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BROUWER *VERSUS* WITTGENSTEIN ON THE INFINITE AND THE LAW OF EXCLUDED MIDDLE*

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Abstract: Wittgenstein and Brouwer were agreed that some of the higher mathematics of their day rested upon a projection into the infinite of methods that legitimately apply only within finite domains. In this paper I compare and assess the different treatments the two philosophers give of problematic cases involving infinity. For Brouwer, certain claims about infinite sequences provide exceptions to the law of excluded middle; while Wittgenstein argues that the same claims are without sense, since for him the law of excluded middle is a criterion of being a proposition. I end the paper by outlining how the intuitionist might respond to Wittgenstein's arguments.

According to Herbert Feigl, who was with him on the day, Wittgenstein was provoked into returning to philosophy by hearing L.E.J. Brouwer's lecture, 'Mathematik, Wissenschaft und Sprache', in Vienna on 10 March 1928 (see the quotation from Feigl in Pitcher 1964, 8*n*). While Wittgenstein's later writings reject several central Brouwerian theses, a comparison between these thinkers is instructive. As I hope this paper will show, Wittgenstein accepts one of Brouwer's key negative contentions—namely, that some of the higher mathematics of their day rests upon an illegitimate projection into the infinite of methods that properly apply only within finite domains. While they differ over the remedy, agreement on that negative point and Wittgenstein's close engagement with Brouwer's positive theory

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belie the widespread view—inspired by a notorious *obiter dictum* in the transcript of a 1939 lecture—that, for Wittgenstein, 'Intuitionism is all bosh—entirely' (*LFM*, 237).\[1]/

1. The intuitionists on infinity

Nowadays, under the influence of the late Sir Michael Dummett, we are apt to associate the intuitionist critique of classical mathematics and logic with the adoption of verificationist semantic theories, in which the meaning of a declarative sentence (henceforth, a statement) is given by specifying the conditions in which a speaker would be entitled to assert it, rather than by specifying the conditions under which it would be true. It is important to set these associations aside in reading the early intuitionists, for the founding fathers of the school were not verificationists. In a paper of 1923, Brouwer wrote that 'a complete empirical corroboration of the inferences drawn [about the "world of perception"] is usually materially excluded *a priori* and there cannot be any question of even a partial corroboration in the case of (juridical and other) inferences about the past' (Brouwer 1923, 336). A verificationist would conclude from that claim that talk about the past is meaningless; Brouwer, though, expressly holds that it is meaningful. Indeed, he allows that the laws of classical logic, including Excluded Middle, may validly be applied in reasoning about the world of perception, so long as we are able to think of the 'objects and mechanisms of [that] world…as (*possibly partly unknown*) finite discrete systems' (*ibid.*, emphasis in the original). More exactly, it is the possibility of projecting 'a finite discrete system upon the objects in question' that is the 'condition of the applicability' of Excluded Middle to judgements concerning those objects. We see here a fundamental difference between Brouwer and Dummett. For Dummett, the basic mistake of the classical mathematicians is that they apply a realist or truth-conditional semantic theory to the language of mathematics. For Brouwer, by contrast, their error was to apply distinctively classical logical rules 'even in the mathematics of infinite systems', where the rules' condition of applicability does not obtain. A.N. Kolmogorov, another pioneer of intuitionism, agreed with Brouwer. He understood Brouwer's

[1] The account of intuitionism that directly precedes this dictum in the lecture notes (which were taken down by some students) is in any case eccentric.

writings to have ‘revealed that it is illegitimate to use the principle of excluded middle in the domain of transfinite argument’ (Kolmogorov 1925, 416).

As Brouwer’s reference to ‘infinite systems’ implies, the early intuitionists did not impugn as unintelligible expressions, such as ‘the sequence of natural numbers’, that purport to designate infinite mathematical structures. They did, however, claim that talk about such structures, if it makes sense at all, is disguised talk about the mathematical principles that characterize them. Thus, to say that the natural number sequence has a property is to say that the property in question is entailed by the laws of Heyting Arithmetic, these laws (the intuitionistic analogue of the Peano Postulates) being the principles that characterize that structure. This marks a fundamental contrast with the finite case. A finite structure might be characterized by certain mathematical principles but, even when it is so characterized, it may still have properties that are not entailed by the principles. As one might put it, in the finite case the *extension* of certain mathematical principles will have mathematical properties over and above those consequent on the principles themselves. According to the intuitionist, this is conceptually impossible in the infinite case. A finite initial segment of an infinite sequence may have properties over and above those entailed by the principles that generate the sequence. But if we speak of the infinite sequence as a whole, we must be referring (perhaps elliptically) to the generating principles themselves. For the intuitionist, one might say, infinite structures cannot be conceived purely extensionally. So to conceive them is illegitimately to project into the infinite a notion that only makes sense in the finite case.

