathbb{B}/\$} \frac{\left(S_T^{\$} - K^{\$}\right)^+}{S_T^{\$}}, \tag{26}$$

where $\bar{X}^{\mathbb{B}/\$}$ is a predetermined exchange rate that transforms the bitcoin denominated point value of an standard inverse option to a USD payoff. In the following, we simplify notation and omit the superscript, so we set $\bar{X}^{\mathbb{B}/\$} = \bar{X}$. Similar to CME options, these would be USD-settled options, on the exchange rate of any token, or a reference rate or a futures contract. In this example, we focus on the USD value of one bitcoin. Using the same notation as in Section 3.4, we denote the underlying of this product at maturity by $S_T^{\$}$, and highlight its USD-denomination via the superscript. Without losing generality, we assume that the dividend yield is zero, and the US money market provides a risk-free interest rate r. The risk-neutral $\mathbb{Q}$-dynamics of the underlying are given by

$$\frac{dS_t^{\$}}{S_t^{\$}} = r dt + \sigma dW_t,$$

$$S_T^{\$} = S_t^{\$} \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(W_T - W_t)\right\}.$$

Now let $Y_t^{\$} = (S_t^{\$})^{-1}$. By Itô's lemma, the $\mathbb{Q}$-dynamics of $Y_t^{\$}$ are

$$\frac{dY_t^{\$}}{Y_t^{\$}} = \left(\sigma^2 - r\right)dt - \sigma dW_t. \tag{27}$$

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The value of the discounted price process is, therefore,

$$\widetilde{Y}_t = e^{-r\tau}\,\mathbb{E}^{\mathbb{Q}}\left[Y_T^{\$} \mid \mathcal{F}_T\right] = Y_T^{\$} e^{(\sigma^2-2r)\tau}$$

where $\tau = T - t$. We note that $\widetilde{Y}_t$ is a martingale under the $\mathbb{Q}$-measure and hence we perform a change of numéraire from the risk-neutral measure $\mathbb{Q}$ to an EMM $\widetilde{\mathbb{Q}}$ where $\widetilde{Y}_t$ is the new numéraire. We can rewrite the dynamics of $S_t^{\$}$ as

$$\frac{dS_t^{\$}}{S_t^{\$}} = \left(r-\sigma^2\right)dt + \sigma d\widetilde{W}_t,$$

$$S_t^{\$} = S_0^{\$}\exp\left\{\left(r-\frac{3}{2}\sigma^2\right)t + \sigma\widetilde{W}_t\right\},$$

where $\widetilde{W}_T = W_t + \sigma t$ is a Wiener process under the martingale measure $\widetilde{\mathbb{Q}}$. Under these assumptions, we can express the Radon–Nikodym derivative by

$$\left.\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\right|_t = \exp\left\{\frac{1}{2}\sigma^2 t + \sigma W_t\right\}.$$

Denoting the price of the quanto inverse call at t by C_t^{q}, its valuation under the new measure is trivial since it is a plain vanilla option and follows the exact same steps as the risk-neutral derivation of the Black–Scholes formula:

$$C_t^{\mathrm{q}} = \frac{B_t}{B_T}\,\mathbb{E}_t^{\mathbb{Q}}\left[\bar{X}\left(\frac{S_T^{\$}-K^{\$}}{S_T^{\$}}\right)^+\right] = \mathbb{E}_t^{\widetilde{\mathbb{Q}}}\left[\left(\left.\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\right|_T\right)^{-1}\frac{\bar{X}}{S_T^{\$}}\left(S_T^{\$}-K^{\$}\right)^+\right]$$

$$= e^{(\sigma^2-2r)\tau}\frac{\bar{X}}{S_t^{\$}}\,\mathbb{E}_t^{\tilde{\mathbb{Q}}}\left[\left(S_T^{\$}-K^{\$}\right)^+\right] = e^{(\sigma^2-2r)\tau}\frac{\bar{X}}{S_t^{\$}}\int_{-\infty}^{\infty}\left(S_T^{\$}(z)-K^{\$}\right)^+\phi(z)dz, \tag{28}$$

where $S_T^{\$}(z) = S_t^{\$}\exp\{\frac{1}{2}\sigma^2\tau + \sigma\sqrt{\tau}z\}$, z is drawn from a standard normal distribution, and $\phi(\cdot)$ is the corresponding probability density function. Note that

$$\left(S_T^{\$}(z)-K^{\$}\right)^+ = 0 \Leftrightarrow z \geq \frac{\ln\left(\frac{S_t^{\$}}{K^{\$}}\right)+\left(r-\frac{3}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} = d_3.$$

Thus, using $\Phi(\cdot)$ to denote the standard normal distribution function, we can calculate the integral on the RHS of Equation (28):

$$\int_{-\infty}^{d_3}\left(S_T^{\$}(z)-K^{\$}\right)\phi(z)dz = S_t^{\$}\int_{-\infty}^{d_3}\exp\left\{\left(r-\frac{3}{2}\sigma^2\right)\tau+\sigma\sqrt{\tau}z\right\}\phi(z)dz - K^{\$}\Phi(d_3).$$

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Evaluating the integral yields the GBM price of the inverse call with strike $K^{\$}$ as

$$C_t^{q} = e^{-r\tau}\bar{X}\left[\Phi(d_2) - e^{(\sigma^2-r)\tau}Y_t^{\$}K^{\$}\Phi(d_3)\right], \tag{29}$$

where $d_2 = d_3 + \sigma\sqrt{\tau}$ is the same as we see in the Black–Scholes FX pricing formula, that is, $d_2 = d_2^{\$}$ under the assumption $r^{\mathbb{B}} = 0$. A similar argument yields the time t GBM price of an inverse put with strike $K^{\$}$ USD and maturity T as

$$P_t^{q} = e^{-r\tau}\bar{X}\left[e^{(\sigma^2-r)\tau}Y_t^{\$}K^{\$}\Phi(-d_3) - \Phi(-d_2)\right]. \tag{30}$$

The two likelihood functions $\frac{d\mathbb{Q}^{\mathbb{B}}}{d\mathbb{Q}^{\$}}$ and $\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}$, albeit very similar, result in different pricing and hedging properties for the two functions, driven by the difference in denomination between these two contracts. In the quanto inverse option case it is possible, and indeed more convenient, to start from the risk-neutral dynamics of the underlying because the payoff is denominated in USD. One may define the EMM for the quanto inverse of the underlying denominated in USD, and price the option accordingly. However, this approach is inappropriate for an inverse option when the payoff is denominated in BTC. Due to Siegel's exchange paradox, denomination conversion from USD to BTC should be performed first under $\mathbb{P}$. Then a risk-neutral measure in BTC can be established, which is symmetrical to the risk-neutral measure in USD. This would be the appropriate measure for pricing derivatives denominated in BTC.

Figure 5 compares (a) the USD-denominated inverse BSM prices and payoffs (above) with (b) the prices and payoffs for quanto inverse options (below). These are displayed above as a function of $S_T^{\$}$, and we display prices for different maturities ranging from 10 days to a year. The panels on the left show calls (blue) and on the right, we have the puts (red). The inverse option payoffs have the familiar convex structure of the BSM pricing function, with an increasing gamma and a positive vega, and the price approaches the payoff as the option approaches expiry. The longer the time to expiration, the more valuable the options, that is, the theta is positive.

The quanto inverse option pricing functions depicted in the lower panels of Figure 5 behave very differently. The payoff to a quanto inverse call is capped above at $\bar{X}$. Deep ITM quanto inverse calls decrease in value as the time to maturity increases, where they are valued below their intrinsic values. Indeed, very deep ITM quanto inverse call prices could be much lower than one would think by simply looking at ATM option prices.[21] Furthermore, the convexity of the quanto inverse call pricing function changes as the underlying price increases: it starts as a convex function but then changes into a concave function as the option moneyness increases. Therefore the delta of a quanto inverse call is not a monotonic increasing function with respect to the underlying price, as it is for vanilla options. There exists a global maximum at which gamma changes from being positive to being negative. For deep OTM and ATM options, the term structure of quanto inverse calls resembles that of inverse calls but for deep ITM options, this pivots. Counter-intuitively, the price of a short-term deep ITM option exceeds the prices of their long-term counterparts, indicating a *negative* theta for these moneyness levels.

