Decision Problems for Linear Logic with Least and Greatest Fixed Points

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DOI:
10.4230/LIPIcs.FSCD.2022.20

Decision Problems for Linear Logic with Least and Greatest Fixed Points

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Abstract

Linear logic is an important logic for modelling resources and decomposing computational interpretations of proofs. Decision problems for fragments of linear logic exhibiting "infinitary" behaviour (such as exponentials) are notoriously complicated. In this work, we address the decision problems for variations of linear logic with fixed points (μMALL), in particular, recent systems based on "circular" and "non-wellfounded" reasoning. In this paper, we show that μMALL is undecidable.

More explicitly, we show that the general non-wellfounded system is Π^0_1-hard via a reduction to the non-halting of Minsky machines, and thus is strictly stronger than its circular counterpart (which is in Σ^0_1). Moreover, we show that the restriction of these systems to theorems with only the least fixed points is already Σ^0_1-complete via a reduction to the reachability problem of alternating vector addition systems with states. This implies that both the circular system and the finitary system (with explicit (co)induction) are Σ^0_1-complete.

2012 ACM Subject Classification Theory of computation → Linear logic; Theory of computation → Proof theory; Theory of computation → Complexity theory and logic

Keywords and phrases Linear logic, fixed points, decidability, vector addition systems

Digital Object Identifier 10.4230/LIPIcs.FSCD.2022.20

Related Version *Full Version*: `https://hal.archives-ouvertes.fr/hal-03655651`

Funding *Anupam Das*: This author is supported by a UKRI Future Leaders Fellowship, *Structure vs. Invariants in Proofs*, project reference MR/S035540/1.
Abhishek De: This author has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 754362.
Alexis Saurin: This author has been partially supported by ANR project *RECIPROG*, project reference ANR-21-CE48-019-01.

Acknowledgements We would like to thank anonymous reviewers for their valuable comments that enhanced the clarity and presentation of this paper. We also thank Sylvain Schmitz for his insights on alternating vector addition systems.

1 Introduction

Fixed point theory occurs in just about every field of computer science, including program analysis [30], game theory [10, 46], automata theory [33, 47], and programming language theory [53]. In the setting of fixed point logics, the (multi)modal μ-calculus (the extension of basic modal logic K with least and greatest fixed point operators) is probably the most well-studied. The most important result in this direction is the obtention of completeness of Hilbert-style axiomatisations for the logic [33, 59, 58]. Another relevant case study is that of

Kleene Algebra (and extensions) where basic algebraic structures are extended by a "Kleene star" modelling iteration. Such theories have similarly received axiomatisations that have been proved complete (over relational and language models) [32, 35, 9, 22].

These formalisations employ inference rules that express an *explicit* (co)induction scheme *i.e.* the induction invariant must be provided explicitly. However, more recently both these settings have received proof-theoretic developments allowing for an implicit treatment of (co)induction, by way of "non-wellfounded" and "circular" reasoning [48, 1, 16]. Such systems admit greater proof-theoretic expressivity while, at the same time, reinforcing connections between these logics and automata theory. Evidence of their utility has been duly provided in recent works that recover the aforementioned completeness results using entirely proof-theoretic (as opposed to automata theoretic) methods, in particular [1], building on [48], in the case of the μ-calculus, and [15], building on [16], in the case of Kleene algebra.

In a parallel direction, the extension of fragments of *linear logic* by fixed points has become increasingly developed in the last 15 years. Baelde and Miller [5, 2] developed a finitary deductive system for first-order linear logic with least and greatest fixed points. Santocanale was the first to propose a circular system in this area, in particular for an extension of "lattice logic" by fixed points in [51], and later with Fortier proved a form of cut-elimination [25]. Baelde, Doumane and Saurin in [4] extended both the system and the cut-elimination result to the full propositional fragment of Baelde and Miller's logic, now yielding three systems: $\mu\mathsf{MALL}^{\mathsf{ind}}$(based on explicit (co)induction), $\mu\mathsf{MALL}^{\infty}$ (based on non-wellfounded reasoning) and $\mu\mathsf{MALL}^{\omega}$ (based on circular reasoning).

In terms of expressivity, $\mu\mathsf{MALL}$ can be seen as an amalgamation of the properties of μ-calculus and Kleene Algebras. Like Kleene Algebras, $\mu\mathsf{MALL}$ is also "resource-conscious" (indeed, Kleene Algebra and extensions are just fragments of a non-commutative $\mu\mathsf{MALL}$); and like the μ-calculus, $\mu\mathsf{MALL}$ also allows for unrestricted interleaving of fixed points.

In this work, we study systems for $\mu\mathsf{MALL}$ in terms of proof-theoretic strength, in particular asking whether a system is *conservative* over another. A pertinent observation at this juncture is that the aforementioned techniques for the μ-calculus and Kleene algebra for comparing such systems seem to break down in the more general setting of substructural logics. Indeed, in this work, we shall show that they do *not* hold *per se*, by addressing the complexity of deciding theorems of $\mu\mathsf{MALL}^{\infty}$ and $\mu\mathsf{MALL}^{\omega}$. In particular, we show that provability in $\mu\mathsf{MALL}^{\infty}$ is Π^0_1-hard, *i.e.* at least co-recursively enumerable. Our proof method is based on an encoding of Minsky machines that is inspired by a previous work of Kuznetsov [37]. Since $\mu\mathsf{MALL}^{\omega}$ is a calculus of finite (recursively checkable) proofs, and so is in Σ^0_1, this in particular implies that it proves strictly fewer theorems than $\mu\mathsf{MALL}^{\infty}$.

Our second main result is that $\mu\mathsf{MALL}^{*}$, the fragment of $\mu\mathsf{MALL}$ restricted to only least fixed points (on which all three aforementioned systems coincide), is Σ^0_1-complete and consequently, undecidable. We use Lincoln's idea of encoding alternating vector addition systems which he originally employed to prove the undecidability of full linear logic [41]. However, in the absence of exponentials, we have had to reinvent the encoding.

The resulting relationships between the systems we consider in this work are summarised in Figure 1.

Organization of the paper. This paper is organised as follows. In Section 2 we motivate our work from the point of view of "regularisation", i.e. the transformation of a non-wellfounded proof into a circular one. We describe the syntax and relevant properties of $\mu\mathsf{MALL}$ in Section 3 and show regularisation in the additive fragment. In Section 4.1, we prove our first main result that $\mu\mathsf{MALL}^{\infty}$ is Π^0_1-hard (and thus, in general, non-regularisable). In Section 5,

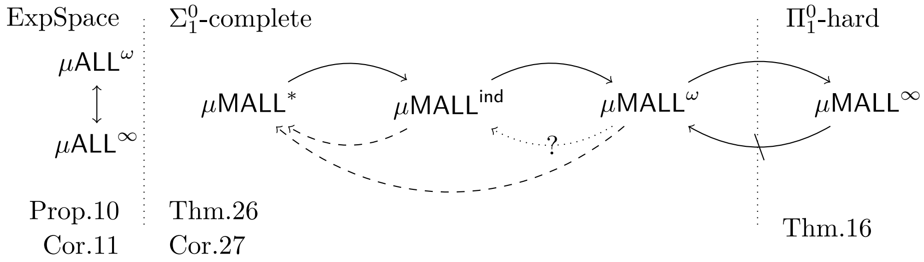


Figure 1 Relationships between systems in this work. Solid arrows $\rightarrow$ denote inclusion, dashed arrows denote conservative extensions, negated arrows $\not\rightarrow$ denote non-inclusion.

we give our second main result, that μMALL* is Σ^0_1-complete. Finally, in Section 6, we conclude and discuss some directions of future work. Additional material, discussions and proof details can be found in an extended version of this paper [14].

2 Motivation: regularisation techniques are not logic-independent

Non-wellfounded systems for logics such as the μ-calculus [1, 17, 21, 48], and in our case μMALL [4], handle least ("μ") and greatest ("ν") fixed points by identical rules:

$$\textbf{Fixed point rules:} \qquad \frac{\Gamma, \phi(\mu X.\phi(X))}{\Gamma, \mu X.\phi(X)}\,\mu \qquad\qquad \frac{\Gamma, \phi(\nu X.\phi(X))}{\Gamma, \nu X.\phi(X)}\,\nu \tag{1}$$

Here Γ (a "sequent") is a list, set or multiset of formulas and the comma is to be read as a form of disjunction, all depending on the logic at hand. To distinguish the two fixed points, non-wellfounded proofs impose a certain global correctness condition; informally speaking, each infinite branch must have a "critical" ν-formula that is unfolded infinitely often (a formal definition is given in the next section). This corresponds to a sort of "infinite descent" argument that mimics inductive reasoning on the fixed point. At least one motivation for our work is to understand when, in general, we can transform a non-wellfounded proof tree into one that is *regular*, i.e. one that has finitely many distinct subtrees, and so may be written as a finite directed (cyclic) graph.

