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# ON THE CHARACTER DEGREES OF SUZUKI $p$ -GROUPS OF TYPE $A$ AND $C$

TUNG LE, KAY MAGAARD AND JAMSHID MOORI

ABSTRACT. In this note, we construct the irreducible characters of Suzuki  $p$ -groups of types  $A_p(m, \theta)$  and  $C_p(m, \theta, \varepsilon)$ .

## 1. INTRODUCTION AND PRELIMINARY

In 1963, Higman [5] classified all Suzuki 2-groups; i.e., those 2-groups which possess an automorphism that cyclically permutes all of its involutions. He showed that they fall into four families which can be explicitly described as families of matrix groups over finite fields of even order. A few years later, Shult [8] showed that for odd primes  $p$  the hypothesis that the subgroups of order  $p$  of a  $p$ -group  $G$  are permuted cyclically by some automorphism of  $G$  implies that  $G$  is abelian. However the definitions of the families of Suzuki 2-groups also make sense for odd primes. Using Higman's terminology we will refer to the groups as  $A_p(m, \theta)$ ,  $B_p(m, \theta, \varepsilon)$ ,  $C_p(m, \theta, \varepsilon)$ , and  $D_p(m, \theta, \varepsilon)$ . Here  $Z(G)$  is isomorphic to the additive group of the field  $\mathbb{F}_q := \mathbb{F}_{p^m}$ ,  $\theta$  denotes a Frobenius homomorphism of  $\mathbb{F}_q$ , and  $\varepsilon$  a suitably chosen element from  $\mathbb{F}_q$ .

Recently Craven and Glesser [4] investigated the fusion systems and automorphism groups of Suzuki 2-groups. From the point of view of fusion systems these groups are minimal. We are interested in the representation theory of Suzuki  $p$ -groups, as they appear as quotients and subgroups of unipotent radicals of twisted Chevalley groups. The character degrees of the groups  $A_2(m, \theta)$  respectively  $A_p(m, \theta)$  were determined in 1999 respectively 2003 by Sagirov [6] respectively [7] and again in 2008 by Berkovich and Janko [2, Chapter 46]. However the character degrees of the other families of Suzuki  $p$ -groups have not yet been determined, even when  $p = 2$ .

Here we reprove Sagirov's result on  $A_p(m, \theta)$  using a different technique which is independent of the prime  $p$ . We describe the structure of  $A_p(m, \theta)$  in more detail and construct all irreducible characters, see Theorem 2.3. In Theorem 3.2 we extend our method to Suzuki  $p$ -groups of type  $C_p(m, \theta, \varepsilon)$ . In particular we show that Suzuki  $p$ -groups of types  $A$  and  $C$  have at most three distinct character degrees. Also we determine the number of characters of each degree as well as the character values of group elements for type  $A$ . We refer the reader to Remark 3.4 for why we do not consider groups of type  $B$  and  $D$  in this paper.

We conclude this section with a few more preparatory definitions and remarks. Let  $\text{Aut}(\mathbb{F}_{p^m})$  be the automorphism group of the field  $\mathbb{F}_{p^m}$ , and  $\theta \in \text{Aut}(\mathbb{F}_{p^m})$ . We

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recall that  $\text{Aut}(\mathbb{F}_{p^m})$  is generated by the map  $a \mapsto a^p$  of order  $m$ . Thus, the order of  $\theta$  is some divisor of  $m$ , call it  $k$ , and set  $n := m/k$ . Throughout this paper, we suppose that  $\theta(a) = a^{p^l}$  for all  $a \in \mathbb{F}_{p^m}$  where  $\gcd(l, m) = n$ . The fixed point set of  $\theta$  is a subfield  $\mathbb{F}_\theta$  of  $\mathbb{F}_{p^m}$  satisfying  $a^{p^l} = a$  so containing  $p^n$  elements, i.e.,  $\mathbb{F}_\theta = \mathbb{F}_{p^n}$ . Let  $\mathbb{F}^\times$  be the (cyclic) multiplicative group of the field  $\mathbb{F}$ . We denote the irreducible character set of a group  $G$  by  $\text{Irr}(G)$ . Put  $\text{cd}(G) := \{\chi(1) : \chi \in \text{Irr}(G)\}$ . Next,  $\text{Irr}_{(t)}(G)$  denotes the members  $\text{Irr}(G)$  of degree  $t$ , and  $\text{Irr}(G)^\times := \text{Irr}(G) - \{1_G\}$ . The other notation is quite standard.

*Definition 1.1.* A  $p$ -group  $G$  is called *extraspecial* if  $[G, G] = \Phi(G) = \text{Z}(G) \cong \mathbb{E}_p$ .

We recall some properties of extraspecial  $p$ -groups as follows.

**Lemma 1.2.** *Let  $G$  be a  $p$ -group of order  $p^{1+2m}$ . The following are true.*

- (i)  $G$  is extraspecial iff  $[x, G] = \text{Z}(G) \cong \mathbb{E}_p$  for all  $x \in G - \text{Z}(G)$ .
- (ii) If  $G$  is extraspecial, then  $\text{cd}(G) = \{1, p^m\}$  and  $|\text{Irr}_{(p^m)}(G)| = p - 1$ . Moreover, for every  $\lambda \in \text{Irr}(\text{Z}(G))^\times$  there exists a unique  $\chi \in \text{Irr}(G)$  such that

$$\chi(g) = \begin{cases} p^m \lambda(g), & \text{if } g \in \text{Z}(G) \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* (i) It suffices to show the converse statement. If  $[G, G] = \text{Z}(G) \cong \mathbb{E}_p$ , then  $[x^p, y] = [x, y]^p = 1$  for all  $x, y \in G$ , which implies that  $G/[G, G]$  is elementary abelian and  $[G, G] = \Phi(G) = \text{Z}(G)$ . From  $[G, G] = \langle [x, G] : x \in G \rangle$  and  $[x, G] = \text{Z}(G)$  for all  $x \in G - \text{Z}(G)$  we have  $[G, G] = \text{Z}(G)$ . So  $G$  is extraspecial.

- (ii) By [1, Theorem 4.7 (d)] and the induction formula of  $\lambda \in \text{Irr}(\text{Z}(G))$  that

$$\lambda^G(g) = \begin{cases} p^{2m} \lambda(g), & \text{if } g \in \text{Z}(G) \\ 0, & \text{otherwise} \end{cases},$$

the claim is clear. □

*Definition 1.3.* Let  $\theta \in \text{Aut}(\mathbb{F}_{p^m})$ . For each  $a \in \mathbb{F}_{p^m}^\times$ , we define  $f_{a, \theta}(t) := t^{p^l} - a^{p^l-1}t$  for all  $t \in \mathbb{F}_{p^m}$  and  $\mathbb{I}_a := \text{im}(f_{a, \theta})$ .

