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# Existences of rainbow matchings and rainbow matching covers 

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## Note

# Existences of rainbow matchings and rainbow matching covers 

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#### Abstract

Let $G$ be an edge-coloured graph. A rainbow subgraph in $G$ is a subgraph such that its edges have distinct colours. The minimum colour degree $\delta^{c}(G)$ of $G$ is the smallest number of distinct colours on the edges incident with a vertex of $G$. We show that every edge-coloured graph $G$ on $n \geq 7 k / 2+2$ vertices with $\delta^{c}(G) \geq k$ contains a rainbow matching of size at least $k$, which improves the previous result for $k \geq 10$.

Let $\Delta_{\text {mon }}(G)$ be the maximum number of edges of the same colour incident with a vertex of $G$. We also prove that if $t \geq 11$ and $\Delta_{\text {mon }}(G) \leq t$, then $G$ can be edge-decomposed into at most $\lfloor$ tn $/ 2\rfloor$ rainbow matchings. This result is sharp and improves a result of LeSaulnier and West.


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## 1. Introduction

Let $G$ be a simple graph, that is, it has no loops or multi-edges. We write $V(G)$ for the vertex set of $G$ and $\delta(G)$ for the minimum degree of $G$. An edge-coloured graph is a graph in which each edge is assigned a colour. We say that an edgecoloured graph $G$ is proper if no two adjacent edges have the same colour. A subgraph $H$ of $G$ is rainbow if all its edges have distinct colours. Rainbow subgraphs are also called totally multicoloured, polychromatic, or heterochromatic subgraphs.

In this paper, we are interested in rainbow matchings in edge-coloured graphs. The study of rainbow matchings began with a conjecture of Ryser [10], which states that every Latin square of odd order contains a Latin transversal. Equivalently, for $n$ odd, every properly $n$-edge-colouring of $K_{n, n}$, the complete bipartite graph with $n$ vertices on each part, contains a rainbow copy of a perfect matching. In a more general setting, given a graph $H$, we wish to know if an edge-coloured graph $G$ contains a rainbow copy of $H$. A survey on rainbow matchings and other rainbow subgraphs in edge-coloured graphs can be found in [3].

For a vertex $v$ of an edge-coloured graph $G$, the colour degree, $d^{c}(v)$, of $v$ is the number of distinct colours on the edges incident with $v$. The smallest colour degree of all vertices in $G$ is the minimum colour degree of $G$ and is denoted by $\delta^{c}(G)$. Note that a properly edge-coloured graph $G$ with $\delta(G) \geq k$ has $\delta^{c}(G) \geq k$.

Li and Wang [8] showed that if $\delta^{c}(G)=k$, then $G$ contains a rainbow matching of size $\lceil(5 k-3) / 12\rceil$. They further conjectured that if $k \geq 4$, then $G$ contains a rainbow matching of size $\lceil k / 2\rceil$. LeSaulnier et al. [6] proved that if $\delta^{c}(G)=k$, then $G$ contains a rainbow matching of size $\lfloor k / 2\rfloor$. The conjecture was later proved in full by Kostochka and Yancey [4].

Wang [11] asked does there exist a function $f(k)$ such that every properly edge-coloured graph $G$ on $n \geq f(k)$ vertices with $\delta(G) \geq k$ contains a rainbow matching of size at least $k$. Diemunsch et al. [1] showed that such function does exist and $f(k) \leq 98 k / 23$. Gyárfás and Sarkozy [2] improved the result to $f(k) \leq 4 k-3$. Independently, Tan and the author [9] showed that $f(k) \leq 4 k-4$ for $k \geq 4$.

[^0]Kostochka, Pfender and Yancey [5] showed that every (not necessarily properly) edge-coloured G on $n \geq 17 k^{2} / 4$ vertices with $\delta^{c}(G) \geq k$ contains a rainbow matching of size $k$. Tan and the author [9] improved the bound to $n \geq 4 k-4$ for $k \geq 4$. In this paper we show that $n \geq 7 k / 2+2$ is sufficient.

Theorem 1.1. Every edge-coloured graph $G$ on $n \geq 7 k / 2+2$ vertices with $\delta^{c}(G) \geq k$ contains a rainbow matching of size $k$.
Moreover if $G$ is bipartite, then we further improve the bound to $n \geq(3+\varepsilon) k+\varepsilon^{-2}$.
Theorem 1.2. Let $0<\varepsilon \leq 1 / 2$ and $k \in \mathbb{N}$. Every edge-coloured bipartite graph $G$ on $n \geq(3+\varepsilon) k+\varepsilon^{-2}$ vertices with $\delta^{c}(G) \geq k$ contains a rainbow matching of size $k$.

We also consider covering an edge-coloured graph $G$ by rainbow matchings. Given an edge-coloured graph $G$, let $\Delta_{\text {mon }}(G)$ be the largest maximum degree of monochromatic subgraphs of $G$. LeSaulnier and West [7] showed that every edge-coloured graph $G$ on $n$ vertices with $\Delta_{\text {mon }}(G) \leq t$ has an edge-decomposition into at most $t(1+t) n \ln n$ rainbow matchings. We show that $G$ can be edge-decomposed into $\lfloor t n / 2\rfloor$ rainbow matchings provided $t \geq 11$.

