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ON DIRECTED VERSIONS OF THE HAJNAL–SZEMERÉDI THEOREM

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Abstract. We say that a (di)graph G has a perfect H-packing if there exists a set of vertex-disjoint copies of H which cover all the vertices in G. The seminal Hajnal–Szemerédi theorem characterises the minimum degree that ensures a graph G contains a perfect K_r-packing. In this paper we prove the following analogue for directed graphs: Suppose that T is a tournament on r vertices and G is a digraph of sufficiently large order n where r divides n. If G has minimum in- and outdegree at least $(1-1/r)n$ then G contains a perfect T-packing.

In the case when T is a cyclic triangle, this result verifies a recent conjecture of Czygrinow, Kierstead and Molla [4] (for large digraphs). Furthermore, in the case when T is transitive we conjecture that it suffices for every vertex in G to have sufficiently large indegree or outdegree. We prove this conjecture for transitive triangles and asymptotically for all $r \geq 3$. Our approach makes use of a result of Keevash and Mycroft [10] concerning almost perfect matchings in hypergraphs as well as the Directed Graph Removal lemma [1, 6].

MSC2000: 5C35, 5C20, 5C70.

1. Introduction

1.1. **Perfect packings in undirected graphs.** Given two (di)graphs H and G, an *H-packing* in G is a collection of vertex-disjoint copies of H in G. An H-packing is called *perfect* if it covers all the vertices of G. Perfect H-packings are also referred to as *H-factors* or *perfect H-tilings*. Note that perfect H-packings are generalisations of perfect matchings (which correspond to the case when H is a single edge). Tutte's theorem characterises all those graphs that contain a perfect matching. On the other hand, Hell and Kirkpatrick [8] showed that the decision problem of whether a graph G has a perfect H-packing is NP-complete precisely when H has a component consisting of at least 3 vertices. Thus, for such graphs H, it is unlikely that there is a complete characterisation of those graphs containing a perfect H-packing. It is natural therefore to ask for simple sufficient conditions which force a graph to contain a perfect H-packing.

A seminal result in the area is the following theorem of Hajnal and Szemerédi [7].

Theorem 1.1 (Hajnal and Szemerédi [7]). *Every graph G whose order n is divisible by r and whose minimum degree satisfies $\delta(G) \geq (1-1/r)n$ contains a perfect K_r-packing.*

It is easy to see that the minimum degree condition here cannot be lowered. In recent years there have been several generalisations of the Hajnal–Szemerédi theorem. Kühn and Osthus [15, 16] characterised, up to an additive constant, the minimum degree which ensures that a graph G contains a perfect H-packing for an *arbitrary* graph H. Kierstead and Kostochka [13] proved an *Ore-type* analogue of the Hajnal–Szemerédi theorem: If G is a graph whose order n is divisible by r, then G contains a perfect K_r-packing provided that $d(x)+d(y) \geq 2(1-1/r)n-1$ for all non-adjacent $x \neq y \in V(G)$. Kühn, Osthus and Treglown [17] characterised, asymptotically, the Ore-type degree condition which ensures that a graph G contains a perfect H-packing for an arbitrary graph H.

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Recently, Keevash and Mycroft [11] proved the following r-partite version of the Hajnal–Szemerédi theorem, thereby tackling a conjecture of Fischer [5] for sufficiently large graphs.

Theorem 1.2 (Keevash and Mycroft [11])**.** *Given $r \in \mathbb{N}$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose G is an r-partite graph with vertex classes $V_1, \dots, V_r$ where $|V_i| = n \geq n_0$ for all $1 \leq i \leq r$. If*

$$\overline{\delta}(G) \geq (1 - 1/r)n + 1$$

then G contains a perfect K_r-packing.

(Here $\overline{\delta}(G)$ denotes the minimum degree of a vertex from one vertex class V_i to another vertex class V_j.) Keevash and Mycroft [11] actually proved a stronger result than Theorem 1.2. Indeed, they showed that the minimum degree condition here can be relaxed to $\overline{\delta}(G) \geq (1-1/r)n$ provided that G is not isomorphic to one special construction. Further, their result extends to perfect K_k-packings where $1 \leq k \leq r$ (see [11] for more details).

1.2. **Packing tournaments in directed graphs.** It is natural to seek analogues of the Hajnal–Szemerédi theorem in the digraph and oriented graph settings. We consider digraphs with no loops and at most one edge in each direction between every pair of vertices. An oriented graph is a digraph without 2-cycles. In this paper we restrict our attention to the problem for digraphs. See [22, 2] for an overview of the known results concerning perfect packings in oriented graphs.

The *minimum semidegree* $\delta^0(G)$ of a digraph G is the minimum of its minimum outdegree $\delta^+(G)$ and its minimum indegree $\delta^-(G)$. Let $\delta(G)$ denote the minimum degree of G, that is, the minimum number of edges incident to a vertex in G. (Note that if both xy and yx are directed edges in G, they are counted as two separate edges.) Denote by $\mathcal{T}_r$ the set of all tournaments on r vertices. Our main result is an analogue of the Hajnal–Szemerédi theorem for perfect tournament packings in digraphs.

Theorem 1.3. *Given an integer $r \geq 3$, there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose $T \in \mathcal{T}_r$ and G is a digraph on $n \geq n_0$ vertices where r divides n. If*

$$\delta^0(G) \geq (1 - 1/r)n$$

then G contains a perfect T-packing.

Notice that the minimum semidegree condition in Theorem 1.3 is tight. Indeed, let $n, r \in \mathbb{N}$ such that r divides n. Let G' be the digraph obtained from the complete digraph on n vertices by removing all those edges lying in a given vertex set of size $n/r + 1$. Then $\delta^0(G') = (1 - 1/r)n - 1$ and G' does not contain a perfect T-packing for any $T \in \mathcal{T}_r$. In general, any digraph G'' on n vertices with an independent set of size $n/r + 1$ (and $\delta^0(G'') = (1 - 1/r)n - 1$) does not contain a perfect T-packing.

In the case when T is the cyclic triangle C_3, Theorem 1.3 verifies a recent conjecture of Czygrinow, Kierstead and Molla [4] for large digraphs. Further, notice that Theorem 1.3 is a 'true generalisation' of the Hajnal–Szemerédi theorem in the sense that the former implies the latter for large graphs.

We remark that, by applying the same probabilistic trick used by Keevash and Sudakov in Section 7 of [12], one can obtain an asymptotic version of Theorem 1.3 from Theorem 1.2. In [4] it was also shown that such an asymptotic version of Theorem 1.3 for $T = C_3$ follows from a result concerning perfect packings in multigraphs.

Similarly to many proofs in the area, our argument splits into 'extremal' and 'non-extremal' cases. When $T \neq C_3$, the extremal case considers digraphs G containing a set of vertices of size n/r that spans an 'almost' independent set (i.e. G is 'close' to an extremal graph G'' as above). Interestingly, when $T = C_3$ we have an extra extremal configuration (see Section 3.2), and thus have two separate extremal cases. In the non-extremal case our proof splits into two main tasks: finding an 'almost' perfect T-packing in G and finding a so-called 'absorbing set' that can be used to cover the remaining vertices with disjoint copies of T (see Section 3.3 for the precise definition of such a set). To obtain the former we apply a result of Keevash and Mycroft [10] concerning almost perfect matchings in hypergraphs. We also make use of the Directed Graph Removal lemma (see e.g. [1, 6]). A substantial proportion of the paper is devoted to obtaining our desired absorbing set. A more detailed overview of the proof is given in Section 2.

1.3. **Degree conditions forcing perfect transitive tournament packings.** Although the minimum semidegree condition in Theorem 1.3 is 'best-possible' one could replace the condition by a weaker one. Indeed, for *transitive* tournaments we conjecture that the following stronger statement is true. Let T_r denote the transitive tournament on r vertices.

Conjecture 1.4. *Let $n, r \in \mathbb{N}$ such that r divides n. Suppose that G is a digraph on n vertices so that for any $x \in V(G)$,*

$$d^+(x) \geq (1-1/r)n \text{ or } d^-(x) \geq (1-1/r)n. \tag{1}$$

Then G contains a perfect T_r-packing.

Conjecture 1.4 would imply the following very recent result of Czygrinow, DeBiasio, Kierstead and Molla [3].

Theorem 1.5 (Czygrinow, DeBiasio, Kierstead and Molla [3])**.** *Let $n, r \in \mathbb{N}$ such that r divides n. Then every digraph G on n vertices with*

$$\delta^+(G) \geq (1-1/r)n$$

contains a perfect T_r-packing.

In Section 4 we give a short proof of Conjecture 1.4 in the case when $r = 3$. We also prove the following asymptotic version of Conjecture 1.4.

Theorem 1.6. *Let $\eta > 0$ and $r \geq 3$. Then there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that G is a digraph on $n \geq n_0$ vertices where r divides n and that for any $x \in V(G)$,*

$$d^+(x) \geq (1-1/r+\eta)\,n \quad \text{or} \quad d^-(x) \geq (1-1/r+\eta)\,n.$$

Then G contains a perfect T_r-packing.

We give a unified approach to proving Theorems 1.3 and 1.6, though the proof of the former is substantially more involved.

For 'most' tournaments T, there does not exist a 'non-trivial' minimum outdegree condition which forces a digraph G to contain a perfect T-packing. Indeed, let $T \in \mathcal{T}_r$ such that every vertex in T has an inneighbour. Let $n \in \mathbb{N}$ such that r divides n. Obtain the digraph G from the complete digraph on $n-1$ vertices by adding a vertex x that sends out all possible edges to the other vertices (but receives none). Then $\delta^+(G) = n-2$ but G does not contain a perfect T-packing since x does not lie in a copy of T.

So certainly Conjecture 1.4 and Theorem 1.5 cannot be generalised to arbitrary tournaments T. It would be interesting to establish whether the degree conditions in Conjecture 1.4 and Theorem 1.5 force a perfect T-packing for some *non-transitive* tournament T on r vertices.

In the next section we give an outline of the proofs as well as details about the organisation of the paper.

2. Overview of the proofs and organisation of the paper

2.1. **Outline of the proof of Theorem 1.3.** Suppose that G and $T \in \mathcal{T}_r$ are as in Theorem 1.3. Further, suppose that there is a 'small' set $M \subseteq V(G)$ with the property that both $G[M]$ and $G[M \cup Q]$ contain perfect T-packings for *any* 'very small' set $Q \subseteq V(G)$ where $|Q| \in r\mathbb{N}$. Then notice that, to find a perfect T-packing in G, it suffices to find an 'almost' perfect T-packing in $G' := G \setminus M$. Indeed, suppose that G' contains a T-packing $\mathcal{M}_1$ covering all but a very small set of vertices Q. Then by definition of M, $G[M \cup Q]$ contains a perfect T-packing $\mathcal{M}_2$. Thus, $\mathcal{M}_1 \cup \mathcal{M}_2$ is a perfect T-packing in G, as desired.

Roughly speaking, we refer to such a set M as an 'absorbing set' (see Section 3.3 for the precise definition of such a set). The 'absorbing method' was first used in [20] and has subsequently been applied to numerous embedding problems in extremal graph theory.

In general, a digraph G as in Theorem 1.3 may not contain an absorbing set. For example, consider the complete 3-partite digraph G_1 with vertex classes V_1, V_2, V_3 of size $n/3$. (So G_1 contains all possible edges with endpoints in different vertex classes.) Then G_1 satisfies the hypothesis of Theorem 1.3 in the case when $T = T_3$ and $r = 3$. However, if $Q \subseteq V_1$ such that $|Q| = 3$ then it is easy to see that there does not exist a set $M \subseteq V(G_1)$ such that both $G_1[M]$ and $G_1[M \cup Q]$ contain perfect T_3-packings. Notice though that G_1 is 'close to extremal' (i.e. G_1 contains an independent set of size $n/3$).

It turns out that being 'close' to an extremal example is the only barrier preventing our digraph G from containing an absorbing set M. Indeed, in the case when $T \neq C_3$ we show that if G does not contain an 'almost' independent set of size n/r then G contains our desired set M. As mentioned in Section 1.2, when $T = C_3$ we have an extra extremal configuration $Ex(n)$ (see Section 3.2). In this case we show that if G is far from $Ex(n)$ and does not contain an 'almost' independent set of size $n/3$ then G contains our desired set M (see Theorem 5.1).

Constructing the absorbing set in the non-extremal case. The crucial idea in proving that a non-extremal digraph G contains an absorbing set M is to first show that G has many 'connecting structures' of a certain type. For example, to find our desired absorbing set it suffices to show that, for any $x, y \in V(G)$, there are 'many' $(r-1)$-sets $X \subseteq V(G)$ so that both $X \cup \{x\}$ and $X \cup \{y\}$ span copies of T in G. In Section 8 we prove a number of so-called connection lemmas that guarantee such connecting structures. This turns out to be quite a subtle process as we prove different connection lemmas depending on the structure and size of T. In particular, we need to deal with the case when $T = C_3$ separately. (This stems from the fact that we now have two extremal cases. See Section 8 for more details.) In Section 9 we use the connection lemmas to construct the absorbing set M.

Covering the remaining vertices of G in the non-extremal case. As mentioned earlier, once we have constructed an absorbing set M in a non-extremal digraph G, it suffices to find an 'almost' perfect T-packing in $G' = G \setminus M$. For this, we translate the problem into one about almost perfect matchings in hypergraphs. Indeed, from G' we construct a hypergraph J on $V(G')$ where, for any $1 \leq i \leq r$, an i-tuple $Y \subseteq V(G')$ forms an edge in J precisely when Y spans a subtournament of T of size i in G'. So one may think of J as consisting of 'layers' $J_1, \dots, J_r$ where J_i contains the edges of size i. For example, if $T = T_3$, then the edge set of J_1 is $V(G')$, the edge set of J_2 consists of all pairs $\{x, y\}$ where $xy \in E(G')$ or $yx \in E(G')$ and the edge set of J_3 consists of all triples $\{x, y, z\}$ that span a copy of T_3 in G'. J is an example of a so-called r-complex (see Section 7 for the precise definition).

Vitally, J has the property that a matching in J_r corresponds to a T-packing in G'. We thus apply a result of Keevash and Mycroft [10] on almost perfect matchings in r-complexes. (In order to apply this result we again use that G is non-extremal.) This ensures an almost perfect matching in J_r and thus an almost perfect T-packing in G', as desired.

The extremal cases. Finally, we deal with the case when G is close to an extremal example. If $T = C_3$ and G is close to $Ex(n)$ then a relatively short argument shows that G must contain a perfect C_3-packing (see Lemma 5.6). On the other hand, the general extremal case when G contains an almost independent set of size n/r is more involved (see Lemma 5.5). (Note that the class of digraphs G on n vertices with an almost independent set of size n/r and $\delta^0(G) \geq (1 - 1/r)n$ is wide.) We draw on ideas from [14] to tackle this case.

The extremal cases are the only parts of the proof where we use the full force of the minimum semidegree condition on G. Indeed, the argument in the non-extremal case holds even if we relax the condition to $\delta^0(G) \geq (1 - 1/r - o(1))n$.

2.2. **The proof of Theorem 1.6.** The proof of Theorem 1.6 follows the same general approach as that of Theorem 1.3 in the *non-extremal* case: Again our two main tasks are to (i) find an absorbing set and (ii) cover almost all of the remaining vertices with a T_r-packing. Thus, where possible, we present the tools for both proofs in a unified way. Indeed, many of our auxiliary results are applied in both proofs.

2.3. **Organisation of the paper.** In the next section we formally introduce the notion of an absorbing set and define the extremal digraph $Ex(n)$. We also introduce other notation and definitions. We prove Conjecture 1.4 in the case of transitive triangles in Section 4. In Section 5 we state the main auxiliary results that we prove in the paper and derive Theorems 1.3 and 1.6 from them. In Section 6 we prove Turán-type results for digraphs. These results will be applied both when constructing our absorbing sets and when finding an almost perfect T-packing in the non-extremal case. Section 7 deals with this latter task. We state and prove the connection lemmas for Theorems 1.3 and 1.6 in Section 8. These are then used to construct our absorbing sets in Section 9. After giving a number of useful results in Section 10 we tackle the extremal cases of Theorem 1.3 in Sections 11 and 12.

3. Notation and preliminaries

3.1. **Definitions and notation.** Given two vertices x and y of a digraph G, we write xy for the edge directed from x to y. We write $V(G)$ for the vertex set of G, $E(G)$ for the edge set of G and define $|G| := |V(G)|$ and $e(G) := |E(G)|$. We denote by $N^+_G(x)$ and $N^-_G(x)$ the out- and the inneighbourhood of x and by $d^+_G(x)$ and $d^-_G(x)$ its out- and indegree. We will write $N^+(x)$ for example, if this is unambiguous. For a vertex $x \in V(G)$ and a set $Y \subseteq V(G)$ we write $d^+_G(x, Y)$ to denote the number of edges in G with startpoint x and endpoint in Y. We define $d^-_G(x, Y)$ analogously. The *minimum semidegree* $\delta^0(G)$ of G is the minimum of its minimum outdegree $\delta^+(G)$ and its minimum indegree $\delta^-(G)$. Let $\delta(G)$ denote the minimum degree of G, that is, the minimum number of edges incident to a vertex in G. (Note that if both xy and yx are directed edges in G, they are counted as two separate edges.)

Given a subset $X \subseteq V(G)$, we write $G[X]$ for the subdigraph of G induced by X. We write $G \setminus X$ for the subdigraph of G induced by $V(G) \setminus X$. For $x_1, \dots, x_m \in V(G)$ we define $G[x_1, \dots, x_m] := G[\{x_1, \dots, x_m\}]$.

Given a set $X \subseteq V(G)$ and a digraph H on $|X|$ vertices we say that X *spans a copy of H in G* if $G[X]$ contains a copy of H. In particular, this does not necessarily mean that X induces a copy of H in G. For disjoint $X, Y \subseteq V(G)$ we let $G[X, Y]$ denote the digraph with vertex set $X \cup Y$ whose edge set consists of all those edges $xy \in E(G)$ with $x \in X$ and $y \in Y$. If G and H are digraphs, we write $G \cup H$ for the digraph whose vertex set is $V(G) \cup V(H)$ and whose edge set is $E(G) \cup E(H)$. If G and H have the same vertex set V then let $G - H$ denote the digraph with vertex set V and edge set $E(G) \setminus E(H)$.

Given digraphs G and H, we say that G is *H-free* if G does not contain H as a subdigraph. Let G be a (di)graph on n vertices and let $\gamma > 0$. We say that a set $S \subseteq V(G)$ is *γ-independent* if $G[S]$ contains at most γn^2 edges. Given two digraphs G and H on n vertices we say that G *γ-contains* H if, after adding at most γn^2 edges to G, the resulting digraph contains a copy of H. More precisely, G γ-contains H if there is an isomorphic copy G' of G such that $V(G') = V(H)$ and $|E(H) \setminus E(G')| \leq \gamma n^2$.

For a (di)graph G and disjoint $A, B \subseteq V(G)$, we write $e_G(A, B)$ for the number of edges in G with one endpoint in A and the other in B. (So $e_G(A, B) = e_G(B, A)$.) Given a (di)graph T, let $2T$ denote the disjoint union of two copies of T.

Recall that T_r denotes the transitive tournament of r vertices. Given $1 \leq i \leq r$, we say a vertex $x \in V(T_r)$ is the *ith vertex of* T_r if x has indegree $i-1$ and outdegree $r-i$ in T_r. Given a set X and $r \in \mathbb{N}$ we denote by $\binom{X}{r}$ the set of all r-subsets of X.

Throughout the paper, we write $0 < \alpha \ll \beta \ll \gamma$ to mean that we can choose the constants α, β, γ from right to left. More precisely, there are increasing functions f and g such that, given γ, whenever we choose some $\beta \leq f(\gamma)$ and $\alpha \leq g(\beta)$, all calculations needed in our proof are valid. Hierarchies of other lengths are defined in the obvious way.

3.2. **The extremal digraph $Ex(n)$.** Suppose that $n \geq 3$ and c are non-negative integers. Define $a_1, a_2, a_3 \in \mathbb{N}$ such that $\lfloor n/3 \rfloor \leq a_1 \leq a_2 \leq a_3 \leq \lceil n/3 \rceil$ where $a_1 + a_2 + a_3 = n$. Let A_1, A_2 and A_3 be disjoint vertex sets of size $a_1 - c, a_2 + c$ and a_3 respectively. Let $Ex_c(n)$ denote the digraph with vertex set $A_1 \cup A_2 \cup A_3$ and whose edge set is defined as follows:

- A_i induces a complete digraph in $Ex_c(n)$ (for all $1 \leq i \leq 3$);
- If $x \in A_i$ and $y \in A_{i+1}$ then $xy \in E(Ex_c(n))$ (for all $1 \leq i \leq 3$, where indices are taken mod 3).

Define $Ex(n) := Ex_0(n)$. (See Figure 1.) We call A_1, A_2 and A_3 the *vertex classes of $Ex(n)$*.

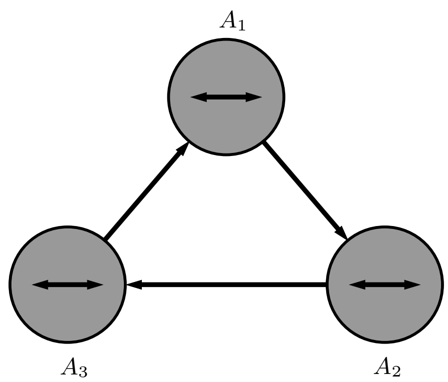

FIGURE 1. The extremal digraph $Ex(n)$

Suppose that n is divisible by 3. Note that $\delta^0(Ex_1(n)) = 2n/3 - 2$ but $Ex_1(n)$ does not contain a perfect C_3-packing. Thus, in the proof of Theorem 1.3 for $T = C_3$ we have two extremal cases to consider: when G contains an 'almost' independent set of size $n/3$ and when G 'almost' contains $Ex(n)$.

3.3. **Absorbing sets.** Let $T \in \mathcal{T}_r$. Given a digraph G, a set $S \subseteq V(G)$ is called a *T-absorbing set for $Q \subseteq V(G)$*, if both $G[S]$ and $G[S \cup Q]$ contain perfect T-packings. In this case we say that *Q is T-absorbed by S*. Sometimes we will simply refer to a set $S \subseteq V(G)$ as a T-absorbing set if *there exists* a set $Q \subseteq V(G)$ that is T-absorbed by S.

When constructing our absorbing sets in Section 9 we will use the following Chernoff bound for binomial distributions (see e.g. [9, Corollary 2.3]). Recall that the binomial random variable with parameters (n, p) is the sum of n independent Bernoulli variables, each taking value 1 with probability p or 0 with probability $1-p$.

Proposition 3.1. *Suppose that X has binomial distribution and $0 < a < 3/2$. Then $\mathbb{P}(|X - \mathbb{E}X| \geq a\mathbb{E}X) \leq 2e^{-\frac{a^2}{3}\mathbb{E}X}$.*

4. Proof of Conjecture 1.4 for transitive triangles

Let $\mathcal{H}$ be a collection of digraphs and G a digraph. We say that G contains a *perfect $\mathcal{H}$-packing* if G contains a collection of vertex-disjoint copies of elements from $\mathcal{H}$ that together cover all the vertices of G. We now prove Conjecture 1.4 in the case of transitive triangles.

Theorem 4.1. *Let $m \in \mathbb{N}$. Suppose that G is a digraph on $n := 3m$ vertices so that for any $x \in V(G)$,*

$$d^+(x) \geq 2n/3 \text{ or } d^-(x) \geq 2n/3. \tag{2}$$

Then G contains a perfect T_3-packing.