We consider the intersection of two finite fields of characteristic  $p$  to be taken in the algebraic closure of the field  $\mathbb{F}_p$ .

**Proposition 1.4.** *Let  $\theta \in \text{Aut}(\mathbb{F}_{p^m})$ . Suppose that  $o(\theta) = k > 1$  and  $m = nk$ . For all  $a, b \in \mathbb{F}_{p^m}^\times$ , the following hold.*

- (i)  $\mathbb{I}_a$  is an  $\mathbb{F}_\theta$ -hyperplane of  $\mathbb{F}_{p^m}$  of dimension  $k - 1$ .
- (ii)  $\mathbb{F}_{p^m}^\times$  acts on  $\{\mathbb{I}_a : a \in \mathbb{F}_{p^m}^\times\}$  by  $\theta(x)\mathbb{I}_a = \mathbb{I}_{ax}$  for all  $x \in \mathbb{F}_{p^m}^\times$ . The stabilizer  $\text{Stab}_{\mathbb{F}_{p^m}^\times}(\mathbb{I}_a) = \mathbb{F}_\theta^\times$  and  $|\{\mathbb{I}_a : a \in \mathbb{F}_{p^m}^\times\}| = (q - 1)/(p^n - 1)$ .
- (iii) For each  $\mathbb{F}_p$ -hyperplane  $H$  of  $\mathbb{F}_{p^m}$ , there is a unique  $\mathbb{I}_a$  such that  $\mathbb{I}_a \leq H$ .

*Proof.* It is clear that  $f_{a, \theta}$  is an  $\mathbb{F}_\theta$ -homomorphism. We shall show  $\ker(f_{a, \theta}) = a\mathbb{F}_\theta$ . We have  $a\mathbb{F}_\theta \subset \ker(f_{a, \theta})$  since  $\theta(au) = \theta(a)u$  for all  $u \in \mathbb{F}_\theta$ . As  $t^{p^l} - a^{p^l-1}t = 0$  for some  $t \neq 0$  implies  $t \in a(\mathbb{F}_{p^l}^\times \cap \mathbb{F}_{p^m}^\times) = a\mathbb{F}_\theta^\times$ ,  $f_{a, \theta}$  has at most  $p^n$  solutions as needed. Now we show (ii) and (iii).

Let  $\mathfrak{S} := \{\mathbb{I}_a : a \in \mathbb{F}_{p^m}^\times\}$ . For all  $a, x \in \mathbb{F}_{p^m}^\times$ ,  $\theta(x)\mathbb{I}_a = \mathbb{I}_{ax}$  since

$$\theta(x)f_{a, \theta}(t) = x^{p^l}(t^{p^l} - a^{p^l-1}t) = (tx)^{p^l} - (ax)^{p^l-1}(tx) = f_{ax, \theta}(tx) \in \text{im}(f_{ax, \theta}) = \mathbb{I}_{ax}.$$

Thus,  $\mathbb{F}_{p^m}^\times$  acts transitively on  $\mathfrak{S}$ . It is clear that  $\mathfrak{S} \neq \emptyset$  and  $\mathbb{F}_\theta^\times \leq \text{Stab}_{\mathbb{F}_{p^m}^\times}(\mathbb{I}_a)$ . So  $0 < |\mathfrak{S}| \leq \frac{p^m-1}{p^n-1}$ . We show  $\text{Stab}_{\mathbb{F}_{p^m}^\times}(\mathbb{I}_a) = \mathbb{F}_\theta^\times$  by computing exactly  $|\mathfrak{S}|$ .

For each  $\mathbb{I}_a$ , the number of  $\mathbb{F}_p$ -hyperplanes of  $\mathbb{F}_{p^m}$  containing  $\mathbb{I}_a$  equals the number of  $\mathbb{F}_p$ -hyperplanes of  $\mathbb{F}_{p^m}/\mathbb{I}_a \cong \mathbb{F}_\theta$ , which is  $\frac{p^n-1}{p-1}$ . There are no  $\mathbb{I}_a \neq \mathbb{I}_b$  contained in the same  $\mathbb{F}_p$ -hyperplane since  $\dim_{\mathbb{F}_\theta}(\mathbb{I}_a + \mathbb{I}_b) \geq \dim_{\mathbb{F}_\theta}(\mathbb{I}_a) + 1 = k = \dim_{\mathbb{F}_\theta}(\mathbb{F}_{p^m})$ . Thus, each  $\mathbb{F}_p$ -hyperplane of  $\mathbb{F}_{p^m}$  contains at most one  $\mathbb{I}_a$ . The transitivity of the action by left multiplication of  $\mathbb{F}_{p^m}^\times = \theta(\mathbb{F}_{p^m}^\times)$  on the  $\mathbb{F}_p$ -hyperplane set shows that each hyperplane containing exactly one  $\mathbb{I}_a$ . Since the number of  $\mathbb{F}_p$ -hyperplanes of  $\mathbb{F}_{p^m}$  is  $\frac{p^m-1}{p-1}$ , we obtain  $|\mathfrak{S}| = \frac{p^m-1}{p-1} : \frac{p^n-1}{p-1} = \frac{p^m-1}{p^n-1}$ .  $\square$

**Corollary 1.5.** *For  $a, b \in \mathbb{F}_{p^m}^\times$ ,  $a\mathbb{I}_a = b\mathbb{I}_b$  iff  $a \in b(\mathbb{F}_{p^m}^\times \cap \mathbb{F}_{\theta^2})$ .*

*Proof.* By Proposition 1.4 (ii),  $\mathbb{I}_a = \theta(a)\mathbb{I}_1$  and  $\mathbb{I}_b = \theta(b)\mathbb{I}_1$ , we obtain  $a\mathbb{I}_a = b\mathbb{I}_b$  iff  $ab^{-1}\theta(ab^{-1}) \in \text{Stab}_{\mathbb{F}_{p^m}^\times}(\mathbb{I}_1) = \mathbb{F}_\theta^\times$ . It suffices to show that  $x\theta(x) \in \mathbb{F}_\theta^\times$  iff  $x \in \mathbb{F}_{\theta^2}^\times$  which is clear by  $x\theta(x) = \theta(x\theta(x))$  iff  $x = \theta^2(x)$ .  $\square$

Throughout this paper, we fix  $\phi \in \text{Irr}(\mathbb{F}_{p^m})^\times$ . For each  $r \in \mathbb{F}_{p^m}$ , we define  $\phi_r \in \text{Irr}(\mathbb{F}_{p^m})$  by  $\phi_r(s) := \phi(rs)$  for all  $s \in \mathbb{F}_{p^m}$ . Thus,  $\text{Irr}(\mathbb{F}_{p^m}) = \{\phi_r : r \in \mathbb{F}_{p^m}\}$ .