Theorem 1.3. For all $t \geq 11$, every edge-coloured graph $G$ on $n$ vertices with $\Delta_{\operatorname{mon}}(G) \leq t$ can be edge-decomposed into $\lfloor t n / 2\rfloor$ rainbow matchings.

Note that the bound is best possible by considering edge-coloured graphs, where one colour class induces a $t$-regular graph.

Theorems 1.1 and 1.2 are proved in Section 2. Theorem 1.3 is proved in Section 3.

## 2. Existence of rainbow matchings

We write $[k]$ for $\{1,2, \ldots, k\}$. Let $G$ be a graph with an edge-colouring $c$. We denote by $c(G)$ the set of colours in $G$. We write $|G|$ for $|V(G)|$. Given $W \subseteq V(G), G[W]$ is the induced subgraph of $G$ on $W$. All colour sets are assumed to be finite.

Before proving Theorems 1.1 and 1.2, we consider the following (weaker) question. Suppose that $G$ is an edge-coloured graph and contains a rainbow matching $M$ of size $k-1$. Under what colour degree and $|G|$ conditions can we 'extend' $M$ into a matching of size $k$ with at least $k-1$ colours? We formalise the question below.

Let $g$ be a family of graphs closed under vertex/edge deletions. Define $\gamma(\mathcal{q})$ to be the smallest constant $\gamma$ such that, whenever $k \in \mathbb{N}, G \in \mathcal{\xi}$ is a graph with $|G| \geq \gamma k$ and an edge-colouring $c$ on $G$, the following holds. If for any rainbow matching $M$ of size $k-1$ in $G$, we have $d^{c}(z) \geq k$ for all $z \in V(G) \backslash V(M)$, then $G$ contains a rainbow matching $M^{\prime}$ of size $k-1$ and a disjoint edge. (Note that the colour of the disjoint edge may appear in $M^{\prime}$.) Clearly, $\gamma(\mathcal{q}) \geq 2$ for any family $g$ of graphs. It is easy to see that equality holds if $\mathcal{G}$ is the family of bipartite graphs.

Proposition 2.1. Let $\mathcal{g}$ be the family of bipartite graphs. Then $\gamma(\mathcal{g})=2$.
Proof. Let $G$ be a bipartite graph on at least $2 k$ vertices. Suppose that $M$ is a rainbow matching of size $k-1$ and that $d^{c}(z) \geq k$ for all $z \in V(G) \backslash V(M)$. Since $G$ is bipartite, there exists an edge vertex-disjoint from $M$ and so the proposition follows.

If $g$ is the family of all graphs, we will show that $\gamma(\mathcal{q}) \leq 3$.
Lemma 2.2. Let $G$ be a graph with at least $3(k-1)+1$ vertices. Suppose that $M$ is a rainbow matching of size $k-1$ and that $d^{c}(z) \geq k$ for all $z \in V(G) \backslash V(M)$. Then $G$ contains a rainbow matching $M^{\prime}$ of size $k-1$ and a disjoint edge.
Proof. Let $x_{1} y_{1}, \ldots, x_{k-1} y_{k-1}$ be the edges of $M$ with $c\left(x_{i} y_{i}\right)=i$. Let $W=V(G) \backslash V(M)$. We may assume that $G[W]$ is empty or else the lemma holds easily.

Suppose the lemma does not hold for $G$. By relabelling the indices of $i$ and swapping the roles of $x_{i}$ and $y_{i}$ if necessary, we will show that there exist distinct vertices $z_{1}, \ldots, z_{k-1}$ in $W$ such that for each $1 \leq i \leq k-1$, the following holds:
$\left(\mathrm{a}_{i}\right) y_{i} z_{i}$ is an edge and $c\left(y_{i} z_{i}\right) \notin[i]$.
$\left(\mathrm{b}_{i}\right)$ Let $T_{i}$ be the vertex set $\left\{x_{j}, y_{j}, z_{j}: i \leq j \leq k-1\right\}$. For any colour $j^{\prime}$, there exists a rainbow matching $M_{j^{\prime}}^{i}$ of size $k-i$ on $T_{i}$ such that $c\left(M_{j^{\prime}}^{i}\right) \cap\left([i-1] \cup\left\{j^{\prime}\right\}\right)=\emptyset$.
$\left(c_{i}\right)$ Let $W_{i}=W \backslash\left\{z_{i}, z_{i+1}, \ldots, z_{k-1}\right\}$. For all $w \in W_{i}, N(w) \cap T_{i} \subseteq\left\{y_{i}, \ldots, y_{k-1}\right\}$.
Let $W_{k}=W$ and $T_{k}=\emptyset$. Suppose that we have already found $z_{k-1}, z_{k-2}, \ldots, z_{i+1}$. We find $z_{i}$ as follows.
Note that $\left|W_{i+1}\right| \geq n-2(k-1)-(k-i-1) \geq 1$, so $W_{i+1} \neq \emptyset$. Let $z$ be a vertex in $W_{i+1}$. By the colour degree condition, $z$ must be incident to at least $k$ edges of distinct colours, and in particular, at least $k-i$ distinct coloured edges not using colours in $[i]$. By $\left(\mathrm{c}_{i+1}\right), z$ sends at most $k-i-1$ edges to $T_{i+1}$. So there exists a vertex $u \in V(M) \backslash T_{i+1}=\left\{x_{j}, y_{j}: 1 \leq j \leq i\right\}$ such that $u z$ is an edge with $c(u z) \notin[i]$. Without loss of generality, $u=y_{i}$ and we set $z_{i}=z$. Clearly $\left(\mathrm{a}_{i}\right)$ holds.

