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On the character degrees of Sylow p-subgroups of Chevalley groups $G(p^f)$ of type E

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Abstract. Let $\mathbb{F}_q$ be a field of characteristic p with q elements. It is known that the degrees of the irreducible characters of the Sylow p-subgroup of $\mathrm{GL}(\mathbb{F}_q)$ are powers of q. On the other hand Sangroniz (2003) showed that this is true for a Sylow p-subgroup of a classical group defined over $\mathbb{F}_q$ if and only if p is odd. For the classical groups of Lie type B, C and D the only bad prime is 2. For the exceptional groups there are others. In this paper we construct irreducible characters for the Sylow p-subgroups of the Chevalley groups $D_4(q)$ with $q = 2^f$ of degree $q^3/2$. Then we use an analogous construction for $E_6(q)$ with $q = 3^f$ to obtain characters of degree $q^7/3$, and for $E_8(q)$ with $q = 5^f$ to obtain characters of degree $q^{16}/5$. This helps to explain why the primes 2, 3 and 5 are bad for the Chevalley groups of type E in terms of the representation theory of the Sylow p-subgroup.

Keywords. Irreducible characters, root system, Lie type.

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1 Introduction

Let G be a Chevalley group defined over a field $\mathbb{F}_q$ of order q and characteristic $p > 0$. By α_0 we denote the highest root of the root system Σ of G. It is well known that α_0 is a positive integral linear combination of the fundamental roots of Σ. So without loss of generality, $\alpha_0 = \sum_{i=1}^{r} a_i\alpha_i$ where the α_i are fundamental roots of Σ. Recall that p is a bad prime for G if p is a divisor of some a_i.

It is well known that if G classical, then the only possible bad prime for G is 2. On the other hand if G is exceptional of type E, then the primes 3 and 5 are also bad. The "badness" of the prime evidences itself in the classification of the unipotent conjugacy classes of G. Here we aim to explain why the primes 3 and 5 are bad for groups of type E in terms to the representation theory of the Sylow p-subgroup of $G = E_6(q)$ with prime 3 and $G = E_8(q)$ with prime 5. Let $UE_k(q)$ denote the unipotent radical of the standard Borel subgroup of $E_k(q)$ for $k = 6$ and 8,

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i.e. the subgroup generated by all the positive root groups of G. By U_k we denote the quotient $UE_k(q)/K_{k-1}$, where K_{k-1} is the normal subgroup of $UE_k(q)$ generated by all root groups X_α such that α has height $k-1$ or more. Clearly any character of U_k inflates to a character of $UE_k(q)$. Abusing terminology slightly we call the image under the natural projection of a root group of $UE_k(q)$, a root group of U_k. We observe that $Z(U_k)$ is generated by the root groups of height $k-2$ and hence $|Z(U_k)| = q^{k-1}$. We define the family

$$\mathcal{F}_k := \{\chi \in \mathrm{Irr}(U_k) : X_\alpha \not\subset \ker(\chi) \text{ for all } X_\alpha \subset Z(U_k)\}.$$

Theorem 1.1. *The following statements are true.*

(a) *If $q = 3^f$, then for all $\chi \in \mathcal{F}_6$ we have $\chi(1) \in \{q^7, q^7/3\}$. Moreover $\mathcal{F}_6$ contains exactly $(q-1)^5(q^2-(q-1)/2)$ characters of degree q^7 and exactly $3^2(q-1)^6/2$ characters of degree $q^7/3$.*

(b) *If $q = 5^f$, then for all $\chi \in \mathcal{F}_8$ we have $\chi(1) \in \{q^{16}, q^{16}/5\}$. Moreover $\mathcal{F}_8$ contains exactly $(q-1)^8(q^3+q^2+q+3/4)$ characters of degree q^{16} and exactly $25(q-1)^8/4$ characters of degree $q^{16}/5$.*

We remark that $9(q-1)^6/2$, $(q-1)^5(q^2-(q-1)/2)$, $(q-1)^8(q^3+q^2+q+3/4)$ and $25(q-1)^8/4$ are not in $\mathbb{Z}[q]$. On the other hand we remark also that $|\mathcal{F}_6| = (q-1)^5q^2 \in \mathbb{Z}[q]$ and every character in $\mathcal{F}_6$ has degree q^7 whenever $p \neq 3$, and that $|\mathcal{F}_8| = (q-1)^7q^4 \in \mathbb{Z}[q]$ and every character in $\mathcal{F}_8$ has degree q^{16} whenever $p \neq 5$. Taken together these remarks provide evidence for a generalization of Higman's conjecture for groups of type $UE_i(q)$, $i = 6, 7, 8$, see for example [2], namely that $|\mathrm{Irr}(UE_i(q))| \notin \mathbb{Z}[q]$ if and only if p is a bad prime for $E_i(q)$.

To prove our main theorem we begin by analyzing our construction of the irreducible characters of the Sylow 2-subgroup of $D_4(2^f)$ from [3]. Our starting point is the quotient of $UD_4(q)/K_4$ where $UD_4(q)$ is the unipotent radical of the standard Borel subgroup of the universal Chevalley group $D_4(q)$ and K_4 is the normal subgroup of $UD_4(q)$ generated by the root groups of roots of height 4 and 5. We showed that when $p = 2$, there exists a $UD_4(q)$ family of characters of degree $q^3/2$ of size $4(q-1)^4$. As $UD_4(q)$ is a quotient of $UE_i(q)$ for $i = 6, 7, 8$, we also see families of irreducible characters of degree $q^3/2$ for groups of type $UE_i(q)$, where $i = 6, 7, 8$ and q is even.

Our construction is fairly elementary. Starting with large elementary abelian normal subgroups, we construct our characters via induction, using Clifford theory. To compute the necessary stabilizers we critically use Proposition 1.3 and Lemma 1.5. Throughout this paper we fix a nontrivial homomorphism

$$\phi : (\mathbb{F}_q, +) \to \mathbb{C}^\times.$$

For each $a \in \mathbb{F}_q$, we define $\phi_a(x) := \phi(ax)$ for all $x \in \mathbb{F}_q$, and denote

$$\mathbb{F}_q^\times := \mathbb{F}_q - \{0\}.$$

Hence, $\{\phi_a : a \in \mathbb{F}_q^\times\}$ are all non-principal irreducible characters of $\mathbb{F}_q$.

Definition 1.2. For $a \in \mathbb{F}_q$, we define $\mathbb{T}_a := \{t^p - a^{p-1}t : t \in \mathbb{F}_q\}$.

We note that $\mathbb{T}_0 = \mathbb{F}_q$.

Proposition 1.3. *The following statements are true.*

(a) $t^p - a^{p-1}t = \prod_{c \in \mathbb{F}_p}(t - ca)$.

(b) *If $a \in \mathbb{F}_q^\times$, then $\mathbb{T}_a$ is an additive subgroup of $\mathbb{F}_q$ of index p.*

(c) *For each $a \in \mathbb{F}_q^\times$, there exists $b \in \mathbb{F}_q^\times$ such that $b\mathbb{T}_a = \ker(\phi)$. Furthermore, $cb\mathbb{T}_a = \ker(\phi)$ iff $c \in \mathbb{F}_p^\times$.*

(d) $\{\mathbb{T}_a : a \in \mathbb{F}_q^\times\} = \{\ker(\phi_a) : a \in \mathbb{F}_q^\times\}$ *are all subgroups of index p in $\mathbb{F}_q$.*

Proof. See Section 5.1. □

Definition 1.4. For each $a \in \mathbb{F}_q^\times$, we pick a_ϕ such that $a_\phi\mathbb{T}_a = \ker(\phi)$.

By Proposition 1.3 (c), a_ϕ exists and but is only determined up to a scalar in the prime field. In the definition above we make an arbitrary choice which is fixed throughout the paper.

Throughout we fix notation as follows. Let G be a group. Set $G^\times := G - \{1\}$, denote by $\mathrm{Irr}(G)$ the set of all complex irreducible characters of G, and $\mathrm{Irr}(G)^\times := \mathrm{Irr}(G) - \{1_G\}$. For $H, K \le G$, and $\xi \in \mathrm{Irr}(H)$, define

$$\mathrm{Irr}(G/K) := \{\chi \in \mathrm{Irr}(G) : K \subset \ker(\chi)\},$$
$$\mathrm{Irr}(G, \xi) := \{\chi \in \mathrm{Irr}(G) : (\chi, \xi^G) \ne 0\},$$
$$\mathrm{Irr}(G/K, \xi) := \mathrm{Irr}(G/K) \cap \mathrm{Irr}(G, \xi).$$

Furthermore, for a character χ of G, we denote its restriction to H by $\chi|_H$.

Lemma 1.5. *Let $N \trianglelefteq G$ and $1 \in X$ be a transversal of N in G. Suppose N is of the form $N = ZYM$ where $Y \trianglelefteq N$, $Z \subset Z(N)$, $M \le N$ and $X \subset N_G(ZY)$. If there is $\lambda \in \mathrm{Irr}(ZY)$ such that $Y \subset \ker(\lambda)$, and ${}^u\lambda \ne {}^v\lambda$ for all $u \ne v \in X$, then the following are true.*

(a) *For all $\chi \in \mathrm{Irr}(N/Y, \lambda)$, $\chi^G \in \mathrm{Irr}(G)$. Moreover, if $\chi_1 \ne \chi_2 \in \mathrm{Irr}(N/Y, \lambda)$, then ${\chi_1}^G \ne {\chi_2}^G$.*

(b) *The induction map from $\mathrm{Irr}(N/Y, \lambda)$ to $\mathrm{Irr}(G, \lambda)$ is bijective.*

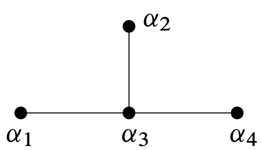
Proof. See Section 5.2. □

We recall that a p-group P is monomial, i.e. for each $\chi \in \mathrm{Irr}(P)$, there exist a subgroup H of P and a linear character λ of H such that $\chi = \lambda^P$. To construct irreducible characters whose degrees are not powers of $q = p^f$, $f > 1$, we construct subgroups $H \trianglelefteq P$ and $T \leq P$ such that T is a transversal of H. Then we find a linear character λ of H such that the order of the stabilizer $\mathrm{Stab}_T(\lambda)$ of T is not a power of q. Moreover we insure that λ is extendable to the inertial group $I_P(\lambda) = H\,\mathrm{Stab}_T(\lambda)$. Let λ_I denote some extension of λ to $I_P(\lambda)$. By Clifford theory the induction of λ_I to P is irreducible and of degree not a power of q. The existence of a suitable pair (H, λ) is based on Proposition 1.3. The reason being that a polynomial of the form $x^p + a^{p-1}x$, with $a \neq 0$, appears in the formulae of the action of elements of T on the characters of H.

We will now highlight the main steps of the constructions of our characters. We have deferred all of our proofs to Section 5.

2 Sylow 2-subgroups of the Chevalley groups $D_4(2^f)$

Let $\mathbb{F}_q$ be a field of order q and characteristic 2. Let $\Sigma := \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ be the root system of type D_4, see Carter [1, Chapter 3]. The Dynkin diagram of Σ is



The positive roots are those roots which can be written as positive integral linear combinations of the simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. We write Σ^+ for the set of positive roots. We use the notation

$$\begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \end{array}$$

for the root $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$ and we use a similar notation for the remaining positive roots. The 12 positive roots of Σ are given in Table 1.

For $\alpha \in \Sigma$ we denote the corresponding root subgroup of the Chevalley group G by X_α whose elements we label by $x_\alpha(t)$ where $t \in \mathbb{F}_q$. Note that $X_\alpha \cong (\mathbb{F}_q, +)$.

We recall the commutator formula

$$[x_\alpha(r), x_\beta(s)] = \begin{cases} x_{\alpha+\beta}(-C_{\alpha,\beta} rs), & \text{if } \alpha + \beta \in \Sigma, \\ 1, & \text{otherwise,} \end{cases}$$

see Carter [1, Theorem 5.2.2]. In $\mathbb{F}_q$ it is the case that $1 = -1$, since $p = 2$, and thus all non-zero coefficients $C_{\alpha,\beta}$ are equal to 1. For positive roots, we use the ab-

Height	Roots			
5	$\alpha_{12} := \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix}$			
4	$\alpha_{11} := \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}$			
3	$\alpha_8 := \begin{smallmatrix} & 1 & \\ 1 & 1 & 0 \end{smallmatrix}$	$\alpha_9 := \begin{smallmatrix} & 0 & \\ 1 & 1 & 1 \end{smallmatrix}$	$\alpha_{10} := \begin{smallmatrix} & 1 & \\ 0 & 1 & 1 \end{smallmatrix}$	
2	$\alpha_5 := \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \end{smallmatrix}$	$\alpha_6 := \begin{smallmatrix} & 1 & \\ 0 & 1 & 0 \end{smallmatrix}$	$\alpha_7 := \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \end{smallmatrix}$	
1	α_1	α_2	α_3	α_4

Table 1. Positive roots of the root system Σ of type D_4.

breviation $x_i(t) := x_{\alpha_i}(t)$, $i = 1, 2, \ldots, 12$. All nontrivial commutators are given in Table 2.

$$
\begin{aligned}
&[x_1(t), x_3(u)] = x_5(tu), && [x_1(t), x_6(u)] = x_8(tu),\\
&[x_1(t), x_7(u)] = x_9(tu), && [x_1(t), x_{10}(u)] = x_{11}(tu),\\
&[x_2(t), x_3(u)] = x_6(tu), && [x_2(t), x_5(u)] = x_8(tu),\\
&[x_2(t), x_7(u)] = x_{10}(tu), && [x_2(t), x_9(u)] = x_{11}(tu),\\
&[x_3(t), x_4(u)] = x_7(tu), && [x_3(t), x_{11}(u)] = x_{12}(tu),\\
&[x_4(t), x_5(u)] = x_9(tu), && [x_4(t), x_6(u)] = x_{10}(tu),\\
&[x_4(t), x_8(u)] = x_{11}(tu), && [x_5(t), x_{10}(u)] = x_{12}(tu),\\
&[x_6(t), x_9(u)] = x_{12}(tu), && [x_7(t), x_8(u)] = x_{12}(tu).
\end{aligned}
$$

Table 2. Commutator relations for type D_4.

The group UD_4 generated by all X_α for $\alpha \in \Sigma^+$ is a Sylow 2-subgroup of the Chevalley group $D_4(q)$. Each element $u \in UD_4$ can be written uniquely as

$$u = x_1(t_1)x_2(t_2)x_4(t_4)x_3(t_3)x_5(t_5)\cdots x_{12}(t_{12}) \quad \text{where } x_i(t_i) \in X_i.$$

So we write $\prod_{i=1}^{12} x_i(t_i)$ as this order. We note that our ordering of the roots is slightly non-standard as the positions of x_3 and x_4 are reversed.

We define

$$\mathcal{F}_4 := \{\chi \in \mathrm{Irr}(UD_4(q)) : \chi|_{X_i} = \chi(1)\phi_{a_i} \text{ for each } a_8, a_9, a_{10} \in \mathbb{F}_q^\times\}.$$

If Ψ is a representation affording $\chi \in \mathcal{F}_4$, then

$$\Psi([x_8(t_8), x_4(t_4)]) = [\Psi(x_8(t_8)), \Psi(x_4(t_4))]$$
$$= [\phi_{a_8}(t_8)\Psi(1), \Psi(x_4(t_4))] = \Psi(1) \quad \text{for all } t_4, t_8 \in \mathbb{F}_q.$$

Therefore, $X_{11} = [X_8, X_4] \subset \ker(\chi)$, and similarly $X_{12} = [X_8, X_7] \subset \ker(\chi)$. Thus only the factor group $U = UD_4/X_{12}X_{11}$ acts on a module affording χ. Therefore, we may work with U which has order q^{10}, and $Z(U) = X_8X_9X_{10}$.

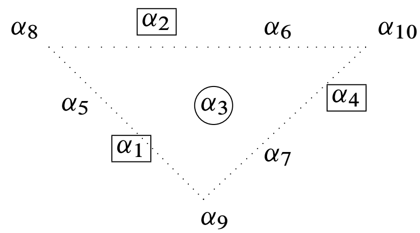


Figure $UD_4(q)$. Relations of roots.

Let $H := [U, U] = X_5X_6X_7X_8X_9X_{10}$, and $T := X_1X_2X_4$. It is clear that H, HX_3 and T are elementary abelian. The group U can be visualized as in the figure above. The roots in boxes are in T, α_3 which is neither in T nor in H is in a circle, whereas all other roots are in H. The broken lines indicate where the hooks, as defined in [3], centered at central roots are; for example $\alpha_2 + \alpha_5 = \alpha_6 + \alpha_1 = \alpha_8$. The hooks centered at α_8, α_9 and α_{10} intersect pairwise in sets of size two so as to form a triangle.

To study the characters $\chi \in \mathcal{F}_4$ we start with a linear character λ of H such that $\lambda|_{X_i} \neq 1_{X_i}$ for $i = 8, 9, 10$.

Definition 2.1. For $a_8, a_9, a_{10} \in \mathbb{F}_q^\times$ and $b_5, b_6, b_7 \in \mathbb{F}_q$, we define

(a) $\lambda^{a_8,a_9,a_{10}}_{b_5,b_6,b_7}(\prod_{i=5}^{10} x_i(t_i)) := \phi(\sum_{i=5}^{7} b_i t_i + \sum_{j=8}^{10} a_j t_j)$.

(b) $S_{567} := \{x_{567}(t) := x_5(a_{10}t)x_6(a_9t)x_7(a_8t) : t \in \mathbb{F}_q\}$.

(c) $S_{124} := \{x_{124}(t) := x_1(a_{10}t)x_2(a_9t)x_4(a_8t) : t \in \mathbb{F}_q\}$.

(d) $A := a_8a_9a_{10}$ and $t_0 := \frac{1}{A}(b_5a_{10} + b_6a_9 + b_7a_8)$.

(e) $F_{124} := \{1, x_{124}(t_0)\}$.

(f) $F_3 := \{1\}$ if $t_0 = 0$, and $F_3 := \{1, x_3(\frac{(t_0)_\phi}{A})\}$ otherwise.

It is easy to check that S_{567}, S_{124}, F_{124}, F_3 are subgroups of U. If $t_0 = 0$, then $F_{124} = F_3 = \{1\}$, otherwise $F_{124} \cong F_3 \cong (\mathbb{F}_2, +)$. Since $S_{124}, S_{567} \cong (\mathbb{F}_q, +)$, their linear characters are of the form

$$\phi_{b_i}(x_i(t)) = \phi(b_i t) \quad \text{where } i \in \{124, 567\} \text{ for all } b_i, t \in \mathbb{F}_q.$$

For each $\xi \in \mathrm{Irr}(F_{124})$, $\xi = \phi_{b_{124}}|_{F_{124}}$ for some $\phi_{b_{124}} \in \mathrm{Irr}(S_{124})$, $b_{124} \in \mathbb{F}_q$. If F_{124} is nontrivial, we choose $b_{124} \in \{0, a_{124}\} \cong (\mathbb{F}_2, +)$ where $\phi(a_{124}t_0) = -1$. The same for $F_3 \le X_3$, for each $\xi \in \mathrm{Irr}(F_3)$, $\xi = \phi_{b_3}|_{F_3}$ for some $\phi_{b_3} \in \mathrm{Irr}(X_3)$ and $b_3 \in \{0, a_3\} \cong (\mathbb{F}_2, +)$ such that

$$\phi\left(a_3 \frac{(t_0)_\phi}{A}\right) = -1$$

if $(t_0)_\phi$ exists.

For each $a_8, a_9, a_{10} \in \mathbb{F}_q^\times$, there are q^3 linear characters $\lambda^{a_8,a_9,a_{10}}_{-,-,-}$ of H. By definition of t_0, there are q^2 of them such that $t_0 = 0$ and $q^2(q-1)$ such that $t_0 \neq 0$. Therefore, there are q^2 cases where F_{124}, F_3 are trivial and $q^2(q-1)$ cases where F_{124}, F_3 are of order 2.

For all $x_1(t_1)x_2(t_2)x_4(t_4) \in T$, we have

$$^{x_1(t_1)x_2(t_2)x_4(t_4)}(\lambda^{a_8,a_9,a_{10}}_{b_5,b_6,b_7}) = \lambda^{a_8,a_9,a_{10}}_{b_5+a_8t_2+a_9t_4, b_6+a_8t_1+a_{10}t_4, b_7+a_9t_1+a_{10}t_2}.$$

Hence, T acts on the set of linear characters $\{\lambda^{a_8,a_9,a_{10}}_{-,-,-}\}$. It is easy to check that t_0 is invariant under this action. The following lemma establishes some facts concerning $\lambda^{a_8,a_9,a_{10}}_{b_5,b_6,b_7}$.

Lemma 2.2. *Set* $\lambda := \lambda^{a_8,a_9,a_{10}}_{b_5,b_6,b_7}$. *The following statements are true.*

(a) $S_{124} = \mathrm{Stab}_T(\lambda)$ *and* $S_{567} = \{x \in X_5X_6X_7 : |\lambda^U(x)| = \lambda^U(1)\}$. *Moreover,* $\lambda^U|_{S_{567}} = \lambda^U(1)\phi_{At_0}$.

(b) λ *extends to* HX_3F_{124} *and* HF_3S_{124}. *Let* λ_1 *and* λ_2 *be extensions of* λ *to* HX_3F_{124}. *The inertia groups* $I_U(\lambda_1) = HX_3F_{124}$.

(c) ${\lambda_1}^U = {\lambda_2}^U \in \mathrm{Irr}(U)$ *iff* $\lambda_1|_{F_3} = \lambda_2|_{F_3}$ *and* $\lambda_1|_{F_{124}} = \lambda_2|_{F_{124}}$.

Proof. See Section 5.3.1. □

Remark. When q is odd, both sets $\{x \in X_5X_6X_7 : |\lambda^U(x)| = \lambda^U(1) = q^4\}$ and $\mathrm{Stab}_T(\lambda)$ are trivial. Thus, λ extends to HX_3 and each extension induces irreducibly to U of degree q^3.

When $t_0 \neq 0$, the statement in Lemma 2.2 (c) makes sense since the dihedral subgroup $\langle F_{124}, F_3\rangle \subset I_U(\lambda_1)$. By Lemma 2.2 (b), $X_3, S_{124} \subset I_U(\lambda)$ but λ does not extend to HX_3S_{124} as $[X_3, S_{124}] \not\subseteq \ker(\lambda)$.

By Lemma 2.2 (a), the action of T acts on the set of q^3 linear $\lambda^{a_8,a_9,a_{10}}_{-,-,-}$ has q orbits, each of size q^2. By Lemma 2.2 (b), all q^3 linears $\lambda^{a_8,a_9,a_{10}}_{-,-,-}$ extend to HX_3 and thus we obtain q^4 linear extensions. For q^3 of these $t_0 = 0$ and whereas $t_0 \neq 0$ for the other $q^3(q-1)$ characters.

If $t_0 = 0$, F_{124} is trivial. By Lemma 2.2 (b), λ extends to $I_U(\lambda) = HX_3 \trianglelefteq U$, as η. The group T is a transversal of HX_3 in U and acts regularly on these q^3 linears η with $t_0 = 0$. Therefore, the character $\eta^U \in \operatorname{Irr}(U)$ of degree q^3 only depends on a_8, a_9, a_{10}, so we denote it by $\chi_{8,9,10,q^3}^{a_8,a_9,a_{10}} \in \operatorname{Irr}(U)$. This character is the unique $\chi \in \mathcal{F}_4$ of degree q^3 such that $\chi|_{X_i} = \chi(1)\phi_{a_i}$ where $i = 8, 9, 10$. Furthermore, by Lemma 2.2 (a), this is the unique constituent χ of $(\lambda|_{X_8X_9X_{10}})^U$ such that $S_{567} \subset \ker(\chi)$.

If $t_0 \neq 0$, then F_{124} and F_3 are isomorphic to $\mathbb{F}_2$. By Lemma 2.2 (b), λ extends to HX_3F_{124} as λ_1, and ${\lambda_1}^U \in \operatorname{Irr}(U)$ of degree $\frac{q^3}{2}$. For each $t_0 \neq 0$, by Lemma 2.2 (c), all constituents ${\lambda_1}^U$ of λ^U only depend on the restrictions of λ_1 to F_{124} and F_3. Therefore, we denote these constituents of λ^U by

$$\chi_{8,9,10,\frac{q^3}{2}}^{b_{124},b_3,t_0,a_8,a_9,a_{10}} \quad \text{where } b_{124}, b_3 \in \mathbb{F}_2,\ t_0, a_8, a_9, a_{10} \in \mathbb{F}_q^\times.$$

For each $a_8, a_9, a_{10} \in \mathbb{F}_q^\times$, there are $4(q-1)$ characters $\chi \in \mathcal{F}_4$ of degree $\frac{q^3}{2}$ such that $\chi|_{X_i} = \chi(1)\phi_{a_i}$ where $i = 8, 9, 10$.