Wittgenstein understood and heeded Brouwer’s warning not to treat infinite collections as though they were large finite ones. In §19 of Part V of the *Remarks on the Foundations of Mathematics* (*RFM*), which its editors date to between 1942 and 1944, he asks:

> Isn’t it like this? The concepts of infinite decimals in mathematical propositions are not concepts of series, but of the unlimited technique of expansion of series.
>
> We learn an endless technique: that is to say, something is done for us first, and then we do it; we are told rules and we do exercises in following them; perhaps some expression like ‘and so on *ad inf.*’ is also used, but what is in question is not some gigantic extension (278-9).

A little later, in §36, he says:

> Our difficulty really already begins with the infinite straight line; although we learn even as children that a straight line has no end, and I do not know that this idea has ever given anyone any difficulty…But the straight line is a *law* for producing further (290).

Remarks such as these—which are typical of Part V of *RFM*—nicely express Brouwer's basic objection to the conception of the infinite that prevailed in his day and still prevails in ours. While Wittgenstein and Brouwer differ over the best prophylactic against this popular misconception, they are at one in perceiving a deep problem in the standard view of the infinite, and as such they are allies against the majority of mathematicians.

2. Brouwer against the Law of Excluded Middle

According to classical logic, we are entitled to assert $\ulcorner A \vee \neg A \urcorner$ no matter what meaningful statement A might be. Brouwer argues, though, that there are meaningful mathematical statements A for which an assertion of $\ulcorner A \vee \neg A \urcorner$ conflicts with a correct view of the infinite. Accordingly, a correct view of the infinite forces us to revise classical logic. In particular, it forces us to restrict the Law of Excluded Middle.\[2]/ Since this revisionist claim is one that Wittgenstein rejects, it will be worth setting out Brouwer's grounds for it carefully.

In the Vienna lecture that Wittgenstein heard, Brouwer introduced the notion of a *Pendelzahl*—a pendulum number or (as he Englished his term) a 'binary oscillatory shrinking number'. He then argued that we are not entitled to assert that such a number is either identical with or distinct from zero (Brouwer 1928, 1183).\[3]/ Wittgenstein evidently remembered the example, for in the *Philosophical Remarks* of 1929-31 he wrote:

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[2] The restriction consists in our not being entitled to assert certain instances of Excluded Middle. For the intuitionist, no such instance is false, i.e. has a true negation. For in intuitionistic logic $\neg(A \vee \neg A)$ entails the patently contradictory $\neg A \wedge \neg\neg A$.

[3] Brouwer actually wrote that 'this binary oscillatory shrinking number is neither equal to zero, nor different from it—in violation of the principle of the excluded middle'. As William Ewald remarks (*op. cit.*, *n.t*), these words need to be read charitably if Brouwer is not to find himself in the contradiction identified in the previous footnote.

> Brouwer is right when he says that the properties of his *Pendelzahl* are incompatible with the law of the excluded middle. But, saying this doesn't reveal a peculiarity of propositions about infinite aggregates. Rather, it is based on the fact that logic presupposes that it cannot be *a priori*—i.e. logically—impossible to tell whether a proposition is true or false. For, if the question of the truth or falsity of a proposition is *a priori* undecidable, the consequence is that the proposition loses its sense, and the consequence of this is precisely that the propositions of logic lose their validity for it (*PR*, 210).

In the light of the developments initiated by Gödel's great paper of 1931, philosophers and logicians will demand a great deal of argument before they can be persuaded to take seriously, let alone accept, Wittgenstein's claim that undecidable propositions lack sense.\[4]/ For present purposes, though, we need not address that large issue. For in other writings from the 1920s, Brouwer presents rather simpler instances of Excluded Middle which (as he thinks) we are not entitled assert and to which Wittgenstein responded with a detailed analysis, not a sweeping denial of sense to all undecidable statements.

