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A REPRESENTATION THEOREM FOR END SPACES OF INFINITE GRAPHS

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Abstract. End spaces of infinite graphs sit at the interface between graph theory, group theory and topology. They arise as the boundary of an infinite graph in a standard sense generalising the theory of the Freudenthal boundary developed by Freudenthal and Hopf in the 1940's for infinite groups.

A long-standing quest in infinite graph theory with a rich body of literature seeks to describe the possible end structures of graphs by a set of low-complexity representatives. In this paper we propose a solution to this fifty-year-old problem by showing that every end space is homeomorphic to the end space of some (*uniform* graph on a) special order tree.

1. Introduction

This paper studies end spaces of infinite graphs, i.e. topological spaces that arise as the boundary of an infinite graph in a natural sense generalising the theory of the Freudenthal boundary developed by Freudenthal and Hopf in the 1940's [18, 19, 27]. A long-standing, influential quest in infinite graph theory is the problem of how to represent or capture the end structure of a given graph. The purpose of this paper is to present a way how to resolve this fifty-year-old problem by showing that all end spaces arise from (certain canonical graphs that we call *uniform* graphs on) special order trees.

1.1. **Ends and end spaces.** Intuitively, the ends of a graph capture the different directions in which the graph expands towards infinity. The simplest case is given by the class of (rooted) trees. Here, the ends correspond to the different rays starting at the root, and their end spaces are precisely the completely ultrametrizable spaces. See [28] for a thorough treatment of end spaces of trees.

The concept of an end generalises naturally from trees to arbitrary graphs. Following Halin [22], an end of a graph G is an equivalence class of rays where two rays are equivalent if for every finite set X of vertices they eventually belong to the same component of $G - X$. For example, infinite complete graphs or grids have a single end, while the infinite binary tree has continuum many ends, one for every rooted ray. The applications of ends in group theory or infinite graph theory are numerous, and reach from Stallings' theorem about the structure of finitely generated groups with more than one end [49, 50] to Halin's seminal paper on automorphisms of infinite graphs [24] – in fact, ends even play a role in the study of automorphism groups of *finite* graphs [1, 21] – to the recent development of infinite matroid theory [5] inspired by topological spanning trees in infinite graphs with ends [12].

The set of ends is equipped with a natural topology, in which for every finite set of vertices X, each component C of $G - X$ induces a basic open set formed by all ends whose rays eventually belong to C. Note that when considering end spaces, we may always assume that the underlying graph is connected, as adding one new vertex and choosing a neighbour for it in each component does not affect the end space. It turns out that end spaces of graphs are significantly more complex than end spaces of trees: For a typical example, start from an uncountable complete graph (which has a unique end) and glue onto each vertex a fresh ray. The resulting end space will be compact, yet no longer metrizable. Going back to the inception of the combinatorial definition of ends by Halin [22] in 1964, the fundamental problem of providing a structural description of end spaces has remained unsolved.

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1.2. **History of the problem.** Before explaining our solution in detail, we recall the four most influential attempts in the history of the problem. In each case we indicate how these attempts advanced our understanding of end spaces, but also why they eventually proved insufficient to capture the full complexity of end spaces.

The first attempt was Halin's conjecture from 1964 [22] that every connected graph has an end-faithful spanning tree, i.e. a rooted spanning tree whose rooted rays form a representation system of the ends of the ambient graph. Positive results included Halin's theorem that every countable graph has an end-faithful spanning tree, as well as Polat's result that every graph without a subdivision of the $\aleph_1$-regular tree contains an end-faithful spanning tree [44], see also [6]. However, Halin's conjecture was refuted in the early 1990's independently by Seymour and Thomas [47] and by Thomassen [51], who constructed one-ended graphs in which every spanning tree has uncountably many ends.

However, even in examples where end-faithful spanning trees exist, they are generally insufficient to capture the topological information of end spaces: End spaces of trees are always metrizable, whereas end spaces of arbitrary graphs generally are not. The question which graphs admit end-faithful trees that respect the topology of the end space then sparked the highly successful theory of normal (spanning) trees of Jung, infinite generalisations of the 'depth-first search trees'. Since their invention due to Jung [30], normal trees have developed to be perhaps the single most useful structural tool in infinite graph theory. Positive results include a characterisation due to Jung when normal spanning trees exist [31], Halin's result that graphs without a (fat) subdivision of a countable clique have normal spanning trees [14, 25, 39, 36] as well as results about forbidden minors for normal spanning trees [3, 16, 26, 38, 37]. For applications of normal trees that are not necessarily spanning, see [7, 35].

The third attempt was initiated by Diestel in 1990 [10]. Diestel asked whether one could use tree-decompositions to capture the end-space of a given graph. Roughly speaking, a tree-decomposition of a graph G is a blueprint which shows how G can be built from smaller graphs by gluing them in the shape of a tree T. The smaller graphs are called parts, the splices are referred to as adhesion sets, and T is the decomposition tree. Tree-decompositions play a central role in the proof of the graph minor theorem by Robertson and Seymour [45], but also in modern proofs of Stallings' theorem about the structure of finitely generated groups [17, 33]. If a connected graph G admits a tree-decomposition with finite adhesion sets, then intuitively every ray in G must either have infinitely many vertices in a unique part or follow the course of a ray of the decomposition tree T. Rays which belong to the same end of G determine the same part or ray of T, hence every end of G either lives in a part or corresponds to an end of T. Diestel asked which connected graphs G admit a tree-decomposition with finite adhesion sets such that the ends of G correspond bijectively to the ends of the decomposition tree, except for those ends of G which are represented by some infinitely many pairwise disjoint rays: these 'thick' ends should live in parts so that no two of them live in the same part [10, Problem 4.3]. However, similar to Halin's end-faithful spanning tree conjecture, also this attempt to capture the end space was too ambitious to be true. In [9], Carmesin constructed counterexamples to Diestel's question, but more importantly in a breakthrough result established Diestel's suggestion in the following amended form: Every graph G admits a tree-decomposition with finite adhesion sets such that the ends of the decomposition tree are in 1-1 correspondence with the undominated ends of G. Here, a vertex v *dominates* a ray R in G if there are infinitely many v–R paths in G which pairwise only meet in v, and v *dominates* an end of G if it dominates one (and hence every) ray in it. An end is called *dominated* or *undominated* accordingly. Furthermore, from this result Carmesin also shows the existence of spanning trees which are end-faithful for the undominated ends, proving Halin's conjecture in amended form. For a detailed treatment of end spaces and tree-decompositions we refer the reader to [32].

The fourth and last attempt to get a hold on the general question is to investigate purely topological properties of the end space. Despite significant advances in our topological understanding of end spaces, for example that end spaces are (hereditarily) ultra-paracompact [35] or admit discrete expansions of length at most ω_1 [43] and numerous further results [11, 15, 34, 42, 48], we still had no complete topological characterisation of end spaces up to now, an open problem formally posed by Diestel [10, Problem 5.1].

1.3. **The representation theorem.** As announced above, the purpose of this paper is to offer a solution to this longstanding quest in a fifth and final attack by establishing a representation result for end spaces that captures both combinatorial and topological properties of end spaces. A major part of the difficulty lies in singling out the correct type of object that allows one to capture end spaces of graphs. The starting point of this paper was the observation that the counterexamples for normal spanning trees and for end-respecting tree-decompositions are almost always based on graphs on certain order trees. Halin [26] observed that binary trees where we add a dominating vertex above every ray (called 'tops') do not admit normal spanning trees, and Carmesin [9] showed that the very same binary tree with tops does not admit a tree decomposition of finite adhesion that displays all the ends. Further, Diestel and Leader [16] presented a graph based on a special Aronszajn tree that admits no normal spanning tree. More examples based on order trees with tops can be found in Carmesin [9] and in the second author's [37] for normal spanning trees.

Our main result shows that this behaviour is no coincidence, and that in fact any end space can be obtained from a special order tree (an order tree is called *special* if it can be partitioned into countably many antichains).

Theorem 1. *Every end space is homeomorphic to the end space of a special order tree.*

The use of order trees for structural descriptions in infinite graph theory is by no means uncommon: They play a role in Halin's *simplicial decompositions* [25], in the *well-founded tree-decompositions* of Robertson, Seymour and Thomas [46], and the *normal tree orders* of Brochet and Diestel [4]. We may think of their results as saying that the class of all graphs is roughly of the same complexity as the class of all order trees. What is highly surprising is that, in contrast, our result shows that the class of all end spaces is only as complex as the class of *special* order trees, a class of significantly lower complexity. (Generally, fewer set-theoretic issues arise for this class of trees.)

We still need to explain what we mean by the end space of a special order tree. First, recall that there is a standard method, systematically introduced by Brochet and Diestel in [4], of turning arbitrary order trees T into graphs reflecting that tree structure: A *graph on* T, or for short a T*-graph*, is a graph whose vertices are the nodes of the tree and in which edges run between comparable vertices only, so that successor nodes are adjacent to their immediate predecessor and limit nodes are adjacent to cofinally many of their predecessors. For a survey on applications of T-graphs see [40].

Now given a special order tree T with antichain partition $\{U_n : n \in \mathbb{N}\}$, there is a canonical way due to Diestel, Leader and Todorcevic [16, Theorem 6.2] to build an even stronger type of T-graph G called a *uniform* graph on T, as follows: Successors are connected to their predecessors, and given a limit $t \in T$, recursively pick down-neighbours $t_0 < t_1 < t_2 < \cdots < t$ with $t_i \in U_{n_i}$ so that each n_i is smallest possible subject to $t_{i-1} < t_i < t$. See Theorem 4.5 below for a thorough discussion. While G depends on the exact choice of $\{U_n : n \in \mathbb{N}\}$, we show that any two uniform graphs on the same order tree have naturally homeomorphic end spaces (Proposition 5.4), thus obtaining a well-defined concept of the end space of a special order tree.

