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A Ramsey bound on stable sets in Jordan pillage games*

Manfred Kerber[†] Colin Rowat[‡]

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Abstract

Jordan [2006] defined ‘pillage games’, a class of cooperative games whose dominance operator is represented by a ‘power function’ satisfying coalitional and resource monotonicity axioms. In this environment, he proved that stable sets must be finite. We provide a graph theoretical interpretation of the problem which tightens the finite bound to a Ramsey number. We also prove that the Jordan pillage axioms are independent.

Key words: pillage, cooperative game theory, stable sets

JEL classification numbers: C71; P14

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Jordan [2006] introduced ‘pillage games’, a class of cooperative games whose dominance relation derived from a contest of power between competing coalitions. Power was formalised via three monotonicity axioms in coalitional membership and resource holdings. Exploiting resource monotonicity, he proved that stable sets in pillage games are necessarily finite, a property that distinguishes pillage games from many other classes of cooperative games.¹ We tighten Jordan’s argument by noting an immediate graph theoretical interpretation of the monotonicity argument. This interpretation eases exploitation of transitive consequences

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¹Most famously, the Shapley [1959] ‘signature result’ showed that uncountable stable sets could be built around arbitrary closed components.

of resource monotonicity unused by Jordan, thereby tightening his finite bound to a Ramsey number. Only for two agents is the Ramsey bound tight; otherwise, a huge gap exists between tight bounds in given examples [q.v. Kerber and Rowat, 2009] and the Ramsey bound; closing this remains an open question.² Graph and cooperative game theory have a long common history;³ to our knowledge, though, this is the first application of Ramsey theory to game theory. Other branches of economics drawing on graph theory – such as network economics – may therefore find this useful.

Let $I = \{1, \dots, n\}$ be a finite set of *agents*. An *allocation* divides a unit resource among them, so that the feasible set is a compact, continuous $n - 1$ dimensional simplex:

$$X \equiv \left\{ \{x_i\}_{i \in I} \mid x_i \geq 0, \sum_{i \in I} x_i = 1 \right\}.$$

Following Jordan [2006], a *power function* is defined over subsets of agents and allocations, so that $\pi : 2^I \times X \rightarrow \mathbb{R}$ satisfies:

(WC) if $C \subset C' \subseteq I$ then $\pi(C', \mathbf{x}) \geq \pi(C, \mathbf{x}) \forall \mathbf{x} \in X$;

(WR) if $y_i \geq x_i \forall i \in C \subseteq I$ then $\pi(C, \mathbf{y}) \geq \pi(C, \mathbf{x})$; and

(SR) if $C \neq \emptyset \subseteq I$ and $y_i > x_i \forall i \in C$ then $\pi(C, \mathbf{y}) > \pi(C, \mathbf{x})$.

Axiom WC requires weak monotonicity in coalitional inclusion; WR requires weak monotonicity in resources; SR requires strong monotonicity in resources.

An allocation \mathbf{y} dominates an allocation \mathbf{x} , written $\mathbf{y} \varepsilon \mathbf{x}$ iff

$$\pi(W, \mathbf{x}) > \pi(L, \mathbf{x});$$

where $W \equiv \{i \mid y_i > x_i\}$ and $L \equiv \{i \mid x_i > y_i\}$. By the strict inequality, domination is irreflexive; by axiom SR, it is asymmetric.

For $Y \subset X$, let

$$D(Y) \equiv \{ \mathbf{x} \in X \mid \exists \mathbf{y} \in Y \text{ s.t. } \mathbf{y} \varepsilon \mathbf{x} \};$$

²Saxton [2010] extends a theorem of Erdős and Szekeres to exploit a second of Jordan's axioms; his result, while tighter than the Ramsey bound, remains doubly exponential and relies on a much more sophisticated construction than that here.

