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ORIGINAL ARTICLE

Sarkisov links for index 1 Fano 3-folds in codimension 4

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Abstract
We classify Sarkisov links from index 1 Fano 3-folds anticanonically embedded in codimension 4 that start from so-called Type I Tom centres. We apply this to compute the Picard rank of many such Fano 3-folds.

KEYWORDS
Fano 3-fold, Picard rank, Sarkisov link, unprojection

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1 | INTRODUCTION

The construction of sequences of birational maps linking algebraic varieties to one another is a crucial part in the Minimal Model Program (MMP). In the framework of the MMP, the most fundamental such sequences go under the name of Sarkisov links ([16, 21]). In this context, the notions of birational rigidity and pliability for Fano 3-folds, and for Mori fibre spaces (Mfs) more generally, are important, and relate to the uniqueness or otherwise of outputs of the MMP. The pliability (see [17, Definition 1.5(4)]) is the number of different Mori fibre spaces W that are birational to a given Mfs X. If the pliability is 1, then X is said to be birationally rigid; if it is 2 or more, then by [16] it is known that the birational transformation between X and any W can be factorised into a sequence of Sarkisov links. The literature often considers Sarkisov links from X according to the codimension of X in its anticanonical embedding. Corti, Pukhlikov, Reid and others ([15, 18]) show that quasi-smooth members of the 95 index 1 terminal Fano 3-fold weighted hypersurfaces of [22, 29] are birationally rigid. In codimension 2, 19 families of Fano 3-folds are birationally rigid, and 66 are non-rigid ([3, 23, 25]); in codimension 3, Brown and Zucconi prove birational non-rigidity whenever there is a Type I centre [13]. Codimension 3 is completed by Ahmadinezhad and Okada [2], where they prove that an index 1 terminal Fano 3-fold in codimension 3 is birationally rigid if and only if it does not have any Type I or Type II_1 centres (this happens for 3 of the 70 Hilbert series). The expectation is that as the codimension increases, rigidity becomes more rare.

We follow ideas of [13, 17] and we focus on terminal $\mathbb{Q}$-factorial Fano 3-folds in codimension 4 having at least one Type I centre that are listed in the Graded Ring Database [10]. In particular we examine those deformation families arising from Type I unprojections of pfaffian Fano 3-folds in Tom format (see [11, Section 3]): we call these *Fano 3-folds of Tom type*.

In our Main Theorem 2.3 we give a description of birational links for Fano 3-folds of Tom type based on the weights of their ambient space and their basket of singularities, in a similar flavour to the main theorems in [13, 15, 17].

Theorem 2.3 is is also related to other works in the literature, such as Takagi's [30], and a comparison with that can be found in Subsection 5.4. The explicit results are given in detail in [14]. Some important remarks regarding Theorem 2.3 are in Section 2.

In Section 6 we apply Theorem 2.3 to compute the Picard rank of some Fano 3-folds in codimension 4.

2 | THE MAIN THEOREM

We work over the field of complex numbers $\mathbb{C}$. A Fano 3-fold is a normal projective 3-dimensional variety X with ample anticanonical divisor $-K_X$ and at worst terminal singularities.

Definition 2.1. The *Fano index* of a Fano 3-fold X is defined to be

$$\iota_X := \max\left\{ q \in \mathbb{Z}_{\geq 1} \ : \ -K_X = qA \text{ for some } A \in \mathrm{Cl}(X) \right\}.$$

Our focus will be on those having Fano index $\iota_X = 1$ and codimension 4 in their total anticanonical embedding (cf [5, 11, Section 1]). A complete description of Type I unprojections is provided in [11], giving a tool to produce families of Fano 3-folds in codimension 4. These realise 115 of the possible Hilbert series, and present at least two distinct deformation families of quasi-smooth Fano 3-folds for each, called *Tom* and *Jerry*. By construction, all such Fano varieties have at least one Type I centre. The list of all possibilities for Hilbert series can be found on the Graded Ring Database [10]: each is identified with an ID number preceded by #.

Definition 2.2. Let X be a codimension 4 index 1 Fano 3-fold X listed in the table [12]. We say X is *of Tom Type* if it is obtained as Type I unprojection of the codimension 3 pair $Z \supset D$ in a Tom family (see [11, 27] for background and examples; see 3.2 for details). The image of $D \subset Z$ in X is called *Tom centre*: it is a cyclic quotient singularity $p \in X$. In the unprojection setup $D \subset Z$, D is a complete intersection of four linear forms of weight $d_1, \dots, d_4$: we refer to $d_1, \dots, d_4$ as the *ideal weights*. Such X of Tom type is said to be *general* if $Z \supset D$ is general in its Tom family.

We prove the following main theorem. Here X is a $\mathbb{Q}$-factorial Fano 3-fold. At this stage, we do not assume that X is a Mori fibre space. However, the explicit construction that we carry out shows a posteriori that the endpoint of each birational link described in Theorem 2.3 is a Mori fibre space.

Theorem 2.3. *Let X be a general codimension 4 Fano 3-fold of Tom type and let $p \in X$ be a Tom centre. Then:*

(A) X admits a birational link to a Mori fibre space $Y \to S$. The link is initiated by the Kawamata blow-up of $p \in X$. Let $d_1 \geq d_2 \geq d_3 \geq d_4$ be the four ideal weights for the Tom centre $p \in X$. In each case the Kawamata blow-up is followed by an algebraically irreducible flop of finitely many smooth rational curves, and proceeds as follows according to $d_1 \geq d_2 \geq d_3 \geq d_4$:
  - *(i) $\underline{d_1 > d_2 > d_3 > d_4}$: a composition of two flips, followed by a divisorial contraction Φ' of (2,0)-type to another (non-isomorphic) Fano 3-fold X';*
  - *(ii) $\underline{d_1 > d_2 = d_3 > d_4}$: a flip (missed in cases #1218 and #1413) followed by a divisorial contraction Φ' of (2,1)-type to another Fano 3-fold X';*
  - *(iii) $\underline{d_1 = d_2 > d_3 > d_4}$: two simultaneous flips, followed by a divisorial contraction Φ' of (2,0)-type to another Fano 3-fold X';*
  - *(iv) $\underline{d_1 > d_2 > d_3 = d_4}$: a composition of two hypersurface flips, followed by a del Pezzo fibration: Φ' is of (3,1)-type;*
  - *(v) $\underline{d_1 = d_2 > d_3 = d_4}$: two simultaneous flips followed by a del Pezzo fibration: Φ' of (3,1)-type;*
  - *(vi) $\underline{d_1 > d_2 = d_3 = d_4}$: a toric flip (missed in case #6865) to a conic bundle: Φ' is of (3,2)-type;*
  - *(vii) $\underline{d_1 = d_2 = d_3 > d_4}$: a divisorial contraction Φ' of (2,1)-type to another Fano 3-fold X';*
  - *(viii) $\underline{d_1 = d_2 = d_3 = d_4}$: a conic bundle over a quadric surface in $\mathbb{P}^3$: Φ' is of (3,2)-type.*

(B) In every birational link in (A), the resulting Mfs $Y \to S$ is not isomorphic to X.

(C) If in addition the Picard rank of X is $\rho_X = 1$, then the link produced in (A) is a Sarkisov link, and so X is not birationally rigid.

The notation on fibrations and divisorial contractions in the above theorem is: (m, n) where m is the dimension of the exceptional locus of Φ' in Y (where applicable) and n is the dimension of its image. For instance, if Φ' is of (2,0)-type it contracts a $w\mathbb{P}^2$ to a point in X'.

Note that the flip in case (iii) and (v) is algebraically irreducible: in this situation, we say that we have two *simultaneous flips*. We expand this in Remark 4.1. Moreover, Theorem 2.3 does not apply to the Fano 3-folds considered in [9] (see Remark 4.2).

3 | THE INPUT DATA

3.1 | Construction and notation

The Definition 3.1 of Sarkisov link stems from the notion of 2-ray game, especially in the context of toric varieties (see [13] for a description in terms of graded rings). A birational link for a codimension 4 Fano 3-fold $X \ni p$ is partially subject to the behaviour of the link for its ambient space $w\mathbb{P}^7 \supset X$. Call $x_1, x_2, x_3, y_1, y_2, y_3, y_4, s$ the coordinates of $w\mathbb{P}^7 = \mathbb{P}^7\left(a, b, c, d_1, d_2, d_3, d_4, r\right)$, and suppose to blow up the cyclic quotient singularity at $p = P_s \in w\mathbb{P}^7$. Call this blow-up $\mathbb{F}_1$: this is a rank 2 toric variety whose bi-grading defining the $\mathbb{C}^\times \times \mathbb{C}^\times$ action on $\mathbb{F}_1$ we will deduce below. In toric geometry (cf. [20]), this corresponds to adding a new lattice vector ρ_t to the 1-skeleton of $w\mathbb{P}^7$ given by the lattice vectors $\rho_s, \rho_{x_i}, \rho_{y_j}$ that satisfy the relation

$$r\rho_s + a\rho_{x_1} + b\rho_{x_2} + c\rho_{x_3} + \sum_{j=1}^{4} d_j \rho_{y_j} = 0\,.$$

The new vector ρ_t must be inside the fan constituted by the convex cone $\sigma_s := \left\langle \rho_{x_1}, \rho_{x_2}, \rho_{x_3}, \rho_{y_1}, \rho_{y_2}, \rho_{y_3}, \rho_{y_4} \right\rangle$; that is, an integer multiple of $\omega\rho_t$ of ρ_t is the integer positive sum of all rays other than ρ_s: there are many possible choices to choose the coefficients for this positive sum, and we will identify a particular one. For $\omega, \omega_i, \delta_j > 0$ and $i \in \{1,2,3\}$, $j \in \{1,2,3,4\}$, the relation involving ρ_t is

$$-\omega\rho_t + \sum_{i=1}^{4} \omega_i \rho_{x_i} + \sum_{j=1}^{4} \delta_j \rho_{y_j} = 0\,. \tag{3.1}$$

In other words, $\mathbb{F}_1$ is the variety with Cox ring $\mathbb{C}\left[t, s, x_1, x_2, x_3, y_1, y_2, y_3, y_4\right]$ having the grading and the irrelevant ideal shown below. In the language of the graded Cox rings, the bottom weights of the bi-grading of $\mathbb{F}_1$ are the coefficient of the rays in the definition of ρ_t. Since ρ_s does not appear in the expression for ρ_t, its bottom weight is 0. Thus the bi-grading of $\mathbb{F}_1$ looks like

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & r & a & b & c & d_1 & d_2 & d_3 & d_4 \\ -\omega & 0 & \omega_1 & \omega_2 & \omega_3 & \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{array}\right) \tag{3.2}$$

and its irrelevant ideal is $(t, s) \cap \left(x_1, x_2, x_3, y_1, y_2, y_3, y_4\right)$, as indicated by the vertical bar between s and x_1. We will determine the values of the bottom weights $\omega, \omega_1, \omega_2, \omega_3, \delta_1, \dots, \delta_4$ later in this section (also refer to the Appendix of [7] for further details). Note that this is not well-formed: we come back to this later.

The 2-ray game for $w\mathbb{P}^7$ is determined by the ray-chamber structure of the Mori cone of $\mathbb{F}_1$. Each bi-degree in (3.2) represent a character of the $\mathbb{C}^* \times \mathbb{C}^*$ action on $\mathbb{F}_1$ which correspond to the rays $\rho_s, \rho_{x_i}, \rho_{y_j}$; these in particular represent the linear systems associated to each of the coordinates of $w\mathbb{P}^7$. This induces the ray-chamber structure in Figure 1.

The variation of GIT on $\mathbb{F}_1$ corresponds to the wall-crossing in the picture above. This induces a 2-ray game for $w\mathbb{P}^7 \ni p$. Given $X \ni p$ of a Tom-type Fano 3-fold and of a Type I centre $p \in X \subset w\mathbb{P}^7$, we want to embed the 2-ray game for $X \ni p$ into the 2-ray game for $w\mathbb{P}^7, \ni p$: this is achieved by finding the appropriate weights $\omega, \omega_1, \omega_2, \omega_3, \delta_1, \dots, \delta_4$ for the grading of the toric variety $\mathbb{F}_1$.

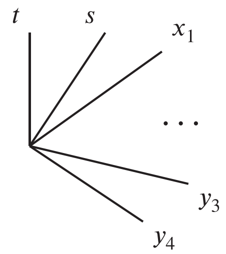

FIGURE 1 The Mori cone of $\mathbb{F}_1$

The objects of the *Mori category* are projective, $\mathbb{Q}$-factorial terminal 3-folds. The Fano 3-folds of Definition 2.2 are in the Mori category.

Definition 3.1. A *Sarkisov link* for $X \ni p$ is a birational map between the Mori fibre spaces $X \to S$ and $X' \to S'$ that factors as

$$\begin{tikzcd}[column sep=small] & Y_1 \arrow[rr, dashed, "\Psi_1"] \arrow[dl, "\Phi"'] \arrow[dr, "\alpha_1"] & & Y_2 \arrow[rr, dashed, "\Psi_2"] \arrow[dl, "\beta_1"'] \arrow[dr, "\alpha_2"] & & Y_3 \arrow[rr, dashed, "\Psi_3"] \arrow[dl, "\beta_2"'] \arrow[dr, "\alpha_3"] & & Y_4 \arrow[dl, "\beta_3"'] \arrow[dr, "\Phi'"] & \\ w\mathbb{P} \supset X & & Z = Z_1 & & Z_2 & & Z_3 & & X' \subset w\mathbb{P}' \end{tikzcd} \tag{3.3}$$

The birational maps Ψ_1, Ψ_2, Ψ_3 are isomorphisms in codimension 1, that is, antiflips, flops, flips in this order (cf. [6, Remark 3.5]). The map Φ is a divisorial extraction, and Φ' can be either a divisorial contraction or a fibration (del Pezzo fibration or conic bundle, in which case the second Mori fibre space is $Y_4 \to w\mathbb{P}'$). Call $\mathbb{G}_i$ the image of α_i (or β_i), and Z_i the image of α_i restricted to Y_i; this is the same as the image of the restriction of β_i to Y_{i+1}. A Sarkisov link takes place in the Mori category if it satisfies the properties listed in Definition 2.2 of [13].