The payoff to the quanto inverse put in Figure 5 is uncapped with the put price tending towards ∞ as the underlying price falls, whereas the payoff to a nonquanto inverse put converges to K. For all moneyness and maturities, the put pricing curves are convex decreasing with the strongest price sensitivity for ATM options. Unlike the inverse put delta, the quanto inverse put delta decreases monotonically with the underlying price, but it has no lower bound. The theta for a

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

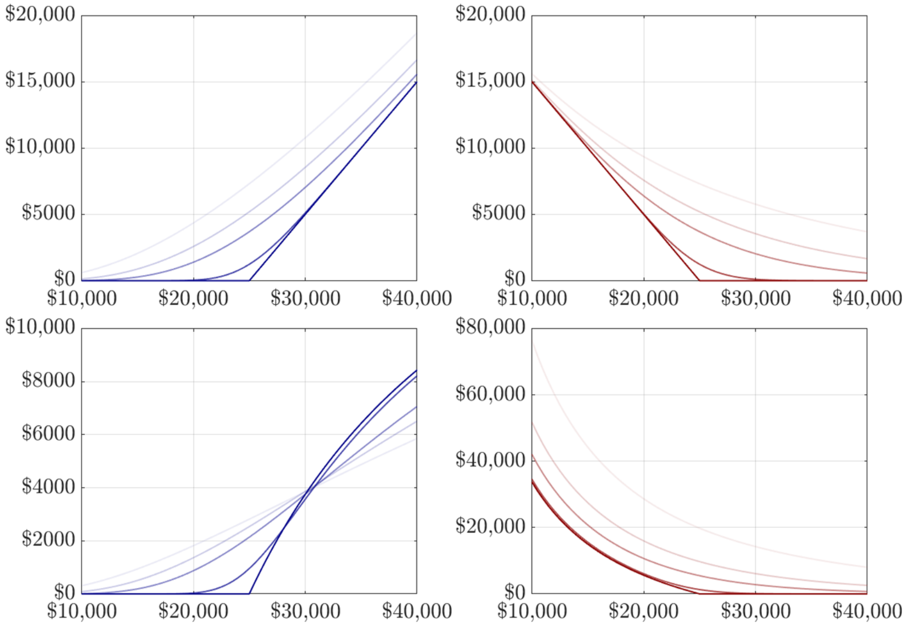

FIGURE 5 Inverse and quanto inverse option prices under GBM. Option payoffs and prices obtained using the Black–Scholes formula for the inverse options and our formula (28) for the quanto inverse options. Prices are represented as a function of the underlying price with a thicker line as the option approaches expiry. Time to maturities of 10 days, 3 months, 6 months, and 1 year are shown. In these plots, the first column displays calls in blue and the second displays puts in red; the upper row shows the inverse option prices and payoffs, and the lower shows the quanto inverse option prices and payoffs. All four plots are calculated using the same $K = \$25,000$, $r = 0\%$, and $\sigma = 75\%$ for all maturities but with different USD-denominated contingent claims (3) and (8), respectively. For the quantos, we set $\bar{X} = \$22,500$. [Color figure can be viewed at wileyonlinelibrary.com]

quanto inverse put is positive, like its inverse option counterpart, that is, the longer the time to maturity, the more valuable the option. It is interesting that the roles of calls and puts are now reversed, in that an inverse call and a quanto inverse put can—theoretically—pay an infinite amount to the holder, whereas the inverse put and quanto inverse call are capped at K and $\bar{X}$, respectively.

Now we investigate the quanto inverse option price dependence on the prefixed conversion factor $\bar{X}$. Figure 6 compares the maturity payoffs to inverse and quanto inverse options for two different values of $\bar{X}$. Suppose an agent who is optimistic, but not euphoric, about future returns enters a long position in a quanto inverse call. She has the greatest interest in negotiating a quanto factor as high as possible because this increases the slope of the terminal payoff. For instance, the bottom left graph in Figure 6 illustrates how the quanto inverse call payoff can exceed that of the inverse call when $\bar{X}$ is high. The advantage of this long position depends on the underlying having a maturity value between K and $\bar{X}$: if the underlying ends below the strike at maturity the option expires worthless; and above $\bar{X}$, the agent would profit more by entering a direct (or standard) option. Market makers could earn a premium by offering these two alternatives to traders seeking to profit from high future volatility. Now consider the inverse versus quanto inverse put, on the right in Figure 6. Here, it is the writer not the buyer of the option who can use quanto inverses to their advantage. Assuming the quanto inverse put ends up ITM with $\bar{X} << K$ (top right plot), then the sell side would reduce their losses up to $\bar{X}$ but exponentially increase them afterwards.

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

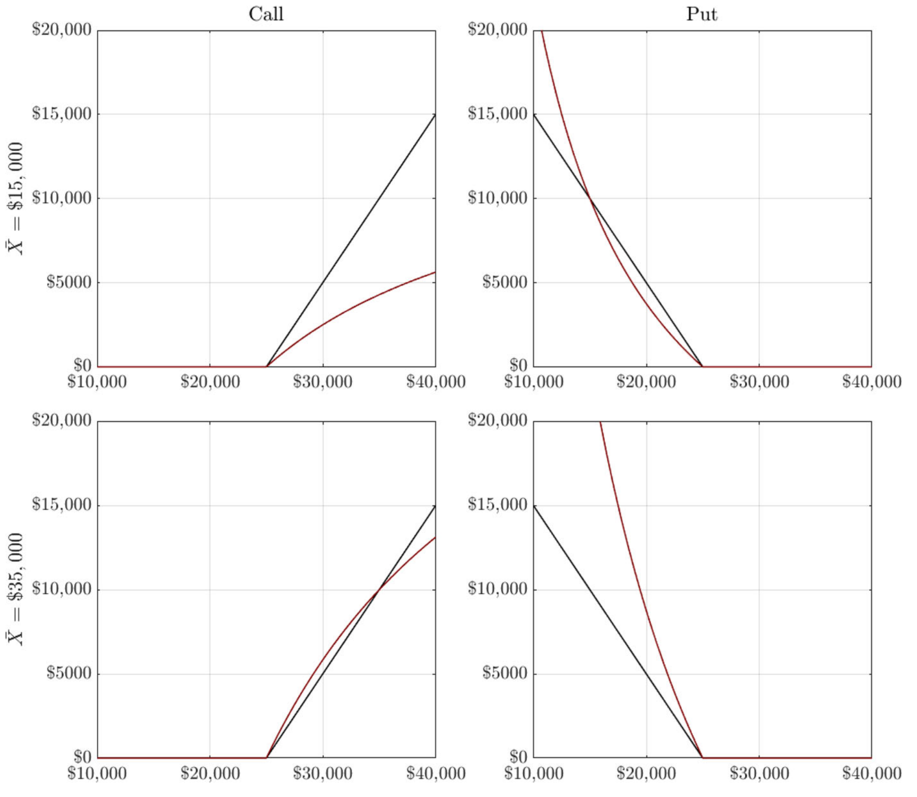

FIGURE 6 Payoff sensitivity with respect to $\bar{X}^{B/\$}$. Payoff comparison of inverse and quanto inverse call (left column) and put (right column) with different predetermined exchange factor $\bar{X}$. The underlying at maturity ranges between \$10,000 and \$40,000 with a strike price at $K^{\$} = \$25{,}000$. We distinguish between inverse (black) and quanto inverse (red) options and compare different $\bar{X}$ between each column, that is, $\bar{X} = \$15,000$ (upper row) and $\bar{X} = \$35,000$ (lower row). [Color figure can be viewed at wileyonlinelibrary.com]

We conclude that the premium on the quanto inverse options could be relatively low to write a call, but would need to be exceptionally high to write a put because, in theory, there is a nonzero probability of the asset price reaching zero resulting in an infinite loss for the writer of quanto inverse puts.

Figure 7 illustrates inverse and quanto inverse call prices as functions of the underlying, for a given fixed strike and interest rate, and for different levels of volatility and time to maturity. Inverse option prices display the familiar pattern of increasing with either time to maturity or volatility. But quanto inverse call prices can be decreasing with volatility (negative vega) as well as maturity (negative theta) as previously discussed. For instance, with a fixed volatility at 200%, the fair price of a 10-day quanto inverse option with an ITM strike level at $K^{\$} = \$30,000$ is \$4123, but this decreases as maturity increases to 90 days (\$4080) and to 180 days (\$3490). But the OTM option with strike $K^{\$} = \$20,000$ is strictly increasing with maturity. The 90-day OTM quanto inverse option has positive vega whereas the ITM option has a vega, which changes sign from positive to negative as the option moves deep ITM; the price increases from \$4020 ($\sigma = 50\%$) to \$4270 ($\sigma = 100\%$) and decreases afterwards to \$4080 ($\sigma = 200\%$). The difference between ITM inverse and quanto inverse option prices is more pronounced than for OTM options, and it also increases

