Groups of even type which are not of even characteristic, II

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DOI:
10.1007/s11856-016-1314-9

GROUPS OF EVEN TYPE WHICH ARE NOT OF EVEN CHARACTERISTIC,II

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1. Introduction

In this second part of the paper we continue the investigation started in the first part and finish the classification of the groups of even type, which are not of even characteristic. More precisely we prove:

Theorem 1.1. *Let G be a simple $\mathcal{K}_2$-group of even type. Then either G is of even characteristic or $G \cong J_1$, Co_3, $M(23)$, A_{12}, $\Omega_7(3)$ or $\Omega_8^-(3)$.*

Let us recall the notation used in the statement of the theorem.

Definition 1.2. A group G is said to be of even type if the following hold:

(i) $\mathcal{L} \subseteq \mathcal{C}_2$, where $\mathcal{L}$ is the set of all components of $C_G(x)$ for all involutions $x \in G$.
(ii) $O(C_G(x)) = 1$ for every involution $x \in G$.
(iii) G has 2-rank at least 3.

Here we denote by $\mathcal{C}_2$ the following set of components of G:

Definition 1.3. [GoLyS1, Definition (12.1)(1)] The set $\mathcal{C}_2$ consists of simple and quasisimple groups.

- The simple groups in $\mathcal{C}_2$ are $K \in \mathrm{Chev}(2)$, $L_2(9)$, $L_2(p)$, p a Fermat or Mersenne prime, $L_3(3)$, $L_4(3)$, $U_4(3)$, $G_2(3)$, M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_2, J_3, J_4, HiS, Suz, Ru, Co_1, Co_2, $M(22)$, $M(23)$, $M(24)'$, Th, F_2, F_1.
- The groups $K \in \mathcal{C}_2$ with K not simple are those for which $K/O_2(K)$ is a simple group in $\mathcal{C}_2$. But the following quasisimple groups are deleted, i.e. are not in $\mathcal{C}_2$: $SL_2(q)$, q odd, $2A_8$, $SL_4(3)$, $SU_4(3)$, $Sp_4(3)$, and $[X]L_3(4)$, with $X \cong \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Date: February 9, 2015.

Furthermore we call a group G a $\mathcal{K}_2$–group if any simple factor of any nontrivial 2–local subgroup of G is either cyclic, a group of Lie type, an alternating group or one of the 26 sporadic groups.

We now define even characteristic.

Definition 1.4. A group G is said to be of even characteristic, if for a Sylow 2-subgroup S and all nontrivial 2-local subgroups H of G with $S \leq H$, we have that $C_G(O_2(H)) \subseteq O_2(H)$.

The main result of the first part of this paper was:

Theorem 1.5. *Let G be a simple $\mathcal{K}_2$-group of even type. Then one of the following holds*

- *G is of even characteristic; or*
- *$G \cong \Omega_7(3)$, $\Omega_8^-(3)$ or A_{12}; or*
- *There is a 2-central involution z such that $C_G(z)$ possesses a standard subgroup L. Furthermore $C_G(L)$ is cyclic.*

In this second part of the paper we start with the statement of Theorem 1.5. We assume that there is some 2-central involution $z \in G$ such that $C_G(z)$ possesses a standard subgroup A_z. Furthermore we assume that G is not isomorphic to J_1, Co_3 or $M(23)$. We then first show that $Z(A_z) = 1$ and then that A_z is a group of Lie type in characteristic two or is one out of a small list of sporadic groups (Proposition 5.1 and Proposition 5.2). For this we use some classifications of groups by standard subgroups. At this point our analysis moves away from $C_G(z)$ and we construct in Lemma 6.4 and Lemma 6.5 a subgroup N of G such that N and $N_G(A_z)$ share a Sylow 2–subgroup S of G, $C_N(O_2(N)) \leq O_2(N)$ and $N \not\leq N_G(A_z)$. By choosing N minimal with these properties we achieve that N is a minimal parabolic subgroup in the sense that we now describe.

We call a subgroup P of a group X a parabolic (subgroup) of X if $1 \neq |X : P|$ is odd. A maximal parabolic is a parabolic which is maximal in the set of parabolics. In contrast a minimal parabolic P is a parabolic which is not 2-closed such that there is exactly one class of maximal subgroups M of P such that $|P : M|$ is odd.

Now using the action of $O_2(C_{A_z}(x))$ on $\Omega_1(Z(O_2(N)))$ for some 2-central involution $x \neq z$ in A_z, we get results about the action of the group $N/C_N(\Omega_1(Z(O_2(N))))$ on $\Omega_1(Z(O_2(N)))$. Using this action we eventually are able to prove that there is some involution t which is central in N (Lemma 6.11, Lemma 6.12). This is the key for the final contradiction. We are able to prove some similarity between z and t. In particular in Lemma 6.16 we show that $C_G(t)$ also has a standard subgroup A_t isomorphic to A_z, but $t \not\sim z$. Then we show that the group N constructed above corresponds to a minimal parabolic in A_z and A_t as well. This at the end shows that $A_z = A_t$ is centralized by a unique involution, which would give $z = t$, the final contradiction. Hence for all 2-central involutions z we have that $F^*(C_G(z)) = O_2(C_G(z))$. The theorem then follows from the following fact ([MaStr, Lemma 2.1]): Let G be a group and S be a Sylow 2-subgroup. Then G is of even characteristic if and only if $C_G(O_2(C_G(x))) \leq O_2(C_G(x))$ for all involutions $x \in Z(S)$.

2. Preliminaries

In this chapter we collect some results in group theory of general nature and some properties of the groups involved in the proof of the main theorem. For convenience of the reader we will also state some of the preliminary lemmas from the first part, which are used quite frequently in this second part.

Lemma 2.1. [Glau] *Let G be a nonabelian simple group, z an involution and $z \in S \in \mathrm{Syl}_2(G)$. Then $z^G \cap S \neq \{z\}$.*

Lemma 2.2. (Thompson transfer)[GoLyS2, Lemma 15.16]. *Let G be a group, $S \in \mathrm{Syl}_2(G)$, $T \trianglelefteq S$ with $S = TA$, $A \cap T = 1$, A cyclic. If G has no subgroup of index two and u is the involution in A, then there is some $g \in G$ with $u^g \in T$ and $C_S(u^g) \in \mathrm{Syl}_2(C_G(u^g))$. In particular $|C_S(u)| \leq |C_S(u^g)|$.*

Lemma 2.3. [GoLyS2, Lemma 24.1] *Let R be a p–group , p odd, and E be an elementary abelian 2-group, acting faithfully on R. Then there is a subgroup U in RE, such that U is a direct product of dihedral groups of order $2p$ and E is a Sylow 2-subgroup of U.*

Lemma 2.4. *Let Q be an extraspecial subgroup of a group G, which is normalized by some element $t \in G$. If $|Q : C_Q(t)| = 2$, then either $t \in QC_G(Q)$ or $[t, Q]$ is cyclic.*

Proof. Assume $t \notin QC_G(Q)$. Let $[t, Q] = \langle s, Z(Q)\rangle$ be elementary abelian. In particular $Q \ncong Q_8$ and so Q is generated by involutions. Let s_1 be some involution in $Q \setminus C_Q(s)$. Assume $t^{s_1} = ts$. Then $t = t^{s_1^2} = tss^{s_1}$ and so $[s, s_1] = 1$, a contradiction. So t centralizes modulo $Z(Q)$ any involution in $Q \setminus C_Q(s)$. As Q is generated by such involutions, together with s, we get that $[Q, t] \leq Z(Q)$. Then t induces an inner automorphism and so $t \in QC_G(Q)$, a contradiction. $\square$

Lemma 2.5. *Let $G \cong L_2(p)$, $p = 2^n \pm 1 > 5$ a prime, A_6, $L_3(3)$ or M_{11}. Then a Sylow 2-subgroup of G is dihedral of order at least 8 or semidihedral of order 16.*

Proof. This is [GoLyS3, Lemma 4.10.5] and [GoLyS3, Table 5.3a] for M_{11}. □

Lemma 2.6. [MaStr, Lemma 2.19] *Let $L = L_4(3)$, $U_4(3)$ or $2U_4(3)$. Then the following holds:*

(i) *If $z \in L \setminus Z(L)$ is a 2-central involution, then $O_2(C_L(z)) \cong Q_8 * Q_8$ or $O_2(C_L(z)) \cong \mathbb{Z}_2 \times Q_8 * Q_8$ in case of $L \cong 2U_4(3)$. In all cases $O_3(C_L(z)/O_2(C_L(z)))$ is elementary abelian of order 9 and $C_L(z)/O_2(C_L(z))$ acts faithfully on $O_2(C_L(z))$.*

(ii) *$\mathrm{Out}(U_4(3)) \cong D_8$ and $\mathrm{Out}(L_4(3))$ is elementary abelian of order 4.*

(iii) *If $G \cong \mathrm{Aut}(L)$, $L \cong U_4(3)$ and x is an involution in G such that $2^6 \cdot 3^2$ divides $|C_L(x)|$ then one of the following holds:*
 (α) *x is contained in L and 2-central,*
 (β) *$C_L(x) \cong PSp_4(3)$, or*
 (γ) *$O_2(C_L(x))$ is elementary abelian and $|C_L(x)/O_2(C_L(x))| = 36$ and $C_L(x)/O_2(C_L(x))$ acts faithfully on $O_2(C_L(x))$.*

(iv) *Let $L \cong L_4(3)$ or $U_4(3)$. Then $|Z(T)| = 2$ for T a Sylow 2-subgroup of L. Let G be a subgroup of $\mathrm{Aut}(L)$ containing L and T_1 be a Sylow 2-subgroup of G. If $|\Omega_1(Z(T_1))| > 2$, then $L \cong L_4(3)$ and $|G : L| = 2$. Furthermore some element $t \in \Omega_1(Z(T_1)) \setminus L$ centralizes $PSp_4(3) : 2$ in L.*

Lemma 2.7. *Let $G \cong G_2(2)'$, $G_2(3)$ or M_{22}. Then G has exactly one conjugacy class or involutions with representative t and we have:*

(i) *$O^2(C_G(t)) \cong SL_2(3)$ for $G \cong G_2(2)'$;*
(ii) *$O^2(C_G(t)) \cong SL_2(3) * SL_2(3)$ for $G \cong G_2(3)$ and*
(iii) *$O^2(C_G(t)) \cong 2^{1+4}\mathbb{Z}_3$ for $G \cong M_{22}$.*
(iv) *If i is an outer automorphism of G, then $C_G(i) \cong SL_2(3)$ in case of $G \cong G_2(2)'$ and $L_2(8) : 3$ in case of $G_2(3)$.*
(v) *If i is an outer automorphism of $G = M_{22}$, then $C_G(i) \cong 2^3L_3(2)$ or 2^4F_{20}.*

Proof. As $G_2(2)' \cong U_3(3)$, we get (i), (ii) and (iv) from [GoLyS3, Table 4.5.1]. The assertions (iii) and (v) follow from [GoLyS3, Table 5.3c]. □

Lemma 2.8. *Let $G = M_{12}$. Then the following holds:*

(i) *G possesses two conjugacy classes of involutions with representatives t and u.*
(ii) *$O^2(C_G(t)) \cong 2^{1+4}\mathbb{Z}_3$, $C_G(u) \cong \mathbb{Z}_2 \times \Sigma_5$.*

(iii) $E(C_G(u))$ *contains conjugates of* t.
(iv) *If* i *is an outer automorphism of* G*, then* $C_G(i) \cong \mathbb{Z}_2 \times A_5$.

Proof. (i), (ii), (iv) follow from [GoLyS3, Table 5.3b]. To prove (iii) let T be a Sylow 2-subgroup of $C_G(u)$ and $T_1 \leq G$ with $|T_1 : T| = 2$. As $T' \leq E(C_G(u))$, we get that $Z(T_1) \cap E(C_G(u)) \neq 1$ and so $E(C_G(u))$ contains a 2-central involution. □

Lemma 2.9. *Let* $G = {}^2F_4(2)$ *and* i *be an involution of* G *which is not* 2*-central. Then* $C_G(i)$ *is of order* $2^{10} \cdot 3$*. If* T *is a Sylow* 2*-subgroup of* $C_G(i)$*, then* $|\Omega_1(Z(T))| = 4$.

Proof. By [Shi, Corollary 2] we just have two classes of involutions in G and so i is uniquely determined. By [Shi, Theorem 2.1] we see that $|C_{F_4(2)}(i)| = 2^{20} \cdot 3^2$ and so $|C_G(i)| = 2^{10} \cdot 3$. By the Borel-Tits-Theorem [MaStr, Lemma 2.15] we have that $C_G(i)$ is contained in the parabolic P_1 of G, with $P_1/O_2(P_1) \cong \Sigma_3$. Application of [MaStr, Lemma 2.31] shows that $i \in Z_3(S)$, where S is a Sylow 2-subgroup of P_1 and so $|C_{O_2(P_1)}(i)| = 2^9$. Furthermore $Z_3(S) = Z(O_2(C_G(i)))$. As by [MaStr, Lemma 2.31] $C_G(i)$ induces Σ_3 on $Z_3(S)$, we see that $|\Omega_1(Z(T))| = 4$. □

Lemma 2.10. *Let* $K \in \mathcal{C}_2$ *be a sporadic simple group and* N *be a subgroup of* K*,* $N \cong L_2(p)$*,* p *a Fermat or Mersenne prime,* $p > 5$*,* $L_2(9)$*,* $L_3(3)$ *or* $L_4(3)$*. Suppose that for a Sylow* 2*-subgroup* S *of* K *we have* $S \leq M < K$ *such that* $F^*(M) = N$*, then* $N \cong L_2(9)$ *and* $K \cong M_{11}$.

Proof. If $N \cong L_2(p)$, then as M is an automorphism group of N, we have that S is dihedral. But there is no such sporadic group. Let $N \cong L_4(3)$. Then Lemma 2.6 implies $2^6 \leq |S| \leq 2^8$. Furthermore 3^6 divides the order of K. Inspection of the list in [GoLyS3, Table 5.3] gives a contradiction. So we have $N \cong L_2(9)$ or $L_3(3)$ and then $|S| \leq 2^5$. As $K \in \mathcal{C}_2$, we see $K \cong M_{11}$. As 13 does not divide $|M_{11}|$, we get $N \cong L_2(9)$. □

Lemma 2.11. *Let* $F^*(G) \cong M(22)$ *and* $t \in F^*(G)$ *be a* 2*–central involution. Set* $Q_t = O_2(C_{F^*(G)}(t))$*. Then* $C_G(Q_t) = Z(Q_t)$*. Furthermore* $O_2(C_G(Z(Q_t))) = Q_t$.

Proof. This follows from [GoLyS3, Table 5.3t]. □

Lemma 2.12. *Let* $G \cong M_{11}$*,* M_{23}*,* J_3*,* Th*,* Ru*,* M_{24}*,* J_4*,* Co_1*,* Co_2*,* F_2 *or* F_1*, then* $G = \mathrm{Aut}(G)$.

Proof. This can be found in [GoLyS3, Table 5.3]. □

Lemma 2.13. *If $G \cong L_2(p)$, p a Fermat or Mersenne prime, $p \neq 5$, $G \cong A_6$, $L_3(3)$ or $L_4(3)$ and T is a Sylow 2-subgroup of G, then $|\Omega_1(Z(T))| = 2$.*

Proof. For $G \cong L_4(3)$ this follows by Lemma 2.6. For the remaining groups it follows by Lemma 2.5 □

Lemma 2.14. *Suppose that either $G \cong J_2$ or $G \cong M(24)'$. Let S be a Sylow 2-subgroup of G. Then $N_G(Z_2(S))$ induces Σ_3 on $Z_2(S)$.*

Proof. For $G \cong J_2$ the statement can be found in [GoLyS3, Table 5.3g]. So we assume $G \cong M(24)'$. Then by [Asch, chapter 19] there is a 2-local subgroup $P \cong 2^{11}M_{24}$ of G, where $O_2(P)$ is the irreducible part of the Todd-module. We may assume that $S \leq P$. Let r be a 2-central involution in S, then by [GoLyS3, Table 5.3v] $C_G(r) \cong 2^{1+12}3U_4(3) : 2$. In particular by [Asch, (19.10)] we have that $C_P(r) \cong 2^{11}2^6 3\Sigma_6$. According to [GoLyS3, Table 5.3e] there is some parabolic P_1 of P containing S with $P_1 \cong 2^{11}2^{1+6}L_3(2)$. Hence there is some minimal parabolic $P_2 \leq P_1$ such that $P_2/O_2(P_2) \cong \Sigma_3$ and $P_2 \not\leq C_G(r)$. Now $|\Omega_1(Z(O_2(P_2)))| = 4$, as $|\Omega_1(Z(S))| = 2$ by [MaStr, Lemma 2.33]. Hence $\Omega_1(Z(O_2(P_2))) = Z_2(S)$ by [MaStr, Lemma 2.35], the assertion follows. □

Let us repeat the definition of a group of Lie type.

Definition 2.15. A genuine group of Lie type in characteristic p is a group isomorphic to $O^{p'}(C_{\bar{K}}(\sigma))$, where $\bar{K}$ is a semisimple $\overline{\mathrm{GF}(p)}$-algebraic group, $\overline{\mathrm{GF}(p)}$ is the algebraic closure of $\mathrm{GF}(p)$, and σ is the Steinberg endomorphism of $\bar{K}$, see [GoLyS3, Definition 2.2.2] for details. A simple group of Lie type in characteristic p is a non-abelian composition factor of a genuine group of Lie type in characteristic p.

Hypothesis 2.16. [MaStr, Hypothesis 2.27] Let $G = G(q)$, $q = 2^n$, be a simple group of Lie type, $G \not\cong Sz(q)$, $L_2(q)$ or ${}^2F_4(q)'$. Let R be a long root subgroup of G if $G \not\cong Sp_{2n}(q)$, and a short root subgroup if $G \cong Sp_{2n}(q)$. Set $X_R = C_G(R)$ and $Q_R = O_2(X_R)$.

Lemma 2.17. [MaStr, Lemma 2.28] *Assume Hypothesis 2.16 with $G \not\cong L_3(q)$, $U_3(q)$, $Sp_4(2)'$ or $G_2(2)'$. Let L be a Levi complement in $N_G(R)$. Then Q_R/R has the following L–module structure:*

- (i) *$G \cong L_n(q)$, $O^{2'}(L) \cong SL_{n-2}(q)$, $Q_R/R = V_1 \oplus V_2$, V_1 is the natural L–module and V_2 its dual.*
- (ii) *$G \cong \Omega^{\pm}_{2n}(q)$, $O^{2'}(L) \cong \Omega^{\pm}_{2n-4}(q) \times L_2(q) = L_1 \times L_2$, $Q_R/R = V_1 \oplus V_2$, V_i, $i = 1, 2$, are natural L_1–modules and $[Q_R, L_2] = Q_R$.*
- (iii) *$G \cong U_n(q)$, $O^{2'}(L) \cong SU_{n-2}(q)$, Q_R/R is the natural module.*

(iv) $G \cong E_6(q)$, $O^{2'}(L) \cong L_6(q)$, Q_R/R *is an irreducible module with* $|Q_R/R| = q^{20}$.

(v) $G \cong {}^2E_6(q)$, $O^{2'}(L) \cong U_6(q)$, Q_R/R *is an irreducible module with* $|Q_R/R| = q^{20}$.

(vi) $G \cong E_7(q)$, $O^{2'}(L) \cong \Omega^+_{12}(q)$, Q_R/R *is an irreducible module with* $|Q_R/R| = q^{32}$.

(vii) $G \cong E_8(q)$, $O^{2'}(L) \cong E_7(q)$, Q_R/R *is an irreducible module with* $|Q_R/R| = q^{56}$.

(viii) $G \cong F_4(q)$, $O^{2'}(L) \cong Sp_6(q)$, Q_R/R *is an extension of the natural module by a spin module, where the natural module is contained in* $Z(Q_R)$, *where the natural module is contained in* $Z(Q_R)$. *Finally* $Z(Q_R)$ *does not split over* R.

(ix) $G \cong {}^3D_4(q)$, $O^{2'}(L) \cong L_2(q^3)$, Q_R/R *is the* 8*–dimensional* GF(q)*–module for* L.

Lemma 2.18. [MaStr, Lemma 2.29] *Let* $K \cong Sp_{2n}(q)$, $n \geq 3$, $q = 2^m$. *We have two root groups* R_1 *and* R_2, *with*

(1) *The Levi factor of* $N_K(R_1)$ *is* $Sp_{2n-2}(q)$, $O_2(N_K(R_1))$ *is elementary abelian and* $O_2(N_K(R_1))/R_1$ *is the natural module.*

(2) *The Levi factor* L *of* $N_K(R_2)$ *is* $Sp_{2n-4}(q) \times L_2(q)$, *furthermore* $Z(O_2(N_K(R_2)))/R_2$ *is the natural* $L_2(q)$*-module, and for* $n > 2$, $O_2(N_K(R_2))' = R_2$, *and* $O_2(N_K(R_2))/Z(O_2(N_K(R_2)))$ *is a tensor product of the two natural modules for the two factors of* L. *If* $q > 2$, *then* $Z(O_2(N_K(R_2)))$ *does not split over* R_2 *as an* $N_K(R_2)$*-module.*

Lemma 2.19. [DeSte, 10.10 and page 238] *Assume Hypothesis* 2.16 *with* $K \cong G_2(2^e)$, $e \neq 1$. *Let* P *be the normalizer of the root group* R. *Then* $O'(P) \cong (2^e)^{1+4} : SL_2(2^e)$. *If* $e \neq 2$, *then* $O'(P)/Q_R$ *acts irreducibly on* Q_R/R. *If* $e = 2$, *then* P *acts irreducibly on* Q_R/R *but* $O'(P)/Q_R$ *induces a direct sum of two permutation modules for* A_5 *on* Q_R/R.
Let S *be a Sylow* 2 *subgroup of* P, *then* $Z_2(S) \leq Q_R$ *and* K *induces the natural* $L_2(q)$*-module on* $Z_2(S)$.

Lemma 2.20. [MaStr, Lemma 2.40] *Let* $G = L_3(q)$, $q = 2^n$, *and* T *be a Sylow* 2*-subgroup of* G. *Then* G *possesses two parabolics* P_1, P_2 *which contain* T, *such that* $U_i = O_2(P_i)$ *is elementary abelian of order* q^2 *and* $O^{2'}(P_i/U_i) \cong L_2(q)$, *for* $i = 1, 2$. *Furthermore* P_i *induces the natural module on* U_i, $i = 1, 2$, $T = U_1U_2$ *and any involution of* T *is contained in* $U_1 \cup U_2$. *Finally there is an automorphism* α *of* G, *which normalizes* T *with* $P_1^\alpha = P_2$.

Lemma 2.21. [MaStr, Lemma 2.48] *Let $G = Sp_4(q)$, $q = 2^n > 2$, and T be a Sylow 2-subgroup of G. Then G possesses two parbolics P_1, P_2 which contain T, such that $U_i = O_2(P_i)$ is elementary abelian of order q^3 and $P_i/U_i \cong GL_2(q)$, for $i = 1, 2$. We have that U_i is an indecomposable module for P_i an $Z(O^{2'}(P_i)) = R_i$ is a root group. Furthermore $Z(T) = R_1R_2 = T'$, $T = U_1U_2$ and any involution in T is contained in $U_1 \cup U_2$. There is an automorphism α of G with $R_1^\alpha = R_2$ and $P_1^\alpha = P_2$.*

Lemma 2.22. [GoLyS3, Theorem 2.5.1.] *Let K be a group of Lie type over $\mathrm{GF}(p^e)$ and $x \in \mathrm{Out}(K)$. Then $x = dfg$ with:*

(a) *d is a diagonal automorphism. In particular $p \nmid o(d)$.*
(b) *f is a field automorphism. In particular if S is a Sylow p-subgroup of K normalized by f, then $X(t)^f = X(t^\sigma)$, where σ is a field automorphism of $\mathrm{GF}(p^e)$ and $X(t)$ is a root group in S. This implies that f also induces a field automorphism on any parabolic containing S and any Levi complement. Recall that twisted groups are not defined over $\mathrm{GF}(p^e)$ but over $\mathrm{GF}(p^{2e})$ or $\mathrm{GF}(p^{3e})$ and σ is an automorphism of this larger field, in particular f might be trivial on Levi factors, which are defined over $\mathrm{GF}(p^e)$.*
(c) *g is a graph automorphism, which comes from a symmetry of the corresponding Dynkin diagram. We have $o(g) = 2$ or 3. The case $o(g) = 3$ just occurs for $K \cong \Omega_8^+(p^e)$. Further $g = 1$, if K is twisted.*

Lemma 2.23. [MaStr, Lemma 2.25] *Let G be a group and $L = F^*(G)$ be a group of Lie type in characteristic two.*

(1) *If there is an outer automorphism of order 2 of L, which centralizes a Sylow 2–subgroup of L, then $L \cong Sp_4(2)'$.*
(2) *Assume that L is a central extension of $Sp_{2n}(q)$, $F_4(q)$, ${}^2F_4(q)'$ or $Sz(q)$, $q = 2^n$, and t is an involution in $G \setminus Z(L)$.*
    (i) *If $C_L(t)/O(C_L(t))$ has a component K, then K is a central extension of $Sp_{2n}(s)$, $F_4(s)$, ${}^2F_4(s)'$, $s = 2^b$, or in case of $Sp_4(q)$ also $Sz(q)$ is possible. Further $F^*(L) \not\cong Sz(q)$ or ${}^2F_4(2)$.*
    (ii) *A Sylow 2–subgroup T of $C_{F^*(G)}(t)$ is not abelian.*
(3) *Let $L \cong PSL_3(4)$ and $t \in G$ be an involution, which induces an outer automorphism on L. Then $C_L(t) \cong 3^2 : Q_8$, $PSL_2(7)$ or A_5.*

Lemma 2.24. *Let G be an automorphism group of a group $H = G(q)$ of Lie type in characteristic two, $G \not\cong L_2(q)$, $Sp_{2n}(q)$, $F_4(q)$, ${}^2F_4(q)'$ or*

$G_2(2)'$. *Let* S *be a Sylow* 2*-subgroup of* G. *If* $O_2(C_G(\Omega_1(Z(S)))) \not\leq H$, *then* $H \cong L_3(q)$ *or* $L_4(q)$.

Proof. We assume $H \not\cong L_3(q)$. Set $R = \Omega_1(Z(S \cap H))$. Then we have that $|R| = q$. By Lemma 2.23 we have $\Omega_1(Z(S)) \leq R$. Let now $t \in O_2(C_G(\Omega_1(Z(S))))$. Then we have that $[C_H(R), t] \leq O_2(C_H(R)) = Q_R$. If $H \cong U_3(q)$, then there is some element ω of order $q+1$ in H, which centralizes R and so also $\Omega_1(Z(S))$. As by Lemma 2.22 a Sylow 2-subgroup of the outer automorphism group of H is cyclic and induces just field automorphism, we see that no such automorphism would centralize ω and so $S \cap H = O_2(C_G(\Omega_1(Z(S))))$. So we may assume that $H \not\cong U_3(q)$. Suppose that also $H \not\cong L_n(q)$. Then $N_H(R)$ is a maximal parabolic in H, whose structure is given by Lemma 2.17 or Lemma 2.19 in case of $H \cong G_2(q)$. Again by Lemma 2.22 we see that field automorphisms induce nontrivial automorphisms on the Levi factor of $N_H(R)$. As no graph automorphism can centralize the Levi factor, we have the assertion.

So we are left with $H \cong L_n(q)$. We now must have a graph automorphism, which centralizes the Levi factor, i.e. the Levi factor admits no nontrivial graph automorphism, which gives that it has to be $L_2(q)$ and so $H = L_4(q)$, the assertion. $\square$

Lemma 2.25. [MaStr, Lemma 2.45] *Assume Hypothesis* 2.16 *with* $G \not\cong G_2(2)'$. *Let* t *be a* 2*-element which induces an automorphism of* G *such that* $[t, Q_R] \leq Z(Q_R)$, *then* t *is induced by some element from* Q_R, *or* $G \cong Sp_4(q)'$.

Lemma 2.26. *Suppose Hypothesis* 2.16 *with* $G \cong Sp_4(q)$ *or* $F_4(q)$, $q = 2^n$. *Let* S *be a Sylow* 2*–subgroup of* G *with* $R \leq Z(S)$. *If* t *is an automorphism of* G *which normalizes* S *with* $R^t \neq R$ *then* $[Q_R, t]$ *is not elementary abelian.*

Proof. If $G \cong Sp_4(q)$, then by Lemma 2.21 Q_R and Q_{R^t} are the only maximal elementary abelian subgroups of S, so we are done.

Assume $G \cong F_4(q)$. Then t normalizes $N_G(RR^t)$. We have that $Q_RQ_{R^t} = O_2(N_G(RR^t))$. Further $Q_R \cap Q_{R^t}$ is elementary abelian and $Q_RQ_{R^t}/Q_R \cap Q_{R^t}$ is a direct sum of two $Sp_4(q)$–modules which are both extensions of the trivial module by a natural module. Take the preimage U of the two trivial modules. Then we have that $U = (Q_R \cap Q_{R^t})Z(Q_R)Z(Q_{R^t})$ and $Z(U) = Q_R \cap Q_{R^t}$. Further $Z(Q_{R^t})$ induces a group of GF(q)–transvections on $Z(U)Z(Q_R)$. This shows that $C_{Z(U)Z(Q_R)}(t) = Z(U)$ for all $t \in Z(Q_{R^t}) \setminus Z(U)$. In particular all involution are either in

$Z(U)Z(Q_R)$ or $Z(U)Z(Q_{R^t})$. But then $(Q_R \cap Q_{R^t})Z(Q_R)$ and $(Q_R \cap Q_{R^t})Z(Q_{R^t})$ are the only maximal elementary abelian subgroups in U, which again gives that $[Q_R, t]$ is not elementary abelian. $\square$

Lemma 2.27. *Assume Hypothesis* 2.16. *Assume further that* $G \not\cong G_2(2)'$, $L_3(2)$, $L_3(4)$, $L_3(16)$ *or* $L_4(2)$. *If* $t \in \mathrm{Aut}(G)$ *is an involution with* $[t, X_R] \leq Q_R$, *then* $t \in G$.

Proof. Suppose that t induces an outer automorphism on G. Suppose further that X_R/Q_R has a normal subgroup L_R, which is a group of Lie type in characteristic 2. Then t cannot induce a field automorphism or a graph/field automorphism, as this has to be nontrivial on L_R. If t induces a graph automorphism, L_R must be of Lie rank at most 1. So we have that $G \cong L_4(q)$, $L_3(q)$ or $U_3(q)$. In case of $L_4(q)$ we have a cyclic group of order $q-1$, which is normal in X_R/Q_R. As $q > 2$ by assumption, we have that graph automorphisms act nontrivially on this group. So assume $G \cong L_3(q)$. Now X_R/Q_R is cyclic of order $(q-1)/d$, where $d = gcd(3, q-1)$. Suppose $d \neq q-1$. Then both field- and graph automorphisms act nontrivially on X_R/Q_R. By [AschSe, (19.1)] graph automorphisms t invert X_R/Q_R and field automorphisms t centralize a subgroup of order $r-1$ for $r^2 = q$. Hence we see that t must be a graph/field automorphism. Then t centralizes a group of order $r+1$ and inverts a group of order $r-1/d$. In particular we must have $d = r-1$, which is $r = 4$, so $q = 16$, a contradiction as $G \not\cong L_3(4)$ or $L_3(16)$. Assume now $G = U_3(q)$. Then X_R/Q_R is cyclic of order $q+1/d$, where $d = gcd(3, q+1)$. As we now have $q > 2$, we have that X_R/Q_R is nontrivial. Further by [AschSe, (19.8)] we see that $[t, X_R] \not\leq Q_R$. $\square$

Lemma 2.28. *Let* $G = L_n(q)$ *or* $U_n(q)$, S *a Sylow* 2*-subgroup of* G, *with center* R. *Let* V *be normal in* S *with* $|V \cap O_2(C_G(R))| = q^3$. *If* $|S : C_S(V)| \leq q^2$ *then* $V = Z(C_S(C_Q(Z_2(S))))$, *where* $Q = O_2(C_G(R))$.

Proof. We start to prove that $V = Z(C_S(C_Q(Z_2(S))))$. If $G \cong U_n(q)$, then by Lemma 2.17 $Q/Z(Q)$ is a module over $\mathrm{GF}(q^2)$. In particular $|[V, Q]/Z(Q)| \geq q^2$. This shows $[V, Q] = V \cap Q$. If $G \cong L_n(q)$, then again by Lemma 2.17 we have that Q/R is a direct sum of two irreducible modules over $\mathrm{GF}(q)$ and so again $[V, Q] = V \cap Q$. Hence in both cases we have that $|Q : C_Q(V)| = q^2$. Furthermore as VQ/Q is normal in S/Q, we see that S acts on $[V, Q]/R$ and so $[S, [V, Q]] \leq R$. This shows $V \cap Q = Z_2(S)$. In particular $V \leq C_S(Z_2(S))$. As $C_Q(Z_2(S))/Z_2(S)$ is irreducible respectively the direct sum of two irreducible modules for $N_{N_G(Q)/Q}(VQ/Q)$, we see that $[V, C_Q(Z_2(S))] \leq R$. But as $|Q : C_Q(V)| = q^2$, we get that $[C_Q(Z_2(S)), V] = 1$. So $V \leq C_S(C_Q(Z_2(S)))$. Let now $t \in S$, with $[t, C_Q(Z_2(S))] = 1$. If $t \notin Q$, then t induces a transvection to $Z_2(S)/R$. But this group of transvections in $U_{n-2}(q)$ and $L_{n-2}(q)$ is of order q and so $t \in VQ$. Hence $V = Z(C_S(C_Q(Z_2(S))))$, the assertion. □

A 2-local N can fail to be of characteristic 2 in one of two ways. Either $E(N) \neq 1$ or $O(N) \neq 1$. The next lemma will become important when we show in Chapter 6 that $O(N) = 1$ for all 2-locals containing a Sylow 2-subgroup.

Lemma 2.29. *Let $F^*(G)$ be a simple group, $F^*(G) \in \mathcal{C}_2$ and let T be a Sylow 2-subgroup of G. Assume that T normalizes a non-trivial subgroup U of G of odd order. Then $G \cong L_3(3)$ or M_{11} and $|U| = 9$.*

Proof. We always have that T contains a fours group V. Then by coprime action we have that

(1) $$U = \langle C_U(v) \mid 1 \neq v \in V \rangle.$$

This we will use in what follows.

If $F^*(G)$ is a sporadic simple group we see that $F^*(G) \cong M_{11}$ by going over the groups in [GoLyS3, Table 5.3]. Suppose now that $F^*(G)$ is a group of Lie type in odd characteristic. If $F^*(G) \cong L_2(p)$, p a Fermat or Mersenne prime, or $L_2(9)$, we have that the centralizer of an involution is a 2-group, and so by (1) T cannot act on U. If $F^*(G) \cong L_4(3)$, $U_4(3)$ or $G_2(3)$, then by Lemma 2.6 and Lemma 2.7 centralizer of involutions are $\{2,3\}$-groups. So by (1) U is a 3-group. Then the Borel-Tits-Theorem [MaStr, Lemma 2.15] implies that UT is contained in some parabolic subgroup. But obviously none of them contains a full Sylow 2-subgroup. Hence as $F^*(G) \in \mathcal{C}_2$ we are left with $F^*(H) \cong L_3(3)$.