For a subgroup  $P \leq (\mathbb{F}_{p^m}, +)$ , we define  $\text{Irr}(P)_{\mathbb{F}_{p^m}}$  as the set of all class functions of  $\mathbb{F}_{p^m}$  of the form

$$\bar{\tau}(x) := \begin{cases} \tau(x), & \text{if } x \in P \\ 0, & \text{otherwise} \end{cases}$$

for each  $\tau \in \text{Irr}(P)$ .

## 2. SUZUKI $p$ -GROUPS $A_p(m, \theta)$

**Definition 2.1.** The Suzuki  $p$ -group  $A_p(m, \theta)$  is the set  $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$  with  $m \geq 1$  and  $\theta \in \text{Aut}(\mathbb{F}_{p^m})$  with the multiplication defined as follows:

$$(a, b)(c, d) := (a + c, b + d + a\theta(c)).$$

Let  $G := A_p(m, \theta)$ . So  $|G| = p^{2m}$ . It follows from the definition that the multiplication is associative, the identity element of  $G$  is  $(0, 0)$ , and  $(a, b)^{-1} = (-a, -b + a\theta(a))$ . So  $G$  is a group.

It is easy to prove, by induction that  $(a, b)^i = (ia, ib + \binom{i}{2}a\theta(a))$ . Hence, if  $p = 2$  then  $\exp(G) = 4$ ; otherwise, if  $p > 2$  then  $\exp(G) = p$ .

**Remark 2.2.** By [3, Proposition 13.6.4 (vi)],  $A_2(2f+1, \theta)$  where  $\theta(x) = x^{2^f+1}$  for all  $x \in \mathbb{F}_{2^{2f+1}}$  is isomorphic to a Sylow 2-subgroup of the Suzuki group  ${}^2B_2(2^{2f+1})$ .

We have  $[(a, b), (c, d)] = (0, a\theta(c) - c\theta(a))$ . If  $\theta = 1$ , then  $G$  is abelian. So we assume without loss of generality that  $\theta \neq 1$ . Thus,  $1 < m$  and  $n < m$ . If there is  $(c, d) \in Z(G)$  with  $c \neq 0$ , then  $\theta(a/c) = a/c$  for all  $a \in \mathbb{F}_{p^m}$ , which implies  $a \in c\mathbb{F}_\theta$  for all  $a \in \mathbb{F}_{p^m}$ , this contradicts  $|\mathbb{F}_\theta| = p^n < |\mathbb{F}_{p^m}|$ . So  $Z(G) = \{(0, d) : d \in \mathbb{F}_{p^m}\}$ .

For each  $a \in \mathbb{F}_{p^m}^\times$ , we define

$$\varphi_a : \mathbb{F}_{p^m} \rightarrow \mathbb{F}_{p^m}, u \mapsto a\theta(u) - u\theta(a) = au^{p^i} - ua^{p^i},$$

$$\text{and } O_a := \{(0, t) : t \in \text{im}(\varphi_a)\}.$$

It is clear that  $[(a, -), (t, -)] = (0, \varphi_a(t))$  and  $\text{im}(\varphi_a) = a\mathbb{I}_a$ . Thus, by Proposition 1.4 (i) we have  $O_a = [(a, -), G] \leq Z(G)$  of order  $p^{n(k-1)}$ .

**Theorem 2.3.** *Let  $G := A_p(m, \theta)$ , where  $m > 1$  and  $\theta \in \text{Aut}(\mathbb{F}_{p^m})$  of order  $k > 1$  and  $n := m/k$ . One of the following holds:*

- (i) *If  $k$  is odd, then  $[G, G] = \Phi(G) = Z(G)$  and the irreducible characters of  $G$  are parameterized in Table 1. We have*

$$\chi_{p^{(m-n)/2}}^{r,\gamma}((a, b)) = p^{(m-n)/2} \phi_r(b) \gamma(a), \quad \text{and} \quad \chi_1^r((a, b)) = \phi_r(a).$$

- (ii) *If  $k = 2$ , then  $[G, G] = \Phi(G)$  is a subgroup of  $Z(G)$  of index  $p^{m/2}$  and the irreducible characters are parameterized in Table 2. We have*

$$\chi_{p^{(m-n)/2}}^\beta((a, b)) = \delta_{a,0} p^{m/2} \beta(b), \quad \text{and} \quad \chi_1^{r,\gamma}((a, b)) = \phi_r(a) \gamma(b).$$

- (iii) *If  $k > 2$  is even, let  $J_1 := \{\phi_r : O_s \not\subset \ker(\phi_r) \forall s \in \mathbb{F}_{p^m}^\times\}$  and  $J_2 := \text{Irr}(\mathbb{F}_{p^m}^\times) - J_1$ , then  $[G, G] = \Phi(G) = Z(G)$ ,  $|J_1| = \frac{p^n(p^m-1)}{p^n+1}$ ,  $|J_2| = \frac{p^m-1}{p^n+1}$ , and the irreducible characters are parameterized in Table 3. We have*

$$\chi_{p^{m/2}}^\beta((a, b)) = \delta_{a,0} p^{m/2} \beta(b),$$

$$\chi_{p^{(m-2n)/2}}^{\beta,\gamma}((a, b)) = p^{(m-2n)/2} \beta(b) \gamma(a), \quad \text{and} \quad \chi_1^r((a, b)) = \phi_r(a).$$

TABLE 1. Irreducible characters of  $A_p(m, \theta)$  where  $\text{o}(\theta) > 2$  odd.

Family	Notation	Parameter set	Condition	Number	Degree
$\mathcal{F}$	$\chi_{p^{(m-n)/2}}^{r,\gamma}$	$\mathbb{F}_{p^m}^\times \times \text{Irr}(s\mathbb{F}_\theta)_{\mathbb{F}_{p^m}}$	$O_s \leq \ker(\phi_r)$	$p^n(p^m - 1)$	$p^{(m-n)/2}$
$\mathcal{F}_{lin}$	$\chi_1^r$	$\mathbb{F}_{p^m}$	-	$p^m$	1

TABLE 2. Irreducible characters of  $A_p(m, \theta)$  where  $\text{o}(\theta) = 2$ .

Family	Notation	Parameter set	Condition	Number	Degree
$\mathcal{F}$	$\chi_{p^{m/2}}^\beta$	$\text{Irr}(\mathbb{F}_{p^m})$	$\mathbb{D} \not\subset \ker(\beta)$	$p^m - p^{m/2}$	$p^{m/2}$
$\mathcal{F}_{lin}$	$\chi_1^{r,\gamma}$	$\mathbb{F}_{p^m} \times \text{Irr}(\mathbb{F}_{p^m})$	$\mathbb{D} \subset \ker(\beta)$	$p^{3m/2}$	1

where  $\mathbb{D} := \{u \in \mathbb{F}_{p^m} : \theta(u) = -u\}$ .