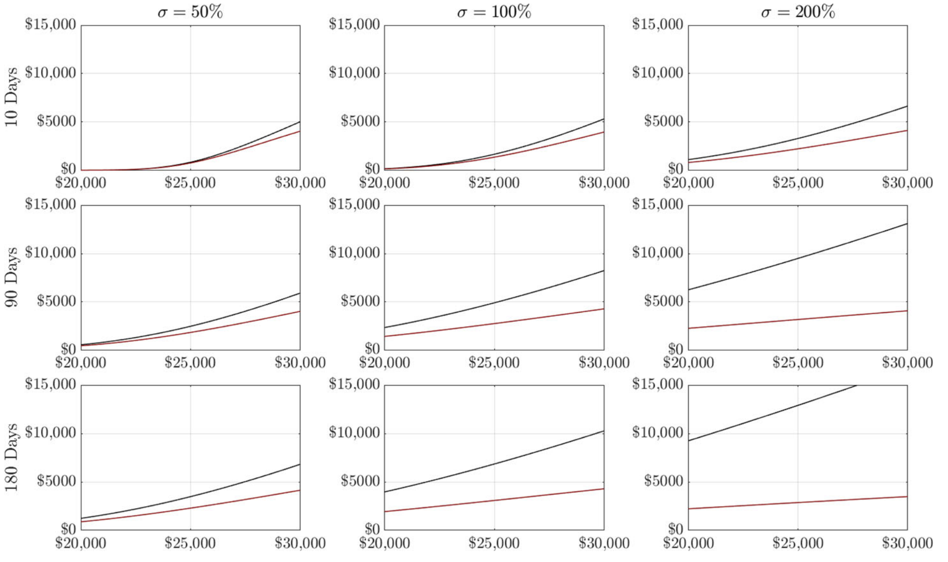

FIGURE 7 Volatility-maturity price sensitivity. Comparison of call prices as a function of the underlying using the inverse pricing formula (blue) and the quanto inverse pricing formula (red), given in USD. The figures depict different times to maturity (ranging from 10 days to 6 months) under different constant volatilites (ranging from 50 to 200%). We assume constant zero interest rate and fix both the underlying and $\bar{X}$ at \$25,000. We display only the 20% area around the strike price. [Color figure can be viewed at wileyonlinelibrary.com]

with maturity and volatility. Especially for long-term options and/or during periods of high volatility, a quanto inverse call provides a very affordable alternative to a standard inverse call.

5 | HEDGE RATIOS

We use an Appendix to display the results discussed in this section. Table A.1 summarizes the pricing formulae for inverse and quanto inverse options and their hedge ratios, under the GBM assumption. As in Section 4.1, we assume $r^{\mathbb{B}} = 0$ and set $r^{\$} = r$ for brevity, this way, we have the same interest rate for both inverse and quanto inverse options. Figures C.1 and C.2 illustrate the delta, gamma, vega, theta, volga, and vanna of inverse and quanto inverse calls as a function of strike $K^{\$}$ for fixed S, σ, and r and for different time to maturity. We set the underlying volatility to 75%, and we compare the inverse options hedge ratios in blue with the corresponding ratios for the quanto inverse option in red, for options of different maturities (10, 30, and 90 days). Figures C.3 and C.4 illustrate the same for puts.

First compare the deltas. The inverse call (put) delta has the usual shape of a monotonically increasing (decreasing) normal distribution function. But the quanto inverse call delta is not monotonic but has a maximum when the underlying price just exceeds the strike and declines thereafter eventually becoming zero for deep ITM options. This is because the payoff is capped at $\bar{X}$, that is, even for large price movements, the change in the payoff is limited, as has already been discussed above when commenting on the shift from convexity to concavity in the pricing function. Comparing the term structure in the two call deltas, short-term OTM (ITM) inverse call deltas are smaller (larger) than their long-term counterparts. The quanto inverse call deltas are similar except that long-term deltas exceed the short-term for deep ITM options. Very frequent

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

rebalancing of a delta-hedged position can have an adverse impact on the volatility of the underlying, especially for short-dated ITM standard calls or puts where the delta is close to ± 1 (Golez & Jackwerth, 2012; Ni et al., 2021). Now, the delta is so much greater on ITM direct (or standard) calls than it is for quanto inverse calls of the same moneyness (indeed the delta approaches zero for very very deep ITM quanto inverse calls). Therefore, the unwanted volatility impact of frequent rebalancing on a delta-hedged position will be very much less for quanto inverse calls.

By contrast with deep ITM quanto inverse calls, a deep ITM quanto inverse put delta can be very much *greater* than the delta of a direct (or standard) put of the same moneyness, so the adverse volatility impact of rapid delta-hedge rebalancing referred to above would be exacerbated. And a delta hedge of a short position on such an option could require buying more of the underlying than the option's notional. However, if this buying pressure causes the underlying price to rise the put would become less ITM and its delta would then decline. Finally, we note that long-term quanto inverse put deltas are generally lower than their short-term counterparts, irrespective of moneyness, except for short- and mid-term near-ATM options.

The inverse call gamma and vega are both positive, following the standard normal density and are identical for call and puts. But this is not the case for quanto inverse options because the nonmonotonicity in the quanto inverse call delta influences the shape of its gamma. Specifically, quanto inverse call gammas have a similar shape to inverse call gamma at high strikes but the gamma becomes negative at lower strikes, before eventually converging to zero as the strike tends to zero. This is due to the change from a convex to concave pricing curve. Both inverse and quanto inverse put deltas *are* monotonic, but the quanto inverse deltas decrease faster as a function of strike, so for strikes above ATM, the quanto inverse gamma is the greater of the two. Just as the standard option gamma decreases as time to expiry increases, the quanto inverse call gamma gradually decreases with maturity, without a lower boundary. The consequences become severe for hedging, as to be delta-neutral, the issuer would need to buy more and more units of the underlying, eventually exceeding the notional of the option.

We have already discussed the negative theta for inverse and quanto inverse options. Figures C.2 and C.4 show that the only inverse options with a positive theta are in fact low strike quanto inverse calls. Another notable feature from these figures is that the volga for longer-maturity quanto inverse puts can very large and positive, and that the vanna can take either sign. For inverse calls and puts, it is negative for low-strike options and positive for high-strike options. For low-strike quanto inverse calls and puts, the vanna is negative but for high-strike calls, it is positive and for most high-strike puts, it is negative.

6 | CONCLUSIONS

Developments in blockchain are driven by computer scientists who bring a fresh approach to traditional solutions and, as use of this technology grows, traditional finance institutions are increasingly drawn towards the new and original products and trading tools now available in centralized and decentralized financial markets. The crypto options market in particular has been expanding extremely rapidly during the last few years. Well over 90% of global trading volumes are on inverse options, a new type of product, which provides the solution for nonfiat platforms, that wish to list crypto-fiat trading pairs.

We have highlighted basis risk and averaging in a volatile environment as sources of market incompleteness for inverse options, and described the issues arising under an indifference pricing framework. The absence of a hedge portfolio leads to major difficulties so instead the pricing

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

problem becomes that of finding an optimal trading strategy. But prices and hedge ratios then depend entirely on the trader's preferences, and whether the position is long or short. This is an interesting avenue for further research but it is not the focus of this paper. Instead, we analyze the pricing and hedging of these options under the GBM assumption. At the time of writing, all inverse options are for trading bitcoin, ether, or solana against the US dollar, so the GBM assumption is certainly not realistic. Yet a similar comment applies to almost *every* options market—and this does not mean that Black–Scholes-type pricing formulae are irrelevant. Not only they are used to back out implied volatilities, many professional traders use Black–Scholes hedge ratios to balance their options books for delta–gamma–vega neutrality. This is especially true for bitcoin options where empirical research has already demonstrated that such hedge rations are difficult to improve upon consistently.

We have discussed the attractions of quanto direct options, which allow USD-denominated traders to hedge the risk of a stablecoin decoupling from its USD peg. Of course, stablecoin prices are much less variable than other crypto assets, and their depegging from USD is more of an operational risk than a market risk. Nevertheless, the liquidity rug pull, which precipitated the Terra collapse in May 2022, shows how easy it is to attack any stablecoin that has substantial liquidity in pools of decentralized exchanges. The risks of denominating crypto trades in stablecoins have become very apparent to fiat-based traders, as well as their regulators. Quanto direct options offer a protection against stablecoin collapse which should therefore be attractive to traders on Binance and other nonfiat exchanges offering direct products, but no standard options.

As well as deriving Black–Scholes hedge ratios for inverse options, we compare their pricing and hedging formulae to those of a new type of currency protected option, which we call a quanto inverse option, based on the assumption that the underlying price follows a GBM. These allow USD-denominated traders to hedge the currency risk of bitcoin, ether, or indeed any other crypto asset, by fixing the price of the crypto asset–USD trading pair up front. We argue that quanto inverse options provide an attractive alternative to standard options for USD-denominated traders. Particularly when the underlying is volatile, as it is bitcoin and even more so for other cryptocurrencies and crypto assets, the profits from both long and short positions on quanto inverse puts can be much greater than they are for standard or direct puts of similar moneyness and maturity. Of course, this depends on moneyness, but also on the fix for the quanto factor.