Proof. Let G be a digraph as in the statement of the theorem. Remove as many edges from G as possible so that (2) still holds. Let G' denote the graph on $V(G)$ where $xy \in E(G')$ if and only if $xy \in E(G)$ or $yx \in E(G)$. So $\delta(G) \geq 2n/3$ by (2). Thus, Theorem 1.1 implies that G' contains a perfect K_3-packing and so G contains a perfect $\{T_3, C_3\}$-packing. Let $\mathcal{M}$ denote the perfect $\{T_3, C_3\}$-packing in G that contains the most copies of T_3.

Suppose for a contradiction that $\mathcal{M}$ is not a perfect T_3-packing. Then there is a copy C'_3 of C_3 in $\mathcal{M}$. Let $V(C'_3) = \{x, y, z\}$ where $xy, yz, zx \in E(C'_3)$. Suppose that $d^-_G(w) < 2n/3$ for some $w \in V(C'_3)$. Without loss of generality assume that $w = x$. Then (2) implies that $d^+_G(x) \geq 2n/3$. If $d^+_G(z) < 2n/3$ then we may remove the edge zx from G and still (2) holds, a contradiction to the minimality of G. So $d^+_G(z) \geq 2n/3$. An identical argument implies that $d^+_G(y) \geq 2n/3$. This shows that $d^-_G(w) \geq 2n/3$ for all $w \in V(C'_3)$ or $d^+_G(w) \geq 2n/3$ for all $w \in V(C'_3)$.

Without loss of generality assume that $d^+_G(w) \geq 2n/3$ for all $w \in V(C'_3)$. (The other case is analogous.) Note that $G[x, y, z]$ contains precisely three edges (else $V(C'_3)$ spans a copy of T_3, a contradiction to the maximality of $\mathcal{M}$). In particular, there are at least $2n-3 = 6m-3 > 6(|\mathcal{M}|-1)$ edges in G with startpoint in $V(C'_3)$ and endpoint in $V(G) \setminus V(C'_3)$. This implies that there is an element $T \in \mathcal{M} \setminus \{C'_3\}$ that receives at least 7 edges from $V(C'_3)$ in G.

So there is a vertex, say x, in $V(C'_3)$ such that $d^+_G(x, V(T)) = 3$. Furthermore, y and z have a common outneighbour in G that lies in $V(T)$. Together this implies that $V(C'_3) \cup V(T)$ spans a copy of $2T_3$ in G. This yields a perfect $\{T_3, C_3\}$-packing in G containing more copies of T_3 than $\mathcal{M}$, a contradiction. So the assumption that $\mathcal{M}$ is not a perfect T_3-packing is false, as desired. $\square$

5. Deriving Theorems 1.3 and 1.6 from the auxiliary results

In this section we state a number of auxiliary results that we will prove in the paper. We then combine these results to prove Theorems 1.3 and 1.6. Roughly speaking, the following result states that if G is as in Theorem 1.3 (namely has large semi-degree) and is *non-extremal* then G contains a 'small' absorbing set that absorbs *any* 'very small' set of vertices in G.

Theorem 5.1. *Let $0 < 1/n \ll \varepsilon \ll \xi \ll \gamma, \alpha \ll 1/r$ where $n, r \in \mathbb{N}$ and $r \geq 3$, and let $T \in \mathcal{T}_r$. Suppose that G is a digraph on n vertices so that*

$$\delta^0(G) \geq (1 - 1/r - \varepsilon)\, n.$$

Further suppose that

- *G does not contain any γ-independent set of size at least n/r;*
- *If $T = C_3$ then G does not α-contain $Ex(n)$.*

Then $V(G)$ contains a set M so that $|M| \leq \xi n$ and M is a T-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in r\mathbb{N}$ and $|W| \leq \xi^2 n$.

The next result is an analogue of Theorem 5.1 that will be applied in the proof of Theorem 1.6.

Theorem 5.2. *Let $0 < 1/n \ll \varepsilon \ll \xi \ll \gamma \ll 1/r$ where $n, r \in \mathbb{N}$ and $r \geq 3$. Suppose that G is a digraph on n vertices so that, for any $x \in V(G)$,*

$$d^+(x) \geq (1 - 1/r - \varepsilon)\, n \quad \text{or} \quad d^-(x) \geq (1 - 1/r - \varepsilon)\, n.$$

Further suppose that G does not contain any γ-independent set of size at least n/r. Then $V(G)$ contains a set M so that $|M| \leq \xi n$ and M is a T_r-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in r\mathbb{N}$ and $|W| \leq \xi^2 n$.

We prove Theorems 5.1 and 5.2 in Section 9. The crucial tools used in these proofs are so-called 'connection lemmas' which we introduce in Section 8.

Theorem 5.3. *Let $0 < 1/n \ll 1/\ell \ll \varepsilon \ll \gamma \ll 1/r$ and $T \in \mathcal{T}_r$ for some $r \geq 3$. Suppose that G is a digraph on n vertices such that*

$$\delta^0(G) \geq (1 - 1/r - \varepsilon)\, n. \tag{3}$$

Then at least one of the following properties holds:

(i) *G contains a T-packing that covers all but at most ℓ vertices;*
(ii) *G contains a γ-independent set of size at least n/r.*

Theorems 5.1 and 5.3 together ensure that a non-extremal digraph G in Theorem 1.3 contains a perfect T-packing. The following result is an analogue of Theorem 5.3 that will be applied in the proof of Theorem 1.6.

Theorem 5.4. *Let $0 < 1/n \ll 1/\ell \ll \varepsilon \ll \gamma \ll 1/r$ where $n, r \in \mathbb{N}$ and $r \geq 3$. Suppose that G is a digraph on n vertices such that, for any $x \in V(G)$,*

$$d^+(x) \geq (1 - 1/r - \varepsilon)\, n \text{ or } d^-(x) \geq (1 - 1/r - \varepsilon)\, n. \tag{4}$$

Further suppose that, given any $x, y \in V(G)$, if $d^+(x) < (1 - 1/r - \varepsilon)\, n$ and $d^-(y) < (1 - 1/r - \varepsilon)\, n$ then $xy \notin E(G)$. Then at least one of the following properties holds:

(i) *G contains a T_r-packing that covers all but at most ℓ vertices;*
(ii) *G contains a γ-independent set of size at least n/r.*

In Section 7 we deduce Theorems 5.3 and 5.4 from a result of Keevash and Mycroft [10] concerning almost perfect matchings in hypergraphs. The next two results cover the extremal cases of Theorem 1.3.

Lemma 5.5. *Let $r \in \mathbb{N}$ such that $r \geq 3$. There exist $\gamma > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that $T \in \mathcal{T}_r$ and G is a digraph on $n \geq n_0$ vertices where n is divisible by r. If*

$$\delta^0(G) \geq (1 - 1/r)\, n \tag{5}$$

and G contains a γ-independent set of size n/r then G contains a perfect T-packing.

Lemma 5.6. *There exist $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that G is a digraph on $n \geq n_0$ vertices where n is divisible by 3. If*

- *$\delta^0(G) \geq 2n/3 - 1$ and*
- *G α-contains $Ex(n)$,*

then G contains a perfect C_3-packing.

Lemmas 5.5 and 5.6 are proved in Sections 11 and 12 respectively. We now deduce Theorem 1.3 from Theorems 5.1 and 5.3 and Lemmas 5.5 and 5.6.

Proof of Theorem 1.3. Define constants $\varepsilon, \xi, \gamma, \alpha$ and integers n_0, ℓ such that

$$0 < 1/n_0 \ll 1/\ell \ll \varepsilon \ll \xi \ll \gamma, \alpha \ll 1/r.$$

Let $T \in \mathcal{T}_r$ and suppose that G is a digraph on $n \geq n_0$ vertices such that r divides n and $\delta^0(G) \geq (1 - 1/r)n$. By Lemmas 5.5 and 5.6 we may assume that

(i) G does not contain any γ-independent set of size n/r;

(ii) If $T = C_3$ then G does not α-contain $Ex(n)$.

(Otherwise G contains a perfect T-packing, as desired.) Thus, we can apply Theorem 5.1 to obtain a set $M \subseteq V(G)$ so that $|M| \leq \xi n$ and M is a T-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in r\mathbb{N}$ and $|W| \leq \xi^2 n$. Set $G' := G \setminus M$ and let $n' := |G'| \geq (1 - \xi)n$. Since n is divisible by r and M is a T-absorbing set, n' is also divisible by r. Further,

$$\delta^0(G') \geq (1 - 1/r)n - \xi n \geq (1 - 1/r - \xi)n'.$$

Notice that G' does not contain any $\gamma/2$-independent set of size at least n'/r. (Otherwise G contains a γ-independent set of size n/r, a contradiction to (i).) Therefore, by applying Theorem 5.3 with $G', n', \xi, \gamma/2$ playing the roles of $G, n, \varepsilon, \gamma$, we obtain a T-packing $\mathcal{M}_1$ in G' that covers all but at most ℓ vertices. Let W denote the set of vertices in G' that are not covered by $\mathcal{M}_1$. So $|W| \leq \ell \leq \xi^2 n$ and, since n' is divisible by r, $|W| \in r\mathbb{N}$. Thus, by definition of M, $G[M \cup W]$ contains a perfect T-packing $\mathcal{M}_2$. Therefore, $\mathcal{M}_1 \cup \mathcal{M}_2$ is a perfect T-packing in G, as desired. $\square$

Similarly we deduce Theorem 1.6 from Theorems 5.2 and 5.4.

Proof of Theorem 1.6. Define additional constants ε, ξ, γ and integers n_0, ℓ such that

$$0 < 1/n_0 \ll 1/\ell \ll \varepsilon \ll \xi \ll \gamma \ll 1/r, \eta.$$

Suppose that G is a digraph on $n \geq n_0$ vertices where r divides n and:

(i) For any $x \in V(G)$, $d^+(x) \geq (1 - 1/r + \eta)\, n$ or $d^-(x) \geq (1 - 1/r + \eta)\, n$.

Suppose that for some $x, y \in V(G)$, $d^+(x) < (1 - 1/r + \eta)n$, $d^-(y) < (1 - 1/r + \eta)n$ and $xy \in E(G)$. Then if we remove the edge xy from G, (i) still holds. In particular, this implies that we may assume:

(ii) Given any $x, y \in V(G)$, if $d^+(x) < (1 - 1/r + \eta)\, n$ and $d^-(y) < (1 - 1/r + \eta)\, n$ then $xy \notin E(G)$.

Note that (i) implies that:

(iii) G does not contain any γ-independent set of size n/r.

Apply Theorem 5.2 to obtain a set $M \subseteq V(G)$ so that $|M| \leq \xi n$ and M is a T_r-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in r\mathbb{N}$ and $|W| \leq \xi^2 n$. Set $G' := G \setminus M$ and let $n' := |G'| \geq (1 - \xi)n$. Since n is divisible by r and M is a T_r-absorbing set, n' is also divisible by r. Further, (i) implies that for any $x \in V(G')$,

$$d^+_{G'}(x) \geq \left(1 - \frac{1}{r} - \varepsilon\right) n' \quad \text{or} \quad d^-_{G'}(x) \geq \left(1 - \frac{1}{r} - \varepsilon\right) n'.$$

Suppose that for some $x, y \in V(G')$, $d^+_{G'}(x) < (1 - 1/r - \varepsilon)n'$ and $d^-_{G'}(y) < (1 - 1/r - \varepsilon)n'$. Then $d^+_G(x) < (1 - 1/r - \varepsilon)n' + \xi n \leq (1 - 1/r + \eta)n$ and $d^-_G(y) < (1 - 1/r + \eta)n$. Thus, by (ii), $xy \notin E(G')$. Notice that G' does not contain any $\gamma/2$-independent set of size at least n'/r. (Otherwise G contains a γ-independent set of size n/r, a contradiction to (iii).) Therefore, by applying Theorem 5.4 with $G', n', \gamma/2$ playing the roles of G, n, γ, we obtain a T_r-packing $\mathcal{M}_1$ in G'

that covers all but at most ℓ vertices. Let W denote the set of vertices in G' that are not covered by $\mathcal{M}_1$. So $|W| \leq \ell \leq \xi^2 n$ and, since n' is divisible by r, $|W| \in r\mathbb{N}$. Thus, by definition of M, $G[M \cup W]$ contains a perfect T_r-packing $\mathcal{M}_2$. Hence, $\mathcal{M}_1 \cup \mathcal{M}_2$ is a perfect T_r-packing in G, as desired. □

Suppose that G is a digraph on n vertices that satisfies (1). Suppose that for some $x, y \in V(G)$, $d^+(x) < (1-1/r)n$, $d^-(y) < (1-1/r)n$ and $xy \in E(G)$. Then if we remove the edge xy from G, (1) still holds. Thus, to prove Conjecture 1.4 it suffices to consider digraphs G with the following additional assumption: Given any $x, y \in V(G)$, if $d^+(x) < (1-1/r)\,n$ and $d^-(y) < (1-1/r)\,n$ then $xy \notin E(G)$. The next result states that such a digraph G contains a perfect T_r-packing or contains an 'almost' independent set of size n/r.

Theorem 5.7. *Given any $\gamma > 0$ and an integer $r \geq 3$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that G is a digraph on $n \geq n_0$ vertices where r divides n and that, for any $x \in V(G)$,*

$$d^+(x) \geq (1-1/r)\,n \quad \text{or} \quad d^-(x) \geq (1-1/r)\,n.$$

Further suppose that, given any $x, y \in V(G)$, if $d^+(x) < (1-1/r)\,n$ and $d^-(y) < (1-1/r)\,n$ then $xy \notin E(G)$. Then at least one of the following properties holds:

(i) *G contains a perfect T_r-packing;*
(ii) *G contains a γ-independent set of size at least n/r.*

Proof. The proof is almost identical to that of Theorem 1.6 so we omit it. □

So Theorem 5.7 implies that to prove Conjecture 1.4 for large digraphs it suffices to prove the extremal case.

6. Turán-type stability results for embedding tournaments

6.1. **The Turán result for Theorem 1.3.** The aim of this subsection is to prove Proposition 6.4 which, roughly speaking, states that a digraph G on n vertices of sufficiently large semidegree (i) contains many copies of a fixed $T \in \mathcal{T}_r$ or (ii) contains an 'almost' independent set of size n/r. Proposition 6.4 will be applied in the proof of both Theorem 5.1 and Theorem 5.3.

The next result is an immediate consequence of Turán's theorem.

Proposition 6.1. *Let $n, r \in \mathbb{N}$ where $r \geq 2$. Suppose that G is a digraph on n vertices such that*

$$e(G) > \left(1 - \frac{1}{r-1}\right)\frac{n^2}{2} + \binom{n}{2}. \tag{6}$$

Then G contains a copy of K_r.

Proof. Let G' be the graph on $V(G)$ whose edge set consists of all pairs xy where $xy, yx \in E(G)$. Then (6) implies that $e(G') > (1-1/(r-1))n^2/2$ and thus G' contains a copy of K_r by Turán's theorem. Hence, $K_r \subseteq G$ as required. □

Proposition 6.2. *Let $1/n \ll \alpha \ll 1/r$ with $n, r \in \mathbb{N}$ and $r \geq 3$, and let $T \in \mathcal{T}_r$. Suppose that G is a digraph on n vertices such that*

$$\delta^0(G) \geq \left(1 - \frac{1}{r-1} - \alpha\right)n. \tag{7}$$

If G is T-free then G contains an independent set of size at least $\left(\frac{1}{r-1} - 2r^2\alpha\right)n$.

Proof. Let $V(T) = \{v_1, \dots, v_{r-2}, a, b\}$ and set $T' := T[v_1, \dots, v_{r-2}]$. Using (7), greedily construct a copy T'' of T' in G. To simplify notation, for each $1 \leq i \leq r-2$, we will refer to the vertex in T'' (and thus G) corresponding to the vertex v_i in T' as v_i.

We say that a vertex $v \in V(G)$ is a *candidate for a in G* if the following conditions hold:

- If $av_i \in E(T)$ then $vv_i \in E(G)$ (for each $1 \leq i \leq r-2$);
- If $v_i a \in E(T)$ then $v_i v \in E(G)$ (for each $1 \leq i \leq r-2$).

We give an analogous definition of a *candidate for b in G*. Let A denote the set of candidates for a in G and let B denote the set of candidates for b in G. Thus, (7) implies that

$$|A|, |B| \geq \left(\frac{1}{r-1} - (r-2)\alpha \right) n. \tag{8}$$

Without loss of generality, suppose that $ab \in E(T)$. Since G is T-free, there is no edge in G whose startpoint lies in A and whose endpoint lies in B. In particular, $A \cap B$ is an independent set.

Set $A' := A \setminus B$. Suppose for a contradiction that $|A'| \geq 2(r-1)^2 \alpha n$. Given any vertex $x \in A'$, since x sends no edges to B, (7) and (8) imply that there are at most

$$\left(\frac{1}{r-1} + \alpha \right) n - \left(\frac{1}{r-1} - (r-2)\alpha \right) n = (r-1)\alpha n$$

vertices in A' that x does not send an edge to (including itself). Thus,

$$\delta^+(G[A']) \geq |A'| - (r-1)\alpha n \geq \left(1 - \frac{1}{2(r-1)} \right) |A'|$$

and so

$$e(G[A']) \geq \left(1 - \frac{1}{2(r-1)} \right) |A'|^2 > \left(1 - \frac{1}{r-1} \right) \frac{|A'|^2}{2} + \binom{|A'|}{2}.$$

Hence, Proposition 6.1 implies that $K_r \subseteq G[A']$ and so G contains a copy of T, a contradiction. Therefore, $|A'| < 2(r-1)^2 \alpha n$. Together with (8) this implies that the independent set $A \cap B$ is of size at least $\left(\frac{1}{r-1} - (r-2)\alpha \right) n - 2(r-1)^2 \alpha n \geq \left(\frac{1}{r-1} - 2r^2 \alpha \right) n$, as required. $\square$

To prove Proposition 6.4 we will apply Proposition 6.2 together with the following directed version of the Removal lemma (see e.g. [1, 6]).

Lemma 6.3 (Directed Graph Removal lemma). *Let $\gamma > 0$ and $t \in \mathbb{N}$. Given any digraph H on t vertices, there exists $\alpha = \alpha(H, \gamma) > 0$ and $n_0 = n_0(H, \gamma) \in \mathbb{N}$ such that the following holds. Suppose that G is a digraph on $n \geq n_0$ vertices such that G contains at most αn^t copies of H. Then G can be made H-free by deleting at most γn^2 edges.*

Proposition 6.4. *Let $0 < 1/n \ll \alpha \ll \varepsilon \ll 1/r$ where $r, n \in \mathbb{N}$ and $r \geq 2$, and let $T \in \mathcal{T}_r$. Suppose that G is a digraph on n vertices such that*

$$\delta^0(G) \geq \left(1 - \frac{1}{r-1} - \varepsilon \right) n$$

and so that G contains at most αn^r copies of T. Then G contains a $\sqrt{\varepsilon}$-independent set of size at least $n/(r-1)$.

Proof. The case when $r = 2$ is trivial thus we may assume that $r \geq 3$. Define an additional constant γ so that $\alpha \ll \gamma \ll \varepsilon$. Suppose that G is as in the statement of the proposition. Since G contains at most αn^r copies of T, Lemma 6.3 implies that one can remove at most γn^2 edges from G to obtain a spanning subdigraph G' that is T-free. So at most $\sqrt{\gamma} n$ vertices in G are incident to more than $2\sqrt{\gamma} n$ of the edges in $G - G'$. Therefore, since $\gamma \ll \varepsilon$, there exists an induced subdigraph G'' of G' such that $n'' := |G''| \geq (1-\varepsilon)n$ and $\delta^0(G'') \geq (1 - 1/(r-1) - 2\varepsilon)n''$.

Since G'' is T-free, Proposition 6.2 implies that G'' contains an independent set S of size at least

$$\left(\frac{1}{r-1} - 4r^2\varepsilon\right) n'' \geq \left(\frac{1}{r-1} - 5r^2\varepsilon\right) n.$$

By construction of G'', S is a γ-independent set in G. By adding at most $5r^2\varepsilon n$ arbitrary vertices to S we obtain a $\sqrt{\varepsilon}$-independent set in G of size at least $n/(r-1)$, as desired. $\square$

6.2. **The Turán result for Theorem 1.6.** In this section we give an analogue of Proposition 6.4 which will be applied in the proof of both Theorem 5.2 and Theorem 5.4. The next result is an analogue of Proposition 6.2.

Proposition 6.5. *Let $1/n \ll \alpha \ll 1/r$ where $n, r \in \mathbb{N}$ and $r \geq 3$. Suppose that G is a digraph on n vertices such that, for any $x \in V(G)$,*

$$d^+(x) \geq \left(1 - \frac{1}{r-1} - \alpha\right) n \text{ or } d^-(x) \geq \left(1 - \frac{1}{r-1} - \alpha\right) n. \tag{9}$$

If G is T_r-free then G contains an independent set of size at least $\left(\frac{1}{r-1} - r\alpha\right) n$.

Proof. Let $r' \in \mathbb{N}$. Suppose that T' is a copy of $T_{r'}$ in G. Let $V(T') = \{x_1, \dots, x_{r'}\}$ where x_i plays the role of the ith vertex of $T_{r'}$. We say that T' is *consistent* if there exists $0 \leq s' \leq r'$ such that

- $d^+(x_i) \geq (1 - 1/(r-1) - \alpha)n$ for all $i \leq s'$;
- $d^-(x_i) \geq (1 - 1/(r-1) - \alpha)n$ for all $i > s'$.

We call s' a *turning point of* T'. (Note that T' could have more than one turning point.)

(9) implies that every copy of T_1 in G is consistent. Suppose that, for some $1 \leq r' < r-2$, we have found a consistent copy T' of $T_{r'}$ in G. As before, let $V(T') = \{x_1, \dots, x_{r'}\}$ where x_i plays the role of the ith vertex of $T_{r'}$ and let s' denote a turning point of T'. Set $N' := \bigcap_{i \leq s'} N^+(x_i) \cap \bigcap_{i > s'} N^-(x_i)$. Since T' is consistent with turning point s' and $r' < r-2$,

$$|N'| \geq \left(1 - \frac{r'}{r-1} - r'\alpha\right) n > 0.$$

Consider any $x \in N'$. Then $V(T') \cup \{x\}$ spans a consistent copy of $T_{r'+1}$ in G where x plays the role of the $(s'+1)$th vertex in $T_{r'+1}$. (This is true regardless of whether $d^+(x) \geq (1 - 1/(r-1) - \alpha)n$ or $d^-(x) \geq (1 - 1/(r-1) - \alpha)n$.)

This observation implies that we can greedily construct a consistent copy T of T_{r-2} in G. Let $V(T) = \{y_1, \dots, y_{r-2}\}$ where y_i plays the role of the ith vertex of T_{r-2} and let s denote a turning point of T. Set $N := \bigcap_{i \leq s} N^+(y_i) \cap \bigcap_{i > s} N^-(y_i)$. Since T is consistent with turning point s,

$$|N| \geq \left(1 - \frac{r-2}{r-1} - (r-2)\alpha\right) n \geq \left(\frac{1}{r-1} - r\alpha\right) n.$$

Suppose that there is an edge $xy \in E(G[N])$. Then $V(T) \cup \{x, y\}$ spans a copy of T_r in G where x and y play the roles of the $(s+1)$th and $(s+2)$th vertices in T_r respectively. This is a contradiction, so N is an independent set in G, as required. $\square$

Proposition 6.6. *Let $0 < 1/n \ll \alpha \ll \varepsilon \ll 1/r$ where $n, r \in \mathbb{N}$ and $r \geq 2$. Suppose that G is a digraph on n vertices such that, for any $x \in V(G)$,*

$$d^+(x) \geq \left(1 - \frac{1}{r-1} - \varepsilon\right) n \text{ or } d^-(x) \geq \left(1 - \frac{1}{r-1} - \varepsilon\right) n.$$

Further suppose that G contains at most αn^r copies of T_r. Then G contains a $\sqrt{\varepsilon}$-independent set of size at least $n/(r-1)$.