This sets our nomenclature. By a little abuse of notation, we call the coordinates of each $\mathbb{F}_i$ in the same way. Following [13], each ray of the ray-chamber structure is associated to the linear system defined by the bi-degree of the variable(s) generating it and induces a map of toric varieties. Each ray corresponds to one of the toric varieties in the bottom row of the 2-ray game (the ambient spaces of the Z_i) in (3.3), while each chamber corresponds to one of the $\mathbb{F}_i$ at the top row of (3.3). Transitioning from one chamber to another adjacent chamber performs the isomorphism $\Psi_i : \mathbb{F}_i \to \mathbb{F}_{i+1}$ in codimension 1. Approaching the ray in between the two chambers from one side or another indicates the two maps $\alpha_i : \mathbb{F}_i \to \mathbb{G}_i$ and $\beta_i : \mathbb{F}_{i+1} \to \mathbb{G}_i$ (defined by the same linear system). In the language of Geometric Invariant Theory, this is a variation of GIT quotient on $\mathbb{F}_1$.

3.2 | The unprojection setup: construction of *X*

The starting point to construct X is the following type of data, coming from [10, 11].

- A fixed projective plane $D := \mathbb{P}^2(a, b, c) \subset \mathbb{P}^6(a, b, c, d_1, \dots, d_4)$ with coordinates $x_1, x_2, x_3, y_1, \dots, y_4$ respectively and $d_1 \geq d_2 \geq d_3 \geq d_4$. So D is defined by the ideal $I_D := \mathrm{Span}\left[y_1, y_2, y_3, y_4\right]$.
- A family $\mathcal{Z}_1$ of codimension 3 Fano 3-folds $Z \subset w\mathbb{P}^6$, each defined by maximal pfaffians of a skew-symmetric 5×5 syzygy matrix M whose entries $\left(a_{i,j}\right)$ have weights

$$\begin{pmatrix} m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} \\ & m_{2,3} & m_{2,4} & m_{2,5} \\ & & m_{3,4} & m_{3,5} \\ & & & m_{4,5} \end{pmatrix}.$$

Here we use the notation of [11] for skew-symmetric matrices: we omit the principal diagonal, whose entries are all zero, and the lower-left triangle, which is the symmetric of the upper-right triangle with opposite signs. The matrix M is graded, i.e. each of its entries is occupied by a polynomial in the given degree. A list of the grading of M is in [12].

The plane is a divisor $D \cong \mathbb{P}^2_{x_1,x_2,x_3}(a,b,c)$ of $Z_1 \in \mathcal{Z}_1$ if the equations of the latter are the maximal pfaffians of a matrix M in either Tom or Jerry format.

Definition 3.2 ([11], Definition 2.2). A 5×5 skew-symmetric matrix M is in Tom$_k$ format if and only if each entry $a_{i,j}$ for $i, j \neq k$ is in the ideal I_D.

Not all possible formats can be realised: [12] records exactly the data of $D \subset w\mathbb{P}^6$, the weights of the syzygy matrix, and the successful formats (and why the others fail). Each of them corresponds to a distinct deformation family of Z_1 (cf. [11]): there can be at most four different deformation families, with at most two realised as Tom formats, and at most two realised as Jerry formats. In this paper we only focus on the Tom case: the aim is to construct M in this general setting by filling its entries with homogeneous polynomials in the x_i and y_j subject to the Tom constraints. It is often possible to place some of the variables in a matrix position having the same degree. The following lemma highlights a key feature of M, that is, the presence of certain quasilinear monomials in the ideal variables. It is a direct observation on the weights $m_{k,l}$ of M. By generality of Z_1, y_j and x_j appear linearly in suitable entries. This is in a similar spirit to [13, Section 3].

Lemma 3.3. *Let $Z_1 \supset D$ be a general member of a Tom$_i$ family in [12] where $i \in \{1, \dots, 5\}$. Then there are at least three entries $a_{k,l}$ of M with $k \neq i$, $l \neq i$ such that $d_j = m_{k,l}$, that is, y_j appears linearly in $a_{k,l}$. Except for #12960 in [10], there is an entry $a_{k,l}$ of M with $k = i$ or $l = i$ such that $m_{k,l}$ is equal to $a, b,$ or c, i.e. linear in at least one of the orbinates x_j.*

Once this set-up in codimension 3 is done and the equations of Z_1 are found, the unprojection of Z_1 at the divisor D is a birational transformation that produces a new Fano 3-fold $X \subset w\mathbb{P}^7$ in codimension 4. In particular, X inherits the five pfaffian equations of Z_1 and gains four extra equations from the unprojection process, to which we will refer as unprojection equations in the rest of this paper. The unprojection equations are of the form $sy_i = g_i(x_1, x_2, x_3, y_1, y_2, y_3, y_4)$ where s is the additional coordinate of $w\mathbb{P}^7$ and the right-hand side is a homogeneous polynomial of the same degree as sy_i. In the unprojection the divisor $D \subset Z_1$ is contracted to the Type I centre $P_s \in X$. In this paper we study birational links from $X \ni P_s$. In Appendix A we present a brief summary about the explicit construction of the unprojection equations based on [26].

3.3 | The bi-grading of $\mathbb{F}_1$

Consider $X \ni P_s$. To perform a blow-up $\Phi : \mathbb{F}_1 \to w\mathbb{P}^7$ at P_s we choose a suitable grading $\omega, \omega_1, \omega_2, \omega_3, \delta_1, \delta_2, \delta_3, \delta_4$ for $\mathbb{F}_1$ in (3.2). We follow a similar method to [4]. Recall the following theorem.

Theorem 3.4 (Kawamata blow-up, [24]). *Let X be a 3-fold, and let $p \in X$ be a terminal cyclic quotient singularity $\frac{1}{r}(a,b,c)$. Suppose that $\phi : (E \subset Y) \to (\Gamma \subset X)$ is a divisorial contraction with $p \in \Gamma$ and Y terminal. Then, $\Gamma = \{p\}$ and ϕ is the weighted blow-up of p with weights (a,b,c) and therefore the exceptional divisor is $E \cong \mathbb{P}(a,b,c)$.*

The blow-up map Φ is defined by the linear system $\left|\mathcal{O}\binom{1}{0}\right|$. Explicitly,

$$\Phi \ : \ \mathbb{F}_1 \longrightarrow w\mathbb{P}^7$$

$$(t,s,x_1,x_2,x_3,y_1,y_2,y_3,y_4) \longmapsto \left(t^{\frac{\omega_1}{\omega}} x_1, t^{\frac{\omega_2}{\omega}} x_2, t^{\frac{\omega_3}{\omega}} x_3, t^{\frac{\delta_1}{\omega}} y_1, t^{\frac{\delta_2}{\omega}} y_2, t^{\frac{\delta_3}{\omega}} y_3, t^{\frac{\delta_4}{\omega}} y_4, s \right).$$

The blown-up point is the cyclic quotient singularity of index r at P_s, so $\omega = r$. In [13] it is shown that the exceptional locus E of Φ is given by the vanishing of the coordinates y_1, y_2, y_3, y_4; thus, $E \cong \mathbb{P}^2(\omega_1, \omega_2, \omega_3)$. In order for Φ to be a Kawamata blow-up the weights $\omega_1, \omega_2, \omega_3$ must be a, b, c respectively.

The equations of X come into play to determine the value of the δ_j. The key point is the definition of the Fano 3-fold Y_1. In the pull-back $\Phi^*(X)$, every monomial in each equation of X picks up a suitable power of t. We aim at defining Y_1 as the saturation over t of the total pull-back of X (see Definition 3.9). On the other hand, we want to embed the link for (X, P_s) into the link for $(w\mathbb{P}^7, P_s)$ in such a way that the birational transformations to which $\mathbb{F}_1$ is subject restrict to Y_1. Thus, we want the leading terms of the unprojection equations to be sy_j, as opposed to sy_jt^τ for an exponent $\tau > 1$. We give a constructive definition of the δ_j starting with δ_4: the analysis for δ_1, δ_2, and δ_3 is done analogously. The fourth unprojection equation is of the form $sy_4 = g_4(x_1, x_2, x_3, y_1, y_2, y_3, y_4)$, where g_4 is a homogeneous polynomial of degree $r + d_4$. Its pull-back via Φ is

$$t^{\frac{\delta_4}{r}} sy_4 = g_4\left(t^{\frac{a}{r}} x_1, t^{\frac{b}{r}} x_2, t^{\frac{c}{r}} x_3, t^{\frac{\delta_1}{r}} y_1, t^{\frac{\delta_2}{r}} y_2, t^{\frac{\delta_3}{r}} y_3, t^{\frac{\delta_4}{r}} y_4\right).$$

By construction, every monomial in g_4 picks up a t factor because there is no pure monomial in s in g_4. Define h_4 to be the polynomial constituted by all the monomials of g_4 containing y_4, except for the term sy_4. For $g_4' := g_4 - h_4$ and $\varkappa$ the minimum exponent that can be factorised from h_4, the equation above becomes

$$t^{\frac{\delta_4}{r}}\left(sy_4 + t^{\frac{\varkappa-\delta_4}{r}} h_4\right) = g_4'\left(t^{\frac{a}{r}} x_1, t^{\frac{b}{r}} x_2, t^{\frac{c}{r}} x_3, t^{\frac{\delta_1}{r}} y_1, t^{\frac{\delta_2}{r}} y_2, t^{\frac{\delta_3}{r}} y_3\right). \tag{3.4}$$

Lemma 3.5. *It holds that $\delta_4 \geq d_4$.*

Proof. By construction of ρ_t, each δ_j is a strictly positive integer. We divide this proof in different cases according to the different types of monomials in g_4. We indicate by $\underline{x}^l$ the multiplication of pure powers of x_1, x_2 and x_3, not necessarily all together, with different multiplicities, summarised by the multi-index l at the exponent, and similarly for $\underline{y}^{l'}$. In the following, l and l' vary from case to case. Monomials $\underline{x}^l$ with $|l| = \deg(g_4) = r + d_4$ pick up a t factor with exponent $k = |l| = r + d_4$ in the pull-back. Monomials $\underline{x}^l\underline{y}^{l'}$ with $|l + l'| = \deg(g_4) = r + d_4$ pick up a t factor with exponent $k \geq |l + l'| = r + d_4$ in the pull-back because $\delta_1, \delta_2, \delta_3 \geq 1$. Monomials $\underline{x}^l y_4^\lambda$ with $l + \lambda = \deg(g_4) = r + d_4$ pick up a t factor with power $k \geq l + \lambda\delta_4 \geq r + d_4$. Monomials $\underline{y}^{l'} y_4^\lambda$ with $l' + \lambda = \deg(g_4) = r + d_4$ pick up a t factor with power $k \geq l' + \lambda\delta_4 \geq r + d_4$. Monomials $\underline{x}^l\underline{y}^{l'} y_4^\lambda$ with $l + l' + \lambda = \deg(g_4) = r + d_4$ pick up a t factor with power $k \geq l + l' + \lambda\delta_4 \geq r + d_4$. In conclusion, every monomial in g_4 picks up a t^k factor with $k \geq (r + d_4)/r$: then, $\delta_4 \geq d_4$. $\square$

Then, for τ_l positive integers and m_l monomials of g_4', the pullback of the unprojection equation for y_4 is

$$t^{\frac{\delta_4}{r}}\left(sy_4 + t^{\frac{\varkappa}{r}} h_4\right) = t^{\frac{\tau_1}{r}} m_1 + \cdots + t^{\frac{\tau_{k_4}}{r}} m_{k_4}.$$

Definition 3.6. Define δ_4 as $\delta_4 := \min\{\tau_l \ : \ 1 \leq l \leq k_4\}$.

Since g_4' does not contain y_4, δ_4 is well-defined. The definition of δ_1, δ_2, and δ_3 is analogous. The grading for $\mathbb{F}_1$ that we just obtained might not be well-formed (cf [1]), but a manipulation on the rows of (3.2) makes it well-formed (3.5).

Proposition 3.7. *Let X be a codimension 4 index 1 Fano 3-fold of Tom type. Then the Kawamata blow-up of X at the Tom centre P_s is contained in a rank 2 toric variety $\mathbb{F}_1$ with grading*

$$\left(\begin{array}{cc|ccccccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 0 & r & a & b & c & d_1 & d_2 & d_3 & d_4 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right). \tag{3.5}$$

Note that x_1, x_2, x_3 generate the same linear system and, therefore, the same ray in the ray-chamber structure of $\mathbb{F}_1$. Since the Fano index of X is 1, then one of the x_i has weight 1. To fix ideas, let the weight of x_1 be 1. To prove Proposition 3.7 we need the following.

Lemma 3.8. *Let Z be a codimension 3 Fano 3-fold defined by pfaffians of a 5×5 skew-symmetric matrix M in Tom format. Consider the Type I unprojection of Z at a divisor D. Then each unprojection equation contains at least one monomial purely in x_1, x_2, x_3.*

Proof. Since x_1 has weight 1, using the notation in Appendix A, p_j contains a monomial of the form $x_1^{\deg(p_j)}$. There are different possibilities to fill the ideal entries $a_{k,l}$. If an ideal entry has the same weight as of one of the y_j, then it contains such ideal variable linearly, i.e. $\alpha^j_{k,l}$ is constant. Otherwise, it contains multiplications of y_j by the x_i, that is $\alpha^j_{k,l}$ is a polynomial containing a term in the x_i. We assume this without loss of generality. Therefore, each N_j has at least one entry that is either a constant or a monomial in the x_i.