We conclude by summarizing the main points to take away from this paper:

1. Inverse options, which account for over 90% of trading on crypto options, have the same payoff structure as a standard FX option, and should, therefore, be priced as such (Garman & Kohlhagen, 1983) even if one side of the trading pair is regarded as a security;
2. Nevertheless, the Deribit options market, which virtually monopolizes the trading on inverse options, is theoretically incomplete. Accounting for this incompleteness requires a preference-based approach, under which the option price would depend on the trader's utility, risk tolerance, and perhaps even his initial wealth;
3. We explain how quanto direct options can offer all traders protection against decoupling of a stablecoin from its 1:1 USD peg;
4. Another use case for quanto crypto options is to provide exposure to another crypto without traders needing to change their base currency. For instance, a BTC-based trader can profit from a change in the XRP price by fixing the BTC/USDT rate in a direct quanto on XRP/USDT, whereby the direct option payoff in USDT is paid out in BTC. This way, there is also no need for fiat onboarding;

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

5. Quanto inverse options pose a new type of exotic option, which allow USD-denominated traders to gain exposure to the expanding crypto market without taking any crypto on the balance sheet. For instance, a trader could fix a BTC/USD quanto factor, so that any options that are settled in BTC have all profit and loss automatically converted to USD.
6. The concave (call) and convex (put) payoff structure of quanto inverse options possess features that are attractive to both buyers and sellers. The uncapped USD put payoff is a perfect insurance against crypto price crashes. The capped USD payoff for a quanto inverse call results in prices that are lower than standard (or direct) calls of the same moneyness and maturity, depending on the conversion factor;
7. All Deribit options are accounted in bitcoin, ether, or solana, and only these currencies can be used for deposits and withdrawals. But it takes time for USD-based traders to convert between USD and these crypto, perhaps using a US-based exchange like Coinbase or Kraken. This friction induces nonzero liquidity premium on the prices of Deribit inverse options. The quanto inverse option avoids such a premium.

For pricing and hedging practices, however, one may wish to deviate from the simple geometric Brownian motion setting and opt for a more sophisticated modeling framework, able to capture the unique features of crypto. The prices of bitcoin and other crypto assets are very prone to jumps, and their volatilities are highly stochastic, so the data generation processes used to model conventional assets are unlikely to fit crypto (Scaillet et al., 2020). It is also evident that, unlike any other types of options, the level of the bitcoin implied volatility surface moves largely independently of the skew—in both moneyness and time-to-maturity dimensions (Alexander & Imeraj, 2023) but there is no consensus on the best framework for capturing the dynamics of the bitcoin implied volatility surface, as yet. Besides these technical issues, another important open question is whether the unique risk factors that drive cryptocurrencies (especially those derived from on-chain data) play key roles in the pricing and hedging of cryptocurrency options, or whether the efficiency of this market is hampered by the commonly observed, lottery-like trades, especially in short-term bitcoin options.

ACKNOWLEDGMENTS

The authors have nothing to report.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author, Carol Alexander, upon reasonable request.

ORCID

Carol Alexander https://orcid.org/0000-0003-1247-0184
Arben Imeraj https://orcid.org/0000-0002-4137-3896

ENDNOTES

[1] For example, ether, the native token of Ethereum is the unit of account for all nonfungible tokens minted onto Ethereum. And although Bitcoin is not a smart contract blockchain, its native token bitcoin is a common crypto unit of account for new token offerings.

[2] Recently, BlackRock, Fidelity, and Charles Schwab started offering their clients investment opportunities in bitcoin, see https://www.ft.com/content/3261f919-ca98-41d2-b950-bc3a670f994cFT.com. Top tier banks like

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Goldman Sachs or J.P. Morgan and proprietary trading houses such as Jump and Cumberland Capital have been active in crypto spot and derivatives market for many years.

[3] For instance, for the month of August 2022, Deribit's ETH options volume was \$11.7bn compared with their BTC options volume of \$9.26bn, see https://www.theblock.co/data/crypto-markets/optionstheblock.com for detailed volume data.

[4] Or indeed an asset swap. Crypto–crypto asset swaps are most heavily traded in decentralized liquidity pools, recorded on blockchains, that is, "on-chain." Not dissimilar to over-the-counter (OTC) agreements in traditional markets, except that on-chain transactions are fully transparent. But block production is rather slow so it could be a relatively long time before crypto–crypto on-chain swap derivatives are developed.

[5] At the other end of the spectrum, Decentralized Finance (DeFi) protocols like https://dydx.exchange/dYdX or https://www.ribbon.finance/Ribbon Finance trade futures and option strategies without an intermediary. The dYdX exchange provides a decentralized alternative platform for futures trading and mimics a limit order book though liquidity pools. On the options side, Ribbon Finance and Thetanuts currently allow traders to invest in smart contracts mimicking strategies like covered calls or cash-covered puts. Traders deposit/stake collateral into vaults and market makers pay a premium which is distributed among the investors. However, with $\sim$ \$160-m total volume in March 2023, this market is too small to consider at present, see https://defillama.com/optionsDeFi Llama.

[6] Liquidations of bitcoin positions alone commonly reach \$100 million per day. See https://www.theblock.co/data/crypto-markets/futures/btc-liquidationsBitcoin Liquidations data from The Block.

[7] See https://www.coindesk.com/markets/2021/12/02/goldman-sachs-sees-crypto-options-markets-as-next-big-step-for-institutional-adoption/Goldman Sachs Bitcoin Options (accessed May 01, 2023).

[8] We define moneyness m as the strike over underlying values and focus only on out-of-the-money (OTM) options, that is, $m < 1$ for puts and $m \geq 1$ for calls as these show much more trading activity and liquidity.

[9] In a recent https://www.sec.gov/news/speech/gensler-sec-speaks-090822SEC speech, Gary Gensler points out that "*the vast majority [of crypto tokens] are securities*" but excludes bitcoin in particular in an earlier https://www.cnbc.com/video/2022/06/27/sec-chair-gary-gensler-discusses-potential-crypto-regulation-and-stablecoins.html?&qsearchterm=gary20genslerinterview. The argument is that native token of other blockchains that are not smart-contract compliant, such as Dogecoin can be thought of a security because their primary use is as a "meme" token. Just recently, the https://uk.finance.yahoo.com/news/ethereum-price-drops-sec-declares-control-093528955.htmlSEC announced to classify Ethereum and all its subsidiary tokens as a security. Thus, every project deployed on top of the Ethereum blockchain could be claimed to be a security and within SEC jurisdiction.

[10] But the public information on Derbit employs confusing terminology, specifying neither how the index is weighted nor how frequently the underling price is monitored during the 30-min interval. For example: "*Exercise of an options contract will result in a settlement in BTC immediately after the expiry. The exercise-settlement value is calculated using the average of the Deribit BTC index over the last 30 min before the expiry. The settlement amount in USD is equal to the difference between the exercise value and the strike price of the option. The exercise value is the 30 min average of the BTC index as calculated before the expiry. The settlement amount in BTC is calculated by dividing this difference by the exercise value*." They have hardly any market share compared with Deribit, but the CME Group are crystal clear: "*The underlying for CME options on Bitcoin futures is one CME Bitcoin futures contract. As you know, The CME Bitcoin futures contract represents five bitcoins and cash settles to the CME CF Bitcoin Reference Rate (BRR)*." See https://www.cmegroup.com/education/courses/introduction-to-bitcoin/get-to-know-options-on-bitcoin-futures.htmlCME Group.

[11] The risks are huge. They include market risk (because the stablecoin is only pegged to the value of USD and can deviate very far from the peg, in May 2022), but also operational risks (stablecoins are held on blockchains and so can be hacked) and regulatory risks (e.g., the European Markets in Crypto Assets (MiCA) directive places very firm caps on stablecoin trading volumes, to try to limit their capitalization).

[12] This follows financial market convention, although economists and physicists would naturally regard the notation $\mathbb{Y}/\mathbb{Z}$ to mean the number of units of $\mathbb{Y}$ per unit of $\mathbb{Z}$. But for currency quotes, it is the other way around.

[13] On the other side of the spectrum, more decentralized options exchanged are emerging. For instance, Ribbon Finance is a decentralized cryptocurrency option exchange built on the Ethereum blockchain. It allows traders to lock their investments into "Theta Vaults" or "DeFi Option Vaults," and generate weekly yields. These vaults

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

run various automatic option strategies, for example, covered calls or short puts. The contract specifications as well as the premium is set via auction, see https://docs.ribbon.finance/theta-vault/theta-vaultTheta Vaults. Other protocols offering DeFi cryptocurrency options exposure are https://www.thetanuts.finance/Thetanuts or Opyn's own https://squeeth.opyn.co/?ct=GBSqueeth.

