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THE GEOMETRIC PHASE MADE SIMPLE

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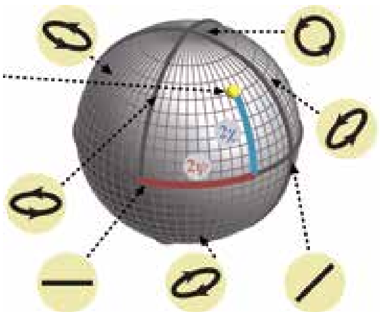

We give a simple description of the Pancharatnam-Berry geometric phase and some of its applications in optics. Geometric phases are a universal phenomenon, but we focus here on the case of geometric phases caused by changes in polarization. The geometric origin of this phase is explained by analogy with the motion of an imaginary creature living on a small planet, inspired by Saint-Exupéry's "The Little Prince".

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The geometric phase is a universal phenomenon that appears in many areas across wave physics. It was first discovered within the context of optics in 1956 by S. Pancharatnam [1] as a net phase difference between two beams which have undergone different sequences of polarization, depending only on these sequences. A few decades later, Berry [2,3] recognized this phenomenon's universality and underlying geometric nature, which goes well beyond electromagnetic waves. In optics and photonics, geometric phases have attracted significant attention and found many practical applications. While extensive reviews exist (for example [4]), the goal of this short article is to provide a simple description of the Pancharatnam-Berry (PB) phase for optical polarization and explain why it is so useful in applications such as wavefront shaping. As one learns in any basic course in optics, a propagating optical beam is described by its intensity and phase, the latter being accumulated proportionally to the beam's optical path length. In addition, a light beam has "internal" degrees of freedom such as polarization, which can also change as the beam propagates. Such evolution can lead to an extra contribution to the phase unrelated to the overall optical path length. This phase is called "geometric" because it is directly related to the geometry in the natural abstract space that best describes the internal degrees of freedom in question.

CURVED SPACES, PARALLEL TRANSPORT, AND THE LITTLE PRINCE

The phenomenon of the geometric phase requires that the space that describes the internal degrees of freedom be curved, and that the evolution over such space follows the rules of what is known as *parallel transport*. Let us explain the basic concept of parallel transport by imagining an episode inspired by Saint-Exupéry's *Le Petit Prince*:

On his way back from the Sahara to his planet (Asteroid B-612), the little prince stopped by another small planet (Asteroid PB-56-84), inhabited only by a strange creature, Par-Tra.

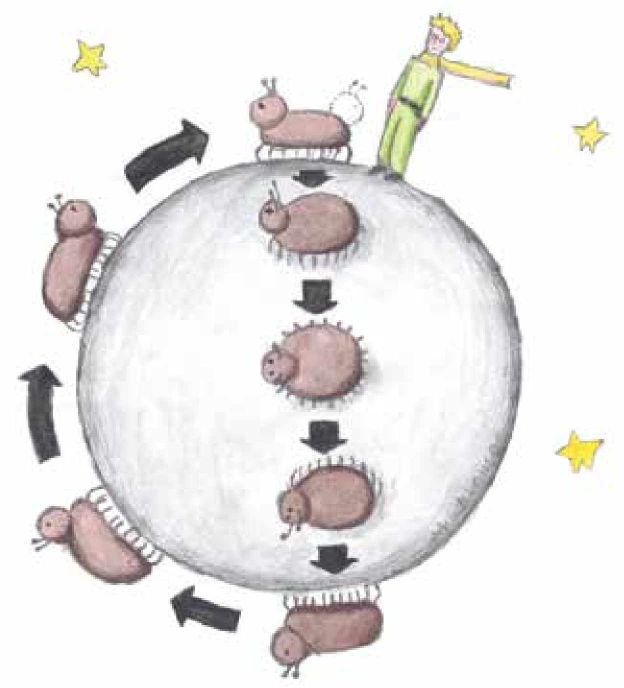

Figure 1. Par-Tra's manoeuvre to turn towards the little prince: starting from the position at the top, facing left, Par-Tra first walks sideways (down) along a geodesic to the antipodal point (bottom), and then along another geodesic (left) facing forward, back to her initial position, where the final orientation of her head is shown in dotted lines.