1.4. **Applications.** To understand the end spaces of arbitrary graphs, it thus suffices to deal with the end spaces of the class of uniform graphs on special order trees instead. In Section 5, we give an exact description of the end space of G in terms of the order tree T, both combinatorially as well as topologically. This has a number of noteworthy consequences.

Corollary 2. *Forbidding uncountable clique minors does not reduce the complexity of end spaces, i.e., every end space is homeomorphic to the end space of a graph without an uncountable clique minor.*

Proof. This follows from our main result by the fact that uniform graphs on special order trees do not contain uncountable clique minors, Corollary 4.7. □

In contrast, forbidding a subdivision of a countably infinite clique gives a normal spanning tree by the aforementioned result of Halin [25, Theorem 10.1], and so in this case the end space is metrizable.

The *degree* of an end is the supremum of the sizes of collections of pairwise disjoint rays in it; Halin [23] showed that this supremum is always attained, see also [13, Theorem 8.2.5].

Corollary 3. *Forbidding ends of uncountable degree does not reduce the complexity of end spaces, i.e., every end space is homeomorphic to the end space of a graph in which every end has countable degree.*

Proof. This follows from our main result by the fact that all ends of uniform graphs on order trees have just countable degree, Corollary 5.2. □

Next to these new consequences, our main result also allows us to give a unified explanation of a number of results in the literature. As a case in point, we present in Section 9 short derivations both of Carmesin's result [9, 5.17] that every end space admits a nested set of open bipartitions which distinguishes all the ends, as well as Polat's [43] result that every end space admits a discrete expansion of length at most ω_1.

Finally, our representation theorem for end spaces will form the key step towards a solution of [10, Problem 5.1] of finding a purely topological characterisation of end spaces, see the second author's [41] for details.

1.5. **Organisation of this paper.** This paper is organised as follows. First, in Section 2, we agree on general notation and recall all relevant preliminaries about ends, end spaces, order trees, normal trees and T-graphs. In Section 3 we introduce a concept called *enveloping* a given set of vertices, which intuitively enables one to maximally expand any set of vertices without changing the number of ends in its closure. In Section 4 we discuss an alternative path to uniform T-graphs and establish an equivalence between uniformity and special order trees. In Section 5 we give a complete description of end spaces of uniform T-graphs in terms of the underlying order tree T.

Sections 6, 7 and 8 contain the main body of work of this paper, at the end of which we prove Theorem 1. In Section 6 we introduce a framework called *partition trees* – a decomposition technique inspired by the normal order trees from [4] – and establish some core properties of it which abstractly relate structural properties of the partition tree to the end structure of the underlying graph. Section 7 gives our central existence result that all graphs G admit partition trees $(T, \mathcal{V})$ that display all ends of G with further desirable properties aimed towards capturing the topology of the underlying end structure. While our proof is formally self-contained, its strategy fuses and subsumes a number of hitherto unconnected ideas present in Jung's theory of normal trees [31], Brochet and Diestel's normal partition trees [4], and Polat's theory of multi-endings [43]. Finally, in Section 8, starting from a suitable partition tree say $(T, \mathcal{V})$ for a given graph G, we modify it by splitting up certain limit nodes to finally obtain a special order tree T' with uniform graph structure whose end space agrees with the end space of G not only combinatorically but also topologically.

Lastly, in Section 9 we show how to obtain Carmesin's nested bipartition theorem and Polat's discrete expansion theorem from our main result.

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2. End spaces and T-graphs: a reminder

For graph theoretic terms we follow the terminology in [13], and in particular [13, Chapter 8] for ends of graphs and the end spaces $\Omega(G)$. A function $f\colon X \to Y$ is *finite-to-one* if $f^{-1}(y)$ is finite for all $y \in Y$.

2.1. **End spaces.** A 1-way infinite path is called a *ray* and the subrays of a ray are its *tails*. Two rays in a graph $G = (V, E)$ are *equivalent* if no finite set of vertices separates them; the corresponding equivalence classes of rays are the *ends* of G. The set of ends of a graph G is denoted by $\Omega = \Omega(G)$. Usually, ends of graphs are denoted by ω, but here we will denote ends by ε or η to avoid confusion with ordinals such as ω_1. If $X \subseteq V$ is finite and $\varepsilon \in \Omega$ is an end, there is a unique component of $G - X$ that contains a tail of every ray in ε, which we denote by $C(X, \varepsilon)$. Then ε *lives* in the component $C(X, \varepsilon)$. If C is any component of $G - X$, we write $\Omega(X, C)$ for the set of ends ε of G with $C(X, \varepsilon) = C$, and abbreviate $\Omega(X, \varepsilon) := \Omega(X, C(X, \varepsilon))$. Finally, if $\mathscr{C}$ is any collection of components of $G - X$, we write $\Omega(X, \mathscr{C}) := \bigcup \{\, \Omega(X, C) \colon C \in \mathscr{C} \,\}$.

The collection of all sets $\Omega(X, C)$ with $X \subseteq V$ finite and C a component of $G - X$ forms a basis for a topology on Ω. This topology is Hausdorff, and it is *zero-dimensional* in that it has a basis consisting of closed-and-open sets.

A crucial property of end spaces is that they are *Fréchet-Urysohn*: This means that closures are given by convergent sequence, i.e. $x \in \overline{X}$ for some subset $X \subseteq \Omega(G)$ if and only if there are $x_n \in X$ with $x_n \to x$ (as $n \to \infty$). In particular, this implies that a function f between two end spaces is continuous if and only if f is sequentially continuous; later in this paper, we will therefore only check for sequential continuity.

Note that when considering end spaces $\Omega(G)$, we may always assume that G is connected; adding one new vertex and choosing a neighbour for it in each component does not affect the end space.

Recall that a *comb* is the union of a ray R (the comb's *spine*) with infinitely many disjoint finite paths, possibly trivial, that have precisely their first vertex on R. The last vertices of those paths are the *teeth* of this comb. Given a vertex set U, a *comb attached to* U is a comb with all its teeth in U, and a *star attached to* U is a subdivided infinite star with all its leaves in U. Then the set of teeth is the *attachment set* of the comb, and the set of leaves is the *attachment set* of the star.

Lemma 2.1 (Star-comb lemma)**.** *Let U be an infinite set of vertices in a connected graph G. Then G contains either a comb attached to U or a star attached to U.*

Let us say that an end ε of G is contained *in the closure* of M, where M is either a subgraph of G or a set of vertices of G, if for every finite vertex set $X \subseteq V$ the component $C(X, \varepsilon)$ meets M. Equivalently,

ε lies in the closure of M if and only if G contains a comb attached to M with its spine in ε. We write $\partial_\Omega M$ for the subset of Ω that consists of the ends of G lying in the closure of M.

2.2. **Normal trees.** Given a graph H, we call a path P an *H-path* if P is non-trivial and meets H exactly in its endvertices. In particular, the edge of any H-path of length 1 is never an edge of H. A rooted tree $T \subseteq G$ is *normal* in G if the endvertices of every T-path in G are comparable in the tree-order of T, cf. [13]. The rays of a normal tree T that start at its root are its *normal rays*.

Lemma 2.2. *Let G be any connected graph and let $v \in G$ be any vertex. Then there exists an inclusionwise maximal normal tree $T \subseteq G$ rooted at v. Every component of $G - T$ has infinite neighbourhood, and the down-closures in T of each such neighbourhood determines a normal ray of T.*

Proof. Inclusionwise maximal normal trees $T \subseteq G$ rooted at v exist by Zorn's lemma. By the defining property of a normal tree, the neighbourhood of every component of $G - V(T)$ forms a chain in the tree-order on T. We claim that all such neighbourhoods are in fact infinite, thus determining a unique normal ray of T. Indeed, suppose for a contradiction that some component C of $G - V(T)$ has finite neighbourhood. Let t be maximal in the tree-order on T amongst all neighbours of C. Then we can extend T to a larger normal tree rooted in v by adding any t–C edge, a contradiction. $\square$

2.3. ***T*-graphs.** A partially ordered set $(T, \leqslant)$ is called an *order tree* if it has a unique minimal element (called the *root*) and all subsets of the form $\lceil t \rceil = \lceil t \rceil_T := \{t' \in T : t' \leqslant t\}$ are well-ordered. Write $\lfloor t \rfloor = \lfloor t \rfloor_T := \{t' \in T : t \leqslant t'\}$. We abbreviate $\lceil \mathring{t} \rceil := \lceil t \rceil \setminus \{t\}$ and $\lfloor \mathring{t} \rfloor := \lfloor t \rfloor \setminus \{t\}$. For subsets $X \subseteq T$ we abbreviate $\lceil X \rceil := \bigcup_{t \in X} \lceil t \rceil$ and $\lfloor X \rfloor := \bigcup_{t \in X} \lfloor t \rfloor$.

A maximal chain in T is called a *branch* of T; note that every branch inherits a well-ordering from T. The *height* of T is the supremum of the order types of its branches. The *height* of a point $t \in T$ is the order type of $\lceil \mathring{t} \rceil$. The set T^i of all points at height i is the ith *level* of T, and we write $T^{<i} := \bigcup \{T^j : j < i\}$ as well as $T^{\leqslant i} := \bigcup \{T^j : j \leqslant i\}$. If $t < t'$, we use the usual interval notation such as $(t, t') = \{s : t < s < t'\}$ for nodes between t and t'. If there is no point between t and t', we call t' a *successor* of t and t the *predecessor* of t'; if t is not a successor of any point it is called a *limit*.