The next theorem lists the generic character values of all $\chi \in \operatorname{Irr}(U)$ such that $\chi|_{X_i} = \chi(1)\phi_{a_i}$ where $i = 8, 9, 10$.

Theorem 2.3. *For $a_8, a_9, a_{10} \in \mathbb{F}_q^\times$, suppose $\chi \in \operatorname{Irr}(U)$ such that $\chi|_{X_i} = \chi(1)\phi_{a_i}$ where $i = 8, 9, 10$. Set $Z = F_{124}S_{567}X_8X_9X_{10}$ and the Kronecker*

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

The following statements are true.

(a) *If $\chi(1) = q^3$, then*

$$\chi = \chi_{8,9,10,q^3}^{a_8,a_9,a_{10}}$$

and

$$\chi\left(\prod_{i=1}^{10} x_i(t_i)\right) = \delta_{0,t_1}\delta_{0,t_2}\delta_{0,t_4}\delta_{0,t_3}\delta_{a_8t_5,a_{10}t_7}\delta_{a_8t_6,a_9t_7}q^3\phi\left(\sum_{i=8}^{10} a_it_i\right).$$

(b) *If $\chi(1) = \frac{q^3}{2}$, then*

$$\chi = \chi_{8,9,10,\frac{q^3}{2}}^{b_{124},b_3,t_0,a_8,a_9,a_{10}} \quad \text{for some } b_{124}, b_3 \in \mathbb{F}_2,\ t_0 \in \mathbb{F}_q^\times$$

and

$$\chi\left(\prod_{i=1}^{10} x_i(t_i)\right) = \frac{q^3}{2}\phi\left(b_{124}\frac{t_1}{a_{10}} + At_0\frac{t_7}{a_8} + \sum_{i=8}^{10} a_it_i\right)$$

if $\prod_{i=1}^{10} x_i(t_i) \in Z$, *and*

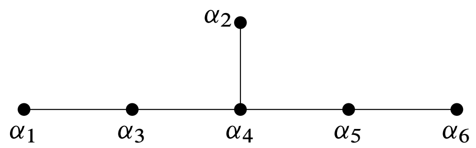
$$\chi\left(\prod_{i=1}^{10} x_i(t_i)\right) = \delta_{a_8 t_1, a_{10} t_4} \delta_{a_8 t_2, a_9 t_4} \delta_{t_3, t_0^{\phi}} \frac{q^2}{2} \times \phi\left(b_{124}\frac{t_1}{a_{10}} + b_3 t_3 + A t_0 \frac{t_7}{a_8} + (*) + \sum_{i=8}^{10} a_i t_i\right),$$

otherwise, where $t_0^{\phi} = \frac{(t_0)_\phi}{A}$ *and* $(*) = \frac{A^2}{(t_0)_\phi}(\frac{t_5}{a_{10}} + \frac{t_7}{a_8})(\frac{t_6}{a_9} + \frac{t_7}{a_8})$.

Proof. See Section 5.3.2. □

3 Sylow 3-subgroups of the Chevalley groups $E_6(3^f)$

Let $\mathbb{F}_q$ be a field of order q and characteristic 3. We study $E_6(q)$ from the point of view of its Lie root system. Let $\Sigma := \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \rangle$ be the root system of E_6, see Carter [1, Chapter 3]. The Dynkin diagram of Σ is



The positive roots are those roots which can be written as nonnegative integral linear combinations of the simple roots $\alpha_1, \alpha_2, \dots, \alpha_6$. We write Σ^+ for the set of positive roots. Here, $|\Sigma^+| = 36$. We use the notation

$$\begin{array}{ccccc} & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{array}$$

for the root $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ and we use a similar notation for the remaining positive roots. Let $X_\alpha := \langle x_\alpha(t) : t \in \mathbb{F}_q \rangle$ be the root subgroup corresponding to $\alpha \in \Sigma$. The group generated by all X_α for $\alpha \in \Sigma^+$ is a Sylow 3-subgroup of the Chevalley group $E_6(q)$, which we call UE_6.

In this section, we will construct irreducible characters of degree $\frac{q^7}{3}$ which are members of the following family of irreducible characters of UE_6 which is defined as follows:

$$\mathcal{F}_6 := \{\chi \in \mathrm{Irr}(UE_6) : \chi|_{X_\alpha} = \chi(1)\phi_a,\ \mathrm{ht}(\alpha) = 4,\ a \in \mathbb{F}_q^\times\}.$$

Let ψ be a representation affording some $\chi \in \mathcal{F}_6$. As in Section 2 we see that $X_\alpha \subset \ker(\chi)$ for all positive roots α with height greater than 4. Let K_5 be the normal subgroup of UE_6 generated by all root subgroups of height greater than 4.

Thus only the factor group $U := UE_6/K_5$ acts on a module affording χ. If α is a root of hight at most 4, then the restriction of the canonical projection from UE_6 to U is an injective and thus we may identify the root group X_α with its image in U. There are 21 roots $\alpha \in \Sigma^+$ with $\mathrm{ht}(\alpha) \leq 4$. These 21 positive roots are given in Table 3. Therefore, the group U has order q^{21} and $Z(U) = X_{17}X_{18}X_{19}X_{20}X_{21} = \langle X_\beta : \mathrm{ht}(\beta) = 4\rangle$.

For positive roots, we use the abbreviation $x_i(t) = x_{\alpha_i}(t)$, $i = 1, 2, \ldots, 21$. Each element $u \in U$ can be written uniquely as

$$u = x_2(t_2)x_1(t_1)x_3(t_3)x_4(t_4)x_5(t_5)\cdots x_{21}(t_{21}) \quad \text{where } x_i(t_i) \in X_i.$$

So we write $\prod_{i=1}^{21} x_i(t_i)$ in the order as above. We note that in our order the term x_2 precedes x_1.

Height	Roots		
4	$\alpha_{20} := \begin{smallmatrix} & & 1 & & \\ 0 & 0 & 1 & 1 & 1 \end{smallmatrix}$	$\alpha_{21} := \begin{smallmatrix} & & 0 & & \\ 0 & 1 & 1 & 1 & 1 \end{smallmatrix}$	
	$\alpha_{17} := \begin{smallmatrix} & & 1 & & \\ 1 & 1 & 1 & 0 & 0 \end{smallmatrix}$	$\alpha_{18} := \begin{smallmatrix} & & 0 & & \\ 1 & 1 & 1 & 1 & 0 \end{smallmatrix}$	$\alpha_{19} := \begin{smallmatrix} & & 1 & & \\ 0 & 1 & 1 & 1 & 0 \end{smallmatrix}$
3	$\alpha_{15} := \begin{smallmatrix} & & 0 & & \\ 0 & 1 & 1 & 1 & 0 \end{smallmatrix}$	$\alpha_{16} := \begin{smallmatrix} & & 0 & & \\ 0 & 0 & 1 & 1 & 1 \end{smallmatrix}$	
	$\alpha_{12} := \begin{smallmatrix} & & 0 & & \\ 1 & 1 & 1 & 0 & 0 \end{smallmatrix}$	$\alpha_{13} := \begin{smallmatrix} & & 1 & & \\ 0 & 1 & 1 & 0 & 0 \end{smallmatrix}$	$\alpha_{14} := \begin{smallmatrix} & & 1 & & \\ 0 & 0 & 1 & 1 & 0 \end{smallmatrix}$
2	$\alpha_{10} := \begin{smallmatrix} & & 0 & & \\ 0 & 0 & 1 & 1 & 0 \end{smallmatrix}$	$\alpha_{11} := \begin{smallmatrix} & & 0 & & \\ 0 & 0 & 0 & 1 & 1 \end{smallmatrix}$	
	$\alpha_7 := \begin{smallmatrix} & & 0 & & \\ 1 & 1 & 0 & 0 & 0 \end{smallmatrix}$	$\alpha_8 := \begin{smallmatrix} & & 1 & & \\ 0 & 0 & 1 & 0 & 0 \end{smallmatrix}$	$\alpha_9 := \begin{smallmatrix} & & 0 & & \\ 0 & 1 & 1 & 0 & 0 \end{smallmatrix}$
1	α_2 α_1	α_3 α_4	α_5 α_6

Table 3. Positive roots of the root system Σ of type E_6.

For all $\alpha, \beta \in \Sigma$ the length of an α-chain through β is at most 1. Thus the Chevalley commutator formula, see Cater [1, Theorem 5.2.2], yields

$$[x_\alpha(r), x_\beta(s)] = \begin{cases} x_{\alpha+\beta}(-C_{\alpha,\beta}rs), & \text{if } \alpha + \beta \in \Sigma, \\ 1, & \text{otherwise.} \end{cases}$$

For each extraspecial pair (α, β), we choose the coefficient $C_{\alpha,\beta} := -1$. By computing directly or using MAGMA [5] with the following codes, all nontrivial commutators are given in Table 4.

```
W:=RootDatum("E6");
R:=PositiveRoots(W); A:=R[1..21];
for i in [7..21] do
  for j in [1..(i-1)] do
    if (R[i]-R[j]) in A then
      k:=RootPosition(W,R[i]-R[j]);
      if k le j then print k,"+",j,"=",i,"(",LieConstant_C(W,1,1,k,j),")"; end if;
    end if;
  end for;
end for;
```

$$
\begin{array}{ll}
[x_1(t), x_3(u)] = x_7(tu), & [x_2(t), x_4(u)] = x_8(tu), \\
[x_3(t), x_4(u)] = x_9(tu), & [x_4(t), x_5(u)] = x_{10}(tu), \\
[x_5(t), x_6(u)] = x_{11}(tu), & [x_1(t), x_9(u)] = x_{12}(tu), \\
[x_4(t), x_7(u)] = x_{12}(-tu), & [x_2(t), x_9(u)] = x_{13}(tu), \\
[x_3(t), x_8(u)] = x_{13}(tu), & [x_2(t), x_{10}(u)] = x_{14}(tu), \\
[x_5(t), x_8(u)] = x_{14}(-tu), & [x_3(t), x_{10}(u)] = x_{15}(tu), \\
[x_5(t), x_9(u)] = x_{15}(-tu), & [x_4(t), x_{11}(u)] = x_{16}(tu), \\
[x_6(t), x_{10}(u)] = x_{16}(-tu), & [x_1(t), x_{13}(u)] = x_{17}(tu), \\
[x_7(t), x_8(u)] = x_{17}(tu), & [x_2(t), x_{12}(u)] = x_{17}(tu), \\
[x_1(t), x_{15}(u)] = x_{18}(tu), & [x_7(t), x_{10}(u)] = x_{18}(tu), \\
[x_5(t), x_{12}(u)] = x_{18}(-tu), & [x_2(t), x_{15}(u)] = x_{19}(tu), \\
[x_3(t), x_{14}(u)] = x_{19}(tu), & [x_5(t), x_{13}(u)] = x_{19}(-tu), \\
[x_2(t), x_{16}(u)] = x_{20}(tu), & [x_8(t), x_{11}(u)] = x_{20}(tu), \\
[x_6(t), x_{14}(u)] = x_{20}(-tu), & [x_3(t), x_{16}(u)] = x_{21}(tu), \\
[x_9(t), x_{11}(u)] = x_{21}(tu), & [x_6(t), x_{15}(u)] = x_{21}(-tu).
\end{array}
$$

Table 4. Commutator relations for type E_6.

Let $H := \langle X_\alpha : \alpha_4 \neq \alpha \in \Sigma^+, (\alpha, \alpha_4) > 0 \rangle = H_4 H_3 H_2$ where

$$H_4 := Z(U), \quad H_3 := \prod_{i=12}^{16} X_i, \quad H_2 := \prod_{i=8}^{10} X_i,$$

and

$$T := \langle X_2, X_1, X_3, X_5, X_6 \rangle = X_2 X_1 X_3 X_7 X_5 X_6 X_{11}.$$

It is clear that $|H| = q^{13}$, $|T| = q^7$, H_k is generated by all root groups of root height k in H, and T is a transversal of HX_4 in U. Both H and HX_4 are elementary abelian and normal in U, and T is isomorphic to $UA_2(q) \times UA_2(q) \times UA_1(q)$, where $UA_k(q)$ is the unipotent subgroup of the standard Borel subgroup of the general linear group $\mathrm{GL}_{k+1}(q)$. We can visualize the group U in the following figure. The roots in boxes are in T, the others outside are in H, and α_4 not in both H and T is in a circle. The dotted lines demonstrate the relations between roots to give a sum root in center, e.g. $\alpha_7 + \alpha_{10} = \alpha_{18}$, $\alpha_7 + \alpha_8 = \alpha_{17}, \ldots$. In addition, we have two triangles, as same as in Section 2 of $UD_4(q)$, namely $(\alpha_{17}, \alpha_{18}, \alpha_{19})$ and $(\alpha_{19}, \alpha_{20}, \alpha_{21})$. These two triangles share a common pair of roots (α_2, α_{15}) where $\alpha_2 + \alpha_{15} = \alpha_{19}$.

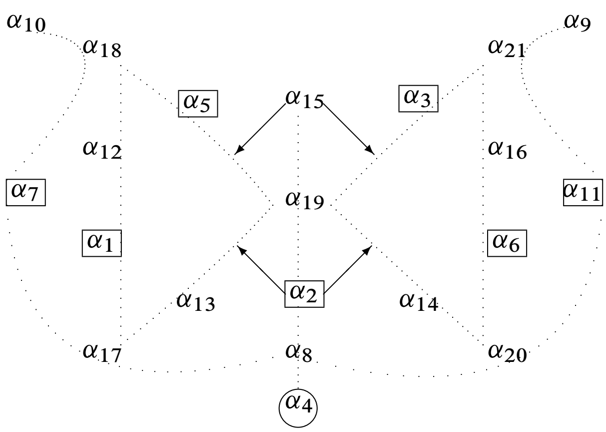


Figure $UE_6(q)$. Relations of roots.

We consider $\lambda \in \mathrm{Irr}(H)$ such that $\lambda|_{X_i} = \phi_{a_i} \neq 1_{X_i}$ for $17 \leq i \leq 21$. Since the maximal split torus of $E_6(q)$ acts transitively on $\bigoplus_{i=17}^{21} \mathrm{Irr}(X_i)^\times$, we may assume that $\lambda|_{X_i} = \phi$ for $17 \leq i \leq 21$. So we set

$$\lambda = \lambda_{b_8, b_9, b_{10}}^{b_{12}, b_{13}, b_{14}, b_{15}, b_{16}} \in \mathrm{Irr}(H)$$

such that $\lambda|_{X_i} = \phi_{b_i}$ where $b_i \in \mathbb{F}_q$ for all $8 \leq i \leq 16, i \neq 11$.

Definition 3.1. For $b_8, b_9, b_{10}, b_{12}, b_{13}, b_{14}, b_{15}, b_{16} \in \mathbb{F}_q$, we define

(a) $S_1 := \{s_1(t,r,s) := x_2(t)x_1(t)x_3(-t)x_5(t)x_6(-t)x_7(r)x_{11}(s) : t, r, s \in F_q\}$.

(b) $S_2 := \{s_2(t) := s_1(t, 2t^2, 2t^2) : t \in \mathbb{F}_q\}$.

(c) $R_3 := \{r_3(t) := x_{12}(t)x_{13}(-t)x_{14}(-t)x_{15}(t)x_{16}(t) : t \in \mathbb{F}_q\}$.

(d) $R_2 := \{r_2(t) := x_8(-t)x_9(t)x_{10}(t) : t \in \mathbb{F}_q\}$.

(e) $B_3 := b_{12} - b_{13} - b_{14} + b_{15} + b_{16}$.

(f) $B_2 := b_{10} + b_9 - b_8$.

(g) If $B_2 = c^2 \in \mathbb{F}_q^\times$, $F_2 := \{1, s_2(\pm c)\}$ and $F_4 := \{1, x_4(\pm c_\phi)\}$.

We note that $R_k \leq H_k$ for $k = 2, 3$, $F_2 \leq S_2 \leq S_1 \leq T$, and $F_4 \leq X_4$. Since $R_k \cong \mathbb{F}_q$, for each $a \in \mathbb{F}_q$ we define $\phi_a(r_k(t)) = \phi_a(t)$ for all $r_k(t) \in R_k$. Hence, $\mathrm{Irr}(R_k) = \{\phi_a : a \in \mathbb{F}_q\}$. Since $S_2 \cong \mathbb{F}_q$, we can define $\phi_a(s_2(t)) = \phi_a(t)$ for all $s_2(t) \in S_2$. When $B_2 = c^2 \in \mathbb{F}_q^\times$, for each linear character $\xi \in \mathrm{Irr}(F_2)$ there is $b_2 \in \{0, \pm a_2\} \cong (\mathbb{F}_3, +)$ such that

$$\xi = \phi_{b_2}|_{F_2} \quad \text{where } \phi_{b_2} \in \mathrm{Irr}(S_2) \text{ and } \phi(a_2 c) \neq 1.$$

Using the same argument for F_4, we find that for each character $\xi \in \mathrm{Irr}(F_4)$ there is $b_4 \in \{0, \pm a_4\} \cong (\mathbb{F}_3, +)$ such that

$$\xi = \phi_{b_4}|_{F_4}, \quad \text{where } \phi_{b_4} \in \mathrm{Irr}(X_4) \text{ and } \phi(a_4 c_\phi) \neq 1.$$

We first outline the induction process of λ up to U, thereby explaining some of the notation in Definition 3.1. Later we give the detailed conditions that are necessary for each step of our construction.

$$\begin{array}{lccccc}
H: & \lambda & \xrightarrow{B_3 = 0} & HX_4S_1 & \xrightarrow{B_2 = c^2 \in \mathbb{F}_q^\times} & HX_5S_2 \\
 & \downarrow B_3 \neq 0 & & \downarrow B_2 \neq c^2 & & \downarrow b_2, b_4 \in \mathbb{F}_3 \\
U: & \chi_{q^7}^{b_4, B_3} & & \chi_{q^7}^{B_2} & & \chi_{\frac{q^7}{3}}^{b_2, b_4, B_2} \\
\text{N}\underline{\text{o}}: & (q-1)q & & (q+1)/2 & & 9(q-1)/2
\end{array}$$

Figure $UE_6(q)$. Summary on the branching rules of λ.

Let $\overline{H_3}$ be the normal closure of H_3 in HX_4S_1. Since HX_4 is abelian, it follows that $X_4 \subset \mathrm{Stab}_U(\lambda)$. The main properties of $\lambda = \lambda_{b_8, b_9, b_{10}}^{b_{12}, b_{13}, b_{14}, b_{15}, b_{16}}$ are as follows.

Lemma 3.2. *The following statements are true.*

(a) $R_3 = \{x \in H_3 : |\lambda^U(x)| = \lambda^U(1)\}$ *and* $S_1 = \mathrm{Stab}_T(\lambda|_{H_4H_3})$. *Moreover, we have* $\lambda^U|_{R_3} = \lambda^U(1)\phi_{B_3}$.

(b) *If* $B_3 \neq 0$, *then* $\mathrm{Stab}_T(\lambda) = \{1\}$. *Hence, if* η *is an extension of* λ *to* HX_4, *then* $I_U(\eta) = HX_4$.

(c) *If* $B_3 = 0$*, then there exists* $x \in T$ *such that*

$$^{x}\lambda = \lambda^{0,0,0,0,0}_{b'_8,b'_9,b'_{10}}$$

for some $b'_8, b'_9, b'_{10} \in \mathbb{F}_q$*. Furthermore,* $\overline{H_3} \subset \ker(^{x}\lambda)^{HX_4S_1}$ *and the induction map from* $\mathrm{Irr}(HX_4S_1, {}^{x}\lambda)$ *to* $\mathrm{Irr}(U, \lambda)$ *is bijective.*

Proof. See Section 5.4.1. □

Remark. If $\gcd(q, 3) = 1$, then $\{x \in H_3 : |\lambda^U(x)| = \lambda^U(1)\}$ and $\mathrm{Stab}_T(\lambda)$ are trivial. Thus λ extends to HX_4 and hence induces up to U irreducibly.

By Lemma 3.2 (a), it is easy to see that $B_3 = B_3(\lambda)$ is T invariant, i.e., we have $B_3(\lambda) = B_3(^{x}\lambda)$ for all $x \in T$. As above we fix the actions of $\lambda|_{X_i} = \phi, 17 \leq i \leq 21$. Now H has q^8 linear characters. On q^7 of these $B_3 = 0$, whereas $B_3 \neq 0$ on the $q^7(q-1)$ remaining characters.

Case $B_3 \neq 0$: By Lemma 3.2 (b), each of the $q^7(q-1)$ linear characters of H with $B_3 \neq 0$ extends to HX_4 in q different ways, yielding $q^8(q-1)$ linear characters. Each of these induces irreducibly thereby partitioning the $q^8(q-1)$ characters into families of size $[U : HX_4] = q^7$. Therefore when $B_3 \neq 0$, there are $\frac{q^8(q-1)}{q^7} = q(q-1)$ irreducible characters of U lying over λ. They are parameterized by (b_4, B_3), and we denote them by $\chi^{b_4,B_3}_{q^7}$, where $b_4 \in \mathbb{F}_q$ and $B_3 \in \mathbb{F}_q^\times$.

Case $B_3 = 0$: As $H \trianglelefteq U$, we have $\lambda, {}^{x}\lambda \in \mathrm{Irr}(H)$ and $\mathrm{Irr}(U, \lambda) = \mathrm{Irr}(U, {}^{x}\lambda)$ for all $x \in T$. Hence, by Lemma 3.2 (c), we may assume that $\lambda := \lambda^{0,0,0,0,0}_{b_8,b_9,b_{10}}$. Since $[U : HX_4S_1] = q^4$ and character induction map from HX_4S_1 to U preserves irreducibility, those q^7 linear characters of H with $B_3 = 0$ are partitioned into q^3 sets each of size q^4. Each of these sets contains a unique HX_4S_1-character of the form $\lambda^{0,0,0,0,0}_{b_8,b_9,b_{10}}$.

Lemma 3.3. *The following statements are true.*

(a) $R_2 = \{x \in H_2 : |\lambda^{HX_4S_1}(x)| = \lambda^{HX_4S_1}(1)\}$ *and* $S_2 = \mathrm{Stab}_{S_1}(\lambda)$*. Moreover,* $\lambda^{HX_4S_1}|_{R_2} = \lambda^{HX_4S_1}(1)\phi_{B_2}$.

(b) *If* $B_2 \notin \{c^2 : c \in \mathbb{F}_q^\times\}$ *and let* η *be an extension of* λ *to* HX_4*, then we have* $I_{HX_4S_1}(\eta) = HX_4$*. Therefore,* S_2 *acts transitively and faithfully on all extensions of* λ *to* HX_4.

(c) *If* $B_2 = c^2 \in \mathbb{F}_q^\times$*, then* λ *extends to* HX_4F_2 *and* HF_4S_2*. Let* λ_1, λ_2 *be extensions of* λ *to* HX_4F_2*. Then* $I_{HX_4S_1}(\lambda_1) = HX_4F_2$*. Moreover,*

$$\lambda_1{}^{HX_4S_1} = \lambda_2{}^{HX_4S_1} \quad \textit{iff} \quad \lambda_1|_{F_2} = \lambda_2|_{F_2} \textit{ and } \lambda_1|_{F_4} = \lambda_2|_{F_4}.$$

Proof. See Section 5.4.2. □

Remark. When $B_2 = c^2 \neq 0$, we see that

$$HX_4F_3 \trianglelefteq U \quad \text{and} \quad HF_4S_2 \ntrianglelefteq U,$$

and both have index $\frac{q^7}{3}$ in U. By Lemma 3.2 (c) and Lemma 3.3 (c) all constituents of λ^U have degree $\frac{q^7}{3}$. Hence, if η is an extension of λ to HF_4S_2, then $\eta^U \in \mathrm{Irr}(U, \lambda)$. We have $X_4, S_2 \subset I_U(\lambda)$ and λ extends to HX_4F_3 and HF_4S_2, but λ does not extend to HX_4S_2.

The group HX_4 has q^4 linear characters λ such that

$$\lambda|_H = \lambda^{0,0,0,0,0}_{b_8,b_9,b_{10}}.$$

Since $\mathbb{F}_q^\times$ is even and cyclic, we see that for $\frac{q^3(q+1)}{2}$ of these $B_2 \notin \{c^2 : c \in \mathbb{F}_q^\times\}$, and for $\frac{q^3(q-1)}{2}$ of them $B_2 \in \{c^2 : c \in \mathbb{F}_q^\times\}$.