[14] See https://www.deribit.com/main/indexesDeribit Index for a more detailed explanation.

[15] Deribit set $N = 1$ bitcoin, or 1 ether or 1 sol, depending on the underlying. See https://www.deribit.com/kb/optionsDeribit Option Specification for example.

[16] Approximate because $\bar{S}_T \neq S_T$, unless there is no trading in 30 min before expiry. However, this difference affects only the payoff and not the fair price at any time prior to expiry.

[17] The derivatives exchange BitMEX was among the first exchanges to offer quanto products in form of futures on various coin/coin or coin/USDT pairs, see https://www.bitmex.com/app/quantoFuturesGuideBitMEX quanto futures.

[18] Recall that (counter-intuitively) the notation $\mathbb{Y}/\mathbb{Z}$ is the number of units of $\mathbb{Z}$ per unit of $\mathbb{Y}$, so multiplication by the quanto factor $\bar{X}^{\mathbb{Y}/\mathbb{Z}}$ converts the units from $\mathbb{Y}$ to $\mathbb{Z}$.

[19] In fact, the quanto inverse option has intuition similar to the self-quanto, sometimes also known as a quadratic option, where a payoff is expressed in units of the foreign currency (Mercurio, 2003).

[20] Risk-free lending platforms are the closest which come to some kind of money market. These companies promised risk-free interest if a client is willing to deposit his/her crypto. However, poor risk management and a dangerous mixture of incompetence and ignorance brought major lenders to their knees just recently. https://www.ft.com/content/8d6dee7d-2cc9-4663-a0a2-e469686baca5Celsius and https://www.ft.com/content/0b5b68d9-85f1-47ce-a9f7-34252e4fe2ceVoyager are the latest big players filing for bankruptcy shattering trader's trust in the whole crypto ecosystem.

[21] The area around ATM and slightly ITM quanto inverse calls is extremely sensitive. At maturity, if the underlying price is 10% higher than the strike, the payoff would be roughly \$2050; if the underlying price is twice the strike the payoff will be \$11,250; and if the underlying price was 10 times the strike, the payoff would be \$20,250.

REFERENCES

Alexander, C., Deng, J., Feng, J., & Wan, H. (2022). Net buying pressure and the information in bitcoin option trades. *Journal of Financial Markets*, *63*, 100764.

Alexander, C., & Imeraj, A. (2023). Delta hedging bitcoin options with a smile. *Quantitative Finance*, *23*(5), 799–817.

Ankirchner, S., Imkeller, P., & Dos Reis, G. (2010). Pricing and hedging of derivatives based on nontradable underlyings. *Mathematical Finance*, *20*(2), 289–312.

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *The Journal of Political Economy*, *81*(3), 637–654.

Cao, M., & Celik, B. (2021). Valuation of bitcoin options. *Journal of Futures Markets*, *41*(7), 1007–1026.

Carr, P., & Wu, L. (2007). Stochastic skew in currency options. *Journal of Financial Economics*, *86*(1), 213–247.

Chen, K.-S., & Huang, Y.-C. (2021). Detecting jump risk and jump-diffusion model for bitcoin options pricing and hedging. *Mathematics*, *9*(20), 2567.

Clark, I. (2011). *Foreign exchange option pricing: A practitioner's guide*. Wiley finance series. Wiley.

CryptoCompare (2022a, July). *Cryptocompare aggregate index constituent reviews*.

CryptoCompare (2022b, August). *Exchange review*.

Davis, M. (2000). *Optimal hedging with basis risk* [Working paper]. Financial and Actuarial Mathematics Group, TU Vienna.

Demeterfi, K. (1998). How to value and hedge options on foreign indexes. *Quantitative strategies research notes* (pp. 1–26).

Duffie, D., Pan, J., & Singleton, K. (2000). Transform analysis and asset pricing for affine jump diffusions. *Econometrica*, *68*(6), 1343–1376.

Garman, M. B., & Kohlhagen, S. W. (1983). Foreign currency option values. *Journal of International Money and Finance*, *2*(3), 231–237.

Golez, B., & Jackwerth, J. (2012). Pinning in the S&P 500 futures. *Journal of Financial Economics*, *106*, 566–585.

Grabbe, J. (1983). The pricing of call and put options on foreign exchange. *Journal of International Money and Finance*, *2*, 239–253.

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Haber, S., & Stornetta, W. (1991). How to time-stamp a digital document. *Journal of Cryptology*, *3*(2), 99–111.

Haber, S., & Stornetta, W. (1997). Secure names for bit-strings. In *Proceedings of the ACM conference on computer and communications security* (pp. 28–35).

Hou, A., Wang, W., Chen, C., & Härdle, W. (2020). Pricing cryptocurrency options. *Journal of Financial Econometrics*, *18*(2), 250–279.

Jalan, A., Matkovskyy, R., & Aziz, S. (2021). The bitcoin options market: A first look at pricing and risk. *Applied Economics*, *53*(17), 2026–2041.

Jamshidian, F. (1993). Option and futures evaluation with deterministic volatilities. *Mathematical Finance*, *3*(2), 149–159.

Jeanblanc, M., Chesney, M., & Yor, M. (2009). *Mathematical methods for financial markets*. Springer Finance.

Levy, E. (1992). Pricing European average rate currency options. *Journal of International Money and Finance*, *11*(5), 474–491.

Lucic, V. (2022). *BTC inverse call and the standard FX framework* [Wokring paper]. SSRN.

Matic, J., Packham, N., & Härdle, W. (2023). Hedging cryptocurrency options. *Review of Derivatives Research*, *26*, 91–133.

McNeil, A., & Frey, R. (2000). Estimation of tail-related risk measures for heteroscedastic financial time series: An extreme value approach. *Journal of Empirical Finance*, *7*(3), 271–300.

Mercurio, F. (2003). *Pricing and static replication of FX quanto options* [Working paper]. Banca IMI.

Monoyios, M. (2004). Performance of utility-based strategies for hedging basis risk. *Quantitative Finance*, *4*(3), 245–255.

Nakamoto, S. (2008). *Bitcoin: A peer-to-peer electronic cash system*. https://bitcoin.org/bitcoin.pdf

Ni, S., Pearson, N., Poteshman, A., & White, J. (2021). Does option trading have a pervasive impact on underlying stock prices. *The Review of Financial Studies*, *34*(4), 1952–1986.

Reiswich, D., & Wystup, U. (2010). A guide to FX options quoting conventions. *Journal of Derivatives*, *18*(2), 58–68.

Rouge, R., & El Karoui, N. (2000). Pricing via utility maximization and entropy. *Mathematical Finance*, *10*(2), 259–276.

Scaillet, O., Treccani, A., & Trevisan, C. (2020). High-frequency jump analysis of the bitcoin market. *Journal of Financial Econometrics*, *18*(2), 209–232.

Sepp, A., & Rakhmonov, P. (2022). *Log-normal stochastic volatility model with quadratic drift: Applications to assets with positive return-volatility correlation and to inverse martingale measures* [Working paper]. SSRN.

Siu, T. K., & Elliott, R. (2021). Bitcoin option pricing with a SETRA-GARCH model. *The European Journal of Finance*, *27*(6), 564–595.

Teng, H., & Härdle, W. (2022). *Financial analytics of inverse BTC options in a stochastic volatility world* [Working paper]. SSRN.