We now show that $\left(\mathrm{b}_{i}\right)$ holds for any colour $j^{\prime}$. If $j^{\prime} \neq i$, then by $\left(\mathrm{b}_{i+1}\right)$, there is a rainbow matching $M_{j^{\prime}}^{i+1}$ of size $k-i-1$ on $T_{i+1}$ such that $c\left(M_{j^{\prime}}^{i+1}\right) \cap\left([i] \cup\left\{j^{\prime}\right\}\right)=\emptyset$. Set $M_{j^{\prime}}^{i}=M_{j^{\prime}}^{i+1} \cup x_{i} y_{i}$. So $M_{j^{\prime}}^{i}$ is a rainbow matching on $T_{i}$ of size $k-i$ and moreover $c\left(M_{j^{\prime}}^{i}\right) \cap\left([i-1] \cup\left\{j^{\prime}\right\}\right)=\emptyset$ as required. If $j^{\prime}=i$, then by $\left(b_{i+1}\right)$, there is a rainbow matching $M_{c\left(y_{i} z_{i}\right)}^{i+1}$ of size $k-i-1$ on $T_{i+1}$ such that $c\left(M_{c\left(y_{i} z_{i}\right)}^{i+1}\right) \cap\left([i] \cup\left\{c\left(y_{i} z_{i}\right)\right\}\right)=\emptyset$. Set $M_{i}^{i}=M_{c\left(y_{i} z_{i}\right)}^{i+1} \cup y_{i} z_{i}$. Note that $M_{i}^{i}$ is the desired rainbow matching.

Let $w t$ be an edge with $w \in W_{i}$ and $t \in T_{i}$. Since $G[W]$ is empty, $t \notin\left\{z_{i}, z_{i+1}, \ldots, z_{k-1}\right\}$. By $\left(c_{i+1}\right), t \notin\left\{x_{i+1}, x_{i+2}, \ldots\right.$, $\left.x_{k-1}\right\}$. Suppose that $t=x_{i}$. By $\left(\mathrm{b}_{i+1}\right)$, there exists a rainbow matching $M_{c\left(y_{i} z_{i}\right)}^{i+1}$ of size $k-i-1$ on $T_{i+1}$ such that $c\left(M_{c\left(y_{i} z_{i}\right)}^{i+1}\right) \cap$ $\left([i] \cup\left\{c\left(y_{i} z_{i}\right)\right\}\right)=\emptyset$. Let $M^{\prime}$ be the matching $\left\{x_{j} y_{j}: j \in[i-1]\right\} \cup M_{c\left(y_{i} z_{i}\right)}^{i+1} \cup\left\{y_{i} z_{i}\right\}$. Note that $M^{\prime}$ is a rainbow matching of size $k-1$ vertex-disjoint from the edge $w x_{i}$. This contradicts the fact that $G$ is a counterexample. Hence we have $t \in\left\{y_{i}, y_{i+1}, \ldots, y_{k-1}\right\}$ implying ( $\mathrm{c}_{i}$ ).

Therefore we have found $z_{1}, \ldots, z_{k-1}$. Let $w \in W_{1} \neq \emptyset$. Recall the $G[W]=\emptyset$, so $N(w) \subseteq\left\{y_{1}, \ldots, y_{k-1}\right\}$ by (c $c_{1}$ ), which implies that $d^{c}(w) \leq d(w) \leq k-1$, a contradiction.

Corollary 2.3. Every family $\mathcal{q}$ of graphs satisfies $\gamma(\mathcal{q}) \leq 3$.
For colour sets $C$ and integers $\ell$, we now define a $(C, \ell)$-adapter below, which will be crucial in the proof of Lemma 2.5. Roughly speaking a $(C, \ell)$-adapter is a vertex subset $W$ that contains a rainbow matching $M$ with $c(M)=C$ even after removing a vertex in $W$.

Given $\ell \in \mathbb{N}$ and a set $C$ of colours, a vertex subset $W \subseteq V(G)$ is said to be a ( $C, \ell$ )-adapter if there exist (not necessarily edge-disjoint) rainbow matchings $M_{1}, \ldots, M_{\ell}$ in $G[W]$ such that $c\left(M_{i}\right)=C$ for all $i \in[\ell]$, and given any $w \in W$, there exists $i \in[\ell]$ such that $w \notin V\left(M_{i}\right)$. We write $C$-adapter for $(C,|C|+1)$-adapter. Note that a ( $C, \ell$ )-adapter is also a ( $\left.C, \ell^{\prime}\right)$-adapter for all $\ell \leq \ell^{\prime}$. The following proposition studies some basic properties of $(C, \ell)$-adapters.