Case $B_2 \notin \{c^2 : c \in \mathbb{F}_q^\times\}$: By Lemma 3.3 (b), there are $\frac{q^3(q+1)}{2|S_1|} = \frac{q+1}{2}$ irreducibles of degree $|S_1| = q^3$ which are parameterized by $B_2 \notin \{c^2 : c \in \mathbb{F}_q^\times\}$. By Lemma 3.2 (c), we obtain $\frac{q+1}{2}$ irreducibles of degree $q^3[U : HX_4S_1] = q^7$, which are denoted by $\chi^{B_2}_{q^7}$ where $B_2 \in \mathbb{F}_q - \{c^2 : c \in \mathbb{F}_q^\times\}$.

Therefore, together with characters $\chi^{b_4,B_3}_{q^7}$ as computed above, $\mathcal{F}_6$ has exactly $(q-1)q + \frac{q+1}{2}$ irreducible characters χ of degree q^7 such that $\chi|_{X_i} = \chi(1)\phi$ for all $X_i \subset Z(U)$.

Case $B_2 \in \{c^2 : c \in \mathbb{F}_q^\times\}$: By Lemma 3.3 (c), let λ_1 be an extension of λ to HX_4F_2, then ${\lambda_1}^{HX_4S_1}$ is irreducible of degree $[HX_4S_1 : HX_4F_2] = \frac{q^3}{3}$. These ${\lambda_1}^{HX_4S_1}$ only depend on B_2 and their restrictions to F_2 and F_4. Hence, by Lemma 3.2 (c), ${\lambda_1}^U \in \mathrm{Irr}(U)$ of degree $\frac{q^7}{3}$ is denoted by $\chi^{b_2,b_4,B_2}_{\frac{q^7}{3}}$ where $b_2, b_4 \in \mathbb{F}_3$ and $B_2 \in \{c^2 : c \in \mathbb{F}_q^\times\}$.

Therefore, $\mathcal{F}_6$ has exactly $\frac{9(q-1)}{2}$ irreducibles of degree $\frac{q^7}{3}$ such that $\chi|_{X_i} = \chi(1)\phi$ for all $X_i \subset Z(U)$.

By the transitivity of the conjugate action of the maximal split torus T_0 of the Chevalley group $E_6(q)$ on $\bigoplus_{i=17}^{21} \mathrm{Irr}(X_i)^\times$, there are $(q-1)^5(q^2 - q + \frac{q+1}{2})$ characters $\chi \in \mathcal{F}_6$ of degree q^7, and $\frac{9(q-1)^6}{2}$ characters $\chi \in \mathcal{F}_6$ of degree $\frac{q^7}{3}$ such that $\chi|_{X_i} = \chi(1)\phi_{a_i}$, where $a_i \in \mathbb{F}_q^\times$, $17 \leq i \leq 21$. This gives the proof for the next theorem.

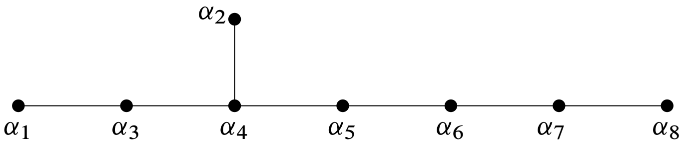
Theorem 3.4. *Let $\chi \in \mathcal{F}_6$. The following statements are true.*

(a) *If $\chi(1) = q^7$, then there exists $t \in T_0$ such that ${}^t\chi$ is either $\chi^{b_4,B_3}_{q^7}$ or $\chi^{B_2}_{q^7}$, for some $b_4 \in \mathbb{F}_q$, $B_3 \in \mathbb{F}_q^\times$, and $B_2 \in \mathbb{F}_q - \{c^2 : c \in \mathbb{F}_q^\times\}$.*

(b) *If $\chi(1) = \frac{q^7}{3}$, then there exists $t \in T_0$ such that ${}^t\chi = \chi^{b_2,b_4,B_2}_{\frac{q^7}{3}}$, for some $b_3, b_4 \in \mathbb{F}_3$ and $B_2 \in \{c^2 : c \in \mathbb{F}_q^\times\}$.*

4 Sylow 5-subgroups of the Chevalley groups $E_8(5^f)$

Let $\mathbb{F}_q$ be a field of order q and characteristic 5. We study $E_8(q)$ from the point of view of its root system. Let $\Sigma := \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \rangle$ be the root system of E_8, see Carter [1, Chapter 3]. The Dynkin diagram of Σ is



The positive roots are certain non-negative integral combinations of the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_8$. We write Σ^+ for the set of positive roots and note that $|\Sigma^+| = 120$. We use the notation

$$\begin{matrix} & & 3 & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 \end{matrix}$$

for the root $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$ and similar notation for the remaining positive roots. Let $X_\alpha := \langle x_\alpha(t) : t \in \mathbb{F}_q \rangle$ be the root subgroup corresponding to $\alpha \in \Sigma$. The group generated by all X_α for $\alpha \in \Sigma^+$ is a Sylow 5-subgroup of the Chevalley group $E_8(q)$, which we call UE_8.

In this section, we are going to construct irreducible characters χ of degree $\frac{q^{16}}{5}$ by considering the following family of irreducible characters of UE_8:

$$\mathscr{F}_8 := \{\chi \in \operatorname{Irr}(UE_8) : \chi|_{X_\alpha} = \chi(1)\phi_a,\ \operatorname{ht}(\alpha) = 6,\ a \in \mathbb{F}_q^\times\}.$$

Let ψ be a representation affording some $\chi \in \mathscr{F}_8$. Using the same argument as in Section 2 we see that $X_\alpha \subset \ker(\chi)$ for all positive roots α with height greater than 6. Let K_7 be the normal subgroup of UE_8 generated by all root subgroups of root heights greater than 6. Clearly the representation ψ is a module for the factor group $U := UE_8/K_7$. The restriction of the canonical projection from UE_8 to U to X_α is injective whenever α has hight 6 or less. Thus, in this case, we may identify X_α with its image in U. To remind the reader that U is a factor group of UE_8 we denoted it by $\widetilde{E_8}$ in the tables below. Recall that $|\Sigma^+|$ has exactly 43 positive roots of height less than or equal 6, which are listed in Table 5.

For positive roots, we use the abbreviation $x_i(t) = x_{\alpha_i}(t)$, $i = 1, 2, \ldots, 43$. Hence, $Z(U) = X_{37}X_{38}X_{39}X_{40}X_{41}X_{42}X_{43} = \langle X_\beta : \operatorname{ht}(\beta) = 6 \rangle$. Each element $u \in U$ can be written uniquely as

$$u = x_2(t_2)x_1(t_1)x_3(t_3)x_4(t_4)x_5(t_5)\cdots x_{43}(t_{43}) \quad \text{where } x_i(t_i) \in X_i.$$

So we write $\prod_{i=1}^{43} x_i(t_i)$ in the order as above. We note that in our order the term x_2 precedes x_1.

Height	Roots		
6	$\alpha_{43} := \begin{smallmatrix} & & 0 & & & & \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{smallmatrix}$		
	$\alpha_{40} := \begin{smallmatrix} & & 1 & & & & \\ 0 & 1 & 2 & 1 & 1 & 0 & 0 \end{smallmatrix}$	$\alpha_{41} := \begin{smallmatrix} & & 1 & & & & \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{smallmatrix}$	$\alpha_{42} := \begin{smallmatrix} & & 1 & & & & \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{smallmatrix}$
	$\alpha_{37} := \begin{smallmatrix} & & 1 & & & & \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 \end{smallmatrix}$	$\alpha_{38} := \begin{smallmatrix} & & 1 & & & & \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{smallmatrix}$	$\alpha_{39} := \begin{smallmatrix} & & 0 & & & & \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{smallmatrix}$
5	$\alpha_{36} := \begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{smallmatrix}$		
	$\alpha_{33} := \begin{smallmatrix} & & 1 & & & & \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{smallmatrix}$	$\alpha_{34} := \begin{smallmatrix} & & 1 & & & & \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{smallmatrix}$	$\alpha_{35} := \begin{smallmatrix} & & 0 & & & & \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{smallmatrix}$
	$\alpha_{30} := \begin{smallmatrix} & & 1 & & & & \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{smallmatrix}$	$\alpha_{31} := \begin{smallmatrix} & & 0 & & & & \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{smallmatrix}$	$\alpha_{32} := \begin{smallmatrix} & & 1 & & & & \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \end{smallmatrix}$
4	$\alpha_{29} := \begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{smallmatrix}$		
	$\alpha_{26} := \begin{smallmatrix} & & 1 & & & & \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{smallmatrix}$	$\alpha_{27} := \begin{smallmatrix} & & 0 & & & & \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{smallmatrix}$	$\alpha_{28} := \begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{smallmatrix}$
	$\alpha_{23} := \begin{smallmatrix} & & 1 & & & & \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{smallmatrix}$	$\alpha_{24} := \begin{smallmatrix} & & 0 & & & & \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{smallmatrix}$	$\alpha_{25} := \begin{smallmatrix} & & 1 & & & & \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{smallmatrix}$

To be continued

Height	Roots		
3	α_{22} := 0 0 0 0 0 1 1 1		
	α_{19} := 0 0 1 1 1 0 0 0	α_{20} := 0 0 0 1 1 1 0 0	α_{21} := 0 0 0 0 1 1 1 0
	α_{16} := 0 1 1 1 0 0 0 0	α_{17} := 1 0 1 1 0 0 0 0	α_{18} := 1 0 0 1 1 0 0 0
2	α_{15} := 0 0 0 0 0 0 1 1		
	α_{12} := 0 0 0 1 1 0 0 0	α_{13} := 0 0 0 0 1 1 0 0	α_{14} := 0 0 0 0 0 1 1 0
	α_{9} := 0 1 1 0 0 0 0 0	α_{10} := 1 0 0 1 0 0 0 0	α_{11} := 0 0 1 1 0 0 0 0
1	α_2 α_1 α_3	α_4 α_5 α_6	α_7 α_8

Table 5. Positive roots of the root system Σ of type $\widetilde{E_8}$.

For all $\alpha, \beta \in \Sigma$ the length of an α-chain through β is at most 1. Thus the Chevalley commutator formula, see Carter [1, Theorem 5.2.2], yields

$$[x_\alpha(r), x_\beta(s)] = \begin{cases} x_{\alpha+\beta}(-C_{\alpha,\beta} rs), & \text{if } \alpha+\beta \in \Sigma, \\ 1, & \text{otherwise.} \end{cases}$$

For each extraspecial pair (α, β), we choose the coefficient $C_{\alpha,\beta} := -1$. By direct computation or using MAGMA [5], we record the nontrivial commutators are in Table 6 below.

$[x_1(t), x_3(u)] = x_9(tu)$, $[x_2(t), x_4(u)] = x_{10}(tu)$,
$[x_3(t), x_4(u)] = x_{11}(tu)$, $[x_4(t), x_5(u)] = x_{12}(tu)$,
$[x_5(t), x_6(u)] = x_{13}(tu)$, $[x_6(t), x_7(u)] = x_{14}(tu)$,
$[x_7(t), x_8(u)] = x_{15}(tu)$, $[x_1(t), x_{11}(u)] = x_{16}(tu)$,
$[x_4(t), x_9(u)] = x_{16}(-tu)$, $[x_2(t), x_{11}(u)] = x_{17}(tu)$,
$[x_3(t), x_{10}(u)] = x_{17}(tu)$, $[x_2(t), x_{12}(u)] = x_{18}(tu)$,
$[x_5(t), x_{10}(u)] = x_{18}(-tu)$, $[x_3(t), x_{12}(u)] = x_{19}(tu)$,
$[x_5(t), x_{11}(u)] = x_{19}(-tu)$, $[x_4(t), x_{13}(u)] = x_{20}(tu)$,
$[x_6(t), x_{12}(u)] = x_{20}(-tu)$, $[x_5(t), x_{14}(u)] = x_{21}(tu)$,
$[x_7(t), x_{13}(u)] = x_{21}(-tu)$, $[x_6(t), x_{15}(u)] = x_{22}(tu)$,
$[x_8(t), x_{14}(u)] = x_{22}(-tu)$, $[x_1(t), x_{17}(u)] = x_{23}(tu)$,
$[x_2(t), x_{16}(u)] = x_{23}(tu)$, $[x_9(t), x_{10}(u)] = x_{23}(tu)$,
$[x_1(t), x_{19}(u)] = x_{24}(tu)$, $[x_5(t), x_{16}(u)] = x_{24}(-tu)$,
$[x_9(t), x_{12}(u)] = x_{24}(tu)$, $[x_2(t), x_{19}(u)] = x_{25}(tu)$,
$[x_3(t), x_{18}(u)] = x_{25}(tu)$, $[x_5(t), x_{17}(u)] = x_{25}(-tu)$,
$[x_2(t), x_{20}(u)] = x_{26}(tu)$, $[x_6(t), x_{18}(u)] = x_{26}(-tu)$,
$[x_{10}(t), x_{13}(u)] = x_{26}(tu)$, $[x_3(t), x_{20}(u)] = x_{27}(tu)$,
$[x_6(t), x_{19}(u)] = x_{27}(-tu)$, $[x_{11}(t), x_{13}(u)] = x_{27}(tu)$,
$[x_4(t), x_{21}(u)] = x_{28}(tu)$, $[x_7(t), x_{20}(u)] = x_{28}(-tu)$,
$[x_{12}(t), x_{14}(u)] = x_{28}(tu)$, $[x_5(t), x_{22}(u)] = x_{29}(tu)$,
$[x_8(t), x_{21}(u)] = x_{29}(-tu)$, $[x_{13}(t), x_{15}(u)] = x_{29}(tu)$,
$[x_1(t), x_{25}(u)] = x_{30}(tu)$, $[x_2(t), x_{24}(u)] = x_{30}(tu)$,
$[x_5(t), x_{23}(u)] = x_{30}(-tu)$, $[x_9(t), x_{18}(u)] = x_{30}(tu)$,
$[x_1(t), x_{27}(u)] = x_{31}(tu)$, $[x_6(t), x_{24}(u)] = x_{31}(-tu)$,
$[x_9(t), x_{20}(u)] = x_{31}(tu)$, $[x_{13}(t), x_{16}(u)] = x_{31}(-tu)$,
$[x_4(t), x_{25}(u)] = x_{32}(tu)$, $[x_{10}(t), x_{19}(u)] = x_{32}(-tu)$,
$[x_{11}(t), x_{18}(u)] = x_{32}(-tu)$, $[x_{12}(t), x_{17}(u)] = x_{32}(-tu)$,
$[x_2(t), x_{27}(u)] = x_{33}(tu)$, $[x_3(t), x_{26}(u)] = x_{33}(tu)$,
$[x_6(t), x_{25}(u)] = x_{33}(-tu)$, $[x_{13}(t), x_{17}(u)] = x_{33}(-tu)$,
$[x_2(t), x_{28}(u)] = x_{34}(tu)$, $[x_7(t), x_{26}(u)] = x_{34}(-tu)$,

To be continued

$[x_{10}(t), x_{21}(u)] = x_{34}(tu)$,	$[x_{14}(t), x_{18}(u)] = x_{34}(-tu)$,
$[x_3(t), x_{28}(u)] = x_{35}(tu)$,	$[x_7(t), x_{27}(u)] = x_{35}(-tu)$,
$[x_{11}(t), x_{21}(u)] = x_{35}(tu)$,	$[x_{14}(t), x_{19}(u)] = x_{35}(-tu)$,
$[x_4(t), x_{29}(u)] = x_{36}(tu)$,	$[x_8(t), x_{28}(u)] = x_{36}(-tu)$,
$[x_{12}(t), x_{22}(u)] = x_{36}(tu)$,	$[x_{15}(t), x_{20}(u)] = x_{36}(-tu)$,
$[x_1(t), x_{32}(u)] = x_{37}(tu)$,	$[x_4(t), x_{30}(u)] = x_{37}(tu)$,
$[x_{10}(t), x_{24}(u)] = x_{37}(-tu)$,	$[x_{12}(t), x_{23}(u)] = x_{37}(-tu)$,
$[x_{16}(t), x_{18}(u)] = x_{37}(-tu)$,	$[x_1(t), x_{33}(u)] = x_{38}(tu)$,
$[x_2(t), x_{31}(u)] = x_{38}(tu)$,	$[x_6(t), x_{30}(u)] = x_{38}(-tu)$,
$[x_9(t), x_{26}(u)] = x_{38}(tu)$,	$[x_{13}(t), x_{23}(u)] = x_{38}(-tu)$,
$[x_1(t), x_{35}(u)] = x_{39}(tu)$,	$[x_7(t), x_{31}(u)] = x_{39}(-tu)$,
$[x_9(t), x_{28}(u)] = x_{39}(tu)$,	$[x_{14}(t), x_{24}(u)] = x_{39}(-tu)$,
$[x_{16}(t), x_{21}(u)] = x_{39}(tu)$,	$[x_4(t), x_{33}(u)] = x_{40}(tu)$,
$[x_6(t), x_{32}(u)] = x_{40}(-tu)$,	$[x_{10}(t), x_{27}(u)] = x_{40}(-tu)$,
$[x_{11}(t), x_{26}(u)] = x_{40}(-tu)$,	$[x_{17}(t), x_{20}(u)] = x_{40}(tu)$,
$[x_2(t), x_{35}(u)] = x_{41}(tu)$,	$[x_3(t), x_{34}(u)] = x_{41}(tu)$,
$[x_7(t), x_{33}(u)] = x_{41}(-tu)$,	$[x_{14}(t), x_{25}(u)] = x_{41}(-tu)$,
$[x_{17}(t), x_{21}(u)] = x_{41}(tu)$,	$[x_2(t), x_{36}(u)] = x_{42}(tu)$,
$[x_8(t), x_{34}(u)] = x_{42}(-tu)$,	$[x_{10}(t), x_{29}(u)] = x_{42}(tu)$,
$[x_{15}(t), x_{26}(u)] = x_{42}(-tu)$,	$[x_{18}(t), x_{22}(u)] = x_{42}(tu)$,
$[x_3(t), x_{36}(u)] = x_{43}(tu)$,	$[x_8(t), x_{35}(u)] = x_{43}(-tu)$,
$[x_{11}(t), x_{29}(u)] = x_{43}(tu)$,	$[x_{15}(t), x_{27}(u)] = x_{43}(-tu)$,
$[x_{19}(t), x_{22}(u)] = x_{43}(tu)$,	

Table 6. Commutator relations for type $\widetilde{E_8}$.

Let $H := \langle X_\alpha : \alpha_4 \neq \alpha \in \Sigma^+,\ (\alpha, \alpha_5) > 0 \rangle = H_6 H_5 H_4 H_3 H_2$ where $(-,-)$ denotes the definite bilinear form of $\mathbb{R}^8$ with respect to which the roots of Σ have length 1,

$$H_6 := Z(U), \quad H_5 := \prod_{i=30}^{36} X_i, \quad H_4 := \prod_{i=24}^{29} X_i,$$

$$H_3 := \prod_{i=18}^{21} X_i, \quad H_2 := X_{12} X_{13}.$$

Let $T := \langle X_1, X_3, X_4, X_2, X_6, X_7, X_8 \rangle = T_4T_3T_2T_1$ where

$$T_4 := X_{23}, \qquad T_3 := X_{16}X_{17}X_{22},$$
$$T_2 := X_9X_{10}X_{11}X_{14}X_{15}, \quad T_1 := X_1X_3X_4X_2X_6X_7X_8.$$

It is clear that $|H| = q^{26}$, $|T| = q^{16}$, H_k is generated by all root groups in H of root height k, just as for T_k generated by all root subgroups in T of height k, and T is a transversal of HX_5 in U. Both H and HX_5 are elementary abelian and normal in U. The group T is isomorphic to $UA_4(q) \times UA_3(q)$, where $UA_k(q)$ is the unipotent subgroup of the standard Borel subgroup of the general linear group $\mathrm{GL}_{k+1}(q)$. We note that if $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ are the simple roots of type A_4, then the map from $\langle X_1, X_3, X_4, X_2 \rangle$ to $UA_4(q)$ that sends $x_1(t)$ to $x_{\beta_1}(t)$, $x_3(t)$ to $x_{\beta_2}(t)$, $x_4(t)$ to $x_{\beta_3}(t)$, and $x_2(t)$ to $x_{\beta_4}(-t)$ for all $t \in \mathbb{F}_q$ induces an isomorphism.

We consider linear characters $\lambda \in \mathrm{Irr}(H)$ such that $\lambda|_{X_i} = \phi_{a_i}$ for $37 \le i \le 43$ and $\lambda|_{X_j} = \phi_{b_j}$ for all appropriate $j \le 36$ where $a_i \in \mathbb{F}_q^\times$ and $b_j \in \mathbb{F}_q$. Since the maximal split torus of the Chevalley group $E_8(q)$ acts transitively on the product $\bigotimes_{i=37}^{43} \mathrm{Irr}(X_i)^\times$, it suffices to suppose that $\lambda|_{X_i} = \phi$ for all $37 \le i \le 43$.

Definition 4.1. For $b_i \in \mathbb{F}_q$ where $i \in [12..13, 18..21, 24..36]$ we define

(a) $B_5 := b_{30} + b_{31} - b_{32} - b_{33} - 2b_{34} + 2b_{35} + 2b_{36}$.

(b) $B_4 := 2b_{24} - 2b_{25} + b_{26} - b_{27} - b_{28} + b_{29}$.

(c) $B_3 := b_{18} - b_{19} - b_{20} + b_{21}$.

(d) $B_2 := b_{12} - b_{13}$.

(e) $R_5 := \{r_5(v) := x_{30}(v)x_{31}(v)x_{32}(-v)x_{33}(-v)x_{34}(-2v)x_{35}(2v)x_{36}(2v) : v \in \mathbb{F}_q\}$.

(f) $R_4 := \{r_4(v) := x_{24}(2v)x_{25}(-2v)x_{26}(v)x_{27}(-v)x_{28}(-v)x_{29}(v) : v \in \mathbb{F}_q\}$.

(g) $R_3 := \{r_3(v) := x_{18}(v)x_{19}(-v)x_{20}(-v)x_{21}(v) : v \in \mathbb{F}_q\}$.

(h) $R_2 := \{r_2(v) := x_{12}(v)x_{13}(-v) : v \in \mathbb{F}_q\}$.

(i) $L_1 := \{l_1(u) := x_2(2u)x_1(u)x_3(-2u)x_4(u)x_6(u)x_7(2u)x_8(-2u) : u \in \mathbb{F}_q\}$, $S_1 := L_1T_2T_3T_4$.

(j) $L_2 := \{l_2(u) := l_1(u)x_9(u^2)x_{10}(-u^2)x_{11}(u^2)x_{14}(-u^2)x_{15}(2u^2) : t \in \mathbb{F}_q\}$, $S_2 := L_2T_3T_4$.

(k) $L_3 := \{l_3(u) := l_2(u)x_{16}(4u^3)x_{17}(2u^3)x_{22}(3u^3) : u \in \mathbb{F}_q\}$, $S_3 := L_3T_4$

(l) $S_4 := \{l_4(u) := l_3(u)x_{23}(3u^4) : u \in \mathbb{F}_q\}$.

(m) If $B_2 = c^4 \in \mathbb{F}_q^\times$, $F_4 := \{s_4(uc) : u \in \mathbb{F}_5\}$ and $F_5 := \{x_5(vc_\phi) : v \in \mathbb{F}_5\}$.

It is easy to check that for $k \in [2..5]$, $R_k \le H_k$ of order q, $S_k \le S_{k-1} \le T$ with $S_5 = \{1\}$, and $F_4 \le S_4$, $F_5 \le X_5$ of order 5. It is noted that all B_i are defined for each λ as above, hence $B_i = B_i(\lambda)$. Since $R_k \cong \mathbb{F}_q$, for each $a \in \mathbb{F}_q$ we define $\phi_a(r_k(t)) = \phi_a(t)$ for all $r_k(t) \in R_k$. Hence, $\mathrm{Irr}(R_k) = \{\phi_a : a \in \mathbb{F}_q\}$. Since $S_4 \cong \mathbb{F}_q$, we can define $\phi_a(s_4(t)) = \phi_a(t)$ for all $s_4(t) \in S_4$. When $B_2 = c^4 \in \mathbb{F}_q^\times$, for each linear character $\xi \in \mathrm{Irr}(F_4)$ there is $b_4 \in \{ta_4 : t \in \mathbb{F}_5\} \cong (\mathbb{F}_5, +)$ such that $\xi = \phi_{b_4}|_{F_4}$ where $\phi_{b_4} \in \mathrm{Irr}(S_4)$ and $\phi(a_4 c) \ne 1$. Use the same argument for $F_5 \le X_5$, for each $\xi \in \mathrm{Irr}(F_5)$ there is $b_5 \in \{ta_5 : t \in \mathbb{F}_5\} \cong (\mathbb{F}_5, +)$ such that $\xi = \phi_{b_5}|_{F_5}$, where $\phi_{b_5} \in \mathrm{Irr}(X_5)$ and $\phi(a_5 c_\phi) \ne 1$.