Regularising μ-calculus is easy. In the case of the modal μ-calculus, a simple proof system is readily obtained by extending (multi)modal logic K (cf., e.g., [6]) by the rules in Equation (1). The induced (cut-free) calculus enjoys a certain generalisation of the subformula property (the "Fischer-Ladner closure") meaning that only finitely many distinct sequents may occur in a proof. As a result, once a particular sequent to be proved is fixed, the aforementioned global correctness criterion becomes an ω-regular property on infinite branches. This allows us to reduce *regular* completeness of the system to *non-wellfounded* completeness of the system, thanks to Rabin's basis theorem [50]. This idea is implicit in Niwinski and Walukiewicz's seminal work [48].

This reduction is, a priori, non-constructive: it asserts the existence of a regular proof but does not tell us how to construct one from a given non-wellfounded one. However it is possible to define a constructive such procedure that "cuts" branches of an infinite proof tree to transform it into a regular one, using automata-theoretic techniques.

Structural rules:	$\dfrac{}{\phi, \phi^{\perp}}\ \mathsf{id}$	$\dfrac{\Gamma_1, \phi \quad \Gamma_2, \phi^{\perp}}{\Gamma_1, \Gamma_2}\ \mathsf{cut}$	$\dfrac{\Gamma, \phi_2, \phi_1, \Delta}{\Gamma, \phi_1, \phi_2, \Delta}\ \mathsf{ex}$	
Logical rules:	$\dfrac{\Gamma, \phi_1, \phi_2}{\Gamma, \phi_1 \parr \phi_2}\ \parr$	$\dfrac{\Gamma_1, \phi_1 \quad \Gamma_2, \phi_2}{\Gamma_1, \Gamma_2, \phi_1 \otimes \phi_2}\ \otimes$	$\dfrac{\Gamma, \phi_i}{\Gamma, \phi_1 \oplus \phi_2}\ \oplus_i$	$\dfrac{\Gamma, \phi_1 \quad \Gamma, \phi_2}{\Gamma, \phi \,\&\, \phi_2}\ \&$
Logical rules (units):	$\dfrac{}{\mathbf{1}}\ \mathbf{1}$	$\dfrac{\Gamma}{\Gamma, \perp}\ \perp$	$\dfrac{}{\Gamma, \top}\ \top$	No rule for $\mathbf{0}$

Figure 2 Inference rules for MALL, where $i \in \{1, 2\}$.

Regularisation not possible (in general) in predicate settings. The situation is considerably different in predicate logics with (co)induction or fixed points, e.g. [7, 8, 54]. There neither is cut eliminable in general, nor are proofs regularisable in general, due to infinitely many choices available when instantiating existentials by terms. Indeed both of these observations have recently been demostrated formally for cyclic systems corresponding to fragments of Peano Arithmetic [13].

The trouble with structural rules. Returning to the propositional setting, at first glance the regularisation argument for the μ-calculus is rather general, relying only on the finitude of sequents occurring in a proof to obtain ω-regularity of the global correctness criterion. However, such finitude of sequents is a rather peculiar property in structural proof theory at large. For the μ-calculus this is a consequence of the underlying classical framework: the admissibility of contraction (duplicating formulas) and weakening (deleting formulas) allows us to limit the number of formula repetitions in a sequent.

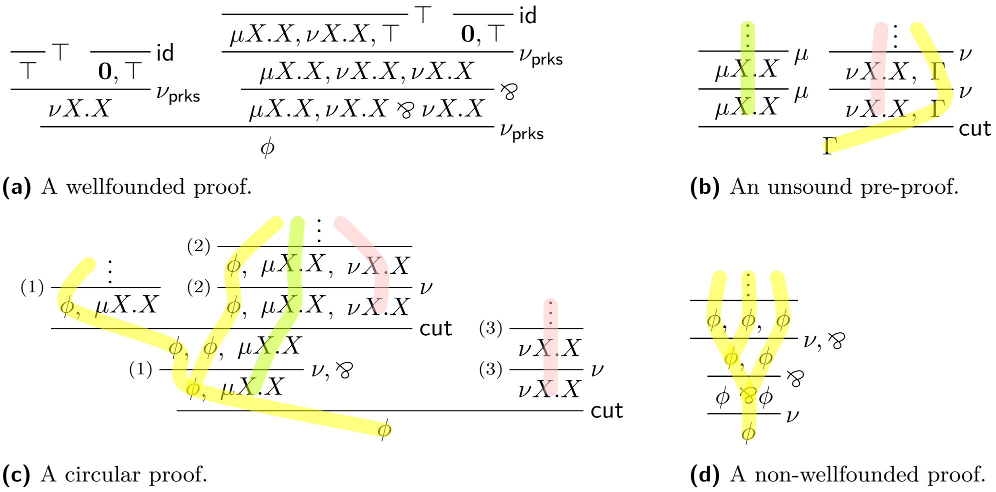
Substructural logics are logics lacking at least one of the usual structural rules. Decidability of substructural logics is often very difficult [34, 57, 38]. In *linear logic* [28], one of the most well-studied substructural logics, sequents are effectively multisets and the use of contraction and weakening is carefully controlled. Conjunction and disjunction each have two versions in linear logic: *multiplicative* and *additive*.

	conjunction	disjunction	true	false
multiplicative	$\otimes$	$\parr$	$\mathbf{1}$	$\perp$
additive	$\&$	$\oplus$	$\top$	$\mathbf{0}$

The logical system thus obtained is called multiplicative-additive linear logic (MALL) and its inference rules are depicted in Figure 2 (sequents being construed as finite lists). Note that, despite the absence of structural rules weakening and contraction, *cut-admissibility* implies that all sequents in a proof have size bounded by that of the conclusion and that the proof search space has only polynomial depth; thus provability is in PSpace (in fact, MALL is PSpace-complete [43]).

Full linear logic extends MALL by incorporating certain "exponential" modalities, written $?\phi$ and, dually, $!\phi$. Structural rules are recovered in the case of $?\phi$, and the resulting logic is undecidable [43]. This is because allowing structural rules only on certain formulas can lead to sequents of unbounded size during proof search. Notably, decidability of multiplicative exponential linear logic (MELL) is still an open question[1] [42, 55].

[1] In [43], non-commutative MELL *i.e.* the fragment without the exchange rule has been proved to be undecidable.



(a) A wellfounded proof.

(b) An unsound pre-proof.

(c) A circular proof.

(d) A non-wellfounded proof.

Figure 3 Various shapes of proof trees for μMALL. Here $\phi = \nu X.X \parr X$. Rules marked (i), for $i \in \{1,2,3\}$, are roots of identical subtrees.

Adding the fixed point rules from Equation (1) to MALL leads to a similar issue, and there is no general way to arrive at such a bound on the set of sequents during proof search. This not only makes decidability of provability non-trivial, but also regularisation of non-wellfounded proofs.

3 Preliminaries

3.1 μMALL: multiplicative additive linear logic with fixed points

In this subsection we recall the system $\mu\mathsf{MALL}^\infty$ introduced in [4].

▶ **Definition 1.** *Fix a countable set of propositional constants $\mathcal{A} = \{A, B, \dots\}$ and variables $\mathcal{V} = \{X, Y, \dots\}$ such that $\mathcal{A} \cap \mathcal{V} = \emptyset$. μMALL* **pre-formulas** *are given by the grammar:*

$$\phi, \psi ::= \mathbf{0} \mid \top \mid \bot \mid \mathbf{1} \mid A \mid A^\perp \mid X \mid \phi \parr \psi \mid \phi \otimes \psi \mid \phi \oplus \psi \mid \phi \,\&\, \psi \mid \mu X.\phi \mid \nu X.\phi$$

where $A \in \mathcal{A}$, $X \in \mathcal{V}$, and μ, ν bind the variable X in ϕ. Free and bound variables, and capture-avoiding substitution are defined as usual. The subformula ordering is denoted $\leq$. When a pre-formula is closed (i.e. no free variables), we simply call it a **formula**.