Since the vector of the g_j is independent on the choice of i in (A.2), it is possible to consider only $\frac{H_1}{p_1}$. Therefore, we exclude from the calculation of g_j all $\mathrm{Pf}_1(N_j)$, that is, all $\mathrm{Pf}_i(N_j)$ involving the top row of the matrices N_j, which are the ones containing pure terms in x_1, x_2, x_3. Thus, each entry of Q in row 2, 3, and 4 contains a polynomial purely in x_1, x_2, x_3. The same holds for the other g_i. □

Proof of Proposition 3.7. By Lemma 3.8, each unprojection equation contains at least one monomial in the orbinates of P_s. By the proof of Lemma 3.5, such monomial realises the minimum value of Definition 3.6. Thus, by Lemma 3.8, we choose $\delta_4 = r + d_4$: so, δ_4 is equal to the degree of g_4. In turn, we can apply this same strategy to δ_1, δ_2, δ_3: the power of t gained by each y_j factor is greater or equal to d_j/r, so $\delta_j \geq d_j$. Since $\delta_j = r + d_j$ for each $j \in \{1, 2, 3, 4\}$, the order in which we determine the δ_j is unimportant. The weights in (3.5) follow by simple manipulation of the rows of (3.2) with the grading we just defined: subtracting the second row to the third row of (3.2) and dividing the third row by $-r$ we obtain an isomorphic rank 2 toric variety whose Cox ring is given by (3.5). □

3.4 | The Kawamata blow-up of a Fano: equations of the blow-up Y_1

As anticipated above, we define the blow-up Y_1 of X at P_s as the following.

Definition 3.9. The ideal of $Y_1 \subset \mathbb{F}_1$ is the saturation over t of the ideal of $\Phi^*(X)$.

This motivates the construction of the bottom weights of $\mathbb{F}_1$ made in Definition 3.6. In relation to the 1-skeleton in (3.5), Φ and α_1 are given by the linear systems $\left|\mathcal{O}\binom{r}{1}\right|$, $\left|\mathcal{O}\binom{1}{0}\right|$ respectively. The next statements make Definition 3.9 more manoeuvrable and explicit.

Proposition 3.10. *The pull-back of the pfaffian equations via Φ and via α_1 are equal up to a t factor.*

More precisely, the evaluation of $\mathrm{Pf}_i(M)$ at the defining monomials of Φ is proportional by a t factor to the evaluation of $\mathrm{Pf}_i(M)$ at those of α_1.

Proof. We prove that $t^{-\frac{1}{r}}\Phi = \alpha_1$. The map $\alpha_1 : \mathbb{F}_1 \to w\mathbb{P}^6$ is

$$(t, s, x_1, x_2, x_3, y_1, y_2, y_3, y_4) \longmapsto (x_1, x_2, x_3, ty_1, ty_2, ty_3, ty_4)\ .$$

Consider a variable w of $\mathbb{F}_1$ among x_1, x_2, x_3, y_1, y_2, y_3, y_4 with bidegree $\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$. Call ζ the exponent of the t factor that w needs to pick up such that the bidegree of wt^ζ is proportional to $\begin{pmatrix} r \\ 1 \end{pmatrix}$. In other words, we need to find ζ such that $\deg wt^\zeta = \begin{pmatrix} \nu_1 \\ \nu_2 + \zeta \end{pmatrix} = \lambda \begin{pmatrix} r \\ 1 \end{pmatrix}$ for some $\lambda > 0$. Since $\lambda = \nu_2 + \zeta$, we have that $\zeta = \frac{\nu_1}{r} - \nu_2$. On the other hand, the exponent ζ' of the t factor needed such that the bidegree of $wt^{\zeta'}$ is proportional to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\deg wt^{\zeta'} = \begin{pmatrix} \nu_1 \\ \nu_2 + \zeta' \end{pmatrix} = \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for some

$\mu > 0$. Here $\zeta' = -\nu_2$. Thus, $\zeta - \zeta' = \nu_1/r = 1/r \deg_{w\mathbb{P}^7} w$. This means that on every variable $x_1, x_2, x_3, y_1, y_2, y_3, y_4$ of $\mathbb{F}_1$ the exponents ζ and ζ' differ only by $1/r$. □

Remark 3.11. If M is in Tom format, it is possible to cancel out from $\alpha_1^*(\mathrm{Pf}(M))$ a t factor with power at least 1.

Let $I_X := \mathrm{Span}\left[f_1, \dots, f_5, f_6, \dots, f_9\right]$ be the ideal of X, generated by polynomials $f_i := \mathrm{Pf}_i(M)$ for $i \in \{1, \dots, 5\}$ and $f_i := sy_i - g_i$ for $i \in \{6, \dots, 9\}$. Recall that Φ is expressed with fractional exponents of t. For $\Phi^*(X)$ to have equation in a polynomial ring, we write an equivalent expression for Φ considering its multiplication by a $t^{(r-a)/r}$ factor. Thus,

$$t^{\frac{r-a}{r}} \cdot \left(t^{\frac{a}{r}} x_1, t^{\frac{b}{r}} x_2, t^{\frac{c}{r}} x_3, t^{\frac{\delta_1}{r}} y_1, t^{\frac{\delta_2}{r}} y_2, t^{\frac{\delta_3}{r}} y_3, t^{\frac{\delta_4}{r}} y_4, s\right)$$
$$= \left(tx_1, t^{\frac{b(r-a)}{r}+\frac{b}{r}} x_2, t^{\frac{c(r-a)}{r}+\frac{c}{r}} x_3, t^{\frac{d_1(r-a)}{r}+\frac{\delta_1}{r}} y_1, t^{\frac{d_2(r-a)}{r}+\frac{\delta_2}{r}} y_2, t^{\frac{d_3(r-a)}{r}+\frac{\delta_3}{r}} y_3, t^{\frac{d_4(r-a)}{r}+\frac{\delta_4}{r}} y_4, t^{r-a} s\right).$$

This expression has integer exponents. Call $I_{\Phi^* X} := \mathrm{Span}\left[\Phi^* f_1, \dots, \Phi^* f_5, \Phi^* f_6, \dots, \Phi^* f_9\right]$. Remark 3.11 guarantees that, up to a t factor, Φ^* and α_1^* coincide on the pfaffian equations. Define the following polynomials

$$h_1 := \frac{\alpha_1^* \, \mathrm{Pf}_1(M)}{t^2} = \frac{\alpha_1^* f_1}{t^2}; \tag{3.6}$$

$$h_i := \frac{\alpha_1^* \, \mathrm{Pf}_i(M)}{t} = \frac{\alpha_1^* f_i}{t}, \qquad \text{for } i \in \{2, \dots, 5\}; \tag{3.7}$$

$$h_i := \frac{\Phi^* f_i}{t^{\delta_{i-5}+r-a}}, \qquad \text{for } i \in \{6, \dots, 9\}; \tag{3.8}$$

and the ideal $I_{Y_1} := \left(I_{\Phi^* X} : t^\infty\right)$ as the saturation of $I_{\Phi^* X}$ over t as in Definition 3.9.

Lemma 3.12. *We have that* $I_{Y_1} = \mathrm{Span}\left[h_1, \dots, h_5, h_6, \dots, h_9\right]$.

Proof. For the saturation algorithm we refer to [19]: we introduce a temporary variable z and define the ideal $J := \mathrm{Span}\left[I_{\Phi^* X}, tz - 1\right] \subset S := R[z]$, where $R := \mathbb{C}\left[t, s, x_1, x_2, x_3, y_1, y_2, y_3, y_4\right]$. We study explicitly the Gröbner basis of J with respect to a complete monomial ordering $\succ$. Then, $\left(I_{\Phi^* X} : t^\infty\right) = J \cap R$ (see [19, Chapter 4, §4]). The monomial ordering is such that z is the largest, s is the second largest, and the monomials containing the least number of y_j follow. In other words, $\succ$ is defined by

$$\begin{pmatrix} z & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 & t \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & c & d_1-1 & d_2-1 & d_3-1 & d_4-1 & 1 \\ 0 & 0 & a & b & c & d_1-1 & d_2-1 & d_3-1 & d_4-1 & 0 \\ 0 & 0 & a & b & c & d_1-1 & d_2-1 & d_3-1 & 0 & 0 \\ 0 & 0 & a & b & c & d_1-1 & d_2-1 & 0 & 0 & 0 \\ 0 & 0 & a & b & c & d_1-1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.9}$$

Consider a polynomial k in which z does not appear. Call $k_1 := \mathrm{LT}(k)$ the leading term of k according to $\succ$, so $k = k_1 + k_2$ is the sum of the monomial k_1 and the polynomial $k_2 := k - k_1$. The least common multiple between the respective leading terms of $t^d k$ and $tz - 1$ is $\mathrm{lcm}\left(\mathrm{LT}(t^d k), \mathrm{LT}(tz-1)\right) = t^{d+1} k_1 z$. Then, following [19], the S-polynomials for $t^d k$ for some $d \geq 1$ are $S\left(t^d k, tz - 1\right) = t^d k$. The leading term of $\Phi^* f_1$ is of the form $\mathrm{LT}\left(\Phi^* f_1\right) = t^2 y_{j_1} y_{j_2}$ for certain $j_1, j_2 \in \{1, 2, 3, 4\}$. Similarly for $i \in \{2, \dots, 5\}$, the leading term is $\mathrm{LT}\left(\Phi^* f_i\right) = t x_{j_i} y_{j_i}$ for certain $j_i \in \{1, 2, 3\}$ and $j_i \in \{1, 2, 3, 4\}$. For

$i \in \{6, \dots, 9\}$ instead, $\mathrm{LT}(\Phi^* f_i) = s y_{i-5} t^{\delta_{i-5}+r-a}$. The monomial ordering $\succ$ is designed to identify as biggest the monomials having the lowest exponent of t. Therefore, for each $i \in \{1, \dots, 9\}$ there is a suitable d such that $\Phi^* f_i = t^d h_i$. So we have that $S(\Phi^* f_i, tz - 1) + S(t^d h_i, tz - 1) = t^d h_1$. Therefore, the Gröbner basis of J is

$$GB_{\succ}(\Phi^* f_1, \dots, \Phi^* f_9, tz - 1) = (t_1^h, th_2, th_3, th_4, th_5, t^{\delta_1+r-a} h_6, t^{\delta_2+r-a} h_7, t^{\delta+r-a} h_8, t^{\delta_4+r-a} h_9) \cup \{tz - 1\}\,.$$

For $i \in \{1, \dots, 9\}$, we have that $\mathrm{LT}(h_i)$ and tz are coprime since the highest common factor $hcf\,(\mathrm{LT}(h_i), tz) = 1$. Thus, $GB_{\succ}(h_1, \dots, h_9, tz - 1) = GB_{\succ}(h_1, \dots, h_9) \cup \{tz - 1\}$. In conclusion,

$$\begin{aligned}(\mathrm{Span}[h_1, \dots, h_9] : t^\infty) &= \mathrm{Span}[GB_{\succ}(h_1, \dots, h_9, tz - 1) \cap R] \\ &= \mathrm{Span}[GB_{\succ}(h_1, \dots, h_9)] \\ &= \mathrm{Span}[h_1, \dots, h_9]\,.\end{aligned}$$

□

4 | DESCRIPTION OF THE LINKS AND PROOF OF THE MAIN THEOREM

We break down every step of the birational links described in Theorem 2.3 and we give a proof of Theorem 2.3. We first mention the following remarks about Theorem 2.3.

Remark 4.1. The flip in case (iii) and (v) is algebraically irreducible (that is, its base is irreducible as an algebraic set and its exceptional locus consists of one connected component), but the intersection between its exceptional locus and Y_2 consists of two disjoint tubular neighbourhoods, that are either both toric or both hypersurface. In other words, the intersection between Y_2 and the contracted locus of the flip is not irreducible, and it is formed of two distinct connected components.

In (vii) the exceptional divisor of Φ' contracts to an irreducible conic $\Gamma \subset \mathbb{P}^2$.

Remark 4.2. This theorem does not consider the Fano 3-folds in $\mathbb{P}^2 \times \mathbb{P}^2$ format of [9] (often stemming from the second Tom deformation families), as they have Picard rank 2. The Hilbert series #12960 is one of them, thus it is not described in (viii) of Theorem 2.3.

Let $X \subset w\mathbb{P}^7$ be a general codimension 4 Fano 3-fold of Tom type and $p \in X$ a Tom centre. We first prove part (B) of Theorem 2.3. Part (C) is an immediate corollary of parts (A) and (B).

Proof of Theorem 2.3. *(B).* Consider a birational link for $X \ni p$ that terminates with a divisorial contraction. Suppose that the endpoint Mori fibre space $Y \to S$ is a Fano 3-fold $X' \to S = \{pt\}$. Let $\mathcal{B}_X$ be the basket of singularities of X. It is possible to track $\mathcal{B}_X$ throughout the link to retrieve the basket $\mathcal{B}_{Y_4}$ of Y_4. The basket $\mathcal{B}_{X'}$ of X' is a subset of $\mathcal{B}_{Y_4}$; that is, $\mathcal{B}_{X'}$ is $\mathcal{B}_{Y_4}$ minus the cyclic quotient singularities sitting inside the exceptional locus $E' := \mathbb{E}' \cap Y_4$. Moreover, if the determinant

$$\det \begin{pmatrix} d_3 & d_4 \\ -1 & -1 \end{pmatrix} = -1 \tag{4.1}$$

then E' is contracted to a Gorenstein point $p' \in X'$, which does not contribute to the basket of X'. We claim that if Φ blows up the cyclic quotient singularity of highest degree, neither the flops nor the flips create a new cyclic quotient singularity of that degree. To prove this we refer to the notation used in the proof of Theorem 4.5 below. First of all, the flop Ψ_1 leaves the basket of Y_1 unchanged, so $\mathcal{B}_{Y_1} = \mathcal{B}_{Y_2}$. Now for simplicity, suppose that $P_{y_2} \in Z_2$, that is Ψ_2 restricted to Y_2 is not an isomorphism. We have that $a, b, c < r$, and, since Φ blows up the cyclic quotient singularity of highest degree, $r \geq d_1$. In Theorem 4.5 we prove that the map β_2 extracts a weighted $\mathbb{P}^1$ or $\mathbb{P}^2$, whose weights are among the following $(d_2 - d_1, d_3 - d_1, d_4 - d_1)$, which are all strictly less than r and d_1. Thus, the flip Ψ_2 never introduces singularities of degree d_1 or higher. A similar argument can be carried out when $d_1 = d_2 > d_3$.