Proof. We follow the same argument as in the proof of Proposition 6.4 except that we apply Proposition 6.5 rather than Proposition 6.2. □

7. k-complexes and almost perfect tournament packings

The key tool in the proofs of Theorems 5.3 and 5.4 is a result of Keevash and Mycroft [10, Theorem 2.3] concerning almost perfect matchings in so-called k-complexes. To state this result we require some more definitions. Let $k \in \mathbb{N}$. A *k-system* is a hypergraph J in which every edge of J contains at most k vertices and $\emptyset \in E(J)$. For $0 \leq i \leq k$, we refer to the edges of size i in J as the *i-edges of J*, and write J_i to denote the i-uniform hypergraph on $V(J)$ induced by these edges. A *k-complex* J is a k-system whose edge set is closed under inclusion. That is, if $e \in E(H)$ and $e' \subseteq e$ then $e' \in E(H)$.

Let J be a k-complex. For any edge $e \in E(J)$, the *degree $d(e)$ of e* is the number of $(|e|+1)$-edges e' of J that contain e as a subset. The *minimum r-degree $\delta_r(J)$* of J is the minimum of $d(e)$ taken over all r-edges $e \in E(J)$. The *degree sequence of J* is defined as $\delta(J) := (\delta_0(J), \delta_1(J), \dots, \delta_{k-1}(J))$. Given a vector $\underline{a} = (a_0, \dots, a_{k-1})$ of positive integers we write $\delta(J) \geq \underline{a}$ to mean that $\delta_i(J) \geq a_i$ for all $0 \leq i \leq k-1$.

Suppose V is a set of n vertices, $1 \leq j \leq k-1$ and $S \subseteq V$. Define $J(S, j)$ to be the k-complex on V in which $J(S, j)_i$ (for $0 \leq i \leq k$) consists of all i-sets in V that contain at most j vertices of S. Let $\beta > 0$. Given k-uniform hypergraphs H, K on the same vertex set of size n we say that K is *β-contained in H* if, by adding at most βn^k edges to H, we can find a copy of K in H. A *matching* in a hypergraph H is a collection of vertex-disjoint edges from H.

Theorem 7.1 (Keevash and Mycroft [10])**.** *Suppose that $1/n \ll 1/\ell \ll \varepsilon \ll \beta \ll 1/k$. Let J be a k-complex on n vertices such that*

$$\delta(J) \geq \left(n, \left(1 - \frac{1}{k} - \varepsilon\right) n, \left(1 - \frac{2}{k} - \varepsilon\right) n, \dots, \left(\frac{1}{k} - \varepsilon\right) n\right).$$

Then at least one of the following properties holds:

(i) *J_k contains a matching that covers all but at most ℓ vertices;*
(ii) *J_k is β-contained in $J(S, j)_k$ for some $1 \leq j \leq k-1$ and $S \subseteq V(J)$ with $|S| = \lfloor jn/k \rfloor$.*

In the following two subsections we apply Theorem 7.1 to prove both Theorem 5.3 and Theorem 5.4.

7.1. Proof of Theorem 5.3.

Define additional constants $\beta, \alpha, \alpha', \varepsilon'$ such that

$$0 < 1/n \ll 1/\ell \ll \varepsilon \ll \beta \ll \alpha \ll \alpha' \ll \varepsilon' \ll \gamma \ll 1/r.$$

Let G be a digraph as in the statement of the theorem.

Our first task is to construct an r-complex J from G so that we can apply Theorem 7.1. Let J be the r-system on $V(G)$ where, for each $0 \leq i \leq r$, J_i is defined as follows:

- For each subtournament T' of T on i vertices, any i-tuple in $V(G)$ that spans a copy of T' in G forms an i-edge in J_i.

So for example, if $T = C_3$, then $E(J_1) = V(J_1)$, $E(J_2)$ is the set of all pairs $\{x, y\}$ where $xy \in E(G)$ or $yx \in E(G)$ and $E(J_3)$ is the set of all triples $\{x, y, z\}$ that span a copy of C_3 in G. (Note though that $\{x, y, z\}$ does not have to *induce* a copy of C_3 in G. For example, we could have $G[x, y, z] = K_3$.)

By construction J is an r-complex. Further, notice that a matching in the r-uniform hypergraph J_r corresponds to a T-packing in G. Clearly $\delta_0(J) = n$. Set $1 \leq i \leq r-1$ and let T' be a subtournament of T on i vertices. If T'' is a copy of T' in G then (3) implies that there are at least $(1 - i/r - i\varepsilon)n \geq (1 - i/r - \varepsilon r)n$ vertices x in G such that $V(T'') \cup \{x\}$ spans a copy of a subtournament of T on $i+1$ vertices. This therefore implies that,

$$\delta(J) \geq \left(n, \left(1 - \frac{1}{r} - \varepsilon r\right)n, \left(1 - \frac{2}{r} - \varepsilon r\right)n, \ldots, \left(\frac{1}{r} - \varepsilon r\right)n\right).$$

Hence, we can apply Theorem 7.1 with $r, \varepsilon r$ playing the roles of k, ε. So at least one of the following conditions holds:

(a) J_r contains a matching that covers all but at most ℓ vertices;
(b) J_r is β-contained in $J(S, j)_r$ for some $1 \leq j \leq r-1$ and $S \subseteq V(J) = V(G)$ with $|S| = \lfloor jn/r \rfloor$.

If (a) holds then this implies that (i) is satisfied. So we may assume that (b) holds. We will show that this implies that (ii) is satisfied.

Let j be as in (b) and consider an arbitrary subtournament T' of T on $j+1$ vertices. Suppose for a contradiction that there are at least αn^{j+1} $(j+1)$-tuples in S that span a copy T'' of T' in G. If $j = r-1$ then $T' = T$ and so this implies that J_r contains at least $\alpha n^{j+1} = \alpha n^r > \beta n^r$ edges that lie in S. This is a contradiction as J_r is β-contained in $J(S, j)_r$. So suppose that $j < r-1$. Then (3) implies that, for each copy T'' of T' in G, there are at least

$$\begin{aligned}
&\frac{1}{(r-j-1)!}\left(1 - \frac{j+1}{r} - (j+1)\varepsilon\right)n \times \left(1 - \frac{j+2}{r} - (j+2)\varepsilon\right)n \times \cdots \times \left(\frac{1}{r} - (r-1)\varepsilon\right)n \\
&\geq \frac{1}{(r-j-1)!} \times \frac{1}{2r^r}n^{r-j-1} \geq \frac{1}{2r^{2r}}n^{r-j-1}
\end{aligned}$$

$(r-j-1)$-tuples X in $V(G)$ such that $V(T'') \cup X$ spans a copy of T in G. Since $\beta \ll \alpha, 1/r$, this implies that there are at least

$$\frac{1}{\binom{r}{j+1}} \times \alpha n^{j+1} \times \frac{1}{2r^{2r}}n^{r-j-1} > \beta n^r$$

r-tuples in $V(G)$ that span a copy of T and which contain at least $j+1$ vertices from S. So J_r contains more than βn^r edges that contain at least $j+1$ vertices from S. This is a contradiction as J_r is β-contained in $J(S, j)_r$.

So there are at most αn^{j+1} $(j+1)$-tuples in S that span a copy of T' in G. Thus, since any $(j+1)$-tuple of vertices spans at most 2^{j+1} copies of T', there are at most $2^{j+1}\alpha n^{j+1} \leq 2^r \alpha n^{j+1} \leq \alpha' |S|^{j+1}$ copies of T' in $G[S]$. Further, since $|S| = \lfloor jn/r \rfloor$, (3) implies that

$$\delta^0(G[S]) \geq |S| - \frac{n}{r} - \varepsilon n \geq \left(1 - \frac{1}{j} - \varepsilon'\right)|S|.$$

Apply Proposition 6.4 with $G[S], T', j+1, \varepsilon', \alpha'$ playing the roles of $G, T, r, \varepsilon, \alpha$. This implies that $G[S]$ contains a $\sqrt{\varepsilon'}$-independent set of size at least $|S|/j \geq n/r - 1$. Since $\sqrt{\varepsilon'}|S|^2 \ll \gamma n^2$, this implies that G contains a γ-independent set of size at least n/r, as required. □

7.2. **Proof of Theorem 5.4.** Define additional constants $\beta, \alpha, \alpha', \varepsilon'$ such that

$$0 < 1/n \ll 1/\ell \ll \varepsilon \ll \beta \ll \alpha \ll \alpha' \ll \varepsilon' \ll \gamma \ll 1/r.$$

Let G be a digraph as in the statement of the theorem.

Let $r' \in \mathbb{N}$ and suppose that T' is a copy of $T_{r'}$ in G. Let $V(T') = \{x_1, \dots, x_{r'}\}$ where x_i plays the role of the ith vertex of $T_{r'}$. We say that T' is *consistent* if there exists $0 \leq s \leq r'$ such that

- $d^+(x_i) \geq (1 - 1/r - \varepsilon)n$ for all $i \leq s$;
- $d^-(x_i) \geq (1 - 1/r - \varepsilon)n$ for all $i > s$.

We call s a *turning point of* T'. (T' could have more than one turning point.)

(4) implies that every copy of T_1 in G is consistent. Suppose, for a contradiction, that T is a copy of $T_{r'}$ in G that is not consistent (for some $r' \geq 2$). Let y_i denote the vertex in T that plays the role of the ith vertex of $T_{r'}$. Let k be the smallest positive integer such that $d^+(y_k) < (1 - 1/r - \varepsilon)n$ (and so $d^-(y_k) \geq (1 - 1/r - \varepsilon)n$ by (4)); such an integer exists else T is consistent with turning point r'. Then there exists $k' > k$ such that $d^-(y_{k'}) < (1 - 1/r - \varepsilon)n$ (otherwise T is consistent with turning point $k - 1$). But then $y_k y_{k'} \in E(T) \subseteq E(G)$ where $d^+(y_k) < (1 - 1/r - \varepsilon)n$ and $d^-(y_{k'}) < (1 - 1/r - \varepsilon)n$. This is a contradiction to the hypothesis of the theorem. Thus, every transitive tournament in G is consistent.

Let J be the r-system on $V(G)$ where, for each $0 \leq i \leq r$, J_i is defined as follows:

- Any i-tuple in $V(G)$ that spans a copy of T_i in G forms an i-edge in J_i.

By construction J is an r-complex. Further, a matching in J_r corresponds to a T_r-packing in G. Clearly $\delta_0(J) = n$. Suppose that T is a (consistent) copy of T_i in G for some $1 \leq i \leq r - 1$ and let s denote a turning point of T. As before, let y_k denote the vertex in T that plays the role of the kth vertex in T_i. Set $N := \bigcap_{k \leq s} N^+(y_k) \cap \bigcap_{k>s} N^-(y_k)$. Since T is consistent with turning point s,

$$|N| \geq \left(1 - \frac{i}{r} - i\varepsilon\right) n \geq \left(1 - \frac{i}{r} - r\varepsilon\right) n.$$

Further, given any $x \in N$, $V(T) \cup \{x\}$ spans a copy of T_{i+1} in G. So $V(T) \cup \{x\}$ is an edge in J. This implies that

$$\delta(J) \geq \left(n, \left(1 - \frac{1}{r} - \varepsilon r\right) n, \left(1 - \frac{2}{r} - \varepsilon r\right) n, \dots, \left(\frac{1}{r} - \varepsilon r\right) n\right).$$

Hence, we can apply Theorem 7.1 with $r, \varepsilon r$ playing the roles of k, ε. So at least one of the following conditions holds:

(a) J_r contains a matching that covers all but at most ℓ vertices;
(b) J_r is β-contained in $J(S, j)_r$ for some $1 \leq j \leq r-1$ and $S \subseteq V(J) = V(G)$ with $|S| = \lfloor jn/r \rfloor$.

If (a) holds then (i) is satisfied. So we may assume that (b) holds. We will show that this implies that (ii) is satisfied.

The argument now closely follows the proof of Theorem 5.3. Indeed, let j be as in (b). Suppose for a contradiction that there are at least αn^{j+1} $(j+1)$-tuples in S that span a copy T of T_{j+1} in G. If $j = r - 1$ then $T_{j+1} = T_r$ and so this implies that J_r contains at least $\alpha n^{j+1} = \alpha n^r > \beta n^r$ edges that lie in S. This is a contradiction as J_r is β-contained in $J(S, j)_r$. So suppose that $j < r - 1$.

Recall that every copy T of T_{j+1} in G is consistent. This implies that there are at least $(1 - \frac{j+1}{r} - (j+1)\varepsilon)n$ vertices $x \in V(G)$ such that $V(T) \cup \{x\}$ spans a copy of T_{j+2} in G. Repeating this process we see that, for every copy T of T_{j+1} in G, there are at least

$$\frac{1}{(r-j-1)!}\left(1-\frac{j+1}{r}-(j+1)\varepsilon\right)n \times \left(1-\frac{j+2}{r}-(j+2)\varepsilon\right)n \times \cdots \times \left(\frac{1}{r}-(r-1)\varepsilon\right)n$$
$$\geq \frac{1}{(r-j-1)!} \times \frac{1}{2r^r}n^{r-j-1} \geq \frac{1}{2r^{2r}}n^{r-j-1}$$

$(r-j-1)$-tuples X in $V(G)$ such that $V(T) \cup X$ spans a copy of T_r in G. Since $\beta \ll \alpha, 1/r$, this implies that there are at least

$$\frac{1}{\binom{r}{j+1}} \times \alpha n^{j+1} \times \frac{1}{2r^{2r}}n^{r-j-1} > \beta n^r$$

r-tuples in $V(G)$ that span a copy of T_r and which contain at least $j+1$ vertices from S. So J_r contains more than βn^r edges that contain at least $j+1$ vertices from S. This is a contradiction as J_r is β-contained in $J(S,j)_r$.

Thus, there are at most αn^{j+1} $(j+1)$-tuples in S that span a copy of T_{j+1} in G. Since any $(j+1)$-tuple of vertices spans at most 2^{j+1} copies of T_{j+1}, there are at most $2^{j+1}\alpha n^{j+1} \leq 2^r \alpha n^{j+1} \leq \alpha' |S|^{j+1}$ copies of T_{j+1} in $G[S]$. Further, since $|S| = \lfloor jn/r \rfloor$, (4) implies that, for every $x \in V(G[S])$,

$$d^+_{G[S]}(x) \geq |S| - \frac{n}{r} - \varepsilon n \geq \left(1 - \frac{1}{j} - \varepsilon'\right)|S| \text{ or } d^-_{G[S]}(x) \geq \left(1 - \frac{1}{j} - \varepsilon'\right)|S|.$$

Apply Proposition 6.6 with $G[S], j+1, \varepsilon', \alpha'$ playing the roles of $G, r, \varepsilon, \alpha$. This implies that $G[S]$ contains a $\sqrt{\varepsilon'}$-independent set of size at least $|S|/j \geq n/r - 1$. Since $\sqrt{\varepsilon'}|S|^2 \ll \gamma n^2$, this implies that G contains a γ-independent set of size at least n/r, as required.

□

8. The connection lemmas

In Section 9 we prove Theorems 5.1 and 5.2. Roughly speaking, in both these theorems, when our digraph G is non-extremal we require a 'small' T-absorbing set in G that absorbs *any* 'very small' set of vertices in G (for some $T \in \mathcal{T}_r$).

The crucial idea in finding such an absorbing set is to first prove that our digraph G has many 'connecting structures' of a certain type. More precisely, to find our desired absorbing set it suffices to show that, for any $x, y \in V(G)$, there are 'many' $(r-1)$-sets $X \subseteq V(G)$ so that both $X \cup \{x\}$ and $X \cup \{y\}$ span copies of T in G. Our main task therefore is to prove so-called 'connection lemmas' that guarantee such sets X. In the case when $T = C_3$ though, we may not be able to find such sets. However, in this case we instead find, for any $x, y \in V(G)$, 'many' 5-sets $X \subseteq V(G)$ so that both $X \cup \{x\}$ and $X \cup \{y\}$ span copies of $2T$ in G.

The approach of proving connection lemmas to find absorbing structures for packing problems has been very fruitful. Indeed, Lo and Markström [18, 19] used such connection lemmas to tackle perfect packing problems for hypergraphs (although their terminology for these 'connecting structures' differs from ours). Further, recently the author [23] applied this method to prove a degree sequence version of the Hajnal–Szemerédi theorem.

A single connection lemma (Lemma 8.1) for Theorem 5.2 is given in Section 8.1. However, for Theorem 5.1, we need to prove a number of separate connection lemmas. In Section 8.3 we prove the connection lemma for Theorem 5.1 for those $T \in \mathcal{T}_r$ where $r \geq 5$ (see Lemma 8.4). The proof relies on T containing T_3 as a subtournament. (So this method certainly cannot be applied in the case when $T = C_3$.) It is easy to see that all tournaments on at least four vertices contain T_3. However for $T \in \mathcal{T}_4$, the minimum semidegree condition on our digraph G is not high enough for the proof method of Lemma 8.4 to go through. Thus, we use a different approach to prove the connection lemma in this case (see Section 8.2). This method makes use of a simple structural property of tournaments on four vertices (see Fact 8.2). The case when $T = T_3$ is covered by Lemma 8.1.

Finally, we need a separate connection lemma for when $T = C_3$ (see Section 8.4). Of all the connection lemmas, this one has the most involved proof. This stems from the fact that we now have two extremal cases. Thus, to find our connecting structures in a non-extremal digraph G on n vertices we must use both the property that G does not contain an 'almost' independent set of size $n/3$ *and* that G does not α-contain $Ex(n)$.

8.1. **The connection lemma for Theorem 5.2.** The following connection lemma is a straightforward consequence of Proposition 6.6.

Lemma 8.1. *Let $0 < 1/n \ll \varepsilon \ll \eta \ll \gamma \ll 1/r$ where $n, r \in \mathbb{N}$ and $r \geq 3$. Suppose that G is a digraph on n vertices such that, for any $z \in V(G)$,*

$$d^+(z) \geq (1 - 1/r - \varepsilon)\, n \text{ or } d^-(z) \geq (1 - 1/r - \varepsilon)\, n. \tag{10}$$

Further, suppose that G does not contain a γ-independent set on at least n/r vertices. Given any $x, y \in V(G)$, there exist at least ηn^{r-1} $(r-1)$-sets $X \subseteq V(G)$ such that both $X \cup \{x\}$ and $X \cup \{y\}$ span copies of T_r in G.

Proof. Let $x, y \in V(G)$. Suppose that $d^+(x) \geq (1 - 1/r - \varepsilon)n$ and $d^-(y) \geq (1 - 1/r - \varepsilon)n$. (The other cases are identical.) Set $S := N^+(x) \cap N^-(y)$. So $|S| \geq (1 - 2/r - 2\varepsilon)n = \frac{r-2-2\varepsilon r}{r} n$. This implies that,

$$|S| - \frac{n}{r} - \varepsilon n \geq |S| - \frac{|S|}{r-2-2\varepsilon r} - 2\varepsilon r|S| \geq |S| - \frac{|S|}{r-2} - 3\varepsilon r|S| - 2\varepsilon r|S| \geq \left(1 - \frac{1}{r-2} - \frac{\gamma^2}{4}\right)|S|.$$

Together with (10) this implies that, for any $z \in S$,

$$d^+_{G[S]}(z) \geq \left(1 - \frac{1}{r-2} - \frac{\gamma^2}{4}\right)|S| \text{ or } d^-_{G[S]}(z) \geq \left(1 - \frac{1}{r-2} - \frac{\gamma^2}{4}\right)|S|.$$

Suppose for a contradiction that $G[S]$ contains at most $2^{3r-3}\eta|S|^{r-1}$ copies of T_{r-1}. Note that $1/|S| \ll 2^{3r-3}\eta \ll \gamma^2/4 \ll 1/(r-1)$. Hence, applying Proposition 6.6 with $G[S], r-1, 2^{3r-3}\eta, \gamma^2/4$ playing the roles of G, r, α and ε respectively, we obtain a $\gamma/2$-independent set S' in $G[S]$ of size at least $|S|/(r-2)$. Note that

$$\frac{|S|}{r-2} \geq \frac{(r-2-2\varepsilon r)}{r(r-2)} n \geq \frac{n}{r} - 2\varepsilon n.$$

Therefore, S' is a $\gamma/2$-independent set in G of size at least $n/r - 2\varepsilon n$. By adding at most $2\varepsilon n$ arbitrary vertices to S', we obtain a γ-independent set in G of size at least n/r, a contradiction.

Thus, there are at least $2^{3r-3}\eta|S|^{r-1} \geq 2^{r-1}\eta n^{r-1}$ copies of T_{r-1} in $G[S]$. (The inequality here follows since $|S| \geq n/4$.) Any $(r-1)$-set in S spans at most 2^{r-1} different copies of T_{r-1} in $G[S]$. So there are at least ηn^{r-1} $(r-1)$-sets $X \subseteq S$ that span a copy of T_{r-1} in $G[S]$. By definition of S, for each such set X, both $X \cup \{x\}$ and $X \cup \{y\}$ span copies of T_r in G, as required. □

8.2. **The connection lemma for tournaments on four vertices.** The following simple fact will be used in the proof of the connection lemma for tournaments on four vertices (Lemma 8.3).

Fact 8.2. *If $T \in \mathcal{T}_4$ then there is a subset $S \subseteq V(T)$ with $|S| \in \{1, 2\}$ and such that given any $s \in S$, either*

- *$ss' \in E(T)$ for all $s' \in V(T) \setminus S$ or*
- *$s's \in E(T)$ for all $s' \in V(T) \setminus S$.*

Lemma 8.3. *Let $0 < 1/n \ll \varepsilon \ll \eta \ll \gamma \ll 1$ and $T \in \mathcal{T}_4$. Suppose that G is a digraph on n vertices such that*

$$\delta^0(G) \geq (3/4 - \varepsilon)n \tag{11}$$

and so that G does not contain any γ-independent set of size at least $n/4$. Then, for any $x, y \in V(G)$, there exist at least ηn^3 3-sets $X \subseteq V(G)$ such that $X \cup \{x\}$ and $X \cup \{y\}$ span copies of T in G.

Proof. By Fact 8.2, T contains a subset $S \subseteq V(T)$ with $|S| \in \{1, 2\}$ and such that given any $s \in S$, either

- $ss' \in E(T)$ for all $s' \in V(T) \setminus S$ or
- $s's \in E(T)$ for all $s' \in V(T) \setminus S$.

We divide the proof into two cases depending on $|S|$.

Case 1: $|S| = 1$
Let $V(T) = \{x_1, x_2, x_3, s\}$ where $S = \{s\}$. Consider the case when $sx_i \in E(T)$ for $i = 1, 2, 3$. (The other case, when $x_is \in E(T)$ for $i = 1, 2, 3$, is analogous.) Set $A := N^+_G(x) \cap N^+_G(y)$ and let $T' := T[x_1, x_2, x_3]$. Our aim is to find ηn^3 3-sets $X \subseteq A$ that span copies of T' in $G[A]$. Then the choice of s and A ensures that each such set X has the property that $X \cup \{x\}$ and $X \cup \{y\}$ span copies of T in G (where x and y respectively play the role of s), as desired.