TABLE 3. Irreducible characters of  $A_p(m, \theta)$  where  $\text{o}(\theta) > 2$  even.

Family	Notation	Parameter set	Condition	Number	Degree
$\mathcal{F}^1$	$\chi_{p^{m/2}}^\beta$	$J_1$	-	$\frac{p^n(p^m-1)}{p^n+1}$	$p^{m/2}$
$\mathcal{F}^2$	$\chi_{p^{(m-2n)/2}}^{\beta,\gamma}$	$J_2 \times \text{Irr}(s\mathbb{F}_{\theta^2})_{\mathbb{F}_{p^m}}$	$O_s \leq \ker(\beta)$	$\frac{p^{2n}(p^m-1)}{p^n+1}$	$p^{(m-2n)/2}$
$\mathcal{F}_{lin}$	$\chi_1^r$	$\mathbb{F}_{p^m}$	-	$p^m$	1

Here is a brief explanation how to read these tables. In Table 1, the characters  $\chi_{p^{(m-n)/2}}^{r,\gamma} \in \mathcal{F}$  have degree  $p^{(m-n)/2}$  and satisfy that  $Z(G) \not\subset \ker(\chi_{p^{(m-n)/2}}^{r,\gamma})$ , where

$r \in \mathbb{F}_{p^m}^\times$  and  $\gamma \in \text{Irr}(s\mathbb{F}_\theta)_{\mathbb{F}_{p^m}}$ . The uniqueness of  $O_s$  and  $s\mathbb{F}_\theta$  corresponding to  $\ker(\phi_r)$  for the condition  $O_s \leq \ker(\phi_r)$  is shown in Proposition 2.4 (v) and Lemma 2.7. There are  $p^n(p^m - 1)$  characters of this type. The rest are read in a usual way. In Table 2, the condition on  $\mathbb{D}$  comes from the derived subgroup  $[G, G]$  in Corollary 2.5. In Table 3, the uniqueness of  $O_s$  and  $s\mathbb{F}_{\theta^2}$  comes from Proposition 2.4 (vi) and Lemma 2.9. To prove Theorem 2.3, we need the following proposition.

**Proposition 2.4.** *Let  $G := A_p(m, \theta)$  where  $\theta$  has order  $k > 1$ , and  $a, b \in \mathbb{F}_{p^m}^\times$ . Set  $n := m/k$ . The following hold.*

- (i)  $\varphi_a$  is an additive homomorphism,  $\ker(\varphi_a) = a\mathbb{F}_\theta$  and  $\text{im}(\varphi_a) = a\mathbb{I}_a \leq \mathbb{F}_{p^m}$ .
- (ii)  $\text{im}(\varphi_a) = \text{im}(\varphi_b)$  iff  $O_a = O_b$  iff  $a \in b(\mathbb{F}_{p^m}^\times \cap \mathbb{F}_{\theta^2})$ .
- (iii) If  $O_a \neq O_b$ , then  $Z(G) = O_a O_b$  and  $|O_a \cap O_b| = p^{n(k-2)}$ .
- (iv) If  $k = 2$  then  $[G, G] = O_a < Z(G)$  of index  $p^{m/2}$ . Otherwise, if  $k > 2$  then  $[G, G] = Z(G)$ .
- (v) If  $k$  is odd, then for each maximal subgroup  $H$  of  $Z(G)$ , there exists a unique subgroup  $O_a$  such that  $O_a \leq H$ .
- (vi) If  $k > 2$  is even, then there are exactly  $\frac{p^n(p^m-1)}{(p^n+1)(p-1)}$  maximal subgroups of  $Z(G)$  containing no  $O_a$ .

*Proof.* It is easy to check directly (i) from their definitions. By (i) and Corollary 1.5, it is clear for (ii). Thus, we shall prove (iii) to (vi). Note that  $O_a$  has index  $p^n$  in  $Z(G)$ .

(iii) By (i) and the definition of  $O_a$ , it suffices to show that if  $a\mathbb{I}_a \neq b\mathbb{I}_b$  then  $a\mathbb{I}_a + b\mathbb{I}_b = \mathbb{F}_{p^m}$  and  $|a\mathbb{I}_a \cap b\mathbb{I}_b| = p^{n(k-2)}$ . By Proposition 1.4 (i) the claim follows by using the properties of  $\mathbb{F}_\theta$ -hyperplanes of  $\mathbb{F}_{p^m}$ .

(iv) Suppose that  $k = 2$ . So  $m = 2n$  and  $\mathbb{F}_{p^m} = \mathbb{F}_{\theta^2}$ . By (ii),  $\text{im}(\varphi_a) = \text{im}(\varphi_b)$  for all  $b \in \mathbb{F}_{p^m}^\times$ . Since  $[G, G]$  is generated by all  $O_a$ ,  $a \in \mathbb{F}_{p^m}^\times$ , we obtain  $[G, G] = O_a < Z(G)$  of index  $p^n = p^{m/2}$  for any  $a \in \mathbb{F}_{p^m}^\times$ .

Suppose that  $k > 2$ . So  $m > 2n$  and  $|\mathbb{F}_{p^m}^\times| > |\mathbb{F}_{\theta^2}|$ . By (iii) it suffices to show that there are at least two distinct  $O_a \neq O_b$ , which is clear by (ii).

(v) If a maximal subgroup of  $Z(G)$  contains an  $O_a$ , then by (iii)  $O_a$  is unique. Now we prove the existence by counting directly the number of maximal subgroups of  $Z(G)$  containing some  $O_a$ . Since  $Z(G)$  is elementary abelian, for a fixed subgroup  $O_a$ , the number of maximal subgroups of  $Z(G)$  containing  $O_a$  equals the number of maximal subgroups of  $Z(G)/O_a \cong E_{p^n}$ , which is  $\frac{p^n-1}{p-1}$ . So by the uniqueness of the property containing  $O_a$ , and  $\mathbb{F}_{\theta^2} \cap \mathbb{F}_{p^m} = \mathbb{F}_\theta$  by the oddity of  $k$ , the number of maximal subgroups of  $Z(G)$  containing some  $O_a$  is  $\frac{p^m-1}{p^n-1} \cdot \frac{p^n-1}{p-1} = \frac{p^m-1}{p-1}$  which is known as the number of maximal subgroups of  $Z(G)$ .