So it remains to deal with groups of Lie type in characteristic 2. Then the assertion follows with [GoLyS3, Corollary 3.1.4]. □

Lemma 2.30. *Let $G \cong L_4(3)$ or $U_4(3)$ and t be a 2-central involution in G. Then $C_G(t)$ has no normal subgroup $Q \cong Q_8$.*

Proof. For both groups the structure of $C_G(t)$ is described in Lemma 2.6. Hence we have that $O_2(C_G(t)) \cong Q_8 * Q_8$. Furthermore there is a subgroup $U \cong SL_2(3) * SL_2(3) = S_1 * S_2$ normal in $C_G(t)$, with $O_2(U) = O_2(C_G(r))$. Suppose Q is normal in $C_G(r)$, then $Q \leq O_2(C_G(r))$ and is normal in U. So Q is one of the two normal quaternion subgroups $O_2(S_i), i = 1, 2$, of $O_2(U)$. But $C_G(t)$ contains some element u with $S_1^u = S_2$, in particular $O_2(S_i)$, $i = 1, 2$, both are not normal in $C_G(t)$. □

Lemma 2.31. *Let $G/Z(G) \in \mathcal{M}$ (see* [MaStr, Definition 2.51]*) with $Z(G) \neq 1$, and assume that G has a 2–central involution z such that $|C_G(z)| = 2^a \cdot 3^b$, with $b \leq 2$. Then $G \cong 2L_3(4)$, $2^2L_3(4)$, $2Sp_6(2)$, $2U_4(3)$, $2M_{12}$, $2M_{22}$, $4M_{22}$, $2Sz(8)$ or $2^2Sz(8)$.*

Proof. We have $z \notin Z(G)$. Hence also $C_{G/Z(G)}(z)$ is a $\{2,3\}$–group. Now we just go over the groups in $\mathcal{M}$. Let us assume that G is not one of the groups listed in the conclusion of the lemma. By inspection of [GoLyS3, Table 5.3] and Lemma 2.17 we see that $5 \big| |C_{G/Z(G)}(z)|$, or $G \cong \Omega_8^+(2)$ and $|C_{G/Z(G)}(z)| = 2^{12} \cdot 3^3$, which contradicts $b \leq 2$. Hence, $G/Z(G)$ is as claimed. □

Lemma 2.32. *Let $G \cong L_2(p)$, p an odd prime, A_6, $L_3(3)$, M_{11}, $L_3(4)$ or $Sz(q)$, $q = 2^m$. Then G possesses exactly one conjugacy class of involutions.*

Proof. If G is isomorphic to $L_2(p)$, A_6, $L_3(3)$ or M_{11}, then by Lemma 2.5 a Sylow 2-subgroup of G is dihedral or semidihedral. Now it is an easy application of Lemma 2.2 to see that these groups have precisely one class of involutions. For $G \cong L_3(4)$ the assertion follows from Lemma 2.20. For $G \cong Sz(q)$, we get the assertion with [GoLyS4, Lemma 4.3.4]. □

Lemma 2.33. *Let $G \cong 2L_3(4)$, $2^2L_3(4)$, $2Sp_6(2)$, $2U_4(3)$, $2M_{12}$, $2M_{22}$, $4M_{22}$, $2Sz(8)$ or $2^2Sz(8)$. If there is an element x of order four in G such that $x^2 \in Z(G)$, then $G \cong 2Sp_6(2)$, $2M_{12}$ or $4M_{22}$.*

Proof. Let S be a Sylow 2–subgroup of G. Suppose $G \not\cong 4M_{22}$. Then $Z(G)$ is elementary abelian. Further it is enough to deal with the case of $|Z(G)| = 2$. As S is not a quaternion group, there are involutions in $S \setminus Z(G)$. Hence $G/Z(G)$ has more than one conjugacy class of involutions. But $U_4(3)$, $L_3(4)$, M_{22} and $Sz(8)$ have just one class of involutions. So $G/Z(G) \cong Sp_6(2)$ or M_{12}. □

Lemma 2.34. *Let G be a group, $L \trianglelefteq G$, $L \cong L_4(3)$. Assume that $C_G(L)$ is a cyclic 2-group. Let S be a Sylow 2–subgroup of G and $|\Omega_1(Z(S))| = 8$. Then $C_G(L) \leq Z(S)$ and $S = C_S(L) \times ((S \cap L)\langle d \rangle)$ with $d \in Z(S)$.*

Proof. By Lemma 2.6 we have that $Z(L \cap S) = \langle t \rangle$ and we may assume that there is $d \in S$, which centralizes in L a group $PSp_4(3) : 2$. Furthermore $|G : LC_G(L)| = 2$ and $td \not\sim d$, as $N_G(S)$ normalizes $S \cap L$ and so centralizes $Z(S)$. Then we have that $S = C_S(L) \times ((S \cap L)\langle d \rangle)$ and so $C_G(L) \leq Z(S)$. □

Lemma 2.35. *Let $G \cong M_{12}$ or M_{22} and let x be a 2-central involution in G. Then $|C_G(x)|$ is divisible by 3 but not by 9. Furthermore* $\mathrm{Out}(G)$ *is of order* 2.

Proof. This follows from [GoLyS3, Table 5.3b] and [GoLyS3, Table 5.3c]. □

Lemma 2.36. *Let $G = Sp_6(2)$ and x be a 2-central involution, which is centralized by an elementary abelian group U of order 9. If there is an elementary abelian subgroup E of order 32 in $C_G(x)$, which is normalized by U, then x is a transvection on the natural module.*

Proof. By the Borel-Tits-Theorem [MaStr, Lemma 2.15] $C_G(x)$ is contained in one of the parabolics $2^5Sp_4(2)$, $(2^2Q_8 * Q_8)(\Sigma_3 \times \Sigma_3)$, or $2^6L_3(2)$. As $|U| = 9$, we have that $C_G(x)$ is contained in one of the first two parabolics. By Lemma 2.18 we see that the centralizer of a group U of order 9 in both case is $U\langle x\rangle$. So we just have to eliminate the second case. Here U normalizes $Q = Q_8 * Q_8$ and so it induces orbits of length 9 on the involutions in $Q \setminus Z(Q)$. In particular U cannot normalize an elementary abelian group of order 32, as this group must contain all involutions of $O_2(C_G(x))$ and so equals to $O_2(C_G(x))$. □

Lemma 2.37. *Let $G = L_2(p)$, p an odd prime, A_6, $L_3(3)$, M_{11}, $Sz(q)$ or $L_3(4)$. Let furthermore t be an involution in $G\langle t\rangle$, which induces an outer automorphism on G and S be a Sylow 2-subgroup of $G\langle t\rangle$. Then $t \sim tx$ for all $x \in \Omega_1(Z(C_S(t)))$, or $G \cong A_6$ and t induces the Σ_6–automorphism.*

Proof. If $G \cong M_{11}$, then by Lemma 2.12 there is no such automorphism t. The same is true for $G \cong Sz(q)$ by Lemma 2.23(2). If $G \cong L_2(p)$ then $\mathrm{Aut}(G) \cong PGL_2(p)$ by [GoLyS3, Table 4.5.3]. Now $\mathrm{Aut}(G)$ has a dihedral Sylow 2-subgroup and so all involutions in $\mathrm{Aut}(G) \setminus G$ are conjugate anyway. If $G \cong L_3(3)$, then by [GoLyS3, Table 4.5.1] , we see that $C_G(t) \cong \Sigma_4$ and so $C_S(t) = \langle t\rangle \times D$, where $D \cong D_8$. As t obviously is not 2-central in S, we see that $t \sim tx$ with $\langle x\rangle = Z(D)$.

Let $G \cong L_3(4)$. Then by Lemma 2.23(3), $C_G(t) \cong L_2(4)$, $L_2(7)$ or 3^2Q_8. In all cases $C_G(t)$ has just one class of involutions and as t is not 2-central, the assertion follows.

So let finally $G \cong A_6$. As t is an involution we get with [GoLyS4, Lemma 4.4.2] that $G\langle t\rangle$ is isomorphic to $PGL_2(9)$ or Σ_6. In the former the assertion follows with [GoLyS4, Lemma 4.4.1]. □

Lemma 2.38. *Let $G = A_8$. There is no subgroup H of G, such that H has abelian Sylow 2-subgroup and $|H|$ is divisible by $3 \cdot 5 \cdot 7$.*

Proof. Assume false. Let L be a Sylow 7-subgroup of H. The normalizer of L in G is of order 21. If $N_H(L) = L$, we have a normal 7–complement in H. But then L centralizes a Sylow 5-subgroup, a contradiction. So we have $|N_H(L)| = 21$. We get with Sylow's theorem that $|H| = 3^2 \cdot 5 \cdot 7$

or $2^3 \cdot 3^2 \cdot 5 \cdot 7$. In the latter $|G : H| = 8$ and so $H \cong A_7$, which does not possess an abelian Sylow 2-subgroup. So we have $|H| = 3^2 \cdot 5 \cdot 7$. Let T be a Sylow 5-subgroup of H, then $N_H(T) = C_H(T)$, as $|H|$ is odd. Hence now H has a normal 5–complement and so again T centralizes a Sylow 7-subgroup, a contradiction. □

Lemma 2.39. [Asch1, Theorem A] *Let G be a finite group with $F^*(G) = L$ a simple group, T a Sylow 2-subgroup of G and $z \in Z(T)$ be an involution. Assume that $M = C_G(z)$ is the unique maximal subgroup of G which contains T. Then one of the following holds:*

(1) *$L \cong L_2(q)$, $q > 5$ odd.*
(2) *$q \equiv -1(\mod 4)$ and $L \cong L_{2^n+1}(q)$, or $q \equiv 1(\mod 4)$ and $L \cong U_{2^n+1}(q)$, and M contains a normal subgroup $SL_{2^n}(q)$, $SU_{2^n}(q)$, respectively. In the first case S acts nontrivially on the Dynkin diagram.l*
(3) *$L \cong \Omega_{2^n+1}(q)$, q odd, $n > 2$, and M contains a normal subgroup $SO_{2^n}(q)$.*
(4) *$q \equiv -1(\mod 4)$ and $L \cong \Omega^+_{2^n+2}(q)$, or $q \equiv 1(\mod 4)$ and $L \cong \Omega^-_{2^n+2}(q)$, and M contains a normal subgroup $SO^+_{2^n}(q)$. Further T is not contained in the group $O_{2^n+2}(q)$ extended by the group of field automorphisms.*

3. Small Modules

In Chapter 6 we will construct a 2-local subgroup N of G, which is not contained in $C_G(z)$ (with z 2-central), such that $N \cap C_G(z)$ contains a Sylow 2-subgroup S of $C_G(z)$ and $N \cap C_G(z)$ is the only maximal subgroup of N which contains S. Finally we will have $F^*(N) = O_2(N)$.

Then we will determine the action of $N/O_2(N)$ on $\Omega_1(Z(O_2(N)))$. This will be a so called small module for $N/O_2(N)$. In this chapter we investigate small modules in generality. The results obtained will be applied to determine the acton of N on $\Omega_1(Z(O_2(N)))$.

Definition 3.1. Let X be a group, V be a faithful module over $\mathrm{GF}(p)$. We call V an

(i) F–module if there is some nontrivial elementary abelian p-subgroup A of X such that $|V : C_V(A)| \le |A|$;
(ii) $F+1$–module if there is some nontrivial elementary abelian p-subgroup A of X such that $|V : C_V(A)| \le 2|A|$;
(iii) $2F$–module if there is some nontrivial elementary abelian p-subgroup A of X such that $|V : C_V(A)| \le |A|^2$.

In all cases the group A is called an offender. We call the module V a sharp F-module, if for any offender A we have that $|V : C_V(A)| = |A|$.

We will call the modules defined in Definition 3.1 small modules. Here is a typical situation in which F-modules show up.

Lemma 3.2. *Let G be a group which acts on a p-group X and S be a Sylow p-subgroup of XG. Assume that G acts faithfully on $W = \Omega_1(Z(X)) \neq \Omega_1(Z(S))$. Then either $J(S) \leq X$, and so $J(S)$ is normal in XG, or W is an F-module for G.*

Proof. We may assume that $J(S) \not\leq X$. Then there is a maximal elementary abelian subgroup A of S, with $A \not\leq X$. Now $|A \cap X||W|/|W \cap A| = |(A \cap X)W| \leq |A|$. This implies that $|W/W \cap A| \leq |A/A \cap X|$. As $W \cap A \leq C_W(A)$, we get that W is an F-module with offender $A/A \cap X$. □

In the next two lemmas we give a classification of some of the small modules for simple groups using the classification of the finite simple groups. By a *full transvection group* we mean the unipotent radical of the stabilizer of a point or hyperplane of the natural module for $SL_n(q)$. Let $X = A_n$ and V be the permutation module over GF(2). We call the non trivial irreducible module involved in V of dimension $n-2$ for n even and dimension $n-1$ for n odd, the reduced permutation module.

Lemma 3.3. *Let X be a group such that $F^*(X)$ is quasisimple and let V be an irreducible $F^*(X)$–module over* GF(2) *which is an F–module for X. Then $F^*(X)$ is classical, $G_2(q)$, A_n, or $3A_6$ and one of the following holds*

(1) *$F^*(X)$ is classical and V is the natural module, or A_n and V is the irreducible reduced permutation module.*
(2) *$F^*(X) \cong SL_n(q)$ and V is the exterior square of the natural module or its dual. Further this module is sharp.*
(3) *$F^*(X) \cong Sp_6(q)$ or $\Omega^+_{10}(q)$ and V is the spin module or half spin module, respectively. If $F^*(X) \cong \Omega^+_{10}(q)$, then this module is sharp.*
(4) *$F^*(X) \cong G_2(q)$ and V is the natural module or $F^*(X) \cong 3A_6$ and V is the 6–dimensional module. In both cases this is sharp.*
(5) *$X \cong A_7$ and V is the 4–dimensional module over* GF(2).

Proof. [GM], [GM1], [GLM]. □

Lemma 3.4. *Let X be a group such that $F^*(X)$ is quasisimple and let V be a faithful irreducible X–module over* GF(2). *Suppose that X is a minimal parabolic (i.e. a Sylow 2-subgroup of X is not normal in*

X but contained in a unique maximal subgroup of X) and V is a $2F$–module with offender A such that $|V : C_V(A)| < |A|^2$. Then one of the following holds

(a) *V is an F–module, $F^*(X) \cong L_2(2^n)$ and V is the natural module, or $F^*(X) \cong A_{2^n+1}$ and V is the irreducible section of the permutation module.*

(b) *V is not an F-module and one of the following holds*

  (1) *$F^*(X) \cong SL_3(2^n)$ and V is the direct sum of the natural module and its dual. Furthermore X contains some element, which induces a graph or graph/field automorphism on $F^*(X)$.*

  (2) *$F^*(X) \cong L_2(2^{2n}) \cong \Omega_4^-(2^n)$ and V is the orthogonal module.*

  (3) *$F^*(X) \cong Sp_4(2^n)$ and V is a direct sum of the two 4–dimensional modules. Furthermore X contains some element, which induces a graph automorphism on $F^*(X)$.*

  (4) *$F^*(X) \cong A_9$ and $|V| = 2^8$, V is the spin module.*

Proof. If V is irreducible for $F^*(X)$ then we get (a), (b)(2) or (b)(4) by [GM], [GM1], [GLM]. If V is not irreducible for $F^*(X)$, then there is a submodule V_1 such that $V = V_1 \oplus \cdots \oplus V_r$, $r > 1$ and V_i are X–conjugate irreducible $F^*(X)$–modules.

We will show:

$$(*) \qquad \begin{array}{l} V_1 \text{ is an } F-\text{module for } F^*(X)\tilde{A}, \text{ where} \\ \tilde{A} \text{ is an offender with } |V_1 : C_{V_1}(\tilde{A})| < |\tilde{A}|. \end{array}$$

For this assume first that A acts on each V_i. Then we see that it induces on at least one V_i an F–module offender $A/C_A(V_i)$ such that $|V_i : C_{V_i}(A)| < |A/C_A(V_i)|$. We may assume $i = 1$. So we can set $\tilde{A} = A/C_A(V_1)$ to get $(*)$. Now let $W = V_1^A = V_1 \oplus \cdots \oplus V_t$, $t > 1$. Then we have that $|A|^2 > |W : C_W(A)| = |V_1 : C_{V_1}(B)||V_1|^t$, where $B = N_A(V_1)$. Assume that $|V_1 : C_{V_1}(B)| \geq |B|$. Then $t^2|B| > |V_1|^{t-1} \geq (2|B|)^{t-1}$. This shows $t = 2$, $B \neq 1$ and $|V_1| = 2|B|$. In particular B induces the full transvection group to a point on V_1. As $A \neq B$ and there is no outer automorphism of $L_n(2)$ centralizing a full transvection group this is not possible. Hence we have $|V_1 : C_{V_1}(B)| < |B|$. Now with $\tilde{A} = B$, we again have $(*)$. This finally proves $(*)$.

Using $(*)$ an application of Lemma 3.3 shows that we have (b)(1) or (3) or $F^*(X) \cong A_n$. In case of A_n, as X is a minimal parabolic, we have n odd. Offenders are transvection groups and so they are sharp. Hence $F^*(X) \ncong A_n$. $\square$

By Thompson replacement [GoLyS2, Theorem 25.2] F-modules are also quadratic modules. Hence we now turn to quadratic modules.

Lemma 3.5. [Cher, Theorem 3] *Let K be a component of a group X, $O_2(K) = 1$ and V be a* GF(2)*–module for X with $[V, K] \neq 1$. Suppose that $A \leq X$ and $[V, a, A] = 1$ for some $1 \neq a \in A$, then one of the following holds:*

(i) $[K, A] \leq K$,
(ii) $K \cong SL_2(2^k)$, $|A/N_A(K)| = 2$ *and* $|A/C_A(K)| > 2$. *Further* $[V, \langle K^A\rangle]$ *is a direct sum of natural* $\Omega_4^+(2^k)$*–modules, or*
(iii) $A \neq N_A(K)$, $|A/C_A(K)| = 2$.

If $[K, A] \not\leq K$, then A does not act as a quadratic F-module offender on $[V, \langle K^A\rangle]$.

Lemma 3.6. [Str2] *Let $X \cong Sp_4(q)'$ or ${}^2F_4(q)'$, $q = 2^n$, and V be an irreducible* GF(2)*-module. Suppose there is a fours group A in X with $[V, A, A] = 1$. If A intersects some root group R nontrivially but $A \not\leq R$, then $X \cong Sp_4(q)'$ and V is a natural module.*

Lemma 3.7. *Let X be a group such that $F^*(X)$ is a perfect central extension of a finite simple group. Suppose there is some elementary abelian 2-subgroup A of $X, |A| \geq 4$, such that for some irreducible nontrivial faithful module V over* GF(2) *we have $[V, A, A] = 1$. Then:*

(i) *If $F^*(X)/Z(F^*(X))$ is sporadic, then $F^*(X)/Z(F^*(X)) \cong M_{12}$, M_{22}, M_{24}, J_2, Co_1, Co_2 or Sz. If $|A| \geq 8$, then $F^*(X) \cong 3{\cdot}M_{22}$.*
(ii) *If $F^*(X)/Z(F^*(X))$ is a group of Lie type in odd characteristic which is not also a group of Lie type in even characteristic, then $F^*(X) \cong 3 \cdot U_4(3)$. Furthermore V is the 12–dimensional module.*
(iii) *If $F^*(X)/Z(F^*(X))$ is alternating, then either V is the reduced permutation module, a spin module or $F^*(X) \cong 3 \cdot A_6$ and V is the 6-dimensional module or $F^*(X) \cong 3 \cdot A_7$ and V is the 12–dimensional module. If $|A| > 8$, then V is natural or $X \cong A_8$ and $|V| = 16$. If V is the spinmodule and $|A| = 4$, then A is conjugate to $\langle(12)(34), (13)(24)\rangle$ or $\langle(12)(34)(56)(78), (13)(24)(57)(68)\rangle$. If $|A| = 8$ then A is conjugate to $\langle(12)(34)(56)(78), (13)(24)(57)(68), (14)(26)(37)(48)\rangle$ in Σ_n.*

Proof. (i) This is [MeiStr2].

(ii) This is [MeiStr1].

(iii) The first assertion is [MeiStr2]. There the group $3 \cdot A_7$ was forgotten. But as J. Hall pointed out there is an embedding $3 \cdot A_7 \leq$

$3 \cdot M_{22} \leq SU_6(2)$, which gives a 6-dimensional module over GF(4) on which a fours group in $3 \cdot A_7$ acts quadratically.
For the proof of the second assertion suppose $|A| \geq 4$. Let $a \in A^\sharp$ and k be the number of fixed points of a. Then there is $K \leq C_X(a), K \cong \Sigma_k$. Furthermore $C_{C_X(a)}(K')'$ is an extension of a 2–group by A_m, $m = (n-k)/2$. Now choose $a \in A$ with m as big as possible. Suppose first $m > 2$. By [MeiStr1, (4.3)] there is no $x \sim (12)(34)$ such that $[[V,a],x] = 1$. In particular $\langle A^{C_X(a)} \rangle$ does not contain such an element x.

Suppose first $[A, C_{C_X(a)}(K')] \neq 1$. If $m \geq 5$, then A_m is nonsolvable and so $C_{C_X(a)}([V,a])$ contains an elementary abelian subgroup of $O_2(C_X(a))$ of order 2^{m-1}. But then this group contains a conjugate t of $(12)(34)$. Now $\langle a, t\rangle$ acts quadratically, a contradiction.

Let $m = 4$. Then $a \sim (12)(34)(56)(78)$. Furthermore as we may assume that no $x \sim (12)(34)$ is contained in $\langle A^{C_X(a)} \rangle$ we see that A is conjugate to a subgroup of $\langle (12)(34)(56)(78), (13)(24)(57)(68), (15)(26)(37)(48)\rangle$.

Let $m = 3$. Then $C_X(K') \leq \Sigma_6$ and $a \sim (12)(34)(56)$. We see that $\langle A^{C_X(a)} \rangle$ has to contain some $x \sim (12)(34)$, a contradiction.

So let $[A, O^{2'}(C_{C_X(a)}(K'))] = 1$. If $[A, K'] \neq 1$, then $[K', [V,a]] = 1$. If $k \geq 4$, then K' contains some $x \sim (12)(34)$, a contradiction. Let $k \leq 3$. As $[A, O^{2'}(C_{C_X(a)}(K'))] = 1$ and $m > 2$, there is $x \sim (12)$ in A. But then xa has fewer fixed points than a, a contradiction. So we are left with $[A, K'] = 1 = [A, C_{C_X(a)}(K')]$. But this is impossible with $m > 2$.

So we have $m \leq 2$ for all $a \in A^\sharp$. As there is no fours group of transpositions we may assume $a = (12)(34) \in A$. Now $A \geq \langle a, b\rangle$, $b = (13)(24)$, $(12)(56)$ or (34). Let $b = (12)(56)$. If $[b, K'] \neq 1$ then K' contains no involutions by [MeiStr1, (4.3)]. This shows $k \leq 3$ and so $A \leq \Sigma_7$. If $[b, K'] = 1$, then even $k \leq 2$ and so $A \leq \Sigma_6$. But for this group $A = \langle (12)(34), (12)(56)\rangle$ does not act quadratically on the four dimensional spin module. Recall that in case of Σ_6 the natural module is defined as the module on which $\langle (12)(34), (12)(56)\rangle$ acts quadratically.

Assume now $b = (34)$. Then $C_X(b) \cong \mathbb{Z}_2 \times \Sigma_{n-2}$. If $n - 2 > 3$, then $(12)(56) \in [a, C_X(b)]$. But then $\langle (34), (12)(56)\rangle$ acts quadratically, a contradiction. So $n = 5$. But $\langle (12), (34) \rangle$ does not act quadratically on the natural $L_2(4)$–module. Hence $b = (13)(24)$, which proves (iii). $\square$

Lemma 3.8. *Let $A \leq \Sigma_6$ be an elementary abelian subgroup of order 8. Then A does not act quadratically on both of the two 4-dimensional modules for Σ_6.*

Proof. As the two 4-dimensional modules are interchanged by an outer automorphism of Σ_6, which also interchanges the two elementary abelian subgroups of order 8, it is enough to show that not both act quadratically on the irreducible part of the permutation module. But the fours group $\langle (12)(34), (13)(24) \rangle$ does not act quadratically on the irreducible permutation module, as the commutator of $(12)(34)$ with the permutation module, which is $\langle v_1 + v_2, v_3 + v_4 \rangle$, is not centralized by $(13)(24)$. $\square$

For later applications we need some information about central extensions of some of the small modules.

Lemma 3.9. *Let $X = A_n, n \geq 5, V$ be a $\mathrm{GF}(2)X$-module with $[V, X]$ the natural irreducible permutation module. Assume $C_V(X) = 1$. Then $|V : [V, X]| \leq 2$, and $V = [V, X]$ if n is odd. Furthermore V is a factor of the reduced permutation module. In particular V is of dimension $n - 1$ or $n - 2$.*

Proof. This will be proved by induction on n. For $n = 5$ this is well known as the permutation module is injective. So let $n > 5, K \cong A_{n-1}, K \leq X$. If $n - 1$ is odd, then $[V, X] = [V, K]$ is the permutation module for K. By induction $V = [V, K] \bigoplus T$. Hence there is $v \in V \setminus [V, X], [v, K] = 1$, i.e. $\langle v^X \rangle = V_1$ is a factor of the permutation module. Let $K_1 \leq K$ such that $K_1 \cong A_{n-2}$. Then $|C_V(K_1) : T| = 2$. Now there is an involution $t \in X$ such that $t \notin K$ but t normalizes K_1. As $\langle K, t \rangle = X$, we get $C_T(t) = 1$ and so $T = \langle v \rangle$, i.e. $V_1 = V$.

Let $n-1$ be even. Then we have a K-chain. $1 < T < T_1 < [V, X] < V$, with $|T| = 2, T_1/T$ the irreducible permutation module for K and $|[V, X]/T_1| = 2$. Now by induction $C_{V/T}(K) \neq 1$. As $C_{V/T}(K) \not\leq [V, X]/T$, we again get some $v \in V \setminus [V, X], [v, K] = 1$, and so V is a factor of the permutation module. $\square$

Lemma 3.10. *Let $F^*(G) = L_2(2^n)$ and V be a faithful F–module over $\mathrm{GF}(2)$ for G such that $C_V(G) = 1$. Then V is irreducible.*

Proof. If $n = 1$, then $V = [V, G'] \oplus C_V(G')$. As $C_V(G) = 1$ also $C_V(G') = 1$ and so $V = [G', V]$ is of order 4. So let $n > 1$. By Lemma 3.3 we have that there is an irreducible submodule V_1 in V which is the natural $L_2(2^n)$–module or $n = 2$ and it is the permutation module for A_5. In both cases we get $|V_1 : C_{V_1}(A)| = |A|$ for an offender A. Hence we see that $V = C_V(A)V_1$. In particular V/V_1 is a trivial $L_2(2^n)$–module. By Lemma 3.9 we may assume that V_1 is the natural $L_2(2^n)$–modules. Now A is a Sylow 2–subgroup of $L_2(2^n)$. Application of [Hu, (I.17.4)] gives $V = V_1$. □

Lemma 3.11. *Let $X = \Omega_4^+(q)$, q even, and V be a module over* GF(2) *with $[V, X]$ the natural module and $C_V(X) = 1$. Then $[V, X] = V$.*

Proof. We have $X = X_1X_2$, $X_i \cong L_2(q)$, $i = 1, 2$. We may assume that $q > 2$, as the assertion is obvious for $q = 2$. There are $\omega_i \in X_i$ with $o(\omega_i) = q + 1$. If $C_{[V,X]}(\omega_1) \neq 1$, then as X_2 acts nontrivially on $C_{[V,X]}(\omega_1)$ we get $|C_{[V,X]}(\omega_1)| = q^2$ and so $|[[V, X], \omega_1]| = q^2$. By Schur's Lemma $[[V, X], \omega_1]$ is a 1-dimensional module over $\mathrm{GF}(q^2)$ for X_2 and so $X_2 \leq GL_1(q^2)$, a contradiction. Hence ω_i act fixed point freely on $[V, X]$ for both $i = 1, 2$. Now choose $v_1 \in V \setminus [V, X]$ with $[v_1, \omega_1] = 1$. Then v_1 is uniquely determined in the coset $[V, X]v_1$. Since ω_1 and ω_2 commute, we have v_1 is centralized by ω_2. So $C_V(\omega_1) = C_V(\omega_2)$ is normalized by $\langle C_X(\omega_1), C_X(\omega_2)\rangle \geq \langle X_2, X_1\rangle = X$, which is a contradiction. □

Lemma 3.12. *Let $F^*(G) = A_{2^n+1}$ and V be a module over* GF(2)*, which is an $2F$–module, with offender A such that $|V : C_V(A)| < |A|^2$. Assume $C_V(G) = 1$ and V involves just trivial and nontrivial irreducible parts of the permutation module. Then we have that V is the irreducible part of the permutation module.*

Proof. If we have just one irreducible part of the permutation module in V, the assertion follows by Lemma 3.9. So we may assume that we have at least two such modules involved. Let W be the irreducible part of the permutation module. Then we have that A is an F–module offender on W with $|W : C_W(A)| < |A|$. Then by Thompson replacement [GoLyS1, Theorem 25.1] there is also a quadratic F–module offender with this property. Take an involution $x \in G$. On W we have that $|[W, x]| = 2^u$, where u is the number of transpositions in the cycle decomposition of x. We may assume that $\{1, 2, \ldots, m\}$ is the support of A. Then there is a subgroup B of A such that $|W : C_W(B)| = |W : C_W(R)|$, where $R = \langle(1, 2), (3, 4), \ldots, (m - 1, m)\rangle$. But then $|W : C_W(R)| = |R|$, a contradiction. □

Lemma 3.13. *Let $G = A_{2^n+1}$ and S be a Sylow 2–subgroup of G. Let V be the irreducible part of the permutation module over* GF(2) *for G. Then $|C_V(S)| = 2$.*

Proof. Let W be the module with basis v_i, $i = 1, \ldots, 2^n + 1$ with natural G-action on W. Then $W = V \oplus W_1$, W_1 the trivial module.

Choose $S \leq X \cong A_{2^n}$, where X is the stabilizer of 1. Then we calculate immediately that $C_W(S) = \langle v_1, v_2 + \cdots + v_{2^n+1}\rangle$. As $v_1 \notin V$, we get the assertion. $\square$

Lemma 3.14. *Let $G = L_2(2^n)$ or A_{2^n+1}, $n \geq 2$. Let H be a Borel subgroup in the first case and a subgroup isomorphic to A_{2^n} in the second case. Let V be a* GF(2)*–module for G such that $[V, G]$ is the natural module, or $G \cong A_9$ and $[V, G]$ is the 8–dimensional spin module. Then one of the following holds:*

(i) *$G = L_2(2^n)$ and $C_V(H) = C_V(G)$.*
(ii) *$G = A_{2^n+1}$ and $C_V(H) = C_V(S)$, S a Sylow 2–subgroup of H.*
(iii) *$G = A_9$, $[V, G]$ is the 8-dimensional spin module and $C_V(H) = C_V(G)$.*

Proof. We may assume that in all cases $V = [V, G]C_V(H)$. As H contains a Sylow 2-subgroup of G we get that $V = [V, G]C_V(S)$. Now application of [Hu, (I.17.4)] shows that $V = [V, G] \oplus C_V(G)$. In case (i) and (iii) we have that $C_{[V,G]}(H) = 1$, so we have that $C_V(G) = C_V(H)$. In case (ii) by Lemma 3.13 we have that $C_{[V,G]}(H) = C_{[V,G]}(S)$, so we get $C_V(H) = C_V(S)$. $\square$

Lemma 3.15. *Let $G = E(G)T$, T a Sylow 2–subgroup of G, $E(G) = G_1 \cdots G_r$, $G_1 \cong L_2(q)$, q even, or A_{2^n+1}. Assume that T acts transitively on the G_i and $C_G(E(G)) = 1$. Let V be an irreducible faithful F–module over* GF(2) *for G. Then $V = V_1 \oplus \cdots \oplus V_r$, V_i the natural module for G_i, $i = 1, \ldots, r$, and $[V_j, G_i] = 1$ for $i \neq j$.*

Proof. Let A be an offender. We may assume $[V, A, A] = 1$ by Thompson replacement. Now choose A with $|A|$ minimal. Set $A_1 = C_A(G_1)$. Then we may assume $A_1 = 1$ or $|V : C_V(A_1)| > |A_1|$. If $[G_1, A] \not\leq G_1$ we get with Lemma 3.5 that $G_1^A = G_1 G_1^a$ and $|A/C_A(G_1)| = 2$. In any case $\langle a\rangle$ has to be an F–module offender on $C_V(A_1)$. This shows $A_1 = 1$ and $\langle a\rangle = A$. But now a inverts some element of prime order $p > 3$ in $E(G)$ and so cannot induce a transvection on V. So we have that $[G_1, A] \leq G_1$. Then G_1 induces an F–module in $C_V(A_1)$. By Lemma 3.3 we have that there is exactly one nontrivial module W involved in $C_V(A_1)$, the natural one.

Assume that $A_1 \neq 1$. Let $B \leq A$ be a complement to A_1 and let $1 \neq a \in A_1$. As A acts quadratically, we see that $[V, a, G_1] = 1$. This implies $[V, G_1] \leq C_V(A_1)$. If $A_1 = 1$, then also $[V, G_1] \leq C_V(A_1)$. Hence in any case $[V, G_1]$ involves just one nontrivial irreducible module. Now we have that $[V, G_1]$ is centralized by $G_2 \times \cdots \times G_r$. As $C_V(E(G)) = 1$, we get that $W = [V, G_1]$ is the natural module. But now $[V, G_i]$ is the natural module for all i, as T acts transitively. Hence $V = V_1 \oplus \cdots \oplus V_r$ with $[V_i, G_j] = 1$ for $i \neq j$ and V_i the natural G_i–module, the assertion. $\square$

The next two lemmas deal with solvable groups having F or $2F$-modules.

Lemma 3.16. *Let G be a solvable group with Sylow 2–subgroup S and $O_2(G) = 1$. Assume that S is contained in a unique maximal subgroup of G. Let V be a faithful* GF(2)*–module for G. If V is an F–module, then $G = O_3(G)S$.*

Proof. If $G \neq F(G)S$, then there are maximal subgroups containing $F(G)S$ and $N_G(S \cap O_{2',2}(G))$, which are different. Hence $G = F(G)S$. Further again by minimality $F(G) = O_p(G)$ for some prime p. By Lemma 2.3 we have a subgroup $D = D_1 \times \cdots \times D_r$ of G such that the D_i are dihedral of order $2p$ and a Sylow 2–subgroup A of D is an F–module offender. Hence we have that $|V/C_V(D)| \leq |A|^2$, as D is generated by two conjugates of A. Now $O_p(D)$ acts faithfully on $V/C_V(D)$ and so $p = 3$. $\square$

Lemma 3.17. *Let G be a group and V be a faithful $2F$–module over* GF(2) *with offender A. Suppose $G = O_p(G)A$ with $O_p(G) = F(G)$ for some odd prime p. Then $p \leq 5$ and in case of $p = 5$, we have that $|V : C_V(A)| = |A|^2$. If A is an F–module offender, then $p = 3$ and $|V : C_V(A)| = |A|$.*

Proof. By the Dihedral Lemma 2.3, we may assume that

$$G = D_1 \times \cdots \times D_r,$$

D_i dihedral of order $2p$. Now as $|V : C_V(A)| \leq |A|^2$ or $|A|$ we have that

$$|V : C_V(G)| \leq |A|^4, |A|^2 \text{ respectively.}$$

Hence $|[V, O_p(G)]| \leq |A|^4 \leq 2^{4r}$, or $|[V, O_p(G)]| \leq 2^{2r}$. In $GL_{4r}(2)$ elementary abelian subgroups of order p^r just exist for $p = 3$ and $p = 5$, while in $GL_{2r}(2)$ they just exist for $p = 3$. This shows that $p \leq 5$. If $p = 5$, then we must have that $|V : C_V(G)| = 2^{4r}$ and so $|V : C_V(A)| = |A|^2$. If $p = 3$ and A is an F–module offender then $|V : C_V(G)| = 2^{2r}$ and so $|V : C_V(A)| = |A|$. $\square$

Lemma 3.18. *Let $X = Sz(q)$ or $L_2(q)$, $q > 2$ even. Suppose that X acts on a 2-group U. Let V be a normal subgroup of U of order 2 and U/V be the natural module for X. In case of $X \cong Sz(q)$ assume additionally that U contains an elementary abelian subgroup U_1 with $|U_1|^2 = 2|U|$. Then U is abelian.*

Proof. If $X \cong L_2(q)$, then X acts transitively on $(U/V)^\sharp$. As $q > 2$ we see that U is not a quaternion group and so there are involutions in $U \setminus V$, so all elements in U are involutions, the assertion.