By (11) we have that $|A| \geq (1/2 - 2\varepsilon)n$ and so

$$\delta^0(G[A]) \geq |A| - (1/4 + \varepsilon)n \geq |A| - (1/4 + \varepsilon)\frac{|A|}{1/2 - 2\varepsilon} \geq |A| - (1/2 + 5\varepsilon)|A| \geq (1/2 - \gamma^2/4)|A|.$$

Suppose for a contradiction that $G[A]$ contains at most $65\eta|A|^3$ copies of T'. Note that $1/|A| \ll 65\eta \ll \gamma^2/4 \ll 1/3$. Hence, applying Proposition 6.4 with $G[A], T', 3, 65\eta, \gamma^2/4$ playing the roles of G, T, r, α and ε respectively, we obtain a $\gamma/2$-independent set A' in $G[A]$ of size at least $|A|/2 \geq (1/4 - \varepsilon)n$. By adding at most εn arbitrary vertices to A', we obtain a γ-independent set in G of size at least $n/4$, a contradiction.

Thus, there are at least $65\eta|A|^3 \geq 2^3\eta n^3$ copies of T' in $G[A]$. Any 3-set in A spans at most 2^3 different copies of T' in $G[A]$. So there are at least ηn^3 3-sets $X \subseteq A$ that span a copy of T' in $G[A]$, as required.

Case 2: $|S| = 2$
Let $V(T) = \{x_1, x_2, s_1, s_2\}$ where $S = \{s_1, s_2\}$. Assume that $x_1, x_2 \in N^+_T(s_1)$ and $x_1, x_2 \in N^-_T(s_2)$. (The other cases can be dealt with analogously.) We may further assume that $s_2s_1 \in E(T)$ (otherwise, we can reset $S = \{s_1\}$ and then follow the argument from Case 1). Finally, we may assume that $x_1x_2 \in E(T)$.

For each of our desired 3-sets X, the vertices x and y will play the role of s_1 in the copy of T that spans $X \cup \{x\}$ and $X \cup \{y\}$, respectively. Let $s'_2 \in V(G)$ such that $s'_2x, s'_2y \in E(G)$. By (11) there are at least $(1/2 - 2\varepsilon)n$ choices for s'_2. Set $A := N^+_G(x) \cap N^+_G(y) \cap N^-_G(s'_2)$. Then (11) implies that

$$|A| \geq \left(\frac{1}{4} - 3\varepsilon\right)n.$$

Suppose that $G[A]$ contains at most $\gamma n^2/2$ edges. Then by adding at most $3\varepsilon n$ vertices to A, we obtain a γ-independent set in G of size at least $n/4$, a contradiction. So $G[A]$ contains at least $\gamma n^2/2$ edges.

Given any $x'_1x'_2 \in E(G[A])$, set $X := \{s'_2, x'_1, x'_2\}$. By construction, $X \cup \{x\}$ spans a copy of T in G where x, s'_2, x'_1, x'_2 play the roles of s_1, s_2, x_1 and x_2 respectively and $X \cup \{y\}$ spans a copy of T in G where y, s'_2, x'_1, x'_2 play the roles of s_1, s_2, x_1 and x_2 respectively.

Recall that there are at least $(1/2 - 2\varepsilon)n$ choices for s_2' and at least $\gamma n^2/2$ choices for $x_1'x_2'$. Overall, this implies that there are at least

$$(1/2 - 2\varepsilon)n \times \frac{\gamma n^2}{2} \times \frac{1}{3!} \geq \eta n^3$$

choices for X, as desired. □

8.3. **The connection lemma for tournaments on at least five vertices.**

Lemma 8.4. *Let $0 < 1/n \ll \varepsilon \ll \eta \ll \gamma \ll 1/r$ where $n, r \in \mathbb{N}$ and $r \geq 5$, and let $T \in \mathcal{T}_r$. Suppose that G is a digraph on n vertices such that*

$$\delta^0(G) \geq (1 - 1/r - \varepsilon)\, n \tag{12}$$

and so that G does not contain a γ-independent set on at least n/r vertices. Given any $x, y \in V(G)$, there exist at least ηn^{r-1} $(r-1)$-sets $X \subseteq V(G)$ such that both $X \cup \{x\}$ and $X \cup \{y\}$ span copies of T in G.

Proof. Define γ' such that $\eta \ll \gamma' \ll \gamma$. Since $|T| \geq 5$, T contains a copy of T_3. Let $V(T) = \{x_1, \dots, x_{r-3}, s_1, s_2, s_3\}$ where $T[s_1, s_2, s_3] = T_3$ so that $s_i s_j \in E(T)$ for $i < j$.

Consider any $x, y \in V(G)$. We now explain how we construct our desired $(r-1)$-sets X. For each such X, x will play the role of x_1 in the copy of T spanning $X \cup \{x\}$ and y will play the role of x_1 in the copy of T spanning $X \cup \{y\}$. When constructing each X we introduce a special vertex x^* that will play the role of s_1, s_2 or s_3 in the copies of T spanned by $X \cup \{x\}$ and $X \cup \{y\}$.

Let $x^* \in V(G)$ be such that $xx^*, x^*x, yx^*, x^*y \in E(G)$. By (12) there are at least $(1-4/r-4\varepsilon)n \geq n/2r$ choices for x^*. (Note that we could not guarantee such a vertex x^* exists if $r = 4$. This is the reason why we cannot generalise this proof to work for $r \geq 4$.)

Next we iteratively choose vertices $x_2', \dots, x_{r-3}' \in V(G)$ such that, for each $2 \leq i \leq r-3$, the following conditions hold:

- (a_i) $x^* x_i', x_i' x^* \in E(G)$;
- (b_i) $xx_i', yx_i' \in E(G)$ if $x_1 x_i \in E(T)$ and $x_i' x, x_i' y \in E(G)$ if $x_i x_1 \in E(T)$;
- (c_i) $x_{i'}' x_i' \in E(G)$ if $x_{i'} x_i \in E(T)$ and $x_i' x_{i'}' \in E(G)$ if $x_i x_{i'} \in E(T)$ (for each $2 \leq i' < i$).

Suppose that, for some $2 \leq j \leq r-4$, we have already chosen $x_2', \dots, x_j'$ so that (a_i)–(c_i) hold for $2 \leq i \leq j$. Then (12) implies that there are at least

$$\left(1 - \frac{j+3}{r} - (j+3)\varepsilon\right) n \geq \left(1 - \frac{r-1}{r} - (r-1)\varepsilon\right) n \geq \frac{n}{2r}$$

choices for x_{j+1}' so that (a_{j+1})–(c_{j+1}) are satisfied.

Conditions (a_i)–(c_i) ensure that x_i' can play the role of x_i for each $2 \leq i \leq r-3$. Since there are double edges between x^* and $x, y, x_2', \dots, x_{r-3}'$ there is currently freedom as to whether x^* will play the role of s_1, s_2 or s_3.

Our next task is to construct sets S_1, S_2, S_3 of 'candidates' to play the role of s_1, s_2 and s_3 respectively. More precisely, we say that a vertex $z \in V(G)$ is a *candidate for s_1 in G* if:

- $zx^* \in E(G)$;
- $xz, yz \in E(G)$ if $x_1 s_1 \in E(T)$ and $zx, zy \in E(G)$ if $s_1 x_1 \in E(T)$;
- $x_i' z \in E(G)$ if $x_i s_1 \in E(T)$ and $zx_i' \in E(G)$ if $s_1 x_i \in E(T)$ (for each $2 \leq i \leq r-3$).

We say that a vertex $z \in V(G)$ is a *candidate for s_2 in G* if:

- $x^* z \in E(G)$;
- $xz, yz \in E(G)$ if $x_1 s_2 \in E(T)$ and $zx, zy \in E(G)$ if $s_2 x_1 \in E(T)$;
- $x_i' z \in E(G)$ if $x_i s_2 \in E(T)$ and $zx_i' \in E(G)$ if $s_2 x_i \in E(T)$ (for each $2 \leq i \leq r-3$).

Similarly we say that a vertex $z \in V(G)$ is a *candidate for s_3 in G* if:

- $x^*z \in E(G)$;
- $xz, yz \in E(G)$ if $x_1s_3 \in E(T)$ and $zx, zy \in E(G)$ if $s_3x_1 \in E(T)$;
- $x'_iz \in E(G)$ if $x_is_3 \in E(T)$ and $zx'_i \in E(G)$ if $s_3x_i \in E(T)$ (for each $2 \leq i \leq r-3$).

(Note that it is important that the first condition in the definition of a candidate for s_1 differs from the first condition in the definitions of candidates for s_2 and s_3.)

Let S_1, S_2, S_3 denote the set of candidates for s_1, s_2 and s_3 respectively. (12) implies that

$$|S_1|, |S_2|, |S_3| \geq \left(1 - \frac{r-1}{r} - (r-1)\varepsilon\right) n \geq \frac{n}{r} - r\varepsilon n. \tag{13}$$

Case 1: $|S_1 \cup S_2| \geq n/r + \gamma' n$.
In this case (12) implies that every $z \in S_3$ receives at least $\gamma' n - \varepsilon n \geq \gamma' n/2$ edges from $S_1 \cup S_2$ in G. So (13) implies that there are at least $\gamma' n^2/3r$ edges in G with startpoint in $S_1 \cup S_2$ and endpoint in S_3. Without loss of generality assume that there are at least $\gamma' n^2/6r$ edges in G with startpoint in S_1 and endpoint in S_3. Let $s'_1s'_3$ be such an edge. Notice that, by definition of candidates for s_1 and s_3, $\{s'_1, x^*, s'_3\}$ spans a copy of T_3 in G with s'_1, x^* and s'_3 playing the roles of s_1, s_2 and s_3 respectively. Set $X := \{x^*, x'_2, \dots, x'_{r-3}, s'_1, s'_3\}$. By construction $X \cup \{x\}$ spans a copy of T in G where x plays the role of x_1, x'_i plays the role of x_i (for $2 \leq i \leq r-3$), s'_1 plays the role of s_1, x^* plays the role of s_2 and s'_3 plays the role of s_3. Similarly, $X \cup \{y\}$ spans a copy of T in G where y plays the role of x_1, x'_i plays the role of x_i (for $2 \leq i \leq r-3$), s'_1 plays the role of s_1, x^* plays the role of s_2 and s'_3 plays the role of s_3.

Case 2: $|S_1 \cup S_2| < n/r + \gamma' n$.
In this case,

$$|S_1 \cap S_2| = |S_1| - |S_1 \setminus S_2| \overset{(13)}{\geq} \left(\frac{n}{r} - r\varepsilon n\right) - \left(\gamma' n + r\varepsilon n\right) \geq \frac{n}{r} - 2\gamma' n.$$

Note that there must be at least $\gamma n^2/2 \geq \gamma' n^2/6r$ edges in $S_1 \cap S_2$ otherwise, by adding at most $2\gamma' n$ arbitrary vertices to $S_1 \cap S_2$, we obtain a γ-independent set in G of size at least n/r, a contradiction. Consider any edge $s'_1s'_2 \in E(G[S_1 \cap S_2])$.

By definition of candidates for s_1 and s_2, $\{s'_1, s'_2, x^*\}$ spans a copy of T_3 in G with s'_1, s'_2 and x^* playing the roles of s_1, s_2 and s_3 respectively. (Indeed, by definition of $S_1 \cap S_2$ there is a double edge from x^* to both s_1 and s_2 in G. Further, $s'_1s'_2 \in E(G)$.) Set $X := \{x^*, x'_2, \dots, x'_{r-3}, s'_1, s'_2\}$. By construction $X \cup \{x\}$ spans a copy of T in G where x plays the role of x_1, x'_i plays the role of x_i (for $2 \leq i \leq r-3$), s'_1 plays the role of s_1, s'_2 plays the role of s_2 and x^* plays the role of s_3. Similarly, $X \cup \{y\}$ spans a copy of T in G.

Recall that there are at least $n/2r$ choices for x^*, at least $n/2r$ choices for each x'_i and at least $\gamma' n^2/6r$ choices for the edges selected in Cases 1 and 2. Overall, this implies that there are at least

$$\frac{n}{2r} \times \left(\frac{n}{2r}\right)^{r-4} \times \frac{\gamma' n^2}{6r} \times \frac{1}{(r-1)!} \geq \eta n^{r-1}$$

choices for X, as desired. □

8.4. **The connection lemma for cyclic triangles.**

Lemma 8.5. *Let $0 < 1/n \ll \varepsilon \ll \eta \ll \gamma, \alpha \ll 1$. Suppose that G is a digraph on n vertices such that*

$$\delta^0(G) \geq (2/3 - \varepsilon)n. \tag{14}$$

Further suppose that

- *G does not contain any γ-independent set of size at least $n/3$, and*

- *G does not α-contain $Ex(n)$.*

Then, given any $x, y \in V(G)$, at least one of the following conditions holds.

(i) *There are at least ηn^2 2-sets $X \subseteq V(G)$ such that $X \cup \{x\}$ and $X \cup \{y\}$ span copies of C_3 in G.*

(ii) *There are at least ηn^5 5-sets $X \subseteq V(G)$ such that $X \cup \{x\}$ and $X \cup \{y\}$ span copies of $2C_3$ in G.*

Proof. Define additional constants $\varepsilon', \varepsilon''$ so that $\eta \ll \varepsilon' \ll \varepsilon'' \ll \gamma, \alpha$. Consider any $x, y \in V(G)$. Set $A' := N_G^+(x) \cap N_G^+(y)$ and $B' := N_G^-(x) \cap N_G^-(y)$. Note that (14) implies $|A'|, |B'| \geq (1/3 - 2\varepsilon)n$. Further, define $A := A' \setminus B'$, $B := B' \setminus A'$, $C := V(G) \setminus (A' \cup B')$ and $D := A' \cap B'$. (So A, B, C, D is a partition of $V(G)$.)

Note that given any $ab \in E(G)$ with $a \in A'$ and $b \in B'$, $\{x, a, b\}$ and $\{y, a, b\}$ span copies of C_3 in G. Thus, if there are at least $2\eta n^2$ such edges $ab \in E(G)$, then we obtain at least ηn^2 2-sets $X \subseteq V(G)$ such that $X \cup \{x\}$ and $X \cup \{y\}$ span copies of C_3 in G, as desired. Therefore, we may assume that

(α) there are at most $2\eta n^2$ edges $ab \in E(G)$ with $a \in A'$ and $b \in B'$.

Suppose that $|B'| > (1/3 + 8\eta)n$. Then by (14), every vertex in A' sends out at least $8\eta n - \varepsilon n \geq 7\eta n$ edges to B' in G. Hence, there are at least

$$7\eta n|A'| \geq 7\eta n \times (1/3 - 2\varepsilon)n > 2\eta n^2$$

edges $ab \in E(G)$ with $a \in A'$ and $b \in B'$, a contradiction to (α). Together with an analogous argument, this implies that

$$\left(\frac{1}{3} - 2\varepsilon\right) n \leq |A'|, |B'| \leq \left(\frac{1}{3} + 8\eta\right) n \tag{15}$$

and so

$$\left(\frac{1}{3} - 16\eta\right) n \leq |C|. \tag{16}$$

Suppose that $|D| > \left(\frac{1}{3} - \varepsilon''\right) n$. Then by ($\alpha$), $e(G[D]) \leq 2\eta n^2$. By adding at most $\varepsilon'' n$ arbitrary vertices to D we obtain a γ-independent set in G of size at least $n/3$, a contradiction. Hence,

$$|D| \leq \left(\frac{1}{3} - \varepsilon''\right) n.$$

We now split the proof into two cases depending on the size of D.

Case 1: $|D| < \varepsilon' n$.

In this case,

$$\left(\frac{1}{3} - 2\varepsilon'\right) n \leq \left(\frac{1}{3} - 2\varepsilon\right) n - \varepsilon' n \overset{(15)}{\leq} |A|, |B| \overset{(15)}{\leq} \left(\frac{1}{3} + 8\eta\right) n \tag{17}$$

and so

$$\left(\frac{1}{3} - 16\eta\right) n \overset{(16)}{\leq} |C| \overset{(17)}{\leq} \left(\frac{1}{3} + 4\varepsilon'\right) n. \tag{18}$$

By (α), all but at most $2\sqrt{\eta} n$ vertices $a \in A$ send out at most $\sqrt{\eta} n$ edges to B in G. So each such vertex a sends out at least $(2/3 - \varepsilon)n - \sqrt{\eta} n - \varepsilon' n - |C| \geq (1/3 - 6\varepsilon')n$ edges to A in G and at least $(2/3 - \varepsilon)n - \sqrt{\eta} n - \varepsilon' n - |A| \geq (1/3 - 2\varepsilon')n$ edges to C in G. Altogether, this implies that

$$e(G[A]) \geq (|A| - 2\sqrt{\eta} n)(1/3 - 6\varepsilon')n \overset{(17)}{\geq} (|A| - 2\sqrt{\eta} n)(|A| - 7\varepsilon' n) \overset{(17)}{\geq} |A|^2 - 3\varepsilon' n^2 \tag{19}$$

and

$$(20)\quad e(G[A,C]) \geq (|A|-2\sqrt{\eta}n)(1/3-2\varepsilon')n \overset{(18)}{\geq} (|A|-2\sqrt{\eta}n)(|C|-6\varepsilon' n) \overset{(17),(18)}{\geq} |A||C|-3\varepsilon' n^2.$$

An analogous argument implies that

$$(21)\qquad e(G[B]) \geq |B|^2 - 3\varepsilon' n^2 \ \text{ and } \ e(G[C,B]) \geq |C||B| - 3\varepsilon' n^2.$$

Suppose that $d^-_G(x,A) \geq \varepsilon'' n$ and $d^-_G(y,A) \geq \varepsilon'' n$. Then since $e(G[A]) \geq |A|^2 - 3\varepsilon' n^2$, there are at least

$$\frac{\varepsilon'' n(\varepsilon'' n - 1)}{2} - 3\varepsilon' n^2 \geq \eta n^2$$

pairs of distinct vertices a, a' where $a \in (N^-_G(x) \cap A)$, $a' \in (N^-_G(y) \cap A)$ and $aa', a'a \in E(G)$. For each such pair a, a', $\{x, a, a'\}$ and $\{y, a, a'\}$ both span copies of C_3 in G (in fact, they both span copies of K_3^-). This implies that (i) is satisfied. Similarly, (i) holds if both $d^+_G(x,B) \geq \varepsilon'' n$ and $d^+_G(y,B) \geq \varepsilon'' n$.

We may therefore assume that $d^-_G(x,A) < \varepsilon'' n$ or $d^-_G(y,A) < \varepsilon'' n$. Without loss of generality assume that

$$d^-_G(x,A) < \varepsilon'' n.$$

This implies that

$$(22)\qquad d^-_G(x,C) \overset{(14),(17)}{\geq} (2/3-\varepsilon)n - \varepsilon' n - \varepsilon'' n - (1/3+8\eta)n \geq (1/3 - 2\varepsilon'')n \overset{(18)}{\geq} |C| - 3\varepsilon'' n.$$

Furthermore, we may assume that $d^+_G(x,B) < \varepsilon'' n$ or $d^+_G(y,B) < \varepsilon'' n$. We now deal with these two subcases separately.

Case 1a: $d^+_G(x,B) < \varepsilon'' n$.

In this case we will show that (ii) is satisfied. Note that

$$(23)\qquad d^+_G(x,C) \overset{(14),(17)}{\geq} (2/3-\varepsilon)n - \varepsilon'' n - \varepsilon' n - (1/3+8\eta)n \overset{(18)}{\geq} |C| - 3\varepsilon'' n.$$

If $d^+_G(y,C) > 3\varepsilon'' n$ then (23) implies that there is a vertex $c \in (N^+_G(x) \cap N^+_G(y) \cap C) = A' \cap C$. But by definition $A' \cap C = \emptyset$, a contradiction. Thus,

$$(24)\qquad d^+_G(y,C) \leq 3\varepsilon'' n.$$

Claim 8.6. *If $e(G[B,C]) \geq 6\varepsilon'' n^2$ then (ii) is satisfied.*

Proof. Suppose that $e(G[B,C]) \geq 6\varepsilon'' n^2$. This implies that there are at least $5\varepsilon'' n$ vertices $c \in C$ that receive at least $\varepsilon'' n$ edges from B in G. By (21), all but at most $3\sqrt{\varepsilon'}n$ vertices $c \in C$ send out at least $|B| - \sqrt{\varepsilon'}n$ edges to B in G. Together with (23) this implies that there are at least $5\varepsilon'' n - 3\sqrt{\varepsilon'}n - 3\varepsilon'' n - 1 \geq \varepsilon'' n$ vertices $c \in C \setminus \{y\}$ such that

- $c \in N^+_G(x)$;
- $d^-_G(c,B) \geq \varepsilon'' n$ and $d^+_G(c,B) \geq |B| - \sqrt{\varepsilon'}n$.

Fix such a vertex c. By the choice of c and (21) there are at least $\varepsilon'' n - \sqrt{\varepsilon'}n - 3\sqrt{\varepsilon'}n \geq \varepsilon'' n/2$ vertices $b_1 \in B$ so that

- $b_1 c, c b_1 \in E(G)$;
- $d^-_G(b_1, B) \geq |B| - \sqrt{\varepsilon'}n$.

Fix such a vertex b_1. By definition of B, $b_1 \in N^-_G(x)$. Thus, $\{x, c, b_1\}$ spans a copy of C_3 in G.

(14), (17) and (24) imply that

$$d^+_G(y,B) \geq (2/3-\varepsilon)n - 3\varepsilon'' n - \varepsilon' n - (1/3+8\eta)n \geq |B| - 4\varepsilon'' n.$$

Together with (21) this implies that there are at least $|B| - 5\varepsilon'' n$ vertices $b_2 \in B \setminus \{b_1\}$ so that

- $b_2 \in N_G^+(y)$;
- $d_G^+(b_2, B), d_G^-(b_2, B) \geq |B| - \sqrt{\varepsilon'}n$.

Fix such a vertex b_2. Next fix a vertex $b_3 \in B \setminus \{b_1\}$ such that

- $b_3 \in N_G^+(b_2)$;
- $d_G^+(b_3, B) \geq |B| - \sqrt{\varepsilon'}n$.

There are at least $|B| - 5\sqrt{\varepsilon'}n$ choices for b_3. By definition of B, $b_3 \in N_G^-(y)$. Thus, $\{y, b_2, b_3\}$ spans a copy of C_3 in G.

Finally, choose a vertex $b_4 \in B$ such that

- $b_4 \in N_G^+(c) \cap N_G^-(b_1) \cap N_G^-(b_2) \cap N_G^+(b_3)$.

The choice of c, b_1, b_2 and b_3 ensures that there are at least $|B| - 4\sqrt{\varepsilon'}n$ choices for b_4. Set $X := \{c, b_1, b_2, b_3, b_4\}$. By construction both $X \cup \{x\}$ and $X \cup \{y\}$ span copies of $2C_3$ in G (see Figure 2).