Brouwer presents the sort of case I have in mind in subtly different ways in different places, but the exposition in his 1923 lecture and paper, 'On the significance of the Principle of Excluded Middle in mathematics', is characteristic. He begins §2 of that paper by identifying two 'fundamental properties'—propositions which are foundational for the current 'mathematics of infinity' and which follow from Excluded Middle. The second of these propositions is that every mathematical species is either finite or infinite. He then presents an example to show that both propositions are incorrect:

> Let d_ν be the νth digit to the right of the decimal point in the decimal expansion of π, and let $m = k_n$ if, as the decimal expansion of π is progressively written, it happens at d_m for the nth time that the segment $d_m\ d_{m+1} \ldots d_{m+9}$ of this decimal expansion forms the sequence

[4] For Gödel—as, I take it, for Wittgenstein in *PR*—a statement is decidable (with respect to a theory *T*) if and only if either it or its negation is deducible from *T* (Gödel 1931, 597). A statement may be decidable in this sense with respect to the whole currently corpus of accepted mathematical theory even though there is no decision procedure for determining its truth-value.

> 0123456789…That the second fundamental property is incorrect is seen from the example provided by the species of the positive integers k_n defined above (Brouwer 1923, 337).

In other words, we cannot assert that the species of integers k_n is either finite or infinite.

Brouwer's species is surely well defined. This is because, for any integers m and n, there is a finite procedure that decides whether $m = k_n$. For suppose we wish to find out whether 538,763 = k_2. To do this, it suffices to calculate π to the first 538,772 decimal places. If the last 10 digits in the expansion are 0123456789, and if that segment occurs precisely once earlier in the expansion, then 538,763 = k_2; otherwise, it is not. A Turing machine could be programmed to apply this test, and it would report an answer in a finite time. For these reasons, it seems clear that Brouwer has identified a mathematically well-defined species of integers.

Why, though, does Brouwer maintain that we cannot assert that the species is either finite or infinite? While he is not fully explicit, I think the reason is clear. The species of k_n's is finite if and only if there are only finitely many segments of the form 0123456789 in the decimal expansion of π; and it is infinite if and only if there are infinitely many such segments. Accordingly, if we were entitled to assert 'Brouwer's species is either finite or infinite', we would also be entitled to assert 'Either (1) there are only finitely many segments 0123456789 in the decimal expansion of π or (2) there are infinitely many such segments'. Given Brouwer's strictures on the meaning of talk about the infinite, however, it is clear that we are not entitled to assert that either (1) or (2) obtains. According to those strictures, a statement about an infinite sequence must be cashed out in terms of the principle or rule that generates the sequence. Given that, alternative (1) can only mean that the rule for expanding π entails that there are only finitely many segments of the form 0123456789 in the expansion. *Pari passu*, alternative (2) can only mean that the rule entails that no bound can be set on the number of such segments. In our present state of knowledge, we are not entitled to assert that either (1) or (2) obtains. Of course, our knowledge might expand in such a way that we become entitled to assert this. For example, a mathematician might prove, on the basis of the rule for expanding π, that there could be at most three occurrences of the segment 0123456789 in its decimal expansion; we would then know that alternative (1) obtains. In our present state of knowledge, however, we are not entitled to assert that either (1) or (2) obtains, and so we cannot assert that Brouwer's species is either finite or infinite.

In fact, it will help to work with a slightly simpler example. At the time of writing, π has been calculated to the first ten trillion (10^{13}) digits. I do not know whether those ten trillion digits include a segment 0123456789, but let us suppose that they do not. (If they do, one could easily change the designated segment to one that does not appear in the largest expansion of π that we currently have.) Let us now consider the statement 'Either Brouwer's species of k_n's is inhabited or it is not'. Given our supposition, we are not entitled to assert this instance of Excluded Middle. Brouwer's species is inhabited if and only if the segment 0123456789 occurs somewhere in the decimal expansion of π, and it is uninhabited (i.e. empty) if and only if no such segment occurs. So we would be entitled to assert 'Either Brouwer's species is inhabited or it is not' only if we were also entitled to assert 'Either 0123456789 occurs somewhere in the expansion of π or it does not'. Given Brouwer's strictures on what statements about the infinite can mean, the latter instance of Excluded Middle means 'Either (1) the rule for expanding π entails that the segment 0123456789 occurs somewhere in the expansion, or (2) the rule for expanding π entails that no such segment occurs anywhere'. In our current state of knowledge, we are not entitled to assert this disjunction. As before, this might change. In calculating π to the first twenty trillion digits, we might find a segment 0123456789; we would then know that alternative (1) obtains. Equally, a mathematician might prove that (2) obtains. In our present state of knowledge, though, we cannot assert that either (1) or (2) obtains; hence we cannot assert that Brouwer's species is either inhabited or not.