We first outline the induction process of λ up to U, thereby explaining some of the notation in Definition 4.1. Later we give the detailed conditions that are necessary for each step of our construction.

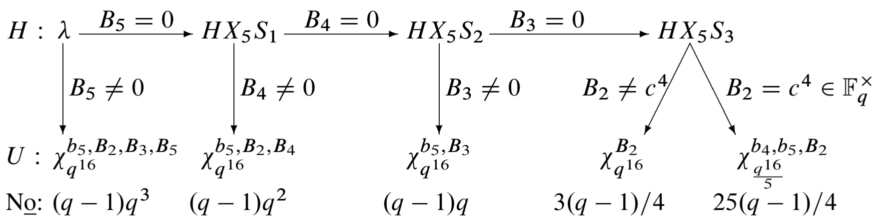


Figure $UE_8(q)$. Summary on the branching rules of λ.

Let $\overline{H_5}$ be the normal closure of H_5 in HX_5S_1. Clearly $X_5 \subset \mathrm{Stab}_U(\lambda)$. The important properties of the λ's are the following.

Lemma 4.2. *The following statements are true.*

(a) $R_5 = \{x \in H_5 : |\lambda^U(x)| = \lambda^U(1)\}$ *and* $S_1 = \mathrm{Stab}_T(\lambda|_{H_6H_5})$. *Moreover, we have* $\lambda^U|_{R_5} = \lambda^U(1)\phi_{B_5}$.

(b) *If* $B_5 \ne 0$, *then* $\mathrm{Stab}_T(\lambda) = \{1\}$. *Hence, if* η *is an extension of* λ *to* HX_5, *then* $I_U(\eta) = HX_5$. *Furthermore, if* η, η' *are two extensions of* $\lambda|_{H_6H_5H_4}$ *to* HX_5, *then* $\eta^U = \eta'^U$ *iff* $B_i(\eta) = B_i(\eta')$ *for* $i = 2, 3$ *and* $\eta|_{X_5} = \eta'|_{X_5}$.

(c) *If* $B_5 = 0$, *then there exists* $x \in T$ *such that* ${}^x\lambda|_{X_i} = 1_{X_i}$ *for all* $X_i \subset H_5$. *Furthermore, we have the inclusion* $\overline{H_5} \subset \ker({}^x\lambda)^{HX_5S_1}$ *and the induction map from* $\mathrm{Irr}(HX_5S_1, {}^x\lambda)$ *to* $\mathrm{Irr}(U, \lambda)$ *is bijective.*

Proof. See Section 5.5.1. □

Remark. When $(q, 5) = 1$, both R_5 and $\mathrm{Stab}_T(\lambda)$ are trivial. Hence, λ extends to HX_5 and each extension induces irreducibly to U, yielding a family of characters of degree $[U : HX_5] = q^{16}$.

Lemma 4.2 (a) can be deduced from the following figure.

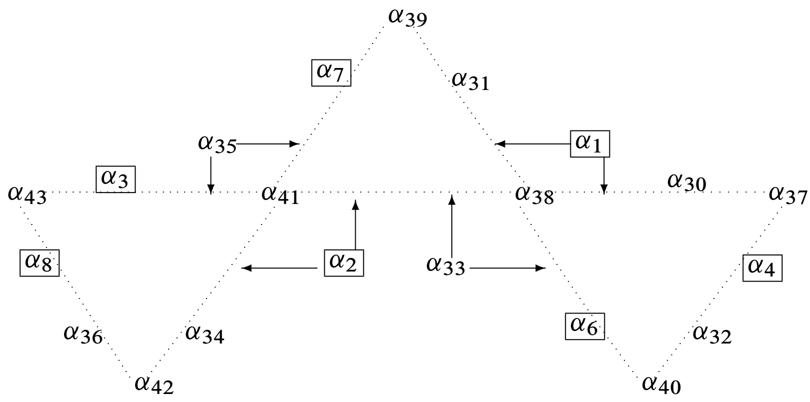


Figure $UE_8(q)$. Relations of between root heights 5 in H and 1 in T.

We have q^{19} linear characters λ of H such that $\lambda|_{X_i} = \phi$ for all $X_i \subset Z(U)$. In these, there are q^{18} linears with $B_5 = 0$ and $q^{18}(q-1)$ linears with $B_5 \neq 0$.

Case $B_5 \neq 0$: Lemma 4.2 (a) implies that $B_5 = B_5(\lambda)$ is invariant under the action of the group T. Therefore, by Lemma 4.2 (b), these $q^{18}(q-1)$ linears with $B_5 \neq 0$ extend to HX_5 and each extension induces irreducibly to U. Thus, we obtain $\frac{q^{19}(q-1)}{q^{16}} = q^3(q-1)$ irreducible characters of U of degree q^{16} which are parameterized by (b_5, B_2, B_3, B_5) where $b_5, B_2, B_3 \in \mathbb{F}_q$ and $B_5 \in \mathbb{F}_q^\times$. We denote them by $\chi_{q^{16}}^{b_5,B_2,B_3,B_5}$.

Case $B_5 = 0$: Since $[U : HX_5S_1] = q^6$, Lemma 4.2 (c) implies that the q^{18} linear characters of H lying over λ with $B_5 = 0$ are partitioned into q^12 families each of size q^6. Each family contains a unique member $\lambda \in \mathrm{Irr}(H)$ such that $\lambda|_{X_i} = \phi$ for all $X_i \subset H_6$, $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5$, and $\lambda|_{X_i} = \phi_{b_i}$ for all $X_i \subset H_4H_3H_2$ where $b_i \in \mathbb{F}_q$. Let $\lambda \in \mathrm{Irr}(H)$ be one of these q^{12} representatives. Now we describe how λ induces up to HX_5S_1. Let $\overline{H_5H_4}$ be the normal closure of H_5H_4 in HX_5S_2.

Lemma 4.3. *The following statements are true.*

(a) $R_4 = \{x \in H_4 : |\lambda^{HX_5S_1}(x)| = \lambda^{HX_5S_1}(1)\}$ *and* $S_2 = \mathrm{Stab}_{S_1}(\lambda|_{H_6H_5H_4})$. *Moreover,* $\lambda^{HX_5S_1}|_{R_4} = \lambda^{HX_5S_1}(1)\phi_{B_4}$.

(b) *If* $B_4 \neq 0$, *then* $\mathrm{Stab}_{S_1}(\lambda) = \{1\}$. *Hence, if* η *is an extension of* λ *to* HX_5, *then* $I_{HX_5S_1}(\eta) = HX_5$. *Furthermore, if* η, η' *are two extensions of* $\lambda|_{H_6H_5H_4H_3}$ *to* HX_5, *then* $\eta^{HX_5S_1} = \eta'^{HX_5S_1}$ *iff* $B_2(\eta) = B_2(\eta')$ *and* $\eta|_{X_5} = \eta'|_{X_5}$.

(c) *If* $B_4 = 0$, *then there exists* $x \in S_1$ *such that* ${}^x\lambda|_{X_i} = 1_{X_i}$ *for all* $X_i \subset H_5H_4$. *Furthermore, we have the inclusion* $\overline{H_5H_4} \subset \ker({}^x\lambda)^{HX_5S_2}$ *and the induction map from* $\mathrm{Irr}(HX_5S_2, {}^x\lambda)$ *to* $\mathrm{Irr}(HX_5S_1, \lambda)$ *is bijective.*

Proof. See Section 5.5.2. □

The main idea of Lemma 4.3 (a) can be visualized in the following figure.

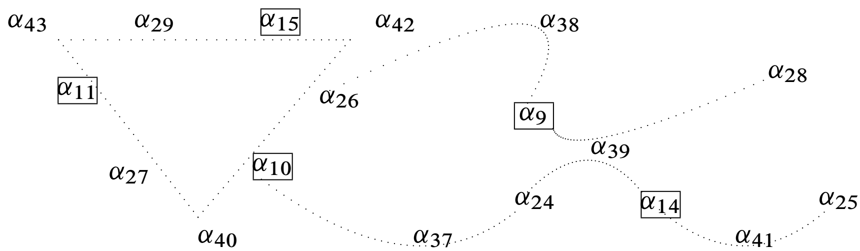


Figure $UE_8(q)$. Relations of between root heights 4 in H and 2 in T.

Recall that we have q^{12} linear characters λ of H such that $\lambda|_{X_i} = \phi$ for all $X_i \subset Z(U)$ and $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5$. For q^{11} of these we have $B_4 = 0$ whereas $B_4 \neq 0$ for the remaining $q^{11}(q-1)$.

Case $B_4 \neq 0$: Lemma 4.3 (a) implies that $B_4 = B_4(\lambda)$ is invariant under the action of S_1. Therefore, by Lemma 4.3 (b), these $q^{11}(q-1)$ linear characters with $B_4 \neq 0$ extend to HX_5 and each extension induces irreducibly to HX_5S_1. Thus, we obtain $\frac{q^{12}(q-1)}{|S_1|} = q^2(q-1)$ irreducible characters of HX_4S_1 of degree $|S_1| = q^{10}$ which are parameterized by (b_5, B_2, B_4) where $b_5, B_2 \in \mathbb{F}_q$ and $B_4 \in \mathbb{F}_q^{\times}$. Now Lemma 4.2 (c) implies that we obtain $q^2(q-1)$ characters of U of degree q^{16}. We denote these by $\chi_{q^{16}}^{b_5,B_2,B_4}$.

Case $B_4 = 0$: By Lemma 4.3 (c) the induction map from HX_5S_2 to HX_5S_1 is bijective. Thus our q^{11} linear characters of H with $B_4 = 0$ can be partitioned into q^6 families each of which has a unique representative λ such that $\lambda|_{X_i} = \phi$ for all $X_i \subset H_6$, $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4$, and $\lambda|_{X_i} = \phi_{b_i}$ for all $X_i \subset H_3H_2$ where $b_i \in \mathbb{F}_q$. Let $\lambda \in \mathrm{Irr}(H)$ be one of above q^6 linears of H. Thus it suffices to describe how λ induces up to HX_5S_2. Let $\overline{H_5H_4H_3}$ be the normal closure of $H_5H_4H_3$ in HX_5S_3.

Lemma 4.4. *The following statements are true.*

(a) $R_3 = \{x \in H_3 : |\lambda^{HX_5S_2}(x)| = \lambda^{HX_5S_2}(1)\}$, $S_3 = \mathrm{Stab}_{S_2}(\lambda|_{H_6H_5H_4H_3})$. *Moreover,* $\lambda^{HX_5S_2}|_{R_3} = \lambda^{HX_5S_2}(1)\phi_{B_3}$.

(b) *If* $B_3 \neq 0$, *then* $\mathrm{Stab}_{S_2}(\lambda) = \{1\}$. *Hence, if* η *is an extension of* λ *to* HX_5, *then* $I_{HX_5S_2}(\eta) = HX_5$. *Furthermore, if* η *and* η' *are two extensions of* λ *to* HX_5, *then* $\eta^{HX_5S_2} = \eta'^{HX_5S_2}$ *iff* $\eta|_{X_5} = \eta'|_{X_5}$.

(c) *If* $B_3 = 0$, *then there exists* $x \in S_2$ *such that*

$$^{x}\lambda|_{X_i} = 1_{X_i} \quad \text{for all } X_i \subset H_5H_4H_3.$$

Furthermore, we have the inclusion $\overline{H_5H_4H_3} \subset \ker({}^{x}\lambda)^{HX_5S_3}$ *and the induction map from* $\mathrm{Irr}(HX_5S_3, {}^{x}\lambda)$ *to* $\mathrm{Irr}(HX_5S_2, \lambda)$ *is bijective.*

Proof. See Section 5.5.3. □

The main idea of Lemma 4.4 (a) can be described as follows.

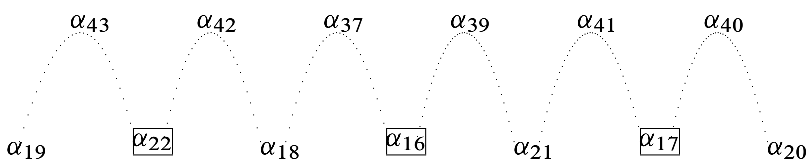


Figure $UE_8(q)$. Relations of between root heights 3 in H and 3 in T.

Recall that we have q^6 linear characters λ of H such that $\lambda|_{X_i} = \phi$ for all $X_i \subset Z(U)$ and $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4$. Of these, there are q^5 with $B_3 = 0$ and $q^5(q-1)$ with $B_3 \neq 0$.

Case $B_3 \neq 0$: Lemma 4.4 (a) implies that $B_3 = B_3(\lambda)$ is invariant under the action of S_2. Therefore, by Lemma 4.4 (b), these $q^5(q-1)$ linears with $B_3 \neq 0$ extend to HX_5 and each extension induces irreducibly to HX_5S_2. Thus, we obtain $\frac{q^6(q-1)}{|S_2|} = q(q-1)$ irreducible characters of HX_4S_2 of degree $|S_2| = q^5$ which are parameterized by (b_5, B_3) where $b_5 \in \mathbb{F}_q$ and $B_3 \in \mathbb{F}_q^{\times}$. Thus using Lemma 4.3 (c) and Lemma 4.2 (c), we obtain $q(q-1)$ characters of U of degree q^{16}. We denote these by $\chi_{q^{16}}^{b_5,B_3}$.

Case $B_3 = 0$: Lemma 4.4 (c) implies that q^5 linear characters of H with $B_3 = 0$ can be partitioned in q^2 families of characters each of which is represented by a λ of H such that $\lambda|_{X_i} = \phi$ for all $X_i \subset H_6$, $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4H_3$, and $\lambda|_{X_i} = \phi_{b_i}$ for all $X_i \subset H_2$ where $b_i \in \mathbb{F}_q$. Let $\lambda \in \mathrm{Irr}(H)$ be one of above q^2 linears of H. It suffices to consider how λ induces up to HX_5S_3.

Lemma 4.5. *The following statements are true.*

(a) $R_2 = \{x \in H_2 : |\lambda^{HX_5S_3}(x)| = \lambda^{HX_5S_3}(1)\}$ *and* $S_4 = \mathrm{Stab}_{S_3}(\lambda)$. *Moreover,* $\lambda^{HX_5S_3}|_{R_2} = \lambda^{HX_5S_3}(1)\phi_{B_2}$.

(b) *If* $B_2 \notin \{c^4 : c \in \mathbb{F}_q^\times\}$ *and let* η *be an extension of* λ *to* HX_5*, then we have* $I_{HX_5S_3}(\eta) = HX_5$*. Therefore,* S_4 *acts transitively and faithfully on all extensions of* λ *to* HX_5*.*

(c) *If* $B_2 = c^4 \in \mathbb{F}_q^\times$*, then* λ *extends to* HX_5F_4 *and* HF_5S_4*. Let* λ_1, λ_2 *be two extensions of* λ *to* HX_5F_4*. Then* $I_{HX_5S_3}(\lambda_1) = HX_5F_4$*. Moreover,*

$$\lambda_1{}^{HX_5S_3} = \lambda_2{}^{HX_5S_3} \quad \textit{iff} \quad \lambda_1|_{F_4} = \lambda_2|_{F_4} \textit{ and } \lambda_1|_{F_5} = \lambda_2|_{F_5}.$$

Proof. See Section 5.5.4. □

It is noted that $\mathrm{Stab}_{S_3}(\lambda) = \mathrm{Stab}_T(\lambda)$. The main idea of Lemma 4.5 (a) can be visualized in the following figure.

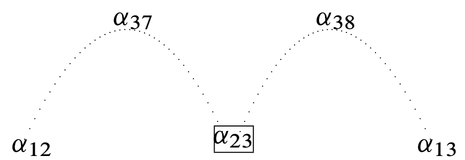


Figure $UE_8(q)$. Relations of between root heights 2 in H and 4 in T.

Recall that we have q^2 linear characters λ of H such that $\lambda|_{X_i} = \phi$ for all $X_i \subset Z(U)$ and $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4H_3$. Lemma 4.5 (a) implies that $B_2 = B_2(\lambda)$ is invariant under the action of S_3. Since $\mathbb{F}_q^\times$ is cyclic, we have

$$|\{c^4 : c \in \mathbb{F}_q^\times\}| = \frac{q-1}{4}.$$

Therefore, there are $\frac{q(q-1)}{4}$ linears with $B_2 \in \{c^4 : c \in \mathbb{F}_q^\times\}$, and there are $\frac{3q(q-1)}{4}$ linears with $B_2 \notin \{c^4 : c \in \mathbb{F}_q^\times\}$.

Case $B_2 \notin \{c^4 : c \in \mathbb{F}_q^\times\}$: These linears with $B_2 \notin \{c^4 : c \in \mathbb{F}_q^\times\}$ extend to HX_5 and each extension induces irreducibly to HX_5S_3 of degree $|S_3| = q^2$. By Lemma 4.4 (c), Lemma 4.3 (c) and Lemma 4.2 (c), we obtain $\frac{3(q-1)}{4}$ characters of U of degree q^{16} which we denote by $\chi_{q^{16}}^{B_2}$.

The irreducibles of degree q^{16} lie in the families

$$\chi_{q^{16}}^{b_5,B_2,B_3,B_5}, \quad \chi_{q^{16}}^{b_5,B_2,B_4}, \quad \chi_{q^{16}}^{b_5,B_3}, \quad \text{and} \quad \chi_{q^{16}}^{B_2}.$$

Therefore, $\mathcal{F}_8$ contains exactly $q^3(q-1)+q^2(q-1)+q(q-1)+\frac{3(q-1)}{5}$ characters χ of U of degree q^{16} such that $\chi|_{X_i} = \chi(1)\phi$ for all $X_i \subset Z(U)$.

Case $B_2 \in \{c^4 : c \in \mathbb{F}_q^\times\}$: By Lemma 4.5 (c), if λ_1 is an extension of the character λ to HX_5F_4, then $\lambda_1{}^{HX_5S_3}$ is irreducible of degree $[HX_5S_3 : HX_5F_4] = \frac{q^2}{5}$. These $\lambda_1{}^{HX_5S_3}$ only depend on B_2 and their restrictions to F_4, F_5. Hence, Lemma 4.4 (c), Lemma 4.3 (c) and Lemma 4.2 (c) imply that $\lambda_1{}^U \in \mathrm{Irr}(U)$ is of degree $\frac{q^{16}}{5}$. We denote the characters so obtained by

$$\chi_{\frac{q^{16}}{5}}^{b_4,b_5,B_2} \quad \text{where } b_4, b_5 \in \mathbb{F}_5 \text{ and } B_2 \in \{c^4 : c \in \mathbb{F}_q^\times\}.$$

Therefore, $\mathcal{F}_8$ has exactly $\frac{25(q-1)}{4}$ irreducibles of degree $\frac{q^{16}}{5}$ such that

$$\chi|_{X_i} = \chi(1)\phi \quad \text{for all } X_i \subset Z(U).$$

The maximal split torus T_0 of the Chevalley group $E_8(q)$ acts transitively via conjugation on $\bigoplus_{i=37}^{43} \operatorname{Irr}(X_i)^\times$, and thus there are $(q-1)^8(q^3+q^2+q+\frac{3}{4})$ characters $\chi \in \mathcal{F}_8$ of degree q^{16}, and $\frac{25(q-1)^8}{4}$ characters $\chi \in \mathcal{F}_8$ of degree $\frac{q^{16}}{5}$ such that $\chi|_{X_i} = \chi(1)\phi_{a_i}$, where $a_i \in \mathbb{F}_q^\times$, $37 \le i \le 43$. This proves our next theorem.

Theorem 4.6. *Let $\chi \in \mathcal{F}_8$. The following statements are true.*

(a) *If $\chi(1) = q^{16}$, then there exists $t \in T_0$ such that ${}^t\chi$ is an element of*

$$\{\chi_{q^{16}}^{b_5,B_2,B_3}, \chi_{q^{16}}^{b_5,B_2}, \chi_{q^{16}}^{b_5}, \chi_{q^{16}}^{B_2}\}.$$

(b) *If $\chi(1) = q^{16}/5$, then there exists $t \in T_0$ such that ${}^t\chi = \chi_{\frac{q^{16}}{5}}^{b_4,b_5,B_2}$.*

5 All proofs

In all proofs, we use the following technique:

(a) For all the decomposition of the commutator formula into product, we apply the formula $[a, bc] = [a, c][a, b]^c$.

(b) For $H \le G$ and $L \trianglelefteq G$, for each $\lambda \in \operatorname{Irr}(L)$, $\operatorname{Stab}_G(\lambda) := \{x \in G : {}^x\lambda = \lambda\}$, and $\operatorname{Stab}_G(\lambda) \subset \operatorname{Stab}_G(\lambda|_H) =: K$, hence, $\operatorname{Stab}_G(\lambda) = \operatorname{Stab}_K(\lambda)$.

(c) For $K \le G$ and $H \trianglelefteq G$, to extend a linear character λ of H to HK, we check if $[HK, HK] \subset \ker(\lambda)$.

5.1 Proof of Proposition 1.3

Let $a \in \mathbb{F}_q^\times$. Part (a) is clear since the degree of the polynomial $t^p - a^{p-1}t$ is p and the $\mathbb{F}_p$-multiples of a are clearly zeros. As $\mathbb{F}_q$ is of characteristic p, the map $\rho_a : \mathbb{F}_q \to \mathbb{F}_q$ defined by $\rho_a(t) = t^p - a^{p-1}t$ is $\mathbb{F}_p$-linear. By part (a) the kernel of the map is 1-dimensional and thus (b) follows. Evidently (d) follows from (c). Now we are going to prove (c).

To prove (c) we note that the set $\mathbb{T}_a$ defined before Proposition 1.3 is the image of the above $\mathbb{F}_p$-homomorphism ρ_a. The kernel of ρ_a is $a\mathbb{F}_p$, and $\mathbb{T}_a$ is an $\mathbb{F}_p$-hyperplane of $\mathbb{F}_q$. Since $\gcd(q-1,p)=1$, for $b\in\mathbb{F}_q^\times$ there exists $s\in\mathbb{F}_q^\times$ such that $b=s^p$. We have

$$b(t^p-a^{p-1}t)=s^p(t^p-a^{p-1}t)=(st)^p-(sa)^{p-1}st\in\mathbb{T}_{sa}=\operatorname{im}(\rho_{sa}).$$

Hence, $\mathbb{F}_q^\times$ acts on $\{\mathbb{T}_a : a\in\mathbb{F}_q^\times\}$. The first claim follows as left multiplication of $\mathbb{F}_q^\times$ on itself is transitive on $\mathbb{F}_q^\times$, hence on $\mathbb{F}_p$-one spaces, and thus by duality also on $\mathbb{F}_p$-hyperplanes. The second claim follows as the stabilizer of each $\mathbb{F}_p$-hyperplane in this action of $\mathbb{F}_q^\times$ is $\mathbb{F}_p^\times$. □

5.2 Proof of Proposition 1.5

(a) Suppose $\chi\in\operatorname{Irr}(N/Y,\lambda)$, we are going to show that $\chi^G\in\operatorname{Irr}(G)$ by showing that the inertia group $I_G(\chi)=N$.

Since $Y\subset\ker(\chi)$ and $Z\subset Z(N)$, we have $\chi|_{ZY}=\chi(1)\lambda$. As $X\subset N_G(ZY)$, for each $x\in X$, ${}^x\lambda\in\operatorname{Irr}(ZY)$. Hence, for any $u\neq v\in X$ we have

$${}^u\chi|_{ZY}=\chi(1)\,{}^u\lambda\neq\chi(1)\,{}^v\lambda={}^v\chi|_{ZY},\quad \text{i.e. } {}^u\chi\neq{}^v\chi.$$

Therefore, $x\in X$ such that ${}^x\chi=\chi$ iff $x=1$. Since X is a transversal of N in G, this shows that the inertia group $I_G(\chi)=N$.

The above argument also proves that for $\chi_1,\chi_2\in\operatorname{Irr}(N/Y,\lambda)$ and $u\neq v\in X$ we have ${}^u\chi_1\neq{}^v\chi_2$. So by Mackey's formula for the double coset $N\backslash G/N=G/N$ represented by X, we have

$$({\chi_1}^G,{\chi_2}^G)=({\chi_1}^G|_N,\chi_2)=\sum_{x\in X}({}^x\chi_1,\chi_2)=(\chi_1,\chi_2)=\begin{cases}1, & \text{if } \chi_1=\chi_2,\\ 0, & \text{otherwise.}\end{cases}$$

(b) It is enough to show that the induction map is surjective, i.e. for each character $\xi\in\operatorname{Irr}(G,\lambda)$ there exists $\chi\in\operatorname{Irr}(N/Y,\lambda)$ such that $\xi=\chi^G$.