Par-Tra had a round body, and her many tiny legs could only move radially, making her perfectly capable of walking in any direction but completely unable to turn. When the prince arrived, he accidentally stood behind Par-Tra. To face him, Par-Tra did what she always did in order to turn: walk sideways to the opposite side of the planet, and then straight ahead back to her starting point.

In this analogy, the curved space is the planet's surface, and Par-Tra's movement restrictions correspond (as her name suggests) to parallel transport—her only means of turning is by moving on a closed path on the curved surface. The angle of rotation (in radians) between Par-Tra's initial and final orientation coincides precisely with the solid angle (in steradians) enclosed by her trajectory (in the illustration, approximately $-\pi$). Note that the rules of parallel transport are not unique to wave phenomena or imaginary creatures: they also explain mechanical effects such as the precession of Foucault's pendulum, or the fact that when one rolls a rubber ball over a flat surface in a circular motion, each turn makes the ball turn proportionally to the solid angle enclosed by the path of the contact points with the surface.

POLARIZATION AND THE POINCARÉ SPHERE

For a paraxial monochromatic optical beam (e.g. a collimated laser), the electric field vector at any point traces an elliptical path. The global size of this ellipse depends on the intensity, and the specific position of the electric field at a given time depends on the phase. The remaining, purely geometric properties of the ellipse, namely its orientation and ellipticity, constitute *the state of polarization* of the field. These are characterized respectively by i) the angle ψ between the horizontal coordinate axis and the ellipse's major axis, and ii) the angle χ subtended by a minor semi-axis from one of the vertices, whose magnitude encodes ellipticity and whose sign determines the handedness with which the electric field traces the ellipse.

Rotating the ellipse by half a turn (that is, turning ψ through π) brings it back to its initial state. Similarly, the angle χ takes values between $\pm\pi/4$. For the extreme values $2\chi = \pm\pi/2$, the ellipse becomes a circle, and ψ becomes irrelevant. These geometric properties make 2ψ and 2χ reminiscent of the longitude and latitude angles on a sphere: their ranges are the same, and longitude becomes •••

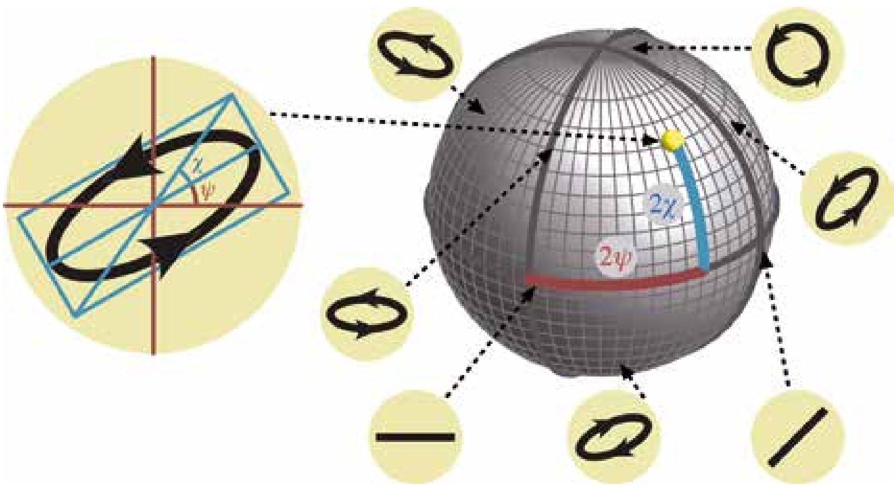

Figure 2. The Poincaré sphere parametrizing states of elliptic polarization.