3. Wittgenstein on unassertible instances of Excluded Middle

I have switched to this simpler example in order to bring Wittgenstein back into the story, for a central question in Part V of *RFM* is precisely whether we are always entitled to assert that a given segment of digits either is or is not to be found somewhere in the decimal expansion of π. The fact that Wittgenstein focuses so intently on this question suggests forcibly that he had studied either the 1923 lecture from which I have quoted, or one of the other papers from the early 1920s in which Brouwer uses the same technique to cast doubt on the Law of Excluded Middle. At any rate, his focus surely refutes the hypothesis that, on Wittgenstein's considered view, intuitionism is 'bosh'—if that means that it is so confused as not to be worth discussing. As we have seen, the question Wittgenstein addresses is central to the intuitionist's critique of classical mathematics, and the paragraphs—from §9 to §23 of Part V—in which he develops his answer to it constitute one of the most sustained passages of argument in the whole of the *Remarks*. In gauging Wittgenstein's attitude to intuitionism, these facts must carry greater weight than a stray remark in a lecture.

In the *Philosophical Remarks* of 1929-31, and in his lectures of 1932-5 (*AWL*), Wittgenstein agrees with Brouwer that we are not entitled to assert certain instances of Excluded Middle. But they offer different diagnoses of why we are not always entitled to make such assertions. On Wittgenstein's view, the unassertible cases are not properly regarded as exceptions to the Law. Rather, statements like 'The segment 0123456789 occurs somewhere in the decimal expansion of π' do not qualify as meaningful propositions. Since the laws of logic apply only to propositions, these statements simply fall outside their ambit:

> I need hardly say that where the law of excluded middle doesn't apply, no other law of logic applies either, because in that case we aren't dealing with propositions of mathematics. (Against Weyl and Brouwer.) (*PR*, 176)

The intuitionists, then, were misguided in seeking a non-classical logic to regulate inferences involving undecidable statements about the infinite: since such statements fail to qualify as propositions, they have no logic. Similarly, in his lectures of the early 1930s, Wittgenstein maintained that a willingness to take $\ulcorner A \vee \neg A \urcorner$ to be a tautology partly defines what it is for A to be a proposition. 'This pattern occurs somewhere in this expansion' is an example of a grammatically well-formed statement that *seems* to qualify as a proposition but in fact does not (*AWL*, 140). On Brouwer's account, we are entitled to assert $\ulcorner A \vee \neg A \urcorner$ when and only when A is decidable, in the sense of being either provable or refutable. As we have seen, the Wittgenstein of the *Remarks* takes decidability to be the test for whether a mathematical statement has a sense, i.e. qualifies as a proposition. So Brouwer and the Wittgenstein of *Philosophical Remarks* will agree as to which instances of Excluded Middle are assertible. When $\ulcorner A \vee \neg A \urcorner$ is not assertible, though, they will offer different explanations of why not. Brouwer will say it is because A is not guaranteed to have a truth-value. Wittgenstein will say it is because A lacks a sense.

In *RFM*, Wittgenstein is less explicit than in *PR* or *AWL* that he wishes to deal with Brouwer's examples in this way. Implicitly, though, he takes the same line. 'In the law of excluded middle', he writes in §12 of Part V, 'we think we have already got something solid, something that at any rate cannot be called in doubt. Whereas in truth this tautology has just as shaky a sense (if I may put it like that), as the question whether *p* or ~*p* is the case' (271).\[5]/ The Wittgenstein of *RFM* clearly regards the question whether 0123456789 occurs somewhere in the expansion of π as 'shaky'. He deems the question 'queer' (*seltsam*) and says we are led to ask it precisely because we are in the grip of 'the false picture of a completed expansion' of an irrational number (§9, 266, 267).