A *top* of a down-closed chain $\mathscr{C}$ in an order tree T is a limit $t \in T \setminus \mathscr{C}$ with $\lceil \mathring{t} \rceil = \mathscr{C}$. Note that a chain $\mathscr{C} \subseteq T$ may have multiple tops.

An order tree T is *normal* in a graph G, if $V(G) = T$ and the two endvertices of any edge of G are comparable in T. We call G a *T-graph* if T is normal in G and the set of lower neighbours of any point t is cofinal in $\lceil \mathring{t} \rceil$. We mention the following standard results about T-graphs, and refer the reader to [4, §2] for details.

Lemma 2.3. *Let $(T, \leqslant)$ be an order tree and G a T-graph.*

(i) *For incomparable vertices t, t' in T, the set $\lceil t \rceil \cap \lceil t' \rceil$ separates t from t' in G.*
(ii) *Every connected subgraph of G has a unique T-minimal element.*
(iii) *If $T' \subseteq T$ is down-closed, the components of $G - T'$ are spanned by $\lfloor t \rfloor$ for t minimal in $T \setminus T'$.*
(iv) *For any two vertices $t \leqslant t'$ in T, the interval $[t, t']$ is connected in G.*

3. Envelopes and concentrated sets of vertices

An *adhesion set* of a set of vertices or a subgraph $U \subseteq G$ is any subset of the form $N(C)$ for a component C of $G - U$. The set or subgraph U is said to have *finite adhesion* in G if all its adhesion sets are finite.

Let G be a connected graph. An *envelope* for a set of vertices $U \subseteq V(G)$ is a set of vertices $U^* \supseteq U$ of finite adhesion such that $\partial_\Omega U^* = \partial_\Omega U$. We will show in Theorem 3.2 below that envelopes exist.

We need a preliminary lemma. A set of vertices U is *concentrated* (in its boundary $\partial_\Omega U$) if for every finite vertex set $X \subseteq V$ only finitely many vertices of U lie outside of $\bigcup_{\varepsilon \in \partial_\Omega U} C(X, \varepsilon)$. We say that a set of vertices U is concentrated in an end ε to mean the case $\partial_\Omega U = \{\varepsilon\}$.

Recall that a subset X of a poset $P = (P, \leqslant)$ is *cofinal* in P, and $\leqslant$, if for every $p \in P$ there is an $x \in X$ with $x \geqslant p$. We say that a rooted tree $T \subseteq G$ contains a set U *cofinally* if $U \subseteq V(T)$ and U is cofinal in the tree-order of T. Note that such trees are easily constructed: any inclusion-minimal subtree T of G with $U \subseteq V(T)$ will contain U cofinally—no matter which vertex of T we choose as root.

Lemma 3.1. *Let G be any graph, let $U \subseteq V(G)$ be a set of vertices, and suppose $T \subseteq G$ is a rooted tree that contains U cofinally. Then the following assertions hold:*

(i) $\partial_\Omega T = \partial_\Omega U$.
(ii) *If U has finite adhesion, then so does T.*
(iii) *If U is concentrated, then so is T.*

Proof. Assertions (i) and (ii) already have been observed in [7, Lemma 2.13] and [38, Lemma 4.3]. For convenience of the reader, we give a self-contained argument for all three properties (i)–(iii) at once. For this, suppose $T \subseteq G$ is a rooted tree that contains U cofinally.

Claim 1. *If X is a finite set of vertices and $\mathscr{C}$ a collection of components of $G - X$ such that $V(T)$ meets $\bigcup \mathscr{C}$ infinitely, then so already does U.*

To see the claim, suppose for a contradiction that for some finite vertex set X and some collection $\mathscr{C}$ of components of $G - X$ we have $\bigcup \mathscr{C}$ meets U meets finitely, but $V(T)$ infinitely. By increasing X, we may assume that $\bigcup \mathscr{C} \cap U = \emptyset$. Pick $t \in (T \setminus \lceil X \rceil) \cap \bigcup \mathscr{C}$. Then $\lfloor t \rfloor \subseteq \bigcup \mathscr{C}$ avoids U, a contradiction to the assumption that U is cofinal in T.

Now to see (i), consider any end $\varepsilon \notin \partial_\Omega U$. Then there is a finite set of vertices X such that $C(X, \varepsilon)$ avoids U. By Claim 1, also $V(T)$ intersects $C(X, \varepsilon)$ finitely, witnessing $\varepsilon \notin \partial_\Omega T$. The argument for (iii) is similar. Finally, for (ii), consider a component C' of $G - T$. Then there is a component C of $G - U$ with $C' \subseteq C$. Assuming that U has finite adhesion, $X = N(C)$ is finite. Claim 1 applied to X and C yields that $V(T)$ intersects C finitely. Then $N(C') \subseteq X \cup (V(T) \cap V(C))$ is finite, so T has finite adhesion. □

Theorem 3.2. *Any [concentrated] set U of vertices in a connected graph G has a connected [concentrated] envelope U^*. Moreover, U^* can be chosen such that $U^* \cap C$ is connected for every component C of $G - U$.*

Proof. Let U be a given set of vertices in a connected graph G. By Zorn's lemma, there is an inclusionwise maximal set of combs attached to U with pairwise disjoint spines. Write $\mathcal{R}$ for this collection of spines. Let S be the set of all centres of (infinite) stars attached to U. We will show that

$$U^* := U \cup \bigcup_{R \in \mathcal{R}} V(R) \cup S$$

is an envelope for U. The verification relies on the following claim:

Claim 2. *If X is a finite set of vertices and $\mathscr{C}$ a collection of components of $G - X$ such that U^* meets $\bigcup \mathscr{C}$ infinitely, then so already does U.*

To see the claim, consider some finite set of vertices X, and assume that $\mathscr{C}$ is a collection of components of $G - X$ such that U meets $\bigcup \mathscr{C}$ finitely. Then S avoids $\bigcup \mathscr{C}$. Consider next a spine $R \in \mathcal{R}$ that meets $\bigcup \mathscr{C}$. Since R eventually lies in a component $C \notin \mathscr{C}$, we have that R meets $\bigcup \mathscr{C}$ finitely. Moreover, R also meets X, and since the spines in $\mathcal{R}$ are pairwise disjoint, there are at most $|X|$ spines from $\mathcal{R}$ that meet $\bigcup \mathscr{C}$, and each does so finitely. Hence $\bigcup_{R \in \mathcal{R}} V(R)$ meets $\bigcup \mathscr{C}$ finitely, too, and the claim follows.

Now to see $\partial_\Omega U^* = \partial_\Omega U$, consider any end $\varepsilon \notin \partial_\Omega U$. Then there is a finite set of vertices X such that $C(X, \varepsilon)$ avoids U. By Claim 2, also U^* intersects $C(X, \varepsilon)$ finitely, witnessing $\varepsilon \notin \partial_\Omega U^*$. The argument that if U is concentrated, then so is U^*, follows similarly from Claim 2.

To see that U^* has finite adhesion, suppose for a contradiction that there is a component C of $G - U^*$ with infinite neighbourhood. Then by a routine application of Lemma 2.1, we either find a star or a comb attached to U^* whose centre v or spine R is contained in C. Then for all finite sets of vertices X (chosen disjoint from v in the star case), the centre v or a tail of R lives in a component of $G - X$ that meets U^* infinitely, and hence also U by Claim 2. But then it is straightforward to inductively construct a star with centre v or a comb with spine R attached to U, violating the choice of S or $\mathcal{R}$ respectively.

To get a connected envelope for U that also satisfies the moreover-part, note that there are inclusionwise minimal subtrees of G containing U^* which satisfy that $T \cap C$ is connected for every component C of $G - U$: Take a spanning tree T_C for every component C of $G - U$, and extend the forest $\bigcup_C T_C$ to a rooted spanning tree T of G. Then the down-closure of U^* in T can serve as the desired envelope by Lemma 3.1. □

For a concept that is somewhat stronger than our notion of envelopes (but whose existence proof is significantly more involved) see Polat's *multiendings* from [43]; a precursor to our notion of envelopes has been used in Bürger and the first author's proof of [8, Theorem 1].

Recall that a sequence $(x_n)_{n \in \mathbb{N}}$ in a topological space X is said to *converge* to a subset $A \subseteq X$, denoted by $x_n \to A$ (as $n \to \infty$) if every open set containing A contains also almost all x_n. Note that if $A = \{x\}$ is a singleton, this reduces to the usual notion of convergence $x_n \to x$.

We will apply the following lemma for a single end only. However, for future reference we state it in its optimal form for a compact collection of ends.

Lemma 3.3. *Let U be a concentrated vertex set of G of finite adhesion. Assume further that $\partial_\Omega U$ is compact. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be any sequence of ends in $\Omega(G) \setminus \partial_\Omega U$, and write D_n for the unique component of $G - U$ in which ε_n lives.*

Then $\varepsilon_n \to \partial_\Omega U$ (as $n \to \infty$) if and only if the map $\mathbb{N} \ni n \mapsto N_G(D_n)$ is finite-to-one.

Proof. First, suppose that for some finite $X \subseteq U$ there are infinitely many $n \in \mathbb{N}$ with $N_G(D_n) = X$. Write $\mathscr{C}$ for the collection of components of $G - X$ containing an end from $\partial_\Omega U$. Then $\Omega(X, \mathscr{C})$ is an open neighbourhood around $\partial_\Omega U$ avoiding infinitely many ε_n, witnessing $\varepsilon_n \not\to \partial_\Omega U$.