Proposition 2.4. Let $G$ be a graph with an edge-colouring $c$.
(i) Let $C=\left\{c_{1}, \ldots, c_{\ell}\right\}$ be a set of distinct colours. Let $W=\left\{x_{i}, y_{i}, z_{i}, w: i \in[\ell]\right\}$ be a vertex set such that $c\left(x_{i} y_{i}\right)=c_{i}=$ $c\left(z_{i} w\right)$ for all $i \in[\ell]$. Then $W$ is a $C$-adapter.
(ii) Let $\ell_{1}, \ldots, \ell_{p} \in \mathbb{N}$ and let $C_{1}, \ldots, C_{p}$ be pairwise disjoint colour sets. Suppose that $W_{j}$ is a $\left(C_{j}, \ell_{j}\right)$-adapter for all $j \in[p]$ and that $W_{1}, \ldots, W_{p}$ are pairwise disjoint. Then $\bigcup_{j=1}^{p} W_{j}$ is $a\left(\bigcup_{j=1}^{p} C_{j}, \max _{j \in[p]}\left\{\ell_{j}\right\}\right)$-adapter.
(iii) Let $C$ be a colour set. Suppose that $W$ is a $(C, \ell)$-adapter. Suppose that $x, y, z \in V(G) \backslash W$ and $w \in W$ such that $x y, z w \in$ $E(G)$ and $c(x y)=c(z w) \notin C$. Then $W \cup\{x, y, z\}$ is $a(C \cup\{c(x y)\}, \ell+1)$-adapter.
Proof. To prove (i), we simply set $M_{i}=\left\{x_{j} y_{j}: j \in[\ell] \backslash\{i\}\right\} \cup\left\{w z_{i}\right\}$ for all $i \in[\ell]$ and $M_{\ell+1}=\left\{x_{j} y_{j}: j \in[\ell]\right\}$.
(ii) Let $\ell=\max \left\{\ell_{j}: j \in[p]\right\}$. Note that each $W_{j}$ is a $\left(C_{j}, \ell\right)$-adapter. For $j \in[p]$, let $M_{1}^{j}, \ldots, M_{\ell}^{j}$ be rainbow matchings in $G\left[W_{j}\right]$ such that $c\left(M_{i}^{j}\right)=C_{j}$ for all $i \in[\ell]$, and given any $w \in W_{j}$, there exists $i \in[\ell]$ such that $w \notin V\left(M_{i}^{j}\right)$. Set $M_{i}=\bigcup_{j=1}^{p} M_{i}^{j}$. So (ii) holds.
(iii) Let $M_{1}, \ldots, M_{\ell}$ be rainbow matchings in $G[W]$ such that $c\left(M_{i}\right)=C$ for all $i \in[\ell]$, and given any $w^{\prime} \in W$, there exists $i \in[\ell]$ such that $w^{\prime} \notin V\left(M_{i}\right)$. Without loss of generality we have $w \notin V\left(M_{1}\right)$. Now set $M_{i}^{\prime}=M_{i} \cup\{x y\}$ for all $i \in[\ell]$ and $M_{\ell+1}^{\prime}=M_{1}^{\prime} \cup\{w z\}$. Hence, $W \cup\{x, y, z\}$ is a $(C \cup\{c(x y)\}, \ell+1)$-adapter.

We prove the following lemma. The main idea of the proof is to consider ( $C, \ell$ )-adapters in $G$ with $\ell$ maximal.
Lemma 2.5. Let $k \in \mathbb{N}$ and let $2<\gamma \leq 3$. Let $\mathcal{q}$ be a family of graphs closed under vertex/edge deletion with $\gamma(\mathcal{q}) \leq \gamma$. Suppose that $G \in$ g with

$$
|G| \geq\left(2+\frac{\gamma}{2}\right) k+\frac{2(4-\gamma)}{(\gamma-2)^{2}}-3+\gamma
$$

and that $G$ contains a rainbow matching of size $k-1$. Further suppose that for all rainbow matchings $M$ of size $k-1$ in $G$, we have $d^{c}(v) \geq k$ for all $v \in V(G) \backslash V(M)$. Then $G$ contains a rainbow matching of size $k$.
Proof. We proceed by induction on $k$. It is trivial for $k=1$, so we may assume that $k \geq 2$.
Let $p \in \mathbb{N} \cup\{0\}$ and let $\ell_{1}, \ldots, \ell_{p} \in \mathbb{N}$ with $\ell_{1} \geq \ldots \geq \ell_{p}$ and $\sum_{i=1}^{p} \ell_{i} \leq k-1$. Let $\mathcal{P}=\left\{W_{1}, \ldots, W_{p}, U\right\}$ be a vertex partition of $V(G)$. We say that $\mathcal{P}$ has parameters $\left(\ell_{1}, \ell_{2}, \ldots, \ell_{p}\right)$ if
(a) there exist $p$ pairwise disjoint colour sets $C_{1}, \ldots, C_{p}$ such that $\left|C_{i}\right|=\ell_{i}$ for all $i \in[p]$;
(b) $W_{i}$ is a $C_{i}$-adapter and $\left|W_{i}\right|=3 \ell_{i}+1$ for all $i \in[p]$;
(c) there exists a rainbow matching $M_{U}$ of size $k-1-\sum_{i=1}^{p} \ell_{i}$ in $G[U]$ with $c\left(M_{U}\right) \cap C_{i}=\emptyset$ for all $i \in[p]$;
(d) $U \backslash V\left(M_{U}\right) \neq \emptyset$.