How to cite this article: Alexander, C., Chen, D., & Imeraj, A. (2023). Crypto quanto and inverse options. *Mathematical Finance*, *33,* 1005–1043. https://doi.org/10.1111/mafi.12410

APPENDIX A: INVERSE AND QUANTO INVERSE OPTION PRICES AND GREEKS

APPENDIX B: DERIVATION OF THE GREEKS

$$d_2 = \frac{\ln\left(\frac{S_T^\$}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \qquad d_3 = \frac{\ln\left(\frac{S_T^\$}{K}\right) + \left(r - \frac{3}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} = d_2 - \sigma\sqrt{\tau}$$

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$\phi(d_2) = \phi(-d_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2}$$

$$\phi(d_3) = \phi\left(d_2 - \sigma\sqrt{\tau}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(d_2 - \sigma\sqrt{\tau}\right)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(d_2^2 - 2d_2\sigma\sqrt{\tau} + \sigma^2\tau\right)}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} e^{d_2\sigma\sqrt{\tau}} e^{-\frac{1}{2}\sigma^2\tau} = \phi(d_2) e^{\ln\left(\frac{S_T^\$}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau} e^{-\frac{1}{2}\sigma^2\tau}$$

$$= \phi(d_2)\frac{S_T^\$}{K} e^{(r-\sigma^2)\tau}$$

$$\frac{\partial d_2}{\partial F} = \frac{\partial d_3}{\partial F} = \frac{1}{F\sigma\sqrt{\tau}}$$

$$\frac{\partial d_2}{\partial \sigma} = \frac{-\ln\left(\frac{S_T^\$}{K}\right) - r\tau - \frac{1}{2}\sigma^2\tau}{\sigma^2\sqrt{\tau}} \qquad \frac{\partial d_3}{\partial \sigma} = \frac{-\ln\left(\frac{S_T^\$}{K}\right) - r\tau - \frac{3}{2}\sigma^2\tau}{\sigma^2\sqrt{\tau}} = \frac{\partial d_2}{\partial \sigma} - \sqrt{\tau}$$

$$\frac{\partial d_2}{\partial \tau} = \frac{-0.5\ln\left(\frac{S_T^\$}{K}\right) + 0.5r\tau - \frac{1}{4}\sigma^2\tau}{\sigma\sqrt{\tau^3}} \qquad \frac{\partial d_3}{\partial \tau} = \frac{-0.5\ln\left(\frac{S_T^\$}{K}\right) + 0.5r\tau - \frac{3}{4}\sigma^2\tau}{\sigma\sqrt{\tau^3}} = \frac{\partial d_2}{\partial \tau} - \frac{\sigma}{2\sqrt{\tau}}$$

TABLE A.1 Inverse and quanto inverse option prices and Greeks.

Name			Inverse	Quanto inverse
Price	f		$\omega[S\Phi(\omega d_1) - e^{-r\tau}K\Phi(\omega d_2)]$	$\omega e^{-r\tau}[\Phi(\omega d_2) - e^{(\sigma^2-r)\tau}S^{-1}K\Phi(\omega d_3)]$
Delta	δ	$\frac{\partial f}{\partial S}$	$\omega\Phi(\omega d_1)$	$\omega e^{(\sigma^2-2r)\tau}\frac{K}{S^2}\Phi(\omega d_3)$
Gamma	γ	$\frac{\partial^2 f}{\partial S^2}$	$\frac{\phi(d_1)}{S\sigma\sqrt{\tau}}$	$e^{(\sigma^2-2r)\tau}\frac{K}{S^3}\left[\frac{\phi(d_3)}{\sigma\sqrt{\tau}} - \omega 2\Phi(\omega d_3)\right]$
Vega	ν	$\frac{\partial f}{\partial \sigma}$	$S\phi(d_1)\sqrt{\tau}$	$e^{-r\tau}\left[\phi(d_2)\sqrt{\tau} - \omega e^{(\sigma^2-r)\tau}\sigma\tau\frac{2K}{S}\Phi(\omega d_3)\right]$
Volga	ν^o	$\frac{\partial^2 f}{\partial \sigma^2}$	$\sqrt{\tau}\phi(d_1)S\frac{d_1 d_2}{\sigma}$	$e^{-r\tau}\sqrt{\tau}\phi(d_2)\frac{d_2 d_1}{\sigma} - \omega 2\frac{K}{S}\tau e^{(\sigma^2-2r)\tau}\times$ $\left[\Phi(\omega d_3)(1+2\sigma^2\tau) - \sigma\phi(d_3)\left(\frac{-d_1}{\sigma} - \sqrt{\tau}\right)\right]$
Vanna	ν^a	$\frac{\partial^2 f}{\partial S\partial\sigma}$	$-\phi(d_1)\frac{d_2}{\sigma}$	$\omega e^{(\sigma^2-2r)\tau}\times$ $\frac{K}{S^2}\left[2\tau\sigma\Phi(\omega d_3) + \omega\phi(d_3)\left(\frac{-d_1}{\sigma} - \sqrt{\tau}\right)\right]$
Theta	ϑ	$-\frac{\partial f}{\partial \tau}$	$\omega[-re^{-r\tau}K\Phi(\omega d_2)]$ $-\frac{\sigma S\phi(d_1)}{2\sqrt{\tau}}$	$-e^{-r\tau}\left(\frac{\phi(d_2)\sigma}{2\sqrt{\tau}} - \omega r\Phi(\omega d_2)\right)$ $+\omega e^{(\sigma^2-2r)\tau}\frac{K}{S}(\sigma^2-2r)\Phi(\omega d_3)$

Note: We assume that the tradable underlying price S follows a GBM with volatility σ, where the drift depends on the USD discount rate r. Furthermore, we assume a zero BTC discount rate. The inverse or quanto inverse call or put have strike K and for the sake if clarity we omit the predefined $\bar{X}$ for the quanto inverse case. Note that this need to be multiplied to the individual Greek. The residual time to maturity τ hold for all types and we use the notation $\omega = \pm 1$ according as the option is a call or a put. Suppressing all time subscripts for simplicity we set $d_1 = \ln(\frac{S}{K})[\sigma\sqrt{\tau}]^{-1} + (r + \frac{1}{2}\sigma)\sqrt{\tau}$, $d_2 = d_1 - \sigma\sqrt{\tau}$ as well as $d_3 = d_2 - \sigma\sqrt{\tau}$.

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$
\begin{aligned}
\delta := \frac{\partial f}{\partial F} &= \frac{\partial}{\partial F} e^{-r\tau}\left[\Phi(d_2) - e^{(\sigma^2-r)\tau}F_t^{-1}K\Phi(d_3)\right] \\
&= e^{-r\tau}\left[\frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial F} - e^{(\sigma^2-r)\tau}\left[-\frac{K}{F^2}\Phi(d_3) + \frac{K}{F}\frac{\partial \Phi(d_3)}{\partial d_3}\frac{\partial d_3}{\partial F}\right]\right] \\
&= e^{-r\tau}\left[\frac{\phi(d_2)}{F\sigma\sqrt{\tau}} - e^{(\sigma^2-r)\tau}\left[-\frac{K}{F^2}\Phi(d_3) + \frac{K}{F^2\sigma\sqrt{\tau}}\phi(d_3)\right]\right] \\
&= e^{-r\tau}\left[\frac{\phi(d_2)}{F\sigma\sqrt{\tau}} + e^{(\sigma^2-r)\tau}\frac{K}{F^2}\Phi(d_3) - e^{(\sigma^2-r)\tau}\frac{K}{F^2\sigma\sqrt{\tau}}\phi(d_2 - \sigma\sqrt{\tau})\right] \\
&= e^{-r\tau}\left[\frac{\phi(d_2)}{F\sigma\sqrt{\tau}} + e^{(\sigma^2-r)\tau}\frac{K}{F^2}\Phi(d_3) - e^{(\sigma^2-r)\tau}\frac{K}{F^2\sigma\sqrt{\tau}}\phi(d_2)\frac{F}{K}e^{-(\sigma^2-r)\tau}\right] \\
&= e^{(\sigma^2-2r)\tau}F^{-2}K\Phi(d_3)
\end{aligned}
$$

$$
\gamma := \frac{\partial \delta}{\partial F} = \frac{\partial}{\partial F}\left[e^{(\sigma^2-2r)\tau}F^{-2}K\Phi(d_3)\right] = e^{(\sigma^2-2r)\tau}F^{-3}K\left[\frac{\phi(d_3)}{\sigma\sqrt{\tau}} - 2\Phi(d_3)\right]
$$

$$
\begin{aligned}
\nu := \frac{\partial f}{\partial \sigma} &= \frac{\partial}{\partial \sigma} e^{-r\tau}\left[\Phi(d_2) - e^{(\sigma^2-r)\tau}F_t^{-1}K\Phi(d_3)\right] \\
&= e^{-r\tau}\left[\frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial \sigma} - \frac{K}{F}\left[\frac{\partial e^{(\sigma^2-r)\tau}}{\partial \sigma}\Phi(d_3) + e^{(\sigma^2-r)\tau}\frac{\partial \Phi(d_3)}{\partial d_3}\frac{\partial d_3}{\partial \sigma}\right]\right] \\
&= e^{-r\tau}\left[\phi(d_2)\frac{\partial d_2}{\partial \sigma} - e^{(\sigma^2-r)\tau}\frac{K}{F}\left[2\tau\sigma\Phi(d_3) + \phi(d_2 - \sigma\sqrt{\tau})\left(\frac{\partial d_2}{\partial \sigma} - \sqrt{\tau}\right)\right]\right] \\
&= e^{-r\tau}\left[\phi(d_2)\frac{\partial d_2}{\partial \sigma} - e^{(\sigma^2-r)\tau}\frac{K}{F}\left[2\tau\sigma\Phi(d_3) + \phi(d_2)e^{d_2\sigma\sqrt{\tau}}e^{-\frac{\sigma^2\tau}{2}}\left(\frac{\partial d_2}{\partial \sigma} - \sqrt{\tau}\right)\right]\right] \\
&= e^{-r\tau}\left[\phi(d_2)\frac{\partial d_2}{\partial \sigma} - e^{(\sigma^2-r)\tau}\frac{K}{F}\left[2\tau\sigma\Phi(d_3) + \phi(d_2)\frac{F}{K}e^{-(\sigma^2-r)\tau}\left(\frac{\partial d_2}{\partial \sigma} - \sqrt{\tau}\right)\right]\right] \\
&= e^{-r\tau}\left[\phi(d_2)\sqrt{\tau} - 2e^{(\sigma^2-r)\tau}F^{-1}K\tau\sigma\Phi(d_3)\right]
\end{aligned}
$$