On the other hand, if Φ blows up a cyclic quotient singularity of a lower degree that is, $r < d_1$, then the flips get rid of the one with higher degree, which will not be generated again for the same reason as above. Still referring to the notation used in the proof of Theorem 4.5, α_2 contracts a weighted $\mathbb{P}^1$ or $\mathbb{P}^2$ with weights among (d_1, a, b, c). The coordinate y_1

appears linearly in one entry of the matrix M so locally analytically at P_{y_1} one of the orbinates is eliminated. In addition, if α_2 contracts a weighted $\mathbb{P}^2$ the hypersurface thta is the intersection of $\mathbb{P}^2$ and Y_2 always contains the coordinate point associated to y_1. Thus, the cyclic quotient singularities with higher degree are contracted by α_2. Therefore, the baskets $\mathcal{B}_X$ and $\mathcal{B}_{X'}$ are different, hence $X \ncong X'$.

If the absolute value of the above determinant is greater or equal than 2, the divisorial contraction Φ' might create a new orbifold singularity with order equal to the absolute value of the determinant. This is a phenomenon that occurred already in [13, Proposition 3.11]: we refer to the latter for the proof of this fact. If S is either a line or $\mathbb{P}^2$, Y is Y_4. We conclude that X cannot be isomorphic to Y because their Picard ranks are different. □

The rest of this section is dedicated to proving part (A) of Theorem 2.3. Following the notation in 3.1, we call Y_i the push-forward $\Psi_{i*}(Y_{i-1}) \subset \mathbb{F}_i$ of Y_{i-1} via Ψ_i. The Cox rings of the $\mathbb{F}_i$ can be naturally identified, as they are isomorphic in codimension 1: similarly holds for the Cox rings of the Y_i, for which we may choose the same generators of the quotient ideal. Throughout this paper we identify these rings and these coordinates, for all $\mathbb{F}_i$ and Y_i.

Theorem 4.3. *The first step $\psi_1 : Y_1 \dashrightarrow Y_2$ of the birational link for X consists of a number n of simultaneous flops, that is equal to the number of nodes on $D \subset Z_1$.*

To fix ideas and without loss of generality, assume X is of Tom_1 type throughout this proof: then, $\mathrm{Pf}_2(M), \mathrm{Pf}_3(M), \mathrm{Pf}_4(M), \mathrm{Pf}_5(M)$ are linear in the generators of I_D and $\mathrm{Pf}_1(M)$ is quadratic in those. The locus $\mathbb{B}_1 \in \mathbb{F}_2$ extracted by β_1 is defined by $\{t = s = 0\}$ (cf. [13]), which is isomorphic to a weighted $\mathbb{P}^3$. Therefore there is a weighted $\mathbb{P}^3$-bundle over $\mathbb{P}^2_{x_1,x_2,x_3} \cong D$. Thus, restricting to $\{t = s = 0\}$ we have that $(\mathrm{Pf}_2, \mathrm{Pf}_3, \mathrm{Pf}_4, \mathrm{Pf}_5)^T = A \cdot (y_1, y_2, y_3, y_4)^T$ where A is a 4×4 matrix defined as $A := (\gamma_i(\mathrm{Pf}_j))_{i=1\ldots4, j=2\ldots5}$ and $\gamma_i(\mathrm{Pf}_j) \in \mathbb{C}[x_1, x_2, x_3]$ is the coefficient of y_i in Pf_j. We need the following technical lemma.

Lemma 4.4. *For each point $p \in D$ the rank of $A_p := ev_p(A)$ is either 2 or 3.*

Proof. The rank is at least 1. There are six syzygies involving the five pfaffians of M; in the notation set in (A.1) one of them is $p_1\mathrm{Pf}_2 + p_2\mathrm{Pf}_3 + p_3\mathrm{Pf}_4 + p_4\mathrm{Pf}_5 = 0$. Therefore, at any point $p \in D$ it is possible to express one of the last four pfaffians in terms of the other three. This means that there are three equations left that are linear on I_D. Thus, $\mathrm{rk}(A_p) \leq 3$. The restriction to D kills all the monomials that come out from the non-linear (in I_D) terms of $\mathrm{Pf}_2, \mathrm{Pf}_3, \mathrm{Pf}_4, \mathrm{Pf}_5$. Since each of the Pf_j has at least one of the y_i appearing at least once, then there are at least two linearly independent column vectors in A. Therefore, the entries of A are all polynomials in $\mathbb{C}[x_1, x_2, x_3]$, so $\mathrm{rk}(A_p) \geq 2$. □

Proof of Theorem 4.3. We first prove that α_1 contracts n smooth rational curves. The locus $\mathbb{A}_1$ contracted by α_1 is defined by $\{y_1 = y_2 = y_3 = y_4 = 0\}$ by [13]. Since Z_1 is in Tom format, $Z_1 \cap \mathrm{Im}(\alpha_1)$ restricted to $\mathbb{A}_1$ depends only on x_1, x_2, x_3, that is, it lies on D. Hence, over every node on D there is a $\mathbb{P}^1$ having coordinates t, s. There are n nodes on D so α_1 contracts n lines.

Now we need to prove that β_1 extracts n lines. Since the locus $\mathbb{B}_1$ is fibred over D with weighted $\mathbb{P}^3$ fibres, then at any point $p \in D$, if $\mathrm{rk}(A_p) = 2$ the image of A is a 2-dimensional space in $\mathbb{P}^3$, which means that β_1 contracts a $\mathbb{P}^1 \subset \mathbb{B}_1 \cap Y_2$ to $p \in D$. Analogously, if $\mathrm{rk}(A_p) = 3$ the map β_1 is an isomorphism in a neighbourhood of a point $p' \subset \mathbb{P}^1 \subset \mathbb{B}_1$ to $p \in D$. This argument uses only the pfaffian equations of X, which contain all the necessary information about the flop. In fact, the unprojection equations do not play any role in the determination of the flop: note that $\mathbb{G}_1$ is a rank 1 toric variety of dimension 10 containing $w\mathbb{P}^6 \supset Z_1$. Its coordinates are $\xi_1 := x_1$, $\xi_2 := x_2$, $\xi_3 := x_3$, $v_1 := y_1 t$, $v_2 := y_2 t$, $v_3 := y_3 t$, $v_4 := y_4 t$, $\sigma_1 := s y_1$, $\sigma_2 := s y_2$, $\sigma_3 := s y_3$, $\sigma_4 := s y_4$. The unprojection equations globally eliminate the variable s on D. In addition, on D, the Jacobian matrix of Z_1 is

$$J(Z_1)|_D = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A \end{array}\right).$$

Therefore we deduce that for each point $p \in D$ we have $\mathrm{rk}(J(Z_1)|_D)_p = \mathrm{rk}(A_p)$.

Lastly, we prove that ψ_1 is a flop. We have just shown that ψ_1 is an isomorphism in codimension 1, so we study the intersections $-K_{Y_1} \cap \mathbb{A}_1$ and $-K_{Y_2} \cap \mathbb{B}_1$. For both i equal to 1 or 2, $-K_{Y_i}$ is $\{x_1 = 0\}$. On the other hand, none of the points in $\mathrm{Sing}(Z_1) \subset D$ satisfies the condition $x_1 = 0$. Hence, $-K_{Y_i} \cdot \mathbb{P}^1_{t,s} = 0$ for $i = 1, 2$. □

This proof is independent on the form of the right-hand side of the unprojection equations: the information about the flop is all encoded in the geometry of Z_1, as expected.

We introduce the following configurations for the matrix M, which appear frequently in the rest of this paper. The argument holds independently on the Tom format. For some suitable positive integers σ and τ, define

(a) The entries $a_{2,4}, a_{2,5}, a_{3,4}, a_{3,5}$ all have weight π. Hence, in order to have homogeneous pfaffians and positive weights, the other weights of M are

$$\begin{pmatrix} \sigma & \sigma & \pi+\sigma-\tau & \pi+\sigma-\tau \\ \hline & \tau & \pi & \pi \\ & & \pi & \pi \\ & & & 2\pi-\tau \end{pmatrix}. \tag{4.2}$$

(b) The entries $a_{2,5}, a_{3,4}$ both have weight $d_1 = d_2$, while $a_{2,4}, a_{3,5}$ are free. Hence, the other weights of M are

$$\begin{pmatrix} \sigma & \pi+\sigma-\upsilon & \pi+\sigma-\tau & 2\pi+\sigma-\tau-\upsilon \\ \hline & \tau & \upsilon & \pi \\ & & \pi & 2\pi-\upsilon \\ & & & 2\pi-\tau \end{pmatrix}. \tag{4.3}$$

4.1 | Proof of (i)

The following theorem describes the flip that occurs when crossing the ray ρ_{y_1}. An identical argument applies when crossing ρ_{y_2} if $d_1 > d_2 > d_3$.

Theorem 4.5. *Suppose $d_1 > d_2$ and that $P_{y_1} \in Z_2$. Then, $\psi_2 : Y_2 \dashrightarrow Y_3$ is a flip.*

Proof. Localise at the point $P_{y_1} \in Z_2$. So, after a row operation, the grading of $\mathbb{F}_2$ becomes

$$\left(\begin{array}{ccccc|cccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ d_1 & r+d_1 & a & b & c & 0 & d_2-d_1 & d_3-d_1 & d_4-d_1 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right).$$

The exceptional locus of α_2 is $\mathbb{A}_2 = \{y_2 = y_3 = y_4 = 0\}$, that is,

$$\mathbb{A}_2 = \left(\begin{array}{ccccc|c} t & s & x_1 & x_2 & x_3 & y_1 \\ d_1 & r+d_1 & a & b & c & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 \end{array}\right) \cong \mathbb{P}^4(d_1, r+d_1, a, b, c)$$

and $\alpha_2(\mathbb{A}_2) = P_{y_1}$. To prove that ψ_2 is a flip for Y_2, we show that the codimension of the intersection $Y_2 \cap \mathbb{A}_2$ is at least 3 in $\mathbb{P}^4(d_1, r+d_1, a, b, c)$. The unprojection equation $sy_1 = g_1$ allows one to eliminate s locally above $P_{y_1} \in Z_2$. Thus, for a hypersurface F isomorphic to the weighted $\mathbb{P}^3(d_1, a, b, c)$ defined by the unprojection equation relative to y_1 in which y_1 has been set at 1, we have $Y_2 \cap \mathbb{A}_2 \subset F \subset \mathbb{P}^4(d_1, r+d_1, a, b, c)$. Hence, $Y_2 \cap \mathbb{A}_2$ has at least codimension 1. By Lemma 3.3, in one of the pfaffian equations there is a monomial of the form $x_i y_1$, that is, locally at P_{y_1}, it is possible to eliminate x_i, i.e.

x_i can be expressed as a function of the other variables: suppose $i = 1$. Thus, $Y_2 \cap \mathbb{A}_2 \subset \mathbb{P}^4(d_1, r + d_1, a, b, c)$ has at least codimension 2. From Lemma 3.8 we deduce that there is another unprojection equation that contains monomials in the x_i and t. Therefore, $Y_2 \cap \mathbb{A}_2 \subset S \subset F \subset \mathbb{P}^4(d_1, r + d_1, a, b, c)$ where $S \cong \mathbb{P}^4(d_1, b, c)$: so, $Y_2 \cap \mathbb{A}_2$ has at least codimension 3 in $\mathbb{P}^4(d_1, r + d_1, a, b, c)$. To prove that the codimension is exactly 3 we need to show that the remaining equations define a curve in S, so we need to exclude the case in which they define a single point or the empty set. The vanishing locus of the remaining equations cannot be empty because $P_{y_1} \in Z_2$, so there must be an intersection between Y_2 and $\mathbb{A}_2$. In addition, $Y_2 \cap \mathbb{A}_2$ cannot be a single point either for the following reason. Since X is quasi-smooth and $\mathbb{Q}$-factorial, the same holds for Y_1. Also Y_2 is quasi-smooth, but it is not isomorphic to Y_1 because $\beta_2 : Y_3 \to Z_2$ contracts the curve defined by the quadratic pfaffian equation (which is Pf_1 if M is in Tom_1 format). Thus, by $\mathbb{Q}$-factoriality, β_2 must also contract a curve.

The last thing to check is that the intersection of $-K_{Y_2}$ with $\mathbb{A}_2$ is positive and that the intersection of $-K_{Y_3}$ with $\mathbb{B}_2$ is negative. This is true because $\{x_1 = 0\} \in |\mathcal{O}(-aK_{Y_2})|$ is relatively ample with respect to α_2, so it meets every curve positively. $\square$

Theorem 4.6. *If the point $P_{y_1} \notin Z_2$, the restriction to Y_2 of the toric flip $\Psi_2 : \mathbb{F}_2 \dashrightarrow \mathbb{F}_3$ is an isomorphism $Y_2 \cong Y_3$.*

Proof. The equations of Z_2 are the same as Z_1, albeit viewed in a different toric variety $\mathbb{G}_2$. If $P_{y_1} \notin Z_2$ then there exists at least one pfaffian equation that is non-zero when evaluated at P_{y_1}. Moreover, $\alpha_2(\mathbb{A}_2) = P_{y_1}$; on the other hand, $\alpha_2(Y_2) = Z_2$. This means that the exceptional locus of the flip at the toric level does not intersect with Y_2, i.e. $\mathbb{A}_2 \cap Y_2 = \emptyset$. $\square$

The nature of the weights of M determine whether the hypotheses of either Theorem 4.5 or Theorem 4.6 are verified.