irrelevant when we are at the North or South poles. Therefore, each state of polarization, determined by ψ and χ, can be represented by a point with coordinates 2ψ and 2χ over a unit sphere, known as the Poincaré sphere, shown in Fig.2. In other words, the natural abstract space for representing polarization is a sphere, as is also confirmed by the fact that the visibility of the intensity fringes resulting from the interference of two nearly parallel beams with different polarizations and equal intensities can be fully determined by the angular separation between the two corresponding points over the Poincaré sphere. Three other aspects of polarization that are important to understand geometric phases are the following: i) two orthogonal polarizations correspond to a pair of antipodal points on the Poincaré sphere, ii) any state of polarization can be expressed as a complex linear superposition of any two orthogonal polarizations, and iii) the angle between any two linear polarizations is half the angle between their corresponding points on the Poincaré sphere.

How can we change the polarization of a beam? Two main types of optical element are often used:
The first corresponds to polarizers, which only transmit a given polarization component and eliminate the orthogonal component (by absorption or reflection). In the Poincaré sphere representation, polarizers take any initial point (corresponding to the beam's initial polarization) and transport it to the point representing the transmitted polarization, which is always the same regardless of the initial polarization. The trajectory for this type of projection corresponds to the shortest geodesic path joining the initial and final points as represented in Fig.3, as this is the path that results from gradually attenuating the orthogonal polarization. This geodesic projection automatically follows the rules of parallel transport.

The second type of element corresponds to wave retarders, such as the waveplates used routinely in optical setups. These elements are typically almost transparent, and hence their action does not rely on eliminating a polarization component; instead they introduce a phase difference δ between two preferred orthogonal polarizations. That is, these two *eigenpolarizations* experience different refractive indices, so one travels faster than the other and accumulates a smaller phase. On the Poincaré sphere, the action of a wave retarder corresponds to a rigid rotation by an angle δ around the axis joining the two eigenpolarizations, as represented in Fig. 3.

PANCHARATNAM-BERRY PHASE

By using a sequence of optical elements like those just discussed, the polarization of the beam can be made to trace a path over the surface of the Poincaré sphere. Imagine a situation in which the final polarization is the same as the initial one, so that the path is closed. What we learn from Par-Tra's story is that a sequence of displacements results in a rotation by an angle equal to the enclosed solid angle. In the small-planet analogy, this rotation can be observed by looking at the change of orientation of Par-Tra's head. In the optical • • •

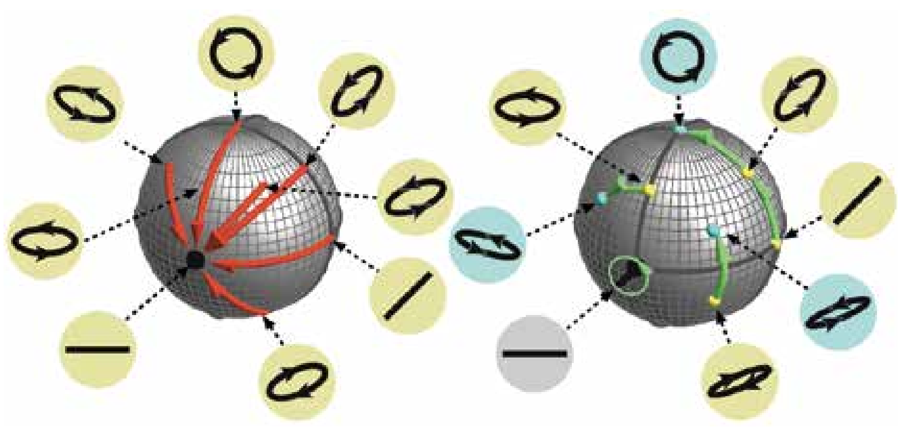

Figure 3. Left: Polarizers carry all states of polarization to a single polarization state here horizontal linear) along red geodesics. Right: Retarders rotate the points on the sphere by angle about an axis given by two orthogonal polarizations (here horizontal and vertical linear).