$$
\begin{aligned}
\vartheta := \frac{\partial f}{\partial \tau} &= \frac{\partial}{\partial \tau} e^{-r\tau}\Phi(d_2) - e^{(\sigma^2-2r)\tau}F_t^{-1}K\Phi(d_3) \\
&= \left[\frac{\partial e^{-r\tau}}{\partial \tau}\Phi(d_2) + e^{-r\tau}\frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial \tau}\right] - \frac{K}{F}\left[\frac{\partial e^{(\sigma^2-2r)\tau}}{\partial \tau}\Phi(d_3) + e^{(\sigma^2-r)\tau}\frac{\partial \Phi(d_3)}{\partial d_3}\frac{\partial d_3}{\partial \tau}\right] \\
&= \left[-re^{-r\tau}\Phi(d_2) + e^{-r\tau}\phi(d_2)\frac{\partial d_2}{\partial \tau}\right] - \frac{K}{F}\left[\left(\sigma^2 - 2r\right)\Phi(d_3)e^{(\sigma^2-2r)\tau} + e^{(\sigma^2-2r)\tau}\phi(d_3)\frac{\partial d_3}{\partial \tau}\right] \\
&= \left[-re^{-r\tau}\Phi(d_2) + e^{-r\tau}\phi(d_2)\frac{\partial d_2}{\partial \tau}\right] - \\
&\quad \frac{K}{F}\left[\left(\sigma^2 - 2r\right)\Phi(d_3)e^{(\sigma^2-2r)\tau} + e^{(\sigma^2-2r)\tau}\phi(d_2 - \sqrt{\tau})\left(\frac{\partial d_2}{\partial \tau} - \frac{\sigma}{2\sqrt{\tau}}\right)\right]
\end{aligned}
$$

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$= \left[-re^{-r\tau}\Phi(d_2) + e^{-r\tau}\phi(d_2)\frac{\partial d_2}{\partial \tau} \right]$$

$$- \frac{K}{F}\left[\left(\sigma^2 - 2r\right)\Phi(d_3)e^{(\sigma^2-2r)\tau} + e^{(\sigma^2-2r)\tau}\phi(d_2)\frac{S_T^{\$}}{K}e^{\left(r-\sigma^2\right)\tau}\left(\frac{\partial d_2}{\partial \tau} - \frac{\sigma}{2\sqrt{\tau}} \right) \right]$$

$$= \left[-re^{-r\tau}\Phi(d_2) + e^{-r\tau}\phi(d_2)\frac{\partial d_2}{\partial \tau} \right] - \left(\sigma^2 - 2r\right)\Phi(d_3)\frac{K}{F}e^{(\sigma^2-2r)\tau} - e^{-r\tau}\phi(d_2)\left(\frac{\partial d_2}{\partial \tau} - \frac{\sigma}{2\sqrt{\tau}} \right)$$

$$= -re^{-r\tau}\Phi(d_2) + e^{-r\tau}\phi(d_2)\frac{\partial d_2}{\partial \tau} - \left(\sigma^2 - 2r\right)\Phi(d_3)\frac{K}{F}e^{(\sigma^2-2r)\tau} - e^{-r\tau}\phi(d_2)\frac{\partial d_2}{\partial \tau} + e^{-r\tau}\phi(d_2)\frac{\sigma}{2\sqrt{\tau}}$$

$$= -re^{-r\tau}\Phi(d_2) - e^{(\sigma^2-2r)\tau}\left(\sigma^2 - 2r\right)\Phi(d_3)\frac{K}{F} + e^{-r\tau}\phi(d_2)\frac{\sigma}{2\sqrt{\tau}}$$

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

APPENDIX C: INVERSE AND QUANTO INVERSE GREEKS

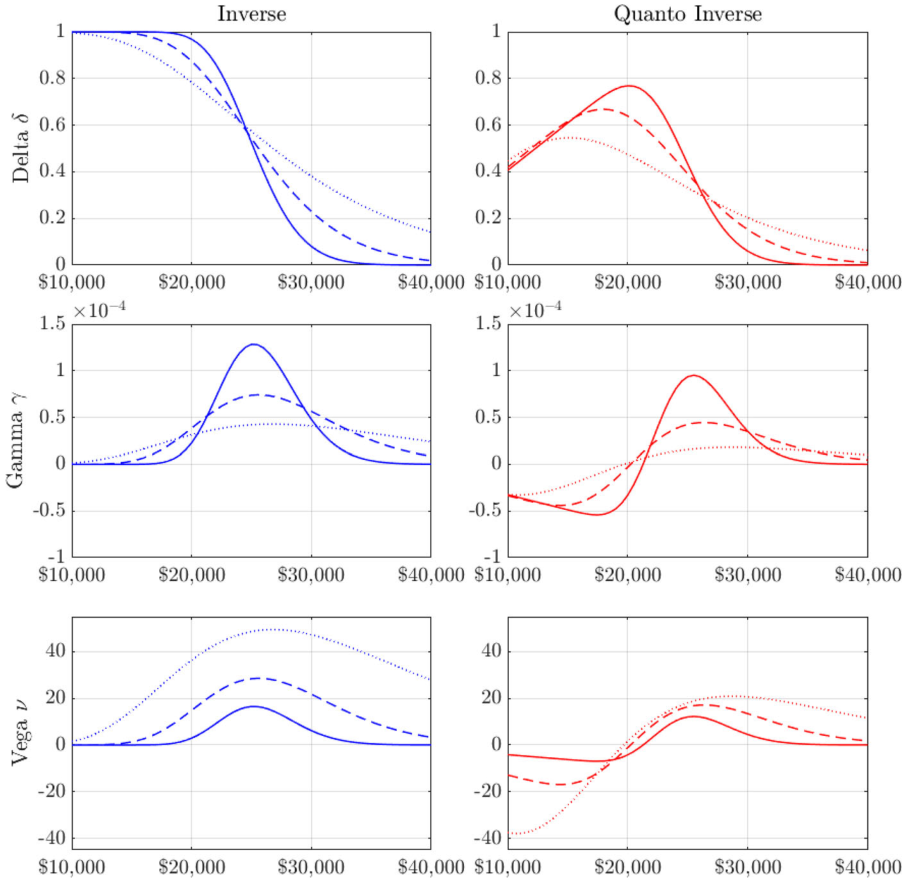

FIGURE C.1 Inverse and quanto inverse call Greeks I. Inverse and quanto inverse Greeks for calls as a function of the strike level $K^{\$}$ with fixed underlying $S_t^{\$}$ and $\bar{X} = S_t^{\$} = \$25,000$, volatility $\sigma = 75\%$ and different times to maturity: from 10 days (bold), 30 days (dashed), and 90 days (dotted). The left, blue column represents the inverse Greeks, while the right, red column shows the quanto inverse Greeks, using the same vertical scale for ease of comparison. [Color figure can be viewed at wileyonlinelibrary.com]