Negation, $(\bullet)^\perp$, defined as a meta-operation on pre-formulas, will be used only on formulas. As it is not part of the syntax, we do not need any positivity condition on the fixed-point expressions. As expected, least and greatest fixed-points are the dual of each other.

▶ **Definition 2.** **Negation** *of a pre-formula is defined inductively as follows.*

$$(\mathbf{0})^\perp = \top; \quad (\top)^\perp = 0; \quad (\bot)^\perp = \mathbf{1}; \quad (\mathbf{1})^\perp = \bot; \quad (A)^\perp = A^\perp; \quad (A^\perp)^\perp = A;$$
$$(X)^\perp = X; \quad (\phi \parr \psi)^\perp = \phi^\perp \otimes \psi^\perp; \quad (\phi \otimes \psi)^\perp = \phi^\perp \parr \psi^\perp; \quad (\phi \oplus \psi)^\perp = \phi^\perp \,\&\, \psi^\perp;$$
$$(\phi \,\&\, \psi)^\perp = \phi^\perp \oplus \psi^\perp; \qquad (\mu X.\phi)^\perp = \nu X.\phi^\perp; \qquad (\nu X.\phi)^\perp = \mu X.\phi^\perp.$$

The system is classical, hence, it is enough to consider a one-sided proof system. However, as discussed in Section 2 it is imperative to allow multiple copies of the same formula in a sequent. A one-sided μMALL sequent is thus a finite list of formulas.

▶ **Definition 3.** *A* **pre-proof** *of* μMALL$^\infty$ *is a possibly infinite tree generated from the inference rules of* MALL *(see Figure 2) and the fixed point rules from Equation* (1)*.*

Let us recall some standard terminology relating to inference rules [11]. The sequent(s) in a rule displayed above the line are **premisse(s)** and the unique sequent below the line is the **conclusion**. In a logical or fixed point rule, the **principal formula** is the distinguished formula occurrence in its conclusion in Equation (1) or Figure 2. **Auxiliary formulas** are the formula occurrences distinguished in the premisse(s). Other formula occurrences in logical or fixed point rules are **side formulas**.

▶ **Definition 4.** *Given a pre-proof π, for all rules r occurring in π, we define the* **immediate ancestor** *relation* $\mathsf{IA}(r)$ *on formula occurrences of r by: $(\phi, \psi) \in \mathsf{IA}(r)$ if ϕ is principal and ψ is auxiliary; or ϕ is a side formula occurrence in the conclusion and ψ is the corresponding side formula occurrence in a premisse; or r is structural and ϕ is a formula occurrence in the conclusion and ψ is the corresponding formula occurrence in a premisse.*

Several examples of pre-proofs can be found in Figure 3. Immediate ancestors are indicated by the same colour (note that immediate ancestors always "go upwards").

One of the key caveats of non-wellfounded pre-proofs is that, unconstrained, they admit inconsistencies: it is possible to derive any sequent, as shown in Figure 3b. For this reason we impose a global criterion on pre-proofs.

▶ **Definition 5** ([4])**.** *Let $\beta = (\Gamma_i)_{i<\omega}$ be an infinite branch of a* μMALL$^\infty$ *pre-proof π and let r_i be the rule with conclusion Γ_i. A* **thread** *of β is given by $k \in \mathbb{N}$ and a sequence of formula occurrences $\{\phi_i\}_{k<i<\omega}$ such that, for $k < i < \omega$, we have $(\phi_i, \phi_{i+1}) \in \mathsf{IA}(r_i)$.*

A thread τ is **progressing** *if: it is infinitely often principal; and, the smallest formula occurring infinitely often in τ is a ν-formula.*[2]

π is called a **proof** *if every infinite branch has a progressing thread.*

For example, in Figure 3b, while the right infinite branch has a progressing thread along $\nu X.X$ (indicated red), the left branch has no progressing thread, so the pre-proof is not a proof. Figure 3d is indeed a proof, assuming, say, each ν step has the left-most ϕ occurrence principal. In this work we shall crucially make use of the *admissibility* of cut in μMALL$^\infty$:

▶ **Theorem 6** ([4, 3, 19])**.** *Every provable* μMALL$^\infty$ *sequent has a cut-free proof.*

Finally, we consider a fragment of pre-proofs that have a finite presentation.

▶ **Definition 7.** *A* μMALL$^\infty$ *pre-proof is said to be* **circular** *(aka* **regular***) if it has finitely many distinct sub-trees.The class of circular proofs is denoted by* μMALL$^\omega$*.*

Figure 3c is a regular pre-proof. In fact, it is a proof; any infinite branch must either loop on one of (1), (2) or (3), whence there is an infinite progressing thread on ϕ (indicated yellow), $\nu X.X$ (indicated red) or $\nu X.X$ (indicated red) respectively, or it alternates between (1) and (2) infinitely often, whence there is an infinite progressing thread (indicated yellow) on ϕ.

Importantly, given a regular pre-proof π, we can decide whether it is a proof by reduction to the universality of non-deterministic parity ω-word automata, cf. [48, 18, 21]. Observe that ν-unfoldings are the source of infiniteness in proofs: with only μ-unfoldings, no infinite branch may have a progressing thread. So ν-free proofs, *i.e.* proofs without any νs, are necessarily finite. Let us call this class of proofs μMALL*; clearly, μMALL$^* \subseteq \mu$MALL$^\omega \subseteq \mu$MALL$^\infty$.

[2] A "smallest" formula must exist along a thread, since immediate ancestry is compatible with the Fischer-Ladner pre-order, cf. [24] (see also [56, 21]). By construction this formula is unique and, furthermore, is a subformula of all other infinitely occurring formulas in τ.

$$\dfrac{\Gamma,\psi \qquad \bullet\,\dfrac{\psi^{\perp},\phi(\psi) \qquad \dfrac{\dfrac{\bullet\,\dfrac{\vdots}{\psi^{\perp},\nu X.\phi(X)}\,\mathsf{cut}}{\phi^{\perp}(\psi^{\perp}),\phi(\nu X.\phi(X))}\,\phi}{\phi^{\perp}(\psi^{\perp}),\nu X.\phi(X)}\,\nu}{\psi^{\perp},\nu X.\phi(X)}\,\mathsf{cut}}{\nu X.\phi(X),\Gamma}\,\mathsf{cut}$$

Figure 4 A (logic-independent) simulation of the Park's rule in circular proofs. The steps marked ϕ are given by "functoriality" or "deep inference" with respect to the positive formula $\phi(X)$.

Although the focus of the paper are these systems, we briefly discuss $\mu\mathsf{MALL}^{\mathsf{ind}}$, the wellfounded system with explicit coinduction. The system has the same set of inference rules as Definition 3 except the rule for the greatest fixed-point which is replaced by the so-called *Park's rule*, implementing a form of (co)induction:

$$\dfrac{\Gamma,\psi \quad \psi^{\perp},\phi(\psi)}{\Gamma,\nu X.\phi(X)}\,\nu_{\mathsf{prks}}$$

We exhibit a $\mu\mathsf{MALL}^{\mathsf{ind}}$ proof in Figure 3a. We still have that $\mu\mathsf{MALL}^{*} \subseteq \mu\mathsf{MALL}^{\mathsf{ind}}$, and it is not hard to see that $\mu\mathsf{MALL}^{\mathsf{ind}} \subseteq \mu\mathsf{MALL}^{\omega}$ as shown in Figure 4.

The opposite direction *i.e.* the question of equiprovability of $\mu\mathsf{MALL}^{\mathsf{ind}}$ and $\mu\mathsf{MALL}^{\omega}$ is a manifestation of the so-called *Brotherston-Simpson conjecture* in the setting of $\mu\mathsf{MALL}$ [8] and is a difficult open question.

3.2 Focusing

In structural proof theory, focused proofs are a family of proofs that have more structure than usual sequent calculus proofs. The additional structure brought by focusing will be crucial in the next sections in order to extract traces of execution in the computational models that we consider. We describe focused proofs as a complete, proper class of $\mu\mathsf{MALL}^{\infty}$ proofs. The starting point of focusing is the classification of the inference rules (*resp.* connectives) of linear logic into two categories: **positive** and **negative**.

The **negative** connectives have ***invertible*** inferences: if the conclusion of the inference is provable, so are its premisses. For example, if a sequent $\Gamma, \phi \parr \psi$ is provable, so is Γ, ϕ, ψ. The negative (resp. positive) connectives of $\mu\mathsf{MALL}^{\infty}$ are $\&, \parr, \perp, \top, \nu$ (resp. $\otimes, \oplus, 1, 0, \mu$).[3]

By assigning arbitrary polarities to atomic variables one can extend the notion to formulas in such a way that each formula is either positive or negative. A sequent is ***positive*** if it contains only positive or atomic formulas, it is ***negative*** otherwise.