Suppose $\xi|_N=\sum_{\chi_i\in S}a_i\chi_i$ where $a_i\in\mathbb{N}^\times$ and $S\subset\operatorname{Irr}(H)$. By Frobenius reciprocity,

$$0\neq(\xi,\lambda^G)=(\xi|_{ZY},\lambda),$$

there exists at least a constituent χ_0 of $\xi|_N$ such that $(\chi_0|_{ZY},\lambda)\neq 0$, i.e. we have $\chi_0\in\operatorname{Irr}(N,\lambda)$.

Since $\lambda|_Y=\lambda(1)1_Y$ and $(\chi_0|_Y,\lambda|_Y)\geq(\chi_0|_{ZY},\lambda)>0$, we have that χ_0 is a constituent of ${1_Y}^N$. Since $Y\trianglelefteq N$, all constituents of ${1_Y}^N$ are $\operatorname{Irr}(N/Y)$. Therefore, $\chi_0\in\operatorname{Irr}(N/Y,\lambda)$. By (a), ${\chi_0}^G\in\operatorname{Irr}(G)$, hence it forces $\xi={\chi_0}^G$. □

5.3 Proofs of Section "Sylow 2-subgroups of $D_4(2^f)$"

5.3.1 Proof of Lemma 2.2

Set $\lambda = \lambda^{a_8,a_9,a_{10}}_{b_3,b_5,b_6,b_7}$ for the whole proof.

(a) First we show that $\mathrm{Stab}_T(\lambda) = S_{124}$. Since ${}^y\lambda(x) = \lambda(x)$ iff $\lambda(x^{-1}x^y) = \lambda([x,y]) = 1$ and $X_8X_9X_{10} \subset Z(U)$, it suffices to check for $[X_5X_6X_7, T]$. For all $t_i, s_j \in \mathbb{F}_q$, we have

$$\begin{aligned}&[x_5(t_5)x_6(t_6)x_7(t_7), x_1(s_1)x_2(s_2)x_4(s_4)]\\ &\qquad = x_8(t_6s_1 + t_5s_2)x_9(t_7s_1 + t_5s_4)x_{10}(t_7s_2 + t_6s_4).\end{aligned}$$

Therefore, $x_1(s_1)x_2(s_2)x_4(s_4) \in \mathrm{Stab}_T(\lambda)$ iff for all $t_5, t_6, t_7 \in \mathbb{F}_q$,

$$\begin{aligned}1 &= \phi(a_8(t_6s_1 + t_5s_2) + a_9(t_7s_1 + t_5s_4) + a_{10}(t_7s_2 + t_6s_4))\\ &= \phi(t_5(a_8s_2 + a_9s_4) + t_6(a_8s_1 + a_{10}s_4) + t_7(a_9s_1 + a_{10}s_2))\end{aligned}$$

iff $a_8s_2 + a_9s_4 = a_8s_1 + a_{10}s_4 = a_9s_1 + a_{10}s_2 = 0$, i.e. $\frac{s_1}{a_{10}} = \frac{s_2}{a_9} = \frac{s_4}{a_8}$. So $\mathrm{Stab}_T(\lambda) = S_{124}$.

We first find all elements of $X_5X_6X_7$ which act scalarly on a module affording λ^U. As X_3T is a transversal of H in U and $[X_3, X_5X_6X_7] = \{1\}$, it is enough to find the ones of $X_5X_6X_7$ which commute with T, i.e. find $x_5x_6x_7 \in X_5X_6X_7$ such that $\lambda([x_5x_6x_7, x_1x_2x_4]) = 1$ for all $x_1x_2x_4 \in T$. This shows that for all $s_i \in \mathbb{F}_q$, we need

$$\begin{aligned}1 &= \phi(a_8(t_6s_1 + t_5s_2) + a_9(t_7s_1 + t_5s_4) + a_{10}(t_7s_2 + t_6s_4))\\ &= \phi(s_1(a_8t_6 + a_9t_7) + s_2(a_8t_5 + a_{10}t_7) + s_4(a_9t_5 + a_{10}t_6))\end{aligned}$$

iff $a_8t_6 + a_9t_7 = a_8t_5 + a_{10}t_7 = a_9t_5 + a_{10}t_6 = 0$, i.e. $\frac{t_5}{a_{10}} = \frac{t_6}{a_9} = \frac{t_7}{a_8}$. Hence, $\prod_{i=5}^{7} x_i(t_i) = x_{567}(\frac{t_7}{a_8}) \in S_{567}$. So

$$S_{567} = \{x \in X_5X_6X_7 : |\lambda^U(x)| = \lambda^U(1)\}.$$

Now, to prove that $\lambda^U|_{S_{567}} = q^4\phi_{At_0}$, it suffices to check that $\lambda(x_{567}(t)) = \phi_{At_0}(t)$. For each $x_{567}(t) = x_5(a_{10}t)x_6(a_9t)x_7(a_8t) \in S_{567}$, we have

$$\lambda(x_{567}(t)) = \phi(t(b_5a_{10} + b_6a_9 + b_7a_8)) = \phi(tAt_0) = \phi_{At_0}(t).$$

(b) We study $\mathrm{Irr}(U, \lambda)$ in two ways. Let

$$K_1 := HX_3F_{124} \quad \text{and} \quad K_2 := HS_{124}F_3.$$

Since $H = [U, U]$, it is clear that $H_1, K_1 \trianglelefteq U$.

$$\begin{array}{ccc} & H & \\ \swarrow & & \searrow \\ HX_3 & & HS_{124} \\ \downarrow & & \downarrow \\ K_1 = HX_3F_{124} & & K_2 = HS_{124}F_3 \\ \searrow & & \swarrow \\ & U & \end{array}$$

Since HX_3 is abelian, λ extends to HX_3 as η_1. By (a), $S_{124} = \mathrm{Stab}_T(\lambda)$, for all $x \in H, x_{124} \in S_{124}$, $\lambda([x, x_{124}]) = 1$, hence λ extends to HS_{124} as η_2. To show that λ extends to K_1 and K_2, we prove that $[K_1, K_1] \subset \ker(\eta_1), [K_2, K_2] \subset \ker(\eta_2)$. We have

$$\begin{aligned}&[x_3(t_3), x_1(s_1)x_2(s_2)x_4(s_4)]\\ &\qquad = x_5(s_1t_3)x_6(s_2t_3)x_7(s_4t_3)x_8(s_1s_2t_3)x_9(s_1s_4t_3)x_{10}(s_2s_4t_3),\end{aligned}$$

and

$$\begin{aligned}&\lambda(x_5(s_1t_3)x_6(s_2t_3)x_7(s_4t_3)x_8(s_1s_2t_3)x_9(s_1s_4t_3)x_{10}(s_2s_4t_3))\\ &\qquad = \phi(t_3(b_5s_1 + b_6s_2 + b_7s_4 + a_8s_1s_2 + a_9s_1s_4 + a_{10}s_2s_4)) = (*).\end{aligned}$$

Plug $s_1 = a_{10}t, s_2 = a_9t, s_4 = a_8t$ into $(*)$, we have

$$(*) = \phi(t_3(t(b_5a_{10} + b_6a_9 + b_7a_8) + t^2a_8a_9a_{10})) = \phi(t_3At(t_0 + t)).$$

Now we distinguish two cases, $t_0 = 0$ and $t_0 \neq 0$. First, if $t_0 = 0$, $\phi(t_3At^2) = 1$ for all t_3 iff $t = 0$, hence, $\mathrm{Stab}_T(\eta_1) = \{1\} = F_{124}$, i.e.

$$I_U(\eta_1) = HX_3.$$

And $\phi(t_3At^2) = 0$ for all t iff $t_3 = 0$, hence, $\mathrm{Stab}_{X_3}(\eta_2) = \{1\} = F_3$, i.e.

$$I_U(\eta_2) = HS_{124}.$$

If $t_0 \neq 0$, then $\phi(t_3At(t_0 + t)) = 1$ for all t_3 iff $t \in \{0, t_0\}$. Therefore, we have $[K_1, K_1] \subset \ker(\lambda)$. For each $\eta \in \mathrm{Irr}(HX_3, \lambda)$, $\mathrm{Stab}_T(\eta) = \{1, x_{124}(t_0)\} = F_{124}$.

We have $\phi(t_3At(t_0 + t)) = 1$ for all t iff $t_3 \in \{0, \frac{(t_0)_\phi}{A}\}$, by Proposition 1.3. Hence, $[K_2, K_2] \subset \ker(\lambda)$. For each $\gamma \in \mathrm{Irr}(HS_{124}, \lambda)$,

$$\mathrm{Stab}_{X_3}(\gamma) = \left\{1, x_3\left(\frac{(t_0)_\phi}{A}\right)\right\} = F_3.$$

So λ extends to K_1 and K_2. Now for each $\lambda_i \in \mathrm{Irr}(K_i, \lambda)$, $I_U(\lambda_i) = K_i, i = 1, 2$.

(c) Let λ_1, λ_2 be extensions of λ to K_1. Let η be an extension of λ to K_2. By (b), we have ${\lambda_1}^U, {\lambda_2}^U, \eta^U \in \mathrm{Irr}(U, \lambda)$.

We choose $1 \in S \subset T$ as a representative set of the double coset $K_1 \backslash U / K_2$. By Mackey's formula, since $K_1 \cap K_2 = HF_3F_{124}$ and $K_1 \trianglelefteq U$, we have

$$\begin{aligned}({\lambda_1}^U, \eta^U) &= \sum_{s \in S} ({}^s\lambda_1|_{{}^sK_1 \cap K_2}, \eta|_{{}^sK_1 \cap K_2}) \\ &= \sum_{s \in S} ({}^s\lambda_1|_{HF_3F_{124}}, \eta|_{HF_3F_{124}}).\end{aligned}$$

For each $s \in S$, if ${}^s\lambda_1|_{HF_3F_{124}} = \eta|_{HF_3F_{124}}$, then ${}^s\lambda_1|_H = \eta|_H$. Since both are extensions of λ from H, we have ${}^s\lambda = \lambda$, i.e. $s \in \mathrm{Stab}_T(\lambda) = S_{124}$. There is a unique $s = 1 \in S \cap S_{124}$ since S is a representative set of $K_1 \backslash U / K_2$. Therefore, $({\lambda_1}^U, \eta^U) = (\lambda_1|_{HF_3F_{124}}, \eta_2|_{HF_3F_{124}}) = 1$ iff $\lambda_1|_{F_i} = \eta|_{F_i}, i \in \{124, 3\}$.

So ${\lambda_1}^U = \eta^U = {\lambda_2}^U \in \mathrm{Irr}(U, \lambda)$ iff $\lambda_1|_{F_i} = \lambda_2|_{F_i}, i \in \{124, 3\}$. □

We remark that since $K_1, K_2 \trianglelefteq U$, the double coset $K_1 \backslash U / K_2$ equals

$$U/K_1K_2 = U/HX_3S_{124}.$$

Hence we see that $S = X_1X_2$ is a transversal of U/K_1K_2.

5.3.2 Proof of Theorem 2.3

Fix $a_8, a_9, a_{10} \in \mathbb{F}_q^\times$ and set $\lambda = \lambda_{b_5,b_6,b_7}^{a_8,a_9,a_{10}}$ for some $b_5, b_6, b_7 \in \mathbb{F}_q$ throughout the whole proof. By Lemma 2.2 and using the same notation, we mostly find the generic character values: in (a)

$$\chi_{8,9,10,q^3}^{a_8,a_9,a_{10}} = {\eta_1}^U \quad \text{where } t_0 = 0,$$

and in (b)

$$\chi_{8,9,10,\frac{q^3}{2}}^{b_{124},b_3,t_0,a_8,a_9,a_{10}} = {\eta_1}^U \quad \text{where } b_{124}, b_3 \in \mathbb{F}_2,\ t_0 \in \mathbb{F}_q^\times.$$

(a) Suppose $t_0 = 0$ and $F_{124} = \{1\}$. Call η an extension of λ to HX_3. By Lemma 2.2 (b), $I_U(\eta) = HX_3$. Therefore, we have $\eta^U \in \mathrm{Irr}(U)$ and $\eta^U(1) = q^3$. By Lemma 2.2 (a), $S_{567}X_8X_9X_{10} \subset Z(\eta^U)$, hence

$$|\eta^U(x)| = q^3 \quad \text{for all } x \in S_{567}X_8X_9X_{10}.$$

We have $|S_{567}X_8X_9X_{10}|q^3q^3 = q^{10} = |U|$. As the scalar product $(\eta^U, \eta^U) = 1$, we see that $\eta^U(x) = 0$ if $x \notin S_{567}X_8X_9X_{10}$. So we have derived the stated formula.

(b) Suppose $t_0 \neq 0$, and $|F_3| = |F_{124}| = 2$. By Lemma 2.2 (b), let η_1, η_2 be extensions of λ to $K_1 := HX_3F_{124}$ and $K_2 := HS_{124}F_3$ respectively such that $\eta_1|_{F_i} = \eta_2|_{F_i} = \phi_{b_i}$, where $b_i \in \mathbb{F}_2$, $i \in \{124, 3\}$. The proof of Lemma 2.2 (c) implies that ${\eta_1}^U = {\eta_2}^U$.

We choose $V \subset T$ as a transversal of K_1 in U, and $1 \in S \subset X_3$ such that SX_1X_2 is a transversal of K_2 in U, so $|S| = q/2$. Since $K_1 \trianglelefteq U$, we have

$$ {\eta_1}^U\left(\prod_{i=1}^{10} x_i\right) = \sum_{x \in V} {}^x\eta_1\left(\prod_{i=1}^{10} x_i\right) = 0 \quad \text{if } x_1x_2x_4 \notin K_1. $$

Since T is abelian, it follows that $[x, y] = 1$ for all $x \in V$ and $y \in F_{124}$. Therefore, $F_{124} \subset Z({\eta_1}^U)$ and hence

$$ {\eta_1}^U\left(\prod_{i=1}^{10} x_i(t_i)\right) = \delta_{a_8t_1, a_{10}t_4}\delta_{a_8t_2, a_9t_4}\phi\left(b_{124}\frac{t_1}{a_{10}}\right){\eta_1}^U\left(x_3(t_3)\prod_{i=5}^{10} x_i(t_i)\right). $$

Since $K_2 \trianglelefteq U$, we have

$$ {\eta_2}^U\left(x_3\prod_{i=5}^{10} x_i\right) = \sum_{x \in SX_1X_2} {}^x\eta_2\left(x_3\prod_{i=5}^{10} x_i\right) = 0 \quad \text{if } x_3 \notin F_3. $$

Since $X_8X_9X_{10} \subset Z(U)$, we need to compute the two following cases:

$$ {\eta_2}^U\left(\prod_{i=5}^{7} x_i\right) \quad \text{and} \quad {\eta_2}^U\left(x_3\prod_{i=5}^{7} x_i\right) \text{ with } x_3 \in F_3^\times. $$

Since $[X_3, X_5X_6X_7] = \{1\}$, we have

$$ {\eta_2}^U(x_5x_6x_7) = \sum_{x \in SX_1X_2} {}^x\eta_2(x_5x_6x_7) = \frac{q}{2}\sum_{x_1x_2 \in X_1X_2} {}^x\eta_2(x_5x_6x_7). $$

Now $(x_5x_6x_7)^{x_1x_2} = x_5x_6x_7[x_5, x_2][x_6, x_1][x_7, x_1][x_7, x_2]$, so substituting with $x_5(t_5)x_6(t_6)x_7(t_7)$ and $x_1(s_1)x_2(s_2)$, yields

$$ \begin{aligned} &{\eta_2}^U\left(\prod_{i=5}^{7} x_i(t_i)\right) \\ &= \frac{q}{2}\sum_{s_1,s_2} \eta_2(x_5(t_5)x_6(t_6)x_7(t_7)x_8(t_5s_2 + t_6s_1)x_9(t_7s_1)x_{10}(t_7s_2)) \\ &= \frac{q}{2}\eta_2(x_5(t_5)x_6(t_6)x_7(t_7))\sum_{s_1,s_2} \phi(a_8(t_5s_2 + t_6s_1) + a_9t_7s_1 + a_{10}t_7s_2) \\ &= \frac{q}{2}\eta_2(x_5(t_5)x_6(t_6)x_7(t_7))\sum_{s_1,s_2} \phi(s_1(a_8t_6 + a_9t_7) + s_2(a_8t_5 + a_{10}t_7)). \end{aligned} $$

Since $\sum_{t\in\mathbb{F}_q}\phi(t)=0$, we obtain non-zero values only if $a_8t_6+a_9t_7=0$ and $a_8t_5+a_{10}t_7=0$. Hence, $\frac{t_5}{a_{10}}=\frac{t_6}{a_9}=\frac{t_7}{a_8}$, and $\prod_{i=5}^{7}x_i(t_i)=x_{567}(\frac{t_7}{a_8})\in S_{567}$. By Lemma 2.2 (a), we have

$$\begin{aligned}{\eta_2}^U\left(\prod_{i=5}^{7}x_i(t_i)\right)&=\delta_{a_8t_5,a_{10}t_7}\delta_{a_8t_6,a_9t_7}\frac{q^3}{2}\eta_2\left(x_{567}\left(\frac{t_7}{a_8}\right)\right)\\&=\delta_{a_8t_5,a_{10}t_7}\delta_{a_8t_6,a_9t_7}\frac{q^3}{2}\phi\left(At_0\frac{t_7}{a_8}\right).\end{aligned}$$

Therefore, we have ${\eta_2}^U(\prod_{i=1}^{10}x_i(t_i))=\frac{q^3}{2}\phi(b_{124}\frac{t_1}{a_{10}}+At_0\frac{t_7}{a_8}+\sum_{i=8}^{10}a_it_i)$ if $\prod_{i=1}^{10}x_i(t_i)\in F_{124}S_{567}X_8X_9X_{10}=Z$, as stated in the theorem.

Now we compute ${\eta_2}^U(x_3\prod_{i=5}^{7}x_i)$ with $x_3\in F_3^\times=\{x_3(t_0^\phi)\}$, $t_0^\phi=\frac{(t_0)_\phi}{A}$. As $I_U(\eta_2)=K_2\trianglelefteq U$ and SX_1X_2 is a representative set of U/K_2,

$$({}^x\eta_2)^U={\eta_2}^U\in\mathrm{Irr}(U)\quad\text{for all } x\in SX_1X_2.$$

For each $x_2(s)\in X_2$, we have

$${}^{x_2(s)}\eta_2(x_5(t))=\eta_2(x_5(t)x_8(ts))=\phi(b_5t+a_8ts)=\phi(t(b_5+a_8s)).$$

So instead of choosing $s=\frac{b_5}{a_8}$, we suppose that η_2 has $b_5=0$, i.e. $\eta_2(x_5)=1$ for all $x_5\in X_5$. It is easy to check that t_0, $\eta_2|_{F_{124}}=\phi_{b_{124}}$ and $\eta_2|_{F_3}=\phi_{b_3}$ are invariant under conjugation.

We have

$$\begin{aligned}&[x_3(t_3)x_5(t_5)x_6(t_6)x_7(t_7),x_1(s_1)x_2(s_2)]\\&\quad=x_3(t_3)x_5(t_5+t_3s_1)x_6(t_6+t_3s_2)x_7(t_7)\\&\qquad\times x_8(t_3s_1s_2+t_5s_2+t_6s_1)x_9(t_7s_1)x_{10}(t_7s_2).\end{aligned}$$

Therefore,

$$\begin{aligned}&{\eta_2}^U(x_3(t_3)x_5(t_5)x_6(t_6)x_7(t_7))\\&\quad=\sum_{x\in SX_1X_2}{}^x\eta_2(x_3(t_3)x_5(t_5)x_6(t_6)x_7(t_7))\\&\quad=\frac{q}{2}\sum_{x\in X_1X_2}{}^x\eta_2(x_3(t_3)x_5(t_5)x_6(t_6)x_7(t_7))\\&\quad=\frac{q}{2}\sum_{s_1,s_2}\eta_2(x_3(t_3)x_5(t_5+t_3s_1)x_6(t_6+t_3s_2)x_7(t_7)\\&\qquad\times x_8(t_3s_1s_2+t_5s_2+t_6s_1)x_9(t_7s_1)x_{10}(t_7s_2))\end{aligned}$$

$$= \frac{q}{2}\eta_2\left(x_3(t_3)\prod_{i=5}^{7}x_i(t_i)\right)$$
$$\times \sum_{s_1,s_2}\phi(b_6t_3s_2 + a_8(t_3s_1s_2 + t_5s_2 + t_6s_1) + a_9t_7s_1 + a_{10}t_7s_2)$$
$$= \frac{q}{2}\eta_2(x_3(t_3)x_6(t_6)x_7(t_7))$$
$$\times \sum_{s_1,s_2}\phi(s_1(a_8t_3s_2 + a_8t_6 + a_9t_7) + s_2(b_6t_3 + a_{10}t_7 + a_8t_5)).$$

Set $C(t_5,t_6,t_7) = \sum_{s_1,s_2}\phi(s_1(a_8t_3s_2+a_8t_6+a_9t_7)+s_2(b_6t_3+a_{10}t_7+a_8t_5))$. We have

$$C(t_5,t_6,0) = \sum_{s_1,s_2}\phi(s_1(t_3s_2+t_6)a_8 + s_2(b_6t_3+a_8t_5))$$
$$= q\sum_{s_2=\frac{t_6}{t_3}}\phi\left(\frac{t_6}{t_3}(b_6t_3+a_8t_5)\right)$$
$$= q\phi\left(b_6t_6 + \frac{a_8t_5t_6}{t_3}\right).$$

Therefore, we get

$$\eta_2{}^U(x_3(t_3)x_5(t_5)x_6(t_6)) = \frac{q^2}{2}\eta_2(x_3(t_3)x_6(t_6))\phi\left(b_6t_6 + \frac{a_8t_5t_6}{t_3}\right)$$
$$= \frac{q^2}{2}\eta_2(x_3(t_3))\phi\left(\frac{a_8t_5t_6}{t_3}\right).$$

Since

$$x_5(t_5)x_6(t_6)x_7(t_7) = x_5\left(t_5 + \frac{a_{10}t_7}{a_8}\right)x_6\left(t_6 + \frac{a_9t_7}{a_8}\right)x_{567}\left(\frac{t_7}{a_{10}}\right),$$

where

$$x_{567}\left(\frac{t_7}{a_8}\right) = x_5\left(a_{10}\frac{t_7}{a_8}\right)x_6\left(a_9\frac{t_7}{a_8}\right)x_7\left(a_8\frac{t_7}{a_8}\right) \in S_{124} \subset Z(\eta_2{}^U)$$

and

$$\eta_2\left(x_{567}\left(\frac{t_7}{a_8}\right)\right) = \phi_{At_0}\left(\frac{t_7}{a_8}\right),$$

we have

$$\begin{aligned}
&{\eta_2}^U(x_3(t_3)x_5(t_5)x_6(t_6)x_7(t_7))\\
&\quad= \phi_{At_0}\left(\frac{t_7}{a_8}\right){\eta_2}^U\left(x_3(t_3)x_5\left(t_5+\frac{a_{10}t_7}{a_8}\right)x_6\left(t_6+\frac{a_9t_7}{a_8}\right)\right)\\
&\quad= \phi\left(At_0\frac{t_7}{a_8}\right)\frac{q^2}{2}\eta_2(x_3(t_3))\phi\left(\frac{a_8}{t_3}\left(t_5+\frac{a_{10}t_7}{a_8}\right)\left(t_6+\frac{a_9t_7}{a_8}\right)\right)\\
&\quad= \frac{q^2}{2}\phi\left(b_3t_3+At_0\frac{t_7}{a_8}+\frac{a_8^2a_9a_{10}}{(t_0)_\phi}\left(t_5+\frac{a_{10}t_7}{a_8}\right)\left(t_6+\frac{a_9t_7}{a_8}\right)\right)\\
&\quad= \frac{q^2}{2}\phi\left(b_3t_0^\phi+At_0\frac{t_7}{a_8}+\frac{A^2}{(t_0)_\phi}\left(\frac{t_5}{a_{10}}+\frac{t_7}{a_8}\right)\left(\frac{t_6}{a_9}+\frac{t_7}{a_8}\right)\right). \qquad \square
\end{aligned}$$

5.4 Proof of Section "Sylow 3-subgroups of $E_6(3^f)$"

5.4.1 Proof of Lemma 3.2

Set $\lambda = \lambda^{b_{12},b_{13},b_{14},b_{15},b_{16}}_{b_8,b_9,b_{10}}$ throughout the whole proof.

(a) Recall $H_3 = \prod_{i=12}^{16} X_i \trianglelefteq H$ is elementary abelian and $H_4H_3 \trianglelefteq U$. First, we show that $R_3 = \{x \in H_3 : |\lambda^U(x)| = \lambda^U(1)\}$ and $\lambda^U(r_3(t)) = q^8\phi_{B_3}(t)$ for all $r_3(t) \in R_3$.