So let $X \cong Sz(q)$. We may assume that U is extraspecial. Now elements of order 5 act fixed point freely on U/V. The existence of U_1 guarantees that U is extraspecial of $+$ type. As $q = 2^{2n+1}$, we get $|U/V| = 2^{8n+4}$ and so U is a central product of $4n+2$ dihedral groups. But as an element of order 5 acts fixed point freely on U/V the number of dihedral groups must be divisible by four by [MaStr, Lemma 2.9], a contradiction. $\square$

Lemma 3.19. *Let $X = L_2(q)$ or $Sz(q)$, $q \geq 4$, q even. Let $S \in Syl_2(X)$ and $A \leq \Omega_1(S)\,, |A| \geq 4$. Then there is some $g \in X$ with $X = \langle A, A^g \rangle$.*

Proof. We have that X acts 2-transitively on a set Ω with $|\Omega| = q+1$, q^2+1, respectively. For $1 \in \Omega$ we have that $X_1 = SK$, where K is cyclic of order $q-1$ and acts transitively on $\Omega_1(S)$. Further $K = X_{1,2}$, the stabilizer of two points. Finally the stabilizer of any three points is trivial.

This has the following consequences. Choose $1 \neq \rho \in K$. Then $\{1, 2\}$ are the two fixed points of ρ. Hence $N_X(\langle \rho \rangle)$ contains K as a subgroup of index two. This shows that $K = C_X(\rho)$. Let $a \in S$ be an involution. Then a has just one fixed point. This shows that $C_X(a) = S$, a 2-group.

Now choose $\langle t, a \rangle \leq A \leq \Omega_1(S)\,, |A| \geq 4$. Choose $g \in X$ such that $N_X(K^g) = \langle a, b \rangle$ for some involution b. Then set $U = \langle a, b, t \rangle$. Let T be a Sylow 2–subgroup of U with $\langle a, t \rangle \leq T$. Then $T \leq C_X(a) = S$, so $T = U \cap S$. If $T = N_U(T)$, we get a normal 2-complement W in U. But then one of $C_W(a)$, $C_W(t)$, $C_W(at)$ must be nontrivial, which contradicts the fact that centralizers of involutions are 2-groups. Hence we have that $K \cap U \neq 1$. Now choose $\rho \in K \cap U$ of prime order p. As $|K|$ is coprime to $|X : K|$ and $K^g \leq U$, there is some $x \in U$ with $\rho^x \in K^g$. Then $K^g = C_U(\rho^x)$. Now $K = C_X(\rho) = K^{gx^{-1}} \leq U$. This shows that $\langle \Omega_1(S), \Omega_1(S)^x \rangle \leq U$. Thus it is enough to show $\langle \Omega_1(S)\,, \Omega_1(S)^x \rangle = X$.

We have that $Y = \langle \Omega_1(S), \Omega_1(S)^g \rangle$ contains at least $q+1$ conjugates of $\Omega_1(S)$. Thus we are done if $X \cong L_2(q)$, as $\langle \Omega_1(S), \Omega_1(S)^b \rangle$ contains all conjugates.

So let $X \cong Sz(q)$. The number of conjugates of $\Omega_1(S)$ in Y is $nq+1$. But then $nq+1 \,\big|\, q(q^2+1)$. Which gives $n = q$ and so $\langle \Omega_1(S)^X \rangle \leq Y$, hence $X = Y$. $\square$

The next two lemmas show how the $2F$-modules will appear later on.

Lemma 3.20. *Let G be a $\mathcal{K}_2$-group with $F^*(G) = O_2(G) \neq 1$, $A \leq G$ be elementary abelian with $A \not\leq O_2(G)$ and $A \trianglelefteq S$ for some Sylow 2–subgroup S of G. Then there is some $g \in G$ such that one of the following holds:*

(i) *$g^2 \in N_G(A)$, $A^g \leq S$, $1 \neq [A^g, A] \leq A \cap A^g$ and $|A : C_A(A^g)| = |A^g : C_{A^g}(A)|$.*

(ii) *With $X = \langle A, A^g \rangle$ the following hold:*
   (1) *$X/O_2(X) \cong L_2(q)$, $Sz(q)$ or $X/O_2(X)$ is a dihedral group of order $2u$, u odd.*
   (2) *$S \cap X$ is a Sylow 2–subgroup of X.*
   (3) *$Y = (A \cap O_2(X))(A^g \cap O_2(X)) \trianglelefteq X$.*
   (4) *$Y \neq A \cap O_2(X)$.*
   (5) *$|A : C_A(Y)| \leq |Y : C_Y(A)|q \leq |Y : C_Y(A)|^2$, where $q = 2$ if $X/O_2(X)$ is dihedral. Further $[Y, a](A \cap A^g) = [Y, A](A \cap A^g)$ for all $a \in A \setminus O_2(X)$.*
   (6) *If $X/O_2(X)$ is not dihedral, then $Y/(A \cap A^g)$ is a direct sum of natural modules for $X/O_2(X)$.*

Proof. We start the proof with some general remarks. Let X be as in (ii) (1) and (2). Then obviously (3) follows. If (4) would be false, then as $[O_2(G), A] \leq O_2(G) \cap A \leq O_2(X) \cap A$, we get that $[O_2(G), X, X] = 1$ and so $[O^2(X), O_2(G)] = 1$, which contradicts $C_G(O_2(G)) \leq O_2(G)$. Hence also (4) holds. Next we see that $C_Y(A) = A \cap Y$ and so we see that $C_{Y/(A \cap A^g)}(A) = (A \cap Y)/(A \cap A^g)$ and $Y/(A \cap A^g) = (Y \cap A)/(A \cap A^g) \oplus (Y \cap A^g)/(A \cap A^g)$. So the first assertion in (5) follows. Further we see that elements of odd order in X act fixed point freely on $Y/(A \cap A^g)$. Hence [Hi] and [Mar] yield (6) and the second assertion in (5). So to prove the lemma we may assume that (i) dos not hold. Then to prove (ii) we just have to prove (1) and (2). In fact when constructing X such that (1) holds, we immediately will see from this construction that also (2) holds.

Set $\bar{G} = G/O_2(G)$. We first prove

(∗) Suppose there is a subgroup L of $\bar{G}$ such that $|\bar{A} : C_{\bar{A}}(L)| = 2$ and $\bar{A} \not\leq O_2(\langle L, \bar{A} \rangle)$ then (ii) holds. In particular (ii) holds if $|\bar{A}| = 2$.

As $\bar{A} \not\leq O_2(\langle L, \bar{A} \rangle)$ there is some $\omega \in \langle L, \bar{A} \rangle$, $o(\omega)$ odd, which is inverted by some $\bar{a} \in \bar{A} \setminus C_{\bar{A}}(L)$. Then $\langle \bar{A}, \omega \rangle / O_2(\langle \bar{A}, \omega \rangle) \cong D_{2u}$, u odd. Set $X = \langle A, \omega \rangle$. Then X satisfies (ii)(1). Of course $S \cap X$ is a Sylow 2-subgroup of X. So (ii)(2) is satisfied. Hence (∗) is proved.

If $[F(\bar{G}), \bar{A}] \neq 1$ then $F(\bar{G}) = \langle C_{F(\bar{G})}(\bar{B}) \mid |\bar{A} : \bar{B}| = 2 \rangle$. Hence there is some $\bar{B}$ with $C_{F(\bar{G})}(\bar{B}) \neq 1$ and $[C_{F(\bar{G})}(\bar{B}), \bar{A}] \neq 1$. So by $(*)$ (ii) holds.

For the remainder of this proof we will assume that $F^*(\bar{G}) = E(\bar{G})$, $\bar{G} = E(\bar{G})\bar{A}$ and $|\bar{A} : C_{\bar{A}}(L)| \geq 4$ for all components L. As $[S, A] \leq A$, we have that A acts quadratically on $O_2(G)$. Hence by Lemma 3.5 we have $[L, \bar{A}] \leq L$ or $L \cong SL_2(q)$, q even. In the latter there is some $a \in \bar{A}$ such that $C_{\langle L^A \rangle}(a) = L_1 \cong L_2(q)$ and as $\bar{A}$ is normal in a Sylow 2–subgroup of $\langle L, \bar{A} \rangle$, we have that $A_1 = L_1 \cap \bar{A}$ is a Sylow 2–subgroup of L_1. So $L_1 = \langle A_1, A_1^g \rangle$ for suitable $g \in L_1$. Hence $X = \langle A, A^g \rangle$ satisfies (ii)(1) and (2). So from now on we assume that $[L, \bar{A}] \leq L$. We collect this in

$(**)$ $L = F^*(\bar{G})$ is a component, $|\bar{A}| \geq 4$ and if $\bar{A} \leq \bar{U} < \bar{G}$, with $S \cap \bar{U}$ a Sylow 2–subgroup of $\bar{U}$, then $\bar{A} \leq O_2(\bar{U})$.

Assume first that L is of Lie type in odd characteristic, which is not also of Lie type in even characteristic. Then by Lemma 3.7 we have that $L/Z(L) \cong U_4(3)$. As $A \trianglelefteq S$, there is some 2–central involution s in $\bar{A}$. By $(**)$ we have $\bar{A} \leq O_2(C_{L\bar{A}}(s))$. As we may generate $C_{L\bar{A}}(s)$ by elements g with $g^2 \in O_2(C_{L\bar{A}}(s))$, then if $\bar{A}$ is not normal in $C_{L\bar{A}}(s)$ there is such a g with $\bar{A}^g \leq O_2(C_L(s))$ and $1 \neq [\bar{A}, \bar{A}^g] \leq \bar{A} \cap \bar{A}^g$. Then also $1 \neq [A, A^g] \leq A \cap A^g$ and $A^{g^2} = A$. Obviously $|A : C_A(A^g)| = |A^g : C_{A^g}(A^{g^2})| = |A^g : C_{A^g}(A)|$. So we may assume $\bar{A} \trianglelefteq C_{L\bar{A}}(s)$. As $C_L(s)$ contains a normal subgroup $U = SL_2(3) * SL_2(3)$ by Lemma 2.6(i) and $O_2(U) = O_2(C_L(s))$, we see that $O_2(C_L(s))$ cannot contain an elementary abelian subgroup of order at least four which is normal in U. So $\bar{A} \not\leq L$. In particular there is some $t \in \bar{A}$ such that $[U, t] \leq O_2(U)$. As $\langle s, t \rangle$ is normal in $C_{L\bar{A}}(s)$, we get that $|C_L(t)|$ is divisible by $2^6 \cdot 3$. Then by Lemma 2.6 we get $C_L(t) \cong PSp_4(3)$, contradicting $(**)$.

Next let $L \cong G(r)$ be a group of Lie type in even characteristic. Suppose first that $\bar{A}$ acts nontrivially on the Dynkin diagram. If the rank is greater than two, then there is a parabolic U of rank two of L such that $\bar{A}$ acts nontrivially on $F^*(U/O_2(U))$. But this contradicts $(**)$. So we may assume that $L/Z(L) \cong L_3(q)$ or $Sp_4(q)'$. Let B be a Borel subgroup of L, which is normalized by $\bar{A}$, then by $(**)$ we have that $[B, \bar{A}] \leq O_2(B)$. This now gives $q = 2$. But then we easily see that $[S \cap L, \bar{A}]$ is not abelian, contradicting $\bar{A} \trianglelefteq \bar{S}$. So we have that $\bar{A}$ acts trivially on the Dynkin diagram.

Let R be a root subgroup in $Z(\bar{S} \cap L)$. By $(**)$ we have that $\bar{A} \leq O_2(N_{\bar{G}}(R))$. If $C_L(R)$ is generated by elements g with $g^2 \in O_2(N_L(R))$, then we either get (i), or $\langle \bar{A}^{N_L(R)} \rangle$ is abelian.

If even $\bar{A} \leq R$, then $\bar{A} \leq \tilde{L} \leq L$, with $\tilde{L} \cong L_2(r)$ or $Sz(r)$ and $S \cap \tilde{L}$ is a Sylow 2–subgroup of $\tilde{L}$.

Now we just have to handle rank 1 groups or groups L in which $N_L(R)$ contains a normal elementary abelian subgroup different from R, in particular $N_L(R)$ does not act irreducibly on $O_2(N_L(R))/R$. Application of Lemma 2.17 shows $L/Z(L) \cong L_n(r)$, $Sp_{2n}(r)'$, $F_4(r)$ or ${}^2F_4(r)$.

Suppose L is a rank 1 group. We have that $\bar{A} \leq O_2(B\bar{A})$ for some Borel subgroup B of L. Hence we have that $\bar{A} \leq L$. Then as $|\bar{A}| \geq 4$, by Lemma 3.19 we get some $g \in L$ such that for $X = \langle A, A^g \rangle$. We have $X/O_2(X) \cong L_2(q)$ or $Sz(q)$ and a Sylow 2-subgroup of X is contained in $\bar{S}$ and we are done. In particular from now on we may assume that $\bar{A} \not\leq R$.

Now assume that $L/Z(L) \cong L_n(r)$, $n \geq 3$. Let P_1, P_{n-1} be the two parabolic subgroups of $L\bar{A}$ containing $\bar{S} \cap L$ which involve $L_{n-1}(r)$. We have that $\bar{A} \leq O_2(P_i)$ for both i. So we have $\bar{A} \leq O_2(P_1) \cap O_2(P_{n-1}) = R$, a contradiction.

Next let $L/Z(L) \cong Sp_{2n}(r)'$, $n \geq 2$. Now $C_L(R)$ is generated by elements g with $g^2 \in O_2(C_L(R))$. By $(**)$ we have $\bar{A} \leq Z(O_2(N_{\bar{G}}(R)))$. We now may embed $\bar{A}$ into some $\tilde{L} \cong Sp_4(r)'$ with $S \cap \tilde{L}$ a Sylow 2–subgroup of $\tilde{L}$. Hence we may assume $L \cong Sp_4(r)'$. We apply Lemma 2.21. So we have two parabolics P_1, P_2 of $L\bar{A}$ containing $\bar{S} \cap L$. By $(**)$ we have $\bar{A} \leq O_2(P_1) \cap O_2(P_2)$. As $\bar{A}$ is not contained in a root subgroup we see that $\langle \bar{A}^{P_i} \rangle = O_2(P_i)$ for $i = 1, 2$. Even in case of $r = 2$ this is true as $|\bar{A}| > 2$. Let H_i be the preimage of P_i, i.e. $H_i/O_2(G) = P_i$. Now suppose that $\langle A^{O^{2'}(H_1)} \rangle$ is not abelian. Then there is some conjugate A^h, $h \in O^{2'}(H_1)$, with $1 \neq [A, A^h] \leq A \cap A^h$. As $O^{2'}(H_1)$ is generated by elements h with $h^2 \in N_{H_1}(A)$, we may even choose h such that $A^{h^2} = A$, so (i) holds. Hence we may suppose that $\langle A^{O^{2'}(H_i)} \rangle$ is abelian for both $i = 1, 2$. Then we see that $O_2(H_i) \leq C_S(A)O_2(G)$. As this is true for both i, we get $S \cap L = C_S(A)O_2(G)/O_2(G)$. As A acts quadratically on $O_2(G)$ we may apply Lemma 3.6. Suppose there is a chief factor V in $O_2(G)$ which is the natural module. We have $|[V, \bar{A}]| = r^2$, while $|C_V(S \cap L)| = r$. As $[V, \bar{A}]$ is covered by A this is a contradiction. So we have that $Z(L)$ is nontrivial and acts faithfully on V. This gives $q = 2$. By Lemma 3.7 we must have $L \cong 3 \cdot A_6$ and the 6–dimensional module is involved in $O_2(G)$. Then by quadratic action we get $\bar{A} \leq L$.

As $\bar{A} \leq O_2(P_1) \cap O_2(P_2)$ and $P_i \cap L \cong \Sigma_4$, this implies $|\bar{A}| = 2$, a contradiction.

Next let $L \cong F_4(r)$. We have two root groups R_1 and R_2 in $Z(\bar{S} \cap L)$ and by $(**)$ $\bar{A} \leq Z(O_2(N_L(R_1))) \cap Z(O_2(N_L(R_2)))$. But this group is contained in some $Sp_4(r)$ as can be seen in [Shi, (1.5), Proposition 2.2 and Theorem 2.1] and we get the assertion by induction.

Next let $L \cong {}^2F_4(r)'$. As $\bar{A}$ acts quadratically we get by Lemma 3.6 that $\bar{A} \leq R$, a contradiction.

Now let $L \cong A_n$, $n \geq 5$. So we may assume $n = 7$ or $n \geq 9$. We have $L\bar{A} \leq \Sigma_n$. If n is odd, then there is $\tilde{L} \leq L$, $\tilde{L} \cong A_{n-1}$, which is normalized by $\bar{S}$. Hence we may assume n to be even right from the beginning. So $n \geq 10$. Let first $n = 2^m$. Then there is a subgroup $\tilde{L} \leq L$ normalized by $\bar{A}$ with $\bar{S} \cap L \leq \tilde{L}$ and $\tilde{L}$ is a subgroup of index at most two in $\Sigma_{\frac{n}{2}} \wr \mathbb{Z}_2$. As $n \geq 16$ we have $O_2(\tilde{L}) = 1$ and so we get a contradiction with $(**)$. Let $m_1, \ldots, m_r$ be the dyadic decomposition of n. Let $\tilde{L}$ be the subgroup of L with $S \cap L \leq \tilde{L} = L \cap \Sigma_{m_1} \times \cdots \times \Sigma_{m_r}$. By $(**)$ $\bar{A}$ centralizes any component X_1 of $\tilde{L}$. So as $|\bar{A}| > 2$ by $(*)$ and $\bar{A}$ acts nontrivially on $\tilde{L}$, we see that $\bar{A} \leq \Sigma_4 \times \mathbb{Z}_2$. Now we can embed $\bar{A}$ into some $X_2 \cong \Sigma_6$ or Σ_5, which contradicts $(**)$.

Finally let L be sporadic. By Lemma 3.7 we get that $L/Z(L) \cong M_{12}$, M_{22}, M_{24}, J_2, Co_1, Co_2, or Suz, recall that by $(*)$ $|\bar{A}| > 2$. Now we choose $s \in Z(\bar{S} \cap L \cap \bar{A})$. By $(**)$ we have $\bar{A} \leq O_2(C_{\bar{G}}(s))$. If there is some involution g in $C_L(s)$ with $[\bar{A}, \bar{A}^g] \neq 1$, we have (i). So we may assume that $\langle \bar{A}^{C_L(s)} \rangle$ is abelian. This gives $L/Z(L) \cong M_i$, $i = 12, 22, 24$. If $L \cong M_{24}$ there is a subgroup $\tilde{L} \leq L$ with $S \cap L \leq \tilde{L}$ and $\tilde{L} \cong 2^4A_8$. Now by $(**)$ we have $\bar{A} \leq O_2(\tilde{L})$. But there is no quadratic foursgroup in $O_2(\tilde{L})$ according to [MeiStr2].

Next let $L/Z(L) \cong M_{22}$. Then $\bar{A}$ normalizes a subgroup P of $\bar{G}/Z(L)$ with $2^4A_6 \leq P \leq 2^4\Sigma_6$. By $(**)$ we have that $\bar{A} \leq O_2(P)$. Hence we may embed $\bar{A}$ into a subgroup $(S)L_3(4)$. But then $(**)$ gives a contradiction.

So we are left with $L \cong M_{12}$. If $\bar{A} \not\leq L$, then with [MeiStr2] we see that $\bar{A}$ cannot be normalized by $S \cap L$, so we have $\bar{A} \leq L$. Now in L there are two parabolics P_1, P_2 such that $P_i/O_2(P_i) \cong \Sigma_3$. By $(**)$ we have that $\bar{A} \leq O_2(P_1) \cap O_2(P_2)$ and so $\langle \bar{A}^{C_L(s)} \rangle$ is elementary abelian of order 8. Then this group contains an involution i which acts fixed point freely on the 12 points moved by L. So $C_L(i) \cong \mathbb{Z}_2 \times \Sigma_5$. Further S contains a Sylow 2–subgroup of $C_L(i)$. As $\bar{A} \leq C_L(i)$, we get a contradiction by $(**)$. $\square$

Lemma 3.21. *Suppose M and H are $\mathcal{K}_2$ – groups with $F^*(M) = O_2(M)$ and $F^*(H) = O_2(H)$, which are subgroups of some group X. Assume further that M contains a Sylow 2–subgroup S of H and $O_2(M) \leq H$. Finally we assume that there is $Z \trianglelefteq M$, $Z \leq \Omega_1(Z(O_2(M)))$ and $Z \not\leq O_2(H)$. Then one of the following holds.*

(1) *There is some $g \in H$, $g^2 \in N_H(Z)$ with $Z^g \leq S \leq M$, $Z \leq M^g$. Further $1 \neq |Z : C_Z(Z^g)| = |Z^g : C_{Z^g}(Z)|$. In particular Z is an F–module.*

(2) *There is some $g \in H$ such that for $L = \langle Z, Z^g\rangle$ we have*
   (i) *$L/O_2(L) \cong L_2(q)$, $Sz(q)$, q even, or D_{2u}, a dihedral group of order $2u$, u odd. Set $q = 2$ in the latter.*
   (ii) *Set $B = Z^g \cap O_2(L) \leq S \leq M$. Then*
      (α) *For the action of B on Z we have $[Z, B, B, B] = 1$. If $x \in Z \backslash O_2(L)$, then $C_B(x) = B \cap Z$, $[x, B](Z \cap Z^g) = [Z, B](Z \cap Z^g)$ and $|Z : C_Z(B)| \leq q|B/(B \cap Z)|$.*
      (β) *In particular Z is a $2F$–module with offender $B/(B \cap Z)$ and an $F{+}1$–module in case of $q = 2$. In all cases we have $|Z : C_Z(B)| < |B/(B \cap Z)|^2$. Moreover if B acts quadratically on Z, then Z is an F–module.*

Proof. Up to the last assertion that $|Z : C_Z(B)| < |B/(B \cap Z)|^2$, we find everything for (1) and (2) in Lemma 3.20 where $G = H$ and $A = Z$.

So assume $|Z : C_Z(B)| = |B/(B \cap Z)|^2$. Then $|(Z \cap O_2(L))(Z^g \cap O_2(L))/Z \cap Z^g| = q^2$. Hence we have that $L/O_2(L) \cong L_2(q)$ or L induces Σ_3 on $(Z \cap O_2(L))(Z^g \cap O_2(L))/Z \cap Z^g$. In both cases L acts transitively on $((Z \cap O_2(L))(Z^g \cap O_2(L))/Z \cap Z^g)^\sharp$ and so $(Z \cap O_2(L))(Z^g \cap O_2(L))$ is abelian. But then $|Z : C_Z(B)| = |B/(B \cap Z)|$, a contradiction.

If B acts quadratically we have that $[B, Z \cap O_2(L)] = 1$. If $L/O_2(L)$ is dihedral, we get that B induces transvections. In the other case we see by Lemma 3.20(ii)(6) that $|B : B \cap O_2(H)| \geq q$. Then Z is an F–module with offender B. $\square$

The last lemma of this chapter is a generalization of Lemma 3.5 to $2F$-modules.

Lemma 3.22. *Let the notation be as in Lemma* 3.21*. Suppose we have the situation of Lemma* 3.21*(*2*). Set $\bar{B} = B/C_B(Z)$ and suppose there is a component K of $M/C_M(Z)$ with $[K, \bar{B}] \not\leq K$. Then $|\bar{B}| > 4$ and $K \cong L_n(2)$ for some n. If $a \in \bar{B}$ with $K^a \neq K$, then $|[Z, a]| = 2^n$ and $\bar{B}$ induces the full transvection group on $[Z, a]$. In particular $|Z^g : B| = 2$. Further $KK^a = K^{\bar{B}}$ and $\bar{B}$ acts faithfully on KK^a.*

Proof. First we show

$$(*) \qquad |\bar{B}| > q.$$

For this assume $|\bar{B}| \leq q$. Then $|(O_2(L) \cap Z)(O_2(L) \cap Z^g)/Z \cap Z^g| \leq q^2$. In particular $\langle Z, Z^g \rangle$ induces $L_2(q)$ on this group, which gives that all elements in the factor group are conjugate. As $(O_2(L) \cap Z)(O_2(L) \cap Z^g)$ is generated by involutions, we get that this group is abelian. Furthermore $|\bar{B}| = q$ and so $\bar{B}$ is a quadratic F-module offender on Z. By Lemma 3.5 we get the contradiction that $\bar{B}$ has to normalize K. This proves $(*)$.

For $b \in \bar{B}$ set $K_b = K$ if $[K, b] = 1$. If $K^b \neq K$ set $K_b = C_{K \times K^b}(b)$. Recall that there is always some K_b as $K = K_b$ for $b = 1$. Hence this notation makes sense.

Suppose first $q > 2$. By Lemma 3.20 we know that $Y := (Z \cap O_2(L))(Z^g \cap O_2(L))/(Z \cap Z^g)$ is a direct sum of natural modules. So let $A_1 \leq Z^g$ such that $A_1 \geq Z \cap Z^g$, $|A_1 : Z \cap Z^g| = q$ and $A_1/Z \cap Z^g$ is contained in one of these modules V_1, say. We have $[Z, A_1, A_1] \leq Z \cap Z^g$.

Let $Z \cap Z^g \leq V_2 \leq O_2(L)$ with $V_2/Z \cap Z^g = V_1$. Let R be any hyperplane in $Z \cap Z^g$. As $|(Z \cap V_2)/R|^2 = 2|V_2/R|$, we have the assumptions of Lemma 3.18, and so V_2/R is abelian. Hence as $[A_1, Z] \leq V_2$, $[Z, A_1, A_1] \leq R$. As this is true for any hyperplane, we have that A_1 acts quadratically on Z. Note that $|\bar{A}_1| = q > 2$, so by Lemma 3.5 we have three possibilities

(1) $[K, \bar{A}_1] \leq K$.
(2) $|\bar{A}_1 : C_{\bar{A}_1}(K)| > 2$, $[K, \bar{A}_1] \not\leq K$ and $K \cong L_2(2^n)$. Further $[Z, \langle K^{\bar{A}_1} \rangle]$ is a direct sum of natural $\Omega_4^+(2^n)$-modules.
(3) $|\bar{A}_1 : C_{\bar{A}_1}(K)| = 2$ and $[K, \bar{A}_1] \not\leq K$.

We first show

(4) $[K_b, \bar{A}_1] \leq K_b$ for all $b \in \bar{B}$. In particular, taking $b = 1$, we have that K is normalized by $\bar{A}_1$.

This will be done in several steps. We fix notation such that $[K_b, \bar{A}_1] \not\leq K_b$ for a certain b. In particular $[K_b, \bar{B}] \not\leq K_b$.

(4.1) $[Z, K_b] \not\leq O_2(L)$.

By way of contradiction assume that $[Z, K_b] \leq O_2(L) \cap Z$. Then $\bar{B}$ acts quadratically on $[Z, \langle K_b^{\bar{B}} \rangle]$. Hence we may apply Lemma 3.5 to K_b and $\bar{B}$. Assume first $|\bar{B} : C_{\bar{B}}(K_b)| = 2$. As $q > 2$, we have $C_{\bar{B}}(K_b) \neq 1$ and $|Z : Z \cap O_2(L)| \geq 4$. Then by Lemma 3.21(2) we see that $Z \cap O_2(L) = [Z, B](Z \cap Z^g) = [Z, C_{\bar{B}}(K_b)](Z \cap Z^g)$. We have that K_b acts on $[Z, C_{\bar{B}}(K_b)]$. By quadratic action we have that $[[Z, K_b], C_{\bar{B}}(K_b), \bar{B}] = 1$. As $K_b \leq \langle \bar{B}^{K_b} \rangle$, we get $[[Z, K_b], C_{\bar{B}}(K_b), K_b] = 1$. Obviously we have $[C_{\bar{B}}(K_b), K_b, [Z, K_b]] = 1$. So by the Three-Subgroups-Lemma we obtain $[[K_b, Z, K_b], C_{\bar{B}}(K_b)] = 1$ and then also $[Z, K_b, C_{\bar{B}}(K_b)] = 1$, which again with the Three-Subgroups-Lemma implies $[Z, C_{\bar{B}}(K_b), K_b] = 1$. As $[B, O_2(L) \cap Z] = [Z, C_{\bar{B}}(K_b)]$, we get $[Z \cap O_2(L), K_b] = 1$. Now $[Z, K_b, K_b] = 1$ and so $[Z, K_b] = 1$, a contradiction.

So we have $\langle K_b^{\bar{B}} \rangle \cong \Omega_4^+(2^n)$. As $[Z, K_b] \leq O_2(L)$ by assumption, we get by Lemma 3.5 and Lemma 3.11 that $Z = [Z, K_b]C_Z(K_b)$. Hence there is some $y \in C_Z(K_b) \setminus O_2(L)$. For this y we see $[y, B](Z \cap B) = Z \cap O_2(L)$. Then $[Z, K_b, B] \leq [Z \cap O_2(L), B] = [y, B, B]$. But as $[y, K_b] = 1$, also $[y, B, K_b] = 1$ Hence $[Z, K_b, B, K_b] = 1$, a contradiction as $[Z, K_b]$ contains natural $\langle K_b^{\bar{B}} \rangle$-modules. So we have shown (4.1).

(4.2) $C_{\bar{A}_1}(K_b) = 1$.

Assume there is $1 \neq a \in C_{\bar{A}_1}(K_b)$. Then $\langle K_b^{\bar{A}_1} \rangle$ acts on $[Z, a]$. By quadratic action of $\bar{A}_1$ we have $[Z, a, K_b] = 1$. By the Three-Subgroups-Lemma we get that $[Z, K_b]$ is centralized by a and so by Lemma 3.21 as $C_Z(a) \leq O_2(L)$, $[Z, K_b] \leq O_2(L)$, a contradiction to (4.1). This proves (4.2).

Now as $|\bar{A}_1| > 2$ we get with (4.2) that we have (2), so $\langle K_b^{\bar{A}_1} \rangle \cong \Omega_4^+(2^n)$. In particular by Lemma 3.5 $[Z, \langle K_b^{\bar{A}_1} \rangle]$ is a direct sum of natural modules for $\Omega_4^+(2^n)$. Let W be the sum of all such modules which are in $O_2(L)$. Then W is a $\langle K_b, \bar{A}_1 \rangle$–module. As $[Z, K_b] \not\leq O_2(L)$ by (4.1) there is some module V for $\langle K_b, \bar{A}_1 \rangle$ in Z such that $V \not\leq O_2(L)$ and V/W is the natural $\Omega_4^+(2^n)$–module. Choose $y \in V \setminus O_2(L)$. We have $[y, A_1](Z \cap Z^g) = [Z, A_1](Z \cap Z^g)$. As $|[V/W, \bar{A}_1]| > |\bar{A}_1|$ by Lemma 3.5, we see that $V \cap B \not\leq W$.

(4.3) $b = 1$ and $C_{\bar{B}}(K_b) = 1$. In particular $K_b = K$.

Suppose there is some $1 \neq a \in C_{\bar{B}}(K_b)$. Then $[a, V \cap B] = 1$. Hence a centralizes some element in $V \setminus W$. As a normalizes $O_2(L)$ and $\langle K_b, \bar{A}_1 \rangle$, we see that a normalizes also $\langle K_b, \bar{A}_1\rangle$-submodules of V, which are in $O_2(L)$. Hence a normalizes W. So we get $[V,a] \leq V$. But now $[V/W,a] < V/W$, and so $[V,a] \leq W$. As $[W,a] \leq Z \cap Z^g$, $[W,a]$ is a sum of natural modules for $\langle K_b^{\bar{A}_1}\rangle$ and $[Z\cap Z^g, A_1] = 1$, we see that $[W,a] = 1$. As $V \not\leq O_2(L)$ and so $V/C_V(a) \cong [V,a]$, we have that $[V,a]$ is a natural module for $\langle K_b^{\bar{A}_1}\rangle$. This gives that $|[V,a] : [[V,a],A_1]| = 2^{2n}$. As $[[V,a],A_1] = [V,a] \cap Z^g$, we see that $|V : V \cap O_2(L)| = 2^{2n}$. In particular $q \geq 2^{2n}$. But as $C_{\bar{A}_1}(K_b) = 1$, we have that $q = |\bar{A}_1| \leq 2^{n+1}$. As $n \geq 2$, this is a contradiction. This proves (4.3).

(4.4) $\langle K^{\bar{B}}\rangle = \langle K^{\bar{A}_1}\rangle$.

Let $B_1 \leq Z$, $Z \cap Z^g \leq B_1$ such that B_1 covers another natural module in $Y/Z\cap Z^g$. Let $b \in \bar{B}_1$. If $K^b \neq K$, then $\bar{A}_1$ normalizes K_b by (4.3) and so $K_b \leq \langle K^{\bar{A}_1}\rangle$. Hence $\bar{B}_1$ normalizes $\langle K^{\bar{A}_1}\rangle$. As $\bar{B}$ is generated by such groups, we get (4.4).

Define W and V as above. If $W \neq 1$, then $|W/C_W(A_1)| \geq 2^{2n}$. So $|Y : C_Y(Z^g \cap Y)| \geq 2^{2n}$ and then also $|Y : C_Y(Z \cap Y)| \geq 2^{2n}$, hence $|\bar{B}| \geq 2^{2n}$. As $C_{\bar{B}}(K) = 1$ by (4.3) we have $|\bar{B}| \leq 2^{n+1}$ and $n > 1$, which is not possible. So we have that $W = 1$. Hence V is the natural $\Omega_4^+(2^n)$–module. Let $a \in \bar{A}_1$ such that $K^a \neq K$. Then $C_{K\times K^a}(a) = K_a \cong K$. Further a Sylow 2–subgroup of K_a together with a acts quadratically on V. As $\bar{A}_1$ acts quadratically we have that $\bar{A}_1$ projects onto $K_a \times \langle a\rangle$. So we have that $\bar{B}$ centralizes a and then acts on K_a. As $C_{\bar{B}}(K) = 1$ by (4.3), we see that $\bar{B}$ contains a subgroup $\tilde{B}$ of index two containing $\bar{A}_1$, which acts quadratically on V. As $V \not\leq O_2(L)$, we have that $|Y : \tilde{B}[\tilde{B},V](Z\cap Z^g)| \leq 4$. As $[[\tilde{B},V]\tilde{B},\tilde{B}] = 1$, we see that $\tilde{B}$ centralizes a subgroup of index at most $2q$ in V. Now as V is not an F–module for $\tilde{B}$ by Lemma 3.5, we get $|\tilde{B}| \leq q$, which gives $|\bar{B}| \leq 2q$. But as $q > 2$, and $|\bar{B}|$ is a power of q, we would get $\bar{A}_1 = \bar{B}$ and then B acts quadratically on Z, which by Lemma 3.21(2) gives that $\bar{B}$ is an F–module offender on V, contradicting Lemma 3.5. So we have proved (4).

From (4) we now get that $\langle K^{\bar{A}_1}\rangle = K$. As $\bar{B}$ is generated by such subgroups $\bar{A}_1$, we have the contradiction $[K,\bar{B}] \leq K$. This shows

(5) $q = 2$.

(5.1) There is some K_b such that $[K_b,\bar{B}] \not\leq K_b$.

Otherwise, if there is no such K_b then for $b = 1$ we have $K_b = K$ and so $[K,\bar{B}] \leq K$, a contradiction.

For the remainder of the proof we fix K_b such that it satisfies (5.1).

(5.2) $[Z, K_b] \not\leq O_2(L)$.

If $[Z, K_b] \leq O_2(L)$, then again B acts quadratically on $[Z, K_b]$. Hence by Lemma 3.5 we have one of the cases (2) or (3) above with $\bar{A}_1$ replaced by $\bar{B}$. Assume $|\bar{B} : C_{\bar{B}}(K_b)| = 2$. As $|\bar{B}| > q = 2$ by $(*)$ we can choose $1 \neq a \in C_{\bar{B}}(K_b)$. Then $\langle K_b^{\bar{B}} \rangle$ acts on $[Z, a]$. As $|Z : Z \cap O_2(L)| = 2$, we have that $|[Z, a] : [Z, a] \cap Z^g| = 2$. Therefore $|[Z, a] : C_{[Z,a]}(B)| \leq 2$. If $\bar{B}$ does not centralize $[Z, a]$, then $\bar{B}$ induces transvections on $[Z, a]$. As $\bar{B}$ does not normalize K_b this is impossible by Lemma 3.5. Hence $\bar{B}$ centralizes $[Z, a]$ and so $[[Z, a], K_b] = 1$ for all $a \in C_{\bar{B}}(K_b)$. We have that $[Z, C_{\bar{B}}(K_b)](Z \cap Z^g) \cap [Z, K_b]$ is a subgroup of index at most four in $[Z, K_b]$. So $\bar{B}$ centralizes a subgroup of index two in $[Z, K_b]/[Z, K_b] \cap [Z, C_{\bar{B}}(K_b)]$, which gives $[K_b, Z] \leq [Z, C_{\bar{B}}(K_b)]$ and then $[Z, K_b] = 1$, a contradiction.