Now assume that the map $\mathbb{N} \ni n \mapsto N_G(D_n)$ is finite-to-one, and let us suppose for a contradiction that the sequence $\varepsilon_0, \varepsilon_1, \ldots$ does not converge to $\partial_\Omega U$ in the end space of G. Then there exists an open set around $\partial_\Omega U$ avoiding infinitely many ε_n, and by compactness of $\partial_\Omega U$ we may find a finite vertex set $X \subseteq V(G)$ such that infinitely many ε_n live outside of $\bigcup_{\varepsilon \in \partial_\Omega U} C(X, \varepsilon)$. Since the map $n \mapsto D_n$ has finite fibres, we may assume without loss of generality that the finite vertex set X meets none of the components D_n in which these ends live. Hence, infinitely many neighbourhoods $N_G(D_n)$ avoid $\bigcup_{\varepsilon \in \partial_\Omega U} C(X, \varepsilon)$. But each of these neighbourhoods consists of vertices in U, and since the fibres of the map $n \mapsto N_G(D_n)$ are finite, the union of these neighbourhoods is an infinite subset of U that lies outside of $\bigcup_{\varepsilon\in\partial_\Omega U} C(X,\varepsilon)$, contradicting the assumption that U is concentrated. $\square$

4. Uniform T-graphs and special order trees

Definition 4.1. A T-graph G has

- *finite adhesion* if for every limit ordinal σ all components of $G - T^{\leqslant\sigma}$ have finite neighbourhoods;
- *uniformly finite adhesion*, or for short, is *uniform* if for every limit $t \in T$ there is a finite $S_t \subseteq \lceil \mathring{t} \rceil$ such that every $t' > t$ has all its down-neighbours below t inside S_t.

The property for a T-graph to be uniform has been introduced by Diestel and Leader in [16], and it has proven useful in [20] as well. Our next two lemmas illuminate the relation between finite and uniformly finite adhesion.

Lemma 4.2. *The following are equivalent for a T-graph G:*

(i) *G has finite adhesion.*
(ii) *For every ordinal σ all components of $G - T^{\leqslant\sigma}$ have finite neighbourhoods.*
(iii) *Every vertex set of the form $\lfloor t \rfloor$ for a successor $t \in T$ has finite neighbourhood in G.*

Proof. (i) $\Rightarrow$ (ii). For limits σ, this holds by assumption, so suppose that σ is a non-limit and consider any component C of $G - T^{\leqslant\sigma}$. Let us assume for a contradiction that C has infinitely many neighbours. Using that these form a well-ordered chain and that σ is a non-limit, we find an ω-chain of infinitely many neighbours $t_0 < t_1 < \cdots$ of C and a limit $t \in T^{<\sigma}$ such that $t_n < t$ for all $n < \omega$. Let μ denote the height of t and note that μ is a limit. Then the component of $G - T^{\leqslant\mu}$ which contains C has infinite neighbourhood, contradicting our assumption that G has finite adhesion.

(ii) $\Rightarrow$ (iii). Suppose t is a successor, and let σ be the height of the predecessor of t. By Lemma 2.3 (iii), the unique component of $G - T^{\leqslant\sigma}$ containing t is spanned by $\lfloor t \rfloor$, so has finite neighbourhood.

(iii) $\Rightarrow$ (i). Let σ be a limit ordinal. By Lemma 2.3 (iii), the components of $G - T^{\leqslant\sigma}$ are spanned by $\lfloor t \rfloor$ for $t \in T^{\sigma+1}$, so have finite neighbourhoods by assumption (iii). $\square$

Lemma 4.3. *The following are equivalent for a T-graph G:*

(i) *G has uniformly finite adhesion.*
(ii) *Every vertex set of the form $\lfloor \underset{\circ}{t} \rfloor$ for a limit $t \in T$ has finite neighbourhood in G.*
(iii) *Every vertex set of the form $\lfloor \underset{\circ}{t} \rfloor$ has finite neighbourhood in G.*

Proof. (i) $\Rightarrow$ (ii). If G has uniformly finite adhesion, then every vertex set of the form $\lfloor \underset{\circ}{t} \rfloor$ for a limit $t \in T$ has all its neighbours in the finite set $S_t \cup \{t\}$.

(ii) $\Rightarrow$ (iii). Let us assume for a contradiction that some $\lfloor \underset{\circ}{t} \rfloor$ has infinitely many neighbours. Then we find an ω-chain of infinitely many neighbours $t_0 < t_1 < \cdots$ of $\lfloor \underset{\circ}{t} \rfloor$ and a limit $\ell \leqslant t$ such that $t_n < \ell$ for all $n < \omega$. But then also $\lfloor \underset{\circ}{\ell} \rfloor \supseteq \lfloor \underset{\circ}{t} \rfloor$ has infinite neighbourhood, a contradiction.

(iii) $\Rightarrow$ (i). Let t be a limit node of T. Then $S_t = N(\lfloor \underset{\circ}{t} \rfloor) \setminus \{t\}$ is as desired. $\square$

Comparing Lemma 4.2 (iii) with Lemma 4.3 (iii) explains the name 'uniform' and in particular shows that uniformly finite adhesion implies finite adhesion.

Lemma 4.4. *If G is a T-graph of finite adhesion and $t \in T$ is a limit, then the down-neighbours of t form a cofinal ω-chain in $\lceil \mathring{t} \rceil$.*

Proof. The down-neighbours of t form an infinite chain $\mathscr{C} \subseteq \lceil \mathring{t} \rceil$. Since G has finite adhesion, below every successor in $\lceil \mathring{t} \rceil$ there are only finitely many elements of $\mathscr{C}$. It follows that $\mathscr{C}$ must be an ω-chain. □

Our next result characterises on which order trees there exist T-graphs of (uniformly) finite adhesion. Recall from [2] that a partial order P is said to *embed* into a partial order Q if there is a map $f\colon P \to Q$ such that $p < p'$ implies $f(p) < f(p')$ (in particular, such maps are not required to be injective).

Theorem 4.5. *Let $(T, \leqslant)$ be an order tree.*

(i) *There exists a T-graph of uniformly finite adhesion if and only if $(T, \leqslant)$ embeds into $(\mathbb{Q}, \leqslant)$.*
(ii) *There exists a T-graph of finite adhesion if and only if $(T, \leqslant)$ embeds into $(\mathbb{R}, \leqslant)$.*

Proof. By a result of Baumgartner and Galvin [2, Section 4.1], T embeds into the rationals if and only if T is special, i.e. has a countable antichain partition, and T embeds into the reals if and only if the set of all successor nodes of T admits a partition into countably many antichains.

(i) Let T be special with antichain partition $\{U_n : n \in \mathbb{N}\}$. Following Diestel, Leader and Todorcevic [16], we construct a T-graph as follows: Successors are connected to their predecessors, and given a limit $t \in T$, pick down-neighbours $t_0 <_T t_1 <_T t_2 <_T \cdots <_T t$ with $t_i \in U_{n_i}$ recursively such that $t_{i-1} < t_i < t$ and each n_i is smallest possible. We claim that the resulting T-graph G has uniformly finite adhesion. Indeed, S_t may be chosen to consist of all elements of $\lceil \mathring{t} \rceil$ contained in an antichain with smaller index than the antichain containing t.

Conversely, suppose that there is a T-graph G of uniformly finite adhesion with root r, say. We argue that there exists a regressive function $f\colon (T - r) \to T$ such that each fibre can be covered by countably many antichains (then T itself is special by a well-known observation of Todorcevic [52, Theorem 2.4]). Indeed, map all successors to their unique predecessor, and map a limit t to any of its neighbours in $(\max S_t, t)$. This regressive map is $\leqslant 2-1$ on branches of T: Indeed, suppose for a contradiction that there are two limits $t < t'$ with $f(t) = f(t')$. Since $t' > t > f(t) = f(t')$ we have $f(t') \in S_t$. But then $f(t) \in S_t$, a contradiction.

(ii) Suppose that the successors of T admit an antichain partition $(U_n)_{n\in\mathbb{N}}$. Following the same procedure as above, we construct a T-graph as follows: Successors are connected to their predecessors, and given a limit $t \in T$, pick down-neighbours $t_0 <_T t_1 <_T t_2 <_T \cdots <_T t$ with $t_i \in U_{n_i}$ recursively such that $t_{i-1} < t_i < t$ and each n_i is smallest possible. We claim that the resulting T-graph G has finite adhesion. Indeed, for any limit ordinal σ and any component C of $G - T^{\leqslant\sigma}$ we have $C = \lfloor s \rfloor$ for a unique successor node s of some limit $t \in T^\sigma$ by Lemma 2.3 (iii). Hence, $N(C)$ is finite, as all elements of $N(C) \cap \lceil \mathring{t} \rceil$ belong to antichains with smaller indices than the antichain containing s.

Conversely, suppose that there is a T-graph G of finite adhesion. Let $L \subseteq T$ consist of all limits of T that do have successors in T, and for every limit $\ell \in L$ let the set $S(\ell)$ consist of all the successors of ℓ in T. We obtain a new order tree T' from T as follows: First, add for each $\ell \in L$ and $s \in S(\ell)$ a new node $v(\ell, s)$ that we declare to be a successor of ℓ and a predecessor of s. Then delete L. Note that there is a natural epimorphism $\varphi\colon T' \to T$ which is the identity on $T \setminus L$ and which sends each $v(\ell, s)$ to ℓ.