(vi) First, we count the number of maximal subgroups of  $Z(G)$  containing some  $O_a$ . With the same argument in (v), the number of maximal subgroups of  $Z(G)$  containing a fixed  $O_a$  is  $\frac{p^n-1}{p-1}$ . Since  $k$  is even, we have  $\mathbb{F}_{\theta^2} = \mathbb{F}_{p^{2n}} \leq \mathbb{F}_{p^m}$ . Hence, by (ii) there are  $\frac{p^m-1}{p^{2n}-1}$  distinct  $O_a$ 's, as sets. So the number of maximal subgroups of  $Z(G)$  containing some  $O_a$  is  $\frac{p^m-1}{(p^n+1)(p-1)}$ , and the number of maximal subgroups of  $Z(G)$  containing no  $O_a$  is  $\frac{p^m-1}{p-1} - \frac{p^m-1}{(p^n+1)(p-1)} = \frac{p^n(p^m-1)}{(p^n+1)(p-1)}$ .  $\square$

Since  $G/Z(G) \cong E_{p^m}$ , we have  $[G, G] \leq \Phi(G) \leq Z(G)$ . By Proposition 2.4 (iv),  $[G, G] = \Phi(G) = Z(G)$  if the order of  $\theta$  is greater than two.

Suppose that  $\theta$  has order two, then  $G/[G, G]$  is abelian of order  $p^{3m/2}$ . If  $p > 2$ , from  $\exp(G) = p$  we have  $[G, G] = \Phi(G)$  of order  $p^{m/2}$ . To obtain a presentation of  $[G, G]$ , we set

$$\mathbb{D} := \{u \in \mathbb{F}_{p^m} : \theta(u) = -u\}.$$

For each  $a \in \mathbb{F}_{p^m}$ , we have  $a = \frac{1}{2}(a + \theta(a)) + \frac{1}{2}(a - \theta(a)) \in \mathbb{F}_\theta \oplus \mathbb{D}$  since  $2, \frac{1}{2} \in \mathbb{F}_\theta$ . Thus,  $\mathbb{F}_{p^m} = \mathbb{F}_\theta \oplus \mathbb{D}$ . It is clear that  $\mathbb{D}$  is a one-dimensional  $\mathbb{F}_\theta$ -subspace of  $\mathbb{F}_{p^m}$ , and invariant under  $\theta$ . From  $[(a, -), (c, -)] = (0, a\theta(c) - c\theta(a))$ , we have

$$[G, G] = \{(0, a) : a \in \mathbb{D}\}.$$

In the case  $p = 2$  we recall that  $\exp(G) = 4$ . We choose  $t \in \mathbb{F}_{p^m} - \mathbb{F}_\theta$  such that  $\mathbb{F}_{p^m} = \mathbb{F}_\theta \oplus t\mathbb{F}_\theta$  and  $t^2 + t + c = 0$  for some  $c \in \mathbb{F}_\theta$ . Suppose that  $\theta(t) = r + st$  for some  $r, s \in \mathbb{F}_\theta$ . Since  $\theta$  has order two, it forces  $s = 1$  and  $\theta(t) = r + t$ . Therefore, it is easy to check directly that  $[G, G] = \{(0, a) : a \in \mathbb{F}_\theta\}$ . For all  $a \in \mathbb{F}_{p^m}$ , the fact that  $(a, -)^2 = (0, a\theta(a)) \in [G, G]$  implies that  $G/[G, G]$  is elementary abelian. Thus,  $[G, G] = \Phi(G)$ . Note that the presentation of  $[G, G]$  when  $p = 2$  also matches with the one with  $\mathbb{D}$  when  $p > 2$ . This argument provides the proof of the following.

**Corollary 2.5.** *The following hold.*

- (i) *If the order of  $\theta$  is greater than two, then  $[G, G] = \Phi(G) = Z(G)$ .*
- (ii) *If the order of  $\theta$  is two, then  $[G, G] = \Phi(G) = \{(0, a) : a \in \mathbb{D}\}$  has index  $p^{m/2}$  in  $Z(G)$ .*

*Remark 2.6.* From the above discussion on the decomposition of  $\mathbb{F}_{p^m}$  into  $\mathbb{F}_\theta$ -subspaces, using an appropriate basis, we see that when  $p > 2$ ,  $\theta$  is the diagonal matrix  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ; when  $p = 2$ ,  $\theta$  is an upper triangular  $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ .

For  $\lambda \in \text{Irr}(Z(G))^\times$ ,  $Z(G)/\ker(\lambda) \cong \text{im}(\lambda) \cong E_p$ . The induction formula of  $\lambda^G$  implies that  $\ker(\lambda) \leq \ker(\chi)$  for all constituents  $\chi$  of  $\lambda^G$ . Using these properties, we shall prove Theorem 2.3 by considering all various possible values of  $k$ .

**Lemma 2.7.** *Let  $G := A_p(m, \theta)$  where  $\theta$  has odd order  $k > 1$ . Let  $n := m/k$ ,  $a \in \mathbb{F}_{p^m}^\times$ ,  $H$  a maximal subgroup of  $Z(G)$  such that  $O_a \leq H$ , and  $\overline{G} := G/H$ . For each  $\lambda \in \text{Irr}(Z(G)/H)^\times$ , the following hold.*

- (i)  $Z(\overline{G}) = \{(au, 0) : u \in \mathbb{F}_\theta\}Z(G)/H$  of order  $p^{n+1}$ .
- (ii) *For each extension  $\eta$  of  $\lambda$  to  $Z(\overline{G})$ ,  $\overline{G}/\ker(\eta)$  is extraspecial of order  $p^{1+m-n}$  and  $\eta^{\overline{G}}$  has a unique irreducible constituent of degree  $p^{(m-n)/2}$ .*

Consequently, Theorem 2.3 (i) holds.

*Proof.* Since  $\mathbb{F}_{p^{2n}} \cap \mathbb{F}_{p^m} = \mathbb{F}_\theta$ , by Proposition 2.4 (v) the claim in (i) is clear. We shall prove (ii). Let  $\eta \in \text{Irr}(Z(\overline{G}))$  be an extension of  $\lambda$ . Note that the order of  $\overline{G}/\ker(\eta)$  is  $p^{m+1}/p^n = p^{1+m-n}$ .

For convenience we write elements in  $\overline{G}/\ker(\eta)$  as  $(r, s)N$ . Here  $O_a \leq N \triangleleft G$  with  $|N| = p^{m+n-1}$  such that  $G/N \cong \overline{G}/\ker(\eta)$ . Since  $\lambda \in \text{Irr}(Z(G)/H)^\times$ , we have  $Z(G) \not\leq N$  and  $\ker(\lambda) \leq Z(G) \cap N$  is a maximal subgroup of  $Z(G)$ . For  $(r, -) \in G$  with  $r \in \mathbb{F}_{p^m} - a\mathbb{F}_\theta$ , we have  $[(r, -), G] = O_r \not\leq N$  by Proposition 2.4 (ii) and (iii). Thus,  $Z(G/N) = Z(G)N/N \cong E_p$ , and  $O_r N = O_r O_a N = Z(G)N$ . Therefore,  $[(r, -)N, G/N] = [(r, -), G]N/N = O_r N/N = Z(G/N) = [G/N, G/N]$ , which shows that  $G/N$  is extraspecial of order  $p^{1+m-n}$ . Hence,  $\eta^{\overline{G}}$  has a unique irreducible constituent of degree  $p^{(m-n)/2}$ .