Hence we are in case (2), i.e. $|\bar{B} : C_{\bar{B}}(K_b)| > 2$. As before by Lemma 3.5 and Lemma 3.11 there is some $y \in C_Z(K_b) \backslash O_2(L)$. This shows $[y, B](Z \cap B) = Z \cap O_2(L)$. Now $[Z, K_b, B] \leq [Z \cap O_2(L), B] = [y, B, B]$. But as $[y, K_b] = 1$, also $[y, B, K_b] = 1$. In particular $[Z, K_b, B, K_b] = 1$, a contradiction. So we have (5.2).

Fix $a \in \bar{B}$ with $K_b^a \neq K_b$.

(5.3) $[Z, a, \bar{B}] \neq 1$.

Assume $[Z, a, \bar{B}] = 1$. Then by Lemma 3.5 either $|\bar{B} : C_{\bar{B}}(K_b)| = 2$, or $K_b \cong L_2(r)$ and $[Z, K_b]$ is a direct sum of orthogonal $\Omega_4^+(r)$–modules, $r = 2^n$.

Suppose the latter. As before let W be the sum of all natural modules in $[Z, K_b]$, which are contained in $O_2(L)$ and V/W be a natural $\Omega_4^+(r)$–module. Then there is $y \in V \setminus O_2(L)$ and $[B, y](B \cap Z) = [Z, B](B \cap Z)$. In particular as $|V : V \cap O_2(L)| = 2$, we see that $V \cap B \not\leq W$. This shows that B normalizes V. Now let $c \in C_{\bar{B}}(K_b)$. Then we have that $[V, c] \leq W$. As $[W, c] \leq Z \cap Z^g$ and $[B, Z \cap Z^g] = 1$, we get that $[W, c] = 1$ or $[W, c]$ is the natural module. But $[\bar{B}, [W, c]] = 1$ and so $[K_b, [W, c]] = 1$, hence $[W, c] = 1$. If $c \neq 1$, then $[V, c]$ is the natural module. But we have that $|[V, c] : [V, c] \cap Z^g| = 2$ and so B induces transvections on $[V, c]$, a contradiction. So we have that $C_{\bar{B}}(K_b) = 1$, i.e. $b = 1$ and $K_b = K$. Assume $W \neq 1$. In the natural module the centralizer of a quadratic fours group is just the commutator of this fours group. Hence we have that $C_W(B) = W \cap Z^g$. So $|W : W \cap Z^g| = |W \cap Z^g| = |W \cap Z|$ and then $|\bar{B}| \geq |W/W \cap Z| \geq r^2$. As the largest quadratic group in $O_4^+(r)$ is of order $2r$ we have $|\bar{B}| \leq 2r$, a contradiction. This implies $W = 1$. So we have V is the natural module and then B acts quadratically on V. But $[y, B](Z \cap Z^g) = Z \cap O_2(L)$. As $[y, B] \leq [V, B]$, $[B, Z \cap O_2(L)] = 1$, and so B induces transvections on Z, a contradiction as $\bar{B}$ does not normalizes K.

So we have $|\bar{B} : C_{\bar{B}}(K_b)| = 2$. As $[Z, a, B] = 1 = [Z, B, a]$ by the Three-Subgroups-Lemma, we see that $[Z, C_{\bar{B}}(K_b), K_b] = 1$. By the Three-Subgroups-Lemma again we get $[K_b, Z, C_{\bar{B}}(K_b)] = 1$. But as $[Z, K_b] \not\leq O_2(L)$ by (5.2) this shows $C_{\bar{B}}(K_b) = 1$ and then $|B : B \cap Z| = 2$. Now B induces transvections on Z and so by Lemma 3.5 B has to normalize K_b, a contradiction. This proves (5.3).

(5.4) We have that $C_{\bar{B}}(K_b) = 1$ and then $b = 1$ and $K_b = K$.

By (5.3) $|\bar{B}| \geq 4$. As $|[Z, a] : [Z, a] \cap Z^g| = 2$, $\bar{B}$ induces transvections on $[Z, a]$ to a hyperplane. Choose $1 \neq c \in C_{\bar{B}}(K_b)$ and assume that $[Z, c, K_b] = 1$. Then also $[Z, K_b, c] = 1$ and so $[Z, K_b] \leq O_2(L)$, a contradiction. Hence K_b acts nontrivially on $[Z, c]$. But a induces a transvection on $[Z, c]$, a contradiction as $K_b^a \neq K_b$. This proves (5.4). In particular we get

(5.5) If $b \neq 1$, then $[K_b, \bar{B}] \leq K_b$.

Let $b \in \bar{B}$ with $(KK^a)^b \neq KK^a$. Then a does not normalize K_b, a contradiction to (5.5). So we have that $KK^a = \langle K^{\bar{B}} \rangle$. As $[Z, a, \bar{B}] \neq 1$ by (5.3), we see that $K_a = C_{K \times K^a}(a) \cong K$ acts faithfully on $[Z, a]$, and so, as $\bar{B}$ induces transvections to a hyperplane, we get by Lemma 3.3 that $K \cong L_n(2)$, $Sp_{2n}(2)$, $\Omega_{2n}^{\pm}(2)$ or A_n. We further have that $C_{\bar{B}}(K_a) = \langle a \rangle$ as $C_{\bar{B}}(K) = 1$ by (5.4).

(5.6) $|\bar{B}| > 4$.

Assume $|\bar{B}| \leq 4$. Then $|[Z, a]| \leq 4$, but K_a has to act nontrivially on $[Z, a]$, a contradiction.

By (5.6) $|\bar{B}| > 4$ and $\bar{B}$ induces at least a fours group of transvections on $[Z, a]$. This gives

(5.7) $K \cong L_n(2)$.

It remains to prove that $[Z, a]$ is the natural module. In fact we know that $[Z, a]/C_{[Z,a]}(K_a)$ is the natural module. We have that $|\bar{B}| \leq 2^n$. Then as $[Z, a, a] = 1$ and $|Z : Z \cap O_2(L)| = 2$ we see that $|[Z, a] \leq |\bar{B}/\langle a \rangle| = |\bar{B}| \leq 2^n$. This shows that $|[Z, a]| = 2^n$ and $\bar{B}$ induces the full transvection group on $[Z, a]$. $\square$

4. Examples

In this chapter we show under which circumstances the examples $M(23)$, Co_3, $\Omega_7(3)$, $\Omega_8^-(3)$ and J_1 in the main theorem appear. The group A_{12} already appeared in [MaStr, Theorem 1.4].

Lemma 4.1. [MaStr, Lemma 4.15] *Let G be a group of even type, which is not of even characteristic. If G has standard subgroup with $L \cong 2M(22)$, then $G \cong M(23)$.*

Lemma 4.2. [Se] *Let G be a group of even type, which is not of even characteristic. Let furthermore $L \in \mathcal{L}$ be a standard subgroup with $L \cong 2Sp_6(2)$. If $C_G(L)$ has cyclic Sylow 2–subgroups, then $G \cong Co_3$.*

Lemma 4.3. *Let G be a group of even type, which is not of even characteristic. Let furthermore L be a standard subgroup of G. Assume that the following hold:*

(1) *$L \cong L_4(3)$, $U_4(3)$ or $2U_4(3)$ and $C_G(L)$ is a cyclic 2-group.*
(2) *$N_G(L)$ contains a Sylow 2-subgroup S of G.*

Then $G \cong \Omega_7(3)$ or $\Omega_8^-(3)$.

Proof. Suppose false. We have that $C_G(L)$ is normal in S, S as in (2), and so contains a 2-central involution z. By Lemma 2.6 we have that for an involution t in $L \setminus Z(L)$ we get $O_2(C_{L\langle z\rangle}(t)) \cong \mathbb{Z}_2 \times Q_8 * Q_8$. Now we choose t such that $t \in O_2(C_L(t))'$. Again by Lemma 2.6 we see that $O_3(N_L(O_2(C_L(t)))/O_2(C_L(t))))$ is elementary abelian of order 9. Let U be the full preimage. Then $[U, O_2(C_L(t))] = Q \cong Q_8 * Q_8$. In particular $Q \trianglelefteq C_{C_G(z)}(t)$ and so we may assume that $[S,t] = 1$, i.e. $t \in Z(S)$.

We have that S centralizes $\langle z, t\rangle$ and so normalizes U. Now the Frattini argument provides us with a Sylow 3–subgroup U_1 of U such that

$$(*) \qquad S = QN_S(U_1).$$

Next we try to determine $O_2(C_G(t))$. For this we assume that $C_G(t) \not\leq N_G(L)$. Furthermore we first assume that $C_{C_G(t)}(O_2(C_G(t))) \leq O_2(C_G(t))$. Suppose additionally that there is some $1 \neq u \in U_1$, with $[u, Q \cap O_2(C_G(t))] = 1$. We have that $O_2(C_G(t)) \leq S$, so $[U_1Q, O_2(C_G(t))] \leq O_2(C_G(t)) \cap U_1Q \leq Q$. Hence $[u, O_2(C_G(t))] \leq Q$ and we get

$$[O_2(C_G(t)), u] = [O_2(C_G(t)), u, u] \leq [Q \cap O_2(C_G(t)), u] = 1,$$

contradicting $C_{C_G(t)}(O_2(C_G(t))) \leq O_2(C_G(t))$.

This shows that U_1 acts faithfully on $Q \cap O_2(C_G(t))$. Then $Q \leq O_2(C_G(t))$. By Lemma 2.6 we have $|C_{L\langle z\rangle}(Q)| = 4$. Furthermore we have that $\mathrm{Out}(L)$ does not contain an elementary abelian group of order 8 by Lemma 2.6. Hence we see that

$$|\Omega_1(Z(O_2(C_G(t))))| \leq 16.$$

We set $Z = \Omega_1(Z(O_2(C_G(t))))$. If $\langle z, t\rangle = Z$, we get $N_G(\langle z, t\rangle) = C_G(\langle z, t\rangle)$ and then the contradiction $C_G(t) \leq C_G(z) \leq N_G(L)$. So we conclude that $|Z| \geq 8$. Then $[Z, U_1] \leq Z$. As $Z \leq S$ we see $[Z, U_1] \leq Q$ and so $[Z, U_1] \leq Q \cap Z$. As $[Q \cap Z, U_1] = 1$, we have $[Z, U_1] = 1$. Furthermore $[O_2(C_G(t)), U_1] \leq Q$. As $C_G(Z) \leq N_G(L)$, we now see that $U_1O_2(C_G(t))/O_2(C_G(t)) \trianglelefteq C_G(t)/O_2(C_G(t))$. Hence

$$U_1O_2(C_G(t)) \trianglelefteq C_G(t).$$

Let $U_2 \leq U_1$, $|U_2| = 3$, with $[U_2, Q] \cong Q_8$. Then $\langle [Q, U_2], U_2\rangle = X \cong SL_2(3)$. Further $t \in Z(X)$. As $[O_2(C_G(t)), U_2] = [U_2, Q]$, we see $X \trianglelefteq U_1O_2(C_G(t)) \trianglelefteq C_G(t)$ and so $X \trianglelefteq \trianglelefteq C_G(t)$ and for $g \in C_G(t)$ we have either $X^g \cap X = \langle t\rangle$ or $X = X^g$. The assertion now follows with [MaStr, Lemma 3.2].

So we may assume that $C_G(O_2(C_G(t))) \not\leq O_2(C_G(t))$. Then

$$F = E(C_G(t)) \neq 1$$

as $O(C_G(t)) = 1$ by the general assumption. Set $T = S \cap F$. Assume there is $1 \neq u \in U_1$ with $[F \cap Q, u] = 1$. Then $Q \neq F \cap Q$ and $F \cap Q$ is normal in $C_L(t)$. By Lemma 2.30 $T \cap Q \leq \langle t\rangle$. We also have $[T, Q] \leq T \cap Q \leq \langle t\rangle$. So $[U_1, S] \leq U_1Q$ and then $[T, U_1] \leq U_1(Q \cap T) \leq U_1\langle t\rangle$. This shows $[T, U_1] = [T, U_1, U_1] \leq [U_1, U_1\langle t\rangle] = 1$. Now $T \leq C_G(\langle U_1^S\rangle) \leq C_G(Q)$ and then again $T/T \cap \langle t\rangle$ has a cyclic normal subgroup $C_L(t) \cap T/\langle t\rangle$ of index at most 4. This shows that F is quasisimple.

Assume first $[C_{C_G(t)}(F), U_1] = 1$. As $C_{C_G(t)}(F)$ is normal in $C_G(t)$, we get that $[Q, C_{C_G(t)}(F)] = 1$. Hence QU_1 induces an outer automorphism group on F, which centralizes a Sylow 2-subgroup, contradicting [GoLyS4, Lemma 4.1.1]. So we have that $[C_{C_G(t)}(F), U_1] \neq 1$. If $[Q, F] \neq 1$ then we get by Lemma 2.30 that $C_Q(F) \leq \langle t\rangle$ and then $Q \cap C_{C_G(t)}(F)F = \langle t\rangle$. But then we have the same contradiction as before. So we have that $[Q, F] = 1$. The Frattini argument now implies that $C_G(t) = FN_{C_G(t)}(T)$. Further we have $(C_S(Q)/\langle t\rangle)' \leq C_S(L)\langle t\rangle/\langle t\rangle$ as $N_S(L)/S \cap L$ is abelian. So if $F\langle t\rangle/\langle t\rangle$ has nonabelian Sylow 2–subgroups, we get that $\langle z, t\rangle \leq F\langle t\rangle$ is centralized by $N_{C_G(t)}(T)$ and so

$$C_G(t) = C_{N_G(L)}(t)F.$$

We are going to prove the same result if $F\langle t\rangle/\langle t\rangle$ has abelian Sylow 2–subgroups. As $F \in \mathcal{C}_2$ we have, that F itself has abelian Sylow 2–subgroups. In particular $t \not\in F$. If $|\Omega_1(Z(S))| = 4$, then again $z \in F\langle t\rangle$ and so $N_{C_G(t)}(T) \leq C_G(z)$, as all involutions in F are conjugate. If $|\Omega_1(Z(S))| > 4$, then by application of Lemma 2.6 we see $L \cong L_4(3)$ and $|\Omega_1(Z(S))| = 8$. By Lemma 2.34 we have that $S \cap C_G(L) \leq Z(S)$ and $S = C_S(L) \times ((S \cap L)\langle u\rangle)$. Hence $C_S(Q) = (S \cap C_G(L))\langle t, u\rangle$. If $C_{C_G(L)}(z) \neq \langle z\rangle$, then $z \in Z(N_G(Z(S)))$ and so $z^G \cap Z(S) = \{z\}$, contradicting Lemma 2.1 and Lemma 2.2. So $Z(S) = \langle z, t, u\rangle = C_S(Q)$. Hence a Sylow 2–subgroup of F is contained in $Z(S)$. Now we have that $N_{C_G(t)}(T) = N_F(T)C_{C_G(t)}(T)$, which gives again

$$(**) \qquad C_G(t) = C_{N_G(L)}(t)F.$$

As $Q \not\leq F$ and $[U_1, S] \leq QU_1$ we have $U_1 \cap F = 1$ and then $C_F(z) = S \cap F$. Hence U_1 cannot induce nontrivial inner automorphisms on F, so $[F, U_1] = 1$. This now gives $[F, U] = 1$. As $C_G(t) = C_{N_G(L)}(t)F$ by $(**)$, we see that $Q \leq O_2(C_{C_G(t)}(F))$ and then U is normal in $C_{C_G(t)}(F)$. Hence as above we construct a subgroup $X \cong SL_2(3)$ in U, with $X \trianglelefteq\trianglelefteq C_G(t)$. Again the assertion follows with [MaStr, Lemma 3.2].

So we may assume

$$Q \leq F.$$

Let first N be a component with $N \cap Q = 1$, then $[S \cap N, Q] \leq S \cap N \cap Q = 1$. As $[S \cap F, Q] \neq 1$, there is at least one component N with $Q \cap N \neq 1$. We now fix such a component N and set $F_1 = \langle N^{U_1}\rangle$. As Q normalizes N we have $F_1 = N_1 * N_2 * \cdots N_x$, where x divides $|U_1| = 9$. If $x = 9$, then, as any N_i has an elementary abelian section of order 4, we have an elementary abelian section of order 2^{18} in F_1, which contradicts the structure of S. Let $x = 3$. As U_1 acts on $S \cap F_1$ and $[S \cap F_1, U_1] \leq [Q, U_1] = Q$, we see by Lemma 2.30 that $Q \leq F_1$. As $x = 3$ there is $1 \neq u \in U_1$, with $[N_i, u] \leq N_i$ for all i. Furthermore we have some element $u_1 \in U_1$, which acts transitively on the N_i and normalizes $S \cap F_1$. As $Q \leq F_1$ we see $[u, N_i \cap S] \neq 1$. As $(S \cap N_i)\langle u_1\rangle$ is a subgroup, we get that $1 \neq [S \cap N_i, u_1] \leq Q$. But then $|\langle (Q \cap N_1)^{\langle u_1\rangle}\rangle| \geq 2^6$, a contradiction. So we have $x = 1$ and then U_1 normalizes $N_1 = N$. Then $(S \cap N)U_1$ is a subgroup of G. As U_1 cannot centralizes all components N with $N \cap Q \neq 1$, we get that there is a component N with $N \cap Q > \langle t\rangle$.

The action of U_1 on Q shows that either $Q \cap N = Q$ or $Q \cap N$ is a quaternion group. Suppose first that $Q \cap N$ is a quaternion group.

Then by Lemma 2.30 there is some $s \in S \cap L$ such that $N^s \neq N$ and $N^s \cap Q$ also is a quaternion group. As $Z(N) \geq \langle t \rangle$ [MaStr, Lemma 2.53] implies that $N \in \mathcal{M}$. In particular the same lemma implies that $|N/Z(N)|_2 \geq 2^6$, and hence $|[s,T]| \geq 2^6$. As $|S \cap L : O_2(C_L(t))| \leq 4$, we now see that $|[T,s] \cap O_2(C_L(t))| \geq 2^5$ and then $|[T,s] \cap Q| \geq 2^4$. Now $Q \cap N$ is a quaternion group, which implies that $[T,s] \cap N > \langle t \rangle$, a contradiction. So we have $Q \leq N$ for some component N. Further $C_{C_G(z)}(t)$ normalizes N as it normalizes Q. Now z induces some automorphism on N, which centralizes a Sylow 2–subgroup and has a solvable centralizer in N of order $2^a \cdot 3^b$, $b \leq 2$. As $N \in \mathcal{M}$ Lemma 2.31 implies that $N \cong 2L_3(4)$, $2^2L_3(4)$, $2Sp_6(2)$, $2U_4(3)$, $2M_{12}$, $2M_{22}$, $4M_{22}$, $2Sz(8)$ or $2^2Sz(8)$. As $Q \leq N$, there are involutions in $N/Z(N)$ which become elements of order 4 in N. So by Lemma 2.33 we are left with $2Sp_6(2)$, $2M_{12}$ or $4M_{22}$. If $N \cong 2M_{12}$ or $4M_{22}$, then by Lemma 2.35 $|U_1|$ induces an inner automorphism of order at most three. On the other hand by the same lemma N has no outer automorphism of order three, so $C_{U_1}(N) \neq 1$, which contradicts $C_{U_1}(Q) = 1$. So we have $N \cong 2Sp_6(2)$. Now U_1 has to induce a group of inner automorphisms of order 9. We have that $[Q, U_1]/\langle t \rangle$ is elementary abelian of order 16. Hence let $\tilde{z}$ be the inner automorphism induced by z, then we see that $\langle [Q, U_1], \tilde{z} \rangle / \langle t \rangle$ is elementary abelian of order 32. By Lemma 2.36 we have that $\tilde{z}$ corresponds to a transvection in $Sp_6(2)$. But then a group isomorphic to Σ_6 would be in $C_G(\langle z, t \rangle)$, a contradiction to $C_G(z) \leq N_G(L)$.

So we have shown that $C_G(t) \leq N_G(L)$. But then $C_G(t)$ has a subnormal subgroup $SL_2(3)$. [MaStr, Lemma 3.2] now yields the assertion. $\square$

Lemma 4.4. *Let G be of even type but not of even characteristic. Let $L \cong L_2(q)$, q even, be a standard subgroup with $C_G(L)$ cyclic. Assume that $C_G(L)$ contains a 2–central involution z. Then $q = 4$ and $G \cong J_1$.*

Proof. Let S be a Sylow 2–subgroup of $N_G(L)$ containing z. In $\Omega_1(Z(S))$ there are three $N_G(L)$–classes of involutions, $\{z\}$, $(\Omega_1(Z(S)) \cap L)^\sharp$ and $z(\Omega_1(Z(S)) \cap L)^\sharp$. Hence either $z^G \cap \Omega_1(Z(S)) = \Omega_1(Z(S))^\sharp$ or $z^G \cap \Omega_1(Z(S)) = \{z\}$. Set $U = L \cap S$. Then there are at most two abelian subgroups of S which have the same order as $E = C_{C_G(L) \cap S}(z) \times U$. In particular conjugacy takes place in $N_G(E)$.

Assume first that $z^G \cap \Omega_1(Z(S)) \neq \{z\}$. Then in particular $C_{C_G(L) \cap S}(z) = \langle z \rangle$. As $\mathrm{Out}(L)$ is cyclic, we have that $z \notin S'$. So we conclude $\Omega_1(Z(S)) \cap S' = 1$ and then $C_G(z) \cong \langle z \rangle \times L_2(q)$. By O'Nan's lemma [MaStr, Lemma 2.6] we obtain $q = 4$ and so $G \cong J_1$ by [Ja].

So we may assume that $z^G \cap \Omega_1(Z(S)) = \{z\}$. As L has just one class of involutions, we have that $z^G \cap (L \times C_{C_G(L)\cap S}(z)) = \{z\}$. By Lemma 2.1 L must possess some outer automorphism u with $u \sim z$ in G. Obviously $u \notin C_S(u)'$. In particular also $z \notin C_S(u)'$. Hence $C_{C_G(L)\cap S}(u) \leq Z(C_S(u))$. As u is not a square in $Z(C_S(u))$, we get that the same holds for z. In particular $C_{C_{C_G(L)\cap S}(z)}(u) = \langle z \rangle$. If $z \notin S'$, then in particular $S \cap C_G(L) = \langle z \rangle$ and we get a contradiction by Lemma 2.2. So we may assume that $z \in S'$. Then $u \sim zu$ by some element in $C_S(L)$. As $C_{S\cap C_G(L)}(u) \leq Z(C_S(u))$, we see that $C_S(u) = \langle u, z, C_U(u) \rangle$. Further $u\langle z, C_U(u) \rangle \subseteq u^G$. Now we may assume that $u \sim z$ in $N_G(C_S(u))$. In particular $C_S(u)$ contains a hyperplane H with $z \notin H$ but $zH \subseteq z^G$. Choose $u_1 \in C_U(u)^\sharp$. Then neither u_1 nor zu_1 are in zH, so both are in H and so $z \in H$, a contradiction. $\square$

5. The central case

In this chapter we fix a Sylow 2–subgroup S of G and assume that G is of even type but not of even characteristic. Furthermore we assume that G is not one of the exceptional groups in the main theorem, i.e.

$$G \not\cong \Omega_7(3), \Omega_8^-(3), A_{12}, Co_3, M(23) \text{ or } J_1.$$

This means by [MaStr, Theorem 1.4] that there is some $1 \neq z \in \Omega_1(Z(S))^\sharp$, which possesses a standard component A_z. Furthermore $C_G(A_z)$ has cyclic Sylow 2–subgroups.

We will prove:

Proposition 5.1. $z \notin A_z$.

and

Proposition 5.2. *A_z is a simple group of Lie type in characteristic two or isomorphic to J_2 or $M(24)'$. Further A_z is not isomorphic to $L_2(q)$, $Sz(q)$, ${}^2F_4(q)'$, q even, $L_3(4)$, $Sp_{2n}(2)$, $G_2(2)'$, $L_4(2)$, $U_4(2)$, A_6 or $L_3(2)$.*

We first are going to prove Proposition 5.1. For this until further notice we assume $z \in A_z$ and aim for a contradiction. By [MaStr, Lemma 2.53] we have that $A_z/Z(A_z) \in \mathcal{M}$. For the proof we consider the various groups in $\mathcal{M}$.

Lemma 5.3. *$A_z/Z(A_z) \not\cong Sz(8)$.*

Proof. Assume $A_z/Z(A_z) \cong Sz(8)$. Let $1 \neq x \in S$, $x^2 = 1$. Then, as $C_S(A_z) \cap A_z = \langle z \rangle$, we see that $x = ab$, $a \in C_S(A_z)$ and $b \in A_z$, where

$a^2, b^2 \in \langle z \rangle$. By Lemma 2.33 z is not a square in $S \cap A_z$. In particular $b^2 = 1$. But then also $a^2 = 1$, which shows that $\Omega_1(S) = \Omega_1(S \cap A_z)$. Hence $\Omega_1(S)$ is elementary abelian of order 16. Furthermore $\Omega_1(S) = J(S)$. So $N_G(J(S))$ controls G-fusion of involutions in S.
If $z^G \cap S \neq \{z\}$, then all involutions in S are conjugate. But then $|N_G(J(S)) : N_{C_G(z)}(J(S))| = 15$, and $N_G(J(S))/C_G(J(S))$ is a subgroup of $GL_4(2) \cong A_8$ of order divisible by $3 \cdot 5 \cdot 7$. As $S/C_S(J(S))$ is abelian, we get a contradiction by Lemma 2.38. So $z^G \cap S = \{z\}$ which contradicts Lemma 2.1. □

Lemma 5.4. [Se] *$A_z/Z(A_z) \not\cong F_4(2)$ or $G_2(4)$.*

Lemma 5.5. [EgaYo] *$A_z/Z(A_z) \not\cong \Omega_8^+(2)$.*

Lemma 5.6. *$A_z/Z(A_z) \not\cong U_6(2)$.*

Proof. [DaSo, Theorem 3.1]. In fact there is shown that $G \cong M(22)$. But then $z \notin Z(S)$. □

Lemma 5.7. *$A_z/Z(A_z) \not\cong {}^2E_6(2)$.*

Proof. [Str1]. In fact in [Str1, (2.2)] it is shown that $z \notin Z(S)$. □

Lemma 5.8. *$A_z/Z(A_z) \not\cong HiS$, M_{12}, M_{22}, J_2, Suz, Co_1 or Ru,*

Proof. Suppose false. Application of [So] shows $A_z/Z(A_z) \not\cong HiS$. In the cases of $A_z/Z(A_z) \cong M_{12}$ or M_{22} we get a contradiction with [HaSo]. The remaining cases are treated in [Fin1] and [Fin2]. □

Lemma 5.9. *$A_z/Z(A_z) \not\cong F_2$.*

Proof. If $A_z \cong 2F_2$ then by [DaSo, (5.5)] we get $z \notin Z(S)$. □

Lemma 5.10. *$A_z/Z(A_z) \not\cong L_3(4)$.*

Proof. Suppose $A_z/Z(A_z) \cong L_3(4)$. As $A_z \in \mathcal{C}_2$ we have by [MaStr, Definition 1.1] that $Z(A_z) = \langle z \rangle$. According to Lemma 2.20 there are exactly two elementary abelian groups of order 16 in $(S \cap A_z)/\langle z \rangle$. Let E be the preimage of such a group. Again by Lemma 2.20 A_5 acts transitively on $(E/\langle z \rangle)^\sharp$. So we see that E is elementary abelian of order 32. Let $C_S(A_z)$ be cyclic of order 2^n, then by Lemma 2.20 there are exactly two abelian subgroups of type $(2, 2, 2, 2, 2^n)$ in $S \cap A_zC(A_z)$. Let F be an elementary abelian group of order 32 in S. Assume there is some $t \in F \setminus A_zC(A_z)$. As $m_2(C_{A_z/\langle z \rangle}(t)) \leq 2$ by Lemma 2.23(3), we get that $|F \cap A_zC(A_z)| \leq 8$. But then F has to induce a fours group of outer automorphisms on $A_z/\langle z \rangle$. Choose $f_1 \in F$ such that f_1 centralizes A_5 in $A_z/\langle z \rangle$. Then F induces an outer automorphism on A_5, which gives the contradiction $m_2(C_{A_z/\langle z \rangle}(F)) \leq 1$. Hence any elementary abelian subgroup of order 32 in S is contained in $A_zC_G(A_z)$. By Lemma 2.20 there are exactly two abelian groups E_1, E_2 of type $(2,2,2,2,2^n)$ contained in S. Set $E_3 = \Omega_1(E_1 \cap E_2)$. As again by Lemma 2.20 E_1E_2 is a Sylow 2–subgroup of $A_zC_G(A_z)$, we have $E_3 = Z(S \cap A_z)$ and $|E_3| = 8$.

Suppose $z^G \cap A_zC(A_z) \neq \{z\}$. Let $t \in A_zC(A_z)$, $t \neq 1$, $t \sim z$ in G. By Lemma 2.20 any involution in $A_zC_G(A_z)$ is conjugate in A_z to some involution in E_i, $i = 1, 2$. On $\Omega_1(E_i)$ we have that $N_{C_G(z)}(E_i)$ induces orbits of length 1, 15 and 15. Hence we see that $z^{N_G(E_i)} \neq \{z\}$. This implies that $C_S(A_z)$ is of order two and so both E_i are elementary abelian. We have that $N_G(E_1) \not\leq C_G(z)$. As $|z^{N_G(E_i)}|$ is odd, this shows that $|N_G(E_1) : N_{C_G(z)}(E_1)| = 31$. Now all involutions in A_z are conjugate in G. As $N_{C_G(z)}(E_1)/E_1 \cong A_5$, $A_5 \times \mathbb{Z}_3$, Σ_5 or $(A_5 \times \mathbb{Z}_3) : 2$, we get that $N_G(E_1)/E_1$ has the order $2^2 \cdot 3 \cdot 5 \cdot 31$, $2^3 \cdot 3 \cdot 5 \cdot 31$, $2^2 \cdot 3^2 \cdot 5 \cdot 31$ or $2^3 \cdot 3^2 \cdot 5 \cdot 31$, respectively. As the normalizer of a Sylow 31–subgroup in $GL_5(2)$ has order $31 \cdot 5$ we get a contradiction with Sylow's theorem. So we have shown

$$z^G \cap A_zC(A_z) = \{z\}. \tag{1}$$

Again let $t \in z^G \cap S$, $z \neq t$ and E_1, E_2 as above. By Lemma 2.20 we have that $N_{A_z}(E_1E_2) = E_1E_2\langle\rho\rangle$, where $o(\rho) = 3$ and ρ acts fixed point freely on $E_i/C_S(A_z)$ for $i = 1, 2$. By Lemma 2.20 we have that t normalizes E_1E_2. By (1) and the Frattini argument we may assume that t normalizes $\langle\rho\rangle$.

Suppose first $\rho^t = \rho^{-1}$. Assume further that $[E_i, t] \leq E_i$ for both $i = 1, 2$. As ρ acts fixed point freely on $E_i/E_1 \cap E_2$ for both i, there is $e_i \in E_i$ with $t^{e_i} = f_it$, $f_i \in \Omega_1(E_i) \setminus E_3$, $i = 1, 2$. So we have that $[f_1f_2, t] = 1$. Now $t^{e_1e_2} = f_1^{e_2}f_2t$. Further $f_1^{e_2} = f_1r$, with $1 \neq r \in E_3 \setminus \langle z\rangle$. By Lemma 2.20 f_1f_2 is of order four. So $1 \neq u = (f_1f_2)^2 = (f_1^{e_2}f_2)^2$. Hence $t \sim ut$. This shows that

$$\begin{array}{l} \text{If } \rho^t = \rho^{-1} \text{ and } E_i^t = E_i, i = 1, 2, \text{ then } \Omega_1(Z(C_S(t))) = \langle z, t, u\rangle \\ \text{with } t \sim tu, u \in \Phi(C_S(t)). \end{array} \tag{2}$$

By Lemma 2.22 we see $\Phi(S) \leq S \cap A_zC_G(A_z)$. So we have that $t, zt \notin \Phi(C_S(t))$. As $z^G \cap \Phi(S) \subseteq \{z\}$ by (1), we see that $z \notin \Phi(C_S(t))$, which shows that

$$\langle z, t, u\rangle \cap \Phi(C_S(t)) = \langle u\rangle. \tag{3}$$

Assume now that $E_1^t = E_2$. We will show that also in this case (2) and (3) hold. Choose $e_1 \in E_1 \setminus E_3$. Then $e_2 = e_1^t \in E_2$. Now $t \sim (e_1e_2)^2t$. This shows that $E_3 = \langle z, (e_1e_2)^2, r\rangle$, with $x = [t, r] \neq 1$, as $[\rho, t] \neq 1$

and $C_E(\rho) = \langle z \rangle$. In particular $x \neq z$.

Suppose that $\langle x, (e_1e_2)^2 \rangle = \langle z, (e_1e_2)^2 \rangle$. Then $t \sim zt$. Again we have that $\Omega_1(Z(C_S(t))) = \langle z, t, (e_1e_2)^2 \rangle$. In G we have that $t \sim tz \sim t(e_1e_2)^2 \sim tz(e_1e_2)^2 \sim z$. Further neither $(e_1e_2)^2$ nor $z(e_1e_2)$ are conjugate to z in G. This shows that $N_G(C_S(t))$ normalizes $\langle (e_1e_2)^2, z(e_1e_2)^2 \rangle$. But as $z^G \cap \langle (e_1e_2)^2, z \rangle = \{z\}$, we have that $N_G(C_S(t)) \leq C_G(z)$, and so $C_S(t)$ is a Sylow 2–subgroup of $C_G(t)$. As $C_S(t) \neq S$, t cannot be conjugate to z in G, a contradiction.

So we have that $(e_1e_2)^2 = x$. Hence $e_1e_2r \in C_S(t)$. As $(e_1e_2)^2 = (e_1e_2r)^2$, we again get (2) and (3) with $u = (e_1e_2r)^2$.

Now we show that $[\rho, t] = 1$. Otherwise (2) and (3) hold. We have that $\langle u \rangle$ is normalized by $N_G(C_S(t))$. Let $T \leq C_G(t)$ with $|T : C_S(t)| = 2$. Then we obtain for $g \in T \setminus C_S(t)$ that $[g, \langle u, t \rangle] = 1$ and so $z^g = zt$ or ztu. But in G we have $zt \sim ztu$. Now $z^G \cap \langle z, u, t \rangle = \{z, t, tu, zt, ztu\}$ and so $\langle u, zu \rangle$ is normal in T, which shows $z \in Z(T)$, a contradiction.

So we have shown that

$$[\rho, t] = 1.$$

Set $E_3 = \langle z, r, s \rangle$, where we choose notation such that $[E_3, \rho] = \langle r, s \rangle$. As t and ρ normalize E_3 and $[t, \rho] = 1$ we force $[E_3, t] = 1$. Set $F = \langle E_3, t \rangle$. Then F is elementary abelian of order 16. Further we have that $N_{A_z \cap S}(F)$ is the preimage of $C_{(S \cap A_z)/E_3}(t)$. Hence $N_{A_z}(F)$ induces A_4 on F. We first show

$$z^{N_G(F)} = \{z\}. \tag{4}$$

Suppose false. We have that $N_{A_z}(F)$ induces orbits of length 1 (z), 3 (r and zr) and length 4 (t, zt). As z is not conjugate to r or zr by (1), we see that z has 5 or 9 conjugates under $N_G(F)$. If z has 9 conjugates, then all the other elements generate $\langle z, s, r \rangle$, a contradiction. So we see that z has 5 conjugates. In particular all $N_G(F)$–orbits have a length divisible by 5, so we must have an orbit of length 10. This shows that $r \sim zr$ in G. As $z, r \in \Omega_1(Z(S))$, we have that $zr \sim r$ in $N_G(S)$. But $\Omega_1(Z(S)) \leq A_z$ and so $z^G \cap \Omega_1(Z(S)) = \{z\}$ by (1). Hence $N_G(S) \leq C_G(z)$, contradicting $zr \not\sim r$ in $C_G(z)$. So we proved (4).