Since λ is linear and $\lambda(x) \in \mathbb{C}$ for all $x \in H$, the induction formula gives that $|\lambda^U(x)| = \lambda^U(1)$ iff ${}^y\lambda(x) = \lambda(x)$ for all $y \in TX_4$, where TX_4 is a transversal of H in U. Since ${}^y\lambda(x) = \lambda(x)$ iff $\lambda([x,y]) = 1$, we are going to find all $x \in H_3$ such that $\lambda([x,y]) = 1$ for all $y \in TX_4$. It is clear that

$$[X_i, X_4] = \{1\} = [X_i, X_7X_{11}] \quad \text{for all } 12 \le i \le 16.$$

Here, we write $\prod_{j=1}^{6} x_i(u_j) \in T$ with $u_4 = 0$. It suffices to check that statement for all $y = \prod_{j=1}^{6} x_j(u_j) \in T$. For $t_i, u_j \in \mathbb{F}_q$, we have

$$\begin{aligned}
&\left[\prod_{i=12}^{16} x_i(t_i), \prod_{j=1}^{6} x_j(u_j)\right]\\
&= [x_{12}(t_{12}), x_2(u_2)][x_{15}(t_{15}), x_2(u_2)][x_{16}(t_{16}), x_2(u_2)][x_{13}(t_{13}), x_1(u_1)]\\
&\qquad\times [x_{15}(t_{15}), x_1(u_1)][x_{14}(t_{14}), x_3(u_3)][x_{16}(t_{16}), x_3(u_3)][x_{12}(t_{12}), x_5(u_5)]\\
&\qquad\times [x_{13}(t_{13}), x_5(u_5)][x_{14}(t_{14}), x_6(u_6)][x_{15}(t_{15}), x_6(u_6)]\\
&= x_{17}(-t_{12}u_2)x_{19}(-t_{15}u_2)x_{20}(-t_{16}u_2)x_{17}(-t_{13}u_1)x_{18}(-t_{15}u_1)x_{19}(-t_{14}u_3)\\
&\qquad\times x_{21}(-t_{16}u_3)x_{18}(t_{12}u_5)x_{19}(t_{13}u_5)x_{20}(t_{14}u_6)x_{21}(t_{15}u_6).
\end{aligned}$$

Since $\lambda(x_i(t)) = \phi(t)$, $17 \leq i \leq 21$, for all $u_j \in \mathbb{F}_q$, we force

$$(-t_{12} - t_{15} - t_{16})u_2 + (-t_{13} - t_{15})u_1 + (-t_{14} - t_{16})u_3 + (t_{12} + t_{13})u_5 + (t_{14} + t_{15})u_6 = 0.$$

So we have a system with variables t_i:

$$\begin{cases} -t_{12} - t_{15} - t_{16} = 0, \\ -t_{13} - t_{15} = 0, \\ -t_{14} - t_{16} = 0, \\ t_{12} + t_{13} = 0, \\ t_{14} + t_{15} = 0. \end{cases}$$

Since $\gcd(q, 3) = 3$, we have

$$t_{12} = t_{16} = t_{15}, t_{13} = t_{14} = -t_{15} \quad \text{for all } t_{15} = t \in \mathbb{F}_q.$$

So $x \in H_3$ satisfies $|\lambda^U(x)| = \lambda^U(1)$ iff

$$x = x_{12}(t)x_{13}(-t)x_{14}(-t)x_{15}(t)x_{16}(t) = r_3(t) \in R_3 \quad \text{for } t \in \mathbb{F}_q,$$

i.e. $R_3 = \{x \in H_3 : |\lambda^U(x)| = \lambda^U(1)\}$.

The computation that shows that $\lambda^U|_{R_3} = \lambda^U(1)\phi_{B_3}$ also shows that it is sufficient to check that $\lambda(r_3(t) = \phi_{B_3}(t)$. For each $r_3(t) \in R_3$, we have

$$\lambda(r_3(t)) = \phi(t(b_{12} - b_{13} - b_{14} + b_{15} + b_{16})) = \phi_{B_3}(t).$$

Now we show that $S_1 = \mathrm{Stab}_T(\lambda|_{H_4 H_3})$. Since $[H_4, T] = [H_3, X_7 X_{11}] = \{1\}$, it suffices to find $y \in X_2 X_1 X_3 X_5 X_6$ such that $\lambda([x, y]) = 1$ for all $x \subset H_3$. Using the computation of $[\prod_{i=12}^{16} x_i(t_i), \prod_{j=1}^{6} x_j(u_j)]$ above, we see that

$$(-u_2 + u_5)t_{12} + (-u_1 + u_5)t_{13} + (-u_3 + u_6)t_{14} + (u_1 + u_2 - u_6)t_{15} + (-u_2 - u_3)t_{16} = 0$$

for all $t_j \in \mathbb{F}_q$. So we have a system with variables u_j:

$$\begin{cases} -u_2 + u_5 = 0, \\ -u_1 + u_5 = 0, \\ -u_3 + u_6 = 0, \\ -u_1 - u_2 + u_6 = 0, \\ -u_2 - u_3 = 0. \end{cases}$$

As $\gcd(q, 3) = 3$, we have $u_1 = u_5 = u_2, u_3 = u_6 = -u_2$ for all $u_2 = t \in \mathbb{F}_q$. So $\prod_{j=1}^{6} x_j(u_j) = x_2(t)x_1(t)x_3(-t)x_5(t)x_6(-t) = s_1(t) \in S_1$.

(b) Since $H = Z(U)H_3X_8X_9X_{10}$, to find $\mathrm{Stab}_T(\lambda) \subset \mathrm{Stab}_T(\lambda|_{H_4H_3}) = S_1$, by (a), it is enough to find $s_1 \in S_1$ such that ${}^{s_1}\lambda(x_i) = \lambda(x_i)$ for $i = 8, 9, 10$. Again, for each $x_i(t_i) \in X_i$, $i = 8, 9, 10$, and $s_1 = s_1(t, r, s) \in S_1$, we compute $[x_i, s_1]$:

$$[x_8(t_8), s_1] = x_{20}(t_8s)x_{17}(-t_8r)x_{14}(t_8t)x_{20}(-t_8t^2)x_{13}(t_8t)x_{19}(t_8t^2),$$

$$[x_9(t_9), s_1] = x_{21}(t_9s)x_{15}(t_9t)x_{21}(-t_9t^2)x_{12}(-t_9t)x_{18}(-t_9t^2)x_{13}(-t_9t) \times x_{19}(-t_9t^2)x_{17}(t_9t^2),$$

$$[x_{10}(t_{10}), s_1] = x_{18}(-t_{10}r)x_{16}(-t_{10}t)x_{15}(t_{10}t)x_{21}(-t_{10}t^2)x_{14}(-t_{10}t) \times x_{20}(t_{10}t^2)x_{19}(-t_{10}t^2).$$

Since $\lambda(x_i(t)) = \phi(b_it)$, $12 \le i \le 16$, and $\lambda(x_i(t)) = \phi(t)$, $17 \le i \le 21$, from $\lambda([x_9(t_9), s_1]) = \lambda([x_{10}(t_{10}), s_1]) = 1$ for all $t_9, t_{10} \in \mathbb{F}_q$, we have

$$s = 2t^2 + b_{12}t + b_{13}t - b_{15}t \quad \text{and} \quad r = 2t^2 - b_{14}t + b_{15}t - b_{16}t.$$

From $\lambda([x_8(t_8), s_1]) = 1$ for all $t_8 \in \mathbb{F}_q$, we have $s - r + b_{14}t + b_{13}t = 0$. Therefore, $s_1 \in \mathrm{Stab}_T(\lambda)$ iff r, s are as above and

$$\begin{aligned} s - r + b_{14}t + b_{13}t &= 2t^2 + b_{12}t + b_{13}t - b_{15}t - (2t^2 - b_{14}t + b_{15}t - b_{16}t) \\ &\quad + b_{14}t + b_{13}t \\ &= t(b_{12} - b_{13} - b_{14} + b_{15} + b_{16}) \\ &= tB_3 = 0. \end{aligned}$$

Therefore, if $B_3 \neq 0$, then $\mathrm{Stab}_T(\lambda) = \{1\}$.

(c) By (a), T/S_1 acts faithfully on the set of all extensions of $\lambda|_{H_4}$ to H_4H_3 with the same B_3. Since $|T/S_1| = q^4 = |H_3/R_3|$, it follows that this action is transitive. Therefore, with $B_3 = 0$, there exists $x \in T$ such that ${}^x\lambda = \lambda^{0,0,0,0,0}_{b'_8,b'_9,b'_{10}}$ for some $b'_8, b'_9, b'_{10} \in \mathbb{F}_q$.

Now set $\lambda = \lambda^{0,0,0,0,0}_{b_8,b_9,b_{10}}$, and $\overline{H_3}$ is the normal closure of H_3 in HX_4S_1. To show that

$$\overline{H_3} \subset \ker(\lambda^{HX_4S_1}) \trianglelefteq HX_4S_1,$$

it suffices to show that $H_3 \subset \ker(\lambda^{HX_4S_1})$. By (a), $\mathrm{Stab}_{TX_4}(\lambda|_{H_4H_3}) = S_1X_4$ which is a transversal of H in HX_4S_1, the claim holds by the induction formula and $H_3 \subset \ker(\lambda)$.

By Lemma 1.5 above for $G = U$ with $N = M = HX_4S_1$, $X = X_1X_3X_5X_6$, $Y = \overline{H_3}$ and $Z = H_4$, the induction map from $\mathrm{Irr}(HX_4S_1/\overline{H_3}, \lambda)$ to $\mathrm{Irr}(U, \lambda)$ is bijective. Since $\overline{H_3} \subset \lambda^{HX_4S_1}$, $\mathrm{Irr}(HX_4S_1/\overline{H_3}, \lambda) = \mathrm{Irr}(HX_4S_1, \lambda)$. □

5.4.2 Proof of Lemma 3.3

Recall that $R_2 = \{r_2(t) := x_8(-t)x_9(t)x_{10}(t) : t \in \mathbb{F}_q\} \leq H_2 = X_8X_9X_{10}$ and

$$\lambda = \lambda^{0,0,0,0,0}_{b_8,b_9,b_{10}}.$$

By Lemma 3.2 (c), it suffices to work with the quotient group $HX_4S_1/\overline{H_3}$.

(a) The fact that $S_2 = \mathrm{Stab}_{S_1}(\lambda)$ is a direct consequence of Lemma 3.2 (b) with $B_3 = 0$. Recall that X_4S_1 is a transversal of H in HX_4S_1 and $[H_2, X_4] = \{1\}$. So to show that $R_2 = \{x \in H_2 : |\lambda^{HX_4S_1}(x)| = \lambda^{HX_4S_1}(1)\}$ we find all $x \in H_2$ such that $\lambda([x, y]) = 1$ for all $y \in S_1$. Since $H \trianglelefteq HX_4S_1$ is abelian, using the computation in Lemma 3.2 (b), for $s_1(t, r, s) \in S_1$ and $x_8(t_8)x_9(t_9)x_{10}(t_10) \in H_2$ we have

$$\begin{aligned}
&[x_8(t_8)x_9(t_9)x_{10}(t_{10}), s_1(t, r, s)]\\
&\quad = [x_8(t_8), s_1(t, r, s)][x_9(t_9), s_1(t, r, s)][x_{10}(t_10), s_1(t, r, s)]\\
&\quad = x_{20}(t_8s)x_{17}(-t_8r)x_{20}(-t_8t^2)x_{19}(t_8t^2)x_{21}(t_9s)x_{21}(-t_9t^2)\\
&\qquad\quad \times x_{18}(-t_9t^2)x_{19}(-t_9t^2)x_{17}(t_9t^2)x_{18}(-t_{10}r)x_{21}(-t_{10}t^2)\\
&\qquad\quad \times x_{20}(t_{10}t^2)x_{19}(-t_{10}t^2).
\end{aligned}$$

Therefore, with $\lambda|_{X_i} = \phi$ for all $17 \leq i \leq 21$, for all $t, r, s \in \mathbb{F}_q$ we need

$$(t_8 + t_9)s - (t_8 + t_{10})r + (t_9 - t_{10})t^2 = 0$$

So $t_9 = t_{10} = u$ and $t_8 = -u$ for all $u \in \mathbb{F}_q$, i.e. $x = r_2(u) \in R_2$.

To show that $\lambda^{HX_4S_1}|_{R_2} = \lambda^{HX_4S_1}(1)\phi_{B_2}$, it is enough to check that

$$\lambda(r_2(t)) = \phi_{B_2}(t).$$

For each $r_2(t) \in R_2$ we have

$$\lambda(r_2(t)) = \phi(t(-b_8 + b_9 + b_{10})) = \phi_{B_2}(t).$$

(b) Suppose that $B_2 \notin \{c^2 : c \in \mathbb{F}_q^{\times}\}$. Let η be an extension of λ to HX_4. By (a), we have $S_2 = \mathrm{Stab}_{S_1}(\lambda)$, hence $\mathrm{Stab}_{S_1}(\eta|_H) = S_2$. Since S_1 is a tranversal of HX_4 in HX_4S_1, to find $\mathrm{Stab}_{S_1}(\eta)$, it is enough to find all $s_2(t) \in S_2$ such that $\eta([x_4, s_2(t)]) = 1$ for all $x_4 \in X_4$. For each $s_2(t) \in S_2$, we have

$$\begin{aligned}
[x_4(t_4), s_2(t)] &= x_{10}(t_4t)x_9(t_4t)x_{21}(2t_4t^3)x_{21}(-t_4t^3)x_8(-t_4t)\\
&\quad \times x_{20}(-2t_4t^3)x_{17}(2t_4t^3)x_{20}(t_4t^3)x_{19}(-t_4t^3).
\end{aligned}$$

Since $\eta(x_i(t)) = \phi(b_it)$, $8 \leq i \leq 10$, and $\eta(x_i(t)) = \phi(t)$, $17 \leq i \leq 21$, for all $t_4 \in \mathbb{F}_q$, $\eta([x_4(t_4), s_3(t)]) = 1$ forces

$$t_4(t^3 - B_2t) = t_4(t^3 - B_2t) \in \ker(\phi).$$

Since $t_4(t^3-B_2t) \in \ker(\phi)$ for all $t_4 \in \mathbb{F}_q$, we have $0 = t^3-B_2t = t(t^2-B_2)$. Since $B_2 \notin \{c^2 : c \in \mathbb{F}_q^\times\}$, the equation $t(t^2-B_2) = 0$ has only the trivial solution $t = 0$ in $\mathbb{F}_q$. Therefore, $s_2(t) = 1$, i.e. $\mathrm{Stab}_{S_1}(\eta) = \{1\}$. Hence,

$$I_{HX_4S_1}(\eta) = HX_4.$$

(c) Suppose $B_2 = c^2 \in \mathbb{F}_q^\times$ and let η be an extension of λ to HX_4. Using the computation in (b), we continue with the analysis for the solutions of t to obtain $t_4t(t^2-B_2) \in \ker(\phi)$ for all $t_4 \in \mathbb{F}_q$. So it forces $t(t^2-B_2) = 0$. This equation has three solutions $\{0, \pm c\}$. Hence, $\mathrm{Stab}_{S_1}(\eta) = \{1, s_2(\pm c)\} = F_2$. So

$$I_{HX_4S_1}(\eta) = HX_4F_2.$$

By the above argument,

$$[HX_4F_2, HX_4F_2] \subset \ker(\eta),$$

hence η extends to $I_{HX_4S_1}(\eta)$.

To show that λ extends to HF_4S_2, we check $[HF_4S_2, HF_4S_2] \subset \ker(\lambda)$. Using the same argument, it is enough to check that $[s_2(t), x_4(t_4)] \in \ker(\lambda)$. By the computation in (b), we need $t_4(t^3-B_2t) = t_4(t^3-c^2t) \in \ker(\phi)$ for all $t \in \mathbb{F}_q$. By Proposition 1.3, since $t_4 \in \{0, \pm c_\phi\}$, the claim holds.

Let λ_1, λ_2 be two extensions of λ to HX_4F_2, and γ an extension of λ to HF_4S_2. Since the degree of all the irreducible constituents of $\lambda^{HX_4S_1}$ is $\frac{q^3}{3}$, we have ${\lambda_1}^{HX_4S_1}, {\lambda_2}^{HX_4S_1}, \gamma^{HX_4S_1} \in \mathrm{Irr}(HX_4S_1, \lambda)$.

Choose $1 \in S \subset S_1$ as a representative set of the double coset

$$HF_4S_2 \backslash HX_4S_1 / HX_4F_2.$$

As $HF_4S_2 \cap HX_4F_2 = HF_4F_2$ and $HX_4F_2 \trianglelefteq HX_4S_1$, by Mackey's formula,

$$\begin{aligned}({\lambda_1}^{HX_4S_1}, \gamma^{HX_4S_1}) &= \sum_{s \in S} ({}^s\lambda_1|_{{}^s(HX_4F_2) \cap HF_4S_2}, \gamma|_{{}^s(HX_4F_2) \cap HF_4S_2}) \\ &= \sum_{s \in S} ({}^s\lambda_1|_{HF_4F_2}, \gamma|_{HF_4F_2}).\end{aligned}$$

For each $s \in S$, if ${}^s\lambda_1|_{HF_4F_2} = \gamma|_{HF_4F_2}$, then ${}^s\lambda_1|_H = \gamma|_H$. Since both are extensions of the character λ, we have ${}^s\lambda = \lambda$, i.e. $s \in \mathrm{Stab}_{S_1}(\lambda) = S_2$. There is a unique $1 \in S \cap S_2$ since S is a representative set of $HF_4S_2 \backslash HX_4S_1 / HX_4F_2$. So

$$({\lambda_1}^{HX_4S_1}, \gamma^{HX_4S_1}) = (\lambda_1|_{HF_4F_2}, \gamma|_{HF_4F_2}) = 1 \text{ iff } \lambda_1|_{F_i} = \gamma|_{F_i},\ i \in \{2, 4\}.$$

Therefore, ${\lambda_1}^{HX_4S_1} = \gamma^{HX_4S_1} = {\lambda_2}^{HX_4S_1}$ iff $\lambda_1|_{F_i} = \lambda_2|_{F_i}, i \in \{2, 4\}$. □

5.5 Proofs of Section "Sylow 5-subgroups of $E_8(5^f)$"

5.5.1 Proof of Lemma 4.2

(a) First we find all $x \in H_5$ such that $|\lambda^U(x)| = \lambda^U(1)$. Since TX_5 is a transversal of H in U, we get $[H_5, X_5] = \{1\} = [H_5, T_k]$ for all $k \geq 2$, and ${}^y\lambda(x) = \lambda(x)$ iff $\lambda([x, y]) = 1$, it suffices to find all $x \in H_5$ such that $\lambda([y, x]) = 1$ where $y \in T_1$. For each $y = \prod_{i=1}^{8} x_i(u_i) \in T_1$ with $u_5 = 0$, and $x = \prod_{j=30}^{36} x_j(v_j) \in H_5$, to abbreviate our notation we write x_i for $x_i(-)$ and plug in the parameters in $(-)$ as need be. We have

$$\begin{aligned}
\left[\prod_{j=30}^{36} x_j(v_j), \prod_{i=1}^{8} x_i(u_i)\right] \\
&= [x_{30}, x_4][x_{30}, x_6][x_{31}, x_2][x_{31}, x_7][x_{32}, x_1][x_{32}, x_6][x_{33}, x_1] \\
&\qquad \times [x_{33}, x_4][x_{33}, x_7][x_{34}, x_3][x_{34}, x_8][x_{35}, x_2][x_{35}, x_1][x_{35}, x_8] \\
&\qquad \times [x_{36}, x_2][x_{36}, x_3] \\
&= x_{37}(-v_{30}u_4)x_{38}(v_{30}u_6)x_{38}(-v_{31}u_2)x_{39}(v_{31}u_7)x_{37}(-v_{32}u_1) \\
&\qquad \times x_{40}(v_{32}u_6)x_{38}(-v_{33}u_1)x_{40}(-v_{33}u_4)x_{41}(v_{33}u_7) \\
&\qquad \times x_{41}(-v_{34}u_3)x_{42}(v_{34}u_8)x_{41}(-v_{35}u_2)x_{39}(-v_{35}u_1) \\
&\qquad \times x_{43}(v_{35}u_8)x_{42}(-v_{36}u_2)x_{43}(-v_{36}u_3).
\end{aligned}$$

Since $\lambda|_{X_i} = \phi$ for all $i \in [37..43]$, for all s_j we need

$$\begin{aligned}
&(-v_{31} - v_{35} - v_{36})u_2 + (-v_{32} - v_{33} - v_{35})u_1 + (-v_{34} - v_{36})u_3 \\
&\quad + (-v_{30} - v_{33})u_4 + (v_{30} + v_{32})u_6 + (v_{31} + v_{33})u_7 \\
&\quad + (v_{34} + v_{35})u_8 = 0.
\end{aligned}$$

Therefore, we obtain a system with variables v_i as follows:

$$\begin{cases}
-v_{31} - v_{35} - v_{36} = 0, \\
-v_{32} - v_{33} - v_{35} = 0, \\
-v_{34} - v_{36} = 0, \\
-v_{30} - v_{33} = 0, \\
v_{30} + v_{32} = 0, \\
v_{31} + v_{33} = 0, \\
v_{34} + v_{35} = 0.
\end{cases}$$

Since $\gcd(q, 5) = 5$, we have

$$(v_{30}, v_{31}, v_{32}, v_{33}, v_{34}, v_{35}, v_{36}) = (v, v, -v, -v, -2v, 2v, 2v)$$

for all $v \in \mathbb{F}_q$. Hence, $x = r_5(v) \in R_5$, i.e. $R_5 = \{x \in H_5 : |\lambda^U(x)| = \lambda^U(1)\}$.

To show that $\lambda^U|_{R_5} = \lambda^U(1)\phi_{B_5}$, it suffices to check that $\lambda(r_5(v)) = \phi_{B_5}(v)$. For each $r_5(v) \in R_5$, we have

$$\lambda(r_5(v)) = \phi(v(b_{30} + b_{31} - b_{32} - b_{33} - 2b_{34} + 2b_{35} + 2b_{36})) = \phi_{B_5}(v).$$

To show that $S_1 = \mathrm{Stab}_T(\lambda|_{H_6H_5})$, we find all $y \in T$ such that $\lambda([x, y]) = 1$ for all $x \in H_6H_5$. Since $H_6 = Z(U)$ and $[H_5, T_k] = \{1\}$ for all $k \geq 2$, it is sufficient to find $y \in T_1$ such that $\lambda([x, y]) = 1$ for all $x \in H_5$. Using the above computation of $[\prod_{j=30}^{36} x_j(v_j), \prod_{i=1}^{8} x_i(u_i)]$, we find u_i such that for all v_j

$$\begin{aligned}(-u_4 + u_6)v_{30} + (-u_2 + u_7)v_{31} + (-u_1 + u_6)v_{32} + (-u_1 - u_4 + u_7)v_{33}\\ + (-u_3 + u_8)v_{34} + (-u_2 - u_1 + u_8)v_{35} + (-u_2 - u_3)v_{36} = 0.\end{aligned}$$

Therefore, we obtain a system with variables u_i as follows:

$$\begin{cases} -u_4 + u_6 = 0, \\ -u_2 + u_7 = 0, \\ -u_1 + u_6 = 0, \\ -u_1 - u_4 + u_7 = 0, \\ -u_3 + u_8 = 0, \\ -u_2 - u_1 + u_8 = 0, \\ -u_2 - u_3 = 0. \end{cases}$$

Since $\gcd(q, 5) = 5$, we have

$$(u_2, u_1, u_3, u_4, u_6, u_7, u_8) = (2u, u, -2u, u, u, 2u, -2u)$$

for all $u \in \mathbb{F}_q$. So $y = l_1(u) \in L_1$, i.e. $S_1 = \mathrm{Stab}_T(\lambda|_{H_6H_5})$.