▶ **Definition 8.** *A $\mu\mathsf{MALL}^{\infty}$ proof is said to be in* **negative normal form** *if every negative sequent occurring in it is the conclusion of a negative inference. A $\mu\mathsf{MALL}^{\infty}$ proof π is said to be* **focused** *if it is in negative normal form and if, for every rule r with a positive sequent s as conclusion, the auxiliary formulas of r are principal in its premisses (the "focus"), unless they are negative atomic formulas.*[4]

[3] Observe that both the μ and ν rules are invertible. See [4, 21] for an explanation of the choice.
[4] As is usual, we neglect the structural rule of exchange in this definition, by working with the exchange built in the other rules.

Note that the focusing constraint enforces that when a positive formula is principal (the "focus"), so are its auxiliary formulas and so on until a negative formula is reached.

▶ **Theorem 9** ([4, 21]). *If a sequent is provable in $\mu\mathsf{MALL}^{\infty}$, it has a focused cut-free proof.*[5]

3.3 Regularising fragments of $\mu\mathsf{MALL}^{\infty}$: the complexity of $\mu\mathsf{ALL}$

While cuts are admissible in $\mu\mathsf{MALL}^{\infty}$, cf. Theorem 6, regularity of proofs is not, in general, preserved by cut-elimination. In other words, the process of cut-elimination on a circular proof produces a non-wellfounded proof which, in general, may not have finitely many distinct sub-proofs. In fact, we can show that the cut-free $\mu\mathsf{MALL}^{\infty}$ and the cut-free $\mu\mathsf{MALL}^{\omega}$ are not equiprovable since $\nu X.X \parr X$ has a unique cut-free $\mu\mathsf{MALL}^{\infty}$ proof (up to choices of principal formulas), given in Figure 3d, that is non-regular. However, there is indeed a circular proof with cuts (see Figure 3c) of the aforementioned theorem. It is natural to ask: *Is every theorem of $\mu\mathsf{MALL}^{\infty}$ also provable in $\mu\mathsf{MALL}^{\omega}$ (possibly with cuts)?* In this paper we formally show that such a regularisation result does not hold.

It is worth pointing out that the argument we mentioned for regularisation in the μ-calculus in Section 2 can in fact be adapted to certain fragments of $\mu\mathsf{MALL}$, in particular the additive fragment. Writing $\mu\mathsf{ALL}^{\infty}$ and $\mu\mathsf{ALL}^{\omega}$ for the restriction of $\mu\mathsf{MALL}^{\infty}$ and $\mu\mathsf{MALL}^{\omega}$, respectively, to only additive connectives, we have:

▶ **Proposition 10.** *If Γ is provable in $\mu\mathsf{ALL}^{\infty}$, then it is also (cut-free) provable in $\mu\mathsf{ALL}^{\omega}$.*

Proof. By the cut-elimination theorem of [25], we may assume that Γ has a cut-free $\mu\mathsf{ALL}^{\infty}$ proof π. Note that each (non-cut) rule of $\mu\mathsf{ALL}$ preserves, bottom-up, the number of formulas in a sequent. Since there are only finitely many formulas that can occur (just those in the Fischer-Ladner closure of Γ, cf. [4]), π may contain only finitely many distinct sequents.

As a result, the set of non-wellfounded proofs of Γ constitutes an ω-regular tree language (since the progressing thread criterion is ω-regular). Since we assumed that this language was non-empty, there must be a regular such proof by Rabin's basis theorem [50]. ◀

Note that this also implies the *decidablility* of $\mu\mathsf{ALL}^{\infty}$ since, after guessing a (exponential-size) pre-proof of Γ, checking that it is a proof is decidable (in space polynomial in the size of the proof).

▶ **Corollary 11.** $\mu\mathsf{ALL}^{\infty}$ *(equivalently $\mu\mathsf{ALL}^{\omega}$) is decidable in exponential space.*

We stop short of attempting to optimise this result since, in particular, it seems sensitive to the precise presentation of $\mu\mathsf{ALL}$. Often $(\mu)\mathsf{ALL}$ is presented with *exactly* two formulas in a sequent, e.g. [51, 25], and this invariant is maintained by the rules of $(\mu)\mathsf{ALL}$. In such a presentation, there are only quadratically many distinct sequents in a $\mu\mathsf{ALL}^{\infty}$ proof.

However note that the calculi $\mu\mathsf{ALL}^{\omega}$ and $\mu\mathsf{ALL}^{\infty}$ make sense with an arbitrary number of formulas in a sequent, since branches need not terminate at an initial step. For instance, it is easy to see that $\mu\mathsf{ALL}^{\infty}$ proves $\Gamma, \nu X.X$, for any Γ, by simply continuously unfolding $\nu X.X$. In this more general setting the number of possible sequents becomes exponential.

[5] The focusing result in [4] is for a logic without atoms but the proof technique can be straightforwardly extended to account for atoms.

4 μMALL$^\infty$ is Π^0_1-hard via (non-halting of) Minsky machines

We prove that the following problem is undecidable by a reduction to the non-halting of Minsky machines.

> Given a sequent Γ does there exist a μMALL$^\infty$ proof of Γ?

Our reduction is inspired by [37] for commutative action logic with Kleene star. At first glance, it seems straightforward to be able to embed this logic in μMALL$^\infty$ via the standard encoding of the Kleene star as $F^* = \mu X.(1 \oplus (F \otimes X))$. However, there are a couple of issues with this.

First, action logic is intuitionistic, requiring an extension of the conservativity of linear logic over intuitionistic linear logic [52] to μMALL$^\infty$. Strictly speaking, this is not possible since $\mathbf{0}$ is itself encodable as a fixed point *viz.* $\mu X.X$, and it is not obvious what language such a conservativity result might hold over.

Moreover, the inference rule for the Kleene star in [37] is omega-branching. Therefore, one would also need to establish translations from the omega-branching μMALL to μMALL$^\infty$ (and vice versa) which seem to be quite nontrivial and require yet further intermediary systems. Therefore, we provide a direct reduction.

4.1 The hardness result

We begin by formally defining a Minsky machine and its corresponding (non)-halting problem.

▶ **Definition 12.** *A* **Minsky machine** $\mathcal{M}$ *is a tuple* (Q, r_1, r_2, I) *where* Q *is a finite set of states,* r_1, r_2 *are two registers, and* I *is a set of instructions of the form* $\mathsf{INC}(\bullet,\bullet,\bullet)$ *and* $\mathsf{JZDEC}(\bullet,\bullet,\bullet,\bullet)$ *that manipulate the current state and the contents of the registers. The operational semantics of* $\mathcal{M}$ *is given by its configuration graph, the vertices of which are configurations of form* $\langle q, a, b\rangle \in Q \times \mathbb{N} \times \mathbb{N}$ *and edges are one of the following forms:*

$$\langle p, a, b\rangle \xrightarrow{\mathsf{INC}(p,r_1,q)} \langle q, a+1, b\rangle \qquad \langle p, a, b\rangle \xrightarrow{\mathsf{INC}(p,r_2,q)} \langle q, a, b+1\rangle$$

$$\langle p, 0, b\rangle \xrightarrow{\mathsf{JZDEC}(p,r_1,q_0,q_1)} \langle q_0, 0, b\rangle \qquad \langle p, a, 0\rangle \xrightarrow{\mathsf{JZDEC}(p,r_2,q_0,q_1)} \langle q_0, a, 0\rangle$$

$$\langle p, a+1, b\rangle \xrightarrow{\mathsf{JZDEC}(p,r_1,q_0,q_1)} \langle q_1, a, b\rangle \qquad \langle p, a, b+1\rangle \xrightarrow{\mathsf{JZDEC}(p,r_2,q_0,q_1)} \langle q_1, a, b\rangle$$

Given a state q_s*, a* **run** *of* $\mathcal{M}$ *is a sequence of configurations* $\{s_i\}_{i\in\mathsf{o}}$ $(\mathsf{o} \in \omega+1)$ *such that* $s_0 = \langle q_s, 0, 0\rangle$ *and for all* $i \in \mathsf{o}$ *with* $i+1 \in \mathsf{o}$, (s_i, s_{i+1}) *is an edge in the configuration graph.*