Proposition 4.7. *Let X be of Tom type. If the weights of M fall in case (b), then either the flip with base at $P_{y_1} \in Z_2$ or the flip with base at $P_{y_2} \in Z_3$ is an isomorphism.*

Proof. In case (b) two ideal entries with the same weight are multiplied in $\mathrm{Pf}_1(M)$. Suppose that $\pi = d_1$. Thus, y_1 occupies both the entries $a_{2,5}$ and $a_{3,4}$. From Theorem 3.12 and since y_1 appears linearly in those entries, we deduce that the monomial y_1^2 is in the equations of Y_1. Therefore, repeating the proof of Theorem 4.6, we have that Ψ_2 restricted to Y_2 is an isomorphism. Same happens for $\pi = d_2$. The weight π is never equal to d_3 or d_4. $\square$

In this argument it is crucial that there is only one ideal generator having weight d_1. The concurrent presence of configuration (a) and of two distinguished ideal generators having the same weight leads to different consequences in (iii) and (v).

Although the majority of the Hilbert series of case (i) falls in configuration (b), it also happens that the weights of M are in configuration neither (a) nor (b). In this situation, both ψ_1 and ψ_2 are flips. In particular, this means that the mobile cone of $\mathbb{F}_1$ coincides with the mobile cone of Y_1. Theorem 4.5 and Proposition 4.7 can be also applied to the crossing of the wall adjacent to $d_2 > d_3$.

Consider the rank 2 toric variety $\mathbb{F}_4$ in case (i), where $d_3 > d_4$. The link terminates with a divisorial contraction.

Lemma 4.8. *Suppose that $\rho_X = 1$. If $d_3 > d_4$, the map $\Phi' : \mathbb{F}_4 \to \mathbb{G}_4$ is a divisorial contraction of Y_4 to a Fano 3-fold $X' \subset \mathbb{P}' \subset \mathbb{G}_4$.*

Proof. Since $\rho_X = 1$, the exceptional divisor $\mathbb{E}'$ of Φ' is irreducible. Thus, $\rho_{X'} = 1$ as well. Moreover, X' is projective. In addition, $-K_{X'}$ is ample. Consider a curve Γ in X' that is not in the image of $\mathbb{E}'$ via Φ' and that is not in the image of the union of the right-hand side contracted loci $\mathbb{B}_i$ of the flips. Such a curve can always be found because the set of curves of X' lying in $\Phi'(\mathbb{E}')$ and the union of the proper transform of the $\mathbb{B}_i$ has codimension 2. The curve Γ can be tracked back down to Y_1. Now, $-K_{Y_1} = \alpha_1^*(-K_{Z_1})$ and every curve in Y_1 is either a flopping curve or strictly positive against $-K_{Y_1}$ and contracted to Z_1. So, the divisor $-K_{Y_1}$ is nef and big, that is Y_1 is a weak Fano. Thus we have $-K_{X'}\Gamma = -K_{Y_1}\tilde{\Gamma} > 0$, where $\tilde{\Gamma}$ is the proper transform of Γ, and is isomorphic to Γ. $\square$

4.1.1 | Identifying the end of the link

By Lemma 4.8, Φ' is a divisorial contraction to another Fano. A crucial observation is the following.

Lemma 4.9. *Let X be a Fano 3-fold of Tom type and let $p \in X$ be a Tom centre such that the link for $X \ni p$ terminates with a divisorial contraction to another Fano 3-fold $X' \subset \mathbb{P}'$. Then,* $\operatorname{codim}(X') < \operatorname{codim}(X)$.

Proof. The map $\Phi' : \mathbb{F}_4 \to \mathbb{G}_4$ is induced by the linear system $\left|\mathcal{O}\binom{d_3}{-1}\right|$. The equations of Y_4 constitute relations among the new coordinates of $\mathbb{G}_4$, that is, some of the equations of Y_4 eliminate (globally) some of the coordinates of $\mathbb{G}_4$. The global elimination of the variable $s' = sy_4^{\varsigma}$ of $\mathbb{G}_4$, for some exponent ς, always happens due to the equation $sy_4 = g_4$. This phenomenon may occur for other coordinates too, depending on each specific case. This shows that $\mathbb{P}'$ is at most a weighted $\mathbb{P}^6$. ☐

It is also possible to track down the evolution of the basket of singularities of X along the link, in order to deduce the one for X'. We refer to the proof of Theorem 2.3, (B) for this. Its basket and its ambient space determine the Hilbert series of X' univocally. We give an example of how to find $\mathcal{B}_{X'}$ in Section 5.1.

The equations of X' can be found by rewriting the equations of Y_4 in terms of the new coordinates of $\mathbb{G}_4$, and by excluding the ones used to perform the global elimination. Usually, X' is a special member in the family associated to its Hilbert series. We show this explicitly in the examples of Section 5.

4.2 | Proof of (ii)

Suppose M has weights as in (b), only for the two Hilbert series #1218 and #1413. For both, the equations of Y_2 have a pure monomial in y_1 (similarly to the phenomenon described in Theorem 4.6). Thus, the following holds.

Theorem 4.10. *Consider the Hilbert series #1218, #1413 and the Fano 3-fold defined by Tom_1 for both. Then, their respective birational links evolve as follows: ψ_1 is a flop, Ψ_2 restricts to an isomorphism ψ_2 on Y_2, ϕ' is a divisorial contraction over $\mathbb{P}^1_{y_2,y_3} \subset X'$.*

Proof. By Theorem 4.3 ψ_1 is a flop. The weights of M of the two Hilbert series are as in (b). Therefore, $\mathrm{Pf}_1(M)$ contains y_1^2. Analogously to Theorem 4.6, ψ_2 is an isomorphism. The last map is a divisorial contraction to X' by Lemma 4.8. Note that $\mathbb{P}^1_{y_2,y_3} \subset X'$ in both cases, so there is one divisorial contraction to $\mathbb{P}^1_{y_2,y_3}$. ☐

None of the other Hilbert series in (ii) comes from M with (b) weights. In this instance, the flip ψ_2 is performed by Y_2 too, and it is followed by a divisorial contraction to X'.

Theorem 4.11. *Let Z_1 be defined by M in Tom format with weights not in (b). Then the birational link for X is formed of: a flop, a flip, and a divisorial contraction to $\mathbb{P}^1_{y_2:y_3} \subset X'$.*

Proof. The point $P_{y_1} \in Z_2$ because the weights of M are not as in (b). We connect this proof to the one of Theorem 4.10. ☐

4.3 | Proof of (iii) and (v)

Now we study the case where $d_1 = d_2$. Both (iii) and (v) share the same behaviour when crossing of the ray ρ_{y_1,y_2}.

Theorem 4.12. *Suppose $d_1 = d_2$. Then, there are two analytic flips* (simultaneous flips *as in Remark 4.1*) *based at two distinct points in Z_2.*

To fix ideas, M is in Tom_1 format throughout this proof. Here the specialisations to (iii) and (v) of (a), (b) still apply, but this time there two different variables that fit the entries with weight $d_1 = d_2$ (which replace the weight π in (a), (b)).

Geometrically, α_2 contracts the locus $\mathbb{A}_2$ to a line $\mathbb{P}^1_{y_1:y_2} \subset \mathbb{G}_2$. So, the intersection $\mathbb{A}_2 \cap Y_2$ is mapped to $\mathbb{P}^1_{y_1:y_2} \cap Z_2$. In Lemma 4.14 and in Remark 4.15 we discuss the nature of the intersection $\mathbb{P}^1_{y_1:y_2} \cap Z_2$ in cases (a) and (b) respectively. The idea is that $\mathbb{P}^1_{y_1:y_2}$ cuts out a rank 2 quadratic form in y_1, y_2, which determines two points in Z_2. Therefore, the variety Y_2 is subjected to two simultaneous flips.

Proposition 4.13. *There exists a rank 2 quadratic form in y_1, y_2 defined on $\mathbb{G}_2$ that determines two distinct points P_1, P_2 in Z_2.*

Proof. Independently on (a) and (b), without loss of generality we assume that y_1 occupies the a_{25} entry and that y_2 occupies the a_{34} entry of M. The equations of Z_2 are in terms of t as well, being the image of Y_2 via α_2. If any of y_1 or y_2 is in one of the entries in the top row of the matrix, it picks up a t factor in the blow up of X, so it vanishes when restricted to $\mathbb{P}^1_{y_1:y_2}$. Moreover, if y_1 and y_2 appear in other entries of M they need to be multiplied by some other variable. Therefore, the quadratic form is to be found in the first pfaffian of M, i.e. it is the restriction of $\mathrm{Pf}_1(M)$ to $\mathbb{P}^1_{y_1:y_2}$. It is of the form $y_1^2 - y_1y_2 + y_2^2$ in case (a), whereas it is $y_1^2 - y_1y_2$ in case (b). No other monomials, also in other equations, survive the restriction to $\mathbb{P}^1_{y_1:y_2}$. For (a), (b) the two quadratic forms describe two distinct points on Z_2. □

Lemma 4.14. *Let Z_1 be defined by a graded matrix M in Tom format having weights as in (a). Then, the two flips have exactly the same weights.*

Proof of Lemma 4.14. Let M have weights as in (a). Place y_1 and y_2 in the entries a_{25} and a_{34} respectively. Locally at P_{y_1} we can eliminate a potential linear term in the entries a_{12} and a_{15}; likewise, locally at P_{y_2} for a linear term in a_{13} and a_{14}. Since a_{12} and a_{13} have the same weights, y_1 and y_2 eliminate the same variable when localising at their respective coordinate points; or otherwise they do not eliminate any variable in those entries at all (same for a_{14} and a_{15}). The variables y_3 and y_4 cannot be eliminated, as they are always multiplied by a t factor on the top row. Therefore, the birational transformations at P_1 and P_2 can only be flips. This proves that α_2 contracts two loci of the same dimension: in fact, those loci are isomorphic. In conclusion, the flip phenomenon is symmetrical over $P_1, P_2 \in Z_2$. □

Remark 4.15. The lemma above does not hold if M is as in (b). In fact, if one of the flips is toroidal/hypersurface, the other one is not necessarily toroidal/hypersurface. This is because the weights in the top row of M are all different, so y_1 and y_2 cannot eliminate the same variables, and therefore the flips at P_1 and P_2 cannot have the same weights. Moreover, suppose that a certain linear variable w occupies the entry a_{14}: it can appear in the a_{15} entry only if multiplied by a polynomial f_{d_1-v} of degree $d_1 - v$. Thus, there is no hope for y_2 to eliminate w, and therefore the two flips can have different numbers of weights.

Proof of Theorem 4.12. Similarly to Theorem 4.5, Ψ_2 is an algebraically irreducible flip. Its restriction to Y_2 is constituted of two distinct components, each contracted to one of the points $P_1, P_2 \in Z_2$ (by Proposition 4.13). Lemma 4.14 and Remark 4.15 clarify the nature of such components. □

Theorem 4.12 holds if $d_1 = d_2 > d_3 = d_4$ and $d_1 = d_2 > d_3 > d_4$, although the continuation of the link is different in the two cases. For the latter, the statements made for item (i) still hold. For the former, we have that

Theorem 4.16. *If $d_2 > d_3 = d_4$, then Φ' is a del Pezzo fibration over $\mathbb{P}^1_{y_3,y_4}$.*

Proof. Consider $\Phi' : \mathbb{F}_3 \to \mathbb{P}^1_{y_3,y_4}$. The grading of $\mathbb{F}_3$ can be written as

$$\left(\begin{array}{ccccccc|cc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ d_3 & r+d_3 & a & b & c & d_2-d_3 & d_2-d_3 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right).$$

This is a weighted $\mathbb{P}^6$-bundle over $\mathbb{P}^1$. The intersection of Y_3 with its general fibre has dimension 2, for y_3 and y_4 now act as parameters. The restriction of K_{Y_3} to such intersection is still ample. Hence, Φ' is a del Pezzo fibration of Y_3 over $\mathbb{P}^1_{y_3,y_4}$. □

Lemma 4.17. *The intersection of Y_3 with the general fibre of the bundle defined by Φ' is smooth.*

Proof. The generic fibre S of Φ' is a surface in Y_3. Suppose S is singular. In particular, its closure inside the 3-fold Y_3 is a line. Therefore, Y_3 would contain a whole singular line, which is a contradiction with Y_3 being terminal. □

In [14] we compute the degree of the general fibre of these del Pezzo fibrations.

4.4 | Proof of (iv)

Similarly to the cases above, the weights of M influence the behaviour of the link, and the distinction of (a), (b) plays a crucial role. The majority of Hilbert series that fall into case (iv) are such that the weights of M are in configuration (b).

Proposition 4.18. *Suppose M has weights in configuration (b). Then, either y_1 appears as a square in the equations of Y_2, or y_2 appears as a square in the equations of Y_3.*

Proof. If the weights of M are in (b), $\mathrm{Pf}_1(M)$ involves the multiplication of the entries a_{25}, a_{34} of same weight (either d_1 or d_2, depending on the specific Hilbert series considered). In contrast to the proof of Proposition 4.13, by hypothesis here only one variable has weight d_1, d_2, i.e. y_1 and y_2 respectively. Therefore, the quadratic form defined on $\mathbb{G}_2$ (or $\mathbb{G}_3$ respectively) is y_1^2 (or y_2^2 in turn). □

Lemma 4.19. *If M has weights in configuration (b), then either Ψ_2 or Ψ_3 is an isomorphism when restricted to Y_2 and Y_3 respectively.*

Proof. Either y_1^2 appears in the equations of Y_2, or y_2^2 appears in the equations of Y_3. We conclude the proof using Proposition 4.7. □

Remark 4.20. In case (iv), only the Hilbert series #20544 has a weight configuration of type (a). Since the only variable with weight d_2 is y_2, it is possible to cancel out y_2 from the entries a_{25} and a_{34} via row/column operations. Therefore the equations of X have the monomial y_2^2. Nonetheless, no flip is missed because, performing the blow-up of X and then saturating over t, the term y_2^2 picks up a t factor.