(b) Suppose $B_5 \neq 0$. To show that $\mathrm{Stab}_T(\lambda) = \{1\}$, we are going to show that

$$\begin{aligned}\mathrm{Stab}_{S_1}(\lambda|_{H_6H_5H_4}) &= T_3T_4,\\ \mathrm{Stab}_{T_3T_4}(\lambda|_{H_6H_5H_4H_3}) &= T_4,\\ \mathrm{Stab}_{T_4}(\lambda) &= \{1\}.\end{aligned}$$

First, we show that $\mathrm{Stab}_{S_1}(\lambda|_{H_6H_5H_4}) = T_3T_4$. By considering root heights, it is clear that $[H_6H_5H_4, T_3T_4] = \{1\}$, hence, $T_3T_4 \subset \mathrm{Stab}_{S_1}(\lambda|_{H_6H_5H_4})$. It suffices to show that $\mathrm{Stab}_{L_1T_2}(\lambda|_{H_6H_5H_4}) = \{1\}$, i.e. there is no nontrivial $y \in L_1T_2$

such that $\lambda([h, y]) = 1$ for all $h \in H_5H_4$. For each

$$y = \prod_{i=1}^{15} x_i(u_i) \in L_1T_2$$

(with $u_5 = u_{12} = u_{13} = 0$ and $\prod_{i=1}^{8} x_i(u_i) = l_1(u)$), and

$$h = \prod_{j=24}^{36} x_j(v_j) \in H_5H_4,$$

we have

$$\begin{aligned}
&\left[\prod_{j=24}^{36} x_j(v_j), \prod_{i=1}^{15} x_i(u_i)\right] \\
&\quad= [x_{24}, x_{10}][x_{24}, x_{14}][x_{25}, x_{14}][x_{26}, x_{15}][x_{26}, x_9][x_{26}, x_{11}][x_{27}, x_{15}] \\
&\qquad\times [x_{27}, x_{10}][x_{29}, x_{11}][x_{24}, x_2][[x_{24}, x_2], x_6][[x_{24}, x_2], x_4] \\
&\qquad\times [x_{24}, x_6][[x_{24}, x_6], x_7][x_{25}, x_1][[x_{25}, x_1], x_4][[x_{25}, x_1], x_6] \\
&\qquad\times [x_{25}, x_4][[x_{25}, x_4], x_6][x_{25}, x_6][[x_{25}, x_6], x_7][x_{26}, x_3] \\
&\qquad\times [[x_{26}, x_3], x_4][[x_{26}, x_3], x_7][x_{26}, x_7][[x_{26}, x_7], x_8][x_{27}, x_2] \\
&\qquad\times [[x_{27}, x_2], x_1][[x_{27}, x_2], x_4][[x_{27}, x_2], x_7][x_{27}, x_1] \\
&\qquad\times [[x_{27}, x_1], x_7][x_{27}, x_7][[x_{27}, x_7], x_8][x_{28}, x_2][[x_{28}, x_2], x_3] \\
&\qquad\times [[x_{28}, x_2], x_8][x_{28}, x_3][[x_{28}, x_3], x_8][x_{28}, x_8][x_{29}, x_4] \\
&\quad= x_{37}(v_{24}u_{10})x_{39}(v_{24}u_{14})x_{41}(v_{25}u_{14})x_{42}(v_{26}u_{15})x_{38}(-v_{26}u_9) \\
&\qquad\times x_{40}(v_{26}u_{11})x_{43}(v_{27}u_{15})x_{40}(v_{27}u_{10})x_{39}(-v_{28}u_9) \\
&\qquad\times x_{42}(-v_{29}u_{10})x_{43}(-v_{29}u_{11})x_{30}(-2v_{24}u)x_{38}(-2v_{24}u^2) \\
&\qquad\times x_{37}(2v_{24}u^2)x_{31}(v_{24}u)x_{39}(v_{24}2u^2)x_{30}(-v_{25}u)x_{37}(v_{25}u^2) \\
&\qquad\times x_{38}(-v_{25}u^2)x_{32}(-v_{25}u)x_{40}(-v_{25}u^2)x_{33}(v_{25}u)x_{41}(2v_{25}u^2) \\
&\qquad\times x_{33}(2v_{26}u)x_{40}(-2v_{26}u^2)x_{41}(4v_{26}u^2)x_{34}(2v_{26}u) \\
&\qquad\times x_{42}(-4v_{26}u^2)x_{33}(-2v_{27}u)x_{38}(2v_{27}u^2)x_{40}(2v_{27}u^2) \\
&\qquad\times x_{41}(-4v_{27}u^2)x_{31}(-v_{27}u)x_{39}(-2v_{27}u^2)x_{35}(v_{27}2u) \\
&\qquad\times x_{43}(-4v_{27}u^2)x_{34}(-2v_{28}u)x_{41}(-4v_{28}u^2)x_{42}(4v_{28}u^2) \\
&\qquad\times x_{35}(2v_{28}u)x_{43}(-4v_{28}u^2)x_{36}(-v_{28}2u)x_{36}(-v_{29}u).
\end{aligned}$$

Since $\lambda|_{X_i} = \phi$ for all $i \in [37..43]$ and $\lambda|_{X_i} = \phi_{b_i}$ for the others, after evaluating the above at λ and setting this to 1, for all v_j, we need

$$\begin{aligned} &v_{24}(u_{10} + u_{14} - 2b_{30}u + b_{31}u + 2u^2) \\ &\qquad + v_{25}(u_{14} - b_{30}u - b_{32}u + b_{33}u + u^2) \\ &\qquad + v_{26}(u_{15} - u_9 + u_{11} + 2b_{33}u - 2u^2 + 2b_{34}u) \\ &\qquad + v_{27}(u_{15} + u_{10} - 2b_{33}u - b_{31}u + 2b_{35}u - u^2) \\ &\qquad + v_{28}(-u_9 - 2b_{34}u + 2b_{35}u + u^2 - 2b_{36}u) \\ &\qquad + v_{29}(-u_{10} - u_{11} - b_{36}u) = 0. \end{aligned}$$

Hence, we have a system with variables u_i and u :

$$\begin{aligned} u_{10} + u_{14} - 2b_{30}u + b_{31}u + 2u^2 &= 0, \\ u_{14} - b_{30}u - b_{32}u + b_{33}u + u^2 &= 0, \\ u_{15} - u_9 + u_{11} + 2b_{33}u - 2u^2 + 2b_{34}u &= 0, \\ u_{15} + u_{10} - 2b_{33}u - b_{31}u + 2b_{35}u - u^2 &= 0, \\ -u_9 - 2b_{34}u + 2b_{35}u + u^2 - 2b_{36}u &= 0, \\ -u_{10} - u_{11} - b_{36}u &= 0, \end{aligned}$$

which is equivalent to

$$\begin{cases} u_9 = u^2 + (3b_{34} + 2b_{35} + 3b_{36})u, \\ u_{10} = -u^2 + (b_{30} - b_{31} - b_{32} + b_{33})u, \\ u_{11} = u^2 + (-b_{30} + b_{31} + b_{32} - b_{33} - b_{36})u, \\ u_{14} = -u^2 + (b_{30} + b_{32} - b_{33})u, \\ u_{15} = 2u^2 + (-b_{30} + 2b_{31} + b_{32} + b_{33} + 3b_{35})u, \\ (b_{30} + b_{31} - b_{32} - b_{33} - 2b_{34} + 2b_{35} + 2b_{36})u = 0. \end{cases} \qquad (*)$$

The last equation is actually $B_5u = 0$. Since $B_5 \neq 0$, we have

$$u = 0 \quad \text{and} \quad u_9 = u_{10} = u_{11} = u_{14} = u_{15} = 0,$$

i.e. $\mathrm{Stab}_{L_1T_2}(\lambda|_{H_6H_5H_4}) = \mathrm{Stab}_{T_1T_2}(\lambda|_{H_6H_5H_4}) = \{1\}$.

Thus T_1T_2 acts faithfully on the set of all extensions of $\lambda|_{H_6}$ to $H_6H_5H_4$ with the same $B_5 \neq 0$, which is invariant under the action of T, i.e. $B_5(\lambda) = B_5({}^x\lambda)$ for all $x \in T$. Since $|H_5H_4/R_5| = q^{12} = |T_1T_2|$, this action is transitive. Therefore, we choose $\lambda|_{X_i} = \phi$ for all $i \in [37..43]$, $\lambda|_{X_{36}} = \phi_{B_5/2}$, and $\lambda|_{X_i} = 1_{X_i}$ for the others $X_i \subset H_5H_4$. By the root heights, we have

$$R_5 \prod_{i=37}^{43} X_i \subset Z(HX_5T_4T_3), \quad HX_5T_4T_3 \trianglelefteq U, \quad H_4 \prod_{i=30}^{35} X_i \trianglelefteq HX_5T_4T_3.$$

By Lemma 1.5 for $G = U$ with $N = M = HX_5T_3T_4$, $X = T_1T_2$, $Z = H_6R_5$ and $Y = H_4 \prod_{i=30}^{35} X_i$, the induction map from $\mathrm{Irr}(HX_5T_4T_3/Y, \lambda)$ to $\mathrm{Irr}(U, \lambda)$ is bijective. As $X_5T_4T_3 = \mathrm{Stab}_{X_5T}(\lambda|_{H_6H_5H_4})$ is a transversal of H in $HX_5T_4T_3$, we have $\lambda^{HX_5T_4T_3}|_Y = [HX_5T_4T_3 : H]\lambda|_Y = |X_5T_4T_3|1_Y$. Hence,

$$\mathrm{Irr}(HX_5T_4T_3/Y, \lambda) = \mathrm{Irr}(HX_5T_4T_3, \lambda).$$

Now we find $\mathrm{Stab}_{T_4T_3}(\lambda|_{H_6H_5H_4H_3})$. Since $[H_6H_5H_4H_3, T_3T_4] = [H_3, T_3]$, we find $y \in T_3$ such that $\lambda([x, y]) = 1$ for all $x \in H_3$. For each

$$y = \prod_{j=16,17,22} x_j(u_j) \in T_3 \quad \text{and} \quad x = \prod_{i=18}^{21} x_i(v_i) \in H_3$$

we have

$$\begin{aligned}[x, y] &= [x_{18}, x_{16}][x_{18}, x_{22}][x_{19}, x_{22}][x_{20}, x_{17}][x_{20}, x_{16}][x_{21}, x_{17}] \\ &= x_{37}(v_{18}u_{16})x_{42}(v_{18}u_{22})x_{43}(v_{19}u_{22})x_{40}(-v_{20}u_{17}) \\ &\qquad \times x_{39}(-v_{21}u_{16})x_{41}(-v_{21}u_{17}).\end{aligned}$$

Since $\lambda|_{X_i} = \phi$ for all $i \in [37..43]$, for all v_i we need

$$v_{18}(u_{16} + u_{22}) + v_{19}u_{22} - v_{20}u_{17} + v_{21}(-u_{17} - u_{16}) = 0.$$

The only solution is $(u_{16}, u_{17}, u_{22}) = (0, 0, 0)$, i.e. $\mathrm{Stab}_{T_4T_3}(\lambda|_{H_6H_5H_4H_3}) = T_4$.

Next, we find $\mathrm{Stab}_{T_4}(\lambda)$. Since $[H, T_4] = [H_2, T_4]$, we find $y \in T_4$ such that $\lambda([x, y]) = 1$ for all $x \in H_2$. For each

$$y = x_{23}(u_{23}) \in T_4 \quad \text{and} \quad x = x_{12}(v_{12})x_{13}(v_{13}) \in H_2$$

we have

$$\begin{aligned}[x_{12}(v_{12})x_{13}(v_{13}), x_{23}(u_{23})] &= [x_{12}, x_{23}][x_{13}, x_{23}] \\ &= x_{37}(-v_{12}u_{23})x_{38}(-v_{13}u_{23}).\end{aligned}$$

Evaluate with λ, for all v_i we need $(-v_{12}-v_{13})u_{23} = 0$. Therefore, the only solution is $u_{23} = 0$, i.e. $\mathrm{Stab}_{T_4}(\lambda) = \{1\}$. So we finish the proof of $\mathrm{Stab}_T(\lambda) = \{1\}$.

Let η, η' be two extensions of $\lambda|_{H_6 H_5 H_4}$ to HX_5. By the bijection of the induction map from $\mathrm{Irr}(HX_5T_4T_3, \lambda)$ to $\mathrm{Irr}(U, \lambda)$, it suffices to show that $\eta^{HX_5T_4T_3} = \eta'^{HX_5T_4T_3}$ iff $\eta|_{R_j} = \eta'|_{R_j}$ for $j = 2, 3$ and $\eta|_{X_5} = \eta'|_{X_5}$. By Mackey's formula and the fact that the double coset $HX_5 \backslash HX_5T_4T_3 / HX_5 = HX_5T_4T_3/HX_5$ is represented by T_4T_3 we have

$$(\eta^{HX_5T_4T_3}, \eta'^{HX_5T_4T_3}) = \sum_{y \in T_4T_3} ({}^y\eta, \eta').$$

Since $[X_5, T_3T_4] \subset H_4 \prod_{i=30}^{35} X_i \subset \ker(\lambda)$, we have ${}^y\eta|_{X_5} = \eta|_{X_5}$. Therefore, the restrictions to X_5 of both η, η' are clear for the proof. To show for the restrictions to R_k with $k = 2, 3$, we are going to prove that

$$R_2R_3 = \{x \in H_2H_3 : |\lambda^{HX_5T_4T_3}(x)| = \lambda^{HX_5T_4T_3}(1)\}$$

and

$$T_4T_3 = \mathrm{Stab}_{T_4T_3}(\lambda|_{R_2R_3}).$$

Then by $\mathrm{Stab}_{T_4T_3}(\lambda) = \{1\}$ and $|T_4T_3| = q^4 = |H_3H_2/R_2R_3|$, the claim holds.

By the above computations of $[H_3, T_3]$ and $[H_2, T_4]$ we find all $x \in H_2H_3$ such that $\lambda([x, y]) = 1$ for all $y \in T_4T_3$. For

$$y = x_{16}(u_{16})x_{17}(u_{17})x_{22}(u_{22})x_{23}(u_{23}) \in T_3T_4$$

and

$$x = x_{12}(v_{12})x_{13}(v_{13}) \prod_{i=18}^{21} x_i(v_i) \in H_2H_3,$$

we solve for v_i in the following:

$$u_{16}(v_{18} - v_{21}) + u_{17}(-v_{20} - v_{21}) + u_{22}(v_{18} + v_{19}) + u_{23}(-v_{12} - v_{13}) = 0.$$

We obtain a system with variables v_i:

$$\begin{cases} v_{18} - v_{21} = 0, \\ -v_{20} - v_{21} = 0, \\ v_{18} + v_{19} = 0, \\ -v_{12} - v_{13} = 0. \end{cases}$$

We obtain solutions

$$(v_{18}, v_{19}, v_{20}, v_{21}) = (v, -v, -v, v) \text{ and } (v_{12}, v_{13}) = (s, -s) \text{ for all } v, s \in \mathbb{F}_q.$$

Therefore, $x \in R_2R_3$. Hence, ${}^y\lambda|_{R_2R_3} = \lambda|_{R_2R_3}$ for all $y \in T_4T_3$.

(c) Suppose that $B_5 = 0$. By (a), T/S_1 acts faithfully on the set of all extensions of $\lambda|_{H_6}$ to H_6H_5 with the same B_5. Since $|H_5|/|R_5| = q^6 = |T/S_1|$, this action is transitive. Hence, there exists $x \in T$ such that ${}^x\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5$. Let λ be this linear. So $S_1X_5 = \mathrm{Stab}_{TX_5}(\lambda|_{H_6H_5})$, a transversal of H in HX_5S_1, and $\lambda^{HX_5S_1}|_{H_5} = \lambda^{HX_5S_1}(1)\lambda|_{H_5}$, i.e. $H_5 \subset \ker(\lambda^{HX_5S_1})$ and so is its normal closure $\overline{H_5}$ in HX_5S_1.

By Lemma 1.5 with $G = U$, $N = M = HX_5S_1$, $X = \prod_{i=1}^{4} X_iX_6X_7$, $Z = H_6$ and $Y = \overline{H_5}$, the induction map from $\mathrm{Irr}(HX_5S_1/\overline{H_5}, \lambda)$ to $\mathrm{Irr}(U, \lambda)$ is bijective. Since $\overline{H_5} \subset \ker(\lambda^{HX_5S_1})$, we have $\mathrm{Irr}(HX_5S_1/\overline{H_5}, \lambda) = \mathrm{Irr}(HX_5S_1, \lambda)$. □

5.5.2 Proof of Lemma 4.3

Recall that λ is a linear character of H such that $\lambda|_{X_i} = \phi$ for all $X_i \subset H_6$, $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5$, and $\lambda|_{X_i} = \phi_{b_i}$ for the others $X_i \subset H_4H_3H_2$ where $b_i \in \mathbb{F}_q$. By Lemma 4.2 (c), we work with the quotient group $HX_5S_1/\overline{H_5}$. Abusing the notation of root groups, we call them root groups in the quotient group.

(a) By computation $(*)$ in Lemma 4.2 (b) with $B_5 = 0$,

$$S_2 = \mathrm{Stab}_{S_1}(\lambda|_{H_6H_5H_4}).$$

Now we show the identity $R_4 = \{x \in H_4 : |\lambda^{HX_5S_1}(x)| = \lambda^{HX_5S_1}(1)\}$. For each $l_1y_2y_3y_4 \in L_1T_2T_3T_4 = S_1$ and $h_4 \in H_4$, we have $[h_4, l_1y_2y_3y_4] = [h_4, l_1y_2]$. Hence, we are going to find all elements $h_4 \in H_4$ such that $\lambda([h_4, l_1y_2]) = 1$ for all $l_1y_2 \in L_1T_2$. Using the computation of $[\prod_{j=24}^{36} x_j(v_j), \prod_{i=1}^{15} x_i(u_i)]$ in Lemma 4.2 (b) with $b_j = 0$ for $j \in [30..36]$, we solve for v_i in the following equation:

$$u_9(-v_{26} - v_{28}) + u_{10}(v_{24} + v_{27} - v_{29}) + u_{11}(v_{26} - v_{29}) + u_{14}(v_{24} + v_{25}) \\ + u_{15}(v_{26} + v_{27}) + u^2(2v_{24} + v_{25} - 2v_{26} - v_{27} + v_{28}) = 0.$$

So we obtain a system with variables v_i:

$$\begin{cases} -v_{26} - v_{28} = 0, \\ v_{24} + v_{27} - v_{29} = 0, \\ v_{26} - v_{29} = 0, \\ v_{24} + v_{25} = 0, \\ v_{26} + v_{27} = 0, \\ 2v_{24} + v_{25} - 2v_{26} - v_{27} + v_{28} = 0. \end{cases}$$

Thus, $(v_{24}, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}) = (2v, -2v, v, -v, -v, v)$ is a solution for all $v \in \mathbb{F}_q$, i.e. $\lambda([h_4, l_1y_2]) = 1$ for all $l_1y_2 \in L_1T_2$ iff $h_4 = r_4(v) \in R_4$.

It is clear that $\lambda^{HX_5S_1}(r_4(v)) = \lambda^{HX_5S_1}(1)\phi_{B_4}(v)$ for all $r_4(v) \in R_4$ by checking directly that $\lambda(r_4(v)) = \phi_{B_4}(v)$.

(b) Suppose that $B_4 \neq 0$. Since $\mathrm{Stab}_T(\lambda|_{H_6H_5H_4}) = S_2 = L_2T_3T_4$, we are going to show that $\mathrm{Stab}_{S_2}(\lambda|_{H_6H_5H_4H_3}) = T_4$, and then $\mathrm{Stab}_{T_4}(\lambda) = 1$ is done by using the same argument in Lemma 4.2 (b). This means that we find all $y \in S_2$ such that $\lambda([x, y]) = 1$ for all $x \in H_3$ since $\lambda([H_6H_5H_4, S_2]) = \{1\}$.

It is clear that $T_4 \subset \mathrm{Stab}_{S_2}(\lambda|_{H_6H_5H_4H_3})$. So by (a) and $|H_3| = q^4 = |L_2T_3|$, it suffices to show that L_2T_3 acts faithfully on the set of all extensions of $\lambda|_{H_6H_5H_4}$ to $H_6H_5H_4H_3$, i.e. $\mathrm{Stab}_{L_2T_3}(\lambda|_{H_6H_5H_4H_3}) = \{1\}$. By the root heights and the fact that H is abelian, $[H_3, L_2T_3] = [H_3, T_3][H_3, L_2]$, where $[H_3, T_3]$ is computed in Lemma 4.2 (b). Since we work with $HX_5S_1/\overline{H_5}$, for each

$$x = \prod_{i=18}^{21} x_i(v_i) \in H_3 \quad \text{and} \quad y = \prod_{j=1}^{11} x_j(u_j) \prod_{j=14}^{17} x_j(u_j) x_{22}(u_{22}) \in L_2T_3,$$

we have

$$\begin{aligned}
[x, y] &= [x, x_{16}x_{17}x_{22}][x_{18}, x_3][[[x_{18}, x_3], x_4], x_6][[[x_{18}, x_3], x_6], x_7] \\
&\quad \times [[x_{18}, x_3], x_{14}][x_{18}, x_6][[[x_{18}, x_6], x_7], x_8][[x_{18}, x_6], x_{15}] \\
&\quad \times [[x_{18}, x_6], x_{11}][[x_{18}, x_6], x_9][x_{19}, x_2][[[x_{19}, x_2], x_1], x_4] \\
&\quad \times [[[x_{19}, x_2], x_1], x_6][[[x_{19}, x_2], x_4], x_6][[[x_{19}, x_2], x_6], x_7] \\
&\quad \times [[x_{19}, x_2], x_{14}][x_{19}, x_1][[[x_{19}, x_1], x_6], x_7][[x_{19}, x_1], x_{10}] \\
&\quad \times [[x_{19}, x_1], x_{14}][x_{19}, x_6][[[x_{19}, x_6], x_7], x_8][[x_{19}, x_6], x_{10}] \\
&\quad \times [[x_{19}, x_6], x_{15}][x_{20}, x_2][[[x_{20}, x_2], x_3], x_4][[[x_{20}, x_2], x_3], x_7] \\
&\quad \times [[[x_{20}, x_2], x_7], x_8][[x_{20}, x_2], x_{15}][[x_{20}, x_2], x_{11}][[x_{20}, x_2], x_9] \\
&\quad \times [x_{20}, x_3][[[x_{20}, x_3], x_7], x_8][[x_{20}, x_3], x_{10}][[x_{20}, x_3], x_{15}] \\
&\quad \times [x_{20}, x_7][[x_{20}, x_7], x_9][x_{21}, x_4][[x_{21}, x_4], x_9][x_{21}, x_8] \\
&\quad \times [[x_{21}, x_8], x_{10}][[x_{21}, x_8], x_{11}] \\
&= x_{37}(v_{18}u_{16})x_{42}(v_{18}u_{22})x_{43}(v_{19}u_{22})x_{40}(-v_{20}u_{17})x_{39}(-v_{21}u_{16}) \\
&\quad \times x_{41}(-v_{21}u_{17})x_{25}(2v_{18}u)x_{40}(-2v_{18}u^3)x_{41}(4v_{18}u^3) \\
&\quad \times x_{41}(-2v_{18}u^3)x_{26}(v_{18}u)x_{42}(-4v_{18}u^3)x_{42}(2v_{18}u^3)x_{40}(v_{18}u^3) \\
&\quad \times x_{38}(-v_{18}u^3)x_{25}(-2v_{19}u)x_{37}(-2v_{19}u^3)x_{38}(2v_{19}u^3) \\
&\quad \times x_{40}(2v_{19}u^3)x_{41}(-4v_{19}u^3)x_{41}(2v_{19}u^3)x_{24}(-v_{19}u) \\
&\quad \times x_{39}(-2v_{19}u^3)x_{37}(v_{19}u^3)x_{39}(v_{19}u^3)x_{27}(v_{19}u)x_{43}(-4v_{19}u^3)
\end{aligned}$$

$$\times x_{40}(-v_{19}u^3)x_{43}(2v_{19}u^3)x_{26}(-2v_{20}u)x_{40}(4v_{20}u^3)$$
$$\times x_{41}(-8v_{20}u^3)x_{42}(8v_{20}u^3)x_{42}(-4v_{20}u^3)x_{40}(-2v_{20}u^3)$$
$$\times x_{38}(2v_{20}u^3)x_{27}(2v_{20}u)x_{43}(-8v_{20}u^3)x_{40}(-2v_{20}u^3)$$
$$\times x_{43}(4v_{20}u^3)x_{28}(2v_{20}u)x_{39}(-2v_{20}u^3)x_{28}(-v_{21}u)x_{39}(v_{21}u^3)$$
$$\times x_{29}(-2v_{21}u)x_{42}(-2v_{21}u^3)x_{43}(2v_{21}u^3).$$

Evaluating at λ and setting the result equal to 1, we see that the following equation is true for all v_i:

$$\begin{aligned} v_{18}(u_{16} + u_{22} + 2b_{25}u + b_{26}u - 2u^3) \\ + v_{19}(u_{22} - b_{24}u - 2b_{25}u + b_{27}u - 3u^3) \\ + v_{20}(-u_{17} - 2b_{26}u + 2b_{27}u + 2b_{28}u - 3u^3) \\ + v_{21}(-u_{16} - u_{17} - b_{28}u - 2b_{29}u + u^3) = 0. \end{aligned}$$

So we obtain a system with variables u_j and u:

$$\begin{cases} u_{16} + u_{22} + 2b_{25}u + b_{26}u - 2u^3 = 0, \\ u_{22} - b_{24}u - 2b_{25}u + b_{27}u - 3u^3 = 0, \\ -u_{17} - 2b_{26}u + 2b_{27}u + 2b_{28}u - 3u^3 = 0, \\ -u_{16} - u_{17} - b_{28}u - 2b_{29}u + u^3 = 0, \end{cases}$$

which is equivalent to:

$$\begin{cases} u_{22} = 3u^3 + (b_{24} + 2b_{25} - b_{27})u, \\ u_{17} = 2u^3 + (3b_{26} + 2b_{27} + 2b_{28})u, \\ u_{16} = 4u^3 + (2b_{26} - 2b_{27} - 3b_{28})u, \\ (2b_{24} - 2b_{25} + b_{26} - b_{27} - b_{28} + b_{29})u = 0. \end{cases}$$

The last equation in the system is actually $B_4u = 0$. Since $B_4 \neq 0$, the only solution of this system is $(u_{16}, u_{17}, u_{22}) = (0, 0, 0)$, i.e.

$$\mathrm{Stab}_{L_2T_3}(\lambda|_{H_6H_5H_4H_3}) = \{1\}.$$

Hence, $\mathrm{Stab}_{S_2}(\lambda|_{H_6H_5H_4H_3}) = T_4$ and $\mathrm{Stab}_{S_1}(\lambda) = \{1\}$.