▶ **Theorem 13** ([45]). *Given a Minsky machine* $\mathcal{M}$ *and an initial state* q_s*, checking that it has an infinite run from* q_s *is* Π^0_1*-hard.*

Fixing a Minsky machine as in the definition above, we construe $\{a, b, z_a, z_b\} \cup Q$ as a set of propositional variables (assuming $\{a, b, z_a, z_b\} \cap Q = \emptyset$). We use a and z_a (*resp.* b and z_b) to represent the contents of the register r_1 (*resp.* r_2). We encode instructions (with any extra 0-ary instruction `zero-check`) as follows:

$$\begin{aligned}
[\mathsf{INC}(p, r_1, q)] &\triangleq p \parr (q^\perp \otimes a^\perp)\\
[\mathsf{JZDEC}(p, r_1, q_0, q_1)] &\triangleq (p \parr (q_0^\perp \oplus z_a^\perp)) \,\&\, ((p \parr a) \parr q_1^\perp)\\
[\texttt{zero-check}] &\triangleq (z_a \otimes z_a^\perp) \oplus (z_b \otimes z_b^\perp)
\end{aligned}$$

For any formula F, define $F^* = \mu X.(\mathbf{1} \oplus (F \otimes X))$ and $F^\omega = \nu X.(\bot \mathbin{\&} (F \parr X))$. Observe that $(F^*)^\perp = (F^\perp)^\omega$. For typographic ease, we use a^n to denote $\overbrace{a, \dots, a}^{n \text{ times}}$ in a sequent.

▶ **Proposition 14.** *For any formula F and any $n \in \mathbb{N}$, $F^n, (F^\perp)^*$ is provable.*

Proof. We proceed by induction on n. We call π_F^n the proof of $F^n, (F^\perp)^*$.

Base Case: $n = 0$. We have

$$\dfrac{\dfrac{\dfrac{}{\mathbf{1}}\,\mathbf{1}}{\mathbf{1} \oplus (F^\perp \otimes (F^\perp)^*)}\,\oplus_1}{(F^\perp)^*}\,\mu$$

Induction Case: $n = m + 1$. We have

$$\dfrac{\dfrac{\dfrac{}{F, F^\perp}\,\mathsf{id} \qquad \dfrac{\mathrm{IH} = \pi_F^m}{F^m, (F^\perp)^*}}{F^{m+1}, F^\perp \otimes (F^\perp)^*}\,\otimes}{F^{m+1}, (F^\perp)^*}\,\mu, \oplus_2$$

◀

In the following, let S be a finite set and $[\bullet] : S \to \mu\mathsf{MALL}^\infty$. We write CH_S for $\bigoplus_{s \in S} [s]^\perp$, the formula that offers a choice of picking the dual of one of the (encoding of) elements of S.

When S is a set of instructions we rely on the above encoding, when S is a set of states, we use the identity encoding benefiting from the fact that states are indeed propositional variables. The reader might be surprised by our use of the logical duality here: it is simply because we are working in the one-sided calculus. Finally, we encode the invariant to be maintained by

$$\boxed{\mathsf{Inv} \triangleq ((a^\perp)^* \otimes (b^\perp)^* \otimes \mathsf{CH}_Q) \oplus ((b^\perp)^* \otimes z_a) \oplus ((a^\perp)^* \otimes z_b).}$$

It checks one of the three following conditions: (i) the control is at a valid configuration (ii) r_1 is zero (iii) r_2 is zero. Note that $[q] = q$ where the left-hand side is the state q and the right-hand side is the propositional variable q.

▶ **Theorem 15.** *A Minsky machine $\mathcal{M}$ has an infinite run from the state q_s iff $\mathsf{CH}_I^\omega, q_s, \mathsf{Inv}$ is derivable in $\mu\mathsf{MALL}^\infty$.*

As a direct consequence of Theorem 15 and Theorem 13 we have the following:

▶ **Theorem 16.** *The set of $\mu\mathsf{MALL}^\infty$-provable sequents is Π_1^0-hard.*

The main technical ingredient of Theorem 15 is the following lemma.

▶ **Lemma 17.** *$\mathcal{M}$ performs n steps starting from $\langle q_s, 0, 0\rangle$ iff $\mathsf{CH}_I^n, q_s, \mathsf{Inv}$ is derivable.*

Before showing how this lemma is proved, let us first see how it allows us to obtain our main result:

Proof of Theorem 15. For the only if part we assume that $\mathcal{M}$ loops. So, $\mathcal{M}$ runs for n steps for all $n \in \mathbb{N}$. Therefore, by Lemma 17, we have that $\Gamma_n = \mathsf{CH}_I^n, q_s, \mathsf{Inv}$ is derivable for all $n \in \mathbb{N}$. Let us call π_n a proof of Γ_n, for $n \in \mathbb{N}$. We have

$$\dfrac{\dfrac{\dfrac{\pi_0}{q_s, \mathsf{Inv}}}{\bot, q_s, \mathsf{Inv}}\,\bot \qquad \dfrac{\dfrac{\dfrac{\pi_1}{\mathsf{CH}_I, q_s, \mathsf{Inv}}}{\mathsf{CH}_I, \bot, q_s, \mathsf{Inv}}\,\bot \qquad \dfrac{\vdots}{\mathsf{CH}_I, \mathsf{CH}_I^\omega, q_s, \mathsf{Inv}}}{\mathsf{CH}_I \parr \mathsf{CH}_I^\omega, q_s, \mathsf{Inv}}\,\parr, \nu, \&}{\mathsf{CH}_I^\omega, q_s, \mathsf{Inv}}\,\nu, \&$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p^{\perp}, p$}
\AxiomC{$\mathrm{IH} = \pi_F^m$}
\UnaryInfC{$q_0, \mathsf{CH}_I^{\ell}, b^n, \mathsf{Inv}$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$z_a, z_a^{\perp}$}
\AxiomC{π_b^n}
\UnaryInfC{$b^n, (b^{\perp})^*$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$z_a, z_a^{\perp}$}
\RightLabel{$\otimes$}
\BinaryInfC{$z_a, b^n, (b^{\perp})^* \otimes z_a^{\perp}$}
\RightLabel{$\oplus_2$}
\UnaryInfC{z_a, b^n, Inv}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{$z_a, \mathsf{CH}_I^{\ell-1}, b^n, \mathsf{Inv}$}
\RightLabel{$\otimes$}
\BinaryInfC{$z_a, z_a \otimes z_a^{\perp}, \mathsf{CH}_I^{\ell-1}, b^n, \mathsf{Inv}$}
\RightLabel{$\oplus_1$}
\UnaryInfC{$z_a, \texttt{zero-check}, \mathsf{CH}_I^{\ell-1}, b^n, \mathsf{Inv}$}
\RightLabel{$\oplus$}
\UnaryInfC{$z_a, \mathsf{CH}_I^{\ell}, b^n, \mathsf{Inv}$}
\RightLabel{$\&$}
\BinaryInfC{$(q_0 \mathbin{\&} z_a), \mathsf{CH}_I^{\ell}, b^n, \mathsf{Inv}$}
\RightLabel{$\otimes$}
\BinaryInfC{$p^{\perp} \otimes (q_0 \mathbin{\&} z_a), \mathsf{CH}_I^{\ell}, p, b^n, \mathsf{Inv}$}
\RightLabel{$\oplus_1$}
\UnaryInfC{$[\mathsf{JZDEC}(p, r_1, q_0, q_1)]^{\perp}, \mathsf{CH}_I^{\ell}, p, b^n, \mathsf{Inv}$}
\RightLabel{$\oplus$}
\UnaryInfC{$\mathsf{CH}_I^{\ell+1}, p, b^n, \mathsf{Inv}$}
\end{prooftree}
$$

Figure 5 A proof that does a zero-check.