Let G' be the T'-graph with $V(G') := T'$ and

$$E(G') := \{tt' : t < t' \in T' \ \wedge\ \varphi(t)\varphi(t') \in E(G)\}.$$

Consider a limit $t = v(\ell, s)$ of T'. Since G has finite adhesion, $\lfloor s \rfloor_T$ has finite neighbourhood say S_s in G. Since every $t' > t$ in T' satisfies $\varphi(t') \in \lfloor s \rfloor_T$, the set $S_t = \varphi^{-1}(S_s) \cap \lceil t \rceil_{T'}$ witnesses that G' has uniformly finite adhesion. It follows from (i) that T' is special, so admits a countable antichain partition. This induces an antichain partition on the successors of T'; since φ acts as isomorphism on this set, it follows that T embeds into $(\mathbb{R}, \leqslant)$. □

Corollary 4.6. *If G is a T-graph of finite adhesion, then all branches of T are countable.*

Proof. An uncountable branch does not embed into the reals. □

Corollary 4.7. *T-graphs of finite adhesion do not contain uncountable clique minors.*

Proof. Suppose some T-graph of finite adhesion contains an uncountable clique minor. Then by a result due to Jung [29], this graph would also contain a subdivision of an uncountable clique. By Corollary 4.6, some pair of branch vertices v, w of this clique must be incomparable in T. By Lemma 2.3 (i) the set $\lceil v \rceil \cap \lceil w \rceil$ is a countable separator between v and w, a contradiction. □

5. End spaces of T-graphs of finite adhesion

A down-closed chain $\mathscr{C}$ in an order tree T is called a *high-ray* of T if its order-type has cofinality ω. We denote the set of all high-rays of T by $\mathcal{R}(T)$. In this section, we show that the end space of a T-graph of finite adhesion can be understood through the high-rays of T.

For this, let G be any T-graph of finite adhesion. Every $t \in T$ induces a bipartition $\{\, \lfloor t \rfloor \,, T \setminus \lfloor t \rfloor \,\}$ of T. Since T is the vertex set of G, it follows that $\Omega(G) = \partial_\Omega \lfloor t \rfloor \cup \partial_\Omega (T \setminus \lfloor t \rfloor)$. For every end ε of G, let us put

$$\sigma(\varepsilon) := \{\, t \in T \mid \varepsilon \in \partial_\Omega \lfloor t \rfloor \setminus \partial_\Omega (T \setminus \lfloor t \rfloor) \,\}.$$

Clearly, $\sigma(\varepsilon)$ is a down-closed chain in T. At the end of the next section, we will see that

Lemma 5.1. *The map σ is a bijection between the end space $\Omega(G)$ and the set $\mathcal{R}(T)$ of all high-rays of T, for each T-graph G of finite adhesion.*

Corollary 5.2. *All ends of T-graphs of finite adhesion have countable degree.*

Proof. Let G be a T-graph of finite adhesion, and let ε be an end of G. Put $\varrho := \sigma(\varepsilon)$, and assume for a contradiction that $\{\, R_i \mid i < \aleph_1 \,\}$ is a collection of uncountably many pairwise disjoint rays in the end ε. Since ϱ has cofinality ω and G has finite adhesion, it follows that all but countably many rays R_i are included in $G[\, \lfloor U \rfloor \,]$ where U is the set of all tops of ϱ. Pick any ray $R_i \subseteq G[\, \lfloor U \rfloor \,]$. Then R_i has a unique T-minimal vertex by Lemma 2.3 (ii), so R_i has a tail R_i' which avoids U. In particular, there is a vertex $s \in T$ that is a successor of a top of ϱ and satisfies $R_i' \subseteq G[\, \lfloor s \rfloor \,]$. But then the finite neighbourhood $N_G(\lfloor s \rfloor)$ separates R_i' from ε, a contradiction. □

We now characterize convergent sequences $\varepsilon_n \to \varepsilon$ for ends of a uniform T-graph in terms of combinatorial behaviour of their corresponding high rays in T.

Lemma 5.3. *Let G be a uniform T-graph. Let ε and ε_n $(n \in \mathbb{N})$ be ends of G, and let $\varrho := \sigma(\varepsilon)$ and $\varrho_n := \sigma(\varepsilon_n)$ be the corresponding high-rays in T. Let $A \subseteq \mathbb{N}$ consist of all numbers n for which $\varrho \subsetneq \varrho_n$, and let $B := \mathbb{N} \setminus A$.*

(i) *We have convergence $\varepsilon_n \to \varepsilon$ in $\Omega(G)$ for $n \in A$ and $n \to \infty$ if and only if A is infinite and for every top t of ϱ there are only finitely many $n \in A$ with $t \in \varrho_n$.*
(ii) *We have convergence $\varepsilon_n \to \varepsilon$ in $\Omega(G)$ for $n \in B$ and $n \to \infty$ if and only if B is infinite and for every successor node $t \in \varrho$ there are only finitely many $n \in B$ with $\varrho \cap \varrho_n \subseteq \lceil \mathring{t} \rceil$.*

Proof. (i) We show the forward implication indirectly. For this, suppose that there is a top t of ϱ such that $t \in \varrho_n$ for infinitely many $n \in A$. Then the finite vertex set $S_t \cup \{t\}$ separates ε from infinitely many ε_n at once, a contradiction.

For the backward implication, let $X \subseteq V(G)$ be any finite vertex set; we have to find a number $N \in \mathbb{N}$ such that ε_n lives in $C(X, \varepsilon)$ for all $n \in A$ with $n \geqslant N$. Let Y be the set of all tops t of ϱ for which $\lfloor t \rfloor$ meets X. By assumption, we find a large enough $N \in \mathbb{N}$ such that all high-rays ϱ_n with $n \in A$ and $n \geqslant N$ avoid Y. We claim that all corresponding ends ε_n live in $C(X, \varepsilon)$. For this, let any $n \in A$ with $n \geqslant N$ be given. Let $t_0' < t_1' < \cdots$ be a cofinal ω-chain in the high-ray ϱ_n such that t_0' is a top of ϱ. Let $t_0 < t_1 < \cdots$ be a cofinal ω-chain in the high-ray ϱ such that X has no vertex on ϱ above or equal to t_0 and t_0 is a neighbour of t_0'. Using that the intervals $[t_k, t_{k+1}]$ and $[t_k', t_{k+1}']$ of the T-graph G are connected for all $k \in \mathbb{N}$ by Lemma 2.3 (iv), we find rays $R \in \varepsilon$ and $R_n \in \varepsilon_n$ which start in t_0 and t_0', respectively, and avoid X. Since $t_0 t_0'$ is an edge of G, it follows that ε_n lives in $C(X, \varepsilon)$.

(ii) We show the forward implication indirectly. For this, suppose that there is a successor node $t \in \varrho$ such that $\varrho \cap \varrho_n \subseteq \lceil \mathring{t} \rceil$ for infinitely many $n \in B$. Then $N_G(\lfloor t \rfloor)$ is a finite vertex set by Lemma 4.2, and it separates infinitely many ε_n from ε simultaneously.

For the backward implication, let any finite vertex set $X \subseteq V(G)$ be given. Let $t \in \varrho$ be any successor such that $X \subseteq V_{\lceil \mathring{t} \rceil}$. By assumption, we find a large enough number $N \in B$ such that $t \in \varrho_n$ for all $n \geqslant N$ (with $n \in B$). Let $t_0 < t_1 < \cdots$ be a cofinal ω-chain in the high-ray ϱ with $t_0 := t$. Since the intervals $[t_n, t_{n+1}] \subseteq \varrho$ are connected in G by Lemma 2.3 (iv), we find a ray $R \in \varepsilon$ in $G - X$ which starts in t. Similarly, we find a ray $R_n \in \varepsilon_n$ in $G - X$ which starts in t, for every $n \geqslant N$. Hence $\varepsilon_n \in \Omega(X, \varepsilon)$ for all $n \geqslant N$. □

Proposition 5.4. *Any two uniform T-graphs on the same order tree T have homeomorphic end spaces.*

Proof. If G and G' are uniform T-graphs on the same order tree T, then Lemma 5.3 translates convergent sequences of ends in $\Omega(G)$ and $\Omega(G')$ into the same combinatorial principles of high rays of T. Since that description only depends on T and not on G and G', it follows that both $f = (\sigma')^{-1} \circ \sigma$ and its inverse f^{-1} are sequentially continuous. Since end spaces are Fréchet-Urysohn, this means that f is a homeomorphism between $\Omega(G)$ and $\Omega(G')$. □

6. High-rays and partition trees that display ends

Inspired by the normal partition trees from [4], in this section we introduce our main structuring tool called 'partition tree', and discuss how it helps in capturing the end space of a graph.

If $\mathcal{V} = (\, V_t \colon t \in T\,)$ is a partition of $V(G)$ whose classes are indexed by the points of an order tree T, then for every subset $T' \subseteq T$ we write $V_{T'} := \bigcup \{\, V_t \mid t \in T' \,\}$. Sometimes, when T' is given by a long formula, we will write $V \restriction T' := V_{T'}$ instead.

Definition 6.1. A *partition tree* of G is a pair $(T, \mathcal{V})$ where T is an order tree and $\mathcal{V} = (\, V_t \colon t \in T\,)$ is a partition of $V(G)$ into connected vertex sets V_t (also called *parts*) such that the following conditions hold:

(PT1) The contraction minor $\dot{G} := G/\mathcal{V}$ (with all arising parallel edges and loops deleted) is a T-graph.
(PT2) $(T, \mathcal{V})$ has *finite adhesion* in that $N_G(V_{\lfloor t \rfloor})$ is finite for all successors $t \in T$.
(PT3) All parts V_t with t not a limit are required to be singletons.