The above argument also proves that $L_1T_2T_3$ acts transitively on the set of all extensions of $\lambda|_{H_6H_5H_4}$ to $H_6H_5H_4H_3$ with the same $B_4 \neq 0$. The number of these extensions is $|H_4H_3|/|R_4|$. Therefore, there exists an element $x \in L_1T_2T_3$ such that ${}^x\lambda|_{X_i} = \phi$ for all $X_i \subset H_6$, ${}^x\lambda|_{X_{29}} = \phi_{B_4}$, ${}^x\lambda|_{X_i} = 1_{X_i}$ for the others $X_i \subset H_5H_4H_3$. Let λ be this linear character. By Lemma 1.5 with $G = HX_5S_1$, $N = M = HX_5T_4$, $X = L_1T_2T_3$, $Z = H_6R_4$, $Y = H_3\prod_{i=24}^{28} X_i$, the induction map from $\mathrm{Irr}(HX_5T_4/Y, \lambda)$ to $\mathrm{Irr}(HX_5S_1, \lambda)$ is bijective. Let η, η' be two extensions of $\lambda|_{H_6H_5H_4H_3}$ to HX_5. We have $\eta^{HX_5S_2}, \eta^{HX_5S_2} \in \mathrm{Irr}(HX_5S_2/Y, \lambda)$. Using the same argument in Lemma 4.2 (b), we obtain $(\eta^{HX_5S_2}, \eta'^{HX_5S_2}) = 1$ iff $\eta|_{R_2} = \eta'|_{R_2}$ and $\eta|_{X_5} = \eta'|_{X_5}$.

(c) Suppose that $B_4 = 0$. By (a), S_1/S_2 acts faithfully on the set of all extensions of $\lambda|_{H_6H_5}$ to $H_6H_5H_4$ with the same B_4. Since $|S_1/S_2| = q^5 = |H_4/R_4|$, it follows that this action is transitive. Hence, with $B_4 = 0$, there exists an element $x \in S_1$ such that ${}^x\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4$. Let λ be this linear character. Since $S_2X_5 = \mathrm{Stab}_{S_1X_5}(\lambda|_{H_6H_5H_4})$ is a transversal of H in HX_5S_2, we have

$$\lambda^{HX_5S_2}|_{H_4} = [HX_5S_2 : H]\lambda|_{H_4} = |X_5S_2|1_{H_4}.$$

So $H_4 \subset \ker(\lambda^{HX_5S_2})$. By Lemma 1.5 for $G = HX_5S_1$ with $N = M = HX_5S_2$, $X = T_2$, $Y = H_4$ and $Z = H_6$, the induction map from $\mathrm{Irr}(HX_5S_2/\overline{H_5H_4}, \lambda)$ to $\mathrm{Irr}(HX_5S_1, \lambda)$ is bijective where $\overline{H_5H_4}$ is the normal closure of H_5H_4 in HX_5S_2. Since $H_5H_4 \subset \ker(\lambda^{HX_5S_2})$, we have

$$\mathrm{Irr}(HX_5S_2/\overline{H_5H_4}, \lambda) = \mathrm{Irr}(HX_5S_2, \lambda). \qquad \square$$

5.5.3 Proof of Lemma 4.4

Recall that λ is a linear character of H such that $\lambda|_{X_i} = \phi$ for all $X_i \subset H_6$, $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4$, and $\lambda|_{X_i} = \phi_{b_i}$ for the others $X_i \subset H_3H_2$ where $b_i \in \mathbb{F}_q$. By Lemma 4.3 (c), we work with the quotient group $HX_5S_2/\overline{H_5H_4}$. Abusing language slightly, we call the images of root groups in a quotient group root groups also.

(a) By the computation in Lemma 4.3 (b) with $B_4 = 0$, it is clear that

$$S_3 = \mathrm{Stab}_{S_2}(\lambda|_{H_6H_5H_4H_3}).$$

Now we show that $R_3 = \{x \in H_3 : |\lambda^{HX_5S_2}(x)| = \lambda^{HX_5S_2}(1)\}$. Since X_5S_2 is a transversal of H in HX_5S_2, we are going to find $x \in H_3$ such that $\lambda([x, y]) = 1$ for all $y \in S_2$. Since $[H_3, X_5] = \{1\} = [H_3, T_4]$, it is enough to work with $x \in H_3$ and $y \in S_2T_3$. For each

$$x = \prod_{i=18}^{21} x_i(v_i) \in H_3 \quad \text{and} \quad y = \prod_{j=1}^{11} x_j(u_j) \prod_{j=14}^{17} x_j(u_j)x_{22}(u_{22}) \in S_2T_3,$$

by the computation in Lemma 4.3, we find $(v_i)_{i\in[18..21]}$ satisfying for all u_j and u in the following equation:

$$u_{16}(v_{18}-v_{21})+u_{17}(-v_{20}-v_{21})+u_{22}(v_{18}+v_{19})$$
$$+u^3(-2v_{18}-3v_{19}-3v_{20}+v_{21})=0.$$

We have a system with variables v_i:

$$\begin{cases} v_{18}-v_{21}=0, \\ -v_{20}-v_{21}=0, \\ v_{18}+v_{19}=0, \\ -2v_{18}-3v_{19}-3v_{20}+v_{21}=0. \end{cases}$$

Its solutions are $(v_{18},v_{19},v_{20},v_{21})=(u,-u,-u,u)$ for all $u\in\mathbb{F}_q$, i.e. we have $x=r_3(u)\in R_3$. Now to show that $\lambda^{HX_5S_2}|_{R_3}=[HX_5S_2:H]\phi_{B_3}$, it is enough to check $\lambda(r_3(t))=\phi_{B_3}(t)$ which is clear.

(b) Suppose that $B_3\neq 0$. By (a) we have $\mathrm{Stab}_{S_2}(\lambda|_{H_6H_5H_4H_3})=S_3=L_3T_4$. To show that $\mathrm{Stab}_{S_2}(\lambda)=\{1\}$, since $|L_3T_4|=q^2=|H_2|$, we show that L_3T_4 acts faithfully on the set of all extensions of $\lambda|_{H_6H_5H_4H_3}$ to H, i.e. proving that there is no nontrivial $y\in L_3T_4$ such that $\lambda([x,y])=1$ for all $x\in H_2$.

By the root heights, $[H_2,L_3T_4]=[H_2,T_4][H_2,L_3]$, where $[H_2,T_4]$ is computed in Lemma 4.2 (b). For

$$x=x_{12}(v_{12})x_{13}(v_{13})\in H_2 \quad\text{and}\quad y=l_3(u)x_{23}(u_{23})\in L_3T_4,$$

we have

$$\begin{aligned}
[x,y]&=[x,x_{23}][x,l_3]\\
&=[x_{12},x_{23}][x_{12},x_2][[[[x_{12},x_2],x_3],x_4],x_6][[[[x_{12},x_2],x_3],x_6],x_7]\\
&\quad\times[[[[x_{12},x_2],x_6],x_7],x_8][x_{12},x_3][[[[x_{12},x_3],x_6],x_7],x_8]\\
&\quad\times[x_{12},x_6][[[x_{12},x_2],x_3],x_{14}][[[x_{12},x_2],x_6],x_9]\\
&\quad\times[[[x_{12},x_2],x_6],x_{11}][[[x_{12},x_2],x_6],x_{15}][[[x_{12},x_3],x_6],x_{10}]\\
&\quad\times[[[x_{12},x_3],x_6],x_{15}][[[x_{12},x_6],x_7],x_9][[x_{12},x_2],x_{22}]\\
&\quad\times[[x_{12},x_2],x_{16}][[x_{12},x_3],x_{22}][[x_{12},x_6],x_{17}]\\
&\quad\times[[x_{12},x_9],x_{10}][[x_{12},x_9],x_{14}][x_{13},x_{23}][x_{13},x_4][x_{13},x_7]\\
&\quad\times[[[x_{13},x_4],x_7],x_9][[[x_{12},x_7],x_8],x_{10}][[[x_{13},x_7],x_8],x_{11}]\\
&\quad\times[[x_{13},x_4],x_{17}][[x_{13},x_7],x_{16}][[x_{13},x_7],x_{17}][[x_{13},x_{10}],x_{15}]\\
&\quad\times[[x_{13},x_{10}],x_{11}][[x_{13},x_{11}],x_{15}]
\end{aligned}$$

$$\begin{aligned} &= x_{37}(-v_{12}u_{23})x_{18}(-2v_{12}u)x_{40}(4v_{12}u^4)x_{41}(-8v_{12}u^4)x_{42}(8v_{12}u^4) \\ &\quad \times x_{19}(2v_{12}u)x_{43}(-8v_{12}u^4)x_{20}(v_{12}u)x_{41}(4v_{12}u^4)x_{38}(2v_{12}u^4) \\ &\quad \times x_{40}(-2v_{12}u^4)x_{42}(-4v_{12}u^4)x_{40}(-2v_{12}u^4)x_{43}(4v_{12}u^4) \\ &\quad \times x_{39}(-2v_{12}u^4)x_{42}(-6v_{12}u^4)x_{37}(-8v_{12}u^4)x_{43}(6v_{12}u^4) \\ &\quad \times x_{40}(-2v_{12}u^4)x_{37}(v_{12}u^4)x_{39}(v_{12}u^4)x_{38}(-v_{13}u_{23}) \\ &\quad \times x_{20}(-v_{13}u)x_{21}(2v_{13}u)x_{39}(2v_{13}u^4)x_{42}(-4v_{13}u^4) \\ &\quad \times x_{43}(4v_{13}u^4)x_{40}(2v_{13}u^4)x_{39}(-8v_{13}u^4)x_{41}(-4v_{13}u^4) \\ &\quad \times x_{42}(2v_{13}u^4)x_{40}(v_{13}u^4)x_{43}(-2v_{13}u^4). \end{aligned}$$

Evaluating λ and setting the result equal to 1, we obtain in the following equation:

$$\begin{aligned} &v_{12}(-s_{23} - 2b_{18}u + b_{20}u + 2b_{19}u - 2u^4) \\ &\qquad + v_{13}(-u_{23} - b_{20}u + 2b_{21}u - 2u^4) = 0. \end{aligned}$$

We have a system with variables u_j and u :

$$\begin{cases} -u_{23} - 2b_{18}u + b_{20}u + 2b_{19}u - 2u^4 = 0, \\ \qquad\qquad -u_{23} - b_{20}u + 2b_{21}u - 2u^4 = 0. \end{cases}$$

It is equivalent to

$$\begin{cases} u_{23} = 3u^4 + (-2b_{18} + b_{20} + 2b_{19})u, \\ (b_{18} - b_{20} - b_{19} + b_{21})u = 0. \end{cases}$$

The last equation is actually $B_3u = 0$. Since $B_3 \neq 0$, it follows that the only solution is $(u_{23}, u) = (0, 0)$, i.e. $\mathrm{Stab}_{L_3T_4}(\lambda) = \{1\}$ or L_3T_4 acts faithfully on the set of all extensions of $\lambda|_{H_6H_5H_4H_3}$ to H. Hence, we also get $\mathrm{Stab}_{S_2}(\lambda) = \{1\}$.

Therefore, there is an element $x \in L_3T_4$ such that ${}^x\lambda|_{X_i} = \phi$ for all $X_i \subset H_6$, ${}^x\lambda|_{X_{21}} = \phi_{B_3}$, ${}^x\lambda|_{X_i} = 1_{X_i}$ for the others $X_i \subset H_5H_4H_3$. Let λ be this linear. By Lemma 1.5 with $G = HX_5S_2$, $N = M = HX_5$, $X = S_2$, $Y = \prod_{i=18}^{20} X_i$ and $Z = H_6X_{21}$, the induction map from $\mathrm{Irr}(HX_5/Y, \lambda)$ to $\mathrm{Irr}(HX_5S_2, \lambda)$ is bijective. Using the same method as in Lemma 4.2 (c), we see that the rest of the statement holds.

(c) Suppose $B_3 = 0$. By (a), S_2/S_3 acts faithfully on the set of all extensions of $\lambda|_{H_6H_5H_4}$ to $H_6H_5H_4H_3$ with the same B_3. Since $|S_2/S_3| = q^3 = |H_3/R_3|$, this action is transitive. Hence, there exists $x \in S_2$ such that ${}^x\lambda|_{X_i} = 1_{X_i}$ for all

$X_i \subset H_5H_4H_3$. Let λ be this linear. Since X_5S_3 is a transversal of H in HX_5S_3 and $S_3 = \mathrm{Stab}_{S_2}(\lambda|_{H_6H_5H_4H_3})$, we have

$$\lambda^{HX_5S_3}|_{H_5H_4H_3} = \lambda^{HX_5S_3}(1)\lambda|_{H_5H_4H_3}.$$

Therefore,

$$H_5H_4H_3 \subset \ker(\lambda^{HX_5S_3}),$$

so is its normal closure $\overline{H_5H_4H_3}$ in HX_5S_3.

By Lemma 1.5 with $G = HX_5S_2$, $N = M = HX_5S_3$, $X = T_3$, $Y = H_3$ and $Z = H_6$, the induction map from $\mathrm{Irr}(HX_5S_3/Y, \lambda)$ to $\mathrm{Irr}(HX_5S_2, \lambda)$. Since we have $Y \subset \ker(\lambda^{HX_5S_3})$, it follows that $\mathrm{Irr}(HX_5S_3/Y, \lambda) = \mathrm{Irr}(HX_5S_3, \lambda)$. □

5.5.4 Proof of Lemma 4.5

Recall that λ is a linear character of the group H such that $\lambda|_{X_i} = \phi$ for all $X_i \subset H_6 = Z(U)$, $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4H_3$, and $\lambda|_{X_i} = \phi_{b_i}$ for the others $X_i \subset H_2$ where $b_i \in \mathbb{F}_q$. By Lemma 4.3 (c), we work with the quotient group $HX_5S_3/\overline{H_5H_4H_3}$. Abusing language slightly, we call the images of root groups in a quotient group root groups also.

(a) By the computation in Lemma 4.4 (b) with $B_3 = 0$, $S_4 = \mathrm{Stab}_{S_3}(\lambda)$. Now we show that $R_2 = \{x \in H_2 : |\lambda^{HX_5S_3}(x)| = \lambda^{HX_5S_3}(1)\}$. Since X_5S_3 is a transversal of H in HX_5S_3, we are going to find $x \in H_2$ such that $\lambda([x, y]) = 1$ for all $y \in S_3$. As $[H_2, X_5] = \{1\}$, it is enough to work with $x \in H_2$ and $y \in S_4$. For each $x = \prod_{i=12}^{13} x_i(v_i) \in H_2$ and $y = l_3(u)x_{23}(u_{23}) \in S_3T_4$, by the computation in Lemma 4.4 (b), we find (v_{12}, v_{13}) satisfying for all u_{23} and u in the following equation:

$$u_{23}(-v_{12} - v_{13}) + 2u^4(-v_{12} - v_{13}) = 0.$$

So $(v_{12}, v_{13}) = (v, -v)$ for all $v \in \mathbb{F}_q$, i.e. $x = r_2(v)$. Since $\lambda(r_2(v)) = \phi_{B_2}(v)$ for all $r_2(v) \in R_2$, we have $\lambda^{HX_5S_3}|_{H_2} = [HX_5S_3 : H]\phi_{B_2}$.

(b) Suppose that $B_2 \in \mathbb{F}_q - \{c^4 : c \in \mathbb{F}_q^\times\}$. Let η be an extension of λ to HX_5. Since $S_4 = \mathrm{Stab}_{S_3}(\lambda)$, to get $I_{HX_5S_3}(\eta) = HX_5$, we show that S_4 acts transitively on the set of all extensions of λ to HX_5. Hence, we find all $l_4 \in S_4$ such that $\lambda([hx_5, l_4]) = 1$ for all $h \in H$ and $x_5 \in X_5$. Since $S_4 = \mathrm{Stab}_{S_3}(\lambda)$, we have $\lambda([h, l_4]) = 1$ for all $h \in H$, $l_4 \in S_4$. Thus we compute $[x_5, l_4]$. Since we work with $HX_5S_3/\overline{H_5H_4H_3}$, for each $x_5(v_5) \in X_5$ and $l_4(u) \in S_4$, we have

$$\begin{aligned}[x_5(v_5), l_4(u)] = {} & [x_5, x_4][[[x_5, x_4], x_9], x_{10}][[[x_5, x_4], x_9], x_{14}] \\ & \times [[[[x_5, x_4], x_6], x_7], x_9][[[x_5, x_4], x_6], x_{17}][[x_5, x_4], x_{23}] \\ & \times [x_5, x_6][[[x_5, x_6], x_{10}], x_{11}][[[x_5, x_6], x_{10}], x_{15}]\end{aligned}$$

$$
\begin{aligned}
&\times [[[x_5,x_6],x_{11}],x_{15}][[[[x_5,x_6],x_7],x_8],x_{10}]\\
&\times [[[[x_5,x_6],x_7],x_8],x_{11}][[[x_5,x_6],x_7],x_{16}]\\
&\times [[[x_5,x_6],x_7],x_{17}][[x_5,x_6],x_{23}][[x_5,x_{14}],x_{16}]\\
&\times [[x_5,x_{14}],x_{17}][[x_5,x_{11}],x_{22}][[x_5,x_{10}],x_{16}]\\
&\times [[x_5,x_{10}],x_{22}]\\
= {}& x_{12}(-v_5u)x_{37}(-v_5u^5)x_{39}(-v_5u^5)x_{39}(2v_5u^5)x_{40}(2v_5u^5)\\
&\times x_{37}(3v_5u^5)x_{13}(v_5u)x_{40}(v_5u^5)x_{42}(2v_5u^5)x_{43}(-2v_5u^5)\\
&\times x_{42}(-4v_5u^5)x_{43}(4v_5u^5)x_{39}(-8v_5u^5)x_{41}(-4v_5u^5)\\
&\times x_{38}(-3v_5u^5)x_{39}(4v_5u^5)x_{41}(2v_5u^5)x_{43}(-3v_5u^5)\\
&\times x_{42}(3v_5u^5)x_{37}(4v_5u^5).
\end{aligned}
$$

Evaluating with λ to get 1, for all v_5 we need

$$v_5(-(b_{12}-b_{13})u+u^5) \in \ker(\phi),$$

which is $v_5(u^5-B_2u) \in \ker(\phi)$ for all v_5. Hence, we solve for u: $u(u^4-B_2)=0$. Since $B_2 \in \mathbb{F}_q - \{c^4 : c \in \mathbb{F}_q^\times\}$, this equation only has one trivial solution $u=0$, i.e. $\mathrm{Stab}_{S_4}(\eta)=\{1\}$, or $I_{HX_5S_3}(\eta)=HX_5$.

(c) Suppose $B_2=c^4 \in \mathbb{F}_q^\times$. Let η be an extension of the character λ to HX_5. Continue the computation in part (b), the equation $u(u^4-B_2)=0$ has five solutions $u \in \{ac : a \in \mathbb{F}_5\}$, i.e. $l_4(u) \in F_4$. Hence, we have $I_{HX_5S_3}(\eta)=HX_5F_4$. Since $[HX_5,F_4] \subset \ker(\eta)$, η extends to HX_5F_4, i.e. λ extends to HX_5F_4.

Since $S_4=\mathrm{Stab}_{S_3}(\lambda) \cong \mathbb{F}_q$, we have

$$[H,S_4] \subset \ker(\lambda).$$

So λ extends to $HS_4 \trianglelefteq HX_5S_3$. Let λ' be an extension of λ to HS_4. We find $I_{HX_5S_3}(\lambda')$. Since $\mathrm{Stab}_{X_5S_3}(\lambda') \subset \mathrm{Stab}_{X_5S_3}(\lambda'|_H)=X_5S_4$, it is enough to find all $x_5 \in X_5$ such that $\lambda'([x_5,hl_4])=1$ for all $hl_4 \in HS_4$. Since HX_5 is abelian, we have

$$[x_5,hl_4]=[x_5,l_4].$$

For each $x_5(v_5) \in X_5$ and $l_4(u) \in S_4$, by the computation in (b), we need

$$v_5(u^5-B_2u) \in \ker(\phi) \quad \text{for all } u \in \mathbb{F}_q.$$

By Proposition 1.3, there are five solutions $v_5 \in \{ac_\phi : a \in \mathbb{F}_5\}$, i.e. $x_5(v_5) \in F_5$. Hence, $I_{HX_5S_3}(\lambda')=HF_5S_4$. Since $[F_5,S_4] \subset \ker(\lambda')$, λ' extends to HF_5S_4, i.e. λ extends to HF_5S_4.

Let λ_1, λ_2 be two extensions of λ to HX_5F_4, and let γ be an extension of λ to HF_5S_3. Since the degree of all irreducible constituents of $\lambda^{HX_5S_3}$ is $\frac{q^2}{5}$, we have

$$\lambda_1{}^{HX_5S_3}, \lambda_2{}^{HX_5S_3}, \gamma^{HX_5S_3} \in \mathrm{Irr}(HX_5S_3, \lambda).$$

Choose $1 \in S \subset S_3$ as a representative set of the double coset

$$HF_5S_4 \backslash HX_5S_3 / HX_5F_4.$$

As $HF_5S_4 \cap HX_5F_4 = HF_5F_4$ and $HX_5F_4 \trianglelefteq HX_5S_3$, by Mackey's formula,

$$\begin{aligned}(\lambda_1{}^{HX_5S_3}, \gamma^{HX_5S_3}) &= \sum_{s \in S} ({}^s\lambda_1|_{{}^s(HX_5F_4) \cap HF_5S_4}, \gamma|_{{}^s(HX_5F_4) \cap HF_5S_4}) \\ &= \sum_{s \in S} ({}^s\lambda_1|_{HF_5F_4}, \gamma|_{HF_5F_4}).\end{aligned}$$

For each $s \in S$, if ${}^s\lambda_1|_{HF_5F_4} = \gamma|_{HF_5F_4}$, then ${}^s\lambda_1|_H = \gamma|_H$. Since both are extensions of λ, we have ${}^s\lambda = \lambda$, i.e. $s \in \mathrm{Stab}_{S_3}(\lambda) = S_4$. There is a unique $1 \in S \cap S_4$ since S is a representative set of $HF_5S_4 \backslash HX_5S_3 / HX_5F_4$. So

$$(\lambda_1{}^{HX_5S_3}, \gamma^{HX_5S_3}) = (\lambda_1|_{HF_5F_4}, \gamma|_{HF_5F_4}) = 1 \text{ iff } \lambda_1|_{F_i} = \gamma|_{F_i},\ i \in \{4, 5\}.$$

Therefore, $\lambda_1{}^{HX_5S_3} = \gamma^{HX_5S_3} = \lambda_2{}^{HX_5S_3}$ iff $\lambda_1|_{F_i} = \lambda_2|_{F_i}, i \in \{4, 5\}$. □

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Bibliography

[1] R. W. Carter, *Simple Groups of Lie Type*, Pure and Applied Mathematics 28, Wiley, London, 1972.

[2] G. Higman, Enumerating p-groups. I. Inequalities, *Proc. Lond. Math. Soc. (3)* **10** (1960), 24–30.

[3] F. Himstedt, T. Le and K. Magaard, Characters of the Sylow p-subgroups of the Chevalley groups $D_4(p^n)$, *J. Algebra* **332** (2011), no. 1, 414–427.

[4] I. M. Isaacs, Characters of groups associated with finite algebras, *J. Algebra* **177** (1995), no. 3, 708–730.

[5] MAGMA Computational Algebra System, `http://magma.maths.usyd.edu.au/magma/`.

[6] J. Sangroniz, Character degrees of the Sylow p-subgroups of classical groups, in: *Groups St. Andrews 2001 in Oxford*, Volume II, London Mathematical Society Lecture Note Series 305, Cambridge University Press, Cambridge (2003), 487–493.

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