What the discussion in Part V adds to the earlier doctrine is some explanation of why this question and others like it fail to make sense. Explanation is surely needed here for, at first blush, the question seems to be entirely intelligible. I think we may distinguish two main strands in Wittgenstein's attempt to show that it is not.

(1) In the first strand, Wittgenstein tries to undermine the most obvious source of confidence that our question makes sense—namely, that we can easily envisage finding ourselves in circumstances where we would return a positive answer to it. We look down a computer print-out of the first one million digits in the expansion of π and—lo and behold—we spot a segment 0123456789. So, to the question 'Does that segment occur somewhere in the expansion of π?', we confidently answer 'yes'. Wittgenstein allows that we would answer the question affirmatively in such a circumstance, but he insists that this does not show that the question possesses a determinate sense:

> If someone says: 'But you surely know what "this pattern occurs in the expansion" means, namely *this*'—and points to a case of occurring,—then I can only reply that what he shows me is capable of illustrating a *variety* of facts. For that reason I can't be said to know what the proposition means just from knowing that he will certainly use it in this case (§13, 271).

The immediate point here is may be Wittgenstein's familiar observation that a single case underdetermines a rule. But his discussion later in Part V of the difference between constructive and non-constructive existence proofs provides more substantial supports for the thesis that there are

[5] Section and page references in the rest of this section are to Part V of *RFM*.

genuinely different interpretations of 'This pattern occurs somewhere in the expansion'.\[6]/ On one interpretation, the only possible ground for asserting the statement would be the identification of the pattern at a specific place in the expansion, as when we spot 0123456789 on the print-out. But there is another interpretation under which the statement also admits of non-constructive proof:

> A proof that 777 occurs in the expansion of π, without showing where, would have to look at this expansion from a totally new point of view, so that it showed e.g. properties of regions of the expansion about which we only knew that they lay very far out. Only the picture floats before one's mind of having to assume as it were a dark zone of indeterminate length very far on in π, where we can no longer rely on our devices for calculating; and then still further out a zone where in a *different* way we can once more see something (§27, 284).

The classical mathematician allows non-constructive existence proofs, so he is committed to trying to make sense of the possibility (or apparent possibility) that Wittgenstein sketches in §27. According to Wittgenstein, though, the conditions for making sense of a mathematical proposition are exacting. One needs to 'command a clear view of its applications' (§25, 283)—clearly a tall order in the present case. Moreover, the statement in question is liable to engender an *illusion* of understanding. 'This pattern occurs somewhere in the expansion' has the form of an existentially quantified statement, and one is apt to think one understands it because one understands the existential quantifier and understands the relevant matrix instances (in this case, statements of the form 'An instance of the pattern is found starting at the *n*th place'). However, 'the understanding of a mathematical proposition is not guaranteed by its verbal form, as is the case with most non-mathematical propositions', for 'the mathematical general does not stand in the same relation to the mathematical particular as elsewhere

[6] I pass over Wittgenstein's suggestion (in §9) that the question is indeterminate in sense because 'the further expansion of an irrational number is a further expansion of mathematics' which calls for 'decisions' about how inherently indeterminate mathematical concepts and rules are to be determined or interpreted. Some mathematical concepts are indeterminate, and as a result some apparently well posed mathematical questions may well lack a determinate sense. For example, it is plausible to maintain that further determination of the concept *set* (or *real number*) is needed before the Generalized Continuum Hypothesis (or the Riemann Hypothesis) qualifies as a well-defined mathematical problem. In these cases, we should agree with Wittgenstein that 'the question…changes its status, when it becomes decidable. For a connection is made then, which formerly *was not there*' (266-7). It is, however, implausible to hold that a conceptual advance of this kind is involved in expanding an irrational number. The rule for writing down the expansion of π is clear and straightforward—a computer may be programmed to follow it—so it is misleading for Wittgenstein to describe this case as one where the 'ground for the decision…has yet to be invented' (*ibid.*).