The weights of M relative to the Hilbert series #5516, #5867, #11437 are neither in configuration (a) nor (b). Thus, Ψ_2, Ψ_3 are flips for Y_2, Y_3 respectively.

The last map Φ' of the link in (iv) is a del Pezzo fibration, as in Theorem 4.16.

4.5 | Proof of (vi)

There are six Hilbert series falling in case $d_1 > d_2 = d_3 = d_4$.

Proposition 4.21. *The birational link starting from the Hilbert series #6865 is such that the restriction to Y_2 of the birational map Ψ_2 is an isomorphism.*

Proof. The weights of M are as in (b), so y_1^2 appears in the equations of Y_2. □

The other five Hilbert series falling in this case behave as expected.

Proposition 4.22. *Consider the birational link starting from X as in one of the five Hilbert series left in case (vi). Then, the restriction to the variety Y_2 of Ψ_2 is a flip for Y_2.*

Proof. The weights of M are neither in case (a) nor (b). Thus, none of the ideal variables appears as a pure power in the equations of Y_2. □

The end of the link is a conic bundle over a plane $\mathbb{P}^2$ with coordinates y_2, y_3, y_4.

Proposition 4.23. *The map Φ' is a conic bundle over the projective plane $\mathbb{P}^2_{y_2,y_3,y_4}$.*

Proof. Localise $\mathbb{F}_3$ at the plane $\mathbb{P}^2(d_2, d_2, d_2)_{y_2,y_3,y_4}$. Eliminate s globally and discard the unprojection equations. We exclude s from the grading of $\mathbb{F}_3$.

$$\left(\begin{array}{ccccc|ccc} t & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ d_2 & a & b & c & d_1 - d_2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right),$$

so $\mathbb{F}_3$ is a weighted $\mathbb{P}^4$-bundle over $\mathbb{P}^2$. Above each point of $\mathbb{P}^2(d_2, d_2, d_2)_{y_2,y_3,y_4}$ we can locally eliminate two variables among t, x_1, x_2, x_3, y_1 via two of the pfaffian equations. The remaining three equations lie in the same principal ideal generated by one of them, which is a conic in the three surviving variables of the fibre with coefficients in the base $\mathbb{P}^2_{y_2,y_3,y_4}$. □

4.6 | Proof of (vii)

In this case there are no flips occurring, and the links evolve as follows: ψ_1 is n simultaneous flops by Theorem 4.3, followed by a divisorial contraction Φ' to a Fano 3-fold X' (by Lemma 4.8 and because $d_4 - d_1 < 0$).

4.7 | Proof of (viii)

Here the first n flops are followed by a conic bundle over the base $\mathbb{P}^3(d_1, d_1, d_1, d_1)_{y_1,y_2,y_3,y_4}$, and a similar statement to Proposition 4.23 holds.

5 | EXAMPLES

In this section we present some explicit examples, highlighting the main phenomena described in Theorem 2.3. Recall that all the Fano 3-folds X in this paper, and in particular in the examples of this section, can be explicitly constructed by means of Type I unprojections and are $\mathbb{Q}$-factorial following [11].

5.1 | Example for (i): #10985, Tom_1

Consider $X \ni p$ where X is the Tom type Fano 3-fold associated to the Hilbert series #10985 and $p \in X$ is the Tom centre $\frac{1}{2}(1,1,1)$ in the basket of singularities $\mathcal{B}_X = \left\{\frac{1}{2}(1,1,1), \frac{1}{6}(1,1,5)\right\}$. The ambient space of X is $\mathbb{P}^7(1^3, 2, 3, 4, 5, 6)$, with coordinates $x_1, x_2, x_3,\ s,\ y_4, y_3, y_2, y_1$ respectively. The divisor $D \cong \mathbb{P}_{x_1,x_2,x_3}(1,1,1)$ is defined by the ideal $I_D = \text{Span}[y_1, y_2, y_3, y_4]$ and $D \subset Z_1$ for M in Tom_1. There are 24 nodes on $D \subset Z_1$ (cf [12]). To summarise, we are looking at the following varieties:

#10985	$X \subset \mathbb{P}^7(1^3, 2, 3, 4, 5, 6)$	codimension 4	$\left\{\frac{1}{2}(1,1,1), \frac{1}{6}(1,1,5)\right\}$,
#10962	$Z_1 \subset \mathbb{P}^6(1^3, 3, 4, 5, 6)$	codimension 3	24 nodes on D.

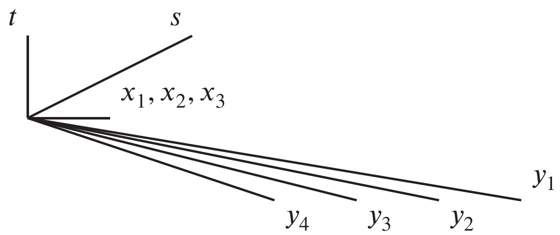

FIGURE 2 Mori cone of $\mathbb{F}_1$ for #10985, Tom_1

We aim to put ideal variables in an ideal entry having their same weight, and do analogously for the orbinates. The rest of the entries can be occupied by general polynomials in the given degrees, accordingly to the Tom_1 constraints. In this specific case, we end up with the following explicit matrix as in Section 3.2

$$M = \begin{pmatrix} x_1 & -x_2x_3 & -x_2^3 + y_4 & -x_3^4 + y_3 \\ & y_4 & y_3 & y_2 \\ & & x_2^2y_4 - y_2 & y_1 \\ & & & x_1^4y_4 \end{pmatrix}. \tag{5.1}$$

The unprojection algorithm produces nine equations defining X. The blow-up Y_1 of X at the Tom centre $p = P_s$ is contained in the rank 2 toric variety $\mathbb{F}_1$ with grading as in Proposition 3.7, whose ray-chamber structure is described in Figure 2.

The Kawamata blow-up of the Tom centre P_s is

$$\begin{aligned} \Phi : \mathbb{F}_1 &\longrightarrow \mathbb{P}^7\left(1^3, 2, 3, 4, 5, 6\right) \\ (t, s, x_1, x_2, x_3, y_1, y_2, y_3, y_4) &\longmapsto \left(x_1t^{\frac{1}{2}}, x_2t^{\frac{1}{2}}, x_3t^{\frac{1}{2}}, y_4t^{\frac{5}{2}}, y_3t^{\frac{6}{2}}, y_2t^{\frac{7}{2}}, y_1t^{\frac{8}{2}}, s\right). \end{aligned} \tag{5.2}$$

We record here only the pfaffian equations of Y_1.

$$\begin{cases} ty_4^2 + x_1x_2^2y_4 - x_1y_2 - x_2^3y_4 + x_2x_3y_3 = 0, \\ -ty_4y_3 - x_1y_1 - x_2x_3y_2 + x_3^4y_4 = 0, \\ -ty_4y_2 + ty_3^2 + x_1^5y_4 + x_2^3y_2 - tx_3^4y_3 = 0, \\ ty_4y_1 + ty_3y_2 + x_1^4x_2x_3y_4 + x_1x_2^2y_1 + x_2^3x_3y_2 - x_2^3y_1 - x_3^4y_2 = 0, \\ x_1^4y_4^2 + x_2^2y_4y_2 - y_3y_1 - y_2^2 = 0. \end{cases} \tag{5.3}$$

From Theorem 4.3, crossing the ray ρ_{x_i} gives that Ψ_1 consists of 24 simultaneous flops based at the 24 nodes of Z_1. Since the weights of M are in configuration (b), then either ψ_2 or ψ_3 is an isomorphism by Proposition 4.7; y_2 appears as a pure power in (5.3), so ψ_3 is an isomorphism. To study ψ_2 we need to localise at $P_{y_1} \in Z_2$, so we look at Equations 5.3 locally analytically in a neighbourhood of the point $P_{y_1} \in Z_2$. Practically, y_1 is a local coordinate and we perform row operations on $\mathbb{F}_2$ in order to write the weight of y_1 as either $\binom{\pm 1}{0}$ or $\binom{0}{\pm 1}$. So, the grading of $\mathbb{F}_2$ becomes

$$\left(\begin{array}{ccccc|cccc} t & s & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & y_4 \\ 6 & 8 & 1 & 1 & 1 & 0 & -1 & -2 & -3 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right).$$

The flip Ψ_2 has weights $(6, 8, 1, 1, 1, -1, -2, -3)$; this stands for the contraction by α_2 of $\mathbb{P}^4_{t,s,x_1,x_2,x_3}(6, 8, 1, 1, 1)$ to the point $P_{y_1} \in Z_2$, and the extraction by β_2 of $\mathbb{P}^2_{y_2,y_3,y_4}(1, 2, 3)$ from P_{y_1}. However, the intersection $\mathbb{P}^4_{t,s,x_2,x_3}(6, 8, 1, 1, 1) \cap Y_2$ can be a projective space smaller than $\mathbb{P}^4$. Analogously, this might hold for $\mathbb{P}^1_{y_2,y_4}(1, 2, 3) \cap Y_3$. The study of these intersections is done via the following argument.

Localising at the base of the isomorphism in codimension 1, Ψ_i, it is possible to write some of the variables as function of the others using the equations of Y_i. Examining the equations of Y_2 locally analytically at a neighbourhood of $P_{y_1} \in Z_2$ and considering y_1 as a local coordinate, we can set $y_1 = 1$ in Equations (5.3). Some linear monomials will emerge in the equations of Y_2 evaluated at $y_1 = 1$: those variables appearing linearly in $Y_2|_{y_1=1}$ can be expressed in terms of the other variables locally analytically. Thus, we can locally eliminate them. In this specific case, the evaluation of (5.3) at $y_1 = 1$ shows that s, x_1, y_3 appear linearly. Therefore, the weights of the flip for Y_2 are $(6, 1, 1, -1, -3)$, associated to the remaining variables t, x_2, x_3, y_2, y_4 respectively. Observe that it looks like that α_2 contracts a 2-dimensional locus inside Y_2 to the point P_{y_1}, thus α_2 does not seem like a small contraction, as required in flips. However, among the equations left after the local elimination process there is one involving t and y_4: that is $\mathrm{Pf}_2 = 0$. This means that there is an equation cutting out the contracted locus by one dimension.

In conclusion, ψ_2 is a flip with weights $(6, 1, 1, -1, -3; 3)$, where the last 3 in this notation tracks the degree of the equation involving the monomial ty_4. In other words, a weighted projective space $\mathbb{P}_{t,x_2,x_3}(6, 1, 1)$ containing a hypersurface of degree 3 with coefficients in $\mathbb{P}_{y_2,y_4}(1, 3)$ is flipped to $\mathbb{P}_{y_2,y_4}(1, 3)$. In particular, a $\frac{1}{6}(1, 1, 5)$ singularity in Y_2 is contracted to P_{y_1} via α_2, and a $\frac{1}{3}(1, 1, 2)$ is extracted in Y_3 from P_{y_1} via β_2. This is a hypersurface flip. Despite the fact that there are three surviving equations after the elimination process, the equation cutting out $\mathbb{P}_{t,x_2,x_3}(6, 1, 1)$ is only one: the other two are multiples of it, that is, Pf_2 is the generator of the principal ideal of Y_2 on $\mathbb{P}_{t,x_2,x_3}(6, 1, 1)$. The map Ψ_3 based at P_{y_2} defines a flip from $\mathbb{F}_3$ to $\mathbb{F}_4$, but its exceptional locus does not intersect Y_3, which is therefore not affected by Ψ_3. The last map of the link is $\Phi' : \mathbb{F}_4 \to \mathbb{G}_4$, defined by the linear system $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$: it contracts the exceptional locus $\mathbb{E}' = \{y_4 = 0\}$ to the point $P_{y_3} \in \mathbb{G}_4$. Explicitly, it is

$$\begin{aligned} \Phi' : \mathbb{F}_4 &\longrightarrow \mathbb{G}_4 = \mathbb{P}^7(1, 1, 1, 1, 2, 3, 3, 5) \\ (t, s, x_1, x_2, x_3, y_1, y_2, y_3, y_4) &\longmapsto (x_1y_4, x_2y_4, x_3y_4, y_3, y_2y_4, y_1y_4^2, ty_4^3, sy_4^6). \end{aligned} \tag{5.4}$$

The exceptional locus $\mathbb{E}'$ is isomorphic to $\mathbb{P}^7(4, 6, 1, 1, 1, 2, 1)$ with coordinates $t, s, x_1, x_2, x_3, y_1, y_2$ respectively: their weights are retrieved performing a localisation at P_{y_3} as before. However, the intersection $\mathbb{E}' \cap Y_4$ is $\mathbb{P}^3(1, 1, 1, 1)$, as we can eliminate the variables t, s, y_1 locally analytically in a neighbourhood of P_{y_3}.

We call X' the push-forward $\Phi'_*(Y_4)$ of Y_4 via Φ'. Practically, y_4 plays for Φ' the role that t played for Φ, being the extra variable needed to perform a blow-up: in this case, Φ' blows up the point $P_{y_3} \in X'$. The equations of X' are therefore given by evaluating the equations of Y_4 at $y_4 = 1$. Observe that this shows that the variables t and s can be algebraically expressed as functions of the other variables: two equations of $Y_4|_{y_4=1}$ are removed in order to perform this global elimination.