Since $G$ contains a rainbow matching $M$ of size $k-1$, such a vertex partition exists ( $p=0$ and $U=V(G)$ say). We now assume that $\mathcal{P}$ is chosen such that the string $\left(\ell_{1}, \ldots, \ell_{p}\right)$ is lexicographically maximal. (Here, we view $\left(a_{1}, a_{2}, \ldots, a_{p}\right)$ as $\left(a_{1}, a_{2}, \ldots, a_{p}, 0, \ldots, 0\right)$, e.g. $(3,2,2) \leq(4,1) \leq(4,1,1)$.)

Let $C_{1}, \ldots, C_{p}$ be the sets of colours guaranteed by (a)-(c). Set $W=W_{1} \cup \cdots \cup W_{p}$ and $C=\bigcup_{i=1}^{p} C_{i}$. Let $\ell_{0}=k-1-$ $\sum_{i=1}^{p} \ell_{i}$. By (b) and Proposition 2.4(ii), $W$ is a ( $C, \ell_{1}+1$ )-adapter. The following claim gives some useful properties of the rainbow matchings in $G[U]$ and $G \backslash W$. This will be needed to finish the proof of the lemma.

Claim 2.6. (i) Let $M_{U}$ be a rainbow matching of size $\ell_{0}$ in $G[U]$ with $c\left(M_{U}\right) \cap C=\emptyset$. If $|U| \geq 2 \ell_{0}+2$ and there is an edge $w z \in E(G)$ with $w \in W$ and $z \in U \backslash V\left(M_{U}\right)$, then we have $c(w z) \in C$.
(ii) Let $M^{\prime}$ be a rainbow matching of size $k-1-\ell_{1}$ in $G \backslash W$ with $c\left(M^{\prime}\right) \cap C_{1}=\emptyset$. If $w x \in E(G)$ with $w \in W_{1}$ and $x \in$ $V(G) \backslash\left(W_{1} \cup V\left(M^{\prime}\right)\right)$, then $c(w x) \in C_{1}$.

Proof of Claim. Suppose that (i) is false. There exists an edge $w z \in E(G)$ such that $c(w z) \notin C, w \in W_{i}$ for some $i \in[p]$ and $z \in U \backslash V\left(M_{U}\right)$. Note that there exists a rainbow matching $M_{W}$ in $G[W \backslash w]$ such that $c\left(M_{W}\right)=C$ since $W$ is a $C$-adapter. If $c(w z) \notin C \cup c\left(M_{U}\right)$, then $M_{U} \cup M_{W} \cup\{w z\}$ is a rainbow matching of size $k$, so we are done. If $c(w z) \in c\left(M_{U}\right)$, then let $x y$ be the edge in $M_{U}$ such that $c(x y)=c(w z)$. Set $W_{i}^{\prime}=W_{i} \cup\{x, y, z\}, W_{j}^{\prime}=W_{j}$ for all $j \in[p] \backslash\{i\}$ and $U^{\prime}=U \backslash\{x, y, z\}$. Let $\ell_{i}^{\prime}=\ell_{i}+1$ and let $\ell_{j}^{\prime}=\ell_{j}$ for all $j \in[p] \backslash\{i\}$. Set $C_{i}^{\prime}=C_{i} \cup\{c(x y)\}$ and $C_{j}^{\prime}=C_{j}$ for all $j \in[p] \backslash\{i\}$. By Proposition 2.4(iii), $W_{j}^{\prime}$ is a $C_{j}^{\prime}$-adapter for all $j \in[p]$. Note that $M_{U^{\prime}}=M_{U}-x y$ is a rainbow matching in $G\left[U^{\prime}\right]$ with $c\left(M_{U^{\prime}}\right) \cap C_{j}^{\prime}=\emptyset$ for all $j \in[p]$. Also $U^{\prime} \backslash V\left(M_{U^{\prime}}\right)=U \backslash\left(V\left(M_{U}\right) \cup\{z\}\right) \neq \emptyset$. By relabelling the sets $W_{j}^{\prime}$ and $C_{j}^{\prime}$ if necessary, we deduce that the vertex partition $\mathcal{P}^{\prime}=\left\{W_{1}^{\prime}, \ldots, W_{p}^{\prime}, U^{\prime}\right\}$ has parameters $\left(\ell_{1}^{\prime}, \ldots, \ell_{p}^{\prime}\right)>\left(\ell_{1}, \ldots, \ell_{p}\right)$, which contradicts the maximality of $\mathcal{P}$. Hence (i) holds.