We have that $F \cap A_z = C_{S \cap A_z}(t)$ and so $F \cap A_z = \Omega_1(C_{S \cap A_z C_G(A_z)}(t))$. As $N_G(C_S(t)) \not\leq C_G(z)$, we conclude from (4) that $N_G(C_S(t)) \not\leq N_G(F)$. Hence we get that $|C_S(t) : C_S(F)| = 2$ and $C_S(t) = C_S(F)F^g$, for some $g \in N_G(C_S(t))$. So we have that $\Omega_1(Z(C_S(t))) = \langle t, z, u\rangle$, where $\langle u\rangle = \Omega_1(Z(C_S(t))) \cap \Phi(C_S(t))$. Further it shows that there are exactly two conjugates of F in $C_S(t)$. In particular $O^2(N_G(C_S(t)))$ normalizes F and so is contained in $C_G(z)$. Hence $|z^{N_G(C_S(t))}|$ is a power of two. Now we may assume that $z \sim t$ in $N_G(C_S(t))$. As $z \not\sim zu$ in G by (1), we have that also $t \not\sim tu$ in $N_G(C_S(t)) \leq C_G(u)$. As $N_{C_G(z)}(C_S(t)) \not\leq C_G(t)$, we obtain that $t \sim tz$ or tzu in $N_{C_G(z)}(C_S(t))$. So as $|z^{N_G(C_S(t))}|$ is even and $z \not\sim u$, we get that both zt and ztu have to be conjugate to z in $N_G(C_S(t)) \leq C_G(u)$, but this again would imply $z \sim zu$, a contradiction to (1). This final contradiction proves the lemma. □

We are going to prove Proposition 5.1. By [MaStr, Lemma 2.53] we have that $A_z/Z(A_z) \in \mathcal{M}$. The groups in $\mathcal{M}$ are given in [MaStr, Definition 2.51(a)]. According to Lemma 5.3 through Lemma 5.10 we are left with $A_z/Z(A_z) \cong Sp_6(2)$, $M(22)$ or $U_4(3)$. By Lemma 4.2 $A_z/Z(A_z) \not\cong Sp_6(2)$, by Lemma 4.1 $A_z/Z(A_z) \not\cong M(22)$ and finally by Lemma 4.3 $A_z/Z(A_z) \not\cong U_4(3)$. This proves Proposition 5.1.

Next we will prove Proposition 5.2. For this we first go over all components A_z, which are not of Lie type in characteristic two or J_2 or $M(24)'$. We furthermore show that the groups of Lie type in characteristic two, which were excluded in Proposition 5.2 also do not appear. The main ingredient of the proof is the interplay between Glauberman's Z^*-theorem and Thompson's transfer lemma.

We begin by eliminating the sporadic groups and some groups in characteristic three.

Lemma 5.11. *$A_z \not\cong M_{23}$, J_3, Th, Ru, M_{24}, J_4, Co_1, Co_2, F_2 or F_1.*

Proof. By Lemma 2.12 in all cases we have $\mathrm{Out}(A_z) = A_z$. So $C_G(z) = C_{C_G(A_z)}(z) \times A_z$. Further by [MaStr, Lemma 2.34] $|Z(S \cap A_z)| = 2$. Hence either $z^G \cap Z(S) = \{z\}$ or all involutions in $Z(S)$ are conjugate in G. But $z \notin S'$ and as S is not abelian, we have $Z(S) \cap S' \neq 1$. So we get $z^G \cap Z(S) = \{z\}$. As $S = (S \cap A_z)C_S(A_z)$ this by Lemma 2.2 contradicts the simplicity of G. □

Lemma 5.12. *$A_z \not\cong HiS$, Suz or $M(22)$.*

Proof. Suppose false. Let S be a Sylow 2–subgroup of $N_G(A_z)$. Then we have by [MaStr, Lemma 2.34] $\Omega_1(Z(S)) = \langle z, t\rangle$, with $t \in A_z$. Further by [GoLyS3, Table 5.3m, 5.3o, 5.3t] we see that

$$C_G(\Omega_1(Z(S)))/O_2(C_G(\Omega_1(Z(S)))) \cong \Sigma_5, U_4(2) \text{ or } U_4(2):2.$$

So $\langle t\rangle = C_G(\Omega_1(Z(S)))^{(\infty)} \cap \Omega_1(Z(S))$. In particular

$$(*) \qquad z^G \cap \Omega_1(Z(S)) = \{z\}.$$

Next we show that

$$(1) \qquad z^G \cap A_z C_G(A_z) = \{z\}.$$

Let first $A_z \cong HiS$ or Suz. Choose $u \in A_z$, $u \not\sim t$. Then by [GoLyS3, Table 5.3m], [GoLyS3, Table 5.3o] we have that $C_{A_z}(u) = \langle u\rangle \times P\Gamma L_2(9)$ or $(V_4 \times L_3(4)) : 2$, respectively. As again by [GoLyS3, Table 5.3m] or [GoLyS3, Table 5.3o] no outer automorphism of HiS centralizes $P\Gamma L_2(9)$ and no outer automorphism of Suz centralizes $L_3(4)$ we see that $\Omega_1(Z(C_S(u))) = \langle z, t, u\rangle$. Assume that z is conjugate to u or zu in G. We will denote this element by v. So let $g \in G$ with $z^g = v$. Then obviously z centralizes in A_v a subgroup $P\Gamma L_2(9)$, $L_3(4)$, respectively. So $z \in A_v C(A_v)$. Hence $E(C_G(z) \cap C_G(v)) = F$ is normalized by g. We now show that we may assume $t \in F$. For this we choose a Sylow 2-subgroup T of $C_{A_z}(v)$ and $T_1 \le A_z$ with $|T_1 : T| = 2$. In the first case, $A_z \cong HiS$, we have that $T' \le F$, and so we have a 2-central involution in F, in particular we can assume that $t \in F$. In the second case, $A_z \cong Suz$, we have by Lemma 2.20 exactly two elementary abelian subgroups F_1, F_2 of order 64 in T and $[F_1, F_2] \le F$. Hence again F contains a 2-central involution and we may assume $t \in F$. As all involutions in A_6 and $L_3(4)$ are conjugate, we may assume that $t^g = t$. But in $C_G(z)$ we have that $u \sim ut$ and so $v \sim vt$, while $z \not\sim zt$ by $(*)$, a contradiction. This proves (1) in these cases.

Assume finally $A_z \cong M(22)$. By [GoLyS3, Table 5.3t] we have a subgroup $H \cong 2^{10}M_{22}$ in A_z. By Lemma 3.3 the group M_{22} does not possess an F-module. Hence $O_2(H) = J(S \cap A_z)$ and so $J(S \cap A_z)$ is the only elementary abelian subgroup of order 2^{10} in $S \cap A_z$. In particular in S there is exactly one abelian subgroup E, which is a direct product of an elementary abelian group of order 2^{10} and a cyclic group of order 2^n, where $|C_S(A_z)| = 2^n$. We see from [GoLyS3, Table 5.3t] that involutions of type $2A$ of A_z are centralized by $L_3(4)$ in the group M_{22} in H above. Hence H induces one orbits of length 22. The product of two involutions in this orbit gives an orbit of length 231. As A_z has exactly three classes of involutions and H controls fusion in $J(A_z \cap S)$, we have a third orbit of length 770. Further any involution of $A_z C_G(A_z)$ is conjugate to one inside of E. So we get that $N_G(E)$ controls fusion in $A_z C_G(A_z)$. In particular if $\langle z\rangle \ne C_S(A_z)$ then $\Phi(E) = \langle z\rangle$ and we have (1). So we may assume that $\langle z\rangle = C_S(A_z)$. By Lemma 2.2 and $(*)$ we have that $C_G(z)/\langle z\rangle \cong \mathrm{Aut}(M(22))$. We now obtain that $z \notin C_G(z)'$

and so $z \not\sim u \in A_z$ with $C_{A_z}(u) \cong 2U_6(2)$. In particular there is at least one orbit of length 22, which cannot be fused with z. As $|z^{N_G(E)}|$ is odd, we get, just by checking all possibilities, that $|z^{N_G(E)}| = 23$, 771, 793, 1541 or 1563. As $N_G(E)/E$ is a subgroup of $GL_{11}(2)$ and 771, 7793, 1541 and 1563 do not divide the order of $GL_{11}(2)$, we conclude $|z^{N_G(E)}| = 23$. As $N_{N_G(A_z)}(E)/E \cong \mathrm{Aut}(M_{22})$ and $|z^{N_G(E)}| = 23$, we obtain $|N_G(E)/E| = 2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. As $2^{11} - 1 = 23 \cdot 89$, we see that a Sylow 23 subgroup of $N_G(E)/E$ is just centralized by itself. Now with Sylow's theorem we receive that a Sylow 23–subgroup is normalized by a cyclic group of order 22. Hence this group acts on $z^{N_G(E)}$ by fixing a point. In particular $N_{N_G(A_z)}(E)/E$ contains a cyclic group of order 22. Then $\mathrm{Aut}(M_{22})$ contains a cyclic group of order 22, which contradicts [GoLyS3, Table 5.3c]. Hence (1) holds.

By Lemma 2.1 there is some involution $u \sim z$, which induces an outer automorphism on A_z. If $C_S(A_z) = \langle z \rangle$ we get a contradiction with Lemma 2.2. Hence

$$C_S(A_z) > \langle z \rangle. \tag{2}$$

As $|\mathrm{Aut}(A_z)| = 2$, we have that $\Phi(C_S(u)) \leq A_z C_S(A_z)$ and $u \notin \Phi(C_S(u))$. As $N_G(C_S(u)) \not\leq C_G(z)$, we see by (1) that $z \notin \Phi(C_S(u))$. This implies $C_{C_S(A_z)}(u) = \langle z \rangle$. In particular $u \sim uz$ by (2). Now there is a fours group $V = \langle z, s \rangle \leq C_S(u)$, $s \in A_z$, not containing u such that $uV \subseteq u^G$. Hence there must be another fours group W such that $z \notin W$ and all involutions in zW are conjugate. We see that $W \cap C_G(A_z)A_z \neq 1$, which contradicts $z^G \cap A_z C_G(A_z) = \{z\}$. This contradiction combined with Lemma 2.1 proves the lemma. $\square$

Lemma 5.13. *$A_z \not\cong G_2(3)$, $G_2(2)'$, M_{12} or M_{22}.*

Proof. By [MaStr, Lemma 2.35] we have $\Omega_1(Z(S)) = \langle z, t \rangle$, with $t \in A_z$. We first show that

$$z^G \cap \Omega_1(Z(S)) = \{z\}. \tag{1}$$

Otherwise under $N_G(C_G(\Omega_1(Z(S))))$ all elements in $\Omega_1(Z(S))^\sharp$ are conjugate. Let P be a Sylow 3-subgroup of $C_G(\Omega_1(Z(S)))$. By Lemma 2.7 and Lemma 2.8 we have that $t \in W = [O_2(C_G(\Omega_1(Z(S)))), P]$ while $z \notin W$ as $W \leq A_z$, a contradiction. This proves (1).

Next we show

$$z^G \cap (A_z C_{C_G(A_z)}(z)) = \{z\}. \tag{2}$$

Assume false. If $A_z \not\cong M_{12}$ then by Lemma 2.7 all involutions in $C_G(A_z) \times A_z$ are conjugate to z, $t \in A_z$ or zt and so are conjugate into $\Omega_1(Z(S))$. Hence (2) follows from (1). So we may assume $A_z \cong M_{12}$. Let $i \in C_S(A_z)A_z$, $i \neq z$ and $i \sim z$ in G. We have $\Omega_1(Z(C_S(i))) = \langle z, i, t\rangle$. By (1) and Lemma 2.8(ii) we see $C_{A_z}(i) \cong \mathbb{Z}_2 \times \Sigma_5$. Let $z^g = i$. Then z is some involution in $C_G(i)$ which centralizes Σ_5 there. By Lemma 2.8iv) this shows that $z \in \langle i, A_i\rangle$. And so $i \sim z$ in $N_G(E(C_G(\langle i, z\rangle)))$, i.e. g normalizes $E(C_G(\langle i, z\rangle))$. By Lemma 2.8(iii) we may assume $t^g = t$. Further $i \sim it$ under the action of S, while $z \not\sim zt$ by (1), a contradiction.

Suppose now that there is some outer automorphism i of A_z with $i \sim z$ in G. As $i \notin \Phi(C_S(i))$, we get by (2) that also $z \notin \Phi(C_S(i))$, which implies that $\langle z\rangle = C_{C_S(A_z)}(i)$. Further by Lemma 2.2 $C_{C_S(A_z)}(z) > \langle z\rangle$. Hence $i \sim iz$.

Let now first $A_z \cong G_2(2)'$ or M_{22}. Then application of Lemma 2.7 shows $C_{A_z}(i) \cong SL_2(3)$, $E_8L_3(2)$, or 2^4F_{20}. So in all cases we see $\Omega_1(Z(C_S(i))) = \langle i, z, t\rangle$. Furthermore we notice that $i \sim iz \sim it \sim itz$. Now $\langle t, zt\rangle$ is generated by the involutions in $\Omega_1(Z(C_S(i)))$ which are not conjugate to z in G. Then $\langle z, t\rangle \trianglelefteq N_G(C_S(i))$. By application of (1) we get $N_G(C_S(i)) \leq C_G(z)$ but $C_S(i) \neq S$, contradicting $i \sim z$.

So we have $A_z \cong G_2(3)$ or M_{12}. By Lemma 2.7 and Lemma 2.8 we get $C_{A_z}(i) \cong L_2(8) : 3$ or $\mathbb{Z}_2 \times A_5$, respectively. In both cases $C_S(i)$ is elementary abelian and all involutions in $i(C_S(i) \cap (C_G(A_z)A_z))$ are conjugate to i in $C_G(z)$. As $z \sim i$ in $N_G(C_S(i))$ there is some elementary abelian group $E \leq C_S(i)$ of order 8 with $z^G \cap zE = zE$, $z \notin E$. Hence we have that $|E \cap A_z| \geq 2$. But this contradicts $z^G \cap zA_z = \{z\}$ by (2). This final contradiction by Lemma 2.1 proves the lemma. □

We now start to exclude the exceptional cases in Proposition 5.2.

Lemma 5.14. $A_z \not\cong {}^2F_4(q)'$.

Proof. Suppose false and assume first $O^{2'}(C_G(z)) = A_z \times C_G(A_z)$. By [MaStr, Lemma 2.31] we see that $Z(S) \cap A_z \leq S'$. In particular $z^G \cap \Omega_1(Z(S)) = \{z\}$ as $z \notin S'$. But then by Lemma 2.2 we get $z \notin G'$, a contradiction.

So we have $O^{2'}(C_G(z)) \neq A_z \times C_G(A_z)$. If $q \neq 2$ then by [MaStr, Lemma 2.24] A_z has just outer automorphisms of odd order. Hence we have $q = 2$. Further we have that $N_G(A_z)/C_G(A_z) \cong {}^2F_4(2)$. By [MaStr, Lemma 2.24] we know that there are no involutions in ${}^2F_4(2) \setminus {}^2F_4(2)'$. In particular $Z(S) \cap A_z \leq \Omega_1(S)'$, while $z \notin \Omega_1(S)'$. Hence again

$$(*) \qquad z^G \cap \Omega_1(Z/S) = \{z\}.$$

As $|\Omega_1(Z(S))| = 4$ and fusion in this group is controlled by $N_G(\Omega_1(Z(S)))$, we get with $(*)$

(1) No two involutions in $\Omega_1(Z(S))$ are conjugate in G.

Let $i \in C_G(z) \setminus \langle z \rangle$, $i \sim z$ in G. Then $i \in C_G(A_z)A_z$. Furthermore by Lemma 2.9 $\Omega_1(Z(C_S(i))) = \langle z, i, r\rangle$, where r is 2-central in A_z. In the notation of [MaStr, Lemma 2.31] we have that $C_{A_z}(i) \leq P_1$. This shows

$$\Omega_1(Z(O_2(C_G(i)))) = \langle z, i, r, r_1\rangle, \text{ where } \langle r, r_1\rangle = Z_2(S \cap A_z).$$

Thus

$$(2) \qquad r \sim r_1 \sim rr_1 \text{ in } A_z.$$

Additionally

$$(3) \qquad i \sim ir \sim ir_1 \sim irr_1 \text{ and } zi \sim zir \sim zir_1 \sim zirr_1.$$

Let now $g \in G$ with $z^g = i$. We have that z is an involution in $C_G(A_i)A_i$, which is centralized by $C_{C_G(z)}(i)$. Furthermore i^g also is contained in $C_G(A_z)A_z$ and centralized by $C_{C_G(z)}(i)^g$. As all involutions in A_i centralizing a subgroup isomorphic to $C_{A_i}(z)$ are conjugate, we may choose g such that

$$C_G(\langle i, z\rangle)^g = C_G(\langle i, z\rangle).$$

Hence we have that $i \sim z$ in $H = N_G(\langle i, z, r, r_1\rangle)$. Application of (1), (2) and (3) show that $|z^H| = 5$ or 9. In the latter case $\langle zr, r, r_1\rangle$ is the subgroup generated by all those involutions, which are not conjugate to z. But then $(*)$ implies $H \leq C_G(z)$, a contradiction.

Thus $|z^H| = 5$. Let ω be an element of order 5 in H. Then ω acts fixed point freely on $\langle z, r, r_1, i\rangle$. Hence all orbits have a length divisible by 5. Now by (2) and (3) there are $H \cap C_G(z)$-orbits of length 3,3,4 left. This shows that we must have an orbit of length 10. But then $r \sim rz$ in G, which contradicts (1).

So we have shown that $z^G \cap C_G(z) = \{z\}$, which contradicts Lemma 2.2. This proves the lemma. $\square$

Lemma 5.15. *$A_z \not\cong Sp_{2n}(2)$, $n \geq 3$.*

Proof. Suppose $A_z \cong Sp_{2n}(2)$. Then by [MaStr, Lemma 2.21] and Lemma 2.22 we see that $C_G(z) = C_S(A_z) \times A_z$. By Lemma 2.18 we see that $\Omega_1(Z(S)) \cap A_z \leq S'$. As $z \notin S'$, we have that $z^G \cap \Omega_1(Z(S)) \cap A_z = \emptyset$. Application of Thompson transfer Lemma 2.2, now yields the contradiction $z \notin G'$. $\square$

Lemma 5.16. *$A_z \ncong A_8$ or $U_4(2)$.*

Proof. Suppose false. As a Sylow 2-subgroup of $\mathrm{Aut}(U_4(2))$ is isomorphic to one of Σ_8, treat A_8 and $U_4(2)$ using similar argument. Set $\langle t \rangle = Z(S \cap A_z)$, then $\Omega_1(Z(S)) = \langle z, t \rangle$. We have that $C_G(\Omega_1(Z(S))) \cong (S \cap C_G(A_z)) \times ((Q_8 * Q_8)\Sigma_3)$ or $((S \cap C_G(A_z)) \times ((Q_8 * Q_8)\Sigma_3)) \cdot 2$ depending on whether $C_G(z)/C_S(A_z) \cong A_8$ or Σ_8 and $C_G(\Omega_1(Z(S))) \cong (S \cap C_G(A_z)) \times ((Q_8 * Q_8)(\Sigma_3 \times \mathbb{Z}_3))$ or $((S \cap C_G(A_z)) \times ((Q_8 * Q_8))(\Sigma_3 \times \mathbb{Z}_3)) \cdot 2$ depending on whether $C_G(z)/C_S(A_z) \cong U_4(2)$ or $U_4(2) : 2$. Now $z \notin [C_G(\Omega_1(Z(S))), O_2(C_G(\Omega_1(Z(S))))]'$ while t is. This shows

$$(*) \qquad z^G \cap \Omega_1(Z(S)) = \{z\}.$$

If $C_G(z) \cong C_G(A_z) \times A_z$ or $C_G(A_z) \times A_z : 2$, we get a contradiction by application of Lemma 2.2. So we have

(1) $C_G(z)/C_G(A_z) \cong \Sigma_8$ or $U_4(2) : 2$. Furthermore there is no involution in $C_G(z) \setminus A_z C_G(A_z)$, which centralizes $C_G(A_z)$.

Let F be the elementary abelian subgroup of $S \cap A_z$ corresponding to $\langle (12)(34), (13)(24), (56)(78), (57)(68) \rangle$. Then this is the only elementary abelian subgroup of order 16 in $S \cap A_z$. Set $E = (S \cap C_G(A_z)) \times F$, then E is an abelian subgroup of S of type $(2^n, 2, 2, 2, 2)$, where $2^n = |C_S(A_z)|$. As Σ_8 and $U_4(2) : 2$ possess no elementary abelian subgroups of order 32, and no involution in $C_G(z) \setminus A_z C_G(A_z)$ centralizes $C_S(A_z)$, we see that E is the only abelian subgroup of this type in S. Hence $N_G(E)$ controls fusion in E. As all involution in $A_z C_G(A_z)$ are conjugate into E in $C_G(z)$, we see that $N_G(E)$ controls fusion of involutions in $A_z C_G(A_z)$. We are going to show

$$(2) \qquad z^G \cap C_S(A_z) A_z = \{z\}.$$

If $n > 1$, then we have that $\langle z \rangle = \Phi(E)$ and so $N_G(E) \leq C_G(z)$, which implies (2). So we may assume that $C_S(A_z) = \langle z \rangle$ and so E is elementary abelian. We have that $N_{A_z}(F)$ induces two orbits on $F^\sharp$ of length 6 and 9 in case of A_8 and of length 5 and 10 in case of $U_4(2)$. Hence $N_{A_z}(E)$ induces orbits of length 1,6,6,9,9 or 1,5,5,10,10 on $E^\sharp$. As $N_G(E)/E$ is a subgroup of $GL_5(2)$ and neither 11 nor 13 divides the order of $GL_5(2)$, we see from $(*)$ that z has one or seven conjugates under $N_G(E)$ in the case of A_8 and one or 21 conjugates in the case of $U_4(2)$. So assume first that z has 7 conjugates. Then $|N_G(E)/E| = 2^3 \cdot 3^2 \cdot 7$.

As this is a subgroup of $GL_5(2)$, and as the normalizer of a Sylow 7–subgroup in $GL_5(2)$ is isomorphic to $\Sigma_3 \times F_{21}$, we see that a Sylow 7–subgroup of $N_G(E)$ is centralized by some element of order three in $N_G(E)$. As $|z^{N_G(E)}| = 7$, we see that this 3–element has to centralize $z^{N_G(E)}$. But this orbit generates E, a contradiction. So we have again $N_G(E) \le C_G(z)$, which proves (2). If z has 21 conjugates, then by $(*)$ we have two orbits of length 5 under $N_G(E)$. But one of theses orbits generates F and so F is normal in $N_G(E)$. This contradicts the fact that z is conjugate to elements in the orbit of length 10 in F. Hence also in this case we have (2).

Suppose now that $z^G \cap S \ne \{z\}$. Then there is some i, $i \sim z$ which induces an outer automorphism on A_z. From (1) we get that $C_S(A_z) > \langle z \rangle$ and so $i \sim iz$. Now conjugacy happens in $N_G(E_1)$, where $E_1 = \langle z \rangle \times \langle (1,2),(3,4),(5,6),(7,8)\rangle$. In both case A_z induces a group of order $2^b \cdot 3$ on E_1. We have that $N_{C_G(z)}(E_1)$ induces two orbits of length 8 on $E_1 \setminus A_zC_G(A_z)$. Hence by (2) we get that $|z^{N_G(E_1)}| = 9$. Then $|N_G(E_1)/C_G(E_1)| = 2^a \cdot 3^3$, but 3^3 does not divide the order of $GL_5(2)$. This shows $z^G \cap S = \{z\}$, contradicting Lemma 2.1, which proves the lemma. □

Lemma 5.17. *We have $A_z \not\cong L_2(p)$, p a prime, $p > 5$, A_6, $Sz(q)$, q even, $L_3(4)$, $L_3(3)$ or M_{11}.*

Proof. Suppose false. If $\Omega_1(Z(S)) \le C_S(A_z)A_z$, then $\langle z, \Omega_1(Z(S)) \cap A_z\rangle = \Omega_1(Z(S))$. If $\Omega_1(Z(S)) \not\le C_S(A_z)A_z$, then A_z possesses an involutory outer automorpism, which centralizes a Sylow 2-subgroup of A_z. Application of [MaStr, Lemma 2.26] shows $C_G(z) \cong C_G(A_z) \times \Sigma_6$. In this case we have $\Omega_1(Z(S)) = \langle z, x, t\rangle$, with $x \in A_z$, where t induces the Σ_6–automorphism.

First we show

$$(1) \qquad z^G \cap (\Omega_1(Z(S)) \cap A_zC_G(A_z)) = \{z\}.$$

If $\Omega_1(Z(S)) = \langle z, x, t\rangle$, then $C_G(z) \cong C_{C_G(z)}(A_z) \times \Sigma_6$ and so $\langle x \rangle = S' \cap \Omega_1(Z(S))$. In particular $z \not\sim x$ in $N_G(S)$, as $z \notin S'$. This shows that $|z^G \cap \Omega_1(Z(S))| = 1$ or 3, as $|z^{N_G(\Omega_1(Z(S)))}|$ has to be odd. Suppose that we have three conjugates. Let ρ be some element in $N_G(\langle z, t, x\rangle)$ which induces an element of order three. Then $\langle x \rangle$ is fixed by ρ and so ρ acts fixed point freely on $\langle z, x, t\rangle / \langle x \rangle$. This implies that $z \not\sim zx$. In particular $z^G \cap \langle z, x\rangle = \{z\}$, which is (1). Of course (1) also holds if $z^G \cap \Omega_1(Z(S)) = \{z\}$.

So we may assume that $\Omega_1(Z(S)) = \langle z, \Omega_1(Z(S)) \cap A_z\rangle$. By Lemma 2.32

all involutions in $\Omega_1(Z(S)) \cap A_z$ are conjugate in A_z. Hence we may assume that $z^G \cap \Omega_1(Z(S)) = \Omega_1(Z(S))^\sharp$.

First let $A_z \cong L_2(p)$, A_6, $L_3(3)$, or M_{11}. Lemma 2.5 implies that $\Omega_1(Z(S)) = \langle x, z\rangle$. Suppose that there is some automorphism g of S of order three, with $z^g \in A_z$. We have $S \cap A_z \trianglelefteq S$. Hence $(S \cap A_z)^g \cap (S \cap A_z) \trianglelefteq S \cap A_z$. Assume that $(S \cap A_z)^g \cap (S \cap A_z) \neq 1$. Then $\Omega_1(Z(S)) \cap A_z = \langle x\rangle \leq (S \cap A_z)^g$ and the same applies to x^g. In particular $\Omega_1(Z(S)) \leq S \cap A_z$, a contradiction. So we have that $(S \cap A_z)^g \cap (S \cap A_z) = 1$. Now we get $(S \cap A_z)^g \leq C_G(S \cap A_z)$. As there is no subgroup isomorphic to $(S \cap A_z) \times (S \cap A_z)^g$ in $A_z C_G(A_z)$, recall that $C_S(A_z)$ is cyclic, we have that $(S \cap A_z)^g$ contains some outer automorphism of A_z which centralizes $S \cap A_z$. By [MaStr, Lemma 2.26] we get $A_z \cong A_6$ and this automorphism is a Σ_6–automorphism. As $S \cap A_z \cong D_8$, we now get that $C_S(A_z) \cong \mathbb{Z}_4$ and then $S \cong D_8 \times D_8$, but this group has no automorphism of order three and the order of g was three.

Let now $A_z \cong Sz(q)$, $q = 2^{2n+1}$. Then by Lemma 2.22 we get $S = (S \cap A_z) \times C_S(A_z)$. But $\Omega_1(Z(S)) \cap A_z \leq S'$, as S is not abelian. As $z \notin S'$ we get (1).

Let now finally $A_z \cong L_3(4)$. By Lemma 2.23(3) any elementary abelian subgroup of order 16 in $\mathrm{Aut}(L_3(4))$ is contained in $L_3(4)$. According to Lemma 2.20 there are exactly two elementary abelian subgroups U_1, U_2 of order 16 in $S \cap A_z$. Hence in S there are exactly two abelian subgroups of type $(2^n, 2, 2, 2, 2)$, where $|C_S(A_z)| = 2^n$. Then the conjugacy in $\Omega_1(Z(S))$ takes place in the normalizer of $C_S(A_z) \times U_1$. As A_z induces an orbit of length 15 on the involutions of U_1 and $|z^{N_G(C_S(A_z) \times U_1)}|$ is odd, we may assume that z possesses 31 conjugates. This then would imply that $|N_G(\langle z, U_1\rangle)/C_G(\langle z, U_1\rangle)| = 2^a \cdot 3 \cdot 5 \cdot 31$, where $a = 2$ or 3. So by Sylow's theorem $N_G(\langle z, U_1\rangle)/C_G(\langle z, U_1\rangle)$ must have a normal subgroup of order 31, a contradiction to the structure of $GL_5(2)$. So we have proved we have that $z^G \cap C_S(A_z) \times U_1 = \{z\}$, which is (1). In particular (1) holds in all cases.

As A_z has just one class of involutions, we have that

$$(2) \qquad z^G \cap (A_z \times C_{C_G(A_z)}(z)) = \{z\}.$$

By Lemma 2.1 we get some $t \in S$, $t \neq z$ with $t \sim z$ in G. So by (2) t has to induce an outer automorphism on A_z. By Lemma 2.22 and Lemma 2.12 we see that $A_z \not\cong M_{11}$ or $Sz(q)$.

Let first t induces the Σ_6-automorphism on A_z. Then we have that

$$C_S(t) = C_{C_S(A_z)}(t) \times (S \cap A_z) \times \langle t \rangle.$$

As $z \sim t$ and $t \notin \Phi(C_S(t))$, we see that $z \notin \Phi(C_S(t))$, so

$$(*) \qquad C_{C_S(A_z)}(t) = \langle z \rangle.$$

This now shows that $C_S(t) = E_1E_2$, where E_i are elementary abelian of order 16, $i = 1, 2$, and $C_{A_z}(t) \cong \Sigma_4$. We choose notation such that

$$E_1 = \langle z, t, r, s \rangle, \langle r, s \rangle = E_1 \cap A_z, E_1 \not\trianglelefteq C_{C_G(z)}(t).$$

Then $N_{A_z}(E_1)$ induces in $E_1^\sharp$ orbits of length 1,3,3,3,3,1,1 with representatives z, t, zt, r, zr, tr, ztr, respectively.

Suppose first that $z^{N_G(E_1)} \neq \{z\}$. Assume further that $t \sim zt$ in $C_G(z) \cap N_G(E_1)$. As $N_{Aut(A_z)}(E_1) \leq \Sigma_6$, there is some $u \in C_S(A_z)$ with $t^u = tz$. This shows that $N_{C_G(z)}(E_1)$ induces on E_1 orbits of length 1,2,3,3,6, with representatives z, t, r, zr, tr, respectively. By (2) we have that $z \not\sim r$ and $z \not\sim zr$. Hence $\langle z, r, s \rangle$ is generated by involutions which are not conjugate to z in G. As $z^G \cap \langle z, r, s \rangle = \{z\}$, we see that $\langle z, r, s \rangle$ must not be $N_G(E_1)$–invariant. In particular $tr \not\sim z$, too. Now z has seven conjugates under $N_G(E_1)$. As this number is odd, we have that $N_S(E_1)$ is a Sylow 2–subgroup of $N_G(E_1)$. In particular as $r \in Z(N_S(E_1))$, we have that both $|r^{N_G(E_1)}|$ and $|(zr)^{N_G(E_1)}|$ are odd. As $z \in \langle zr, zs, zrs \rangle$, we see that $|r^{N_G(E_1)}| = 3$ and so $\langle r, s \rangle$ is normal in $N_G(E_1)$. Let ν be an element of order 7 in $N_G(E_1)$. Then $[\nu, \langle r, s \rangle] = 1$. Furthermore also $[E_1/\langle r, s \rangle, \nu] = 1$, which gives the contradiction $[E_1, \nu] = 1$, but $z^\nu \neq z$. But as $z^G \cap z\langle r, s \rangle = \{z\}$ and $t^G \cap t\langle r, s \rangle$ contains t and ts, we get a contradiction.

So we have that $t \not\sim tz$ in $N_{C_G(z)}(E_1)$. In particular $[t, C_S(A_z)] = 1$ and so $C_S(A_z) = \langle z \rangle$ by $(*)$. This again shows that $C_G(z)$ contains a Sylow 2–subgroup of $N_G(E_1)$, i.e. $|z^{N_G(E_1)}|$, $|r^{N_G(E_1)}|$ and $|(zr)^{N_G(E_1)}|$ are all odd. Further $z \not\sim r \not\sim zr \not\sim z$ by (2). In particular t is not conjugate to r or zr. Counting orbits we see again that either $|r^{N_G(E_1)}| = 3$ or $|(zr)^{N_G(E_1)}| = 3$. As above we see the later is not possible, so zr has 5 or 7 conjugates and $\langle r, s \rangle \trianglelefteq N_G(E_1)$. But then p–elements, $p = 5$ or 7, have to centralize E_1, a contradiction.

So we have shown that

$$z^{N_G(E_1)} = \{z\}.$$

Assume now that $N_G(C_S(t)) \not\leq C_G(z)$. Then we have that $N_G(C_S(t)) \not\leq N_G(E_1)$. This means that there is some $g \in N_G(C_S(t))$ with $E_1^g = E_2$. In particular $(z^g)^{N_G(E_2)} = \{z^g\}$. We have that E_2 is normal in $C_{C_G(z)}(t)$. So $N_{A_z}(E_2)$ induces orbits of length three and exactly three orbits of length 1 with representatives z, t and zt. If $t \sim zt$ in $N_{C_G(z)}(t)$, then there is exactly one $N_G(E_2)$–orbit of length 1, which is $\{z\}$. But then $z^g = z$, a contradiction. This again shows that $[t, C_S(A_z)] = 1$ and then $C_S(A_z) = \langle z \rangle$. As there are exactly two elementary abelian subgroups of order 16 in S, we see that $o(g)$ cannot be odd. This implies $t \notin Z(S)$. Hence we get that there is some $u \in C_G(z)$, which induces an additional outer automorphism on A_6, in particular we may assume that $E_2^u = E_1$. Now $gu \in N_G(E_1) \leq C_G(z)$. But then $g \in C_G(z)$, a contradiction. So we have shown

(3) $\quad$ If $A_z \cong A_6,$ then t does not induce a $\Sigma_6 -$ automorphism.

Now (3) together with (1) imply that

(4) $$z^G \cap \Omega_1(Z(S)) = \{z\}.$$

Again by Lemma 2.1 we get

(5) $$z \in S'.$$

By Lemma 2.22 or Lemma 2.12 we have that t is not a square in $C_S(t)$. Hence we also have that z is not a square in $C_S(t)$. This gives that

(6) $$C_{C_S(A_z)}(t) = \langle z \rangle.$$

We next show

(7) $$t \sim tz \text{ in } C_G(z).$$

This is true if $C_S(A_z) > \langle z \rangle = C_{C_S(A_z)}(t)$ by (6). So we may assume that $C_S(A_z) = \langle z \rangle$. By (5) we have $z \in S'$. In particular there is some $s \in S$ with $t^s = tzj$, where $j \in A_z \cap S$. As by (3) all involutions in $A_z t$ are conjugate to t, there is some $g \in A_z$ with $(tj)^g = t$. Hence (7) holds.

We now come to the final contradiction. We have that $\Omega_1(Z(C_S(t))) = \langle z, t, X \rangle = F$, where $X \leq A_z$. Assume first that $z^{N_G(F)} = \{z\}$. Then $C_S(t)$ is a Sylow 2–subgroup of $C_G(t)$ and so $t \in Z(S)$, as $t \sim z$ in G. But then $A_z \cong A_6$ and t induces a Σ_6–automorphism, contradicting (3).

So we have $z^{N_G(F)} \neq \{z\}$. By (2) we have that $\langle z, X \rangle$ is generated by involutions which are not conjugate to z in G. Hence $t\langle z, X \rangle$ must also contain such involutions, as $z^G \cap \langle z, X \rangle = \{z\}$. But by (3) and

Lemma 2.37 all involutions in tX are conjugate and by (7) $t \sim tz$ in $C_G(z)$, so all involutions in $t\langle z, X\rangle$ are conjugate to t and thus to z, a contradiction. This final contradiction proves the lemma. □

Now we are going to prove Proposition 5.2. Besides the groups we have excluded in Lemma 5.14 through Lemma 5.17 we just have to exclude the groups $A_z \cong L_4(3)$, $U_4(3)$, $L_2(q)$, q even and $M(23)$. The first three cases have been handled in Lemma 4.3 and Lemma 4.4 where groups show up which are in the statement of our theorem, so G is not a counterexample. The last has been handled in [MaStr, Lemma 4.14].