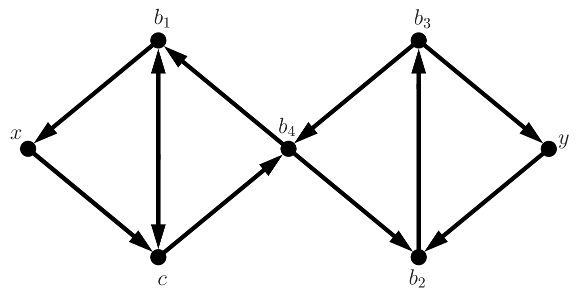

FIGURE 2. The connecting structure in Case 1a

Recall that there are at least $\varepsilon''n$ choices for c, at least $\varepsilon''n/2$ choices for b_1, at least $|B| - 5\varepsilon''n$ choices for b_2, at least $|B| - 5\sqrt{\varepsilon'}n$ choices for b_3 and at least $|B| - 4\sqrt{\varepsilon'}n$ choices for b_4. Overall, this implies that there are at least

$$\varepsilon''n \times \frac{\varepsilon''n}{2} \times (|B| - 5\varepsilon''n) \times (|B| - 5\sqrt{\varepsilon'}n) \times (|B| - 4\sqrt{\varepsilon'}n) \times \frac{1}{4!} \geq \eta n^5$$

choices for X. So indeed (ii) is satisfied. This proves the claim. $\square$

Assume for a contradiction that $e(G[B, C]) < 6\varepsilon''n^2$. This implies that

$$\begin{aligned} e(G[C]) &\geq \delta^-(G)|C| - e(G[B, C]) - e(G[A, C]) - e(G[D, C]) \\ &\overset{(14)}{\geq} (2/3 - \varepsilon)n|C| - 6\varepsilon''n^2 - |A||C| - \varepsilon'n|C| \\ &\overset{(17),(18)}{\geq} |C|(2/3 - \varepsilon - 24\varepsilon'' - 1/3 - 8\eta - \varepsilon')n \geq |C|(1/3 - 25\varepsilon'')n \\ &\overset{(18)}{\geq} |C|^2 - \sqrt{\varepsilon''}n^2. \end{aligned} \tag{25}$$

Similarly,

$$e(G[B,A]) \geq \delta^+(G)|B| - e(G[B]) - e(G[B,C]) - e(G[B,D]) \tag{26}$$
$$\stackrel{(14)}{\geq} (2/3-\varepsilon)n|B| - |B|^2 - 6\varepsilon''n^2 - \varepsilon' n|B|$$
$$\stackrel{(17)}{\geq} |B|(2/3-\varepsilon-1/3-8\eta-24\varepsilon''-\varepsilon')n \geq |B|(1/3-25\varepsilon'')n$$
$$\stackrel{(17)}{\geq} |B||A| - \sqrt{\varepsilon''}n^2.$$

Let A'', B'', C'' be a partition of $V(G)$ such that

- $\lfloor n/3 \rfloor \leq |A''| \leq |B''| \leq |C''| \leq \lceil n/3 \rceil$;
- $|A'' \setminus A|, |B'' \setminus B|, |C'' \setminus C| \leq 3\varepsilon' n$.

Such a partition exists by (17) and (18). Further,

- $e(G[A'',C'']) \stackrel{(20)}{\geq} |A''||C''| - \alpha n^2/6$;
- $e(G[C'',B'']) \stackrel{(21)}{\geq} |C''||B''| - \alpha n^2/6$;
- $e(G[B'',A'']) \stackrel{(26)}{\geq} |B''||A''| - \alpha n^2/6$;
- $e(G[A'']) \stackrel{(19)}{\geq} |A''|^2 - \alpha n^2/6$;
- $e(G[B'']) \stackrel{(21)}{\geq} |B''|^2 - \alpha n^2/6$;
- $e(G[C'']) \stackrel{(25)}{\geq} |C''|^2 - \alpha n^2/6$.

This implies that G α-contains $Ex(n)$, a contradiction. So $e(G[B,C]) \geq 6\varepsilon''n^2$. Claim 8.6 therefore implies that (ii) holds, as required.

Case 1b: $d^+_G(y,B) < \varepsilon''n$.

In this case we will show that (i) is satisfied. Since $d^+_G(y,B) < \varepsilon''n$,

$$d^+_G(y,C) \stackrel{(14),(17)}{\geq} (2/3-\varepsilon)n - \varepsilon''n - \varepsilon'n - (1/3+8\eta)n \geq (1/3-2\varepsilon'')n \stackrel{(18)}{\geq} |C| - 3\varepsilon''n. \tag{27}$$

If $d^-_G(y,C) > 3\varepsilon''n$ then (22) implies that there is a vertex $c \in (N^-_G(x) \cap N^-_G(y) \cap C) = B' \cap C$. But by definition $B' \cap C = \emptyset$, a contradiction. Thus,

$$d^-_G(y,C) \leq 3\varepsilon''n.$$

This implies that

$$d^-_G(y,A) \stackrel{(14),(17)}{\geq} (2/3-\varepsilon)n - 3\varepsilon''n - \varepsilon'n - (1/3+8\eta)n \geq (1/3-4\varepsilon'')n \stackrel{(17)}{\geq} |A| - 5\varepsilon''n. \tag{28}$$

Claim 8.7. *If $e(G[C,A]) \geq 14\varepsilon''n^2$ then (i) is satisfied.*

Proof. Suppose that $e(G[C,A]) \geq 14\varepsilon''n^2$. This implies that there are at least $8\varepsilon''n$ vertices $c \in C$ that send out at least $6\varepsilon''n$ edges to A in G. By (20) all but at most $3\sqrt{\varepsilon'}n$ vertices $c \in C$ receive at least $|A| - \sqrt{\varepsilon'}n$ edges from A in G. Together with (22) and (27) this implies that there are at least $8\varepsilon''n - 3\sqrt{\varepsilon'}n - 6\varepsilon''n \geq \varepsilon''n$ vertices $c \in C$ so that

- $c \in N^-_G(x) \cap N^+_G(y)$;
- $d^+_G(c,A) \geq 6\varepsilon''n$ and $d^-_G(c,A) \geq |A| - \sqrt{\varepsilon'}n$.

Fix such a vertex c. Let $a \in A$ such that

- $a \in N^-_G(y) \cap N^+_G(c) \cap N^-_G(c)$.

The choice of c together with (28) implies that there are at least $6\varepsilon''n - \sqrt{\varepsilon'}n - 5\varepsilon''n \geq \varepsilon''n/2$ choices for a. Since $a \in A$, $a \in N^+_G(x)$. Set $X := \{a, c\}$. By construction $X \cup \{x\}$ and $X \cup \{y\}$ both span copies of C_3 in G. In total there are at least

$$\varepsilon''n \times \varepsilon''n/2 \geq \eta n^2$$

choices for X. Therefore (i) is satisfied. This proves the claim. $\square$

Assume for a contradiction that $e(G[C, A]) < 14\varepsilon''n^2$. This implies that

$$\begin{aligned}(29)\qquad e(G[C]) &\geq \delta^+(G)|C| - e(G[C, A]) - e(G[C, B]) - e(G[C, D]) \\ &\overset{(14)}{\geq} (2/3 - \varepsilon)n|C| - 14\varepsilon''n^2 - |C||B| - \varepsilon'n|C| \\ &\overset{(17),(18)}{\geq} |C|(2/3 - \varepsilon - 50\varepsilon'' - 1/3 - 8\eta - \varepsilon')n \geq |C|(1/3 - 51\varepsilon'')n \\ &\overset{(18)}{\geq} |C|^2 - \sqrt{\varepsilon''}n^2.\end{aligned}$$

Similarly,

$$\begin{aligned}(30)\qquad e(G[B, A]) &\geq \delta^-(G)|A| - e(G[A]) - e(G[C, A]) - e(G[D, A]) \\ &\overset{(14)}{\geq} (2/3 - \varepsilon)n|A| - |A|^2 - 14\varepsilon''n^2 - \varepsilon'n|A| \\ &\overset{(17)}{\geq} |A|(2/3 - \varepsilon - 1/3 - 8\eta - 50\varepsilon'' - \varepsilon')n \geq |A|(1/3 - 51\varepsilon'')n \\ &\overset{(17)}{\geq} |B||A| - \sqrt{\varepsilon''}n^2.\end{aligned}$$

By arguing precisely as in Case 1a, (19)–(21), (29) and (30) imply that G α-contains $Ex(n)$, a contradiction. So $e(G[C, A]) \geq 14\varepsilon''n^2$. Claim 8.7 therefore implies that (i) holds, as required.

Case 2: $\varepsilon'n \leq |D| \leq (1/3 - \varepsilon'')n$.
In this case we will show that (ii) is satisfied. Set $d := |D|/n$. So

$$(31)\qquad \varepsilon' \leq d \leq 1/3 - \varepsilon''.$$

This implies that

$$(32)\qquad \varepsilon''n/2 \overset{(31)}{\leq} (1/3 - 2\varepsilon - d)n \overset{(15)}{\leq} |A|, |B| \overset{(15)}{\leq} (1/3 + 8\eta - d)n.$$

Thus,

$$(33)\qquad |C| \overset{(32)}{\leq} n - dn - 2(1/3 - 2\varepsilon - d)n = (1/3 + 4\varepsilon + d)n$$

and

$$(34)\qquad |C| \overset{(32)}{\geq} n - dn - 2(1/3 + 8\eta - d)n = (1/3 - 16\eta + d)n \overset{(31)}{\geq} (1/3 + \varepsilon'/2)n.$$

Hence,

$$(35)\qquad d^+_G(x, C), d^+_G(y, C) \overset{(14),(34)}{\geq} (2/3 - \varepsilon)n - (2/3 - \varepsilon'/2)n \geq \varepsilon'n/3.$$

By (α), all but at most $2\sqrt{\eta}n$ vertices $b \in B$ receive at most $\sqrt{\eta}n$ edges from $A \cup D = A'$ in G. So each such vertex b receives at least

$$(2/3 - \varepsilon)n - \sqrt{\eta}n - |B| \overset{(32)}{\geq} (1/3 - 2\sqrt{\eta} + d)n \overset{(33)}{\geq} |C| - 3\sqrt{\eta}n$$

edges from C in G. This implies that

(36) $$e(G[C,B]) \geq (|B| - 2\sqrt{\eta}n)(|C| - 3\sqrt{\eta}n) \geq |C||B| - 5\sqrt{\eta}n^2.$$

By (α),

(37) $$e(G[D]) + e(G[A,D]) \leq 2\eta n^2 \quad \text{and} \quad e(G[D]) + e(G[D,B]) \leq 2\eta n^2.$$

Therefore,

(38)
$$e(G[B,D]) \geq \delta^-(G)|D| - e(G[A,D]) - e(G[D]) - e(G[C,D])$$
$$\overset{(14),(37)}{\geq} (2/3 - \varepsilon)n|D| - 2\eta n^2 - |C||D| \overset{(33)}{\geq} (1/3 - \sqrt{\eta} - d)n|D| \overset{(32)}{\geq} |B||D| - \sqrt{\eta}n^2$$

and

(39)
$$e(G[D,C]) \overset{(14),(37)}{\geq} (2/3 - \varepsilon)n|D| - 2\eta n^2 - |D||A| \overset{(32)}{\geq} (1/3 - \sqrt{\eta} + d)n|D| \overset{(33)}{\geq} |C||D| - \sqrt{\eta}n^2.$$

Fix $c_1 \in C \setminus \{y\}$ such that

- $xc_1 \in E(G)$;
- $d^+_G(c_1, B) \geq |B| - \eta^{1/4}n$;
- $d^-_G(c_1, D) \geq |D| - \eta^{1/4}n$.

(35), (36) and (39) imply that there are at least $\varepsilon' n/3 - 6\eta^{1/4}n - 1 \geq \varepsilon' n/4$ choices for c_1. Next fix $b_1 \in B$ such that

- $c_1 b_1 \in E(G)$;
- $d^+_G(b_1, D) \geq |D| - \eta^{1/4}n$.

The choice of c_1 together with (38) implies that there are at least $|B| - 2\eta^{1/4}n \geq \varepsilon'' n/3$ choices for b_1. Further, $b_1 x \in E(G)$ by definition of B. Thus, $\{x, b_1, c_1\}$ spans a copy of C_3 in G.

Fix $c_2 \in C \setminus \{c_1, x\}$ such that

- $yc_2 \in E(G)$;
- $d^+_G(c_2, B) \geq |B| - \eta^{1/4}n$;
- $d^-_G(c_2, D) \geq |D| - \eta^{1/4}n$.

Again (35), (36) and (39) imply that there are at least $\varepsilon' n/4$ choices for c_2. Next fix $b_2 \in B \setminus \{b_1\}$ such that

- $c_2 b_2 \in E(G)$;
- $d^+_G(b_2, D) \geq |D| - \eta^{1/4}n$.

There are at least $|B| - 2\eta^{1/4}n - 1 \geq \varepsilon'' n/3$ choices for b_2. Since $b_2 y \in E(G)$, $\{y, b_2, c_3\}$ spans a copy of C_3 in G.

Finally let $d \in (N^+_G(b_1) \cap N^+_G(b_2) \cap N^-_G(c_1) N^-_G(c_2) \cap D)$. There are at least $|D| - 4\eta^{1/4}n \geq \varepsilon' n/2$ choices for d. Set $X := \{b_1, b_2, c_1, c_2, d\}$. By construction $X \cup \{x\}$ and $X \cup \{y\}$ both span copies of $2C_3$ in G (see Figure 3).

In total there are at least

$$\frac{\varepsilon' n}{4} \times \frac{\varepsilon'' n}{3} \times \frac{\varepsilon' n}{4} \times \frac{\varepsilon'' n}{3} \times \frac{\varepsilon' n}{2} \times \frac{1}{5!} \geq \eta n^5$$

choices for X. Therefore (ii) is satisfied, as desired. □

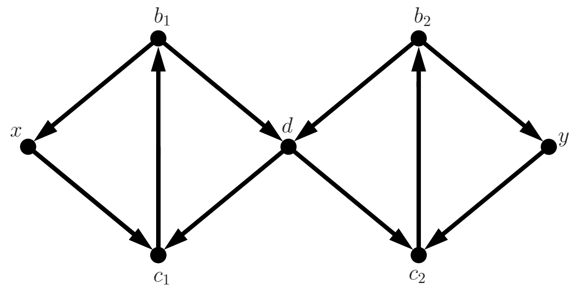

FIGURE 3. The connecting structure in Case 2

9. Proof of Theorems 5.1 and 5.2

In this section we apply our connection lemmas to prove Theorems 5.1 and 5.2. Following the ideas in [20, 21], we first show in Lemma 9.1 that in order to find the absorbing set described in Theorems 5.1 and 5.2, it suffices to prove that there are at least ξn^{2r^2} T-absorbing $2r^2$-sets for every fixed r-set from $V(G)$.

Lemma 9.1 (Absorbing lemma)**.** *Let $0 < \xi \ll 1$ and let $r \geq 2$. Then there exists an $n_0 \in \mathbb{N}$ such that the following holds. Let $T \in \mathcal{T}_r$. Consider a digraph G on $n \geq n_0$ vertices. Suppose that any r-set of vertices $Q \subseteq V(G)$ can be T-absorbed by at least ξn^{2r^2} $2r^2$-sets of vertices from $V(G)$. Then $V(G)$ contains a set M so that*

- *$|M| \leq \xi n$;*
- *M is a T-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \in r\mathbb{N}$ and $|W| \leq \xi^2 n$.*

The proof of Lemma 9.1 follows the same ideas as other such absorbing lemmas in the area. In particular, the proof of Lemma 9.1 follows the proof of Lemma 5.2 in [24] very closely. For completeness, we give the proof in Section 9.1.

Lemma 9.2. *Let $0 < 1/n \ll \varepsilon \ll \xi \ll \gamma, \alpha \ll 1/r$ where $n, r \in \mathbb{N}$ and $r \geq 3$, and let $T \in \mathcal{T}_r$. Suppose that G is a digraph on n vertices so that*

$$\delta^0(G) \geq (1 - 1/r - \varepsilon)\, n. \tag{40}$$

Further suppose that

- *G does not contain any γ-independent set of size at least n/r, and*
- *If $T = C_3$ then G does not α-contain $Ex(n)$.*

Then there are at least ξn^{2r^2} T-absorbing $2r^2$-sets in $V(G)$ for every r-subset of $V(G)$.

Theorem 5.1 follows immediately from Lemmas 9.1 and 9.2. Similarly, Theorem 5.2 follows immediately from Lemma 9.1 and the following result.

Lemma 9.3. *Let $0 < 1/n \ll \varepsilon \ll \xi \ll \gamma \ll 1/r$ where $n, r \in \mathbb{N}$ and $r \geq 3$. Suppose that G is a digraph on n vertices so that, for any $x \in V(G)$,*

$$d^+(x) \geq (1 - 1/r - \varepsilon)\, n \quad \text{or} \quad d^-(x) \geq (1 - 1/r - \varepsilon)\, n. \tag{41}$$

Further suppose that G does not contain any γ-independent set of size at least n/r. Then there are at least ξn^{2r^2} T_r-absorbing $2r^2$-sets in $V(G)$ for every r-subset of $V(G)$.

The rest of the section is devoted to the proofs of Lemmas 9.1–9.3.

9.1. **Proof of Lemma 9.1.** Given an r-set $Q \subseteq V(G)$, let L_Q denote the family of all T-absorbing $2r^2$-sets for Q in $\binom{V(G)}{2r^2}$. By assumption, $|L_Q| \geq \xi n^{2r^2}$. Let F be the family of $2r^2$-sets obtained by selecting each of the $\binom{n}{2r^2}$ elements of $\binom{V(G)}{2r^2}$ independently with probability $p := \xi/n^{2r^2-1}$. Then

$$\mathbb{E}(|F|) = p\binom{n}{2r^2} < \frac{\xi}{(2r^2)!}n \text{ and } \mathbb{E}(|L_Q \cap F|) \geq p\,\xi n^{2r^2} = \xi^2 n$$

for every set $Q \in \binom{V(G)}{r}$.

Since n is sufficiently large, Proposition 3.1 implies that with high probability we have

$$|F| \leq 2\mathbb{E}(|F|) < \frac{2\xi}{(2r^2)!}n, \tag{42}$$

$$|L_Q \cap F| \geq \frac{1}{2}\,\mathbb{E}(|L_Q \cap F|) \geq \frac{\xi^2}{2}n \quad \text{for all } Q \in \binom{V(G)}{r}. \tag{43}$$

Let Y be the number of intersecting pairs of members of F. Then

$$\mathbb{E}(Y) \leq p^2\binom{n}{2r^2}2r^2\binom{n}{2r^2-1} \leq \frac{\xi^2 n}{(2r^2-1)!(2r^2-1)!}.$$

By Markov's bound, the probability that $Y \leq \frac{2\xi^2}{(2r^2-1)!(2r^2-1)!}n$ is at least $\frac{1}{2}$. Therefore we can find a family F of $2r^2$-sets satisfying (42) and (43) and having at most $\frac{2\xi^2}{(2r^2-1)!(2r^2-1)!}n$ intersecting pairs. Removing all non-absorbing $2r^2$-sets and one set from each of the intersecting pairs in F, we obtain a family F' of disjoint T-absorbing $2r^2$-sets such that $|F'| \leq |F| \leq \frac{2\xi}{(2r^2)!}n \leq \xi n/2r^2$ and for all $Q \in \binom{V(G)}{r}$,

$$|L_Q \cap F'| \geq \frac{\xi^2}{2}n - \frac{2\xi^2}{(2r^2-1)!(2r^2-1)!}n > \frac{\xi^2}{r}n. \tag{44}$$

Let M denote the disjoint union of the sets in F'. Then $|M| = |F'|2r^2 \leq \xi n$. Since F' consists of disjoint T-absorbing sets and each T-absorbing set is covered by a perfect T-packing, $G[M]$ contains a perfect T-packing. Now let $W \subseteq V(G)\backslash M$ be a set of at most $\xi^2 n$ vertices such that $|W| = r\ell$ for some $\ell \in \mathbb{N}$. We arbitrarily partition W into r-sets $Q_1, \dots, Q_\ell$. Because of (44), we are able to T-absorb each Q_i with a different $2r^2$-set from $L_{Q_i} \cap F'$. Therefore $G[M \cup W]$ contains a perfect T-packing, as desired. $\square$

9.2. **Proof of Lemma 9.2.** Define η such that $\xi \ll \eta \ll \gamma$. Note that (40) implies that there are at least

$$n \times \left(1 - \frac{1}{r} - \varepsilon\right) n \times \left(1 - \frac{2}{r} - 2\varepsilon\right) n \times \cdots \times \left(1 - \frac{r-1}{r} - (r-1)\varepsilon\right) n \times \frac{1}{r!} \geq 2\eta^2 n^r \tag{45}$$

r-sets in $V(G)$ that span copies of T in G.

Claim 9.4. *For any $x, y \in V(G)$ there are at least $\eta^4 n^{2r-1}$ $(2r-1)$-sets $X \subseteq V(G)$ such that both $X \cup \{x\}$ and $X \cup \{y\}$ span copies of $2T$ in G.*

Proof. Suppose for a contradiction that Claim 9.4 is false. Then if $T = C_3$, certainly Lemma 8.5(ii) does not hold. In particular, Lemmas 8.1, 8.3–8.5 imply that, for any $x, y \in V(G)$ there are at least ηn^{r-1} $(r-1)$-sets $X' \subseteq V(G)$ such that both $X' \cup \{x\}$ and $X' \cup \{y\}$ span copies of T in G. Fix such a set X'. (45) implies that there are least

$$2\eta^2 n^r - (r+1)\binom{n}{r-1} \geq \eta^2 n^r$$

r-sets $X'' \subseteq V(G)$ that span copies of T in G and that are disjoint from $X' \cup \{x, y\}$. Fix such a set X'' and define $X := X' \cup X''$. By construction both $X \cup \{x\}$ and $X \cup \{y\}$ span copies of $2T$ in G. Further, since $\eta \ll 1/r$, there are at least

$$\eta n^{r-1} \times \eta^2 n^r \times \frac{1}{\binom{2r-1}{r-1}} \geq \eta^4 n^{2r-1}$$

choices for X, a contradiction. This proves the claim. $\square$

Consider any r-subset $Q := \{x_1, x_2, \dots, x_r\}$ of $V(G)$. Fix some r-subset $Y := \{y_1, y_2, \dots, y_r\}$ of $V(G)$ that spans a copy of T in G and that is disjoint from Q. (45) implies that there are least

$$2\eta^2 n^r - r\binom{n}{r-1} \geq \eta^2 n^r$$

choices for Y. Next fix a $(2r-1)$-set $X_1 \subseteq V(G)$ such that both $X_1 \cup \{x_1\}$ and $X_1 \cup \{y_1\}$ span copies of $2T$ in G and so that X_1 is disjoint from $Q \cup Y$. Claim 9.4 implies that there are at least

$$\eta^4 n^{2r-1} - 2r\binom{n}{2r-2} \geq \eta^4 n^{2r-1}/2$$

choices for X_1. Similarly, Claim 9.4 implies that we can iteratively choose $(2r-1)$-sets $X_2, \dots, X_r \subseteq V(G)$ such that, for each $2 \leq i \leq r$:

- Both $X_i \cup \{x_i\}$ and $X_i \cup \{y_i\}$ span copies of $2T$ in G;
- X_i is disjoint from $Q \cup Y$;
- X_i is disjoint from X_j for all $1 \leq j < i$;
- There are at least $\eta^4 n^{2r-1}/2$ choices for X_i.

Set $S := Y \cup \bigcup_{1 \leq i \leq r} X_i$. Then S is a T-absorbing $2r^2$-set for Q. Indeed, $G[X_i \cup \{y_i\}]$ contains a perfect T-packing for all $1 \leq i \leq r$ so $G[S]$ contains a perfect T-packing. Furthermore, $G[X_i \cup \{x_i\}]$ contains a perfect T-packing for all $1 \leq i \leq r$ and Y spans a copy of T in G so $G[S \cup Q]$ contains a perfect T-packing.