the general to the particular' (§25, 282, 284). At least, this is so in classical mathematics. The classical mathematician allows that someone may prove that a given segment occurs somewhere in an infinite series even when there is no possibility of finding out where. A thinker understands a mathematical proposition to the extent that he knows 'what to do with it', and what one can do with the conclusion of a non-constructive existence proof is very different from what one can do with the conclusion of a constructive proof (§46, 299). These differences are disguised by the fact that the existential quantifier 'somewhere' figures in both 'This pattern occurs somewhere in the expansion' and 'The mug is somewhere in the cupboard'. But this common 'verbal expression…is a mere shadow [which] keeps mum about the important things (*Hauptsache*)' (§25, 282). The logician's use of the symbol '∃' to formalize both of these quantifiers reinforces the illusion of understanding and is a signal illustration of the 'disastrous invasion' of mathematics by logic (§24, 281). The common 'logical notation suppresses the structure' of two very different sorts of statement (§25, 284).\[7]/

(2) The strand of argument that I have just traced out is designed to shake our confidence that we do understand such statements as '0123456789 occurs somewhere in the expansion of π'. In the second strand, Wittgenstein argues that the claim that we always understand such statements can be maintained only at the price of assimilating the infinite to the finite—the very mistake that both he and Brouwer discern in the higher mathematics of their day. As we have seen, we have a clear apprehension of one sort of ground for asserting our statement—viz., the sort of ground we acquire when we spot 0123456789 in the expansion of π. In §12 of Part V, though, Wittgenstein puts his finger on another reason why this sort of knowledge does not give us the understanding that we seek:

> For how do I know what it means to say: the pattern…occurs in the expansion? Surely by way of examples: which show me what it is like for…[to occur]. But these examples do not show me what it is like for this pattern *not* to occur in the expansion!\[8]/

[7] Cfr. §46 again: 'The curse of the invasion of mathematics by mathematical logic is that now any proposition can be represented in a mathematical symbolism, and this make us feel obliged to understand it. Although of course this method of writing is nothing but the translation of vague ordinary prose' (299).

[8] The italicized 'not', although clearly present in Wittgenstein's manuscript, is erroneously omitted from both the German and English editions of *RFM*. (I am very grateful to Professor Joachim Schulte for pointing this out to me.)

> Might one not say: if I really had a right to say that these examples tell me what it is like for the pattern to occur in the expansion, then they would have to show me what the opposite means (§12, 271).

This suggests the following argument. In order to attain a clear conception of what it is for P to be the case, one needs to attain a conception of what it is for *P not* to be the case. *Eadem est scientia oppositorum*, as the medieval logicians put it. In the present case, though, we seem to lack the negative side of the story. Or rather, quoting §11 this time,

> To say of an unending series that it does *not* contain a particular pattern makes sense only under quite special conditions.
>
> That is to say: this proposition has been given a sense for certain cases.
>
> Roughly, for those where it is in the *rule* for this series, not to contain the pattern…(268-9)

What happens if we try to make sense of the hypothesis that 0123456789 appears nowhere in the expansion of π when these 'special conditions' do not obtain? Well, that would mean entertaining the hypothesis that no occurrence of 0123456789 is to be found in the entire expansion, even though such an occurrence is not precluded by the rule for expanding π. And that hypothesis is incoherent on the view of the infinite that Brouwer and Wittgenstein share. It amounts to the absurd hypothesis that the expansion merely *happens* not to contain any instance of 0123456789. In the words of §18:

> Does it make sense to say: 'While there isn't any rule forbidding the occurrence, as a matter of fact the pattern does not occur?'—And if this does not make sense, how can the opposite make sense, namely, that the pattern does occur?
>
> Well, when I say it occurs, a picture of the series from its beginning up to the pattern floats before my mind—but if I say that the pattern does *not* occur, then no such picture is of any use to me, and my supply of pictures gives out…
>
> The queer thing about the alternative 'ϕ occurs in the infinite series or it does not', is that we have to imagine the two possibilities individually, that we look for a distinct idea of each, and that *one* is not adequate for the negative and for the positive cases, as it is elsewhere (278).

Thus, when Wittgenstein's 'special conditions' do not obtain, we can attain no clear conception of what is involved in the negative case's being true.

4. How an intuitionist should reply

On Wittgenstein's view, then, the conception of the infinite that he and Brouwer share exposes as senseless statements saying that this or that pattern occurs in an infinite decimal expansion, except in the special case when the hypothesis that it does may be proved or refuted. Since Brouwer holds that his intuitionistic logic applies to such statements, he is committed to ascribing a sense to them. How might an intuitionist reply to Wittgenstein's arguments?