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

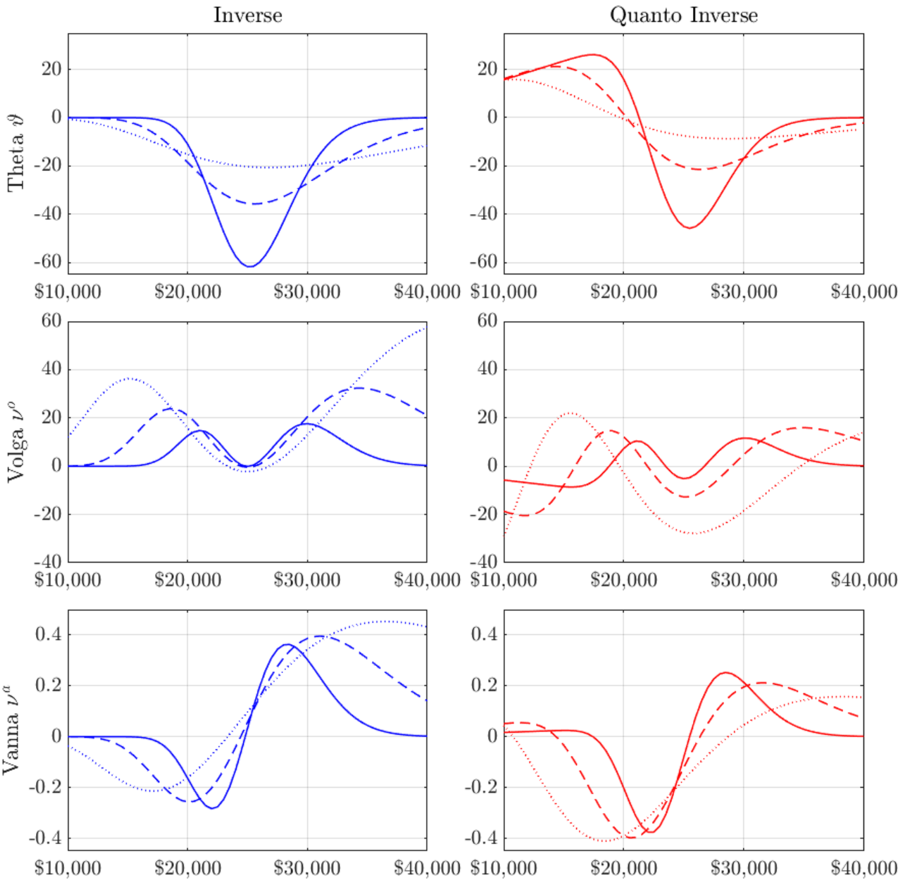

FIGURE C.2 Inverse and quanto inverse call Greeks II. Inverse and quanto inverse Greeks for calls as a function of the strike K with fixed underlying and predetermined conversion factor $S_t^{\$} = \bar{X} = \$25,000$, volatility $\sigma = 75\%$ and different times to maturity: from 10 days (bold), 30 days (dashed), and 90 days (dotted). The left, blue column represents the inverse Greeks, while the right, red column displays the quanto inverse Greeks, using the same vertical scale for ease of comparison. [Color figure can be viewed at wileyonlinelibrary.com]

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

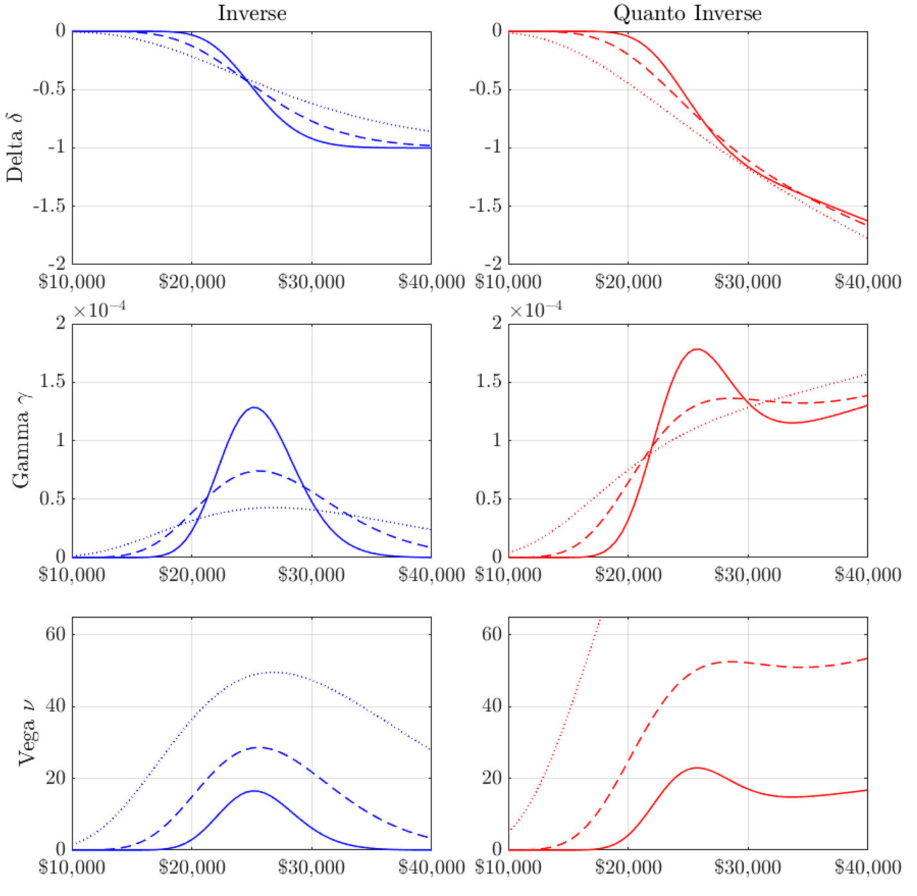

FIGURE C.3 Inverse and quanto inverse put Greeks I. Inverse and quanto inverse Greeks for puts as a function of the strike K with fixed underlying and predetermined conversion factor $S_t^{\$} = \bar{X} = \$25,000$, volatility $\sigma = 75\%$ and different times to maturity: from 10 days (bold), 30 days (dashed), and 90 days (dotted). The left, blue column represents the inverse Greeks, while the right, red column displays the quanto inverse Greeks, using the same vertical scale for ease of comparison. [Color figure can be viewed at wileyonlinelibrary.com]

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

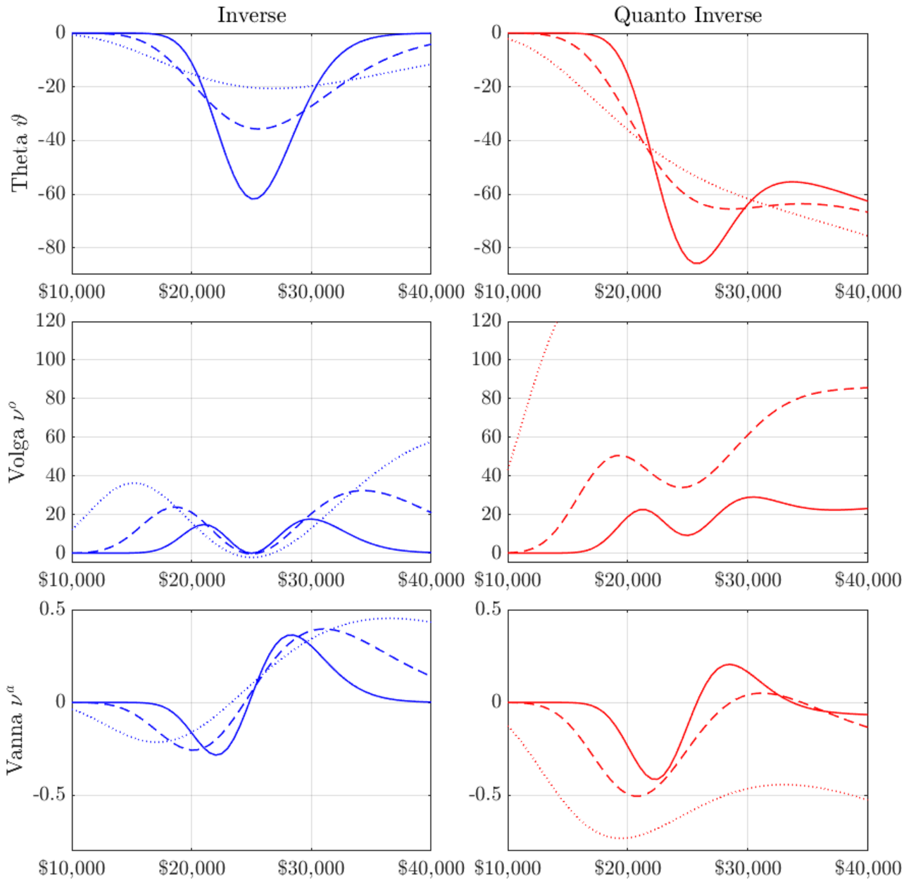

FIGURE C.4 Inverse and quanto inverse put Greeks II. Inverse and quanto inverse Greeks for puts as a function of the strike K with fixed underlying and predetermined conversion factor $S_t^{\$} = \bar{X} = \$25,000$, volatility $\sigma = 75\%$ and different times to maturity: from 10 days (bold), 30 days (dashed), and 90 days (dotted). The left, blue column represents the inverse Greeks, while the right, red column displays the quanto inverse Greeks, using the same vertical scale for ease of comparison. [Color figure can be viewed at wileyonlinelibrary.com]

14679965, 2023, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/mafi.12410 by University Of Birmingham Eresources And Serials Team, Wiley Online Library on [04/10/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License