Since there are  $p^n$  extensions  $\eta$  of  $\lambda$  to  $Z(\overline{G})$ ,  $\lambda^G$  has  $p^n$  irreducible constituents. For each maximal subgroup  $H$  of  $Z(G)$ , there are exactly  $p - 1$  nontrivial linear characters of  $Z(G) \cong E_{p^m}$  whose kernels contain  $H$ . By Proposition 2.4 (v) and Lemma 1.2 (ii) the rest of the statement is clear.  $\square$

**Lemma 2.8.** *Let  $G := A_p(m, \theta)$  where  $\theta$  has order 2. For all  $\lambda \in \text{Irr}(Z(G))^\times$  such that  $[G, G] \not\subseteq \ker(\lambda)$ , we have  $G/\ker(\lambda)$  is extraspecial of order  $p^{1+m}$  and  $\lambda^G$  has a unique irreducible constituent of degree  $p^{m/2}$ . Consequently, Theorem 2.3 (ii) holds.*

*Proof.* Let  $H := \ker(\lambda)$  and  $\overline{G} := G/H$ . By Proposition 2.4 (iv), for all  $a \in \mathbb{F}_{p^m}^\times$  we have  $O_a = [(a, -), G] = [G, G]$ . Thus, if  $[G, G] \not\subseteq H$ , then  $Z(\overline{G}) = Z(G)/H \cong E_p$  and  $[(a, -)H, G/H] = [(a, -), G]H/H = O_aH/H = Z(\overline{G})$ . So  $\overline{G}$  is extraspecial of order  $p^{1+m}$ , and  $\lambda^{\overline{G}}$  has only one irreducible constituent of degree  $p^{m/2}$ . Since  $Z(G)$  has  $(p^{m/2} - 1)p^{m/2}$  non trivial linear characters whose kernels do not contain  $[G, G]$ , the rest of the statement is clear by Corollary 2.5 and Lemma 1.2 (ii).  $\square$

**Lemma 2.9.** *Let  $G := A_p(m, \theta)$  where  $\theta$  has even order  $k > 2$ . Let  $n := m/k$ ,  $H$  a maximal subgroup of  $Z(G)$ , and  $\overline{G} := G/H$ . For each  $\lambda \in \text{Irr}(Z(G)/H)^\times$ , the following hold.*

- (i) *If there is  $O_a \leq H$ , then  $Z(\overline{G}) = \{(au, 0) : u \in \mathbb{F}_{p^{2n}}\}Z(G)/H$  is of order  $p^{2n+1}$ . For each extension  $\eta$  of  $\lambda$  to  $Z(\overline{G})$ ,  $\overline{G}/\ker(\eta)$  is extraspecial of order  $p^{1+m-2n}$  and  $\eta^{\overline{G}}$  has a unique irreducible constituent of degree  $p^{(m-2n)/2}$ .*
- (ii) *If there is no  $O_a \leq H$ , then  $\overline{G}$  is extraspecial of order  $p^{1+m}$ , and  $\lambda^{\overline{G}}$  has a unique irreducible constituent of degree  $p^{m/2}$ .*

*Consequently, Theorem 2.3 (iii) holds.*

*Proof.* (i) Suppose that  $O_a \leq H$  for some  $a \in \mathbb{F}_{p^m}^\times$ . By Proposition 2.4 (ii),  $[(b, -), G] = O_b \leq H$  for all  $b \in a\mathbb{F}_{p^{2n}}^\times$ , and by Proposition 2.4 (iii),  $[(r, -), G] = O_r \not\subseteq H$  for all  $r \in \mathbb{F}_{p^m} - a\mathbb{F}_{p^{2n}}$ . Hence,  $Z(\overline{G}) = \{(au, -) : u \in \mathbb{F}_{p^{2n}}\}Z(G)/H$  of order  $p^{2n+1}$ . Let  $\eta \in \text{Irr}(Z(\overline{G}))$  be an extension of  $\lambda$ . Since  $|\ker(\eta)| = p^{2n}$ , the quotient group  $\overline{G}/\ker(\eta)$  is of order  $p^{1+m-2n}$ .

As in Lemma 2.7 we write elements in  $\overline{G}/\ker(\eta)$  as  $(r, s)N$  where  $O_a \leq N \triangleleft \overline{G}$  with  $|N| = p^{m+2n-1}$  such that  $G/N \cong \overline{G}/\ker(\eta)$ . With the same argument we obtain that  $G/N$  is extraspecial. Therefore,  $\eta^{\overline{G}}$  has a unique irreducible constituent of degree  $p^{(m-2n)/2}$ .

Since there are  $p^{2n}$  extensions  $\eta$  of  $\lambda$ ,  $\lambda^{\overline{G}}$  has  $p^{2n}$  irreducible constituents. Since there are exactly  $p-1$  nontrivial linear characters of  $Z(G)$  having the same kernel, by Proposition 2.4 (vi) we obtain  $\frac{p^{2n}(p^m-1)}{p^n+1}$  irreducible characters of degree  $p^{(m-2n)/2}$ .

(ii) Suppose that there is no  $O_a \leq H$ . It is clear that  $Z(G/H) = [G/H, G/H] = Z(G)/H \cong E_p$ . For each  $a \in \mathbb{F}_{p^m}^\times$ , we have  $[(a, -)H, G/H] = [(a, -), G]H/H = O_aH/H = Z(G)/H = Z(G/H)$ . Hence,  $G/H$  is extraspecial of order  $p^{1+m}$ . So  $\lambda^{\overline{G}}$  has a unique irreducible constituent of degree  $p^{m/2}$ . By Proposition 2.4 (vi) we obtain  $\frac{p^n(p^m-1)}{p^n+1}$  irreducible characters of degree  $p^{m/2}$ . Now Theorem 2.3 (iii) holds by Lemma 1.2.  $\square$

3. SUZUKI  $p$ -GROUPS  $C_p(m, \theta, \varepsilon)$ 

*Definition 3.1.* The Suzuki  $p$ -group  $C_p(m, \theta, \varepsilon)$  is the set  $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$  with  $m \geq 1$ ,  $\theta \in \text{Aut}(\mathbb{F}_{p^m})$  and  $\varepsilon \in \mathbb{F}_{p^m}$  with the multiplication defined as follows:

$$(a, b, c)(d, e, f) := (a + d, b + e, c + f + a\theta(d) + \varepsilon a^{\frac{1}{p}}\theta(e^p) + be).$$

Note that the map  $a \mapsto a^{\frac{1}{p}}$  is the inverse of the automorphism  $a \mapsto a^p$ , and it can be written as  $a \mapsto a^{p^{m-1}}$ . Therefore, this map is a field automorphism of  $\mathbb{F}_{p^m}$ .