What he needs to do is to *attach* a coherent sense to statements of the problematical kind. As Wittgenstein in effect concedes, there is no great difficulty attaching sense to a statement 'ϕ occurs in the infinite series', so long as we understand it in such a way that its ultimate grounds are constructive proofs. So understood, we know in what circumstances we shall be entitled to assert the statement (viz., when we know that ϕ occurs at such-and-such a point in the series) and we also know 'what to do' with such an assertion (viz., look at the proof to discover where ϕ occurs). This method does not extend to attach a sense to our statement, if it is also supposed to admit of a non-constructive proof; but that is not a problem for an intuitionist.

How, though, may we attach sense to 'ϕ does *not* occur in the series'? The key to the intuitionist's answer is his denial that *eadem est scientia* invariably constrains the relation between a statement and its negation. One does not always need a conception of what would be the case if not *P* in order to have a conception of what would be the case if *P*. Rather, one's knowledge of what would be the case if not *P* may draw upon prior knowledge of what would be the case if *P*. So it is in the present case. *Ex hypothesi*, we have a conception of what it would be for ϕ to occur at some identifiable place in the series—identifiable, that is, by means of a mathematical construction. Drawing upon that conception, we can then form the notion of a proof that establishes that no such construction is possible. Such a proof will be the ground for asserting 'ϕ does *not* occur in the series'.

Moreover, we know what to do with such an assertion: on its strength, we can set aside for ever any possibility of finding ϕ in the series.

This, in outline, is how the intuitionist should answer the arguments sketched in §3. The answer also shows how to reply to some of Wittgenstein's additional criticisms. Like many critics since, he worries that what the intuitionist refuses to assert is not the 'real' Law of Excluded Middle—i.e., is not Excluded Middle as the classical logician understands it. On the intuitionist's understanding of the statements, 'ϕ occurs in the series' is tantamount to 'It follows from the laws of mathematics that ϕ occurs in the series', and 'ϕ does not occur in the series' is tantamount to 'It follows from the laws of mathematics that ϕ does not occur'. And yet: 'The opposite of "there exists a law that *p*" is not: "there exists a law that ~*p*". But if one expresses the first by means of *P*, and the second by means of ~*P*, one will get into difficulties' (§13, 272). Or again: 'If "you do it" means: you must do it, and "you do not do it" means: you must not do it—then "Either you do it, or you do not" is not the law of excluded middle (§17, 275). It is certainly not the Law of Excluded Middle as the classical logician understands it, but that cannot be a legitimate criticism. Wittgenstein agrees with Brouwer that any attempt to apply classical negation to 'ϕ occurs in the series' will result in nonsense. So the intuitionist cannot be faulted for trying to articulate a non-classical conception of negation, which in turns yields a non-classical reading of the Law of Excluded Middle. On that conception, $\ulcorner \neg A \urcorner$ is inherently a more complex statement than *A*, so it should be no surprise that $\ulcorner \neg\neg A \urcorner$ does not always entail *A*, or that $\ulcorner A \vee \neg A \urcorner$ is not always assertible.

Our analysis also brings out the depth of the gulf that separates Brouwer's case for intuitionism from Dummett's. On Brouwer's view, we are driven to interpret mathematical statements in terms of constructions because the attempt to apply a classical interpretation, which respects *eadem est scientia*, leads ineluctably to an incoherent view of the infinite. His case, then, is specific to higher mathematics. It is not, and cannot be, the harbinger of a general argument in favour of casting semantic theories in terms of assertability conditions rather than truth-conditions.

5. A lasting legacy of the *Tractatus*

At the heart of the dispute between Brouwer and Wittgenstein lies a disagreement about the conditions that a form of words must satisfy in order to qualify as a proposition—that is, to be an intelligible statement to which the laws of logic apply. The following formulation of the disagreement may be helpful. Let us assume that *denying* a proposition is logically equivalent to asserting its negation: both classical and intuitionist logicians will grant this assumption. Let us then say that a statement has a *back* when an assertion of it *ipso facto* amounts to a denial of some other statement. Both classical and intuitionist logicians assume that any statement *has* a negation. A statement with a back will also *be* a negation, or be equivalent to one. That is, A has a back if and only if, for some statement B, A is equivalent to $\ulcorner\neg B\urcorner$; to assert A will be to deny B. The locus of dispute between Brouwer and Wittgenstein is then the following thesis:

(B) Every proposition has a back, i.e., every proposition is the negation of some other proposition.