6. Some 2-local subgroups

We continue with the assumption that G is a counterexample to the main theorem. Hence there is some $z \in \Omega_1(Z(S))$ such that A_z is standard. By Proposition 5.1 we have that A_z is simple. Furthermore by Proposition 5.2 we have that A_z is a group of Lie type of characteristic two or J_2 or $M(24)'$. Remember that among the groups of Lie type in characteristic two the group A_z is not isomorphic to one of $L_2(q)$, $Sz(q)$, ${}^2F_4(q)'$, q even, $L_3(4)$, $G_2(2)'$, $L_4(2)$, A_6 or $L_3(2)$. The aim of this chapter is to derive a contradiction, which then proves the main theorem.

For this chapter we fix the following notation. We denote by S a Sylow 2–subgroup of G with $z \in Z(S)$. By R we denote a fixed root group in $\Omega_1(Z(S \cap A_z))$ if A_z is of Lie type and just $\Omega_1(Z(S \cap A_z))$ if A_z is sporadic. By Q_R we denote $O_2(C_{A_z}(R))$. As A_z is normal in $C_G(z)$ we see that

Lemma 6.1. $|\Omega_1(Z(S))| \geq 4$.

The first step towards deriving a contradiction it to show the existence of a group N such that $S \leq N$, $N \not\leq C_G(z)$ and $F^*(N) = O_2(N)$ (Lemma 6.5 and Lemma 6.6). Among these groups we choose N minimal with this property. In Lemma 6.11 and Lemma 6.12 we determine the structure of N. Here Lemma 3.20 and Lemma 3.21 come into the game. The key fact for us will be to show that there is some $t \in Z(N)$, $t \neq z$ and $t \notin A_z$. Furthermore we will see that A_z is one of the two sporadic groups or defined over GF(2). In particular we get that Q_R is extraspecial.

At this point we turn our attention to $C_G(t)$. We show that also $C_G(t)$ has a standard component A_t. Then we can show that $Q_R \leq A_z \cap A_t$.

This is sufficient to show that eventually A_t will be isomorphic to A_z. With this information we get that N is isomorphic to a minimal parabolic in A_z and A_t as well. Now both of these groups induce some action on $\Omega_1(Z(O_2(N)))$. This together with the fact that $t \not\sim z$ in G eventually yields the desired contradiction.

Now we are going to show the existence of a suitable N. But first a technical lemma.

Lemma 6.2. *Let $x \in \Omega_1(Z(S)) \setminus C_S(A_z)$ and $K \leq C_{C_G(z)}(x)$ such that $K = D_1 \times D_2 \times \cdots \times D_m$, $m \geq 1$, D_1 dihedral of order 2^n, quaternion of order 8 or isomorphic to $SL_2(3) * SL_2(3)$ and there are $s_2, \ldots s_m \in S$ such that $D_i = D_1^{s_i}$, $i = 2, \ldots, m$. Then K is not normal in $C_{C_G(z)}(x)$.*

Proof. Suppose false. We first will treat the case of $A_z \cong Sp_4(q)$, $q > 2$, as in this group there is some x such that $C_{A_z}(x)$ is a 2-group. We fix the following notation. According to Lemma 2.21 there are two elementary abelian subgroups E_1, E_2 in $A_z \cap S$ of order q^3 such that $E_1E_2 = S \cap A_z$ and $E_1 \cup E_2 = \Omega_1(S \cap A_z)$. Furthermore $E_1 \cap E_2 = R_1R_2$, where R_1, R_2 are the two root subgroups such that $R_1R_2 = \Omega_1(Z(S \cap A_z))$. We now set

$$F_i = \langle z, E_i \rangle, i = 1, 2, \text{ and } S_1 = S \cap A_z C_S(A_z).$$

We first show

$$D_1 \text{ is dihedral.}$$

Obviously $D_1 \not\cong SL_2(3) * SL_2(3)$. So let D_1 be quaternion. Then $\Omega_1(K) = Z(K)$. Assume $D_1 \leq S_1$. Then $[E_1, D_1] \leq E_1$ and so $[D_1, E_1] \leq Z(K)$. As $|D_1 : D_1 \cap E_1| \geq 4$, we see with [MaStr, Lemma 2.67] that $Z(S \cap A_z) \leq [D_1, E_1]$. As $|K| = |Z(K)|^3$, we now get $|K| \geq q^6$. But $|\Omega_2(S_1)| \leq 4q^4$, a contradiction. So we have that $D_1 \not\leq S_1$. Choose $u \in D_1 \setminus S_1$. If $[u, E_1] \leq E_1$, then by Lemma 2.21 and Lemma 2.22 we see that u induces a field automorphism on A_z and so $|[E_1, u]| = r^3$, where $q = r^2$. Again $[E_1, u] \leq Z(K)$ and so $K \cap A_z \leq C_{A_z}([E_1, u])$. As $[E_1, u] \not\leq Z(A_z \cap S)$, we see that $C_{S \cap A_z}([E_1, u]) = E_1$. Hence we have that $[E_1, u] = K \cap A_z$. But the same applies to E_2. So $[E_1, u] = [E_2, u]$, which is impossible as $E_1 \cap E_2 = Z(S \cap A_z)$. This shows that $E_1^u = E_2$. Then $|[E_1, u] : [E_1, u] \cap Z(S \cap A_z)| = q$. Again by [MaStr, Lemma 2.67] we get that $Z(S \cap A_z) \leq [[E_1, u], S \cap A_z]$ and so $Z(S \cap A_z) \leq Z(K)$. But then $[R_1, u] = 1$, while we have $R_1^u = R_2$, a contradiction.

So we have shown that D_1 is dihedral. We fix the following notation:

$$D_1 = \langle x_1, x_2 \rangle, \text{ where } x_1^2 = x_2^2 = 1.$$

Let first $m = 1$. We may assume that $[E_1, x_1] \neq 1$. In particular $x_1 \notin E_1$. If $[E_1, x_1] \leq E_1$. then $|\langle [E_1, x_1], x_1\rangle| \geq 2q \geq 8$. But there are no elementary abelian subgroups of order 8 in D_1. So we have that $R_1^{x_1} = R_2$ and again $|\langle [R_1R_2, x_1], x_1\rangle| \geq 2q$ and this group is elementary abelian.

So we have ptoved that $m > 1$. Now we set

$$D_2 = \langle y_1, y_2\rangle, \text{ where we choose notation such that } x_i^{s_2} = y_i, i = 1, 2.$$

Suppose first that $D_1 \leq S_1$. Then as S_1 is normal in S, we have $K \leq S_1$. As $\Omega_1(S_1) = F_1 \cup F_2$, we may assume that $x_1y_1 \in F_1$. As $[x_1, x_2] \neq 1$, we get $x_2y_2 \in F_2$. Now we consider the involution $x_2y_1 \in K$. We have $[x_1y_1, x_2y_1] \neq 1 \neq [x_2y_2, x_2y_1]$, so $x_2y_1 \notin F_1 \cup F_2$, a contradiction.

So we may assume that $x_1 \notin S_1$. By Lemma 2.22 we have that S/S_1 is abelian. Hence $x_1x_1^{s_2} = x_1y_1 \in S_1$. Furthermore also $x_2y_2 \in S_1$. So we may assume that $x_1y_1 \in F_1$ and $x_2y_2 \in F_2$. As $[x_1y_1, x_2y_2] \neq 1$, we see that x_1y_1, x_2y_2 both are not in $Z(S_1)$. As $[x_1, x_1y_1] = 1$, we see that x_1 normalizes E_1 and induces a field automorphism on A_z. In particular it also normalizes E_2 and so we get that K normalizes E_i, $i = 1, 2$. As the group of field automorphisms is cyclic, we get $|K : K \cap S_1| = 2$. We consider the involution x_1y_2. As above we get that $x_1y_2 \notin S_1$. But then $y_2 = x_1x_1y_2 \in S_1$ and so also $x_2 \in S_1$. In particular $\langle x_2, Z(D_2)\rangle$ is normal in D_2, which shows that D_1 is dihedral of order 8. As $[x_2, x_2y_2] = 1$, we have $x_2, y_2 \in E_2$. As $[x_1y_1, x_2] \neq [x_1y_1, y_2]$,we see that $|\langle x_2, y_2, Z(S_1)\rangle/Z(S_1)| = 4$. Now application of [MaStr, Lemma 2.67] shows that $[\langle x_2, y_2\rangle, E_1] = R_1R_2$ and so $R_1R_2 \leq K$. As x_1 induces a field automorphism on A_z, we have that $|R_1R_2 : C_{R_1R_2}(x_1)| = q > 2$. On the other hand $|K : C_K(x_1)| = 2$, a contradiction. So we have shown

$$A_z \not\cong Sp_4(q). \tag{1}$$

By Lemma 2.23 we have that $x \in C_S(A_z) \times A_z$. Hence $x = z^ir$ where $1 \neq r \in Z(S \cap A_z)$ and $i = 0, 1$.

We assume first that $r \in R$ and show

$$O_2(K) \leq A_zC_S(A_z). \tag{$*$}$$

Suppose false. As $[O_2(K), C_{A_z}(r)] \leq Q_R$, we see from [GoLyS3, Table 5.3] for the two sporadic groups and by application of Lemma 2.27 in the case A_z a group of Lie type that $A_z \cong L_3(16)$ and some element $k \in K$ induces a graph/field automorphism on A_z. In particular $K = O_2(K)$. So $C_{A_z}(k) \cong U_3(4)$. As $Z(K) \leq K'$, we have

$Z(K) \leq C_S(A_z)A_z$ and $Z(K) \leq C(k)$. Hence $|Z(K)| \leq 8$. In $C_{A_z}(k)$ we have some element ω of order 5, which centralizes $Z(K)$ and x. Hence this element has to normalize any quaternion group or dihedral group in K modulo $Z(K)$ and so it has to centralize K. As ω acts fixed point freely on Q_R/R, we see that $K \cap A_zC_S(A_z) \leq Z(K)C_S(A_z)$. But then K cannot be normal in S. So we have $(*)$.

As $Z(K) \leq K'$ and $C_S(A_z)$ is cyclic, we get by $(*)$ that $K \cap C_S(A_z) = 1$. As $C_S(A_z)A_z/C_S(A_z) \cong A_z$, we may assume $x = r$ and $O_2(K)$ is a subgroup of A_z. Now $O_2(K) \leq O_2(C_{A_z}(r))$, as K is normal in $C_{A_z}(R)$, which gives that $O_2(K)$ is of class two and $Z(K) \leq O_2(C_{A_z}(r))' = R$. But then any $O_2(D_i)$ has to be normal modulo R, which gives that $C_{A_z}(r)$ has a normal dihedral group, quaternion group or $Q_8 * Q_8$. For $A_z \not\cong L_3(q)$ we receive from Lemma 2.17, Lemma 2.18, Lemma 2.19 or [MaStr, Lemma 2.10] that $C_{A_z}(r)$ induces at most two nontrivial modules on $O_2(C_{A_z}(r))/Z(O_2(C_{A_z}(r)))$. We conclude that we must have exactly two such modules and $C_{A_z}(r)$ induces $\mathbb{Z}_3$ or Σ_3, or $m = 1$ and $O_2(C_{A_z}(r))$ is dihedral of order eight or isomorphic to $Q_8 * Q_8$. This then implies that we are over GF(2). Hence we just have the groups excluded by Proposition 5.2. In case of $A_z \cong L_3(q)$, $q > 2$, by [MaStr, Lemma 2.39], we have that $R = Z(K)$ and so as $|O_2(K)| \geq |Z(K)|^3$, we see $K = Q_R$. As $q > 2$, we have $m > 1$. But Q_R is not a direct product of m dihedral groups.

So we may assume that r is not a root element. In particular $A_z \cong F_4(q)$ or $Sp_{2n}(q)$. In the latter by (1) we have $n > 2$. We first show that $(*)$ holds again. Set $X_z = C_{A_z}(Z(S \cap A_z))$. Then we have that $[O_2(K), X_z] \leq O_2(X_z)$. Assume that there is some $t \in O_2(K)$ such that t induces an outer automorphism on A_z. As $A_z \not\cong Sp_4(q)$, we have that $E(X_z/O_2(X_z))$ is a nonsolvable group and by Lemma 2.22 any outer automorphism of A_z induces a nontrivial automorphism on this group. Hence $(*)$ holds. So as above we may assume that $\langle x, K\rangle \leq A_z$. As $O_2(X_z)/O_2(C_{A_z}(R))$ is elementary abelian, we see that $Z(K) \leq O_2(C_{A_z}(R))$.

Let first $A_z \cong Sp_{2n}(q)$. Assume furthermore $Z(K) \not\leq Z(O_2(C_{A_z}(R)))$. Then $Z(K)/Z(K) \cap Z(O_2(C_{A_z}(R)))$ is a natural $Sp_{2n-4}(q)$–module. We have that $C_{O_2(C_{A_z}(R))}(Z(K)) = Z(K)Z(O_2(C_{A_z}(R))) \geq K \cap O_2(C_{A_z}(R))$. Hence

$$\begin{aligned}&|O_2(K)Z(O_2(C_{A_z}(R))) : Z(K)Z(O_2(C_{A_z}(R)))|\\ &\leq |O_2(X_z) : O_2(C_{A_z}(R))| = q = 2^t.\end{aligned}$$

This shows that $m \le t$. As $|Z(K)Z(O_2(C_{A_z}(R)))/Z(O_2(C_{A_z}(R)))| = 2^{t(2n-4)} \ge 2^{2t}$, as $n > 2$, and $2^{2t} > 2^m$, we get a contradiction to $|Z(K)| = 2^m$. Hence $Z(K) \le Z(O_2(C_{A_z}(R)))$. But now $O^2(X_z)$ centralizes $Z(K)$, i.e. $O_2(K) \le C_{O_2(X_z)}(O^2(X_z))$, or $O^2(X_z)$ induces a 3-group on $O_2(X_z)$.

Assume first that $[O^2(X_z), O_2(K)] = 1$. Then $K \not\le O_2(C_{A_z}(R))$, as $C_{O_2(C_{A_z}(R))}(O^2(X_z)) = Z(O_2(C_{A_z}(R)))$. Take $u \in K \setminus O_2(C_{A_z}(R))$. Then $[u, O_2(C_{A_z}(R))] \le O_2(C_{A_z}(R)) \cap K \le Z(O_2(C_{A_z}(R)))$. But this contradicts Lemma 2.18.

So assume that $O^2(X_z)$ induces a 3-group on $O_2(X_z)$. Then application of Lemma 2.18 yields $A_z \cong Sp_6(2)$. But this contradicts Proposition 5.2.

So we are left with $A_z \cong F_4(q)$. As there is no 3-group, which centralizes $C_{A_z}(Z(S \cap A_z))/O_2(C_{A_z}(Z(S \cap A_z)))$, we see that $K = O_2(K)$. By Lemma 2.17 $C_{A_z}(O_2(C_{A_z}(R)))/Z(O_2(C_{A_z}(R)))) \le O_2(C_{A_z}(R))$. Now assume that $O_2(C_{A_z}(R)) \cap K \le Z(O_2(C_{A_z}(R)))$. As $[K, O_2(C_{A_z}(R))] \le K \cap O_2(C_{A_z}(R))$, we get $K \le O_2(C_{A_z}(R))$. But then K would be elementary abelian, a contradiction. Hence

$$(**) \qquad O_2(C_{A_z}(R)) \cap K \not\le Z(O_2(C_{A_z}(R)))$$

and so as $O_2(C_{A_z}(R)) \cap K$ is normal in $O_2(C_{A_z}(R))$ we get $R \le K$. But in case of $F_4(q)$ we have two roots with isomorphic centralizers. Then a similar argument shows that $Z(S \cap A_z) \le K$. As $O_2(X_z)/O_2(C_{A_z}(R))$ is elementary abelian and $Z(K) \le K'$, we have $Z(K) \le O_2(C_{A_z}(R))$. Assume first $Z(K) = Z(S \cap A_z)$. Then $O^2(X_z)$ centralizes K. But by Lemma 2.17 we have that $O_2(X_z)/Z(S \cap A_z)$ has a normal subgroup which is a direct sum of two natural $Sp_4(q)$–modules whose factor group is a direct sum of two natural $\Omega_5(q)$–modules. This implies that $C_{O_2(C_{A_z}(R))/Z(C_{A_z}(R))}(O^2(X_z)) = 1$, which contradicts $(**)$. So we have that $Z(K) > Z(A_z \cap S)$. Hence by Lemma 2.17 we have that either $|Z(K)/Z(K) \cap Z(O_2(C_{A_z}(R)))| = q^4$ or $|Z(K) \cap Z(O_2(C_{A_z}(R)))| \ge q^6$. In both cases we have that $|Z(K)| \ge q^6$ and so as $|K| \ge |Z(K)|^3$, we get $|K| \ge q^{18}$. In particular there is some proper normal subgroup of order at least q^{18}. But now the structure of X_z as described before shows that $K = O_2(X_z)$. Then $Z(K) \le Z(O_2(X_z)) = Z(A_z \cap S)$, a contradiction. $\square$

Next we set

$$\mathcal{N} = \{N \mid N \le G, \Omega_1(Z(S)) \le N \not\le C_G(z), 1 \ne O_2(N) \le S\}.$$

The group N we are looking for will be in this set $\mathcal{N}$. So we first show that $\mathcal{N}$ is not empty.

Lemma 6.3. *There exists* $1 \neq S_1 \leq S$ *such that* $N_G(S_1) \not\leq C_G(z)$. *Among those choose* S_1 *such that* $|N_G(S_1) \cap C_G(z)|_2$ *is maximal. Then*

(i) $N_G(S_1) \in \mathcal{N}$, *in particular* $\mathcal{N} \neq \emptyset$.
(ii) $N_G(S_1) \cap S \in Syl_2(C_{N_G(S_1)}(z)) \subseteq Syl_2(N_G(S_1))$.
(iii) *If* $N_G(S_1) \cap S$ *is not a Sylow* 2*-subgroup of* G, *then* $N_G(S \cap N_G(S_1)) \leq C_G(z)$.

Proof. As $C_G(z)$ cannot control fusion in $C_G(z)$ by Lemma 2.1 we have that $C_G(z)$ is not strongly 2-embedded in G. Hence there is some $1 \neq S_1 \leq S$ with $N_G(S_1) \not\leq C_G(z)$. Now we choose S_1 such that $|N_G(S_1) \cap C_G(z)|_2$ is maximal. Obviously $\Omega_1(Z(S)) \leq N_G(S_1)$. Set $T = N_S(S_1)$. Then $S_1 \leq T$. Let T_1 be a Sylow 2-subgroup of $C_{N_G(S_1)}(z)$, which contains T. Then there is some $g \in C_G(z)$ with $T_1^g \leq S$. We have $|S \cap N_G(S_1)^g| \geq |S \cap N_G(S_1)|$. As $S_1^g \leq S$ and $N_G(S_1)^g \not\leq C_G(z)$, we have by the choice of $N_G(S_1)$ that $T = T_1$ is a Sylow 2-subgroup of $C_{N_G(S_1)}(z)$. If $T = S$, we have the assertion (ii). So assume $T \neq S$. In particular $N_S(T) > T$. Hence by the choice of S_1 we have that $N_G(T) \leq C_G(z)$, which is (iii). As T is a Sylow 2-subgroup of $C_{N_G(S_1)}(z)$, this shows that T is a Sylow 2–subgroup of $N_G(S_1)$, which finishes the proof of (ii). In particular $O_2(N_G(S_1)) \leq T \leq S$, which shows that $\mathcal{N} \neq \emptyset$, which proves (i). □

Lemma 6.4. *Set* $\mathcal{N}_1 = \{U \mid U \in \mathcal{N}$ *with* $|U \cap S|$ *maximal*$\}$. *Choose* $N \in \mathcal{N}_1$ *minimal by inclusion. Then* N *is a minimal parabolic where* $C_N(z)$ *is the unique maximal subgroup of* N *containing* $N \cap S$. *Furthermore we have:*

(i) *If* E *is normal in* N *and* $E \leq C_G(z)$, *then* $S \cap E$ *is also normal in* N.
(ii) $E(N) = 1$.
(iii) $O(N) \leq C_G(z)$ *and* $O_{2',2}(N) = O_2(N)O(N)$.

Proof. Recall that by Lemma 6.3 there is such an $N \in \mathcal{N}$. Further we have that $T = S \cap N$ is a Sylow 2–subgroup of N.

The minimality of N then shows, that for $M < N$ and $T \leq M$ we have $M \leq C_G(z)$. Therefore $N \cap C_G(z)$ is the only maximal subgroup of N containing T, which means that

$$N \textit{ is a minimal parabolic with respect to } T.$$

Let now E be normal in N. Then we have that $N = N_N(E \cap T)E$. If $E \leq C_G(z)$, then $N_N(E \cap T) \not\leq C_G(z)$ and so by minimality we have that $N = N_N(E \cap T)$, which is (i).

Assume there is some involution $x \in Z(N)$. Then $C_G(x) \not\leq C_G(z)$. Let $T_1 \leq C_G(x)$ with $|T_1 : T| = 2$. Then as $N_G(T) \leq C_G(z)$ by Lemma 6.3, we see $T_1 \leq C_G(z)$. This implies that there is some $g \in C_G(z)$ with $T_1^g \leq S$. In particular $|S \cap C_G(x^g)| > |S \cap N|$. So we may apply Lemma 6.3 with $\langle x^g \rangle$. This implies the existence of some $S_1 \leq S$ such that $N_G(S_1) \in \mathcal{N}$ and $|N_G(S_1) \cap C_G(z)|_2 \geq |C_S(x^g)| > |N \cap S|$, which contradicts the choice of N. So we have that T is a Sylow 2–subgroup of $C_G(x)$, in particular $O_2(C_G(x)) \leq O_2(N)$. We collect:

(1) If $1 \neq x \in Z(N)$ is an involution then T is a Sylow 2-subgroup of $C_G(x)$.

Assume now $E(N) \neq 1$. Then by (i) we have $E(N) \not\leq C_G(z)$ and so $N = E(N)T$. Let $E(N) = N_1 \cdots N_r$. As $E(N)T$ is a minimal parabolic we have that $N_1 N_T(N_1)$ is a minimal parabolic with respect to $N_T(N_1)$. As $[O_2(N), E(N)] = 1$, we have $z \notin O_2(N)$. So the maximal subgroup containing the Sylow 2-subgroup is $C_{N_1}(z)N_T(N_1)$, the centralizer of an involution. Hence by Lemma 2.39 we get that

(2) N_1 is a group of Lie type in odd characteristic.

Choose $x \in Z(N)$ an involution, which exists as $O_2(N) \neq 1$. By (1) we have that T is a Sylow 2–subgroup of $C_G(x)$, in particular $O_2(C_G(x)) \leq O_2(N)$ and then $[E(N), O_2(C_G(x))] = 1$. This shows that $E(N) \leq E(C_G(x))$ (recall that $O(C_G(x)) = 1$ by the general assumption).

Assume first that N_1 is not conjugate to $L_2(p)$, $L_2(9)$, $L_3(3)$, $L_4(3)$, $U_4(3)$ or $PSp_4(3)$. It follows that N_1 is not a component of $C_G(x)$, as now by (2) $N_1 \notin \mathcal{C}_2$. Furthermore from Lemma 2.39 we get that $C_{N_1}(z)$ has a component K_1, which is a group of Lie type in odd characteristic. Let K be some component of $C_G(x)$ with $N_1 \leq K$. As by $(*)$ we see that N contains a Sylow 2-subgroup of $C_G(x)$, we have that $[K_1, O_2(C_{C_G(x)}(z))] = 1$. This shows that also $C_K(z)$ has a component. As $K \in \mathcal{C}_2$ and z centralizes a Sylow 2-subgroup of K, we get with [MaStr, Lemma 2.26] that either z induces an inner automorphism on K or $K \cong L_4(3)$ and then z has to induce an outer automorphism, which then is a graph automorphism, which centralizes $L \cong PSp_4(3)$ in K. Now $K_1 \leq L$. But as $PSp_4(3) \cong \Omega^-6(2)$ we get with the Borel-Tits-Theorem, that all subgroups of L containing a Sylow 2-subgroup of L are constrained, in particular do not have components, a contradiction. So we may assume that z induces an inner automorphism on

K. With [MaStr, Lemma 2.22] we see that K cannot be a group of Lie type in characteristic two. Further as centralizers of involutions in $L_3(3)$, $G_2(3)$, $U_4(3)$ are solvable by [MaStr, Lemma 2.20], Lemma 2.7, Lemma 2.6 respectively, these groups are also not possible. The centralizers of 2–central involutions in the sporadic groups are given by [GoLyS3, Table 5.3]. From there we see that only $M(23)$ possesses a 2-central involution, whose centralizer has a component. In $M(23)$ this component would be $2M(22)$, which is not a group of Lie type in odd characteristic. Hence $K_1 \neq 2M(22)$. But then $M(22)$ must contain a subgroup L, which contains a Sylow 2-subgroup and a normal subgroup which is a product of groups of Lie type in odd characteristic, contradicting [GoLyS3, Table 5.3].

Hence we have that

(3) $\quad N_1/Z(N_1) \cong L_2(p), p > 5, L_2(9), L_3(3), L_4(3), U_4(3)$ or $PSp_4(3)$.

In particular $N_1 \in \mathcal{C}_2$. By Lemma 2.39 we get that $N_1/Z(N_1) \not\cong U_4(3)$ or $PSp_4(3)$. If $N_1/Z(N_1)$ is isomorphic to $L_2(p)$ or $L_3(3)$, then $\Omega_1(Z(N_T(N_1)/C_T(N_1))) \leq N_1$. If $N_1 \cong A_6$, then as $N_T(N_1)N_1$ is a minimal parabolic there is some element in $N_T(N_1)$, which interchanges the two subgroups isomorphic of Σ_4. So also $\Omega_1(Z(N_T(N_1)/C_T(N_1))) \leq N_1$. In case of $N_1/Z(N_1) \cong L_4(3) \cong \Omega_6^+(3)$, we see from Lemma 2.39 that also a graph automorphism is induced by T. This then again implies $\Omega_1(Z(N_T(N_1)/C_T(N_1))) \leq N_1$. As $N_1 \not\cong L_2(5)$, we have by Lemma 2.13 that $|\Omega_1(Z(N_T(N_1)/C_T(N_1)))| = 2$.

We have $\Omega_1(Z(S)) \leq N$ by the definition of $\mathcal{N}$. Further we have $|\Omega_1(Z(S))| \geq 4$ by Lemma 6.1. As $Z(S)$ centralizes $T \cap N_1$ it normalizes N_1, we get that $\Omega_1(Z(S)) \cap C(N_1) \neq 1$. As T acts transitively on the components of N,we get that $\Omega_1(Z(S)) \cap C(E(N)) \neq 1$.

So we may assume

(4) $\quad x \in Z(S)$ and then $T = S$. Further $N_1 \leq K$ for K some component of $C_G(x)$.

Now we show that

(5) $$K = N_1 \text{ or } K \cong M_{11}.$$

There is $T_1 \leq T$ such that $M_1 = \langle N_1^{T_1} \rangle T_1 \leq K$ and M_1 contains a Sylow 2-subgroup of K. If K is a group of Lie type in characteristic 2, then by [MaStr, Lemma 2.15] we have that $K = N_1$, as M_1 would be a parabolic subgroup. This proves (5). If K is a group of Lie type in odd characteristic, the list $\mathcal{C}_2$ shows that $K/Z(K) \cong L_2(p)$, $L_2(9)$, $L_3(3)$, $L_4(3)$, $U_4(3)$ or $G_2(3)$. As the centralizer of a 2-central involution in K is solvable ([MaStr, Lemma 2.20], Lemma 2.7, Lemma 2.6), we see that $O_2(M_1) = 1$. Now the order of $L_4(3)$ is divisible by $3^6 \cdot 5 \cdot 13$, so we get $N_1 = K$ in case of $N_1 \cong L_4(3)$ or $G_2(3)$. Also the order of $L_3(3)$ is divisible by 13, which shows $N_1 = K$ or $M_1 \leq L_4(3)$ in case of $N_1 \cong L_3(3)$. Suppose the latter. By [MaStr, Lemma 2.21] and Lemma 2.22 we have that $|\mathrm{Aut}(L_3(3)| = 2^5 \cdot 3^3 \cdot 13$, which contradicts $|K|_2 = 2^6$ and $O_2(M_1) = 1$. So it remains $N_1 \cong L_2(q)$. If $N_1 \neq K_1$, we see that $K \cong L_3(3)$, $L_4(3)$, $U_4(3)$ or $G_2(3)$. As p is a Fermat or Mersenne prime and $p > 5$, we see that $N_1 \cong L_2(7)$ or $L_2(9)$. As neither 5 nor 7 divides the order of $L_3(3)$, we get $K \not\cong L_3(3)$. As 2^6 does not divide $|\mathrm{Aut}(N_1)|$, we get that $N_1 \times N_1^t \leq K$. But then 5^2 or 7^2 has to divide $|K|$, which is not the case. This proves (5) in case of K a group of Lie type in odd characteristic.

So we are left with K a sporadic simple group. Suppose first that $C_{M_1}(N_1) = 1$. Then by Lemma 2.10 we get $M_1 \cong M_{10}$ and $K \cong M_{11}$, which is (5). So assume $K \not\cong M_{11}$. Then $C_{M_1}(N_1) \neq 1$. If $M_1 = \langle N_1^{T_1} \rangle T_1$ has 2^n, $n \geq 1$, many components isomorphic to N_1, there is some involution $y \in M_1$, which centralizes in T_1 a subgroup of index two and 2^{n-1} of these components. In particular in both cases $C_K(y)$ possesses a component $\tilde{K}$. By [GoLyS3, Table 5.3] we see that $K \cong M(22)$ or $M(23)$. The same is true if $O_2(M_1) \neq 1$, where $y \in Z(M_1)$. Now the situation in $\tilde{K}$ is the same as in K and so $\tilde{K}/Z(\tilde{K})$ cannot be a group of Lie type in characteristic 2. As $\tilde{K}/Z(\tilde{K}) \cong U_6(2)$ for $K \cong M(22)$, we get $K \cong M(23)$ and $\tilde{K}/Z(\tilde{K}) \cong M(22)$. The odd part of the order of $M(23)$ implies that there are at most two components N_1, N_1^g in M_1. If there are two of them, we have with [GoLyS3, Table 5.3u] and (1) that $N_1 \cong L_2(9)$. Now in any case we see that $|M_1/O_2(M_1)|_2 \leq 2^{11}$. Hence $|O_2(M_1)| \geq 4$. So we may choose $y \in Z(T_1)$ and then we have the same situation in $C_K(y)/\langle y \rangle \cong M(22)$. Now we get a contradiction with the same arguments as for $K \cong M(22)$.

So we have shown that either $K = N_1$ or $K \cong M_{11}$ In any case we have that $C_K(z)$ is dihedral or isomorphic to $GL_2(3)$ or in case of $L_4(3)$ contains a normal subgroup $SL_2(3) * SL_2(3)$. This shows that $C_G(z) \cap C_G(x)$ has a normal subgroup which is a direct product of dihedral, quaternion groups or groups isomorphic to $SL_2(3) * SL_2(3)$, which are permuted by S. This now contradicts Lemma 6.2 and so we have (ii).

Assume now $O(N) \neq 1$. We first show $O(N) \leq C_G(z)$. So assume false. Then by the minimality of N we have that $N = O(N)T$. Again there is some involution $x \in Z(N)$. By $(*)$ T is a Sylow 2-subgroup of $C_G(x)$. As $[O_2(C_G(x)), O(N)] = 1$ and $O(C_G(x)) = 1$, we must have that $E(C_G(x)) \neq 1$. As $[T, O(N)] \leq O(N)$, we see that $O(N)$ normalizes any component K of $C_G(x)$. Further a Sylow 2-subgroup of K has to normalize some nontrivial group of odd order of its automorphism group. As $K \in \mathcal{C}_2$ we get by Lemma 2.29 that $K \cong L_3(3)$ or M_{11}. Set $K_1 = \langle K^T \rangle$ and $K_2 = K_1T$. Then by Lemma 2.29 we have that $\Omega_1(Z(T)) = \Omega_1(Z(T)) \cap C_T(K_1) \times \Omega_1(Z(T)) \cap K_1$. Further $|\Omega_1(Z(T)) \cap K_1| = 2$. As $|\Omega_1(Z(S))| \geq 4$, we see that $\Omega_1(Z(S)) \cap C_T(K_1) \neq 1$. Hence we have that $\Omega_1(Z(S)) \cap O_2(N) \neq 1$, and so we may choose $x \in \Omega_1(Z(S))$, which gives $S = T$. As $C_{K_1}(z)$ is a direct product of groups isomorphic to $GL_2(3)$, we see that $C_{C_G(z)}(x)$ contains a normal subgroup, which is a direct product of quaternion groups, contradicting Lemma 6.2.

So we have that $O(N) \leq C_G(z)$. Further by (i) we get that $T \cap O_{2',2}(N)$ must be normal in N, so we have (iii). $\square$

Lemma 6.5. *There is some subgroup $N \in \mathcal{N}$ with $S \leq N$.*

Proof. Assume false. Then in particular by Lemma 6.3 we see $C_G(x) \leq C_G(z)$ for all $x \in Z(S)^\sharp$. By Lemma 6.3 we can pick some $N \in \mathcal{N}$ with $|N \cap S|$ maximal. Among all such N choose N minimal. Set $T = S \cap N$. By Lemma 6.3 T is a Sylow 2-subgroup of N. As $S \neq T$ and by the maximal choice of $N \cap S$ we see $N_G(T) \leq C_G(z)$ and then that no nontrivial characteristic subgroup of T is normal in N. Set $W = \Omega_1(Z(O_2(N)))$. By Lemma 6.4 we have $C_N(O_2(N)) \leq O_2(N)O(N)$, and so $\Omega_1(Z(T)) \leq W$. Hence $W \neq \Omega_1(Z(T))$ and then as $J(T) \not\leq O_2(N)$, we have by Lemma 3.2 that W is an F–module for N. As by Lemma 6.4 N is a minimal parabolic with respect to T, this now gives with Lemma 3.4 that any component of $N/C_N(W)$ is isomorphic to $L_2(2^n)$ or A_{2^n+1}, for suitable n, or $N/C_N(W)$ is solvable.

First assume that $N/C_N(W)$ is not solvable, i.e. $1 \neq E(N/C_N(W)) = N_1 * \cdots * N_r$. Then by Lemma 3.15 $W/C_W(E(N/C_N(W))) = V_1 \oplus \cdots \oplus V_r$, where each V_i is a natural N_i–module. By the choice of $N \in \mathcal{N}$, we have that the maximal subgroup M in $E(N/C_N(W))$ containing $TC_N(W)/C_N(W)$ centralizes z. If $N_i \cong L_2(2^u)$, then M is the normalizer of a Sylow 2-subgroup in $E(N/C_N(W))$ and so has no fixed point in $V_1 \oplus \cdots \oplus V_r$. This shows $[z, E(N/C_N(W))] = 1$, which contradicts

Lemma 6.4(i).

So let $N_i \cong A_{2^n+1}$. We have that in each module V_i, which is the irreducible part of the permutation module, a Sylow 2-subgroup of N_i just centralizes a 1-space. As T acts transitively on the components N_i, since N is a minimal parabolic, we get that $|C_{W/C_W(N)}(T)| = 2$. As $|\Omega_1(Z(S))| \geq 4$, we must have some $1 \neq t \in \Omega_1(Z(S))$ with $[E(N/C_N(W)), t] = 1$. But for any such t we know that $C_G(t) \leq C_G(z)$ and so again $[E(N/C_N(W)), z] = 1$, a contradiction to Lemma 6.4(i).