A similar argument proves (ii).
Suppose that $|U|>\gamma\left(\ell_{0}+1\right)$, so $|U| \geq 2 \ell_{0}+3$. Let $H$ be the resulting subgraph of $G[U]$ obtained after removing all edges of colours in $C$. Let $M_{U}$ be a rainbow matching in $H$ of size $\ell_{0}$ with $c\left(M_{U}\right) \cap C=\emptyset$, which exists by (c). By Claim 2.6(i), we have for all $z \in V(H) \backslash V\left(M_{U}\right), d_{H}^{c}(z) \geq k-|C|=\ell_{0}+1$. Since $\gamma(g) \leq \gamma, H$ contains a rainbow matching $M_{0}$ of size $\ell_{0}$ and a disjoint edge $e$. If $c(e)=c(x y)$ for some $x y \in M_{0}$, then set $W_{p+1}=V(e) \cup\{x, y\}, C_{p+1}=\{c(x y)\}$, and $U^{\prime}=U \backslash(V(e) \cup\{x, y\})$. Observe that $W_{p+1}$ is a $C_{p+1}$-adapter by Proposition 2.4(i). Note that $M_{0}-x y$ is a rainbow matching of size $\ell_{0}-1$ in $G\left[U^{\prime}\right]$ with $c\left(M_{0}\right) \cap \bigcup_{j \in[p+1]} C_{j}=\emptyset$ and $\left|U^{\prime} \backslash V\left(M_{0}\right)\right|=|U|-2 \ell_{0}-2 \geq 1$. Hence the vertex partition $\mathcal{P}^{\prime}=\left\{W_{1}, \ldots, W_{p+1}, U^{\prime}\right\}$ has parameters $\left(\ell_{1}, \ldots, \ell_{p}, 1\right)$, contradicting the maximality of $\mathcal{P}$. If $c(e) \notin c\left(M_{0}\right)$, then $M_{0} \cup e$ is a rainbow matching with $c\left(M_{0} \cup e\right) \cap C=\emptyset$. Together with (b), $G$ contains a rainbow matching of size $k$ with colours $c\left(M_{0} \cup e\right) \cup C$, so we are done. Therefore we may assume that

$$
\begin{equation*}
|U| \leq \gamma\left(\ell_{0}+1\right) \tag{1}
\end{equation*}
$$

Since $2<\gamma \leq 3$ and $\ell_{0} \leq k-1$, by the assumptions of Lemma 2.5, we have $|G|>(2+\gamma / 2) k>\gamma k \geq|U|$. Therefore, $W \neq \emptyset$ and $\ell_{1} \geq 1$.

Next, suppose that $(\gamma-2) \ell_{1} \geq 2$, so $\left|W_{1}\right|=3 \ell_{1}+1 \leq(2+\gamma / 2) \ell_{1}$. Let $H_{1}$ be the subgraph of $G$ obtained by removing all vertices of $W_{1}$ and all edges of colours in $C_{1}$. By the assumptions of Lemma 2.5 , we then have

$$
\left|H_{1}\right|=|G|-\left|W_{1}\right| \geq\left(2+\frac{\gamma}{2}\right)\left(k-\ell_{1}\right)+\frac{2(4-\gamma)}{(\gamma-2)^{2}}-3+\gamma
$$

By (b) and (c), $H_{1}$ contains a rainbow matching $M^{\prime}$ of size $k-1-\ell_{1}$. By Claim 2.6(ii), $c(w x) \in C_{1}$ for all $w \in W_{1}$ and $x \in$ $V\left(H_{1}\right) \backslash V\left(M^{\prime}\right)$. Hence, $d_{H_{1}}^{c}(z) \geq k-\left|C_{1}\right|=k-\ell_{1}$ for all $z \in V\left(H_{1}\right) \backslash V\left(M^{\prime}\right)$. Note that this statement also holds for any rainbow matchings $M^{\prime}$ of size $k-1-\ell_{1}$ in $H_{1}$. Hence $H_{1}$ satisfies the hypothesis of the lemma with $k=k-\ell_{1}$. By the induction hypothesis, $H_{1}$ contains a rainbow matching $M^{\prime \prime}$ of size $k-\ell_{1}$. By (b), there exists a rainbow matching $M_{1}$ of size $\ell_{1}$ in $G\left[W_{1}\right]$ such that $c\left(M_{1}\right)=C_{1}$. Since $c\left(M_{1}\right) \cap c\left(M^{\prime \prime}\right) \subseteq C_{1} \cap c\left(H_{1}\right)=\emptyset, M_{1} \cup M^{\prime \prime}$ is a rainbow matching of size $k$ as required. Therefore we may assume that

$$
\begin{equation*}
(\gamma-2) \ell_{1}<2 \tag{2}
\end{equation*}
$$

Recall that $W$ is a $\left(C, \ell_{1}+1\right)$-adapter. So there exist rainbow matchings $M_{1}^{*}, M_{2}^{*}, \ldots, M_{\ell_{1}+1}^{*}$ such that $c\left(M_{i}^{*}\right)=C$ for all $i \in\left[\ell_{1}+1\right]$ and

$$
\begin{equation*}
W=\bigcup_{i=1}^{\ell_{1}+1}\left(W \backslash V\left(M_{i}^{*}\right)\right) \tag{3}
\end{equation*}
$$