In summary, there are at least $\eta^2 n^r$ choices for Y and at least $\eta^4 n^{2r-1}/2$ choices for each of the X_i. Since each T-absorbing $2r^2$-set may be counted $\binom{2r^2}{r}\binom{r(2r-1)}{2r-1}\binom{(r-1)(2r-1)}{2r-1}\cdots\binom{2(2r-1)}{2r-1}$ times there are at least

$$\eta^2 n^r \times \left(\frac{\eta^4 n^{2r-1}}{2}\right)^r \times \frac{1}{\binom{2r^2}{r}\binom{r(2r-1)}{2r-1}\binom{(r-1)(2r-1)}{2r-1}\cdots\binom{2(2r-1)}{2r-1}} \geq \xi n^{2r^2}$$

T-absorbing $2r^2$-sets for Q, as desired. $\square$

9.3. **Proof of Lemma 9.3.** Define η such that $\xi \ll \eta \ll \gamma$. By Lemma 8.1, for every vertex $x \in V(G)$, there are at least ηn^{r-1} $(r-1)$-sets $X \subseteq V(G)$ such that $X \cup \{x\}$ spans a copy of T_r in G. Thus, there are at least

$$n \times \eta n^{r-1} \times \frac{1}{r} \geq 2\eta^2 n^r \tag{46}$$

r-sets in $V(G)$ that span copies of T_r in G.

By now following the proof of Lemma 9.2 identically (applying (46) and Lemma 8.1) we conclude that there are at least ξn^{2r^2} T_r-absorbing $2r^2$-sets in $V(G)$ for every r-subset of $V(G)$, as required. $\square$

10. Tools for the proof of Lemma 5.5

In Sections 11 and 12 we deal with the extremal cases of Theorems 1.3. The proof of Lemma 5.5 builds on the ideas from the extremal case in [14]. (Note though that [14] concerns embedding *powers of Hamilton cycles* in *graphs*.) In this section we give a number of results that will be applied in the proof of Lemma 5.5.

10.1. **Perfect T-packings in the non-extremal case.** In the proof of Lemma 5.5 we will apply the following result which is a direct consequence of Theorems 5.1 and 5.3 (its proof is implicit in the proof of Theorem 1.3 given in Section 5).

Theorem 10.1. *Let $0 < 1/n \ll \varepsilon \ll \gamma \ll 1/r$ where $n, r \in \mathbb{N}$ and $r \geq 3$ so that r divides n, and let $T \in \mathcal{T}_r \setminus \{C_3\}$. Suppose that G is a digraph on n vertices so that*

$$\delta^0(G) \geq (1 - 1/r - \varepsilon)\, n.$$

Further suppose that G does not contain any γ-independent set of size at least n/r. Then G contains a perfect T-packing.

10.2. **Perfect K_r-packings in r-partite digraphs.** We will also apply the following immediate consequence of Theorem 1.2.

Theorem 10.2. *Given $r \in \mathbb{N}$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose G is an r-partite digraph with vertex classes $V_1, \dots, V_r$ where $|V_i| = n \geq n_0$ for all $1 \leq i \leq r$. If*

$$\bar{\delta}^+(G), \bar{\delta}^-(G) \geq (1 - 1/2r)n + 1$$

then G contains a perfect K_r-packing.

Here $\bar{\delta}^+(G)$ ($\bar{\delta}^-(G)$) denotes the minimum outdegree (indegree) of a vertex from one vertex class V_i to another vertex class V_j.

10.3. **Matchings in digraphs.** A *matching* in a (di)graph G is a collection of vertex-disjoint edges $M \subseteq E(G)$. We write $V(M)$ for the set of vertices covered by the edges from M. We say that M is a *d-matching* if $|M| = d$. We say that M is a *perfect matching* if $V(M) = V(G)$.

Proposition 10.3. *Let $d, n \in \mathbb{N}$. Suppose that G is a graph on $n \geq 2d$ vertices such that $\delta(G) \geq d$. Let $X \subseteq V(G)$ such that $|X| = d$. Then G contains a d-matching that covers all the vertices in X.*

Proof. It is easy to see that G contains a d-matching. Let M be a d-matching in G that covers the maximum number of vertices from X. Suppose for a contradiction that there is a vertex $x \in X$ uncovered by M. In particular, M covers more vertices in $V(G) \setminus X$ than in X. There exist non-negative integers a, b, c such that $a + b + c = d$ and:

- (i) M contains precisely a edges wz where $w \in X$ and $z \in V(G) \setminus X$;
- (ii) M contains precisely b edges with both endpoints in X;
- (iii) M contains precisely c edges with both endpoints in $V(G) \setminus X$.

Since M covers more vertices in $V(G)\setminus X$ than in X, $b < c$ (and so $c \geq 1$). Suppose x has a neighbour $y \in V(G) \setminus V(M)$. Then add xy to M and delete an edge wz from M such that $w, z \in V(G) \setminus X$. Then M is a d-matching covering more vertices in X than before, a contradiction. So x only has neighbours in $V(M)$.

Suppose wz is an edge in M such that $w \in X$ and $z \in V(G) \setminus X$. If $xw \in E(G)$ then delete wz from M and add xw to M. So again M is a d-matching covering more vertices in X than before, a contradiction. Thus, x is not adjacent to w. A similar argument shows that, if $wz \in M$ with $w, z \in V(G) \setminus X$, then $xw, xz \notin E(G)$. Together with (i)–(iii) this shows that x has at most $a + 2b < a + b + c = d$ neighbours in G, a contradiction, as desired. $\square$

The following immediate consequence of Proposition 10.3 will be applied in the proof of Lemma 5.5.

Proposition 10.4. *Let $d, n \in \mathbb{N}$. Suppose that G is a digraph on $n \geq 2d$ vertices such that, for any $x \in V(G)$, $d^+(x) \geq d$ or $d^-(x) \geq d$. Let $X \subseteq V(G)$ such that $|X| = d$. Then G contains a d-matching that covers all the vertices in X.*

Let $\varepsilon > 0$. Suppose that G is a (di)graph G on n vertices. Then we say that G is *ε-close to $2K_{n/2}$* if there exists a partition A, B of $V(G)$ such that $|A| = \lfloor n/2 \rfloor$, $|B| = \lceil n/2 \rceil$ and $e_G(A, B) \leq \varepsilon n^2$.

Proposition 10.5. *Let $\gamma > 0$ and $n \in \mathbb{N}$ be even such that $1/n \ll \gamma$. Suppose that G is a graph on n vertices so that*

$$\delta(G) \geq (1/2 - \gamma)n. \tag{47}$$

Then at least one of the following conditions holds:

- *G contains a 3γ-independent set of size at least $n/2$;*
- *G is 3γ-close to $2K_{n/2}$;*
- *G contains a perfect matching.*

Proof. Suppose that G does not contain a perfect matching. Let M be a maximal matching in G. So there exists distinct $x, y \in V(G) \backslash V(M)$. The maximality of M implies that $N(x), N(y) \subseteq V(M)$. Define $SN(x) := \{z \in V(M) : wz \in M$ for some $w \in N(x)\}$. Define $SN(y)$ analogously. (47) implies that

$$|SN(x)|, |SN(y)| \geq (1/2 - \gamma)n. \tag{48}$$

Suppose for a contradiction that there is an edge $zz' \in E(G)$ such that $z \in SN(x)$ and $z' \in SN(y)$. If $zz' \in M$ then by definition of $SN(x)$ and $SN(y)$, $xz', yz \in E(G)$. Define $M' := (M \setminus \{zz'\}) \cup \{xz', yz\} \subseteq E(G)$. Thus, M' is a larger matching than M, a contradiction. So $zz' \notin M$. Let $w, w' \in V(M)$ such that $wz, w'z' \in M$. Then by definition of $SN(x)$ and $SN(y)$, $xw, yw' \in E(G)$. Set $M' := (M \backslash \{wz, w'z'\}) \cup \{xw, yw', zz'\} \subseteq E(G)$. Then M' is a larger matching than M, a contradiction. This proves that there is no such edge zz'.

Define $SN(x, y) := SN(x) \cap SN(y)$. Suppose that $SN(x, y) \neq \emptyset$. Consider any $z \in SN(x, y)$. Then in G, z does not have any neighbours in $SN(x) \cup SN(y)$. So (47) implies that $|SN(x) \cup SN(y)| \leq (1/2 + \gamma)n$. So together with (48) this implies that $|SN(x, y)| \geq (1/2 - 3\gamma)n$. Further, $SN(x, y)$ is an independent set in G. By adding at most $3\gamma n$ arbitrary vertices to $SN(x, y)$ we obtain a 3γ-independent set of size at least $n/2$ in G.

Finally, suppose that $SN(x, y) = \emptyset$. So $SN(x)$ and $SN(y)$ are disjoint and $e_G(SN(x), SN(y)) = 0$. Together with (48) this implies that G is 3γ-close to $2K_{n/2}$, as desired. □

We will also apply the following consequence of Proposition 10.5.

Proposition 10.6. *Let $\gamma > 0$ and $n \in \mathbb{N}$ be even such that $1/n \ll \gamma$. Suppose that G is a digraph on n vertices so that, for every $x \in V(G)$,*

$$d^+(x) \geq (1/2 - \gamma)\, n \quad \text{or} \quad d^-(x) \geq (1/2 - \gamma)\, n.$$

Then at least one of the following conditions holds:

- *G contains a 6γ-independent set of size at least $n/2$;*
- *G is 6γ-close to $2K_{n/2}$;*
- *G contains a perfect matching.*

11. Proof of Lemma 5.5

Define constants $\gamma, \gamma_1, \gamma_2, \dots, \gamma_r$ and $n_0 \in \mathbb{N}$ such that

$$0 < 1/n_0 \ll \gamma \ll \gamma_1 \ll \gamma_2 \ll \cdots \ll \gamma_r \ll 1/r.$$

Let $T \in \mathcal{T}_r$ and G be a digraph on $n \geq n_0$ vertices as in the statement of the lemma. By assumption G contains a γ-independent set A_1 of size n/r. (So A_1 is also a γ_1-independent set in G.) Consider $G_1 := G \setminus A_1$. If G_1 contains a γ_2-independent set A_2 of size n/r set $G_2 := G_1 \setminus A_2$. (Note that A_2 is also a γ_2-independent set in G.) Otherwise let $B := V(G_1)$. Repeating this process, for some $1 \leq s \leq r$, we obtain a partition $A_1, \dots, A_s, B$ of $V(G)$ such that:

- A_i is a γ_i-independent set of size n/r in G (for all $1 \leq i \leq s$);
- $|B| = (r-s)n/r$ and $G[B]$ does not contain a γ_{s+1}-independent set of size n/r.

(The latter condition is vacuous if $B = \emptyset$.) If $B = \emptyset$ define additional constants α, β', β so that

$$\gamma_r \ll \alpha \ll \beta' \ll \beta \ll 1/r.$$

If $B \neq \emptyset$ then define $\alpha, \beta', \beta, \eta$ so that

$$\gamma_s \ll \alpha \ll \beta' \ll \beta \ll \eta \ll \gamma_{s+1}.$$

Let $\delta > 0$ and $1 \leq i \leq s$. We now introduce a number of definitions.

- We say that a vertex $x \in A_i$ is (δ, i)-*bad* if $d^+_G(x, A_i) \geq \delta n$ or $d^-_G(x, A_i) \geq \delta n$. Otherwise we say that x is (δ, i)-*good*.
- We say that a vertex $x \in V(G) \setminus A_i$ is (δ, i)-*exceptional* if $d^+_G(x, A_i), d^-_G(x, A_i) \leq \delta n$. Otherwise we say that x is (δ, i)-*acceptable*.
- We say that a vertex $x \in V(G) \setminus A_i$ is (δ, i)-*excellent* if $d^+_G(x, A_i), d^-_G(x, A_i) \geq |A_i| - \delta n$.
- Similarly, we say that a vertex $x \in V(G) \setminus B$ is (δ, B)-*excellent* if $d^+_G(x, B), d^-_G(x, B) \geq |B| - \delta n$.

Later on we will modify the vertex classes $A_1, \dots, A_s, B$. When referring to, for example, (δ, i)-bad vertices, we mean with respect to the current class A_i and not the original class.

For each $1 \leq i \leq s$, since A_i is a γ_i-independent set in G and $\gamma_i \ll \alpha \ll \beta$, there are at most αn vertices in A_i that are (β, i)-bad. Furthermore, (5) implies that there are at least

$$2\delta^0(G)|A_i| - 2\gamma_i n^2 \geq 2|A_i||V(G) \setminus A_i| - 2\gamma_i n^2$$

edges in G with one endpoint in A_i and the other in $V(G) \setminus A_i$. So as $\gamma_i \ll \alpha \ll \beta'$, there are at most αn vertices $x \in V(G) \setminus A_i$ that are not (β', i)-excellent. (This implies that there are at most αn (β, i)-exceptional vertices.)

Modifying the partition $A_1, \dots, A_s, B$. Let t be the largest integer such that there exists both t $(\beta, 1)$-bad vertices $x_1, \dots, x_t \in A_1$ *and* t $(\beta, 1)$-exceptional vertices $y_1, \dots, y_t \in V(G) \setminus A_1$. Note that $t \leq \alpha n$. Move $y_1, \dots, y_t$ into A_1 and remove $x_1, \dots, x_t$ from A_1 so that each x_i replaces y_i in their respective classes. (So if originally $y_i \in A_j$ then we move x_i into A_j. If originally $y_i \in B$ then we move x_i into B.) We call this 'phase' *Step 1.*

If a vertex $x \in A_1$ was initially $(\beta, 1)$-good then after Step 1, x is still $(\beta + \alpha, 1)$-good. Further, each y_i is now $(\beta + \alpha, 1)$-good. So if A_1 initially contained *precisely* t $(\beta, 1)$-bad vertices, then A_1 no longer contains any $(\beta + \alpha, 1)$-bad vertices.

If a vertex $y \in V(G) \setminus A_1$ was initially $(\beta, 1)$-acceptable, then after Stage 1, y is still $(\beta - \alpha, 1)$-acceptable. Further, each vertex x_i is $(\beta - \alpha, 1)$-acceptable. So if initially there were *precisely* t $(\beta, 1)$-exceptional vertices, then after Stage 1 there are no $(\beta - \alpha, 1)$-exceptional vertices.

Thus, after Stage 1 we have that:

- A_i is an α-independent set of size n/r in G (for all $1 \leq i \leq s$);
- If $B \neq \emptyset$ then $G[B]$ does not contain any $(\gamma_{s+1} - 2\alpha r)$-independent set of size n/r;

- There are no $(\beta+\alpha,1)$-bad vertices in A_1 or there are no $(\beta-\alpha,1)$-exceptional vertices in $V(G)\setminus A_1$;
- A_i contains at most $2\alpha n$ $(\beta+\alpha,i)$-bad vertices (for all $1\le i\le s$);
- There are at most $2\alpha n$ vertices in $V(G)\setminus A_i$ that are not $(\beta'+\alpha,i)$-excellent (for each $1\le i\le s$).

Suppose that $s\ge 2$. We now explain *Stage 2* of the switching procedure. Let t' be the largest integer such that there exists both t' $(\beta+\alpha,2)$-bad vertices $x_1,\dots,x_{t'}\in A_2$ and t' $(\beta,2)$-exceptional vertices $y_1,\dots,y_{t'}\in V(G)\setminus A_2$ at the end of Stage 1. Note that $t'\le 2\alpha n$. Move $y_1,\dots,y_{t'}$ into A_2 and remove $x_1,\dots,x_{t'}$ from A_2 so that each x_i replaces y_i in their respective classes.

If a vertex $x\in A_2$ was $(\beta+\alpha,2)$-good after Stage 1 then x is still $(\beta+3\alpha,2)$-good. Further, each y_i is now $(\beta+3\alpha,2)$-good. So if at the end of Stage 1, A_2 contained *precisely* t' $(\beta+\alpha,2)$-bad vertices, then A_2 no longer contains any $(\beta+3\alpha,2)$-bad vertices.

If a vertex $y\in V(G)\setminus A_2$ was $(\beta,2)$-acceptable at the end of Stage 1, then y is still $(\beta-2\alpha,2)$-acceptable. Further, each vertex x_i is $(\beta-2\alpha,2)$-acceptable. So if at the end of Stage 1, there were *precisely* t' $(\beta,2)$-exceptional vertices, then there are now no $(\beta-2\alpha,2)$-exceptional vertices.

Recall that after Stage 1 there are no $(\beta+\alpha,1)$-bad vertices in A_1 or there are no $(\beta-\alpha,1)$-exceptional vertices in $V(G)\setminus A_1$. Suppose that the former holds. Then (5) implies that after Stage 1 every vertex in A_1 is $(\beta+\alpha,2)$-excellent. In particular, A_1 is not modified in Stage 2. Next suppose that after Stage 1 there were no $(\beta-\alpha,1)$-exceptional vertices in $V(G)\setminus A_1$. Suppose that x is a vertex that lies in $V(G)\setminus A_1$ both after Stage 1 and after Stage 2. Then after Stage 2 x is a $(\beta-3\alpha,1)$-acceptable vertex. Suppose that x is a vertex in A_1 after Stage 1 and a vertex in $V(G)\setminus A_1$ after Stage 2. Then $x\in A_2$ after Stage 2 and so was a $(\beta,2)$-exceptional vertex after Stage 1. Together with (5), this implies that, after Stage 2, x is a $(\beta+2\alpha,1)$-excellent vertex (in particular, x is not $(\beta-3\alpha,1)$-exceptional). Overall this implies that, after Stage 2, there are no $(\beta+3\alpha,1)$-bad vertices in A_1 or there are no $(\beta-3\alpha,1)$-exceptional vertices in $V(G)\setminus A_1$.

Therefore, after Stage 2 we have that:

- A_i is a 3α-independent set of size n/r in G (for all $1\le i\le s$);
- If $B\ne\emptyset$ then $G[B]$ does not contain any $(\gamma_{s+1}-6\alpha r)$-independent set of size n/r;
- There are no $(\beta+3\alpha,i)$-bad vertices in A_i or there are no $(\beta-3\alpha,i)$-exceptional vertices in $V(G)\setminus A_i$ for $i=1,2$;
- A_i contains at most $4\alpha n$ $(\beta+3\alpha,i)$-bad vertices (for all $1\le i\le s$);
- There are at most $4\alpha n$ vertices in $V(G)\setminus A_i$ that are not $(\beta'+3\alpha,i)$-excellent (for each $1\le i\le s$).

By applying an analogous switching procedure iteratively for $A_3,\dots,A_s$ we modify the partition $A_1,\dots,A_s,B$ of $V(G)$ such that the following conditions hold:

(α_1) $A_1,\dots,A_s,B$ remains a partition of $V(G)$ so that A_i is a $\sqrt{\alpha}$-independent set of size n/r in G (for all $1\le i\le s$);

(α_2) If $B\ne\emptyset$ then $G[B]$ does not contain any $(\gamma_{s+1}/2)$-independent set of size n/r;

(α_3) There are no $(2\beta,i)$-bad vertices in A_i or there are no $(\beta/2,i)$-exceptional vertices in $V(G)\setminus A_i$ (for all $1\le i\le s$);

(α_4) A_i contains at most $\sqrt{\alpha}n$ $(2\beta,i)$-bad vertices (for all $1\le i\le s$);

(α_5) There are at most $\sqrt{\alpha}n$ vertices in $V(G)\setminus A_i$ that are not $(2\beta',i)$-excellent (for each $1\le i\le s$).

Note that if $B\ne\emptyset$ then (5) implies that

$$\delta^0(G[B])\ge\left(1-\frac{1}{r-s}\right)|B|. \tag{49}$$

11.1. **The case when $|B| \neq 2n/r$ or $G[B]$ is not close to $2K_{n/r}$.** In this subsection we will assume that either (i) $r-s \neq 2$ (and so $|B| \neq 2n/r$) or (ii) $r-s=2$ and $G[B]$ is not η-close to $2K_{n/r}$. The case when $r-s=2$ and $G[B]$ is η-close to $2K_{n/r}$ is considered in Section 11.2.

Covering the exceptional vertices with matchings. Given any $1 \leq i \leq s$, let $V_{ex,i}$ denote the set of $(\beta/2, i)$-exceptional vertices in $V(G) \setminus A_i$. (α_5) implies that $c_i := |V_{ex,i}| \leq \sqrt{\alpha} n$ for all $1 \leq i \leq s$. (5) implies that a vertex cannot be both $(\beta/2, i)$-exceptional and $(\beta/2, j)$-exceptional for $i \neq j$. So $V_{ex,i}$ and $V_{ex,j}$ are disjoint for all $i \neq j$.

Given $1 \leq i \leq s$, if $V_{ex,i} = \emptyset$, set G_i to be the empty digraph. If $V_{ex,i} \neq \emptyset$, set $G_i := G[A_i \cup V_{ex,i}]$. If $V_{ex,i} \neq \emptyset$ then by (α_3), there are no $(2\beta, i)$-bad vertices in A_i. Thus, by (5) every vertex in A_i is $(2\beta, j)$-excellent for all $j \neq i$. In particular, if $x \in A_i$ then $x \notin V_{ex,j}$. Therefore the digraphs $G_1, \dots, G_s$ are vertex-disjoint.

If $V_{ex,i} \neq \emptyset$ then, since $|G_i| = n/r + c_i$, (5) implies that $\delta^0(G_i) \geq c_i$ (for $1 \leq i \leq s$). Further, $|G_i| \geq 2c_i$. So Proposition 10.4 implies that there are disjoint matchings $M_1, \dots, M_s$ in G such that:

(β_1) M_i is a c_i-matching in G_i that covers all the vertices in $V_{ex,i}$ (for all $1 \leq i \leq s$).

Note that if $V_{ex,i} = \emptyset$ then M_i is empty.

Extending the matchings M_i into T-packings. Our next task is to find vertex-disjoint T-packings $\mathcal{M}_1, \dots, \mathcal{M}_s$ in G so that, for each $1 \leq i \leq s$:

(γ_1) $\mathcal{M}_i$ contains precisely c_i disjoint copies of T;
(γ_2) $\mathcal{M}_i$ covers M_i. In particular, each copy of T in $\mathcal{M}_i$ contains a unique edge from M_i;
(γ_3) $\mathcal{M}_i$ covers precisely c_i vertices from A_j (for each $1 \leq j \leq s$) and precisely $(r-s)c_i$ vertices from B.

Suppose that for some $1 \leq i \leq s$ we have found our desired T-packings $\mathcal{M}_1, \dots, \mathcal{M}_{i-1}$. We now construct $\mathcal{M}_i$. If M_i is empty then we set $\mathcal{M}_i = \emptyset$ and then (γ_1)–(γ_3) are vacuously true for $\mathcal{M}_i$. So suppose that M_i is non-empty. Since $|V_{ex,i}| = c_i$, (β_1) implies that there exist non-negative integers a_i, b_i so that $c_i = a_i + 2b_i$ and

(i) M_i contains precisely a_i edges with one endpoint in $V_{ex,i}$ and the other in A_i;
(ii) M_i contains precisely b_i edges with both endpoints in $V_{ex,i}$;
(iii) M_i contains precisely b_i edges with both endpoints in A_i.

Consider any edge e in M_i with one endpoint $x \in A_i$ and one endpoint $y \in V_{ex,i}$. Since $V_{ex,i} \neq \emptyset$, (α_3) implies that x is $(2\beta, i)$-good. In particular, together with (5) this implies that $d^+(x, A_j), d^-(x, A_j) \geq |A_j| - 2\beta n$ for all $j \neq i$ and $d^+(x, B), d^-(x, B) \geq |B| - 2\beta n$. Since y is $(\beta/2, i)$-exceptional, (5) implies that $d^+(y, A_j), d^-(y, A_j) \geq |A_j| - \beta n/2$ for all $j \neq i$ and $d^+(y, B), d^-(y, B) \geq |B| - \beta n/2$. It is easy to see that, together with (49), (α_4) and (α_5), this implies that we can greedily construct a copy T' of T in G such that:

- T' is vertex-disjoint from $\mathcal{M}_1, \dots, \mathcal{M}_{i-1}$ and $M_i \setminus \{e\}, M_{i+1}, \dots, M_s$;
- T' contains e and contains precisely one vertex from each of $A_1 \dots, A_s$ and $r-s$ vertices from B.