So we have that $N/C_N(W)$ is solvable. As $C_N(W) \leq C_G(z)$, we get by application of Lemma 6.4(i) that $T \cap C_N(W)$ is normal in N, so N is solvable. Set $\tilde{N} = N/O(N)$. As $W = \Omega_1(Z(O_2(\tilde{N})))$ is an F–module, we have by Lemma 3.16 that $\tilde{N}/C_{\tilde{N}}(W) = O_{3,2}(\tilde{N})$. As $C_{\tilde{N}}(W)$ is 2-closed and $\tilde{N}$ is a minimal parabolic we get that $\tilde{N} = O_{2,3,2}(\tilde{N})$. Let P be a Sylow 3–subgroup of $\tilde{N}$. Then obviously P is not contained in $C_N(z)/O(N)$. If C is a proper characteristic subgroup of P, then $CT < \tilde{N}$ and so by minimality of N, we have that $[C, z] = 1$. In particular $[\Phi(P), z] = 1$. Set $W_1 = \langle z^N \rangle$. Then $[W_1, \Phi(P)] = 1$. As T acts irreducibly on $P/\Phi(P)$ we see that $P/\Phi(P)$ acts faithfully on W_1 and so also on $W_2 = [C_W(\Phi(P)), P]$. Let A be an F–module offender on W. As A acts faithfully on $P/\Phi(P)$, we see that A also acts faithfully on W_2 and induces an F–module offender there. Then by the Dihedral Lemma 2.3 we get some $\rho \in P \setminus \Phi(P)$ with $|[W_2, \rho]| = 4$. Further $|W_2 : C_{W_2}(A)| = |A|$. So let $|P/\Phi(P)| = 3^n$, then $|W_2| = 4^n$. We also have that $W_2 = U_1 \oplus \cdots \oplus U_n$, where $|U_i| = 4$ and T acts transitively on the U_i. As we may choose $U_1 = [W_2, \rho]$, where ρ is inverted by some element in A, we see that $|C_{W_2}(T)| = 2$. We have $\Omega_1(Z(S)) \leq W$ and $|\Omega_1(Z(S))| \geq 4$ by Lemma 6.1. Further as $C_G(t) \leq C_G(z)$ for all involutions $t \in \Omega_1(Z(S))$ and $P \not\leq C_G(z)$, we have $\Omega_1(Z(S)) \cap C_N(P) = 1$. Assume $[W, \Phi(P)] = 1$. Then $W = W_2 \oplus C_W(P)$. As $|\Omega_1(Z(S)) \cap W_2| \leq 2$ and $|\Omega_1(Z(S))| \geq 4$, we get $C_{\Omega_1(Z(S))}(P) \neq 1$. But as $C_G(x) \leq C_G(z)$ for all $1 \neq x \in \Omega_1(Z(S))$, we now get $N \leq C_G(z)$, a contradiction. Therefore $W = [W, \Phi(P)] \oplus C_W(\Phi(P))$, with $W_3 = [W, \Phi(P)] \neq 1$. As A is an F-module offender and $|W_2 : C_{W_2}(A)| = |A|$, we must have that $[A, W_3] = 1$. Now $[A, P] \leq C_N(W_3)$. But as $A \not\leq O_2(N)$, we have that $[A, P] \not\leq O_2(N)\Phi(P)$. Hence by the irreducible action of T on $P/\Phi(P)$ we get $P = C_P(W_3)\Phi(P) = C_P(W_3)$, which contradicts $[W_3, \Phi(P)] \neq 1$. $\square$

We now set

$$\mathcal{N}_S = \{N \in \mathcal{N} \text{ with } N \text{ is minimal with respect to } S \leq N \not\leq C_G(z)\}$$

By Lemma 6.5 $\mathcal{N}_S$ is not empty.

Lemma 6.6. *For $N \in \mathcal{N}_S$ we have $C_N(O_2(N)) \leq O_2(N)$.*

Proof. By Lemma 6.4(iii) $O(N) \leq C_G(z)$ and is normalized by S. As $O(C_G(z)) = 1$, we get the assertion from Lemma 2.29 as $A_z \not\cong L_3(3)$ or M_{11} by Proposition 5.2. □

We recall some notation which will be maintained until the end of this chapter.

Notation 6.7. If $A_z = G(q)$, $q = 2^f$, is a group of Lie type not isomorphic to $Sp_{2n}(q)$, we denote by R a long root group in $Z(S \cap A_z)$. In the case of $A_z \cong Sp_{2n}(q)$ we take a short root group. If A_z is a sporadic simple group we choose $R = Z(S \cap A_z)$. Further we denote the group $O_2(C_{A_z}(R))$ by Q_R. The structure of Q_R is given in Lemma 2.17 and Lemma 2.19. In all cases but $A_z \cong Sp_{2n}(q)$ or $F_4(q)$ we have that $R = \Omega_1(Z(Q_R))$. If A_z is a sporadic simple group we have by [MaStr, Lemma 2.10] that Q_R is extraspecial. If $A_z \not\cong Sp_4(q)$ then $R = Q_R'$. Finally we always have that $C_{A_z}(Q_R) = Z(Q_R)$ by Lemma 2.11.

Lemma 6.8. *Let $A_z \cong Sp_4(q)$ or $F_4(q)$ and assume that S induces a graph automorphism on A_z. Set $X = \langle z, R_1, R_2 \rangle$, where $R_1 R_2 = Z(S \cap A_z)$. Then $N_G(X) \leq C_G(z)$.*

Proof. We have $O_2(C_G(X)) = C_S(A_z) Q_{R_1} Q_{R_2}$, so $Z(O_2(C_G(X))) = R_1 R_2 C_S(A_z)$. In particular $\Phi(C_S(A_z))$ is invariant under $N_G(X)$. So if $C_S(A_z) > \langle z \rangle$, we get that $N_G(X) \leq C_G(z)$, the assertion.

Assume now $C_S(A_z) = \langle z \rangle$. As $Z(Q_{R_1} Q_{R_2}) = X \cap (\langle z \rangle Q_{R_1} Q_{R_2})'$, we have that $N_G(X)$ acts on $Z(Q_{R_1} Q_{R_2})$. We have that $N_{C_G(z)}(X)$ induces two orbits of length $2(q-1)$ and $(q-1)^2$ in $(R_1 R_2)^\sharp$ (recall that there is a graph automorphism in S, so R_1 is conjugate to R_2 in S). Further $\emptyset = z^{N_G(X)} \cap Z(Q_{R_1} Q_{R_2})$. As the $|z^{N_G(X)}|$ is odd, we get that either $N_G(X) \leq C_G(z)$ or z has precisely $2q-1$ conjugates under $N_G(X)$, which are $zR_1 \cup zR_2$.

By way of contradiction we assume that z has precisely $2q-1$ conjugates. Then $N_G(X)$ acts 2-transitively on $z^{N_G(X)}$. In particular all $z^g z^h$, $g, h \in N_G(X)$, $z^g \neq z^h$, are conjugate. Choose $r_1, \tilde{r}_1 \in R_1$, $r_2, \tilde{r}_2 \in R_2$ with $r_1 r_2 \neq 1 \neq \tilde{r}_1 \tilde{r}_2$. Then $(zr_1)(zr_2)$ is conjugate to $(z\tilde{r}_1)(z\tilde{r}_2)$. This shows that all elements in $Z(Q_{R_1} Q_{R_2})^\sharp$ are conjugate in $N_G(X)$. As

$Z(Q_{R_1}Q_{R_2})$ contains involutions x which are centralized by S, we see that $|x^{N_G(X)}|$ is odd. Hence x has exactly q^2-1 conjugates. This gives that q^2-1 divides $|N_G(X)/C_G(X)|$.

Assume there is a Zsigmondy prime p dividing q^2-1 and let ω be some element in $N_G(X)$ with $\omega \notin C_G(X)$ but $\omega^p \in C_G(X)$. Suppose first that p does not also divide $2q-1$, then we may assume that $[\omega, z] = 1$. But $|N_{C_G(z)}(X)/C_{C_G(z)}(X)|_{2'}$ divides $(q-1)^2u$, where $q = 2^u$. As p is a Zsigmondy prime, it does not divide $(q-1)$. Hence p divides u. By the little Fermat Theorem we have that p divides $2^{p-1}-1$ which is smaller than $q-1 = 2^u-1$, but this contradicts p being a Zsigmondy prime. Hence we may assume that p divides $2q-1$ which gives $q = 2$ and $p = 3$. By Proposition 5.2 we have $A_z \cong F_4(2)$. As $Q_{R_1} \cap Q_{R_2} = (Q_{R_1}Q_{R_2})'$, we have that ω normalizes $Q_{R_1} \cap Q_{R_2}$ Further it acts on $C_{Q_{R_1}Q_{R_2}}(Q_{R_1} \cap Q_{R_2}) = (Q_{R_1} \cap Q_{R_2})Z(Q_{R_1})Z(Q_{R_2}) = Y$. As $q = 2$ and $Z(Q_{R_1})$ induces a transvection on $Z(Q_{R_2})$, we see $|Y'| = 2$, and so $C_{R_1R_2}(\omega) \neq 1$. As $|R_1R_2| = 4$, we get $[\omega, R_1R_2] = 1$, which then gives the contradiction $[X, \omega] = 1$.

So we have that there is no Zsigmondy prime which divides q^2-1. Hence $q = 8$. By Lemma 2.22 we see $|\mathrm{Out}(F_4(8))| = |\mathrm{Out}(Sp_4(8))| = 2 \cdot 3$. This implies $|S : C_S(X)| = 2$. In particular $N_G(X)/C_G(X)$ has a normal 2–complement K. As $|z^{N_G(X)}| = 15$, we get that $|K| = 3 \cdot 5 \cdot 7^2$ or $3^2 \cdot 5 \cdot 7^2$, as $7^2 = |N_{A_z}(X)/C_{A_z}(X)|$. In both cases with the Burnside lemma we get a normal 5–complement in K. Hence a Sylow 5–subgroup centralizes a Sylow 7–subgroup and then we have a normal Sylow 7-subgroup P in K. As 7^2 divides $|N_{A_z}(X)|$, we have $P \leq C_G(z)$ and P acts as the Borel subgroup on X. This gives $C_X(P) = \langle z \rangle$. But then $\langle z \rangle \trianglelefteq N_G(X)$, a contradiction. $\square$

Lemma 6.9. *$N_G(S) \leq C_G(z)$. In particular $z^G \cap \Omega_1(Z(S)) = \{z\}$.*

Proof. Set $N = N_G(S)$ and assume that $N \not\leq C_G(z)$. We first show that

$$Z(Q_R) \neq R. \tag{1}$$

Suppose false. Assume first that $O_2(C_G(\Omega_1(Z(S)))) = Q_R \times C_S(A_z)$. Set $M = N_G(Q_R \times C_S(A_z))$. Then $N \leq M$. If $Z(Q_R) = R$, then M acts on $\langle z, R \rangle$. As all elements in $R^\sharp$ are conjugate in M, and $|z^M|$ is odd, we would get that $z^M = \langle z, R \rangle^\sharp$. But $R \leq (C_S(A_z) \times Q_R)'$, while z is not, a contradiction. So we have that $O_2(C_G(\Omega_1(Z(S)))) \neq Q_R \times C_S(A_z)$. By Lemma 2.24 we get that $A_z \cong L_3(q)$ or $L_4(q)$. By Proposition 5.2 we have $q > 2$. If $A_z \cong L_3(q)$, then by Lemma 2.20 S contains exactly two abelian groups isomorphic to $\mathbb{Z}_{2^n} \times E_{q^2}$, where $|C_S(A_z)| = 2^n$. If $A_z \cong L_4(q)$, then S contains exactly one abelian group isomorphic to $\mathbb{Z}_{2^n} \times E_{q^4}$. This shows that elements of odd order in N normalize these groups. As $N \not\leq C_G(z)$, we see that $n = 1$. If $A_z \cong L_3(q)$, then the product of these two elementary groups is just $\langle z \rangle Q_R$, which now is normal in N. But $z \notin (\langle z \rangle Q_R)'$, a contradiction as before. So assume $A_z \cong L_4(q)$. Then some graph automorphism is contained in $O_2(C_G(\Omega_1(Z(S))))$. In particular this group contains $\langle z \rangle \times Q_R$ of index two. Then again $z \notin O_2(C_G(\Omega_1(Z(S))))'$ but R is, a contradiction as before. This proves (1)

With [MaStr, Definition 2.32] and (1) we now have that $A_z \cong Sp_{2n}(q)$ or $F_4(q)$. We next show

$$(2) \qquad R \cap \Omega_1(Z(S)) = 1.$$

Suppose false and let first $A_z \not\cong Sp_4(q)$, i.e. Q_R is not abelian. Set $X_z = O_2(C_G(\Omega_1(Z(S))))$. Then $X_z = C_S(A_z) \times (X_z \cap A_z)$. Further $Z(X_z \cap A_z)$ is elementary abelian. As N normalizes X_z, we get $C_S(A_z) = \langle z \rangle$ again. We see that $|X_z \cap A_z : Q_R| = q$ in case of $Sp_{2n}(q)$ and $X_z \cap A_z = Q_{R_1}Q_{R_2}$ in case of $F_4(q)$, where R_1, R_2 are the two root groups in $Z(S \cap A_z)$. Now in both cases $Z(X_z \cap A_z) \leq X_z'$, while $z \notin X_z'$. Let K be a 2–complement of S in $N_G(S)$. Then K acts on $\Omega_1(Z(S)) \cap A_z$ and $\Omega_1(Z(S))/\Omega_1(Z(S)) \cap A_z$. If $|\Omega_1(Z(S)) \cap A_z| > 4$, then $q > 2$, and so $\Omega_1(Z(S)) \cap A_z = [\Omega_1(Z(S)), N_{N_G(A_z)}(S)]$. Hence $\Omega_1(Z(S)) \cap A_z = [\Omega_1(Z(S)), K]$ and we see that $\Omega_1(Z(S)) = (\Omega_1(Z(S)) \cap A_z) \times C_{\Omega_1(Z(S))}(K)$. As $C_{\Omega_1(Z(S))}(N_{N_G(A_z)}(S)) = \langle z \rangle$, we get $[z, K] = 1$, a contradiction.

So we have $|\Omega_1(Z(S)) \cap A_z| = 4$. If $A_z \cong Sp_{2n}(q)$, then $q > 2$ by Proposition 5.2. So we receive $O_2(C_G(\Omega_1(Z(S)))^{(\infty)}) \leq Q_R$ is nonabelian. Hence $R = O_2(C_G(\Omega_1(Z(S)))^{(\infty)})'$ and $|R \cap \Omega_1(Z(S))| = 2$. But then $[K, R \cap \Omega_1(Z(S))] = 1$ and so $[K, \Omega_1(Z(S))] = 1$. So we are left with $A_z \cong F_4(q)$. Now with [MaStr, Definition 2.32] and Lemma 2.17 we receive $O_2(C_G(\Omega_1(Z(S)))') = Q_{R_1}Q_{R_2}$. We further have that $Q_{R_1} \cap Q_{R_2}/R_1R_2$ just involves two natural $Sp_4(q)$–modules and $Q_{R_1}Q_{R_2}/Q_{R_1} \cap Q_{R_2}$ is a direct sum of two modules which are non split extensions of the trivial module by the natural module. As $Q_{R_1} \cap Q_{R_2} = (Q_{R_1}Q_{R_2})'$, we have that K normalizes $Q_{R_1} \cap Q_{R_2}$ and then Y_z, where $Y_z/(Q_{R_1} \cap Q_{R_2})$ is the sum of the trivial modules in $Q_{R_1}Q_{R_2}/(Q_{R_1} \cap Q_{R_2})$, i.e. $Y_z = (Q_{R_1} \cap Q_{R_2})Z(Q_{R_1})Z(Q_{R_2})$. Hence $Y_z' = [Z(Q_{R_1}), Z(Q_{R_2})]$. We have $[K, \Omega_1(Z(S))] \leq A_z$. As $|\Omega_1(Z(S)) \cap A_z| = 4$, there is a field automorphism ν of A_z possibly trivial, such that $\bar{Z}_z = C_{Y_z/Q_{R_1} \cap Q_{R_2}}(\nu)$ is of order 4. As then $\bar{Z}_z = C_{Y_z/Q_{R_1} \cap Q_{R_2}}(S)$, we have that K normalizes $\bar{Z}_z$. For the preimage Z_z we have $|Z_z'| = 2$. Hence $[K, Z_z'] = 1$. As $|\Omega_1(Z(S)) \cap A_z| = 4$, this yields $[K, \Omega_1(Z(S)) \cap A_z] = 1$ and so as $|\Omega_1(Z(S)) : \Omega_1(Z(S)) \cap A_z| = 2$, the contradiction $[K, \Omega_1(Z(S))] = 1$.

To complete the proof of (2) we have to treat $A_z \cong Sp_4(q)$. Then by Proposition 5.2 $q > 2$. We have two root groups R_1, R_2 in $Z(S \cap A_z)$. Let $|C_S(A_z)| = 2^n$. By Lemma 2.21 we have exactly two abelian subgroups $C_S(A_z) \times Q_{R_1}$ and $C_S(A_z) \times Q_{R_2}$ of type $\mathbb{Z}_{2^n} \times E_{q^3}$ in S. Hence N normalizes both and so $C_S(A_z) = \langle z \rangle$. Now N normalizes a Sylow 2–subgroup of $A_z \times \langle z \rangle$, which is $\langle z \rangle \times Q_{R_1}Q_{R_2}$.

We have $(Q_{R_1}Q_{R_2})' = R_1R_2$. Let K be as before a 2-complement in N. Then K acts on R_1R_2. If $|Z(S) \cap R_1R_2| > 4$, we may argue as before. So we may assume that $|Z(S) \cap R_1| = |Z(S) \cap R_2| = 2$. Then again we must have some element $\nu \in S$, which induces a field automorphism on A_z such that $|C_R(\nu)| = 2$. By Lemma 2.22 ν acts in the same way on Q_{R_i}/R_1R_2, $i = 1, 2$. Hence $\bar{Z}_z = C_{Q_{R_1}Q_{R_2}/R_1R_2}(\nu)$ is of order 4. This shows $|Z_z'| = 2$, and so $[N, Z_z] = 1$. But then also $[N, Z(S) \cap R_1R_2] = 1$ and so $[\Omega_1(Z(S)), K] = 1$, a contradiction. This proves (2).

By (2) we have that R does not contain 2–central elements of $C_G(z)$. Then A_z admits a graph automorphism in $C_G(z)$. So $A_z \cong Sp_4(q)$ or $F_4(q)$. Set $X = \langle z \rangle Z(Q_{R_1}Q_{R_2})$, R_i as above. Now $\Omega_1(Z(S)) \leq X$. As before we see that $\langle z \rangle = C_S(A_z)$. Now by Lemma 2.21 $\langle z \rangle \times Q_{R_1}Q_{R_2}$ is the group generated by the elementary abelian subgroups of $O_2(N)$ of order $2q^3$ for $A_z \cong Sp_4(q)$ and $\langle z \rangle \times Q_{R_1}Q_{R_2} = O_2(C_G(\Omega_1(Z(S))))$ if $A_z \cong F_4(q)$. Hence N normalizes $\langle z \rangle Q_{R_1}Q_{R_2}$ in both cases. So $N \leq N_G(X)$. By Lemma 6.8 we have $N_G(X) \leq C_G(z)$ and so also $N_G(S) \leq C_G(z)$, the assertion. $\square$

Lemma 6.10. *If $N \in \mathcal{N}_S$, then $Q_R \not\leq O_2(N)$.*

Proof. Suppose $Q_R \leq O_2(N)$. Assume first that we have $\Omega_1(Z(Q_R)) = R$. Then we have that $\Omega_1(Z(O_2(N))) = \langle z, R_1 \rangle$ with $R_1 \leq R$. But then all elements in $\Omega_1(Z(O_2(N)))$ are 2-central in G. By Lemma 6.9 we then have $z^N \cap Z(O_2(N)) = \{z\}$ and so the contraction $N \leq C_G(z)$.

By [MaStr, Definition 2.32] we are left with $A_z \cong Sp_{2n}(q)$ or $F_4(q)$. Then all involutions in $Z(Q_R)$ are 2–central in A_z. If this is also true in $C_G(z)$, then again all involutions in $\Omega_1(Z(O_2(N)))$ are 2-central and so again $N \leq C_G(z)$.

So S must contain some element which induces a graph automorphism on A_z. This implies $A_z \cong Sp_4(q)$ or $F_4(q)$. In both cases we have that Q_{R_1} and Q_{R_2} both are contained in $O_2(N)$, where R_1, R_2 are the two root subgroups with $R_1R_2 = Z(A_z \cap S)$. Set $X = \langle z\rangle Z(Q_{R_1}Q_{R_2})$, $\Omega_1(Z(O_2(N))) \leq X$. If $A_z \cong F_4(q)$, we see that $C_S(A_z) \times Q_{R_1}Q_{R_2} = O_2(C_G(X))$. If $A_z \cong Sp_4(q)$, we see by Lemma 2.22 and Lemma 2.21 that $\langle C_S(A_z), Q_{R_1}\rangle$, $\langle C_S(A_z), Q_{R_2}\rangle$ are the only two abelian subgroups of order 2^nq^3, $|C_S(A_z)| = 2^n$, in S. Hence in any case we see that $C_S(A_z) \times Q_{R_1}Q_{R_2}$ is normal in N. As $N \not\leq C_G(z)$, we get $C_S(A_z) = \langle z\rangle$. Now N normalizes $Z(\langle z, Q_{R_1}, Q_{R_2}\rangle) = \langle z, R_1, R_2\rangle$. Application of Lemma 6.8 gives the final contradiction. □

The next two lemmas are of central importance for the proof of the main theorem. These describe the structure of $N \in \mathcal{N}_S$. Moreover we show that $q = 2$, if A_z is a group of Lie type over $\mathrm{GF}(q)$, and finally that there is some involution $t \in Z(N)$. In what follows we then determine the centralizer of this involution t, which eventually will yield the final contradiction.

Lemma 6.11. *Let $N \in \mathcal{N}_S$ with $U = \Omega_1(Z(O_2(N))) \leq C_G(A_z) \times Q_R$, then $|\Omega_1(Z(S)) \cap (A_z \times C_G(A_z))| = 4$, $|R| = 2$ and there is some $t \in \Omega_1(Z(S)) \setminus \langle z\rangle$ such that $t \notin A_z$ and $t \in Z(N)$. Further one of the following holds:*

(i) *$N/C_N(U) \cong \Sigma_3$ and $Q_R \trianglelefteq S$; or*
(ii) *$N/C_N(U) \cong \Sigma_3 \wr \mathbb{Z}_2$, $A_z \cong F_4(2)$ and $Q_R \ntrianglelefteq S$*

Proof. By Lemma 6.6 $z \in U$, so $C_G(U) \leq C_G(z)$. We have

$$U \not\leq Z(Q_R) \times C_G(A_z). \tag{1}$$

Otherwise $\langle Q_R^N\rangle \leq C_N(U)$, so $\langle Q_R^N\rangle \leq C_G(z)$. By Lemma 6.4(i) we have $N = N_N(S \cap \langle Q_R^N\rangle)$. Hence $\langle Q_R^N\rangle \leq O_2(N)$, contradicting Lemma 6.10.

In particular by (1) Q_R is not abelian, hence $A_z \ncong Sp_4(q)$. As by (1) $[U, Q_R] = R \leq U$, we have that Q_R induces an elementary abelian group $Q_R/Q_R \cap O_2(N)$ on U.

Let H be a hyperplane in $Z(Q_R)$ not containing Q_R'. Then by [MaStr, Lemma 2.36] Q_R/H is extraspecial. Hence we receive that $|Q_R/H : C_{Q_R/H}(UH/(\langle z\rangle H))| \geq |UH/(\langle z\rangle H)|/|U \cap Z(Q_R)/H|$. So we have that $|Q_R : C_{Q_R}(U)| \geq |U : C_U(Q_R)|$. In particular U is an F–module with quadratic offender $A = Q_R/C_{Q_R}(U)$.

Suppose that $N/C_N(U)$ is nonsolvable. We have that $O^2(N/C_N(U)) = N_1 * \cdots * N_r$, S acts transitively on the N_i and by Lemma 6.4 induces on each a minimal parabolic. By Lemma 3.5 A normalizes each N_i and so induces with some N_i an F–module on U. Hence by Lemma 3.3 and [Asch1, Theorem A] we have that $N_i \cong SL_2(2^n)$ or A_{2^n+1}. Let $V_i = [U, N_i]$, then also $[V_i, Q_R] \leq V_i$ and then we may assume that $R \leq V_1$. But as S acts transitively on the N_i, we get $r = 1$ if R is normalized by S. If there is $t \in S$ with $R^t = \tilde{R}R$, then $\tilde{R} \leq V_2$ and we have $r \leq 2$. If $r = 2$ then $[Q_R, V_2] = 1 = [Q_{\tilde{R}}, V_1]$. In any case we have that $V_1/C_{V_1}(N_1)$ is the natural module by Lemma 3.3, Lemma 3.9 and Lemma 3.10. As $[N_1, V_2] = 1$, we get in any case that $U/C_U(N_1)$ is the natural module.

Let $N_1 \cong SL_2(2^n)$. Then first of all, as $|Q_R : C_{Q_R}(U)| \geq q$ and $|[U, Q_R]| = q$, we have $2^n = q$. Then as N is a minimal parabolic such that the unique maximal subgroup containing S is $C_N(z)$, we have that a Borel subgroup B of N_1 centralizes z. But we have that $U/C_U(N_1)$ is the natural module. Now $C_U(B) = C_U(N_1)$ by Lemma 3.14, and so $z \in Z(N)$, a contradiction.

Let $N_1 \cong A_{2^n+1}$, $n > 1$. Then $U/C_U(N_1)$ is the permutation module. We have again that z is centralized by some subgroup $L \cong A_{2^n}$ in N_1. By Lemma 3.13 we see that $\Omega_1(Z(S)) \leq C_U(C_{N_1}(z))$. Hence $[C_N(z), \Omega_1(Z(S))] = 1$. So $C_N(z) \leq C_N(\Omega_1(Z(S)))$ and then $O_2(C_N(z)) \leq O_2(C_N(\Omega_1(Z(S))))$. In particular we have that $Q_R \leq O_2(C_N(z))$. As $Q_R \not\leq C_N(U)$, so $O_2(L) \neq 1$, we get that $2^n = 4$. So we have that $N_1/C_{N_1}(U) \cong A_5$ and Q_R projects onto a subgroup of a Sylow 2-subgroup of N_1. As $[N_1, U]$ is the permutation module now Q_R cannot be an offender.

Assume next that $N/C_N(U)$ is solvable. Set $U_1 = \langle z^N \rangle$. By Lemma 3.16 we have that $C_N(U_1)$ is 2-closed and $N/C_N(U_1) = O_{3,2}(N/C_N(U_1))$. Hence as N is a minimal parabolic, we receive $N = SP$, where P is a Sylow 3–subgroup of N. Further by the minimal choice of N we have that $\Phi(P)$ centralizes z, so $[\Phi(P), U_1] = 1$. Let A be an F–module offender, which normalizes P. As $O_2(N)$ is a Sylow 2–subgroup of $C_N(U_1)$, we have that A exists and $[a, P] \not\leq \Phi(P)$ for $a \in A^\sharp$. Hence A induces an F–module offender on U_1 too. By Lemma 3.17 we get that $|U_1 : C_{U_1}(A)| = |A|$. As $|U : C_U(A)| \leq |A|$, we see that $[U, A] \leq U_1$. As S acts irreducibly on $P/\Phi(P)$, we see that $[U, P] \leq U_1$ and so $[U, \Phi(P)] \leq C_U(\Phi(P))$. Hence $[U, \Phi(P)] = 1$. This shows that $C_N(U_1) = C_N(U)$ and so P induces an elementary abelian group on U.

By Lemma 2.3 we get a direct product $M = M_1 \times \cdots \times M_r$ of dihedral groups M_i of order 6 contained in $N/C_N(U)$ with $Q_R/C_{Q_R}(U)$ as a Sylow 2–subgroup. As $U \le Q_R \times C_S(A_z)$ we see $[U, Q_R] \le R$ and so Q_R acts quadratically on U. We get that $[U, O_3(M)] = V_1 \oplus \cdots \oplus V_r$, with $[O_3(M_i), V_i] = V_i$ and $[O_3(M_i), V_j] = 1$, $i, j = 1, \ldots, r$, $i \neq j$. As $[Q_R, V_i] \le R$ and $[V_1, Q_R] = R$, we get $r = 1$ and $|Q_R/C_{Q_R}(U)| = 2$. Then also $|[U, Q_R]| = 2$. If R is normalized by S, Q_R must invert $P/C_P(U))$, so $|[P/\Phi(P), Q_R]| = 3$ and $N/C_N(U) \cong \Sigma_3$, which is (i). In the other case $P/C_P(U) = [P/C_P(U), Q_R Q_R^t]$ for some t in $S \setminus N_S(Q_R)$. Hence $|P/C_P(U)| = 9$. We have a fours group acting on P and $U/C_U(P)$ is the natural $O_4^-(2)$–module. In both cases $|Q_R : C_{Q_R}(U)| = 2$. This shows that Q_R is extraspecial or isomorphic to $E \times Q$, with Q extraspecial and $E \le Z(Q_R)$. In particular if $Q_R \ntrianglelefteq S$, we get that $A_z \cong F_4(2)$, which is (ii).

Suppose that $|Z(S) \cap E \times Q| > 2$. Then by Proposition 5.2 we have that $A_z \cong F_4(2)$. Further Q_R is normal in S and so $C_G(z) = A_z \times C_G(A_z)$. As $z \notin Z(N)$, we have $C_S(A_z) = \langle z \rangle$. Then Lemma 2.2 and Lemma 6.9 give a contradiction. So we have

$$|\Omega_1(Z(S)) \cap A_z \times C_S(A_z)| = 4.$$

To prove the lemma, we just have to show the existence of the involution t.

In any case we have that $U = [O_{2,3}(N), U] \times C_U(O_{2,3}(N))$. Furthermore $|\Omega_1(Z(S))| = 4$ and $|C_{[O_{2,3}(N),U]}(S)| = 2$. This implies $C_U(O_{2,3}(N)) \neq 1$. Hence there is some $t \in \Omega_1(Z(S)) \setminus \langle z \rangle$ with $[t, N] = 1$. We have that $[[O_{2,3}(N), U], Q_R] \neq 1$. In particular we have that $R \le [O_{2,3}(N), U]$. Hence $\Omega_1(Z(S)) \cap A_z \le [O_{2,3}(N), U]$. As $t \notin [O_{2,3}(N), U]$ we get $t \notin A_z$. □

Lemma 6.12. *Let $N \in \mathcal{N}_S$ with $U = \Omega_1(Z(O_2(N))) \not\le C(A_z) \times Q_R$. Then $|\Omega_1(Z(S)) \cap (A_z \times C_G(A_z))| = 4$, $|R| = 2$ and $Q_R \trianglelefteq S$. Further $E(N/C_N(U)) \cong A_5$ and induces just one nontrivial irreducible module in U, the permutation module, or $N/C_N(U) \cong O_4^+(2)$ and just the natural module is induced in U. Further there is some $t \in \Omega_1(Z(S)) \setminus \langle z \rangle$ such that $t \notin A_z$ and $t \in Z(N)$.*

Proof. We first show

(1) $\qquad$ U normalizes Q_R.

If U does not normalize Q_R we get $A_z \cong Sp_4(q)$ or $F_4(q)$. We have $[U, Q_R] \le U$. In particular $[U, Q_R]$ is abelian. From Lemma 2.26 we get

$A_z \not\cong Sp_4(q)$ or $F_4(q)$. This proves (1).

Next we show

$$O_2(N) \leq N_{N_G(A_z)}(R). \tag{2}$$

Otherwise $R \cap U = 1$. Hence by (1) $[[U, Q_R], Q_R] = 1$, which shows that $[U, Q_R] \leq Z(Q_R)$. As $U \not\leq Q_R \times C_G(A_z)$, we get from Lemma 2.25 that $A_z \cong Sp_4(q)$ and Q_R is elementary abelian. Let R_1, R_2 be the two root groups in $Z(S \cap A_z)$. As U is elementary abelian, we have that $U \cap A_z \times C_S(A_z)$ is contained in $R_1R_2 \times C_S(A_z)$, recall that $R_1R_2 = Q_{R_1} \cap Q_{R_2}$ by Lemma 2.21. Then we also see that U cannot contain elements which induce field automorphisms on A_z, otherwise for such $u \in U$, we have that $1 \neq [u, R] \leq R$, contradicting $R \cap U = 1$. Hence $U \leq R_1R_2 \times C_G(A_z)$, contradicting $U \not\leq Q_R \times C_G(A_z)$. So we have (2).

Now we apply Lemma 3.21 with N in the role of M and $N_{N_G(A_z)}(R)$ in the role of H. Suppose that $X = O_2(N_{N_G(A_z)}(R)) \not\leq C_G(A_z) \times Q_R$. Then there is some $x \in X$ inducing an outer automorphism on A_z. In particular A_z is of Lie type in characteristic two. If x is a field automorphism it acts on a group of order $q-1$ which acts nontrivially on R, so x cannot be contained in X. Hence x acts nontrivially on the Dynkin diagram and so has to centralize the Levi factor. This shows $A_z \cong L_4(q)$. By Proposition 5.2 we have $q > 2$. But then x acts nontrivially on a group $\mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$ in $N_{A_z}(R)$, a contradiction. So we have that $X \leq C_G(A_z) \times Q_R$ and then $U \not\leq X$. Hence from Lemma 3.21 we get

$$U \text{ is a } 2F-\text{module}. \tag{3}$$

We show

(U) If $\tilde{U}$ is a Q_R–invariant submodule of U with $[Q_R, \tilde{U}] \neq 1$. Then either Q_R is abelian or $R \leq \tilde{U}$. Further $O_2(N) \leq N_N(Q_R)$.

Suppose Q_R to be nonabelian. By (1) $[\tilde{U}, Q_R] \leq Q_R$. Then $[\tilde{U} \cap Q_R, Q_R] = 1$ or $[\tilde{U} \cap Q_R, Q_R] = R$. In the latter $R \leq \tilde{U}$. In the former we have $[Q_R, \tilde{U}] \leq Z(Q_R)$. Hence by Lemma 2.25 $\tilde{U} \leq Q_R \times C_S(A_z)$ and so $[\tilde{U}, Q_R] = R$ and we are done. The second statement in (U) follows by (2).

We now first work under the assumption:

(A) Assume N is nonsolvable.

Let $E(N/C_N(U)) = N_1 * \cdots * N_r$. Assume first that for an offender A as a $2F$-module which is given by Lemma 3.21 we have that $[A, N_1] \not\leq N_1$. If A acts quadratically then by Lemma 3.21 we have that A induces an F–module offender. But this contradicts Lemma 3.5. So A cannot be quadratic on U. By Lemma 3.22 we get that $N_1 \cong L_n(2)$, and for some $a \in A$ with $N_1^a \neq N_1$ we have that A induces the full transvection group on $[U, a]$. Hence $C_{N_1 N_1^a}(a)$ induces the natural module on $[U, a]$. As $N_1 N_S(N_1)$ is a minimal parabolic by Lemma 6.4, we have $n = 3$ and with the natural module also the dual module is involved. Hence for $C_{N_1 N_1^a}(a)$ we have a natural and a dual module involved in U, which contradicts that $C_{N_1 N_1^a}(a)$ induces just the natural module.

So we have that

(A.1) The offender A from Lemma 3.21 normalizes all components.

We have $|U : C_U(A)| < |A : C_A(U)|^2$ by Lemma 3.21. Now we choose A minimal such that $|U : C_U(A)| < |A : C_A(U)|^2$. Set $A_1 = C_A(N_1)$. Then we have that $|U : C_U(A_1)| \geq |A_1 : C_{A_1}(U)|^2$. In particular for a complement B of A_1 in A we have that $|C_U(A_1) : C_{C_U(A_1)}(B)| < |B : C_B(C_U(A_1))|^2$, which yields:

(A.2) Let $T = C_S(N_1)$, then we have that $V = C_U(T)$ is a 2F–module for N_1 with offender B such that $|V : C_V(B)| < |B|^2$.

Application of Lemma 6.10 yields $Q_R \not\leq O_2(N)$. By Lemma 6.4(i) we have that $O_2(N)$ is normal in $C_N(U)$. This shows $[Q_R, U] \neq 1$. Hence $[N_1 * \cdots * N_r, Q_R] \neq 1$. So we may assume that $[N_1, Q_R] \neq 1$. Set $U_1 = [N_1, V]$.

Now by (A.2) N_1 and U_1 are given in Lemma 3.4. As $[z, N_1] \neq 1$ we get $C_V(C_{N_1}(z)) \neq C_V(N_1)$. If the irreducible N_1–modules in V are F–modules, we have that $N_1 \cong L_2(2^n)$ and $C_{N_1}(z)$ is a Borel subgroup, a contradiction to Lemma 3.14, or $N_1 \cong A_{2^n+1}$ and $C_{N_1}(z) \cong A_{2^n}$. By Lemma 3.12 we see that $V/C_V(N_1)$ is the permutation module. Then Lemma 3.14 shows that $C_{N_1}(z)$ centralizes $\Omega_1(Z(S))$. So we see that $Q_R \leq O_2(C_N(z))$. We get $N_1 = A_5$ and then U_1 is the permutation module.