Let  $G := C_p(m, \theta, \varepsilon)$ . Then  $|G| = p^{3m}$ . One can check directly from the definition that the multiplication is associative, the identity of  $G$  is  $(0, 0, 0)$ , and the inverse of  $(a, b, c)$  is  $(-a, -b, -c + a\theta(a) + \varepsilon a^{\frac{1}{p}}\theta(b^p) + b^2)$ . So  $G$  is a group.

It is easy to check that  $(a, b, c)^i = (ia, ib, ic + \binom{i}{2}(a\theta(a) + \varepsilon a^{\frac{1}{p}}\theta(b^p) + b^2))$ . If  $p = 2$ , then  $\exp(G) = 4$ ; otherwise, if  $p \geq 3$ , then  $\exp(G) = p$ . By Higman [5] with  $p = 2$ , the conditions  $\varepsilon \notin \{a^{-1} + a\theta^2(a) : a \in \mathbb{F}_{p^m}^\times\}$ , and  $2\theta^2 = 1$  are needed to guarantee that all involutions of  $C_2(m, \theta, \varepsilon)$  are in its center and to distinguish type  $C$  from the other types. Here, we treat  $\theta$  and  $\varepsilon$  as parameters on which we place no restrictions. For every choice of parameters we find all irreducible character degrees of the corresponding group.

We have  $[(a, b, c), (d, e, f)] = (0, 0, a\theta(d) - d\theta(a) + \varepsilon(a^{\frac{1}{p}}\theta(e^p) - d^{\frac{1}{p}}\theta(b^p)))$ . If  $\theta = 1$  and  $\varepsilon = 0$  then  $G$  is abelian. Hence, we suppose that either  $\theta \neq 1$  or  $\varepsilon \neq 0$ .

For  $(a, b, c) \in Z(G)$ , its commutator with  $(0, e, 0)$  equal to  $1_G$  implies  $\varepsilon a^{\frac{1}{p}}\theta(e^p) = 0$  for all  $e \in \mathbb{F}_{p^m}$ , so  $\varepsilon = 0$  or  $a = 0$ . If  $\varepsilon = 0$ , then  $Z(G) = \{(0, b, c) : b, c \in \mathbb{F}_{p^m}\}$ . If  $\varepsilon \neq 0$ , using  $(d, 0, 0) \in G$ , we obtain  $d^{\frac{1}{p}}\theta(b^p) = 0$  for all  $d \in \mathbb{F}_{p^m}$ , which implies  $b = 0$ . Thus,  $Z(G) = \{(0, 0, c) : c \in \mathbb{F}_{p^m}\}$ .

**Theorem 3.2.** *Let  $G := C_p(m, \theta, \varepsilon)$  where  $\theta \in \text{Aut}(\mathbb{F}_{p^m})$  of order  $k$ ,  $\varepsilon \in \mathbb{F}_{p^m}$  and  $n := m/k$ . One of the following holds:*

- (i) *If  $\varepsilon \neq 0$ , then  $\text{cd}(G) = \{1, p^m\}$  and  $|\text{Irr}_{(p^m)}(G)| = p^m - 1$ .*
- (ii) *If  $\varepsilon = 0$  and  $k = 2$ , then  $\text{cd}(G) = \{1, p^{m/2}\}$  and  $|\text{Irr}_{(p^{m/2})}(G)| = p^{2m} - p^{3m/2}$ .*
- (iii) *If  $\varepsilon = 0$  and  $k > 2$  is odd, then  $\text{cd}(G) = \{1, p^{(m-n)/2}\}$  and  $|\text{Irr}_{(p^{(m-n)/2})}(G)| = p^{m+n}(p^m - 1)$ .*
- (iv) *If  $\varepsilon = 0$  and  $k > 2$  is even, then  $\text{cd}(G) = \{1, p^{(m-2n)/2}, p^{m/2}\}$ , and  $|\text{Irr}_{(p^{(m-2n)/2})}(G)| = \frac{p^{m+2n}(p^m-1)}{p^n+1}$ ,  $|\text{Irr}_{(p^{m/2})}(G)| = \frac{p^{m+n}(p^m-1)}{p^n+1}$ .*

To prove Theorem 3.2 we do some investigations first. For all  $a, b \in \mathbb{F}_{p^m}$ , we define

$$\varphi_{a,b}(d, e) := a\theta(d) - d\theta(a) + \varepsilon(a^{\frac{1}{p}}\theta(e^p) - d^{\frac{1}{p}}\theta(b^p)),$$

$$\psi_{a,b}^1(d) := \varphi_{a,b}(d, 0) = a\theta(d) - d\theta(a) - \varepsilon d^{\frac{1}{p}}\theta(b^p),$$

$$\psi_{a,b}^2(e) := \varphi_{a,b}(0, e) = \varepsilon a^{\frac{1}{p}}\theta(e^p),$$

$$\text{and } O_{a,b} := \{(0, 0, c) : c \in \text{im}(\varphi_{a,b})\},$$

$$O_{a,b}^1 := \{(0, 0, c) : c \in \text{im}(\psi_{a,b}^1)\}, \quad O_{a,b}^2 := \{(0, 0, c) : c \in \text{im}(\psi_{a,b}^2)\}.$$

Here,  $[(a, b, c), (d, e, f)] = (0, 0, \varphi_{a,b}(d, e))$ , and  $\varphi_{a,b}(d, e) = \psi_{a,b}^1(d) + \psi_{a,b}^2(e)$ . It is easy to see that  $\psi_{a,b}^2$  is independent of  $b$ , so is  $O_{a,b}^2$ .

**Lemma 3.3.** *Let  $G := C_p(m, \theta, \varepsilon)$  where  $\theta \in \text{Aut}(\mathbb{F}_{p^m})$  of order  $k$ ,  $\varepsilon \in \mathbb{F}_{p^m}$ . The following hold.*

- (i)  $\varphi_{a,b}, \psi_{a,b}^1, \psi_{a,b}^2$  are  $\mathbb{F}_p$ -homomorphisms and  $\text{im}(\varphi_{a,b}) = \text{im}(\psi_{a,b}^1) + \text{im}(\psi_{a,b}^2)$ . Moreover,  $O_{a,b}^1, O_{a,b}^2, O_{a,b} \leq \mathbf{Z}(G)$  and  $O_{a,b} = O_{a,b}^1 O_{a,b}^2$ .
- (ii) If  $\varepsilon = 0$ , then with notations in Section 2,  $\varphi_{a,b}(d, e) = \varphi_a(d)$  and  $O_{a,b} = O_a$ . Moreover,  $[G, G] \leq \mathbf{Z}(G)$  has index  $p^m$  if  $k > 2$ , and has index  $p^{m+n}$  if  $k = 2$ .
- (iii) If  $\varepsilon \neq 0$ , then  $O_{a,b} = \mathbf{Z}(G)$  for all  $(a, b) \neq (0, 0)$ . Moreover, for all  $(a, b) \neq (0, 0)$ ,  $[G, G] = \Phi(G) = \mathbf{Z}(G) = [(a, b, -), G]$ .