Let $M_{U}$ be a rainbow matching of size $\ell_{0}$ in $G[U]$ with $c\left(M_{U}\right) \cap C=\emptyset$ (which exists by (c)). By (d), there exists $z \in U \backslash V\left(M_{U}\right)$. Note that $z$ sends at least $d^{c}(z)-\left|V\left(M_{U}\right)\right| \geq k-2 \ell_{0}$ edges of distinct colours to $V(G) \backslash V\left(M_{U}\right)$. Let $q=\left\lceil\left(k-2 \ell_{0}\right) /\left(\ell_{1}+1\right)\right\rceil$. By (3) and an averaging argument, there exists $i \in\left[\ell_{1}+1\right]$ such that there exist vertices $x_{1}, \ldots, x_{q} \in V(G) \backslash V\left(M_{U} \cup M_{i}^{*}\right)$ such that $c\left(z x_{j}\right)$ is distinct for each $j \in[q]$. By Claim 2.6(i), we have $c\left(z x_{j}\right) \in C=c\left(M_{i}^{*}\right)$ for all $j \in[q]$. Let $e_{1}, \ldots, e_{q}$ be edges of $M_{i}^{*}$ such that $c\left(e_{j}\right)=c\left(z x_{j}\right)$ for all $j \in[q]$. Set $W^{\prime}=\bigcup_{j \in[q]}\left(V\left(e_{j}\right) \cup\left\{x_{j}, z\right\}\right)$ and $C^{\prime}=\left\{c\left(e_{j}\right): j \in[q]\right\}$. By Proposition 2.4(i), $W^{\prime}$ is a $C^{\prime}$-adapter. Set $U^{\prime}=V(G) \backslash W^{\prime}$ and $M_{U^{\prime}}=\left(M_{i}^{*} \cup M_{U}\right) \backslash W^{\prime}$. Note that $V\left(M_{U^{\prime}}\right) \subseteq U^{\prime}$ and $M_{U^{\prime}}$ is a rainbow matching of size $k-1-q$ with $c\left(M_{U^{\prime}}\right) \cap C^{\prime}=\emptyset$. Therefore, the vertex partition $\mathcal{P}^{\prime}=\left\{W^{\prime}, U^{\prime}\right\}$ has parameter $(q)$. By the maximality of $\mathcal{P}$, we have $\ell_{1} \geq q \geq\left(k-2 \ell_{0}\right) /\left(\ell_{1}+1\right)$ and so

$$
\begin{equation*}
\ell_{0} \geq\left(k-\ell_{1}\left(\ell_{1}+1\right)\right) / 2 \tag{4}
\end{equation*}
$$

Recall that $\left|W_{i}\right|=3 \ell_{i}+1 \leq 4 \ell_{i}$ for all $i \in[p]$, that $\sum_{i=1}^{p} \ell_{i}+\ell_{0}=k-1$, and that $2<\gamma \leq 3$. Finally, we have

$$
\begin{aligned}
|G| & =\left|W_{1}\right|+\sum_{i=2}^{p}\left|W_{i}\right|+|U| \stackrel{(1)}{\leq} 3 \ell_{1}+1+4 \sum_{i=2}^{p} \ell_{i}+\gamma\left(\ell_{0}+1\right) \\
& =3 \ell_{1}+1+4\left(k-1-\ell_{1}\right)-(4-\gamma) \ell_{0}+\gamma \\
& \stackrel{(4)}{\leq} 4 k-3-\ell_{1}-\frac{(4-\gamma)\left(k-\ell_{1}\left(\ell_{1}+1\right)\right)}{2}+\gamma
\end{aligned}
$$

$$
\begin{aligned}
& =\left(2+\frac{\gamma}{2}\right) k-3-\ell_{1}+\frac{(4-\gamma) \ell_{1}\left(\ell_{1}+1\right)}{2}+\gamma \\
& <\left(2+\frac{\gamma}{2}\right) k+\frac{(4-\gamma) \ell_{1}^{2}}{2}-3+\gamma \stackrel{(2)}{<}\left(2+\frac{\gamma}{2}\right) k+\frac{2(4-\gamma)}{(\gamma-2)^{2}}-3+\gamma
\end{aligned}
$$

a contradiction. This completes the proof of the lemma.
We are now ready to prove Theorems 1.1 and 1.2.
Proof of Theorems $\mathbf{1 . 1}$ and 1.2. We first prove Theorem 1.1 by induction on $k$. Let $G$ be an edge-coloured graph on $n \geq$ $7 k / 2+2$ vertices with $\delta^{c}(G) \geq k$. This is trivial for $k=1$ and so we may assume that $k \geq 2$. By the induction hypothesis $G$ contains a rainbow matching of size $k-1$. Since $\delta^{c}(G) \geq k$, Corollary 2.3 implies that $G$ satisfies the hypothesis of Lemma 2.5 with $\gamma=3$. Therefore, $G$ contains a rainbow matching of size $k$ as required.