The first condition implies that only connected graphs can have partition trees. The second condition implies by Lemma 4.2 (iii) that the T-graph $\dot{G}$ has finite adhesion, but note that the converse is false: if

$t \in T$ is a successor of a limit ℓ, then (PT2) forbids that the sole vertex in V_t (cf. (PT1)) has infinitely many neighbours in V_ℓ, but in $\dot{G}$ all these neighbours would collapse to a single vertex. However, if G is already a T-graph of finite adhesion, then T defines a partition tree $(T, \mathcal{V})$ of G if we let $V_t := \{t\}$ for all $t \in T$.

Lemma 6.2. *If $(T, \mathcal{V})$ is a partition tree of a graph G, then all branches of T are countable.*

Proof. Since $(T, \mathcal{V})$ has finite adhesion, so does the T-graph $\dot{G}$, and we may apply Corollary 4.6 to $\dot{G}$. □

Recall more generally from the previous section that we already know exactly how the end space of the T-graph $\dot{G}$ looks like. We now investigate the relationship between the ends of G and the ends of $\dot{G}$.

Let G be any graph, and let $(T, \mathcal{V})$ be any partition tree of G. Each $t \in T$ induces a bipartition $\{ V_{\lfloor t \rfloor}, V_{T \setminus \lfloor t \rfloor} \}$ of $V(G)$, hence $\Omega(G) = \partial_\Omega V_{\lfloor t \rfloor} \cup \partial_\Omega V_{T \setminus \lfloor t \rfloor}$ follows for all t. For every end ε of G let us put

$$O(\varepsilon) := \{ t \in T \mid \varepsilon \in \partial_\Omega V_{\lfloor t \rfloor} \setminus \partial_\Omega V_{T \setminus \lfloor t \rfloor} \}.$$

Clearly, $O(\varepsilon)$ is a non-empty down-closed chain in T, and it is countable because all branches of T are countable by Lemma 6.2. If $O(\varepsilon)$ has no maximal element, then $O(\varepsilon)$ is a high-ray of T and we say that ε *corresponds* to this high-ray. Otherwise $O(\varepsilon)$ has a maximal element t. Then we say that ε *lives at* t and *in* the part V_t.

Lemma 6.3. *If $O(\varepsilon)$ has a maximal element t, then t must be a limit such that every ray in ε has a tail in $G[V_{\lfloor t \rfloor}]$ and meets V_t infinitely often.*

Proof. Since t is contained in $O(\varepsilon)$, the end ε does not lie in the closure of $V_{T \setminus \lfloor t \rfloor}$, i.e., every ray in ε has a tail in $G[V_{\lfloor t \rfloor}]$. Let s_i $(i \in I)$ be the successors of t in T. Since no s_i is contained in $O(\varepsilon)$ and $(T, \mathcal{V})$ has finite adhesion, every ray in ε meets each vertex set $V_{\lfloor s_i \rfloor}$ only finitely often. As V_t pairwise separates all vertex sets $V_{\lfloor s_i \rfloor}$ in the subgraph $G[V_{\lfloor t \rfloor}]$ which contains a tail of every ray in ε, it follows that every ray in ε meets V_t infinitely often. In particular, V_t is infinite, so t must be a limit. □

Consider the map $\tau \colon \Omega(G) \to \mathcal{R}(T) \sqcup T$ that takes each end of G to the high-ray or point of T which it corresponds to or lives at, respectively. We say that $(T, \mathcal{V})$ *displays* a set Ψ of ends of G if τ restricts to a bijection $\tau \restriction \Psi \colon \Psi \to \mathcal{R}(T)$ between Ψ and the high-rays of T and maps every end that is not contained in Ψ to some point of T.

Lemma 6.4. *Let $(T, \mathcal{V})$ be a partition tree of G with a high-ray $\varrho \subseteq T$, let $u_0, u_1, \ldots$ be vertices of G, and let $t_0 < t_1 < \cdots$ be a cofinal ω-chain in ϱ such that $u_n \in V_{t_n}$ for all $n \in \mathbb{N}$. Then $G[V \restriction \{t \in \varrho \colon t \geqslant t_0\}]$ contains a comb attached to $\{ u_n \mid n \in \mathbb{N} \}$. Moreover, the spine of this comb belongs to an end of G which corresponds to ϱ.*

Proof. Since $\dot{G}$ is a T-graph, each interval $[t_n, t_{n+1}]$ is connected in it by Lemma 2.3 (iv), so in particular it contains a ray R with $V(R) \subseteq \varrho$ that traverses all t_n in the correct order. Using that the induced subgraphs $G[V_t]$ with $t \in T$ are connected, it is straightforward to construct the desired comb from R. □

Lemma 6.5. *Let $(T, \mathcal{V})$ be a partition tree of G, and let ε be any end of G that corresponds to a high-ray $\varrho \subseteq T$. Then*

$$\bigcap_{t \in \varrho} \partial_\Omega \Big(\bigcup \{ V_s \mid s \in \varrho \text{ and } s \geqslant t \} \Big) = \{\varepsilon\}.$$

In particular, no other end of G corresponds to ϱ.

Proof. First, we show that ε is contained in the intersection on the left side of the equation. For this, let any $t \in \varrho$ be given and put $X := \bigcup \{ V_s \mid s \in \varrho \text{ and } s \geqslant t \}$. Assume for a contradiction that ε is not contained in $\partial_\Omega X$. Then ε contains a ray R which avoids X. Since $t \in \varrho$ implies $\varepsilon \notin \partial_\Omega V_{T \setminus \lfloor t \rfloor}$, a tail (so without loss of generality all) of R avoids V_ϱ. By Lemma 2.3 (iii), the components of $G - V_\varrho$ are of the form $V_{\lfloor r \rfloor}$ for r minimal in $T - \varrho$, and since ε corresponds to ϱ, the component of $G - V_\varrho$ containing R has to be of the form $G[V_{\lfloor r \rfloor}]$ for r a top of ϱ.

This last statement implies $\varepsilon \in \partial_\Omega V_{\lfloor r \rfloor}$. By definition of $O(\varepsilon) = \varrho$, the fact $r \notin \varrho$ implies $\varepsilon \in \partial_\Omega V_{T \setminus \lfloor r \rfloor}$, and so we can extend R to a comb in G which is attached to $V_{T \setminus \lfloor r \rfloor}$. However, since $\dot{G}$ is a T-graph and $(T, \mathcal{V})$ has finite adhesion, it follows that all but finitely many of the paths that have been added to R in order to obtain this comb must pass through X, which gives $\varepsilon \in \partial_\Omega X$, a contradiction.

It remains to show the forward inclusion of the equation. For this, let δ be an element of the intersection on the left. We have just shown that ε is an element of this intersection as well. Pick rays $R \in \varepsilon$ and $S \in \delta$. We show that R and S are equivalent in G. Let $t_0 < t_1 < \cdots$ be a cofinal ω-chain in the high-ray ϱ, and put $X_n := \bigcup \{ V_s \mid s \in \varrho, s \geqslant t_n \}$. We extend both R and S to combs C_R and C_S in G with teeth u_R^n and u_S^n in X_n for each $n \in \mathbb{N}$, respectively. Using that all intervals $[t_n, t_{n+1}]$ of the T-graph $\dot{G}$ are connected by Lemma 2.3 (iv), we find infinitely many pairwise disjoint $\{ u_R^n \mid n \in \mathbb{N} \}$–$\{ u_S^n \mid n \in \mathbb{N} \}$ paths in G, which shows that R and S are equivalent as desired. □

Lemma 6.6. *Every partition tree $(T, \mathcal{V})$ of G displays the ends of G that correspond to the high-rays of T.*

Proof. Every high-ray of T has some ends of G corresponding to it by Lemma 6.4. Every high-ray of T has at most one end of G corresponding to it by Lemma 6.5. □

Lemma 6.7. *Let $(T, \mathcal{V})$ be a partition tree of G and $t \in T$ a limit. Then $N(V_{\lfloor t \rfloor})$ is concentrated in the unique end of G that corresponds to the high-ray $\lceil \mathring{t} \rceil$.*

Proof. Using Lemma 6.6 we may let ε be the unique end of G that corresponds to the high-ray $\lceil \mathring{t} \rceil$. To show that $N(V_{\lfloor t \rfloor})$ is concentrated in ε, it suffices to show that for every infinite set U of neighbours of $V_{\lfloor t \rfloor}$ there exists a comb in G attached to U with its spine in ε. For this, let U be any infinite set of neighbours of $V_{\lfloor t \rfloor}$. Since $\lceil \mathring{t} \rceil$ is a high-ray, it contains a cofinal ω-chain $t_0 < t_1 < \cdots$, and since $(T, \mathcal{V})$ has finite adhesion, by Lemma 4.4 all but finitely many of the vertices of U are contained in $V \restriction \{t' \in T \mid t_n \leqslant t' < t\}$ for each $n \in \mathbb{N}$. Hence we find a cofinal ω-chain $t'_0 < t'_1 < \cdots$ in $\lceil \mathring{t} \rceil$ such that $V_{t'_n}$ contains a vertex u_n of U for all $n \in \mathbb{N}$. Applying Lemma 6.4 to $\{ u_n \mid n \in \mathbb{N} \}$ yields the desired comb. □

Suppose now that T is any order tree and that G is a T-graph of finite adhesion. Then T defines a partition tree $(T, \mathcal{V})$ of G if we let $V_t := \{t\}$ for all $t \in T$. By Lemma 6.3, no end of G can live in a part of this partition tree, so every end of G corresponds to a high-ray of T via τ. Then $(T, \mathcal{V})$ displays all the ends of G by Lemma 6.6. To distinguish this special case from the general case, we denote the bijection $\Omega(G) \to \mathcal{R}(T)$ by σ in this case. This proves Lemma 5.1.