case, on the other hand, polarization is represented by a zero-size point over the sphere with no discernible feature. Nevertheless, it turns out that there is a measurable analogue of the angle of orientation of Par-Tra's head. To see this, consider the case in which the initial/final polarization is circular and is represented by a point at one of the poles. A rigid rotation of the sphere by Φ leaves this point unchanged. However, this initial polarization state, **c**, can be expressed as a linear superposition of two orthogonal states, such as the two horizontal and vertical linear polarizations **h** and **v**, according to $\mathbf{c} = (\mathbf{h} \pm i\mathbf{v})/\sqrt{2}$. While the point for **c** remains at the north pole, the points initially representing **h** and **v** rotate by an angle Φ, and the corresponding polarization directions in physical space rotate by $\Phi/2$. The final state would then be $[(\mathbf{h}\cos\Phi/2 + \mathbf{v}\sin\Phi/2) \pm i(\mathbf{v}\cos\Phi/2 - \mathbf{h}\sin\Phi/2)]/\sqrt{2} = \mathbf{c}\exp(\mp i\Phi/2)$. This last global phase factor, the geometric phase, is the measurable quantity that reveals that a rotation of the point about its axis took place.

As mentioned earlier, the geodesic paths resulting from using polarizers automatically satisfy parallel transport. However, the same is not necessarily true for the circular paths resulting from the rotations enacted by wave retarders. To visualize this fact, let us go back to Asteroid PB-56-84, and imagine that the little prince and Par-Tra go for a stroll around the only tree in the small planet (the axis of their rotational displacement). They both follow circular paths (of different radii in the figure only for illustration purposes) around the tree. The little prince walks always facing the forward motion direction, but Par-Tra's physiology constrains her to gradually rotate at an angle different from that of the path. After completing one full circle, the prince has the same orientation as he did at the beginning, but Par-Tra's orientation changes by an angle equal to the solid angle enclosed by her path. Only had they chosen to walk along the geodesic normal to the tree, would their translational and rotational motions have coincided. The rotations enacted by a phase retarder correspond to the rigid rotational motion of the little prince, and not to the parallel transport of Par-Tra. This can be seen by the fact that a "full-wave plate" (e.g. the cascade of two parallel half-wave plates), corresponding to a full circular path around the axis of rotation, does not write different phases on different polarizations, even though the paths over the Poincaré sphere enclose different solid angles. That is, if a closed path includes non-geodesic segments resulting from using wave retarders, the corresponding phase is not necessarily equal to half the enclosed solid angle (although other geometric interpretations are possible). Therefore, to enact a clean geometric phase with wave retarders, the displacement must be around great circles (geodesics). That is, the polarization entering a wave retarder must correspond to a point that is at $\pi/2$ over the Poincaré sphere from both eigenpolarizations. For birefringent waveplates, with linear eigenpolarizations (i.e. on the Poincaré sphere equator), maximal geometric phase control can be enacted if the initial polarization is circular (i.e. at one of the poles).

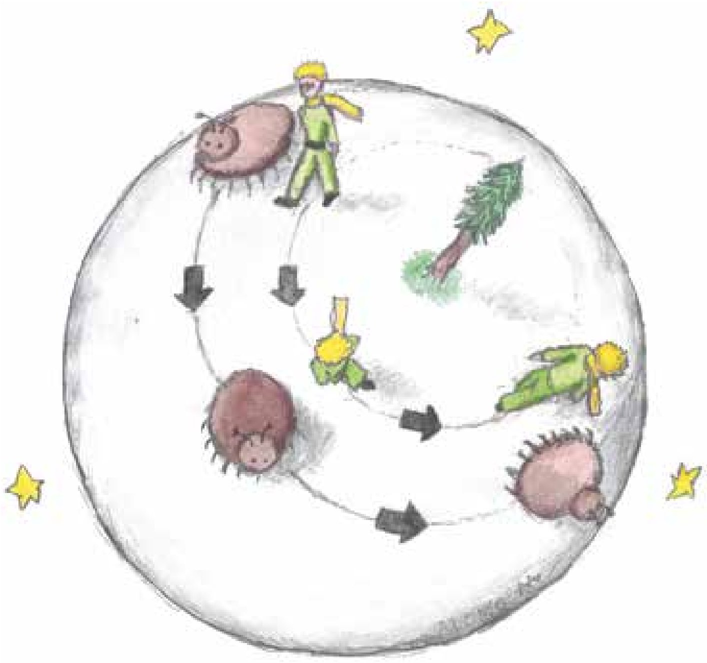

Figure 4. Par-Tra and the little prince going for a stroll around the tree.