Like any intuitionist, Brouwer cannot assert (B): were he to assert it, intuitionistic propositional logic would collapse into classical propositional logic. The reason is this. For any formula B, the triple negation $\ulcorner\neg\neg\neg B\urcorner$ is intuitionistically equivalent to the single negation $\ulcorner\neg B\urcorner$. Suppose, then, that A has a back. Then, for some B, A is equivalent to $\ulcorner\neg B\urcorner$, so that $\ulcorner\neg\neg A\urcorner$ is equivalent to $\ulcorner\neg\neg\neg B\urcorner$. By the result about triple negations, this means that, whenever A has a back, A is intuitionistically equivalent to $\ulcorner\neg\neg A\urcorner$. So, if an intuitionist were to assert (B), he would be committed to taking each proposition to be equivalent to its own double negation. That would suffice to collapse intuitionistic propositional logic into classical logic.

Wittgenstein, by contrast, had a long-standing and deep-seated commitment to (B). When we understand a proposition, he wrote in the 'Notes on Logic' of September 1913, 'we know what is the case if it is true and what is the case if it is false' (*NB*, 94). In this way, any proposition is associated with true and false 'poles'. To accept the true pole is *ipso facto* to reject the false pole. The negation operator, on Wittgenstein's account, simply reverses the poles, so asserting that P is *ipso facto* denying that not P, just as (B) has it. *Eadem est scientia* follows. This is why the *Tractatus* makes no room for doubting the equivalence between a proposition and its double negation. These say the same thing (*TLP* 5.44); indeed, in a fully perspicuous symbolism, double negations would vanish (*TLP* 5.254). In any event, the universal equivalence of a proposition and its double negation suffices (given very weak assumptions about the logic of disjunction) to ensure the validity of every instance of Excluded Middle.\[9]/

But is it really a universal requirement that any fully intelligible statement should have a back? (*B*) has great initial plausibility: it is at first hard to see how a statement could have a determinate content unless it is determinate what it excludes. And our reluctance to deviate from (*B*) explains, I think, why so many reasoners are willing to apply classical logic even to statements whose bivalence they find doubtful. In the previous section, though, we saw reasons why statements involving infinity might be exceptions to (*B*). In asserting '0123456789 occurs somewhere in the expansion of π', there is nothing that one is thereby denying. In particular, one is not thereby denying '0123456789 occurs nowhere in the expansion'. In order to understand a negated statement one must understand its negand, but not necessarily *vice versa*.

In the light of that, I do not think that anyone could claim that (*B*) is *obviously* correct. So Wittgenstein has not shown that any attempt to attach sense to statements of the problematical kind must fail. To say as much, of course, is not to say that Brouwer or any other intuitionist has succeeded in attaching sense to those statements. To show that, one would need to elaborate the putative sense to the point where it clearly provides a coherent alternative to the classical account that Brouwer and Wittgenstein both reject. Like them, I regard that classical account as deeply suspect. So I regard the open question here—whether the intuitionist can succeed in articulating an alternative sense, or whether we must follow Wittgenstein in deeming such statements to be senseless—as one of the most important in the philosophy of mathematics.

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[9] Let A be any statement. Since $\neg(A \vee \neg A)$ intuitionistically entails a contradiction (see *n*.2), $\neg\neg(A \vee \neg A)$ is a theorem of the intuitionistic propositional calculus. So if all double negations were eliminable, we would always have $A \vee \neg A$.

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TLP *Tractatus Logico-Philosophicus*, trans. D.F. Pears and B.F. McGuinness. London: Routledge and Kegan Paul, 1961.

PR *Philosophical Remarks*, ed. R. Rhees, trans. R. Hargreaves and R. White. Oxford: Blackwell, 1975.

AWL *Wittgenstein's Lectures, Cambridge, 1932-35*, ed. A. Ambrose. Oxford: Blackwell, 1979.

LFM *Wittgenstein's Lectures on the Foundations of Mathematics, Cambridge, 1939*, ed. C. Diamond. Ithaca, New York: Cornell University Press, 1975.

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