7. Sequentially faithful partition trees that display all ends

For any uniform T-graph G, Lemma 5.3 allows us to fully understand the topology of $\Omega(G)$ through the combinatorial behaviour of high-rays in T. In this section, we generalise this lemma from uniform T-graphs to arbitrary connected graphs: We show that every connected graph G admits a partition tree $(T, \mathcal{V})$ for which we can prove an analogue of Lemma 5.3. The partition trees that we construct will have two additional properties which are necessary to obtain that analogue of Lemma 5.3. The first property is that the partition tree displays all the ends of the underlying graph. The second property, called 'sequentially faithful', is slightly more technical, but we will see in Lemma 7.1 below that this property precisely captures what we need.

Suppose that G is any graph and that $(T, \mathcal{V})$ is a partition tree of G. As in the previous section, let $\tau\colon \Omega(G) \to \mathcal{R}(T) \sqcup T$ denote the map that takes each end of G to the high-ray or point of T which it corresponds to or lives at, respectively. An end ε of G *lives at* $\lfloor t \rfloor \subseteq T$ and *in* $V_{\lfloor t \rfloor}$ if either the point $\tau(\varepsilon)$ is contained in $\lfloor t \rfloor$ or the high-ray $\tau(\varepsilon)$ contains t. We call $(T, \mathcal{V})$ *sequentially faithful at* an end ε of G if ε corresponds to a high-ray of T and for every sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of ends of G that live in the up-closures $\lfloor s_n \rfloor$ of successors s_n of tops of $\tau(\varepsilon)$ we have convergence $\varepsilon_n \to \varepsilon$ in $\Omega(G)$ as soon as for every finite vertex set $X \subseteq V(G)$ there are only finitely many numbers $n \in \mathbb{N}$ with $N_G(V_{\lfloor s_n \rfloor}) = X$. If $(T, \mathcal{V})$ is sequentially faithful at every end of G that corresponds to a high-ray of T, then we call $(T, \mathcal{V})$ *sequentially faithful*.

Indeed, if a connected graph admits a partition tree which both displays all its ends and is sequentially faithful then we can prove an analogue of Lemma 5.3:

Lemma 7.1. *Let G be any connected graph and let $(T, \mathcal{V})$ be a sequentially faithful partition tree of G that displays all the ends of G. Let ε and ε_n $(n \in \mathbb{N})$ be ends of G, and let $\varrho := \tau(\varepsilon)$ and $\varrho_n := \tau(\varepsilon_n)$ be the corresponding high-rays in T. Let $A \subseteq \mathbb{N}$ consist of all numbers n for which $\varrho \subsetneq \varrho_n$, and let $B := \mathbb{N} \setminus A$. For each $n \in A$ we write s_n for the successor in ϱ_n of the top of ϱ in ϱ_n.*

- (i) *We have convergence $\varepsilon_n \to \varepsilon$ in $\Omega(G)$ for $n \in A$ and $n \to \infty$ if and only if A is infinite and for every finite vertex set $X \subseteq V(G)$ there are only finitely many $n \in A$ such that $X = N_G(V_{\lfloor s_n \rfloor})$.*
- (ii) *We have convergence $\varepsilon_n \to \varepsilon$ in $\Omega(G)$ for $n \in B$ and $n \to \infty$ if and only if for every successor node $t \in \varrho$ there are only finitely many $n \in B$ with $\varrho \cap \varrho_n \subseteq \lceil \mathring{t} \rceil$.*

Proof. (i) The forward implication is evident. The backward implication follows from the assumption that $(T, \mathcal{V})$ is sequentially faithful.

(ii) The forward implication holds because $(T, \mathcal{V})$ is a partition tree and as such has finite adhesion.

For the backward implication, we have to prove that $\varepsilon_n \to \varepsilon$ for $n \in B$ and $n \to \infty$. For this, let $X \subseteq V(G)$ be any finite vertex set. Let $t \in \varrho$ be any successor node such that $X \subseteq V_{\lceil \mathring{t} \rceil}$. By assumption, we find a large enough number $N \in \mathbb{N}$ such that $t \in \varrho_n$ for all $n \in B$ with $n \geqslant N$. We claim that $\varepsilon_n \in \Omega(X, \varepsilon)$ for all $n \in B$ with $n \geqslant N$. Indeed, consider any such n, and pick any vertex $v \in V_t$. Lemma 6.4 yields a ray $R_n \in \varepsilon_n$ that is included in $G[\,V \restriction \{t' \in \varrho_n \colon t' \geqslant t\}\,]$ and starts at v. Similarly, Lemma 6.4 yields a ray $R \in \varepsilon$ that is included in $G[\,V \restriction \{t' \in \varrho \colon t' \geqslant t\}\,]$ and starts at v. Then $R_n \cup R$ is a double ray in G which avoids X with one tail in ε_n and another in ε, yielding $\varepsilon_n \in \Omega(X, \varepsilon)$. □

In the remainder of this section, we prove that every connected graph admits a partition tree with these two properties, Theorem 7.3. For the proof we need the following lemma:

Lemma 7.2. *Let G be any connected graph and let C be any connected induced subgraph of G whose neighbourhood is concentrated in an end ε of G. Then there exists a non-empty connected vertex set $U \subseteq V(C)$ that satisfies all of the following conditions:*

- (i) *U is either finite or concentrated in ε.*
- (ii) *Every component of $C - U$ has finite neighbourhood in G.*
- (iii) *Every sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of ends of G, where each ε_n lives in a component D_n of $C - U$ so that the map $\mathbb{N} \ni n \mapsto N_G(D_n)$ is finite-to-one, converges to ε.*

(iv) $N_G(U) \cap N_G(C)$ *is infinite.*

Proof. Let G_C be the graph that is obtained from $G[C \cup N(C)]$ by removing all the edges that run between any two vertices in $N(C)$. By the star-comb lemma (2.1), G_C contains either a star or a comb Z attached to $N(C)$. Let G_C^+ be the supergraph of G_C in which we make $N(C)$ complete. Let ε^+ be the end of G_C^+ containing the rays from the clique induced by $N(C)$, and note that $N(C) \cup Z$ is concentrated in ε^+ in the graph G_C^+. Now apply Theorem 3.2 inside G_C^+ to find a connected envelope U^* for $N(C) \cup Z$ that is concentrated in ε^+ and so that $U := U^* \cap C$ is connected. We verify that U satisfies (i)–(iv):

(i) Suppose that U is infinite. We need to show that for any finite set X of vertices of G, almost all vertices of U are contained in $C(X, \varepsilon)$. By assumption, almost all vertices of $N(C)$ are contained in $C(X, \varepsilon)$, and by increasing X if necessary, we may assume that $C(X, \varepsilon)$ is the only component of $G - X$ containing vertices of $N(C)$. Now since U^* is concentrated in ε^+, it follows that almost all vertices of U^* can be connected to $N(C)$ in $G_C^+ - X$. Then the same is true in $G - X$, and so all these vertices belong to $C(X, \varepsilon)$. Thus, we have shown that almost all vertices of U^* (and hence of U) belong to $C(X, \varepsilon)$, as desired.

(ii) Every component of $C - U$ is also a component of $G_C - U^*$, and so has finite neighbourhood by the definition of an envelope. Note that it does not matter here whether we take the neighbourhood in G or in G_C, because $N_G(C)$ is included in U^*.

(iii) Given a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ as in the statement of (iii), note first that U must be infinite. Hence U is concentrated in ε by (i), and so the claim follows by Lemma 3.3.

(iv) $N_G(U) \cap N_G(C)$ is infinite by the choice of $Z \subseteq U^*$. □

Theorem 7.3. *Every connected graph has a sequentially faithful partition tree that displays all its ends.*

Proof. Let G be any connected graph. We will use the following notation. Suppose that $(T_1, \mathcal{V}_1)$ and $(T_2, \mathcal{V}_2)$ are two partition trees of G and that T_1 has a final level. Furthermore, suppose that $\mathcal{V}_1 = (\, V_t^1 \mid t \in T_1 \,)$ and that $\mathcal{V}_2 = (\, V_t^2 \mid t \in T_2 \,)$. We write $(T_1, \mathcal{V}_1) \leqslant (T_2, \mathcal{V}_2)$ if the following two conditions are met:

- T_2 extends T_1 so that all the points of $T_2 \setminus T_1$ lie above the final level of T_1;
- $V_t^1 = V_t^2$ for all $t \in T_1$ below the final level, and $V_t^1 = V_{\lfloor t \rfloor}^2$ for all $t \in T_1$ in the final level.

We will transfinitely construct a sequence $(\, (T_i, \mathcal{V}_i) \,)_{i \leqslant \kappa}$ of sequentially faithful partition trees $(T_0, \mathcal{V}_0) < (T_1, \mathcal{V}_1) < \cdots$, where κ will be an ordinal $\kappa \leqslant \omega_1$. Each T_i will have a final level F_i of limit height at least i, and each $(T_i, \mathcal{V}_i)$ will display all the ends of G that do not live at points in F_i. In the construction, we shall ensure that each point $C \in F_i$ is a connected induced subgraph of G and V_C^i is equal to the vertex set of this subgraph, i.e., $V_C^i = V(C)$. We will terminate the construction at the first ordinal κ with $T_{\kappa+1} = T_\kappa$. In the end, we will argue that $(T_\kappa, \mathcal{V}_\kappa)$ is the desired partition tree.