Further, we can repeat this process for all a_i such edges e so that the a_i copies of T thus obtained are vertex-disjoint. Let $\mathcal{M}'_i$ denote the set of these copies of T. So $\mathcal{M}'_i$ covers a_i vertices from each A_j and $a_i(r-s)$ vertices from B.

Next pair off each of the b_i edges from (ii) with a unique edge from (iii). Consider one such pair (e, e') of edges. So the endpoints x, y of e lie in $V_{ex,i}$ and the endpoints x', y' of e' lie in A_i. Suppose that $x, y \in A_{i'}$ for some $i' \neq i$. (The other cases are similar.) Since x, y are $(\beta/2, i)$-exceptional, (5) implies that $d^+(x, A_j), d^-(x, A_j), d^+(y, A_j), d^-(y, A_j) \geq |A_j| - \beta n/2$ for all $j \neq i$ and $d^+(x, B), d^-(x, B), d^+(y, B)$, $d^-(y, B) \geq |B| - \beta n/2$. It is easy to see that, together with (49), (α_4) and (α_5), this implies that we can greedily construct a copy T' of T in G such that:

- T' is vertex-disjoint from $\mathcal{M}_1, \dots, \mathcal{M}_{i-1}, \mathcal{M}'_i$ and $M_i \setminus \{e\}, M_{i+1}, \dots, M_s$;
- T' contains e and contains two vertices from $A_{i'}$ (namely x and y), no vertices from A_i, one vertex from A_j (for $j \neq i, i'$) and $r-s$ vertices from B.

Similarly we can greedily construct a copy T'' of T in G such that:

- T'' is vertex-disjoint from $\mathcal{M}_1, \dots, \mathcal{M}_{i-1}, \mathcal{M}'_i$, T' and $M_i \setminus \{e'\}, M_{i+1}, \dots, M_s$;
- T'' contains e' and contains two vertices from A_i (namely x' and y'), no vertices from $A_{i'}$, one vertex from A_j (for $j \neq i, i'$) and $r-s$ vertices from B.

So together T' and T'' cover precisely two vertices from each A_j and $2(r-s)$ vertices from B. Further, we can repeat this process for all such pairs of edges (e, e') so that the $2b_i$ copies of T thus obtained are vertex-disjoint. Let $\mathcal{M}''_i$ denote the set of these copies of T. Then by construction $\mathcal{M}''_i$ covers precisely $2b_i$ vertices from each A_j and $2b_i(r-s)$ vertices from B. Set $\mathcal{M}_i := \mathcal{M}'_i \cup \mathcal{M}''_i$. So $\mathcal{M}_i$ is a T-packing in G. By construction $\mathcal{M}_i$ is vertex-disjoint from $\mathcal{M}_1, \dots, \mathcal{M}_{i-1}$ and satisfies (γ_1)–(γ_3), as desired.

Covering the remaining vertices. Remove all those vertices covered by $\mathcal{M}_1, \dots, \mathcal{M}_s$ from G (and from the classes $A_1, \dots, A_s, B$). Call the resulting digraph G'. So $n' := |G'| \geq (1 - r^2\sqrt{\alpha})n$ by (γ_1), $|A_i| = n'/r$ for all $1 \leq i \leq s$ and $|B| = (r-s)n'/r$ by (γ_3). Further, (5) and (α_1)–(α_5) imply that the following conditions hold:

(δ_1) $\delta^0(G') \geq (1 - 1/r - r^2\sqrt{\alpha})n \geq (1 - 1/r - r^2\sqrt{\alpha})n'$;
(δ_2) $A_1, \dots, A_s, B$ is a partition of $V(G')$ so that A_i is a $2\sqrt{\alpha}$-independent set of size n'/r in G' (for all $1 \leq i \leq s$);
(δ_3) If $B \neq \emptyset$ then $G'[B]$ does not contain any $(\gamma_{s+1}/3)$-independent set of size n'/r;
(δ_4) Every vertex in $V(G') \setminus A_i$ is $(\beta/3, i)$-acceptable (for each $1 \leq i \leq s$);
(δ_5) There are at most $\sqrt{\alpha}n$ vertices in $V(G') \setminus A_i$ that are not $(2\beta', i)$-excellent (for each $1 \leq i \leq s$).

In particular, note that (δ_4) follows since $\mathcal{M}_i$ contains the vertices in $V_{ex,i}$. If $B \neq \emptyset$ then (δ_1) implies that

$$d^+_{G'}(y, B), d^-_{G'}(y, B) \geq \left(1 - \frac{1}{r} - r^2\sqrt{\alpha}\right)n' - \frac{sn'}{r} \geq \left(1 - \frac{1}{r-s} - \alpha^{1/3}\right)|B| \tag{50}$$

for all $y \in V(G')$.

We now split the proof into cases depending on the size of B.

Case 1: $B = \emptyset$. In this case $s = r$ and $A_1, \dots, A_r$ is a partition of $V(G')$. Then G' contains a T-packing $\mathcal{M}'$ such that:

(ε_1) $\mathcal{M}'$ contains at most $r\sqrt{\alpha}n$ copies of T;
(ε_2) If $x \in V(G') \setminus A_i$ is not $(2\beta', i)$-excellent then x is contained in a copy of T in $\mathcal{M}'$ (for any $1 \leq i \leq r$);
(ε_3) Each copy of T in $\mathcal{M}'$ covers exactly one vertex from A_i (for each $1 \leq i \leq r$).

To see that such a T-packing $\mathcal{M}'$ exists, suppose that we have found a T-packing $\mathcal{M}^*$ in G' that satisfies (ε_1) and (ε_3). Suppose that for some $1 \leq i \leq r$, $x \in V(G) \setminus A_i$ so that x is not $(2\beta', i)$-excellent and x is not covered by $\mathcal{M}^*$. (By (δ_5) there are at most $r\sqrt{\alpha}n$ such vertices x.) Without loss of generality assume that $x \in A_1$. Then by (δ_1) and (δ_4) there exist $2 \leq i' \neq i'' \leq r$ such that:

- $d^+_{G'}(x, A_{i'}) \geq \beta n/3$;
- $d^-_{G'}(x, A_{i''}) \geq \beta n/3$;
- $d^+_{G'}(x, A_j), d^-_{G'}(x, A_j) \geq \beta n/3$ for all $2 \leq j \leq r$ such that $j \neq i', i''$.

Without loss of generality assume that $i' = 2$ and $i'' = 3$. Write $V(T) = \{x_1, \dots, x_r\}$ where $x_1x_2, x_3x_1 \in E(T)$. Since $d^+_{G'}(x, A_2) \geq \beta n/3$, (δ_5) implies that there is a vertex $y_2 \in A_2 \setminus V(\mathcal{M}^*)$ such that $xy_2 \in E(G')$ and y_2 is $(2\beta', i)$-excellent for all $3 \leq i \leq r$. Further, since $d^-_{G'}(x, A_3) \geq \beta n/3$, the choice of y_2 together with (δ_5) ensures that there is a vertex $y_3 \in A_3 \setminus V(\mathcal{M}^*)$ such that $y_3x, y_2y_3, y_3y_2 \in E(G')$ and y_3 is $(2\beta', i)$-excellent for all $4 \leq i \leq r$. In particular, $\{x, y_2, y_3\}$ spans a copy of $T[x_1, x_2, x_3]$ in G' that is vertex-disjoint from $\mathcal{M}^*$. Continuing in this way we obtain vertices $y_2, \dots, y_r$ such that $y_i \in A_i \setminus V(\mathcal{M}^*)$ and $\{x, y_2, \dots, y_r\}$ spans a copy of T in G' where $x, y_2, \dots, y_r$ play the roles of $x_1, \dots, x_r$ respectively. This argument shows that we can indeed find a T-packing $\mathcal{M}'$ that satisfies (ε_1)–(ε_3).

Remove all those vertices covered by $\mathcal{M}'$ from G' (and from the classes $A_1, \dots, A_r$). Call the resulting digraph G''. So $n'' := |G''| \geq (1 - 2r^2\sqrt{\alpha})n$ by (ε_1) and $|A_i| = n''/r$ for all $1 \leq i \leq r$ by (ε_3). Further, (ε_2) implies that, given any $x \in V(G'') \setminus A_i$, x is $(2\beta', i)$-excellent (for all $1 \leq i \leq r$). So Theorem 10.2 implies that G'' contains a perfect K_r-packing and thus a perfect T-packing $\mathcal{M}''$. Set $\mathcal{M} := \mathcal{M}' \cup \mathcal{M}'' \cup \mathcal{M}_1 \cup \dots \cup \mathcal{M}_r$. Then $\mathcal{M}$ is a perfect T-packing in G, as required.

Case 2: $B \neq \emptyset$. In this case $s \leq r - 1$. Given any $1 \leq i \leq s$, (δ_1) and (δ_2) imply that there are at least

$$2|B||A_i| - r^2\sqrt{\alpha}n^2 - 4\sqrt{\alpha}n^2 \geq 2|B||A_i| - \alpha^{1/3}n^2$$

edges in G' with one endpoint in A_i and the other endpoint in B. Since $\alpha \ll \beta \ll 1/r$, this implies that there are at most $\alpha^{1/4}n/r$ vertices in A_i that are not (β, B)-excellent. Let $V_{ex,B}$ denote the set of all those vertices in $V(G') \setminus B$ that are not (β, B)-excellent. So $|V_{ex,B}| \leq \alpha^{1/4}n$.

Then G' contains a T-packing $\mathcal{M}'$ such that:

- (ε'_1) $\mathcal{M}'$ contains $m' \leq 2r\sqrt{\alpha}n + 2\alpha^{1/4}n \leq 3\alpha^{1/4}n$ copies of T;
- (ε'_2) If $x \in V(G') \setminus A_i$ is not (β, i)-excellent then x is contained in a copy of T in $\mathcal{M}'$ (for any $1 \leq i \leq s$). Similarly, if $x \in V(G') \setminus B$ is not (β, B)-excellent then x is contained in a copy of T in $\mathcal{M}'$;
- (ε'_3) $\mathcal{M}'$ covers exactly m' vertices from A_i (for each $1 \leq i \leq s$) and $m'(r - s)$ vertices from B.

To prove that such a T-packing $\mathcal{M}'$ exists, we will use the follow three claims.

Claim 11.1. *Let $x \in V(G') \setminus B$ be such that x is not (β, B)-excellent and let $W \subseteq V(G') \setminus \{x\}$ where $|W| \leq \alpha^{1/5}n$. Then there are two vertex-disjoint copies T', T'' of T in G' so that:*

- (i) *$V(T') \cup V(T'')$ contains two vertices from A_i (for each $1 \leq i \leq s$) and $2(r - s)$ vertices from B;*
- (ii) *$x \in V(T') \cup V(T'')$ and $V(T') \cup V(T'')$ is disjoint from W.*

Proof. To prove the claim consider a vertex $x \in V(G') \setminus B$ that is not (β, B)-excellent. If $s = 1$ then $x \in A_1$. Further, since x is not (β, B)-excellent, (δ_1) implies that

- $d^+_{G'}(x, A_1) \geq \beta n - r^2\sqrt{\alpha}n' \geq \beta n/2$ or $d^-_{G'}(x, A_1) \geq \beta n/2$.

Without loss of generality assume that $d^+_{G'}(x, A_1) \geq \beta n/2$.

Fix a vertex y in A_1 such that

- $xy \in E(G')$;
- y is (β, B)-excellent;
- $y \notin W$.

Note that there are at least $\beta n/2 - \alpha^{1/4}n - \alpha^{1/5}n \geq \beta n/4$ choices for y since $|V_{ex,B}| \leq \alpha^{1/4}n$ and $|W| \leq \alpha^{1/5}n$. Then by repeatedly applying (50) we can greedily extend xy to a copy T' of T in G' containing two vertices from A_1 (namely x and y) and $r - 2$ vertices from B so that T' is disjoint from W.

Next suppose that $s \geq 2$. Since x is not (β, B)-excellent, (δ_1) and (δ_4) imply that there exist $1 \leq i' \neq i'' \leq s$ such that:

- $d^+_{G'}(x, A_{i'}) \geq \beta n/3$;
- $d^-_{G'}(x, A_{i''}) \geq \beta n/3$;
- $d^+_{G'}(x, A_j), d^-_{G'}(x, A_j) \geq \beta n/3$ for all $1 \leq j \leq r$ such that $j \neq i', i''$.

Without loss of generality assume that $x \in A_1$, $i' = 1$ and $i'' = 2$ (the other cases are similar).

Write $V(T) = \{x_1, \dots, x_r\}$ where $x_1x_2, x_3x_1 \in E(T)$. Since $d^+_{G'}(x, A_1) \geq \beta n/3$, $|V_{ex,B}| \leq \alpha^{1/4}n$ and $|W| \leq \alpha^{1/5}n$, (δ_5) implies that there is a vertex $y_2 \in A_1$ such that:

- $xy_2 \in E(G')$;
- y_2 is $(2\beta', i)$-excellent for all $2 \leq i \leq s$;
- y_2 is (β, B)-excellent;
- $y_2 \notin W$.

Since $d^-_{G'}(x, A_2) \geq \beta n/3$, the choice of y_2 together with (δ_5) ensures that there is a vertex $y_3 \in A_2$ such that:

- $y_3x, y_2y_3, y_3y_2 \in E(G')$;
- y_3 is $(2\beta', i)$-excellent for all $3 \leq i \leq s$;
- y_3 is (β, B)-excellent;
- $y_3 \notin W$.

In particular, $\{x, y_2, y_3\}$ spans a copy of $T[x_1, x_2, x_3]$ in G'. Continuing in this fashion and then repeatedly applying (50) we can greedily find a copy T' of T in G' that covers two vertices in A_1 (namely x and y_2), one vertex from A_j (for $2 \leq j \leq s$) and $r - s - 1$ vertices from B so that T' is disjoint from W. So in both cases we have found a copy T' of T in G' that covers two vertices in A_1 (including x), one vertex from A_j (for $2 \leq j \leq s$) and $r - s - 1$ vertices from B.

Let T^* be a subtournament of T of size $r - s + 1$. Let B' denote the set of vertices $x \in B \setminus (V(T') \cup W)$ that are $(2\beta', j)$-excellent for all $1 \leq j \leq s$. Then $|B'| \geq |B| - r\sqrt{\alpha}n - r - \alpha^{1/5}n$ by (δ_5). Together with (50) this implies that

$$\delta^0(G'[B']) \geq \left(1 - \frac{1}{r-s} - \alpha^{1/6}\right)|B'|.$$

Moreover, (δ_3) implies that $G'[B']$ does not contain any $(\gamma_{s+1}/4)$-independent set of size $|B'|/(r-s)$. Proposition 6.4 (with $G'[B']$, $|B'|$, $\alpha^{1/6}$, T^*, $r - s + 1$ playing the roles of G, n, ε, T, r respectively) implies that $G'[B']$ contains a copy T_1^* of T^*. The choice of B' ensures that we can greedily extend T_1^* to a copy T'' of T in G' that is disjoint from $V(T') \cup W$ and that covers no vertices from A_1, one vertex from A_j (for $2 \leq j \leq s$) and $r - s + 1$ vertices from B. So together T' and T'' satisfy (i) and (ii). This completes the proof of Claim 11.1. $\square$

Claim 11.2. *Let $x \in V(G') \setminus (A_i \cup B)$ be such that x is not (β, i)-excellent (for some $1 \leq i \leq s$) and let $W \subseteq V(G') \setminus \{x\}$ where $|W| \leq \alpha^{1/5}n$. Then there are two vertex-disjoint copies T', T'' of T in G' so that:*

(i) *$V(T') \cup V(T'')$ contains two vertices from A_j (for each $1 \leq j \leq s$) and $2(r - s)$ vertices from B;*
(ii) *$x \in V(T') \cup V(T'')$ and $V(T') \cup V(T'')$ is disjoint from W.*

The proof of Claim 11.2 is essentially identical to the proof of Claim 11.1, so we omit it.

Claim 11.3. *Let $x \in B$ be such that x is not (β, i)-excellent (for some $1 \leq i \leq s$) and let $W \subseteq V(G') \setminus \{x\}$ where $|W| \leq \alpha^{1/5}n$. Then there is a copy T' of T in G' so that:*

(i) *$V(T')$ contains one vertex from A_j (for each $1 \leq j \leq s$) and $r - s$ vertices from B;*
(ii) *$x \in V(T')$ and $V(T')$ is disjoint from W.*

It is easy to see that (δ_1), (δ_4) and (50) imply that we can greedily construct a copy T' of T as in Claim 11.3.

Recall that $|V_{ex,B}| \leq \alpha^{1/4}n$. Together with (δ_5) this implies that we can repeatedly apply Claims 11.1–11.3 to obtain a T-packing $\mathcal{M}'$ in G' satisfying (ε_1')–(ε_3').

Remove all those vertices covered by $\mathcal{M}'$ from G' (and from the classes $A_1, \dots, A_s, B$). Call the resulting digraph G''. So $n'' := |G''| \geq (1 - \alpha^{1/5})n$ by (ε_1') and $|A_i| = n''/r$ for all $1 \leq i \leq s$ and $|B| = (r-s)n''/r$ by (ε_3'). Further, (ε_2') implies that, given any $x \in V(G'') \setminus A_i$, x is (β, i)-excellent (for all $1 \leq i \leq s$) and every vertex $y \in V(G'') \setminus B$ is (β, B)-excellent.

Suppose that $|B| = n''/r$. Then as in Case 1, Theorem 10.2 implies that G'' contains a perfect K_r-packing and thus a perfect T-packing $\mathcal{M}''$. Set $\mathcal{M} := \mathcal{M}' \cup \mathcal{M}'' \cup \mathcal{M}_1 \cup \dots \cup \mathcal{M}_s$. Then $\mathcal{M}$ is a perfect T-packing in G, as required.

Next suppose that $|B| \geq 2n''/r$. Let T^* be a subtournament of T on $r-s$ vertices such that $T^* \neq C_3$. (Note that if $r-s = 3$ then $r \geq 4$. Every tournament on at least four vertices contains T_3, so we indeed may choose $T^* \neq C_3$.) By (50) and (ε_1') we have that

$$\delta^0(G''[B]) \geq (1 - 1/(r-s) - \alpha^{1/5})|B|.$$

Moreover, (δ_3) implies that $G''[B]$ does not contain any $(\gamma_{s+1}/4)$-independent set of size $n''/r = |B|/(r-s)$. Further, if $|B| = 2n''/r$ (and so $r-s=2$) then by assumption $G''[B]$ is not $\eta/2$-close to $2K_{n''/r}$. Thus, by Theorem 10.1 and Proposition 10.6, $G''[B]$ contains a perfect T^*-packing $\mathcal{M}^*$.

Define an auxiliary digraph G^* from G'' as follows. G^* has vertex set $A_1 \cup \dots \cup A_s \cup B^*$ where $|B^*| = n''/r$ and each vertex $x \in B^*$ corresponds to a unique copy T^*_x of T^* from $\mathcal{M}^*$. The edge set of G^* consists of every edge $wz \in E(G'')$ such that $w \in A_i$ and $z \in A_j$ for some $i \neq j$ together with the following edges: Suppose that $x \in B^*$ and $y \in V(G^*) \setminus B^*$. Then

- $yx \in E(G^*)$ precisely if y sends an edge to every vertex in T^*_x in G'';
- $xy \in E(G^*)$ precisely if y receives an edge from every vertex in T^*_x in G''.

Note that G^* is an $(s+1)$-partite digraph with vertex classes of size n''/r. Further,

$$\bar{\delta}^+(G^*), \bar{\delta}^-(G^*) \geq n''/r - \beta r n.$$

So by Theorem 10.2, G^* contains a perfect K_{s+1}-packing. By construction of G^* this implies that G'' contains a perfect T-packing $\mathcal{M}''$. Set $\mathcal{M} := \mathcal{M}' \cup \mathcal{M}'' \cup \mathcal{M}_1 \cup \dots \cup \mathcal{M}_s$. Then $\mathcal{M}$ is a perfect T-packing in G, as required.

11.2. **The case when $|B| = 2n/r$ and $G[B]$ is close to $2K_{n/r}$.** In this subsection we consider the case when $|B| = 2n/r$ and $G[B]$ is η-close to $2K_{n/r}$. Thus, there exists a partition B_1, B_2 of B such that $|B_1| = |B_2| = n/r$ and $e_G(B_1, B_2) \leq \eta |B|^2$. (49) implies that $\delta^0(G[B]) \geq |B|/2 = n/r$.

For $i = 1, 2$ and $\delta > 0$ we say that a vertex $x \in B_i$ is *(δ, B_i)-excellent* if $d^+_G(x, B_i), d^-_G(x, B_i) \geq |B_i| - \delta n$. (Later on we will modify the classes B_1, B_2. When referring to, for example, (δ, B_1)-excellent vertices, we mean with respect to the current class B_1 and not the original class.) Note that there are at most $\eta^{1/2}|B|$ vertices $x \in B_i$ that are not $(\eta^{1/2}, B_i)$-excellent for $i = 1, 2$.

Let V^1_{ex} denote the set of vertices $x \in B_1$ that are not $(\eta^{1/2}, B_1)$-excellent. Define V^2_{ex} analogously. Given a vertex $x \in V^1_{ex}$, if $d^+_G(x, B_2) \geq n/2r$ then move x into B_2. Similarly, if $x \in V^2_{ex}$ and $d^+_G(x, B_1) \geq n/2r$ then move x into B_1. Thus, the following conditions hold:

(ζ_1) $n/r - \eta^{1/2} n \leq |B_1|, |B_2| \leq n/r + \eta^{1/2} n$;
(ζ_2) $e_G(B_1, B_2) \leq 5\eta^{1/2}|B|^2$;
(ζ_3) There are at most $2\eta^{1/2}|B|$ vertices $x \in B_i$ that are not $(2\eta^{1/2}, B_i)$-excellent (for $i = 1, 2$);
(ζ_4) Given any $x \in B_i$, $d^+_G(x, B_i) \geq n/3r$ (for $i = 1, 2$).

Actually, there is slack in conditions (ζ_1) and (ζ_2). Indeed, if we move a single vertex from B_2 to B_1 (or vice versa) then (ζ_1) and (ζ_2) still hold.

Since $|B| = 2n/r$ is even, either $|B_1|$ and $|B_2|$ are even or $|B_1|$ and $|B_2|$ are odd. Suppose that $|B_1|$ and $|B_2|$ are odd. Without loss of generality assume that $|B_2| \geq n/r$. Fix a vertex $b_1 \in B_1$ such that

(i) b_1 is $(2\beta', i)$-excellent for all $1 \leq i \leq s$.

(Such a vertex b_1 exists by (α_5).) Then by (5) there is a vertex $b_2 \in B_2$ such that $b_1b_2 \in E(G)$. Further, (5) implies that

(ii) $d^+_G(b_2, A_i), d^-_G(b_2, A_i) \geq \eta n$ for all $1 \leq i \leq s$ or;

(iii) $d^+_G(b_2, B_1) \geq n/3r$ or $d^-_G(b_2, B_1) \geq n/3r$.

If (iii) holds then move b_2 into B_1. Otherwise, we leave the partition B_1, B_2 of B unchanged. Thus, the following conditions hold:

(η_1) $n/r - \eta^{1/2}n \leq |B_1|, |B_2| \leq n/r + \eta^{1/2}n$;

(η_2) $e_G(B_1, B_2) \leq 5\eta^{1/2}|B|^2$;

(η_3) There are at most $3\eta^{1/2}|B|$ vertices $x \in B_i$ that are not $(3\eta^{1/2}, B_i)$-excellent (for $i = 1, 2$);

(η_4) Given any $x \in B_i$, $d^+_G(x, B_i) \geq n/4r$ or $d^-_G(x, B_i) \geq n/4r$ (for $i = 1, 2$).