³As Richardson [1953] observed, the stable set appears as a *Punktbasis zweiter Art* - a point basis of the second type - in the pioneering text on graph theory [König, 1936]. More recently, Brandt et al. [2007] proved that, when the dominance operator is irreflexive and asymmetric, the question of whether an allocation belongs to a stable set is \mathcal{NP} -complete in the number of possible allocations.

be the *dominion* of Y , the set of allocations dominated by an allocation in Y . Then a set of allocations, $S \subseteq X$, is a *stable set*⁴ iff it satisfies *internal stability*,

$$S \cap D(S) = \emptyset; \text{ and} \tag{IS}$$

and *external stability*,

$$S \cup D(S) = X. \tag{ES}$$

Denote by $2c$ the number of non-empty, non-intersecting sets, W and L , induced by distinct allocations. Thus, c , is the the number of distinct, unordered pairs, (W, L) , so induced. We now state our main result:

Theorem 1. *An internally stable set can contain at most $R_c(4) - 1$ allocations, the diagonal multicolour Ramsey number.*

To aid understanding, we first define the graph theoretical terms used.⁵ A *graph* (or *undirected graph*) $G = (V, E)$ with $E \subseteq [V]^2$ is a set of vertices, V , and a set of edges, E , connecting them. Vertices are *adjacent* if an edge connects them. A graph is *complete* if all pairs of its vertices are adjacent; it is *c-coloured* if each edge is represented by one of c colours. An *r-clique* is a complete subgraph of G on r vertices; it is *monochromatic* if all edges in the clique are of the same colour. A *directed graph* (or *digraph*) is a graph in which each edge is oriented from an initial vertex to a terminal vertex. A *path* is an ordered set of distinct vertices, the initial vertex connected to its successor, and so on until the terminal vertex is reached. An *oriented graph* is a digraph without *loops* (edges whose initial and terminal vertex is the same) or multiple edges between two vertices. Ramsey’s theorem guarantees that any sufficiently large, complete c -coloured graph will have a monochromatic r -clique; Ramsey’s number, $R_c(r)$, is the least bound on the number of vertices in the graph required to make it “sufficiently large”.

Proof. Let S be an internally stable set. By Lemma 1, any distinct $x^i, x^j \in S$ induce one of $2c$ distinct associated non-empty sets, W and L . Consequently, x^i and x^j induce one of c distinct unordered pairs, (W, L) . Now form an undirected, c -coloured, complete graph as follows: let each $x^i \in S$ be a vertex; for each $x^i, x^j \in S$, colour the edge between them according to the induced unordered pair, (W, L) .

The proof of Theorem 2.9 [Jordan, 2006] established that there exists no sequence of allocations $\{x^i\}_{i=1}^4$ in S which holds W and L constant as the sequence

⁴The stable sets of combinatorial optimisation [Korte and Vygen, 2006] are game theory’s internally stable sets: connect two nodes (allocations) with an edge if either of the two allocations dominates the other.

⁵See Diestel [2005], from which the definitions here are taken, for more detailed treatment of the terms and theory.

progresses: by axiom SR, $\pi(W, \mathbf{x}^i)$ increases with i and $\pi(L, \mathbf{x}^i)$ decreases, forcing one allocation in the sequence to dominate another, a violation of internal stability. More than this is true: any such sequence induces a transitive relation on S as $i \leq j \leq k$ imply $i \leq k$. Thus, the subsequences $\{\mathbf{x}^1, \mathbf{x}^3\}$, $\{\mathbf{x}^1, \mathbf{x}^4\}$ and $\{\mathbf{x}^2, \mathbf{x}^4\}$ also maintain W and L constant, so that the existence of the sequence implies a monochromatic 4-clique in the graph.

The converse is also true: a monochromatic 4-clique implies a sequence $\{\mathbf{x}^i\}_{i=1}^4$ in S over which W and L are constant. This follows from two general results: first, a complete digraph contains – by definition – an oriented complete graph, or *tournament*; second, every tournament contains a (directed) Hamilton path that visits each vertex exactly once [Diestel, 2005, §10.3].

By definition of a Ramsey number, an undirected, c -coloured, complete graph with $R_c(4)$ vertices guarantees the existence of a monochromatic 4-clique, which cannot exist. The result follows. \square

Figure 1 illustrates a 4-digraph associated with a 4-clique: while there is a direct path from \mathbf{x}^1 to \mathbf{x}^4 , the sequence $\{\mathbf{x}^1, \mathbf{x}^4\}$ may be augmented to include \mathbf{x}^2 and \mathbf{x}^3 while maintaining constant W and L .