Call ς_i for $i \in \{1, \dots, 8\}$ the coordinates of $\mathbb{G}_4$. Since we globally eliminated t, s, then $X' \subset w\mathbb{P}' \subset \mathbb{G}_4$, where $w\mathbb{P}' := \mathbb{P}^5(1, 1, 1, 1, 2, 3)$ with coordinates $\varsigma_1, \dots, \varsigma_6$. So, Φ' restricts to $\phi' : Y_4 \to X' \subset \mathbb{P}^5(1, 1, 1, 1, 2, 3)$. The minimal basis of the ideal generated by the surviving equations of $Y_4|_{y_4=1}$ give the explicit equations of X', both of degree 4, are

$$\begin{cases} \varsigma_1\varsigma_2^2\varsigma_4 - \varsigma_1\varsigma_4\varsigma_5 - \varsigma_1\varsigma_6 - \varsigma_2^3\varsigma_4 + \varsigma_2\varsigma_3\varsigma_4^2 - \varsigma_2\varsigma_3\varsigma_5 + \varsigma_3^4 = 0, \\ \varsigma_1^4 + \varsigma_2^2\varsigma_5 - \varsigma_4\varsigma_6 - \varsigma_5^2 = 0. \end{cases} \tag{5.5}$$

In addition, it is possible to keep track of the singularities throughout the link. That is: X has $\frac{1}{2}(1, 1, 1)$ and $\frac{1}{6}(1, 1, 5)$ singularities. After the blowup Φ, Y_1 has only a singularity of type $\frac{1}{6}$: this holds for Y_2 too, as the basket does not change after the flops. The hypersurface flip Ψ_2 replaces $\frac{1}{6}(1, 1, 5)$ with $\frac{1}{3}(1, 1, 2)$, so Y_3 has one singularity of type $\frac{1}{3}$; same for Y_4, given that Y_3 and Y_4 are actually isomorphic. Lastly, ϕ' contracts a smooth locus, so the $\frac{1}{3}$ singularity of Y_4 is maintained in X'.

Now that we know the equations of X' and their degrees, the basket of X and its ambient space we deduce that X' is a representative of the family #16204 in [10], which is a Fano 3-fold in codimension 2.

5.2 | Example for (v): #20652, Tom_1

Consider $X \ni p$ where X is the Tom type Fano 3-fold associated to #20652 and $p \in X$ is the Tom centre $\frac{1}{2}(1, 1, 1)$. Its ambient space is $\mathbb{P}^7(1^5, 2^3)$, with coordinates $y_1, y_2, x_1, x_2, x_3, y_3, y_4, s$ respectively, $D \cong \mathbb{P}_{x_1,x_2,x_3}(1, 1, 1) \subset Z_1$ has 7 nodes,

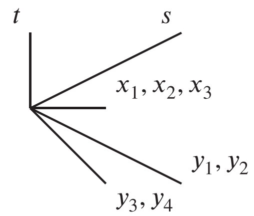

FIGURE 3 Mori cone of $\mathbb{F}_1$ for #20652, Tom_1

and M is in Tom_1 format with entries

$$M = \begin{pmatrix} x_1 & x_2 & x_3 & y_3 \\ & y_1 & y_2 & x_2y_4 - x_3y_3 + y_1 \\ & & x_1y_3 - y_2 & y_4^2 - y_2 \\ & & & x_1y_3 + y_1 \end{pmatrix}. \tag{5.6}$$

$$\begin{array}{llll} \#20652 & X \subset \mathbb{P}^7(1^5, 2^3) & \text{codimension } 4 & \mathcal{B}_X = \left\{3 \times \frac{1}{2}(1,1,1)\right\}, \\ \#20543 & Z_1 \subset \mathbb{P}^6(1^5, 2^2) & \text{codimension } 3 & 7 \text{ nodes on } D. \end{array}$$

This time, the Mori cone of $\mathbb{F}_1$ is given by the fan in Figure 3.

By Theorem 4.3, Ψ_1 is given by 7 simultaneous flops. The weights of M are in configuration (a), so there is a quadratic form determining two points P_1, P_2 in the intersection $Z_2 \cap \mathbb{P}^1_{y_1,y_2}$ (Proposition 4.13). Thus, Lemma 4.14 shows that the pencil of flips along the line $\mathbb{P}^1_{y_1,y_2} \subset \mathbb{G}_2$ restricts to two flips with base P_1 and P_2 respectively. So we look locally analytically in a neighbourhood of $P_1, P_2 \in Z_2$. Carrying out a similar calculation to the previous examples, we localise at $\mathbb{P}^1_{y_1,y_2}Z_2$. The weights of the flip of toric varieties based at $\mathbb{P}^1_{y_1,y_2}$ are $(2,4,1,1,1,-1,-1)$, where α_2 contracts $\mathbb{P}^4_{t,s,x_1,x_2,x_3}(2,4,1,1,1)$ to $\mathbb{P}^1_{y_1,y_2}$, and β_2 extracts $\mathbb{P}^1_{y_3,y_4}$. We study the intersections $\mathbb{P}^4_{t,s,x_1,x_2,x_3}(2,4,1,1,1) \cap Y_2$ and $\mathbb{P}^1_{y_3,y_4} \cap Y_2$ locally analytically at a neighbourhood of P_1 and P_2 respectively. The first and second unprojection equations allow one to globally eliminate s at P_1 and P_2. Similarly happens for x_1 using $\mathrm{Pf}_3\left(\alpha_1^*(M)\right)$. On the other hand, we can use either $\mathrm{Pf}_4\left(\alpha_1^*(M)\right)$ to eliminate x_2 at P_1, or $\mathrm{Pf}_2\left(\alpha_1^*(M)\right)$ to eliminate x_3 at P_2. The intersection $\mathbb{P}^4_{t,s,x_1,x_2,x_3}(2,4,1,1,1) \cap Y_2$ is formed by two disjoint loci, generated by t, x_2 and t, x_3 at P_1 and P_2 respectively. Nonetheless, they determine two projective lines $\mathbb{P}^1(2,1)$. The fact that this elimination process has not excluded y_3 nor y_4 shows that $\mathbb{P}^1_{y_3,y_4} \subset Y_2$. The variable t does not get eliminated because in $\mathrm{Pf}_1\left(\alpha_1^*(M)\right)$ the polynomial $t\left(y_1^2 - y_1y_2 + y_2^2\right)$ appears: the variable t could be eliminated only if $y_1^2 - y_1y_2 + y_2^2 \neq 0$, but P_1 and P_2 are exactly the two solutions of $y_1^2 - y_1y_2 + y_2^2 = 0$.

In conclusion, Ψ_2 restricts to two simultaneous Francia flips $(2,1,-1,-1)$ based at $P_1, P_2 \in Z_2$, as anticipated in Remark 4.1. In particular, two cyclic quotient singularities of type $\frac{1}{2}(1,1,1)$ in Y_2 are contracted to P_1 and P_2 respectively via α_2, and β_2 extracts a smooth locus in Y_3. Therefore, Y_3 has Picard rank 2.

The last map of the link is the fibration $\Phi' : \mathbb{F}_4 \to \mathbb{P}^1_{y_3,y_4}$. Recall that $-K_{Y_3} \sim \mathcal{O}\binom{1}{0}$. If F is a general fibre of Φ', then $K_F = \left(K_{Y_3} + F\right)\big|_F = K_{Y_3}|_F$ by adjunction. Thus, K_F is ample, F a del Pezzo and, as a consequence, Φ' a del Pezzo fibration. The unprojection variable s can be globally eliminated over each general point of $\mathbb{P}^1_{y_3,y_4}$. There is no other elimination that can be made. Therefore, the fibre F of the del Pezzo fibration sits inside a projective space $\mathbb{P}^6$ with coordinates $t, x_1, x_2, x_3, y_1, y_2$. The matrix M has become a matrix of linear forms in these variables. The equations of F are the five (quadratic) maximal pfaffians of M. Therefore, the degree of F, and of the del Pezzo fibration, is 5.

5.3 | Example for (vi): #24097, Tom_1

Consider $X \ni p$ where $X \subset \mathbb{P}^7(1^6, 2^2)$ is the Tom type Fano 3-fold #24097, and $p \in X$ is the Tom centre $\frac{1}{2}(1,1,1)$. The coordinates of $\mathbb{P}^7(1^6, 2^2)$ are $x_1, x_2, x_3, y_2, y_3, y_4, y_1, s$. The unprojection of $D \cong \mathbb{P}_{x_1,x_2,x_3}(1,1,1) \subset Z_1$ in Tom_1 format produces X, and there are 8 nodes on D. Here Z_1 is #24077 defined by M:

$$M = \begin{pmatrix} x_1 & x_2 & x_3 & -y_2^2 - x_3 y_3 \\ & y_2 & y_3 & y_1 \\ & & y_4 & x_1 y_3 - y_4^2 \\ & & & -x_2 y_4 - x_3 y_4 + y_1 \end{pmatrix}.$$

After the 8 simultaneous flops of Ψ_1, the map Ψ_2 is a Francia flip $(2, 1, -1, -1)$, and Φ' is a weighted $\mathbb{P}^5$-bundle over the projective space $\mathbb{P}^2_{y_2,y_3,y_4}(1, 1, 1)$. We show that Y_3 is a conic bundle over that base, and we compute its discriminant Δ. Note that Y_3 is smooth. We record here the five pfaffian equations of Y_3.

$$\begin{cases} x_1 y_3^2 + x_2 y_2 y_4 + x_2 y_3 y_4 - x_1 y_4^2 - t y_3 y_4^2 - y_2 y_1 - y_4 y_1 = 0, \\ x_1 x_3 y_3 + x_2^2 y_4 + x_2 x_3 y_4 + t^2 y_2^2 y_4 + t x_3 y_3 y_4 - t x_3 y_4^2 - x_2 y_1 = 0, \\ t^2 y_2^2 y_3 + t x_3 y_3^2 + x_1 x_2 y_4 + x_1 x_3 y_4 - x_1 y_1 + x_3 y_1 = 0, \\ t^2 y_2^3 - x_1^2 y_3 + t x_2 y_3^2 - t x_1 y_3 y_4 + t x_1 y_4^2 + x_2 y_1 = 0, \\ x_3 y_2 - x_2 y_3 + x_1 y_4 = 0. \end{cases}$$

At a general point in $\mathbb{P}^2_{y_2,y_3,y_4}(1, 1, 1)$, it is possible to globally eliminate the variable s thanks to the unprojection equations.

Now consider the line $\{y_4 = 0\}$ in the base $\mathbb{P}^2_{y_2,y_3,y_4}(1, 1, 1)$, and look at its two affine patches $\{y_2 \neq 0\}$ and $\{y_3 \neq 0\}$. We want to study the conic equations above each of these patches: in fact, they both contribute to the discriminant Δ.

Over the patch $\{y_2 \neq 0\}$, Pf_5 and Pf_1 globally eliminate the variables x_3 and y_1 respectively: hence they are $x_3 = x_2 y_3$ and $y_1 = x_1 y_3^2$. Replace their expressions in the remaining three pfaffian equations, obtaining

$$\begin{cases} t^2 y_3 + t x_2 y_3^3 - x_1^2 y_3^2 + x_2 x_1 y_3^3 = 0, \\ x_1 x_2 y_3^2 - x_2 x_1 y_3^2 = 0, \\ t^2 - x_1^2 y_3 + t x_2 y_3^2 + x_2 x_1 y_3^2 = 0. \end{cases}$$

where Pf_2 is identically zero, and Pf_3 (above) is a multiple of Pf_4 by a y_3 factor. Therefore, the conic that Pf_4 describes is defined by the matrix

$$A_{y_2} = \begin{pmatrix} 1 & 0 & \frac{1}{2} y_3^2 \\ 0 & -y_3 & \frac{1}{2} y_3^2 \\ \frac{1}{2} y_3^2 & \frac{1}{2} y_3^2 & 0 \end{pmatrix}$$

as $(t, x_1, x_2) \cdot A_{y_2} \cdot (t, x_1, x_2)^T$. Its determinant is $\det(A_{y_2}) = -\frac{1}{4} y_3^4 (1 + y_3) = 0$.

On the other hand, over the patch $\{y_3 \neq 0\}$, Pf_1 and Pf_5 globally eliminate the variables x_1 and x_2 respectively: hence they are $x_1 = y_2 y_1$ and $x_2 = x_3 y_2$. Replace their expressions in the remaining three pfaffian equations: similarly to the other patch, the equation of the conic is $t^2 y_2^2 + t x_3 - y_2 y_1^2 + x_3 y_1 = 0$ given by Pf_3. It is defined by the matrix A_{y_3}

$$A_{y_3} = \begin{pmatrix} y_2^2 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -y_2 \end{pmatrix}$$

determinant $\det(A_{y_3}) = -1/4 y_2 (1 + y_2)$ and by the equation $(t, x_3, y_1) \cdot A_{y_3} \cdot (t, x_3, y_1)^T = 0$. Even though the contribution of $\det(A_{y_2})$ and $\det(A_{y_3})$ to the discriminant might look like $5 + 2 = 7$, the solutions to $\det(A_{y_2}) = 0$ and $\det(A_{y_3}) = 0$ overlap at the point $(-1, -1, 0)$ which is counted twice. Therefore, $\Delta = 5 + 7 - 1 = 6$.