Additionally, one of the following conditions holds:

(η_5) $|B_1|$ and $|B_2|$ are even or;

(η_6) $|B_1|$ and $|B_2|$ are odd. Further, there exist $b_1 \in B_1$ and $b_2 \in B_2$ such that
- $b_1b_2 \in E(G)$;
- b_1 is $(2\beta', i)$-excellent for all $1 \leq i \leq s$;
- $d^+_G(b_2, A_i), d^-_G(b_2, A_i) \geq \eta n$ for all $1 \leq i \leq s$.

If (η_6) holds then we will extend the edge b_1b_2 into a copy of T in G. First though we will cover the 'exceptional vertices' in G with T-packings.

Covering the exceptional vertices with T-packings. Given any $1 \leq i \leq s$, let $V_{ex,i}$ denote the set of $(\beta/2, i)$-exceptional vertices in $V(G) \setminus A_i$. (Note that if (η_6) holds then $b_1, b_2 \notin V_{ex,i}$.) (α_5) implies that $c_i := |V_{ex,i}| \leq \sqrt{\alpha}n$ for all $1 \leq i \leq s$. Then there exist vertex-disjoint T-packings $\mathcal{M}_1, \dots, \mathcal{M}_s$ in G so that, for each $1 \leq i \leq s$:

(θ_1) $\mathcal{M}_i$ contains precisely c_i disjoint copies of T;

(θ_2) Each vertex from $V_{ex,i}$ lies in a copy of T in $\mathcal{M}_i$;

(θ_3) $\mathcal{M}_i$ covers precisely c_i vertices from A_j (for each $1 \leq j \leq s$) and precisely $2c_i$ vertices from B;

(θ_4) $\mathcal{M}_i$ covers an even number of vertices from B_1 and an even number of vertices from B_2;

(θ_5) If (η_6) holds then $\mathcal{M}_i$ does not cover b_1 or b_2.

Note that the same argument used to construct $\mathcal{M}_1, \dots, \mathcal{M}_s$ in Section 11.1 shows that we can construct $\mathcal{M}_1, \dots, \mathcal{M}_s$ here so that (θ_1)–(θ_3) hold. It is not difficult to see that we can additionally ensure that (θ_4) and (θ_5) hold.

Extending the edge b_1b_2 to a copy of T. If $|B_1|$ and $|B_2|$ are even set $\mathcal{T} := \emptyset$. Otherwise, (η_6) holds. In this case, we can greedily construct a copy T' of T in G such that:

- T' is vertex-disjoint from $\mathcal{M}_1, \dots, \mathcal{M}_s$;
- T' contains b_1b_2 (and so one vertex from each of B_1 and B_2) and precisely one vertex from each of $A_1, \dots, A_s$.

Set $\mathcal{T} := \{T'\}$.

Covering the remaining vertices. Remove all those vertices covered by $\mathcal{M}_1, \dots, \mathcal{M}_s, \mathcal{T}$ from G (and from the classes $A_1, \dots, A_s, B$ and from B_1, B_2). Call the resulting digraph G'. So $n' := |G'| \geq (1 - 2r^2\sqrt{\alpha})n$ by (θ_1), $|A_i| = n'/r$ for all $1 \leq i \leq s$ and $|B| = 2n'/r$ by (θ_3) and the choice of $\mathcal{T}$.

Further, (5) and (α_1)–(α_5) imply that the following conditions hold:

(ι_1) $\delta^0(G') \geq (1 - 1/r - 2r^2\sqrt{\alpha})n \geq (1 - 1/r - 2r^2\sqrt{\alpha})n'$;

(ι_2) $A_1, \dots, A_s, B$ is a partition of $V(G')$ so that A_i is a $2\sqrt{\alpha}$-independent set of size n'/r in G' (for all $1 \leq i \leq s$);

(ι_3) Every vertex in $V(G') \setminus A_i$ is $(\beta/3, i)$-acceptable (for each $1 \leq i \leq s$);

(ι_4) There are at most $\sqrt{\alpha}n$ vertices in $V(G') \setminus A_i$ that are not $(2\beta', i)$-excellent (for each $1 \leq i \leq s$).

In particular, note that (ι_3) follows from (θ_2). Further, (η_1)–(η_6) and (θ_1)–(θ_5) together with the choice of $\mathcal{T}$ implies that:

(κ_1) $n/r - 2\eta^{1/2}n \leq |B_1|, |B_2| \leq n/r + \eta^{1/2}n$ and $|B_1|$ and $|B_2|$ are even;

(κ_2) There are at most $4\eta^{1/2}|B|$ vertices $x \in B_i$ that are not $(3\eta^{1/2}, B_i)$-excellent (for $i = 1, 2$);

(κ_3) Given any $x \in B_i$, $d^+_{G'}(x, B_i) \geq n/5r$ or $d^-_{G'}(x, B_i) \geq n/5r$ (for $i = 1, 2$).

Note that (ι_1) implies that

$$d^+_{G'}(y, B), d^-_{G'}(y, B) \geq \left(\frac{1}{2} - \alpha^{1/3}\right)|B|$$

for all $y \in V(G')$.

Let $V_{ex,B}$ denote the set of all those vertices in $V(G') \setminus B$ that are not (β, B)-excellent. By arguing as in Case 2 from Section 11.1 we see that $|V_{ex,B}| \leq \alpha^{1/4}n$.

G' contains a T-packing $\mathcal{M}'$ such that:

(λ_1) $\mathcal{M}'$ contains $m' \leq 2r\sqrt{\alpha}n + 2\alpha^{1/4}n \leq 3\alpha^{1/4}n$ copies of T;

(λ_2) If $x \in V(G') \setminus A_i$ is not (β, i)-excellent then x is contained in a copy of T in $\mathcal{M}'$ (for any $1 \leq i \leq s$). Similarly, if $x \in V(G') \setminus B$ is not (β, B)-excellent then x is contained in a copy of T in $\mathcal{M}'$;

(λ_3) $\mathcal{M}'$ covers exactly m' vertices from A_i (for each $1 \leq i \leq s$) and $2m'$ vertices from B. Further, $\mathcal{M}'$ covers an even number of vertices from B_1 and an even number of vertices from B_2.

To prove that such a T-packing $\mathcal{M}'$ exists, we will use the follow three claims.

Claim 11.4. *Let $x \in V(G') \setminus B$ be such that x is not (β, B)-excellent and let $W \subseteq V(G') \setminus \{x\}$ where $|W| \leq \alpha^{1/5}n$. Then there are two vertex-disjoint copies T', T'' of T in G' so that:*

(i) *$V(T') \cup V(T'')$ contains two vertices from A_i (for each $1 \leq i \leq s$) and four vertices from B;*

(ii) *$x \in V(T') \cup V(T'')$ and $V(T') \cup V(T'')$ is disjoint from W.*

(iii) *$V(T') \cup V(T'')$ contains an even number of vertices from B_1 and an even number of vertices from B_2.*

Proof. To prove the claim consider a vertex $x \in V(G') \setminus B$ that is not (β, B)-excellent. Without loss of generality suppose that $x \in A_1$. By arguing precisely as in Claim 11.1 one can find a copy T' of T in G' that covers two vertices in A_1 (including x), one vertex from A_i (for $2 \leq i \leq s$) and one vertex from B so that T' is disjoint from W.

Without loss of generality suppose that T' covers a vertex from B_1. Then by applying (κ_2) and (ι_4) it is easy to see that we can greedily construct a copy T'' of T so that T'' covers three vertices from B_1, no vertices from A_1 and one vertex from A_j (for all $2 \leq j \leq s$) and so that T'' is disjoint from $V(T') \cup W$. Together T' and T'' satisfy (i)–(iii). This completes the proof of Claim 11.4. □

Claim 11.5. *Let $x \in V(G') \setminus (A_i \cup B)$ be such that x is not (β, i)-excellent (for some $1 \leq i \leq s$) and let $W \subseteq V(G') \setminus \{x\}$ where $|W| \leq \alpha^{1/5}n$. Then there are two vertex-disjoint copies T', T'' of T in G' so that:*

(i) *$V(T') \cup V(T'')$ contains two vertices from A_i (for each $1 \leq i \leq s$) and four vertices from B;*

(ii) *$x \in V(T') \cup V(T'')$ and $V(T') \cup V(T'')$ is disjoint from W.*

(iii) *$V(T') \cup V(T'')$ contains an even number of vertices from B_1 and an even number of vertices from B_2.*

The proof of Claim 11.5 is essentially identical to the proof of Claim 11.4, so we omit it.

Claim 11.6. *Let $x \in B$ be such that x is not (β, i)-excellent (for some $1 \leq i \leq s$) and let $W \subseteq V(G') \setminus \{x\}$ where $|W| \leq \alpha^{1/5}n$. Then there is a copy T' of T in G' so that:*

(i) *$V(T')$ contains one vertex from A_j (for each $1 \leq j \leq s$) and two vertices from either B_1 or B_2;*
(ii) *$x \in V(T')$ and $V(T')$ is disjoint from W.*

It is easy to see that (κ_3), (ι_3) and (ι_4) imply that we can greedily construct a copy T' of T as in Claim 11.6.

Recall that $|V_{ex,B}| \leq \alpha^{1/4}n$. Together with (ι_4) this implies that we can repeatedly apply Claims 11.4–11.6 to obtain a T-packing $\mathcal{M}'$ in G' satisfying (λ_1)–(λ_3).

Remove all those vertices covered by $\mathcal{M}'$ from G' (and from the classes $A_1, \dots, A_s, B$ and from B_1 and B_2). Call the resulting digraph G''. So $n'' := |G''| \geq (1 - \alpha^{1/5})n$ by (λ_1) and $|A_i| = n''/r$ for all $1 \leq i \leq s$ and $|B| = 2n''/r$ by (λ_3). (λ_2) implies that, given any $x \in V(G'') \setminus A_i$, x is (β, i)-excellent (for all $1 \leq i \leq s$) and every vertex $y \in V(G'') \setminus B$ is (β, B)-excellent.

Moreover, (κ_1) and (λ_3) imply that $|B_1|$ and $|B_2|$ are even. (κ_1)–(κ_3) and (λ_1) imply that

- $|B_1|, |B_2| \geq n/r - 3\eta^{1/2}n$;
- There are at most $5\eta^{1/2}|B|$ vertices $x \in B_i$ that are not $(3\eta^{1/2}, B_i)$-excellent (for $i = 1, 2$);
- Given any $x \in B_i$, $d^+_{G''}(x, B_i) \geq n/6r$ or $d^-_{G''}(x, B_i) \geq n/6r$ (for $i = 1, 2$).

It is easy to see that this implies that $G''[B]$ contains a perfect matching $\mathcal{P}$.

Define an auxiliary digraph G^* from G'' as follows. G^* has vertex set $A_1 \cup \dots \cup A_s \cup B^*$ where $|B^*| = n''/r$ and each vertex $x \in B^*$ corresponds to a unique edge e_x from $\mathcal{P}$. The edge set of G^* consists of every edge $wz \in E(G'')$ such that $w \in A_i$ and $z \in A_j$ for some $i \neq j$ together with the following edges: Suppose that $x \in B^*$ and $y \in V(G^*) \setminus B^*$. Then

- $yx \in E(G^*)$ precisely if y sends an edge to both vertices on e_x in G'';
- $xy \in E(G^*)$ precisely if y receives an edge from both vertices on e_x in G''.

Note that G^* is an $(r-1)$-partite digraph with vertex classes of size n''/r. Further,

$$\bar{\delta}^+(G^*), \bar{\delta}^-(G^*) \geq n''/r - 2\beta n.$$

So by Theorem 10.2, G^* contains a perfect K_{r-1}-packing. By construction of G^* this implies that G'' contains a perfect T-packing $\mathcal{M}''$. Set $\mathcal{M} := \mathcal{M}' \cup \mathcal{M}'' \cup \mathcal{M}_1 \cup \dots \cup \mathcal{M}_s \cup \mathcal{T}$. Then $\mathcal{M}$ is a perfect T-packing in G, as required. □

12. Proof of Lemma 5.6

Let $0 < 1/n_0 \ll \alpha \ll \beta \ll \gamma \ll 1$. Suppose that G is as in the statement of the lemma and let A_1, A_2, A_3 denote the partition of $V(G)$ corresponding to the vertex classes of $Ex(n)$.

Given any $1 \leq i \leq 3$, any $x \in A_i$ and any $\delta > 0$, we say that x is *externally δ-excellent* if x sends out at least $(1-\delta)|A_{i+1}|$ edges to A_{i+1} in G and receives at least $(1-\delta)|A_{i-1}|$ edges from A_{i-1} in G. (Here indices are taken mod 3.) Otherwise we say that x is *externally δ-bad*. Since G α-contains $Ex(n)$ and $\alpha \ll \beta \ll \gamma$, there are at most βn vertices in G that are externally γ-bad.

We also require analogous definitions corresponding to edges inside our vertex classes. Indeed, given any $1 \leq i \leq 3$, any $x \in A_i$ and any $\delta > 0$, we say that x is *internally δ-excellent* if x sends out at least $(1-\delta)|A_i|$ edges in $G[A_i]$ and receives at least $(1-\delta)|A_i|$ edges in $G[A_i]$. Otherwise we say that x is *internally δ-bad*. Since G α-contains $Ex(n)$ and $\alpha \ll \beta \ll \gamma$, there are at most βn vertices in G that are internally γ-bad. Throughout the proof we will modify the classes A_1, A_2 and A_3. When referring to, for example, internally excellent vertices, we mean with respect to the current classes A_1, A_2 and A_3 rather than the original partition of $V(G)$.

Since $\delta^0(G) \geq 2n/3 - 1$, given any vertex $x \in V(G)$ there is an $1 \leq i_x \leq 3$ such that x sends out at least $n/10$ edges to A_{i_x} in G and receives at least $n/10$ edges from A_{i_x} in G. For each vertex $x \in V(G)$ that is internally γ-bad we move x into the class A_{i_x}. Thus, we now have that:

(a) $|A_i| = n/3 \pm 2\beta n$ for each $1 \leq i \leq 3$;
(b) $\delta^0(G[A_i]) \geq n/20$ for each $1 \leq i \leq 3$;
(c) All but at most βn vertices in G are internally 2γ-excellent;
(d) All but at most $2\beta n$ vertices in G are externally 2γ-excellent.

Actually there is some slack in these conditions. Indeed, (a)–(d) hold even if we remove three vertices from G (and thus from A_1, A_2 and A_3).

Changing the parity of the class sizes. Our next task is to remove (the vertices of) at most one copy of C_3 from G to ensure that $|A_1| \equiv |A_2| \equiv |A_3| \pmod 3$. If $|A_1| \equiv |A_2| \equiv |A_3| \pmod 3$ already then we do not remove a copy of C_3. Recall that $n \equiv 0 \pmod 3$. Therefore, without loss of generality we may assume that $|A_1| \equiv 0$, $|A_2| \equiv 1$ and $|A_3| \equiv 2 \pmod 3$. (The other cases follow analogously.)

In this case there exists a $1 \leq j \leq 3$ such that $|A_j| \geq n/3 + 1$. Suppose that $|A_2| \geq n/3 + 1$. Fix some $a \in A_1$ that is externally 2γ-excellent. (Such a vertex exists by (a) and (d).) Since $d^-(a) \geq 2n/3 - 1$ there exists a vertex $b \in A_2$ such that $ba \in E(G)$. Further, since a is externally 2γ-excellent and $d^-_{G[A_2]}(b) \geq n/20$ by (b), there is a vertex $c \in A_2$ such that $ac, cb \in E(G)$. Thus, a, b and c together span a copy C'_3 of C_3 in G. Remove $V(C'_3)$ from G (and thus from A_1 and A_2). So now $|A_1| \equiv |A_2| \equiv |A_3| \equiv 2 \pmod 3$. In all other cases, we can similarly remove three vertices from G that span a copy C'_3 of C_3 so that $|A_1| \equiv |A_2| \equiv |A_3| \pmod 3$. As outlined earlier, (a)–(d) still hold.

Covering the externally bad vertices and balancing the class sizes. (b)–(d) ensure that we can greedily construct a collection $\mathcal{C}_1$ of at most $2\beta n$ vertex-disjoint copies of C_3 in G that together cover all those vertices in G that are externally 2γ-bad. In particular, (b) and (c) ensure that we can choose each such copy of C_3 to lie in one of $G[A_1]$, $G[A_2]$ and $G[A_3]$. So after removing the vertices in $\mathcal{C}_1$ from G we still have that $|A_1| \equiv |A_2| \equiv |A_3| \pmod 3$. Further, $n/3 - 8\beta n \leq |A_i| \leq n/3 + 2\beta n$ for each $1 \leq i \leq 3$.

These last two properties together with (b) and (c) ensure that we can greedily construct a collection $\mathcal{C}_2$ of at most $7\beta n$ vertex-disjoint copies of C_3 in G such that:

- The copies of C_3 in $\mathcal{C}_2$ are vertex-disjoint from the copies of C_3 in $\mathcal{C}_1$;
- Each copy of C_3 in $\mathcal{C}_2$ lies in one of $G[A_1]$, $G[A_2]$ and $G[A_3]$.
- By removing the vertices in $\mathcal{C}_1 \cup \mathcal{C}_2$ from G we have that $|A_1| = |A_2| = |A_3| \geq n/3 - 8\beta n$.

Covering the remaining vertices. Remove the vertices in $\mathcal{C}_1 \cup \mathcal{C}_2$ from G (and thus from A_1, A_2 and A_3). The choice of $\mathcal{C}_1$ ensures that every vertex now in G is externally 3γ-excellent and the choice of $\mathcal{C}_2$ ensures that now $|A_1| = |A_2| = |A_3| \geq n/3 - 8\beta n$. Let $G' := G[A_1, A_2] \cup G[A_2, A_3] \cup G[A_3, A_1]$. By ignoring the orientations of the edges in G', we can (for example) apply Theorem 1.2 to find a perfect C_3-packing $\mathcal{C}_3$ in G'. (Indeed, the underlying graph of G' satisfies the minimum degree condition in Theorem 1.2 since every vertex in $V(G')$ is externally 3γ-excellent.) The union of $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and C'_3 (if it exists) is a perfect C_3-packing in G, as desired. $\square$

13. Concluding remarks

In this section we raise a number of open questions concerning perfect packings in digraphs.

13.1. **Minimum degree conditions forcing perfect tournament packings.** In [3], Czygrinow, DeBiasio, Kierstead and Molla proved the following minimum degree result for perfect transitive tournament packings.

Theorem 13.1 (Czygrinow, DeBiasio, Kierstead and Molla [3]). *Let $n, r \in \mathbb{N}$ such that r divides n. Then every digraph G on n vertices with*

$$\delta(G) \geq 2(1 - 1/r)n - 1$$

contains a perfect T_r-packing.

Note that Conjecture 1.4 would imply Theorem 13.1. At first sight one may ask whether T_r can be replaced by *any* $T \in \mathcal{T}_r$ in Theorem 13.1. However, the following result of Wang [25] shows that one requires a significantly larger minimum degree condition in the case when $T = C_3$.

Theorem 13.2 (Wang [25]). *Let $n \in \mathbb{N}$ such that 3 divides n. If G is a digraph on n vertices and*

$$\delta(G) \geq \frac{3n-3}{2}$$

then G contains a perfect C_3-packing. Moreover, if $n/3$ is odd, then there is a digraph G' on n vertices with $\delta(G') = \frac{3n-5}{2}$ which does not contain a perfect C_3-packing.

Together with Zhang [26], Wang also characterised the minimum degree threshold that ensures a digraph contains a perfect C_4-packing. (Here C_4 denotes the directed cycle on four vertices.) Czygrinow, Kierstead and Molla [4] showed that the degree condition in Theorem 13.2 can be relaxed to $\delta(G) \geq (4n-3)/3$ if we instead seek a perfect packing consisting of a fixed number of cyclic triangles and at least one transitive triangle.

In light of Theorems 13.1 and 13.2 we ask the following question.

Question 13.3. *Let $n, r \in \mathbb{N}$ such that r divides n. Let $T \in \mathcal{T}_r \setminus \{C_3\}$. Does every digraph G on n vertices with $\delta(G) \geq 2(1 - 1/r)n - 1$ contain a perfect T-packing?*

Czygrinow, DeBiasio, Kierstead and Molla [3] have answered Question 13.3 in the affirmative under the additional assumptions that r is sufficiently large and $\delta(G) \geq 2(1 - 1/r + o(1))n$.

13.2. **Packing other directed graphs.** It is also natural to seek minimum degree conditions which ensure a digraph contains a perfect H-packing where H is some digraph other than a tournament. Let K_r denote the complete digraph on r vertices, and write K_r^- to denote K_r minus an edge. (In the undirected setting we also use K_r to denote the complete graph on r vertices.) The following result is a simple consequence of the Hajnal–Szemerédi theorem.

Proposition 13.4. *Let $n, r \in \mathbb{N}$ such that r divides n. Suppose that G is a digraph on n vertices such that*

$$\delta(G) \geq (2 - 1/r)n - 1.$$

Then G contains a perfect K_r-packing.

Proof. Let G' denote the *graph* on $V(G)$ whose edge set consists of all those pairs xy such that both $xy, yx \in E(G)$. Since $\delta(G) \geq (2 - 1/r)n - 1$, we have that $\delta(G') \geq (1 - 1/r)n$. Thus, Theorem 1.1 implies that G' contains a perfect K_r-packing. By definition of G' this implies that G contains a perfect K_r-packing. □

Note that the minimum degree condition in Proposition 13.4 is best-possible: Let $n \in \mathbb{N}$ such that r divides n. Suppose A and B are disjoint vertex sets where $|A| = n/r + 1$ and $|B| = (1 - 1/r)n - 1$. Let G_1 be the digraph with vertex set $A \cup B$ such that G_1 contains all possible edges between A and B, all edges in B and so that A induces a tournament. Then $\delta(G_1) = (2 - 1/r)n - 2$ and G does not contain a perfect K_r-packing since every copy of K_r in G_1 contains at most one vertex from A.

Proposition 13.4 implies that a digraph G whose order n is divisible by r and with $\delta^0(G) \geq (1 - 1/2r)n - 1/2$ contains a perfect K_r-packing. Further, in the digraph G_1 above, set n/r to be even and $G_1[A]$ to be a regular tournament. Then G_1 does not contain a perfect K_r-packing but $\delta^0(G_1) = (1 - 1/2r)n - 1$. Thus, together with Theorem 1.3, this shows that the minimum semidegree threshold that forces a perfect K_r-packing is much higher than the threshold that forces a perfect T-packing for any tournament T on r vertices. It would be interesting to establish the minimum semidegree threshold that forces a perfect K_r^--packing in a digraph. In particular, is this threshold significantly lower than the corresponding threshold for perfect K_r-packings?

Let $m \in \mathbb{N}$ be divisible by 6 and set $n := 2m + 3$. Suppose that G is a digraph on n vertices with the following properties: $V(G) = V_1 \cup V_2$ where $|V_1| = m + 1$ and $|V_2| = m + 2$; $G[V_1]$ and $G[V_2]$ are complete digraphs; the edges between V_1 and V_2 in G form a bipartite tournament that is as regular as possible. Note that, since $|V_1|$ and $|V_2|$ are not divisible by 3, G does not contain a perfect K_3^--packing. Further, $\delta^0(G) = m/2 + 1 + m = (3n - 5)/4$.

Question 13.5. *Let $n \in \mathbb{N}$ be divisible by 3. Does every digraph G on n vertices with $\delta^0(G) \geq 3n/4$ contain a perfect K_3^--packing?*

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