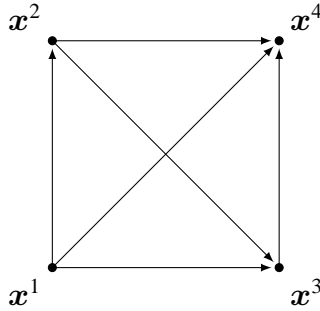


Figure 1: A 4-digraph associated with a sequence, $\{\mathbf{x}^i\}_{i=1}^4$, that maintains constant W and L

Given a fixed number of agents, n , determining c allows determination of the Ramsey bound:

Lemma 1. *There are $2c \equiv 3^n - 2^{n+1} + 1$ non-empty, non-intersecting subsets (W, L) of $I \times I$.*

Proof. There are 3^n functions from $\{1, \dots, n\}$ to $\{W, L, I \setminus (W \cup L)\}$. Of these, 2^n assign no agents to W , so are disqualified. Symmetrically, 2^n assign no agents to L , so are also disqualified. As one function assigns no agents to W or L , it has been disqualified twice; correcting for this, leaves $3^n - 2^{n+1} + 1$ ways of assigning agents to the non-empty, non-intersecting sets W and L . \square

For $n = 2$, $c = 1$; for $n = 3$, $c = \frac{1}{2}(27 - 16 + 1) = 6$; for $n = 4$, it is $\frac{1}{2}(81 - 32 + 1) = 25$. The simplest two examples are:

Example 1. For $n = 2$, the theorem's upper bound is $R_1(4) - 1 = 3$, and therefore tight. This can be demonstrated with the 'wealth is power' function, $\pi_w(C, \mathbf{x}) = \sum_{i \in C} x_i$ [Jordan, 2006].

Example 2. When $n = 3$, the upper bound is $R_6(4) - 1$, which is bounded by

$$16,129 = 127^2 \leq [R_3(4) - 1]^2 < R_6(4) \leq 19,100,738.$$

The bounds are obtained through direct calculation of inequality 5.n and recursive application of inequality 5.a, respectively, in Radziszowski [2006].^{6 7}

For Jordan's wealth is power function, the unique stable set has nine elements. For all power functions satisfying an anonymity, a continuity and a responsiveness axiom, the unique stable set has no more than 15 elements, when it exists [Kerber and Rowat, 2009].

To gain an idea of the numbers involved for larger n , we note the quick bounds $3^c \leq R_c(4) \leq c^{2c+1}$ provided by Saxton [2010].

In closing, note that axioms WC, WR and SR are independent. While perhaps unsurprising, this has not yet been formally established elsewhere, and it may be of interest to see how the 'strong' axiom SR may be satisfied even when the 'weak' axiom WR is violated.

Lemma 2. Axioms WC, WR and SR are independent.

Proof. The 'weakest link' function, $\pi(C, \mathbf{x}) = \min_{i \in C} x_i$, satisfies axioms WR and SR but not WC. The constant function, $\pi(C, \mathbf{x}) = 1$, satisfies WC and WR but not SR. Finally, defining Kronecker's delta, $\delta(x)$, as 0 for all $x \neq 0$ and 1 for $x = 0$, only axiom WR is violated by

$$\pi(C, \mathbf{x}) = v\|C\| + \min_{i \in C} x_i - \delta\left(\min_{i \in C} x_i\right) \left(\sum_{i \in C} x_i - \min_{i \in C} x_i\right);$$

for large v . The first term ensures satisfaction of WC. When x_i strictly increases for all $i \in C$, the first term is unchanged, the second strictly increases, and the third weakly increases. The third term penalises increased resources by members of C if some member still has nothing. \square

⁶The code for the latter may be found at www.cs.bham.ac.uk/~mmk/demos/ramsey-upper-limit.lisp.

⁷Xiaodong et al. [2004, Theorem 2] constructed a graph satisfying the lower bound. The construction in Xiaodong [2002] added a further 595 vertices to the lower bound, for a new lower bound of 16,724.

Transposing an ‘O-ring’ production function [Kremer, 1993] to a pillage game yields $\pi(C, \mathbf{x}) = \prod_{i \in C} x_i$, which, like the weakest link function, also violates only WC. The constant function corresponds to strong property rights. Functions violating only WR are less compelling; in the example used in the proof, $\min_{i \in C} x_i$ can be replaced by $\prod_{i \in C} x_i$ or any other increasing function that remains constant when its least argument is zero.

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