Observe that this pre-proof is indeed a proof as the right-most non-wellfounded branch is validated by a thread on CH_I^{ω}. For the other direction assume that we have a proof π of $\mathsf{CH}_I^{\omega}, q_s, \mathsf{Inv}$. Observe that for all $n \in \mathbb{N}$ we have a proof of $\mathsf{CH}_I^{n}, q_s, \mathsf{Inv}$:

$$
\begin{prooftree}
\AxiomC{$\pi_{\mathsf{CH}_I}^n$}
\UnaryInfC{$\mathsf{CH}_I^n, (\mathsf{CH}_I^{\perp})^*$}
\AxiomC{π}
\UnaryInfC{$\mathsf{CH}_I^{\omega}, q_s, \mathsf{Inv}$}
\RightLabel{cut}
\BinaryInfC{$\mathsf{CH}_I^n, q_s, \mathsf{Inv}$}
\end{prooftree}
$$

By Lemma 17, $\mathcal{M}$ runs at least n steps for all $n \in \mathbb{N}$. We collect all these runs and get a finitely branching infinite tree rooted at $\langle q_s, 0, 0\rangle$. König's lemma ensures that there is an infinite run of $\mathcal{M}$ from q_s. ◀

Proof sketch of Lemma 17. We will prove a stronger statement (stated this way it is easier to apply induction, however, as demonstrated above, we only need the weaker statement to prove the theorem): $\mathcal{M}$ performs k steps from $\langle p, m, n\rangle$ iff $\mathsf{CH}_I^k, p, a^m, b^n, \mathsf{Inv}$ is derivable.

The **only-if** part is proved by induction on k. The base case ensures that the initial configuration is indeed a valid configuration. For the induction case, one examines the first step of the execution and applies the corresponding encoding of the instruction. We will exhibit the case of the decrementation of a zero-valued register.

Suppose the first instruction is $\mathsf{JZDEC}(p, r_1, q_0, q_1)$ and $m = 0$. We have the derivation shown in Figure 5 where we select the appropriate instruction by applying the corresponding $\oplus$ inference. The vertical ellipsis symbolises the repeating pattern decreasing the number of CH_I formulas in the sequent.

For the **if** part we first observe that the proof is necessarily finite and hence we can induct on it. The base case is vacuous. For the induction case, we will first assume (*wlog* by Theorem 6 and Theorem 9) that we have a focused and cut-free proof of $\mathsf{CH}_I^k, p, a^m, b^n, \mathsf{Inv}$. We assign atomic polarities as follows: a, b and q are negative for any state $q \in Q$, z_a, z_b are positive. By careful case-analysis, we get that one of the CH_Is is necessarily the focus. The instruction it chooses, we will execute that on $\mathcal{M}$. Finally, we check that `zero-check` cannot be chosen and after choosing a decrementation one cannot be led astray into the wrong state.

Basically a focused proof is forced to go exactly as exhibited in the only-if part and we will end up in a subproof of the shape $\mathsf{CH}_I^{k-1}, q, a^{m'}, b^{n'}, \mathsf{Inv}$ for some state q and some natural number m', n'. We can then apply the induction hypothesis and get the desired result. ◀

4.2 Separation of regular and non-wellfounded proofs

An immediate consequence of our results is that $\mu\mathsf{MALL}^\omega$ and $\mu\mathsf{MALL}^\infty$ are distinct logics.

▶ **Theorem 18.** *There are theorems of $\mu\mathsf{MALL}^\infty$ that are not provable in $\mu\mathsf{MALL}^\omega$.*

Proof. Any $\mu\mathsf{MALL}^\omega$ pre-proof has only finitely many distinct sequents, and so can be checked for correctness recursively by reduction to universality of non-deterministic parity automata over infinite words. Thus $\mu\mathsf{MALL}^\omega \in \Sigma_1^0$. On the other hand, we showed in Theorem 16 that $\mu\mathsf{MALL}^\infty$ is Π_1^0-hard, and we conclude since $\Pi_1^0 \setminus \Sigma_1^0 \neq \emptyset$. ◀

Observe that this proof is apparently non-constructive in the sense that we do not explicitly exhibit a sequent in $\mu\mathsf{MALL}^\infty \setminus \mu\mathsf{MALL}^\omega$. While it is clear that not all sequents of the form $\mathsf{CH}_I^\omega, q_s, \mathsf{Inv}$ from Section 4 can be derivable in $\mu\mathsf{MALL}^\omega$, it is not clear which particular Minsky machine $\mathcal{M}$ to choose to witness this underivability. In fact the argument can indeed be constructivised using established recursion-theoretic techniques, namely the notion of *productive function*. The application of such techniques to the present situation is explained elegantly by Kuznetsov in [[36], pp. 497], so we shall not recast it here.

5 $\mu\mathsf{MALL}^*$ is Σ_1^0-complete via (reachability in) AVASS

5.1 Towards an upper bound

The works of Palka [49] and Kuznetsov [37] proceed by showing Π_1^0 membership of their various logics (say L) in two stages:

1. The cut-free fragment L^μ with only least fixed points (*i.e.* the Kleene star is only on the right side of the sequent) is decidable.
2. The provability problem for any sequent is in $\Pi_1^0(L^\mu)$, whence it is in Π_1^0 by (1) above.

Usually, the difficult part is (2), requiring some combination of proof-theoretic and logical techniques, typically requiring infinitary wellfounded proof search to obtain the Π_1^0 bound. However, in our case, we are already stuck at (1): μ-only cut-free $\mu\mathsf{MALL}$, i.e. $\mu\mathsf{MALL}^*$, is undecidable. In the absence of greatest fixed-points, all systems (non-wellfounded, circular, inductive) coincide; so the logic $\mu\mathsf{MALL}^*$ is indeed an interesting and robustly defined core of the theory of linear logic with fixed points.

Propositional linear logic was shown to be undecidable [41, 43] by a reduction from the reachability problem in an *and-branching* two counter machine without zero-test. Such machines are essentially equivalent to a particular extension of vector addition systems, called *alternating vector addition systems with states* (or AVASS) [39, 31] (in particular, the fork rule is exactly the same). More recently, other substructural logics have been related with extensions of vector addition systems [20, 39, 40]. Our work is in the spirit of this line of research and we show the undecidability of $\mu\mathsf{MALL}^*$ by a reduction from the reachability problem of AVASS.

One could try using the undecidability of propositional linear logic to prove the undecidability of μMALL* using a standard encoding of the exponential modalities by fixed point formulas of the following form:

$$[?F] = \mu X.\bot \oplus [F] \oplus (X \parr X) \quad ; \quad [!F] = \nu X.\mathbf{1} \,\&\, [F] \,\&\, (X \otimes X)$$

However, this encoding is not known to be faithful. Note that the reduction in [41, 43] uses only ? so the encoding is indeed in μMALL*. We provide a direct proof via the reachability problem of AVASS.

Before defining the reduction in the next subsection (Section 5.2), we conclude this subsection by formally introducing AVASS and the corresponding reachability problem.

▶ **Definition 19.** *An* **AVASS** *is a tuple $\mathcal{B} = (Q, Q_\ell, k, \mathsf{A}, T_u, T_f)$ such that:*
- *Q is a finite set of states with $Q_\ell \subseteq Q$;*
- *$k \in \mathbb{N}$ is called the* **dimension***;*
- *A is a finite subset of $\mathbb{N}^k$ called the set of* **axioms***;*
- *$T_u \subseteq Q \times \mathbb{Z}^k \times Q$ and $T_f \subseteq Q^3$ are finite and called the* **unary** *and* **fork** *rules respectively.*

A configuration of an AVASS $\mathcal{B}$ is a pair $(q, \vec{v}) \in Q \times \mathbb{N}^k$ where Q is the set of states of $\mathcal{B}$ and k is its dimension.

▶ **Definition 20.** *Given an AVASS $\mathcal{B} = (Q, Q_\ell, k, \mathsf{A}, T_u, T_f)$, a configuration $(q, \vec{v}) \in Q \times \mathbb{N}^k$ is said to be* **reachable** *if there is a binary tree labelled by configurations such that:*
- *The root node is labelled by $(q, \vec{v})$.*
- *If a node $(q, \vec{v})$ has a unique child $(q', \vec{v'})$ then $(q, \vec{v} - \vec{v'}, q') \in T_u$.*
- *If a node $(q, \vec{v})$ has children $(q', \vec{v'})$ and $(q'', \vec{v''})$ then $\vec{v} = \vec{v'} = \vec{v''}$ and $(q, q', q'') \in T_f$.*
- *The leaves are labelled by elements of $Q_\ell \times \mathsf{A}$.*

Such that a binary tree is called the **run tree** *of the configuration.*

▶ **Theorem 21** ([43])**.** *The AVASS reachability problem is Σ^0_1-complete.*

5.2 Encoding an AVASS in μMALL*

We fix $k+1$ propositional variables, $a_1, \ldots, a_k, z$, and define below an encoding of integer vectors of dimension up to k (the unique vector of dimension 0 is written ϵ). For the purpose of the encoding vectors will be read from left to right *i.e.* a vector $\vec{v}$ of dimension $l+1$ will be of the form $(n, \vec{u})$ for an integer n and a vector $\vec{u}$ of dimension l.