To prove Theorem 1.2, first note that by Proposition $2.1, \gamma\left(g^{\prime}\right)=2$, where $g^{\prime}$ is the family of all bipartite graphs. Also, for $\gamma=2+2 \varepsilon$, we have

$$
\left(2+\frac{\gamma}{2}\right) k+\frac{2(4-\gamma)}{(\gamma-2)^{2}}-3+\gamma=(3+\varepsilon) k+\frac{2(2-2 \varepsilon)}{4 \varepsilon^{2}}-1+2 \varepsilon \leq(3+\varepsilon) k+\varepsilon^{-2}
$$

Therefore, Theorem 1.2 follows from a similar argument used in the preceding paragraph, where we take $\gamma=2+2 \varepsilon$ and $\xi$ to be the family of all bipartite graphs in the application of Lemma 2.5.

We would like to point out that an improvement of Corollary 2.3 would lead to an improvement of Theorem 1.1. However, we believe that new ideas are needed to prove the case when $2 k<|G|<3 k$.

## 3. Existence of rainbow matching covers

Proof of Theorem 1.3. By colouring every missing edge in $G$ with a new colour, we may assume that $G$ is an edge-coloured complete graph on $n$ vertices with $\Delta_{\text {mon }}(G)=t$ and colours $\{1,2, \ldots, p\}$. For $i \leq p$, let $G^{i}$ be the subgraph of $G$ induced by the edges of colour $i$. Without loss of generality, we may assume that $e\left(G^{1}\right) \geq e\left(G^{2}\right) \geq \cdots \geq e\left(G^{p}\right)$.

For $1 \leq i \leq p$, suppose that we have already found a set $\mathcal{M}=\left\{M_{1}, \ldots, M_{\lfloor t n / 2\rfloor}\right\}$ of edge-disjoint (possibly empty) rainbow matchings such that $\bigcup_{1 \leq j \leq\lfloor t n / 2\rfloor} M_{j}=\bigcup_{j^{\prime}<i} E\left(G^{j^{\prime}}\right)$. We now assign edges of $G^{i}$ to these matchings so that the resulting rainbow matchings $M_{1}^{\prime}, \ldots, M_{\lfloor t n / 2\rfloor}^{\prime}$ contain all edges of $G^{1} \cup \ldots \cup G^{i}$. Define an auxiliary bipartite graph $H$ as follows. The vertex classes of $H$ are $E\left(G^{i}\right)$ and $\mathcal{M}$. An edge $f \in E\left(G^{i}\right)$ is joined to a rainbow matching $M_{j} \in \mathcal{M}$ if and only if $f$ is vertex-disjoint from $M_{j}$. If $H$ contains a matching of size $e\left(G^{i}\right)$, then we assign $f \in E\left(G^{i}\right)$ to $M_{j} \in \mathcal{M}$ according to the matching in $H$. Thus we have obtained the desired rainbow matchings $M_{1}^{\prime}, \ldots, M_{\lfloor t n / 2\rfloor}^{\prime}$. Therefore, to prove the theorem, it is sufficient to show that $H$ satisfies Hall's conditions.

Let $f \in E\left(G^{i}\right)$. Since $f$ is incident to $2(n-2)$ edges in $G, f$ is incident to at most $2(n-2)$ matchings $M_{j} \in \mathcal{M}$. Thus,

$$
\begin{equation*}
\left|N_{H}(f)\right| \geq|\mathcal{M}|-2(n-2) \geq(t-4) n / 2 \tag{5}
\end{equation*}
$$

We divide the proof into two cases depending on the value of $i$.
Case 1: $i \leq \frac{(t-4) n}{4(t+1)}$. Let $S \subseteq E\left(G^{i}\right)$ with $S \neq \emptyset$. Note that each $M_{j} \in \mathcal{M}$ has size at most $i-1$. If $S$ contains a matching of size $2 i-1$, then for every $M_{j} \in \mathcal{M}$, there exists an edge $f \in S$ vertex-disjoint from $M_{j}$. Thus, $N_{H}(S)=\mathcal{M}$ and so $\left|N_{H}(S)\right|=$ $\lfloor$ tn $/ 2\rfloor \geq e\left(G^{i}\right) \geq|S|$.

Therefore, we may assume that $S$ does not contain a matching of size $2 i-1$. By Vizing's theorem, $|S| \leq 2(i-1)\left(\Delta\left(G^{i}\right)+\right.$ $1) \leq 2(i-1)(t+1) . \operatorname{By}(5)$ and the assumption on $i$, we have

$$
\left|N_{H}(S)\right| \geq(t-4) n / 2 \geq 2(i-1)(t+1) \geq|S|
$$

Therefore, Hall's condition holds for this case.
Case 2: $i>\frac{(t-4) n}{4(t+1)}$. Since $e\left(G^{1}\right) \geq e\left(G^{2}\right) \geq \cdots \geq e\left(G^{p}\right)$, we have $e\left(G^{i}\right) \leq\binom{ n}{2} / i<2(t+1) n /(t-4)$. Let $S \subseteq E\left(G^{i}\right)$ with $S \neq \emptyset$. $\operatorname{By}$ (5) and the fact that $t \geq 11$, we have

$$
\left|N_{H}(S)\right| \geq(t-4) n / 2 \geq 2(t+1) n /(t-4)>e\left(G^{i}\right) \geq|S| .
$$

Therefore, Hall's condition also holds for this case. This completes the proof of the theorem.

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