TABLE 1 Fano 3-folds in codimension 4 with Picard rank 1

GRDB ID	Embedding	Format	$e(X)$	$h^{2,1}(X)$	ρ_X
1169	$\mathbb{P}^7(1,2,3,4,5,7^2,9)$	T_1	−38	21	1
1253	$\mathbb{P}^7(1,2,3,4^2,5^2,7)$	T_1	−24	14	1
4925	$\mathbb{P}^7(1^2,3,4,5,6,7^2)$	T_1	−56	30	1
5177	$\mathbb{P}^7(1^2,2,3,4,5^2,6)$	T_1	−48	26	1
5279	$\mathbb{P}^7(1^2,2,3^2,4,5^2)$	$\mathrm{T}_1, \frac{1}{5}$	−38	21	1
5305	$\mathbb{P}^7(1^2,2,3^2,4^2,5)$	$\mathrm{T}_1, \frac{1}{5}$	−36	20	1
5963	$\mathbb{P}^7(1^2,2^2,3^3,5)$	$\mathrm{T}_1, \frac{1}{3}$	−28	16	1
11005	$\mathbb{P}^7(1^3,2,3^2,4,5)$	T_1	−46	25	1
11125	$\mathbb{P}^7(1^3,2^2,3^2,4)$	$\mathrm{T}_1, \frac{1}{2}$	−32	18	1
11125	$\mathbb{P}^7(1^3,2^2,3^2,4)$	$\mathrm{T}_2, \frac{1}{4}$	−32	18	1
11455	$\mathbb{P}^7(1^3,2^3,3^2)$	$\mathrm{T}_1, \frac{1}{3}$	−22	13	1
16339	$\mathbb{P}^7(1^4,2^3,3)$	$\mathrm{T}_1, \frac{1}{2}$	−22	13	1

5.4 | Comparison with Takagi

In [30], the author classifies all possible extremal contractions Φ' appearing in sequences of flops and flips on $\mathbb{Q}$-factorial terminal Fano 3-folds Y of Picard rank $\rho_Y = 2$. These are Sarkisov links from certain $\mathbb{Q}$-Fano 3-folds X with Picard rank 1 enjoying some additional properties (cf. [30, "Main Assumption 0.1"]). In particular, these varieties must have a singularity of type $\frac{1}{2}(1,1,1)$, that is blown up to initiate the link. Six of the varieties falling in Takagi's assumption are in codimension 4 and have a Type I centre. In particular, three of them are of Tom-type, and follow the description of Theorem 2.3. They are: #24097 Tom$_1$ (above in Subsection 5.3, number 4.4 in Takagi's paper) falling in case $d_1 = d_2 = d_3 < d_4$, #20652 Tom$_1$ (above in 5.2, number 5.4) in case $d_1 = d_2 < d_3 = d_4$, and #16645 Tom$_1$ (number 2.2) in case $d_1 < d_2 = d_3 = d_4$.

We have examined them here with our method, and we have showed that the outcomes predicted by Theorem 2.3 match his results. The remaining three Hilbert series indicated by Takagi are of Jerry type. We omit their study from this paper.

6 | THE PICARD RANK OF CERTAIN CODIMENSION 4 TERMINAL FANO 3-FOLDS

The Picard rank of quasi-smooth terminal Fano 3-folds in codimension 4 is unknown, except for some computational results in [8]. The construction carried out so far in this paper provides a tool to compute ρ_X for certain Families of Fano 3-folds X of Tom type in codimension 4.

Theorem 2.3 produces a birational link from each Fano 3-fold of Tom type. Among 161 families of such Fano 3-folds, 96 have a link to another Fano 3-fold X', and moreover X and X' have the same Picard rank. In 12 of these cases, X' is quasi-smooth and so we may compute the rank directly.

Theorem 6.1. *Let X be a general Fano 3-fold of first Tom type and let $p \in X$ be its Tom centre. Suppose that the birational link for $X \ni p$ terminates with a quasi-smooth Mori fibre space $X' \to S$ with $\dim S = 0$, that is, X' is a Fano 3-fold. Then, the Picard rank of X is $\rho_X = 1$.*

Proof. Recall that $\Phi' : Y \to X'$ is an extremal divisorial contraction and $Y = Y_3, Y_4$ is a $\mathbb{Q}$-factorial Fano 3-fold. Hence, X' is $\mathbb{Q}$-factorial.

The Fano 3-fold X' is quasi-smooth if the birational link for $X \ni p$ only involves toric flips and terminates with a divisorial contraction Φ' contracting the singular locus $\mathbb{E}'$ to a quasi-smooth point $p' \in X'$. Since $\mathrm{codim}(X') \leq 3$ by Lemma 4.9 and X' is quasi-smooth, we apply [28, Proposition 2.3], [8, Tables 1, 2, 3] to conclude that $\rho_{X'} = 1$. The birational link extracts and contracts exactly one irreducible divisor, and is otherwise an isomorphism in codimension 1. Therefore $\rho_X = \rho_{X'} = 1$. □

In particular, in the hypotheses of Theorem 6.1 the link is a Sarkisov link. We expect that the hypotheses of quasi-smoothness of X' can be lifted, and that Theorem 6.1 can be generalised to the rest of the 96 Tom families. The Fano 3-fold in codimension 4 having Picard rank 1 are summarised in Table 1 together with their formats, Euler characteristic $e(X)$, and Hodge number $h^{2,1}(X)$, calculated using [8, Theorem 4 and Table 3]. Note that #11125 has two different birational links with ending with a quasi-smooth X', as reported in Table 1. Moreover, some of the Fano 3-folds in Table 1 admit other links to non-quasi-smooth Fano 3-folds, which therefore have Picard rank 1. This constitutes a further evidence that Theorem 6.1 could still hold for non-quasi-smooth X'.

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REFERENCES

[1] H. Ahmadinezhad, *On pliability of del Pezzo fibrations and Cox rings*, J. Reine Angew. Math. **723** (2017), 101–125.
[2] H. Ahmadinezhad and T. Okada, *Birationally rigid Pfaffian Fano 3-folds*, Algebr. Geom. **5** (2018), no. 2, 160–199.
[3] H. Ahmadinezhad and F. Zucconi, *Mori dream spaces and birational rigidity of Fano 3-folds*, Adv. Math. **292** (2016), 410–445.
[4] H. Ahmadinezhad and F. Zucconi, *Circle of Sarkisov links on a Fano 3-fold*, Proc. Edinb. Math. Soc. (2) **60** (2017), no. 1, 1–16.
[5] S. Altınok, G. Brown, and M. Reid, *Fano 3-folds, $K3$ surfaces and graded rings*, Topology and Geometry: Commemorating SISTAG, Contemp. Math., vol. 314, Amer. Math. Soc., Providence, RI, 2002, pp. 25–53.
[6] J. Blanc et al., *Birational self-maps of threefolds of (un)-bounded genus or gonality*, arXiv preprint arXiv:1905.00940, Amer. J. Math. (to appear).
[7] G. Brown, A. Corti, and F. Zucconi, *Birational geometry of 3-fold Mori fibre spaces*, The Fano Conference, Univ. Torino, Turin, 2004, pp. 235–275.
[8] G. Brown and E. Fatighenti, *Hodge numbers and deformations of Fano 3-folds*, Doc. Math. **5** (2020), 267–307.
[9] G. Brown, A. M. Kasprzyk, and M. I. Qureshi, *Fano 3-folds in $\mathbb{P}^2 \times \mathbb{P}^2$ format, Tom and Jerry*, Eur. J. Math. **4** (2018), no. 1, 51–72.
[10] G. Brown, A. M. Kasprzyk et al., *Graded ring database*, Online. Available at http://www.grdb.co.uk.
[11] G. Brown, M. Kerber, and M. Reid, *Fano 3-folds in codimension 4, Tom and Jerry. Part I*, Compos. Math. **148** (2012), no. 4, 1171–1194.
[12] G. Brown, M. Kerber, and M. Reid, *Tom and Jerry table, part of "Fano 3-folds in codimension 4, Tom and Terry. Part I"*, Compos. Math. **148** (2012), no. 4, 1171–1194.
[13] G. Brown and F. Zucconi, *Graded rings of rank 2 Sarkisov links*, Nagoya Math. J. **197** (2010), 1–44.
[14] L. Campo, *Big table links*, Online. Available at http://www.grdb.co.uk/files/fanolinks/BigTableLinks.pdf (2020).
[15] I. Cheltsov and J. Park, *Birationally rigid Fano threefold hypersurfaces*, Mem. Amer. Math. Soc. **246** (2017), no. 1167.
[16] A. Corti, *Factoring birational maps of threefolds after Sarkisov*, J. Algebraic Geom. **4** (1995), no. 2, 223–254.
[17] A. Corti and M. Mella, *Birational geometry of terminal quartic 3-folds. I*, Amer. J. Math. **126** (2004), no. 4, 739–761.
[18] A. Corti, A. Pukhlikov, and M. Reid, *Fano 3-fold hypersurfaces*, Explicit birational geometry of 3-folds, London Math. Soc. Lecture Note Ser., vol. 281, Cambridge Univ. Press, Cambridge, 2000, pp. 175–258.
[19] D. A. Cox, J. Little, and D. O'Shea, *Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebraic*, Undergrad. Texts Math., 4th edn., Springer, 2015.
[20] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, Grad. Stud. Math., vol. 124, Amer. Math. Soc., Providence, RI, 2011.
[21] C. D. Hacon and J. McKernan, *The Sarkisov program*, J. Algebraic Geom. **22** (2013), no. 2, 389–405.
[22] A. R. Iano-Fletcher, *Working with weighted complete intersections*, Explicit birational geometry of 3-folds, London Math. Soc. Lecture Note Ser., vol. 281, Cambridge Univ. Press, Cambridge, 2000, pp. 101–173.
[23] V. A. Iskovskikh and A. V. Pukhlikov, *Birational automorphisms of multidimensional algebraic manifolds*, Algebr. Geom. **1** (1996), 3528–3613.
[24] Y. Kawamata, *Divisorial contractions to 3-dimensional terminal quotient singularities*, Higher-dimensional complex varieties (Trento, 1994), de Gruyter, Berlin, 1996, pp. 241–246.
[25] T. Okada, *Birational Mori fiber structures of $\mathbb{Q}$-Fano 3-fold weighted complete intersections*, Proc. Lond. Math. Soc. (3) **109** (2014), no. 6, 1549–1600.
[26] S. A. Papadakis, *Kustin–Miller unprojection with complexes*, J. Algebraic Geom. **13** (2004), no. 2, 249–268.
[27] S. A. Papadakis and M. Reid, *Kustin–Miller unprojection without complexes*, J. Algebraic Geom. **13** (2004), no. 3, 563–577.
[28] M. Pizzato, T. Sano, and L. Tasin, *Effective nonvanishing for Fano weighted complete intersections*, Algebra Number Theory **11** (2017), no. 10, 2369–2395.
[29] M. Reid, *Canonical 3-folds*, Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980, pp. 273–310.
[30] H. Takagi, *On classification of $\mathbb{Q}$-Fano 3-folds of Gorenstein index 2. I, II*, Nagoya Math. J. **167** (2002), 117–155, 157–216.

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APPENDIX A: PAPADAKIS' ALGORITHM FOR UNPROJECTION

In [26] the author explicitly builds the Type I unprojection equations from a codimension 3 Fano 3-fold Z in Tom format. Here we briefly retrace the steps of Papadakis' construction, combining the two notations. Suppose for simplicity that the matrix M is in format Tom_1. For $D \cong \mathbb{P}_{x_1,x_2,x_3}(a,b,c)$ the divisor in Z, and $I_D := \mathrm{Span}\left[y_1, y_2, y_3, y_4\right]$, the graded matrix M is

$$M = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ & a_{23} & a_{24} & a_{25} \\ & & a_{34} & a_{35} \\ & & & a_{45} \end{pmatrix}. \tag{A.1}$$

Here the a_{ij} are polynomials of the form $a_{ij} := \sum_{k=1}^{4} \alpha_{ij}^k y_k$ for some polynomial coefficients α_{ij}^k. The a_{ij} are in the ideal I_D. Instead, p_j are not in I_D, in accordance to Definition 3.2. Only in this Appendix, we calculate Pf_i by excluding the $(i+1)$-th row and the $(i+1)$-th column for $i \in \{0,1,2,3,4\}$. Only $\mathrm{Pf}_1, \dots, \mathrm{Pf}_4$ are linear in the y_i; hence, there exists a unique matrix Q such that $\left(\mathrm{Pf}_1(M), \dots, \mathrm{Pf}_4(M)\right)^T = Q\left(y_1, \dots, y_4\right)^T$. Explicitly, $Q = \left(\mathrm{Pf}_i\left(N_j\right)\right)_{i,j=1\dots4}$ where

$$N_i = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ & \alpha_{23}^i & \alpha_{24}^i & \alpha_{25}^i \\ & & \alpha_{34}^i & \alpha_{35}^i \\ & & & \alpha_{45}^i \end{pmatrix}$$

and α_{kl}^i is the coefficient of y_i in a_{kl}. Define H_i as the vector of length 4 whose i-th entry is $(-1)^{i+1}$ times the determinant of the submatrix of Q obtained by removing the i-th column and the i-th row. F or all $i, j \in \{1, \dots, 4\}$, the vectors H_i satisfy

$$p_i H_j = p_j H_i \tag{A.2}$$

(cf. Lemma 5.3 of [26]). Thus, the quotient H_i/p_i is independent of i. The polynomials $g_1, \dots, g_4$ are defined via the following equality of vectors of length 4 $\left(g_1, g_2, g_3, g_4\right) = H_i/p_i$. For instance, g_1 is the determinant of the matrix obtained deleting the first column and the first row of Q divided by p_1, i.e.

$$g_1 = \frac{1}{p_1} \det \begin{pmatrix} \mathrm{Pf}_2\left(N_2\right) & \mathrm{Pf}_2\left(N_3\right) & \mathrm{Pf}_2\left(N_4\right) \\ \mathrm{Pf}_3\left(N_2\right) & \mathrm{Pf}_3\left(N_3\right) & \mathrm{Pf}_3\left(N_4\right) \\ \mathrm{Pf}_4\left(N_2\right) & \mathrm{Pf}_4\left(N_3\right) & \mathrm{Pf}_4\left(N_4\right) \end{pmatrix}. \tag{A.3}$$

The g_j are the right-hand sides of the unprojection equations, that is, the unprojection equations defining X are $sy_j = g_j$ for $j = 1, \dots, 4$.