So assume now that we have Lemma 3.4(b). In the first three cases always $C_{N_1}(z)$ is a Borel subgroup, which has a fixed point on the corresponding modules exactly when $r = 2$, so $N_1 \cong L_3(2)$ or $Sp_4(2)'$. In both cases we have that $U_1/C_{U_1}(N_1)$ is a direct sum of a natural module and its dual. Assume now $N_1 \cong A_9$ and $|U_1/C_{U_1}(N_1)| = 2^8$. Then $z \in C_U(N_1)$ by Lemma 3.14, a contradiction.

So we collect:

(A.3) $N_1 \cong L_2(4), L_3(2),$ or A_6. In the last two cases we have $U_1/C_{U_1}(N_1) = U_{11} \oplus U_{12},$ where U_{11} and U_{12} are dual modules for N_1. In the first case we have that U_1 is the permutation module.

Next we show

(A.4) $A_z \not\cong F_4(q)$. Further if $A_z \cong Sp_4(q),$ then Q_R is elementary abelian and Q_R acts quadratically on U.

The second statement follows from (1). So assume $A_z \cong F_4(q)$. Suppose first $\langle U_1^S \rangle \le Q_R \times C_S(Q_R)$. Then $[\langle U_1^S \rangle, Z(Q_R)] = 1$. Hence $[\langle N_1^S \rangle, Z(Q_R)] \le O_2(N)$. This shows that $Z(Q_R) \le O_2(N)$ and so $[U, Z(Q_R)] = 1$, which by Lemma 2.17 shows $U \le Q_R \times C_S(Q_R)$, a contradiction. Thus we may assume that $U_1 \not\le Q_R \times C_S(Q_R)$. Then for $u \in U_1 \setminus Q_R \times C_G(A_z)$, we obtain that $|[Q_R, u]| \ge q^4$. As $R \le U$ by (U) we receive that $Q_R O_2(N)/O_2(N)$ is elementary abelian. Hence we get from (A.3) that $|Q_R : C_{Q_R}(U_1)| \le 8$ if Q_R normalizes N_1, a contradiction to $|[Q_R, u]| \ge q^4 \ge 16$. So we have that Q_R does not normalize N_1. Then at least a subgroup T of index two in $S \cap N_1$ normalizes Q_R, as this is true in $\mathrm{Aut}(F_4(q))$, and then $[Q_R, T]$ is abelian and centralized by Q_R. Hence we get that $|N_1^{Q_R}| = 2$ and then $|Q_R : C_{Q_R}(U_1)| \le 16$. In particular $q = 2$. Further U_1 does not induce transvections on $Z(Q_R)$, as for any transvection $u \in U_1$ we have $|[Q_R/Z(Q_R), u]| = 16$ by Lemma 2.17. This implies $N_1 \cong A_6$ and further $Sp_4(2)$ is induced. Now $Z(Q_R)$ acts quadratically on U and so we have by Lemma 3.5 that $Z(Q_R)$ normalizes N_1. Then it acts quadratically on U_1. As U_1 involves the natural module and the dual as well, we see that $Z(Q_R)$ induces a group of order at most four which is in the center of a Sylow 2–subgroup of $Sp_4(2)$. But then U_1 contains some u which induces a transvection on $Z(Q_R)$, a contradiction. This proves (A.4).

(A.5) $$[Q_R, N_1] \le N_1.$$

Suppose false. If $A_z \cong Sp_4(q)$, then by Proposition 5.2 $q > 2$. Hence $|Q_R : C_{Q_R}(U_1)| \ge 4$. By (A.4) Q_R acts quadratically on U. We get by Lemma 3.5 that $N_1 \cong L_2(4)$. Further $\langle N_1^{Q_R}\rangle$ induces just natural $\Omega_4^+(4)$–modules in U, contradicting the fact that by (A.3) N_1 induces an $\Omega_4^-(2)$–module. So we have that $A_z \not\cong Sp_4(q)$ and by (A.4) $Q_R \trianglelefteq S$. We further have that $R \leq U$ by (U) and so $Q_R O_2(N)/O_2(N)$ is elementary abelian. Hence N_1 has elementary abelian Sylow 2-subgroups, as for $t \in Q_R$, with $N_1^t \neq N_1$, we have that $[N_1, t]$ has a Sylow 2-subgroup contained in $Q_R O_2(N)/O_2(N)$ and so is abelian. Then $N_1 \cong L_2(4)$ again. We further have $|Q_R : N_{Q_R}(N_1)| = 2$. Set $W_1 = [U_1, N_{Q_R}(N_1)]$. As $N_{Q_R}(N_1)$ projects onto a Sylow 2–subgroup of N_1 and N_1 induces an $\Omega_4^-(2)$–submodule, we have $[W_1, N_{Q_R}(N_1)] \neq 1$. As U_1 normalizes Q_R, we have that $W_1 \leq Q_R$ and so $|R \cap U_1| = 2$. Set $W_1 = \langle U_1 \cap R, x_1, y_1\rangle$ and choose $x \in Q_R$ with $N_1^x = N_2$. We have $|Q_R : C_{Q_R}(x_1)| \leq 4$. From Lemma 2.17 we see that $|Q_R : C_{Q_R}(x_1)| \geq q$. Hence $|R| = q \leq 4$. As $[W_1, x] \leq R$ and $|R : R \cap U_1| \leq 2$, we may assume that $[x_1, x] \in U_1 \cap R$. Hence $|Q_R : C_{Q_R}(x_1)| = 2$, which gives $|R| = 2 = q$ and $R \leq U_1$. But then $[Q_R, W_1] \leq R \leq W_1$. This shows $W_1^x = W_1$. Set $M = N_{N_1}(W_1)N_{N_1}(W_1)^x$, which is isomorphic to $A_4 \times A_4$. Then M acts on W_1. Hence there is some element of order three in M which centralizes W_1. But then $O_2(M)$ centralizes W_1 too, which contradicts the action of $N_{Q_R}(N_1)$ on W_1. This proves (A.5)

Next we show

(A.6) $N_1 \cong A_5$ and U_1 is the irreducible part of the permutation module.

According to (A.3) we may assume $N_1 \cong L_3(2)$ or A_6. Assume further $A_z \not\cong Sp_4(q)$. If Q_R normalizes both modules U_{11} and U_{12} given in (A.3) then by (U) $R \leq U_{11} \cap U_{12}$, a contradiction. Hence there is $x \in Q_R$ with $U_{11}^x = U_{12}$. But then x induces an outer automorphism of $L_3(2)$ or Σ_6 and then $[x, S/O_2(N)]$ is not abelian. By (U) we have $R \leq O_2(N)$ and so $Q_R/Q_R \cap O_2(N)$ is elementary abelian. This contradicts $Q_R \trianglelefteq S$ and $[x, S/O_2(N)]$ being not abelian.

So we have that $A_z \cong Sp_4(q)$, $q \geq 4$. Then by (A.4) Q_R is elementary abelian and acts quadratically on U. As $|Q_R/O_2(N) \cap Q_R| \geq 4$, we see that $Q_R \cap N(U_{11}) \not\leq C(U_{11})$. By quadratic action we get that Q_R normalizes U_{11} and U_{12}. This even shows $Q_R/Q_R \cap O_2(N) \cap N_1 \neq 1$. In particular $|U_1 : C_{U_1}(Q_R)| \geq 16$. As by Lemma 3.8 in Σ_6 no subgroup of order 8 acts quadratically on both modules, we get that Q_R induces a foursgroup on N_1 and so $q = 4$. But then $N_{C_G(z)}(Q_R)/C_{C_G(z)}(Q_R)$ is isomorphic to a subgroup of $(A_5 \times \mathbb{Z}_3) : \mathbb{Z}_2$ and so contains no elementary abelian subgroup of order 16, but $U_1/C_{U_1}(Q_R)$ contains such an elementary abelian subgroup. This proves (A.6).

Next we are going to describe the structure of N. We have $N_1 \cong L_2(4)$. Further we have that U_1 is the permutation module. As before we see by (U) that $R \leq U_1$ for Q_R not abelian. This shows that Q_R acts quadratically on U/U_1 in any case. As by Lemma 3.14 $Q_R \leq O_2(C_N(z))$, we see that Q_R projects into a subgroup of $C_{N_1}(z) \cong A_4$. If this projection is of order two, we get that U_1 induces transvections on Q_R. In particular $A_z \not\cong Sp_4(q)$. This now shows that $U_1 \leq Q_R C_G(Q_R)$. But $|U_1 : C_{U_1}(Q_R)| = 4$ and so $|Q_R : C_{Q_R}(U_1)| \geq 4$. Hence $Q_R/Q_R \cap O_2(N)$ acts as a Sylow 2–subgroup of A_5, which is not quadratic on the permutation module. In particular $Q_R \trianglelefteq S$ and $A_z \not\cong Sp_4(q)$ by (A.4). Hence U_1 is the only permutation module for N_1 involved in U. This shows that $[U_1, N_i] = 1$ for $i = 2, \cdots, r$. Choose $s \in S$ with $N_1^s = N_2$. Then by (U) we have that $R \leq U_1 \cap U_2 = 1$, a contradiction. This shows $r = 1$. Now we have that $U = U_1 \oplus U_2$, with some N-module U_2. As $R \leq U_1$, we get from (U) that U_2 is a trivial $E(N/O_2(N))$–module. Hence

$$U = U_1 \oplus C_U(N_1).$$

So we have shown

(A.7) If $N/C_N(U)$ is nonsolvable, then $E(N/C_N(U)) \cong A_5$ and $U = U_1 \oplus C_U(E(N/C_N(U)))$, where U_1 is the permutation module. Further $|R| = 2, R \leq U_1$ and $Q_R \trianglelefteq S$.

Now we assume:

(B) Assume N is solvable.

By Lemma 6.4 $N = O_{2,2',2}(N)$. As by Lemma 3.21(2) offenders are not exact provided U is not an F–module, we get with Lemma 3.17 that $N/C_N(U)$ is a $\{2,3\}$–group. As N is a minimal parabolic we have $N = O_{2,3,2}(N)$. By minimality we have that $\Phi(O_{2,3}(N)/O_2(N)) \leq C_N(z)/O_2(N)$. So $\Phi(O_{2,3}(N)/O_2(N))$ centralizes

$$\langle z^N \rangle = U_1,$$

which gives that S acts irreducibly on $O_3(N/C_N(U_1))$.

We show

(∗) $C_{O_{2,3}(N)}(U) = C_{O_{2,3}(N)}(U_1)$ and so $[O_{2,3}(N)', U] = 1$.

For this let P be a Sylow 3–subgroup of N such that $O_2(N)N_N(P) = N$. In particular $P/C_P(U_1)$ is elementary abelian. Hence we may assume that a $2F$–module offender A with $|U : C_U(A)| < |A|^2$ acts on P. We have that $|U_1 : C_{U_1}(A)| \geq |A|$ by Lemma 3.17. Hence we conclude

$|U/U_1 : C_{U/U_1}(A)| < |A|$. So by Lemma 3.17, we get some $1 \neq a \in A$, which acts trivially on U/U_1. This gives that $C_P(U/U_1) \not\leq \Phi(P)$. As $N_N(P)$ acts irreducibly on $P/\Phi(P)$, we get that $P = \Phi(P)C_P(U/U_1)$ and then $[P, U] \leq U_1$. In particular $[C_P(U_1), U] = 1$. This is $(*)$.

Application of (U) shows that for $A_z \not\cong Sp_4(q)$ we have $Q'_R = R \leq U_1$.

So we have

(B.1) $$Q_R C_N(U)/C_N(U) \text{ is abelian.}$$

Let $|Q_R : C_{Q_R}(U)| = 2$. Then U induces a transvection on Q_R with elementary abelian commutator, so $U \leq Q_R C_S(Q_R)$, a contradiction.

We receive

(B.2) $$|Q_R : C_{Q_R}(U)| \geq 4.$$

By the Dihedral Lemma 2.3 we have a subgroup $D_1 \times \cdots \times D_s$, $s \geq 2$ in $N/C_N(U)$, $D_i = \langle x_i, \rho_i \rangle$, $x_i \in Q_R$, $o(\rho_i) = 3$, $D_i \cong \Sigma_3$, $i = 1, \ldots, s$.

Set $W = [[\rho_1, U], x_1]$. We have $W \leq Q_R$. If $[Q_R, W] = 1$, then $W \leq Z(Q_R)$. As $\langle W^{\rho_1} \rangle = [\rho_1, U]$, we get $[x_i, [\rho_1, U]] = 1$, $i = 2, \ldots, s$, and so $[[\rho_1, U], Q_R] \leq Z(Q_R)$. Now the elements in $[\rho_1, U]$ induce transvections on Q_R, which gives that $A_z \not\cong Sp_4(q)$, $q > 2$. Application of Lemma 2.25 shows $[\rho_1, U] \leq Q_R$ and so $[[\rho_1, U], Q_R] \leq R$. As $[\langle x_2, \ldots, x_s \rangle, [\rho_1, U]] = 1$, we see that $|Q_R : C_{Q_R}([\rho_1, U])| = 2$ and so we have $q = 2$, and $W = R$ is of order 2, further $|[U, \rho_1]| = 4$. Set $T = N_S(Q_R)$ and let $t \in T$. Then $R \leq [U, \rho_1] \cap [U, \rho_1^t]$. But as $[\rho_1, \rho_1^t] \in C_N(U)$ by $(*)$, we have $[U, \rho_1, \rho_1^t] \leq [U, \rho_1]$. This yields $\langle \rho_1 \rangle C_N(U) = \langle \rho_1^t \rangle C_N(U)$. Now also $\langle \rho_1^T \rangle C_N(U)/C_N(U) = \langle \rho_1 \rangle C_N(U)/C_N(U)$. By (B.2) we have that $O_{2,3}(N)/C_N(U)$ contains an elementary abelian group of order 9. So we get that $|S : T| = 2$ and $O_3(N/C_N(U)) = \langle \rho_1, \rho_1^s \rangle C_N(U)/C_N(U)$, for some $s \in S \setminus T$. This shows $|[U, O_{2,3}(N)]| = 16$ and so $N/C_N(U)$ is a subgroup of $GL_4(2)$, which gives that $S/C_S(U)$ is contained in a dihedral group. But as $|Q_R/C_{Q_R}(U)| = 4$, this shows that Q_R is normal in S, a contradiction.

So we have

(B.3) $$[Q_R, [[U, \rho_i], x_i]] \neq 1 \text{ for all } i = 1, \ldots, s.$$

As by (B.3) Q_R does not act quadratically, we have that Q_R is not abelian and so

(B.4) $$A_z \not\cong Sp_4(q).$$

By (B.3) and (U) we have $R \le [U, \rho_1]$. Hence $R \cap C_U(\rho_1) = 1$. So by (U) we get that $[Q_R, C_U(\rho_1)] = 1$. In particular $[C_U(\rho_1), \rho_2] = 1$. By choosing ρ_1 with $C_U(\rho_1)$ maximal we obtain

$$C_U(\rho_1) = C_U(\rho_i) \text{ and } [U, \rho_1] = [U, \rho_i] \text{ for } i = 1, \dots, s. \tag{B.5}$$

Now we consider $\langle \rho_1, \rho_2 \rangle$. We have $(\rho_1\rho_2)^{x_2} = \rho_1\rho_2^{-1}$. Then $[U, \rho_1] = C_{[U,\rho_1]}(\rho_1\rho_2) \times C_{[U,\rho_1]}(\rho_1\rho_2^{-1})$. Set $V_1 = C_{[U,\rho_1]}(\rho_1\rho_2)$. We have that x_1x_2 normalizes V_1 and $[V_1, x_1x_2] \le Q_R$. Set $V_2 = V_1^{x_2}$, then we obtain $1 \ne [[V_1, x_1x_2], x_2] \le R$. Further $|[[V_1, x_1x_2], x_2]| = |[V_1, x_1x_2]|$. As x_1x_2 inverts $\rho_1\rho_2^{-1}$ and $\rho_1\rho_2^{-1}$ acts fixed point freely on V_1, we get that $|V_1| = |[V_1, x_1x_2]|^2 \le |R|^2 = q^2$. This gives

$$|[U, \rho_1]| \le q^4. \tag{B.6}$$

Suppose $s \ge 3$. Now x_3 centralizes $\rho_1\rho_2$ and so normalizes V_1 and $[V_1, x_1x_2]$. This gives $[[V_1, x_1x_2], x_3] \le R \cap V_1$. As $R \cap V_1 = (R \cap V_1)^{x_2} = R \cap V_2$ and $V_1 \cap V_2 = 1$, we get $[[V_1, x_1x_2], x_3] = 1$. But as $[x_3, \rho_1\rho_2^{-1}] = 1$ and $V_1 = \langle [V_1, x_1x_2]^{\rho_1\rho_2^{-1}} \rangle$ we then have $[x_3, V_1] = 1$ and also $[x_3, V_1^{x_2}] = 1$. This gives $[[U, \rho_1], x_3] = 1$. But then $[[U, \rho_1], \rho_3] = 1$, a contradiction to (B.5). So we have

$$s = 2. \tag{B.7}$$

Suppose that $[V_1, x_1x_2] \le C_G(Q_R)$. Then

$$|Q_RC_S(Q_R)/C_S(Q_R) : C_{Q_RC_S(Q_R)/C_S(Q_R)}(V_1)| \le 2.$$

By (B.4) and Lemma 2.4 we see that $V_1 \le Q_RC_S(Q_R)$. Now also $V_2 = V_1^{x_2} \le Q_RC_S(Q_R)$, which gives $[U, \rho_1] \le Q_RC_S(Q_R)$. This shows $[[[U, \rho_1], x_1], Q_R] = 1$, which contradicts (B.3). Hence we have that $[V_1, x_1x_2]$ centralizes a subgroup of index two in Q_R, which implies

$$q = 2. \tag{B.8}$$

Assume now $|S : T| = 2$, $T = N_S(Q_R)$. Then by (B.4) and (B.8) $A_z \cong F_4(2)$. As $[U, \rho_1] \not\in Q_R$, we have for $1 \ne u \in [U, \rho_1]$ that $|Z(Q_R) : C_{Z(Q_R)}(u)| \ge 2$ and $[Q_R/Z(Q_R) : C_{Q_R/Z(Q_R)}(u)| \ge 4$. In particular $|Q_R : C_{Q_R}(u)| \ge 8$, which contradicts $|Q_R : C_{Q_R}(U)| = 4$ by (B.7).

So we have that $Q_R \trianglelefteq S$. Further $[\langle \rho_1, \rho_2 \rangle, U]$ is of order 16 by (B.6) and (B.8). As above we see that $[\langle \rho_1, \rho_2 \rangle, U] = [\langle \rho_1, \rho_1 \rangle^s, U]$ for all $s \in S$. In particular $O_{2,3}(N)/C_N(U)$ is of order 9.

So we have shown

$$\text{(B.9)} \qquad \begin{array}{l} Q_R \trianglelefteq S, |R| = 2, N/C_N(U) \cong O_4^+(2) \text{ and} \\ [U, O_3(N/C_N(U))] \text{ is the natural module.} \end{array}$$

As $R \leq [U, O_3(N/C_N(U))]$ and $[Q_R, O_3(N/C_N(U))] = O_3(N/C_N(U))$, we get

$$\text{(B.10)} \qquad U = [U, O_3(N/C_N(U))] \times C_U(O_3(N/C_N(U))).$$

Hence in both cases, N solvable and nonsolvable, by (A.7) and (B.9) we just need to prove the existence of t and determine the size of $|\Omega_1(Z(S))|$.

For the remainder N might be solvable or not. Assume $|\Omega_1(Z(S)) \cap A_z| > 2$. By (B.8) and (A.7) $q = 2$. So we have that $A_z \cong Sp_{2n}(2)$ or $F_4(2)$. By Proposition 5.2 we have $A_z \not\cong Sp_{2n}(2)$. Now in $[U, O^2(N)]$, we have some x such that $x \notin Q_R$ but $|[Q_R/R, x]| = 4$. As $A_z \cong F_4(2)$, then by Lemma 2.17 Q_R/R involves two non isomorphic modules for $N_{A_z}(R)$ on one there are transvections on the other not, a contradiction. So we have $|\Omega_1(Z(S)) \cap A_z| = 2$.

As $|\Omega_1(Z(S))| \geq 4$, we see $|\Omega_1(Z(S))| = 4$ and from (A.7) and (B.10) we get that $C_U(N) \neq 1$ and so there is some $1 \neq t \in \Omega_1(Z(S))$, which is central in N. By (U) $R \leq [U, F^*(N/C_N(U))]$. As $|\Omega_1(Z(S)) \cap A_z| = 2$ we have that $\Omega_1(Z(S)) \cap A_z \leq \langle R^S \rangle$ and so $\Omega_1(Z(S)) \cap A_z \leq [U, F^*(N/C_N(U))]$. Hence $t \notin A_z$. □

We now can get further restrictions on the structure of A_z.

Lemma 6.13. *$A_z \not\cong F_4(2)$. Further Q_R is extraspecial with center R, normal in S and $N_{N_G(A_z)}(Q_R)$ acts irreducibly on Q_R/R.*

Proof. Suppose $A_z \cong F_4(2)$. By Lemma 6.11 and Lemma 6.12 we have that $|\Omega_1(Z(S)) \cap A_z| = 2$. Hence there is some $u \in C_G(z)$, which induces a graph automorphism on A_z. In particular $Q_R \ntrianglelefteq S$. This shows by Lemma 6.12 that Lemma 6.11(ii) holds. In particular $U = \Omega_1(Z(O_2(N)) \leq Q_R C_S(A_z)$. As $z^G \cap U \neq \{z\}$ also $z^G \cap \langle z \rangle \times Q_R \neq \{z\}$. Let r_1, r_2 be the two root elements such that $\langle r_1, r_2 \rangle = Z(S \cap A_z)$. Then $C_S(\langle z, r_1, r_2 \rangle) = C_S(A_z) \times (S \cap A_z)$. As $Q_R \leq O_2(C_G(\langle z, r_1, r_2 \rangle))$, we get from Lemma 6.10 that $N_G(\langle z, r_1, r_2 \rangle)$ does not contain some element in $\mathcal{N}_S$. Hence $N_G(\langle z, r_1, r_2 \rangle) \leq N_G(A_z)$. Let $v \in \langle z, r_1, r_2 \rangle$ such that $|S : C_S(v)| = 2$. So $\Omega_1(Z(C_S(v))) = \langle z, r_1, r_2 \rangle$. As $N_G(\Omega_1(Z(C_S(v)))) \leq A_z$, we see that $C_S(v)$ is a Sylow 2-subgroup of $C_G(v)$ and so $v \not\sim z$ in G. As $z^G \cap \Omega_1(Z(S)) = \{z\}$ by Lemma 6.9 we get that $z^G \cap \langle z, r_1, r_2 \rangle = \{z\}$. In particular $z^G \cap Z(\langle z, Q_R \rangle) = \{z\}$. On the other hand we have some

$v \in U$, $z \neq v \sim z$. This implies $v \notin Z(Q_R)\langle z\rangle$. By Lemma 6.11 we see $|S : C_S(v)| \leq 8$. As $F_4(2)$ has four classes of involutions by [Shi, Theorem 2.1] three of them are 2-central and the forth has centralizer of order $2^{20} \cdot 3^2$, we see that v must be conjugate to some element in $\langle z, r_1, r_2\rangle$ in $N_G(A_z)$. As $z^G \cap \langle z, r_1, r_2\rangle = \{z\}$, this is impossible. So $A_z \not\cong F_4(2)$.

As by Proposition 5.2 $A_z \not\cong G_2(2)'$ and $A_z \not\cong Sp_{2n}(2)$, we have that Q_R is extraspecial. Further by Proposition 5.2 $A_z \not\cong L_3(2)$ or $L_4(2)$. If $N_{N_G(A_z)}(Q_R)$ is not irreducible on Q_R/R, then by [MaStr, Lemma 2.33] $A_z \cong L_n(2)$ and no graph automorphism is involved. Now $N_G(A_z) = C_G(A_z) \times A_z$. From Lemma 2.2 and Lemma 6.9 we get a contradiction. □

For the remainder of this chapter we fix t as in Lemma 6.11 or Lemma 6.12. We will prove that $C_G(t)$ has a standard subgroup A_t, which is isomorphic to A_z.

Lemma 6.14. *$A_t = E(C_G(t))$ is simple, $Q_R \leq A_t$ and $C_S(A_t)$ is cyclic. In particular A_t is a standard subgroup.*

Proof. By Lemma 6.13 we have that Q_R is extraspecial, $R = \langle r\rangle$ and $C_G(\langle z, t\rangle) = C_G(\langle z, r\rangle)$ acts irreducibly on Q_R/R.

We first prove:

(A) Let $H \leq C_G(t)$ with $N_{C_G(z)}(Q_R) \leq N_G(H)$ and let $T = S \cap H$ be a Sylow 2–subgroup of H, then $Q_R \leq H$, or $T \leq C_S(Q_R)$.

For this suppose $Q_R \not\leq H$. As by Lemma 6.13 $N_{C_G(z)}(Q_R)$ acts irreducibly on Q_R/R, we see that $H \cap Q_R \leq R$. Hence $[T, Q_R] \leq H \cap Q_R \leq R$. Then we have by Lemma 2.25 that $T \leq C_S(Q_R)Q_R$. As $N_{C_G(z)}(Q_R)$ normalizes H and $C_S(Q_R)Q_R$, it also normalizes $T = H \cap C_S(Q_R)Q_R$. As $N_{C_G(z)}(Q_R)$ has no fixed point in Q_R/R we see that $T \leq C_S(Q_R)$, the assertion (A).

Suppose first $C_{C_G(t)}(O_2(C_G(t))) \leq O_2(C_G(t))$. Then set $H = O_2(C_G(t))$. As $N_{C_G(z)}(Q_R) \leq C_G(\langle z, t\rangle)$ we see that $N_{C_G(z)}(Q_R)$ normalizes H. As $t \in Z(S)$, we also have $H \leq S$. Now (A) implies that either $Q_R \leq O_2(C_G(t))$ or $O_2(C_G(t)) \leq C_S(Q_R) \leq C_S(A_z) \times \langle r\rangle$. But the latter contradicts $C_{C_G(t)}(O_2(C_G(t))) \leq O_2(C_G(t))$. So we have $Q_R \leq O_2(C_G(t))$. By Lemma 6.10 we see that $C_G(t)$ contains no $M \in \mathcal{N}_S$. This implies $C_G(t) \leq C_G(z)$, contradicting the choice of N.

So we have that $E(C_G(t)) \neq 1$ (recall that $O(C_G(i)) = 1$ for all involutions $i \in G$). Now set $H = E(C_G(t))$ in (A). If $Q_R \not\leq E(C_G(t))$ then as $C_S(Q_R)/\langle t \rangle$ is cyclic, we get a cyclic Sylow 2–subgroup of $E(C_G(t))$, a contradiction. Hence $Q_R \leq E(C_G(t))$.

Now let N_1 be some component of $C_G(t)$ and set $T = S \cap N_1$. If $[T, Q_R] = 1$, then $T\langle t \rangle/\langle t \rangle$ is cyclic, which cannot be a Sylow 2-subgroup of N_1. So $1 \neq [T, Q_R]$. In particular $R \leq N_1$. If $[R, N_1] = 1$ we get as $\langle t, R \rangle = \langle z, R \rangle$ that $N_1 \leq C_G(z)$. Now N_1 normalizes $O_2(C_{A_z}(\langle z, R \rangle)) = Q_R$. But $[Q_R, N_1] \leq Q_R \cap N_1 \leq R \leq Z(N_1)$, a contradiction. So we have that $R \not\leq Z(N_1)$. In particular $N_{C_G(z)}(Q_R) \leq N_G(N_1)$. Application of (A) now shows that $Q_R \leq N_1$. As this is true for any component N_i, we get that $E(C_G(t))$ is quasisimple.

Next we show

$$E(C_G(t)) \text{ is simple.}$$

Otherwise some $1 \neq u \in Z(S)$ is contained in $Z(E(C_G(t)))$. Suppose $u \neq t$. We then have that $\Omega_1(Z(S)) = \langle r, z \rangle = \langle u, t \rangle$. Hence $E(C_G(t)) \leq C_G(z)$, a contradiction. So we must have $t \in Z(E(C_G(t)))$. Now $C_G(E(C_G(t))) \leq C_{C_G(t)}(Q_R) = C_{C_G(z)}(Q_R)$. As $r \in E(C_G(t)) \setminus Z(E(C_G(t)))$, we see that $C_G(E(C_G(t)))$ has a cyclic Sylow 2–subgroup and so in particular $E(C_G(t))$ is standard. But this contradicts Proposition 5.1. Hence $E(C_G(t))$ is simple.

As $Q_R \leq E(C_G(t))$ we see that $C_S(E(C_G(t))) \leq C_S(Q_R)$ is cyclic. In particular $E(C_G(t))$ is standard. $\square$

We have $\langle z, t \rangle = \Omega_1(Z(S)$ and $r = zt \in A_z \cap A_t$. Now everything we proved for A_z applies for A_t too. This shows that both groups are isomorphic to one of the following groups: J_2, $M(24)'$, $L_n(2)$, $U_n(2)$, $n \geq 5$, $\Omega^{\pm}_{2n}(2)$, $E_6(2)$, $E_7(2)$, $E_8(2)$, ${}^2E_6(2)$, ${}^3D_4(2)$.

Lemma 6.15. *We have that $O_2(C_{A_t}(R)) = O_2(C_{A_z}(R))$. Further let H_t be the preimage of $E(N_{A_t}(O_2(C_{A_t}(R)))/O_2(C_{A_t}(R)))$ and H_z the preimage of $E(N_{A_z}(Q_R)/Q_R)$. Then $H_t = H_z$.*

Proof. By Lemma 6.14 we have $Q_R \leq A_t$. Further we have that $H_z \leq C_G(t)$. As $H_z' Q_R = H_z$ by Lemma 6.13 and $C_G(t)/A_t$ is solvable, we get that $H_z \leq A_t$ (this is also true if $N_{A_z}(O_2(C_{A_z}(R)))$ is solvable, as then $H_z = Q_R$). Similarly we see $H_t \leq A_z$ and then we have that $O_2(N_{A_t}(R)) \leq O_2(C_{A_z\langle z \rangle}(R))$ and $Q_R \leq O_2(C_{A_t\langle t \rangle}(R))$. This shows that $Q_R \leq O_2(C_{A_t}(R)) \leq Q_R$ and so $Q_R = O_2(C_{A_t}(R))$. We further get $H_t \leq H_z \leq H_t$, the assertion. $\square$

Lemma 6.16. *We have $A_z \cong A_t$.*

Proof. Let first A_z be sporadic. By Proposition 5.2 we have that $A_z \cong J_2$ or $M(24)'$. In both cases $N_{A_z}(Q_R)$ is nonsolvable. By Lemma 6.15 we have that $H_z = H_t \leq A_t$ and $H_z \cong 2^{1+4}A_5$ or $2^{1+12}3U_4(3)$. If A_t is sporadic too, then we have that $A_z \cong A_t$. So we may assume that A_t is a group of Lie type over GF(2). As $3U_4(3)$ is not a group of Lie type in characteristic two, we get a contradiction. In the first case we have that $|Q_R| = 2^5$. Then by Lemma 2.17 we get that $A_t \cong L_4(2)$ or $U_4(2)$, which contradicts Proposition 5.2. So we have $A_z \cong A_t$.

Next we assume that both A_z and A_t are groups of Lie type. If $N_{A_z}(Q_R)$ is nonsolvable we may argue as before, i.e. we compare the orders of Q_R and the Levi factors, as given by Lemma 2.17. Then we receive $A_z \cong A_t$ or $A_z \cong L_3(2)$, $L_4(2)$, $\Omega_8^+(2)$, $U_4(2)$, $U_5(2)$. By Proposition 5.2 $A_z \not\cong L_3(2)$, $L_4(2)$ or $U_4(2)$. Now we have symmetry and so also $A_t \cong \Omega_8^+(2)$ or $U_5(2)$. But these groups are determined just by the order of Q_R, which is 2^9, 2^7, respectively, so $A_t \cong A_z$ too. $\square$

Proposition 6.17. *The main theorem holds.*

Proof. Suppose false. Then according to Lemma 6.11 and Lemma 6.12 we have some $t \in \Omega_1(Z(S))$, $t \neq z$, $t \in Z(N)$. By Lemma 6.16 $A_z \cong A_t$ and by Lemma 6.14 both groups are standard. We first show

$$A_t \cong A_z \cong L_n(2) \text{ or } U_n(2). \tag{1}$$

Suppose false. By [MaStr, Lemma 2.33] we have that $N_{A_z}(Q_R)$ acts irreducibly on Q_R/R. Set $V = \Omega_1(Z_2(S \cap A_z))$. We get with [MaStr, Lemma 2.35] that $|V| = 4$. Set $P = N_{A_z}(V)$. Then P is normalized by S and $P/O_2(P) \cong \Sigma_3$. For a group of Lie type this is just a minimal parabolic not in $N_{A_z}(R)$. For the sporadic groups this follows with Lemma 2.14.

Hence $\Omega_1(Z(O_2(P))) = \Omega_1(Z_2(S \cap A_z))$. Then $V \leq Q_R$ and so $V = \Omega_1(Z_2(S \cap A_t))$ by Lemma 6.15. On V both $N_{A_t}(V)$ and $N_{A_z}(V)$ induce Σ_3. Now $\langle N_{A_z}(V), N_{A_t}(V)\rangle$ acts on $\langle z, V\rangle = \langle t, V\rangle$. As $z^G \cap \Omega_1(Z(S)) = \{z\}$ by Lemma 6.9 we have $z^G \cap \langle z, V\rangle = \{z\}$. So $N_{A_t}(V) \leq C_G(z)$. This implies $A_t = \langle N_{A_t}(V), N_{A_t}(Q_R)\rangle \leq C_G(z)$, a contradiction. This proves (1).

By (1) $A_t \cong A_z \cong U_n(2)$, or $L_n(2)$. In the latter by Lemma 6.13 we have some graph automorphism induced. As $[C_S(A_z), Q_R] = 1$, we get $C_S(A_z) \le C_S(A_t) \times R$. This yields $\Omega_1(\Phi(C_S(A_z))) \le \Omega_1(\Phi(C_S(A_t))) \le \langle t\rangle$. As $z \ne t$ this shows that $C_S(A_z) = \langle z\rangle$. By the Thompson transfer lemma (Lemma 2.2) and $z^G \cap \Omega_1(Z(S)) = \{z\}$ by Lemma 6.9, we have that z is a square of some $x \in C_G(z)$, which induces an outer automorphism on A_z. The same of course is true for t. In particular

$$\text{(2)} \qquad \begin{array}{l} \text{All involutions of } C_G(z) \text{ are in } \langle z\rangle \times A_z \\ \text{and all involutions of } C_G(t) \text{ are in } \langle t\rangle \times A_t. \end{array}$$

By Lemma 6.11 and Lemma 6.12 there is some parabolic N in $C_G(t)$, $N \not\le N_{C_G(t)}(R)$. This shows that $N/C_N(\Omega_1(Z(O_2(N)))) \cong \mathrm{O}_4^+(2)$ in case of $A_t \cong L_n(2)$ and $\Omega_4^-(2)$ or $\mathrm{O}_4^-(2)$ in case of $A_t \cong U_n(2)$. Set again $U = \Omega_1(Z(O_2(N)))$ and $V = U \cap A_t$. Then V is the natural module for $N/C_N(U)$. Further we have that $V \cap Q_R = [V, Q_R]$ is of order eight. By (2) and Lemma 2.28 we have that U is uniquely determined in S. But then also there is a corresponding subgroup N_1 of $C_G(z)$ such that N_1 induces $\Omega_4^\pm(2)$ on U. This now implies the following. The orbits of $N \le N_{C_G(t)}(U)$ on $U^\sharp$ are 1,5,5,10,10, or 1,6,6,9,9 and $N_1 \le N_{C_G(z)}(U)$ induces the same orbit sizes. As $|z^{N_G(U)}|$ is odd, we see that under $N_G(U)$ the orbit of z must have length 11 or 21 and 7 or 13, respectively. Recall that $z \not\sim t$ or r. But $|z^{N_G(U)}|$ has to divide the order of $GL_5(2)$, which implies that $|z^{N_G(U)}| = 21$ in the first case and 7 in the second. The same applies for t, i.e. $|t^{N_G(U)}| = 21, 7$, respectively. But there is obviously just one possibility to make up an orbit of length 21 or 7, which implies that $z \sim t$, the final contradiction. □

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