▶ **Definition 22.** *The* **encoding of an integer vector** *$\vec{v}$ of dimension d, relative to propositional variables $b_i, \ldots, b_{d+i-1}, z$, written $[\vec{v}]_{b_i,\ldots,b_{d+i-1},z}$, is defined inductively as follows:*

$$[\vec{v}]_{b_i,\ldots,b_{d+i-1}} \triangleq \begin{cases} z & \text{if } \vec{v} = \epsilon; \\ b_i \parr [\vec{v'}]_{b_i,\ldots,b_{d+i-1},z} & \text{if } \vec{v} = (n, \vec{u}), n \geq 1, \text{ and } \vec{v'} = (n-1, \vec{u}); \\ b_i^\perp \otimes [\vec{v'}]_{b_i,\ldots,b_{d+i-1},z} & \text{if } \vec{v} = (n, \vec{u}), n \leq -1, \text{ and } \vec{v'} = (n+1, \vec{u}); \\ [\vec{u}]_{b_{i+1},\ldots,b_{d+i-1},z} & \text{if } \vec{v} = (0, \vec{u}). \end{cases}$$

We will simply write $[\vec{v}]$ for the encoding of a vector of dimension k relative to $a_1, \ldots, a_k, z$. (We also use this lighter notation for vectors of lower dimension when the dimension and the $\{a_i, \ldots, a_k, z\}$ to be used are clear from the context.)

The encoding is slightly involved, so let us first consider an example:

▶ **Example 23.** Consider the encoding of $(-1, 0, 1)$ relative to b_1, b_2, b_3, z.

$$\begin{aligned}[(-1,0,1)]_{b_1,b_2,b_3} &= {b_1}^{\perp} \otimes [(0,0,1)]_{b_1,b_2,b_3} \\ &= {b_1}^{\perp} \otimes [(1)]_{b_3} = {b_1}^{\perp} \otimes (b_3 \parr [(0)]_{b_3}) \\ &= {b_1}^{\perp} \otimes (b_3 \parr [\epsilon]_{b_3}) = {b_1}^{\perp} \otimes (b_3 \parr z)\end{aligned}$$

Observe that the i^{th} coordinate is represented by the propositional variables b_i. The following lemma shows that the encoding is meaningful with respect to vector equality.

▶ **Lemma 24.** *Let $\vec{u}$ and $\vec{v}$ be vectors of the same dimension. Then $[\vec{u}]^{\perp}, [\vec{v}]$ is derivable if and only if $\vec{u} = \vec{v}$.*

Proof sketch. The "if" direction is trivial. Let us consider the "only if" direction. Assume that $\vec{u} = (n_{k-d+1}, \ldots, n_k)$, $\vec{v} = (m_{k-d+1}, \ldots, m_k)$ and $[\vec{u}]^{\perp}, [\vec{v}]$ is derivable. Let π be a cut-free proof of this sequent. Since π is a MALL proof we can apply the soundness of the sequent calculus wrt. phase semantics [28]. Consider for instance the phase space[6] $((\mathbb{Z}, +, 0), \{0\})$ and for each propositional variable a_i, $1 \leq i \leq k$, consider the valuation ϕ_i such that $\phi_i(a_i) = \{1\}$ and $\phi_i(b) = \{0\}$ for $b \neq a_i$. Soundness ensures that $[\vec{u}]^{\perp}, [\vec{v}]$ is valid in every phase model, that is $[\![[\vec{u}]^{\perp}, [\vec{v}]]\!]^{\phi_i} = m_i - n_i = 0$ for $k-d+1 \leq i \leq k$, that is $\vec{u} = \vec{v}$. ◀

The following technical lemma will allow us to reason by induction on the dimension via the encoding at the provability level, which is crucial to prove our forthcoming theorem.

▶ **Lemma 25.** *Let $1 \leq i \leq k$, $m \geq 0$ and s an integer such that $s + m \geq 0$. Let $\vec{q}$ be an integer vector of dimension $k - i$. If $[\vec{q}], \Gamma, a_i^{s+m}, a_{i+1}^{m_{i+1}}, \ldots, a_k^{m_k}$ is provable, then so is $[\vec{r}], \Gamma, a_i^{m}, a_{i+1}^{m_{i+1}}, \ldots, a_k^{m_k}$ where $\vec{r} = (s, \vec{q})$ and $\vec{v} = (m, \vec{u})$.*

Proof. Let π be a proof of $[\vec{q}]^{\perp}, \Gamma, [\vec{u}], a_i^m$. We have three cases depending on s: when s is positive, negative, or zero.

Case 1: If s is positive.

$$\dfrac{\dfrac{\{\ \dfrac{}{a_i^{\perp}, a_i}\,\mathsf{id}\ \}_s \qquad \dfrac{\pi}{[\vec{q}]^{\perp}, \Gamma, a_i^{m}, [\vec{u}]}}{[\vec{r}]^{\perp}, \Gamma, a_i^{s+m}, [\vec{u}]}\,\otimes^{s}}{[\vec{r}]^{\perp}, \Gamma, [\vec{v}]}\,\parr^{s+m}$$

Case 2: If s is negative.

$$\dfrac{\dfrac{\dfrac{\pi}{[\vec{q}]^{\perp}, \Gamma, [\vec{u}], a_i^{m}}}{a_i^{|s|}, [\vec{q}]^{\perp}, \Gamma, [\vec{v}]}\,\parr^{s+m}}{[\vec{r}]^{\perp}, \Gamma, [\vec{v}]}\,\parr^{|s|} si$$

Case 3: If s is zero, then by the encoding it is simply ignored, hence this case is trivial. ◀

We can now define the encoding of an AVASS. Fix an AVASS $\mathcal{B}$ with $|Q_\ell \times \mathsf{A}| = \alpha$, $|T_u| = \beta$, and $|T_f| = \gamma$.

- For a unary rule $t \in T_u$ of the form $(p, \vec{r}, q)$ we have $[t] \triangleq p \parr (q^{\perp} \otimes [\vec{r}])$.
- For a fork rule $t \in T_f$ of the form (p, q_1, q_2) we have that $[t] \triangleq (p \parr ({q_1}^{\perp} \otimes z)) \oplus (p \parr ({q_2}^{\perp} \otimes z))$.
- For a "final" configuration $(q, \vec{v}) \in Q_\ell \times \mathsf{A}$, we have that $[(q, \vec{v})] \triangleq q \parr [\vec{v}]$.

$\mathcal{B}$ is encoded as $B \triangleq \mu X. \big(\mathsf{CH}_{Q_\ell \times \mathsf{A}} \oplus (\mathsf{CH}_{T_u} \parr (z \otimes X) \oplus (\mathsf{CH}_{T_f} \parr (z \otimes X)\big)$.

[6] The *facts* of this phase space are the singletons, $\mathbb{Z}$ and $\emptyset$. One has $[\![A^{\perp} \otimes B^{\perp}]\!]^{\phi} = [\![A]\!]^{\phi} + [\![B]\!]^{\phi}$, $[\![A \parr B]\!]^{\phi} = [\![A \otimes B]\!]^{\phi\,\perp}$ when $+$ is lifted to sets of integers. In particular, the interpretations of $\otimes$ and $\parr$ coincide on formulas interpreted with singleton facts.

▶ **Theorem 26.** *The configuration* $(q, \vec{v})$ *is reachable in* $\mathcal{B}$ *iff* $B, q, a_1^{v_1}, \ldots, a_k^{v_k}, z$ *is provable in* $\mu\mathsf{MALL}^*$ *where* $\vec{v} = (v_1, \ldots, v_k)$.

Proof sketch. The idea is quite similar to the proof of Lemma 17 *i.e.* given the run tree of $(q, \vec{v})$, we produce a proof tree of the sequent $B, q, a_1^{v_1}, \ldots, a_k^{v_k}, z$ that closely resembles the shape of the run tree, and vice versa. The first direction is relatively simple going through an induction on the run tree. The base case checks that leaves are valid final configurations. For the induction case, we have two cases depending on whether a unary or a fork rule has been applied at the node in question. We exhibit the proof in the case of a unary rule which exploits Lemma 25.