*Proof.* All statements in part (i) follow directly from the definitions. We shall prove parts (ii) and (iii).

(ii) With  $\varepsilon = 0$ ,  $\varphi_{a,b}(d, e) = a\theta(d) - d\theta(a) = \varphi_a(d)$ . Hence,  $O_{a,b} = O_a$ . Since  $\mathbf{Z}(G) = \{(0, b, c) : b, c \in \mathbb{F}_{p^m}\}$ , the rest follows by Proposition 2.4.

(iii) Since  $\{u^p : u \in \mathbb{F}_{p^m}\} = \mathbb{F}_{p^m}$  and  $\theta \in \text{Aut}(\mathbb{F}_{p^m})$ ,  $\psi_{a,b}^2(\mathbb{F}_{p^m}) = \varepsilon a^{\frac{1}{p}} \theta(\mathbb{F}_{p^m}) = \mathbb{F}_{p^m}$  for all  $a \in \mathbb{F}_{p^m}^\times$ . If  $a = 0$  and  $b \neq 0$ , then  $\psi_{a,b}^1(\mathbb{F}_{p^m}) = -\varepsilon \mathbb{F}_{p^m} \theta(b^p) = \mathbb{F}_{p^m}$  by  $\{u^{\frac{1}{p}} : u \in \mathbb{F}_{p^m}\} = \mathbb{F}_{p^m}$ . Hence, by (i),  $\text{im}(\varphi_{a,b}) = \mathbb{F}_{p^m}$  for all  $(a, b) \neq (0, 0)$ . Since  $\mathbf{Z}(G) = \{(0, 0, c) : c \in \mathbb{F}_{p^m}\}$ , all are clear.  $\square$

Now we shall prove Theorem 3.2.

*Proof of Theorem 3.2.* (i) Suppose that  $\varepsilon \neq 0$ . For  $\lambda \in \text{Irr}(\mathbf{Z}(G))^\times$ , let  $H := \ker(\lambda)$ . We shall show that  $G/H$  is extraspecial of order  $p^{1+2m}$ . By Lemma 3.3 (iii), for all  $H \neq (a, b, c)H \in G/H$ , we have  $[(a, b, c)H, G/H] = [(a, b, c), G]H/H = \mathbf{Z}(G)/H$ . Hence,  $\mathbf{Z}(G/H) = [G/H, G/H] \cong E_p$  and  $G/H$  is extraspecial. Thus,  $\lambda^G$  has a unique irreducible constituent of degree  $p^m$ . Since  $|\text{Irr}(\mathbf{Z}(G))^\times| = p^m - 1$  and  $G/\mathbf{Z}(G) \cong E_{p^{2m}}$ , the claim holds.

(ii) Suppose that  $\varepsilon = 0$  and  $k = 2$ . By Lemma 3.3 (ii), there are  $p^{2m} - p^{3m/2}$  linear characters  $\lambda$  of  $\mathbf{Z}(G)$  such that  $[G, G] \not\subseteq \ker(\lambda)$ . Set  $H := \ker(\lambda)$ . Since  $[(a, -, -), G] = O_a = [G, G]$  for all  $a \in \mathbb{F}_{p^m}^\times$ , we obtain  $\mathbf{Z}(G/H) = [G/H, G/H] \cong E_p$ . Thus,  $G/H$  is extraspecial of order  $p^{1+m}$  and  $\lambda^G$  has a unique irreducible constituent of degree  $p^{m/2}$ . So  $\text{cd}(G) = \{1, p^{m/2}\}$  and  $|\text{Irr}_{(p^{m/2})}(G)| = p^{2m} - p^{3m/2}$ .

(iii) Suppose that  $\varepsilon = 0$  and  $k > 2$  is odd. By Lemma 3.3 (ii), there are  $p^{2m} - p^m$  linear characters  $\lambda$  of  $\mathbf{Z}(G)$  such that  $[G, G] \not\subseteq \ker(\lambda)$ . Set  $H := \ker(\lambda)$ . It is easy to see that  $G/H$  is isomorphic to the quotient group  $A_p(m, \theta)/H$  in Lemma 2.7. Thus,  $\lambda$  has  $p^n$  distinct extensions to  $\mathbf{Z}(G/H)$ . Let  $\eta$  be an extension of  $\lambda$ . Then  $(G/H)/\ker(\eta)$  is extraspecial of order  $p^{1+m-n}$  and  $\eta^{G/H}$  has a unique irreducible constituent of degree  $p^{(m-n)/2}$ . Thus,  $\lambda^G$  has  $p^n$  distinct irreducible constituents of degree  $p^{(m-n)/2}$ . So  $\text{cd}(G) = \{1, p^{(m-n)/2}\}$  and  $|\text{Irr}_{(p^{(m-n)/2})}(G)| = p^{m+n}(p^m - 1)$ .

(iv) Suppose that  $\varepsilon = 0$  and  $k > 2$  is even. By Lemma 3.3 (ii), there are  $p^{2m} - p^m$  linear characters  $\lambda$  of  $\mathbf{Z}(G)$  such that  $[G, G] \not\subseteq \ker(\lambda)$ . Set  $H := \ker(\lambda)$ . Here, similar to Lemma 2.9, the proof divides into two cases: where there is some  $O_{a,b} \leq H$  and where there is no  $O_{a,b} \leq H$  for all  $(a, b) \neq (0, 0)$ . With the same argument used in Lemma 2.9, we obtain  $\text{cd}(G) = \{1, p^{(m-2n)/2}, p^{m/2}\}$ , and  $|\text{Irr}_{(p^{(m-2n)/2})}(G)| = \frac{p^{m+2n}(p^m-1)}{p^n+1}$ ,  $|\text{Irr}_{(p^{m/2})}(G)| = \frac{p^{m+n}(p^m-1)}{p^n+1}$ .  $\square$

*Remark 3.4.* Our technique can be applied to the groups of type  $B_p(m, \theta, \varepsilon)$  and  $D_p(m, \theta, \varepsilon)$  and will yield partial results. In these cases however, control over the  $\mathbb{F}_\theta$ -hyperplanes in  $\mathbb{F}_{p^m}$  depends heavily on both parameters  $\theta$  and  $\varepsilon$ . This will require further investigation and will be the subject of a subsequent paper.

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