WAVEFRONT SHAPING WITH GEOMETRIC PHASE ELEMENTS

Consider a half-wave plate ($\delta = \pi$), whose linear fast eigenpolarization is at an angle γ from the horizontal axis. Illuminating this element with circular polarization (corresponding to the north pole), the path over the Poincaré sphere will be a meridian ending at the south pole (circular polarization with the opposite handedness), regardless of γ. However, the phase difference between two such paths corresponding to two values of γ, namely γ_1 and γ_2, will equal one half the solid angle enclosed by them. Since the two meridians are at an angle $2(\gamma_2-\gamma_1)$, the solid angle is $4(\gamma_2-\gamma_1)$ and the phase difference is simply $\pm 2(\gamma_2-\gamma_1)$, the sign depending on the conventions being used. That is, the phase of each emerging circularly polarized beam is (to within an additive constant) $\pm 2\gamma$.

Modern technologies such as liquid crystals or metasurfaces allow the creation of birefringent optical elements where the eigenpolarization orientation can be tailored to change from point to point in any desired way for a range of applications. These devices allow writing arbitrary phase profiles on a wavefront. In the case of liquid-crystal-based devices, these phases can be turned on and off through appropriate electric control, something that is useful in applications such as beam steering. Note that the alternative to using geometric phase is to use dynamic phase, enacted through an increase in optical path length based on refractive index and element thickness, as is the case with standard lenses, prisms, and phase gratings. A very attractive feature of geometric phase elements is that they naturally incorporate the periodicity of the phase in the periodicity of the eigenpolarization angle. This allows, for example, creating optical elements (known as Q-plates) that can induce a vortex on the wavefront without requiring a discontinuity in the element's surface (as is the case in a spiral phase plate), making the resulting element more robust to chromatic changes.

OTHER GEOMETRIC PHASES

Geometric phases are not restricted to optics, and there are many analogies in other quantum phenomena. For instance, the spin of an electron, represented on the Bloch sphere, acquires a phase evolving through closed paths, precisely as on the Poincaré sphere. Other systems have more complicated parameter spaces whose curvature takes more complicated distributions than the constant value on a sphere, and although the final phase might not reduce to a simple enclosed area or solid angle, the underlying principle is the same. Even in optics there are other realizations corresponding to different abstract spaces. One case is that of a specific family of structured beams, where the beam structure can be represented as a point on a sphere, and rotations of this sphere can be enacted with simple combinations of cylindrical lenses. Another one that involves polarization is the so-called (spin) redirection phase: imagine an optical fibre carrying a single mode with circular polarization. If the fibre is bent in 3D so that the local direction of propagation changes, a geometric phase is acquired that equals the solid angle enclosed by the local normalized direction vector over the sphere of possible directions. It turns out that the redirection phase and the standard PB phase can be unified through an appropriate 3D formalism [5].

CONCLUSION

As their name suggests, geometric phases arise from the inherent geometry of the space that naturally describes a property of a wave field. If the laws of parallel transport apply, a series of displacements in this space enact a rotation if the space is curved; if Par-Tra's planet were flat (or a very large sphere), she would be doomed to always look in the same direction. The fact that the natural space that describes polarization (and propagation direction) is a sphere leads to a particularly simple connection between the rotation (and hence the phase) and the enclosed solid angle, and to simple ways to control the phase of a beam through the local orientation of a half-wave retarder.

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