Suppose the unary rule $(p, \vec{r}, q)$ is applied to the node labelled $(p, \vec{v})$ where $\vec{r} = (r_1, \ldots, r_k)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p^\perp, p}\,\mathsf{id} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{z^\perp, z}\,\mathsf{id} \qquad \dfrac{\text{IH}}{q, B, a_1^{v_1+r_1}, \ldots, a_k^{v_k+r_k}, z}}{q, z^\perp, z \otimes B, a_1^{v_1+r_1}, \ldots, a_k^{v_k+r_k}, z}\,\otimes}{q, [\vec{r}], z \otimes B, a_1^{v_1}, \ldots, a_k^{v_k}, z}\,\text{Lemma 25, } k \text{ times}}{q \parr [\vec{r}], z \otimes B, a_1^{v_1}, \ldots, a_k^{v_k}, z}\,\parr}{p^\perp \otimes (q \parr [\vec{r}]), z \otimes B, p, a_1^{v_1}, \ldots, a_k^{v_k}, z}\,\otimes}{\mathsf{CH}_{T_u}, z \otimes B, q, a_1^{v_1}, \ldots, a_k^{v_k}, z}\,\oplus}{\mathsf{CH}_{T_u} \parr (z \otimes B), q, a_1^{v_1}, \ldots, a_k^{v_k}, z}\,\parr}{\dfrac{\mathsf{CH}_{Q_\ell \times \mathsf{A}} \oplus (\mathsf{CH}_{T_u} \parr (z \otimes B)) \oplus (\mathsf{CH}_{T_f} \parr (z \otimes B)), q, a_1^{v_1}, \ldots, a_k^{v_k}, z}{B, q, a_1^{v_1}, \ldots, a_k^{v_k}, z}\,\mu}\,\oplus_2$$

where the β-ary $\oplus$ chooses the (encoding of the) rule $(p, \vec{r}, q)$.

The other direction (*i.e.* given a proof tree produce a run tree) is more involved but as in the proof of Lemma 17 exploits the stringent structure of focused proofs *i.e. wlog* we assume that we are given a cut-free focused proof of sequent $B, q, a_1^{v_1}, \ldots, a_k^{v_k}$. We will induct on the height of the proof. It must have the following prefix.

$$\dfrac{\mathsf{CH}_{Q_\ell \times \mathsf{A}} \oplus (\mathsf{CH}_{T_u} \parr (z \otimes B)) \oplus (\mathsf{CH}_{T_f} \parr (z \otimes B)), q, a_1^{v_1}, \ldots, a_k^{v_k}, z}{B, q, a_1^{v_1}, \ldots, a_k^{v_k}, z}\,\mu$$

Assume that $q \in Q$ and $\{a_1, \ldots, a_k\}$ are negative atoms and z is a positive atom. There are now three cases depending on whether the next rule is $\oplus_1, \oplus_2$ or $\oplus_3$. Based on what is chosen, we know whether the first rule of the run tree is an axiom, or a unary rule, or a binary rule. In fact, the focusing constraint forced one to choose the particular rule as well. We will exemplify the reasoning by exhibiting the situation for the fork rule (*i.e.* $\oplus_3$). The following rule is necessarily $\parr$. Now there are two possibilities, either CH_{T_f} is the focus or $z \otimes B$ is the focus. Suppose the latter happens. Then, the left premisse of the tensor rule with principal formula $z \otimes B$ is of the form z, Γ. As this is a positive sequent, it must be the conclusion of an id rule which is not possible as $z^\perp \notin \Gamma$ and therefore CH_{T_f} is the focus.

Assume that the proof chooses the (encoding of the) fork rule $t = (q', q_1, q_2) \in T_f$ (by applying the correct version of the γ-ary $\oplus$). The premisse is now negative and the next rule must be a $\&$. Therefore, we have two sequents of the form $q'^\perp \otimes (q_i \parr z^\perp), z \otimes B, q, a_1^{v_1}, \ldots, a_k^{v_k}, z$ for $i \in \{1, 2\}$. Let us discuss about the case when $i = 1$. (The other case is symmetric.)

As before, $z \otimes B$ cannot be the focus, so, $q'^\perp \otimes (q_1 \parr z^\perp)$ is the focus and the next rule is a $\otimes$. Since $q'^\perp$ is positive, the left premisse must be the conclusion of an id rule which forces $q' = q$. The right premisse is $q_1 \parr z^\perp, z \otimes B, a_1^{v_1}, \ldots, a_k^{v_k}, z$ which is negative and after a $\parr$ rule, we focus on $z \otimes B$: the left premisse of the tensor must be $z, z^\perp$ as it is the conclusion of a id rule, which leaves us with a subproof on which we can apply the induction hypothesis and we get a run-tree rooted at $(q_1, \vec{v})$. Similarly we get a run-tree rooted at $(q_2, \vec{v})$. Now we have a run tree rooted at $(q, \vec{v})$ where the first rule is the fork rule (q, q_1, q_2). ◀

From Theorem 21 and Theorem 26, we have the following corollaries.

▶ **Corollary 27.** μMALL* *is* Σ^0_1*-complete.*

▶ **Corollary 28.** μMALL$^{\mathsf{ind}}$ *and* μMALL$^\omega$ *are* Σ^0_1*-complete.*

Proof sketch. Σ^0_1-memership is immediate, since both μMALL$^{\mathsf{ind}}$ and μMALL$^\omega$ are systems of finitely presentable proofs that are recursively checkable.

For hardness, note that μMALL$^\infty \supseteq \mu$MALL$^\omega \supseteq \mu$MALL$^{\mathsf{ind}} \supseteq \mu$MALL* satisfies cut-elimination [4]. Since any μMALL$^\infty$ proof of a μ-only sequent is necessarily a finite tree, all these systems are actually conservative over μMALL*, and thus Σ^0_1-hardness is inherited. ◀

It is folklore that if $\phi(X)$ is an LK formula with a free variable X then $\phi(X)$ and $\phi(\phi(\phi(X)))$ are equivalent. This immediately gives us a conservative embedding of μLK (note that this is different from μ-calculus since there are no modalities) in LK with a polynomial blowup. In the same vein, [27, 26] shows that there is a conservative embedding of μLJ in LJ with an exponential blowup. MALL is known to be PSpace-complete [43]. Therefore we have the following corollary.

▶ **Corollary 29.** *There is no effectively computable reduction from* μMALL* *(or* μMALL$^{\mathsf{ind}}$*,* μMALL$^\omega$*) to* MALL*.*

6 Conclusions and future work

In this work we classify the complexity of several systems for fixed point logics (cf. Figure 1). In particular, we proved that the non-wellfounded calculus μMALL$^\infty$ is undecidable (via a reduction to the non-halting of Minsky machines) and prove strictly more theorems than μMALL$^\omega$, its regular counterpart. We further proved that the finite provability for μMALL (in any of our systems) is already undecidable. Namely the problem is Σ^0_1-complete, via a reduction to reachability in alternating vector addition systems. One novelty of our reductions is that they are based on focusing and establishes an isomorphism between proof-trees and run-trees of Minsky machines and AVASSs.

Since its inception, linear logic was advertised as the logic for concurrency [29] and its relation with VASs (or, rather, Petri nets) has been significantly explored from both syntactic [23] and semantic [44] points of view. Our results are also cognate with this line of research. The main open questions that remain from this work is:

Complexity of μMALL$^\infty$. There is a trivial upper bound for μMALL$^\infty$ *viz.* Σ^1_3 in the analytical hierarchy. That leaves a huge gap between our Π^0_1 lower bound proved in Theorem 15. Discerning the exact complexity of μMALL$^\infty$ therefore, amounts to closing this gap. Note that the undecidability of μMALL* shows that established strategies [12, 49, 36, 37] of proving a Π^0_1 upper bound cannot be adapted to μMALL$^\infty$.

Induction vs cycles. Is μMALL$^{\mathsf{ind}}$ equivalent to μMALL$^\omega$? This is a manifestation of the so-called *Brotherston-Simpson conjecture* in the setting of μMALL [8]. Roughly speaking, is induction as powerful as circular reasoning? Note that if the answer is indeed negative, then such a result cannot be obtained using techniques similar to Section 4.2 since in Section 5 we show that they have the same complexities.

We conclude by mentioning another pertinent direction for future work. In Section 3.3 we exhibited a purely multiplicative sequent $\nu X.X \parr X$ which has a circular proof only if we allow cuts (Figure 3c). It would be interesting to further develop regularisation procedures that blend ideas from both automata theory and proof theory, generalising the construction in Figure 3c. Naturally, by Theorem 18, no such procedure can be well-defined for all of $\mu\mathsf{MALL}^\infty$, but it is reasonable to ask if there is a middle ground.

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