To start the construction, we use Lemma 2.2 to let T_0 be the order tree that arises from an inclusionwise maximal normal tree $T_0' \subseteq G$ by declaring each component C of $G - T_0'$ a top in T_0 of the down-closed ω-chain $\lceil N(C) \rceil_{T_0'}$. We let $V_t^0 := \{t\}$ for all $t \in T_0'$ and $V_C^0 := V(C)$ for all components C of $G - T_0'$. Then $(T_0, \mathcal{V}_0)$ is sequentially faithful since T_0 contains no successors of limits. And it displays all the ends of G that do not live in any part V_C^0 by Lemma 6.6.

At a general step $0 < i < \kappa$, suppose we have already constructed $(T_j, \mathcal{V}_j)$ for all $j < i$ such that $(T_j, \mathcal{V}_j) < (T_k, \mathcal{V}_k)$ for all $j < k < i$. Then we put $T := \bigcup \{\, T_j \mid j < i \,\}$. We consider two cases, that i is a successor or a limit.

Case 1. In the first case, i is a successor ordinal $i = j + 1$. Then $T = T_j$ has a final level F_j of limit height. Consider any point $C \in F_j$. The point C is a top of a high-ray ϱ of T, and for the high-ray ϱ there is a unique end ε of G with $\tau(\varepsilon) = \varrho$ by Lemma 6.6. Furthermore, C is a connected subgraph of G whose neighbourhood is concentrated in ε by Lemma 6.7. We apply Lemma 7.2 to $C \subseteq G$ and ε to find a non-empty connected vertex set $U_C \subseteq V(C)$ that satisfies the following conditions:

(i) U_C is either finite or concentrated in ε.
(ii) Every component of $C - U_C$ has finite neighbourhood in G.
(iii) Every sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of ends of G, where each ε_n lives in a component D_n of $C - U$ so that the map $\mathbb{N} \ni n \mapsto N_G(D_n)$ is finite-to-one, converges to ε.
(iv) $N_G(U_C) \cap N_G(C)$ is infinite.

In each component D of $C - U_C$ we use Lemma 2.2 to pick an inclusionwise maximal normal tree $T(C, D)$ rooted in a vertex that sends an edge to U_C. Then the neighbourhood in G of every component K of $D - T(C, D)$ is included in $U_C \cup T(C, D)$, where the proportion $N(K) \cap U_C$ is finite (because $N(D)$ is finite by (ii)) while $N(K) \cap T(C, D)$ is infinite; and in fact the infinitely many neighbours of K in $T(C, D)$ determine a normal ray $R(K)$ of $T(C, D)$.

We obtain the tree T_i from T in two steps, as follows. First, we add for each C in the final level F_j of $T = T_j$ and every component D of $C - U_C$ the order tree defined by $T(C, D)$ directly above the point C so that the root of $T(C, D)$ becomes a successor of C. Second, we add for each C and D every component K of $D - T(C, D)$ as a top of the high-ray $\lceil R(K) \rceil_{T_i}$. The family $\mathcal{V}_i$ is defined as follows.

- For each $t \in T - F_j$ we let $V^i_t := V^j_t$.
- For each $C \in F_j$ we let $V^i_C := U_C$.
- For each $t \in T(C, D)$ we let $V^i_t := \{t\}$.
- For each K in the final level of T^i we let $V^i_K := V(K)$.

Clearly, $(T_j, \mathcal{V}_j) \leqslant (T_i, \mathcal{V}_i)$ if $(T_i, \mathcal{V}_i)$ is a partition tree. We claim that $(T_i, \mathcal{V}_i)$ is a partition tree of G. (PT1) follows with (iv). Condition (ii) ensures that $(T_i, \mathcal{V}_i)$ has finite adhesion (PT2). (PT3) holds by construction.

To see that the partition tree $(T_i, \mathcal{V}_i)$ is sequentially faithful, let any end ε of G be given that corresponds to a high-ray $\tau(\varepsilon)$ of T_i. Here, τ is defined with regard to $(T_i, \mathcal{V}_i)$. If the order-type of the high-ray $\tau(\varepsilon)$ is equal to the height of F_i, then $(T_i, \mathcal{V}_i)$ is sequentially faithful at ε for the trivial reason that all tops of $\tau(\varepsilon)$ lie in the final level F_i and therefore have no successors. Hence we may assume that $\tau(\varepsilon)$ is included in $T_j \subseteq T_i$. If the order-type of $\tau(\varepsilon)$ is less than the height of F_j, then all successors of the tops of $\tau(\varepsilon) \subseteq T_i$ are present in T_j. Hence $(T_i, \mathcal{V}_i)$ is sequentially faithful at ε because $(T_j, \mathcal{V}_j) \leqslant (T_i, \mathcal{V}_i)$ is. Otherwise, the order-type of $\tau(\varepsilon)$ is equal to the height of F_j. Then the successors of the tops of $\tau(\varepsilon) \subseteq T_i$ are missing in T_j. To see that $(T_i, \mathcal{V}_i)$ is sequentially faithful at ε, let any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of ends of G be given that live in the up-closures $\lfloor s_n \rfloor$ of successors of tops of $\tau(\varepsilon)$ such that the map $\mathbb{N} \ni n \mapsto N_G(V_{\lfloor s_n \rfloor})$ is finite-to-one. By construction, each successor s_n is the root of a normal tree $T(C_n, D_n)$. Let $X \subseteq V(G)$ be any finite vertex set. We have to find a number $N \in \mathbb{N}$ such that all ends ε_n with $n \geqslant N$ live in the component $C(X, \varepsilon)$. Only the vertex sets of finitely many tops C of $\tau(\varepsilon)$ meet X, all others must be included in $C(X, \varepsilon)$. This partitions the successors s_n, and hence the sequence ε_n, into finitely many subsequences: the first subsequence is formed by all ends ε_n that already live in $C(X, \varepsilon)$, and the other subsequences are formed by all ends ε_n that live in C for a common top C of $\tau(\varepsilon)$. Applying (iii) to all these subsequences except the first yields N.

To show that $(T_i, \mathcal{V}_i)$ displays all the ends of G that do not live at points in F_i, it suffices by Lemma 6.6 to show that every end ε of G with $\tau(\varepsilon) \in T_i$ lives at some $K \in F_i$ (i.e. satisfies $\tau(\varepsilon) = K$). Here, τ is defined with regard to $(T_i, \mathcal{V}_i)$. Let any end ε of G be given with $\tau(\varepsilon) \in T_i$, and recall that $\tau(\varepsilon)$ must be a limit. Then $\tau(\varepsilon)$ cannot be a point of $T - F_j$, because $(T_j, \mathcal{V}_j)$ displays all the ends of G that do not live at points in F_j. And $\tau(\varepsilon)$ cannot be a point $C \in F_j \subseteq T_i$, because this would imply $\varepsilon \in \partial_\Omega U_C$ (by Lemma 6.3) where by (i) the end ε would then be sent to the high-ray $\lceil \mathring{C} \rceil_{T_i}$ instead of C. Therefore, $\tau(\varepsilon) \in F_i$ is the only possibility, as desired.

Case 2. In the second case, i is a limit, so T has no final level (or else the construction would have terminated for some $j < i$). Then for every $t \in T$ there is a least ordinal $j(t) < i$ for which t is contained in $T_{j(t)}$ but not in the final level $F_{j(t)}$, so $V_t^j = V_t^{j(i)}$ for all $j \in [j(t), i)$. We let $V_t^i := V_t^{j(t)}$ for all $t \in T$. If C is any component of $G - \bigcup_{t\in T} V_t^i$, then for each $j < i$ there is a unique point $C_j \in F_j$ with $C \subseteq C_j$, and $B(C) := \{ C_j \mid j < i \}$ is a branch T. We obtain T_i from T by adding each component C as a top of the branch $B(C)$ of T. Letting $V_C := V(C)$ for all these tops completes the definition of $(T_i, \mathcal{V}_i)$.

To ensure that $(T_i, \mathcal{V}_i)$ is a partition tree, we have to show that each C has cofinal down-neighbourhood in $G/\mathcal{V}_i$. Since $(T_i, \mathcal{V}_i)$ has finite adhesion, it suffices to show that the collection T_C of all points of T whose parts contain neighbours of C is an infinite subset of $B(C)$. The inclusion $T_C \subseteq B(C)$ is immediate from the fact that each $G/\mathcal{V}_j$ with $j < i$ is a T_j-graph in which C_j is a maximal vertex. So assume for a contradiction that T_C is finite and consider any $j < i$ with $T_C \subseteq T_j - F_j$. Then $C = C_j$ since otherwise $C \subsetneq C_j$ has a neighbour in C_j that is contained in no part V_t with $t \in T_C$, contradicting the definition of T_C. But then C_j has finite neighbourhood, contradicting the fact that $C_j \in F_j$ is a vertex at limit height in the T_j-graph $G/\mathcal{V}_j$.

To see that $(T_i, \mathcal{V}_i)$ is sequentially faithful, consider any end ε of G for which $\tau(\varepsilon)$ is a high-ray of T_i. If $\tau(\varepsilon)$ is a high-ray of any T_j with $j < i$, then we are done because $(T_j, \mathcal{V}_j)$ is sequentially faithful at ε by assumption. Otherwise $\tau(\varepsilon)$ is a branch of T with all tops in F_i, so $(T_i, \mathcal{V}_i)$ is sequentially faithful at ε for the trivial reason that these tops have no successors in T_i.

To show that $(T_i, \mathcal{V}_i)$ displays all the ends of G that do not live at points in F_i, it suffices by Lemma 6.6 to show that every end ε of G with $\tau(\varepsilon) \in T_i$ satisfies $\tau(\varepsilon) \in F_i$. Let us assume for a contradiction that G has an end ε such that $\tau(\varepsilon) =: t \in T_i$ is a point below F_i. Then t lies below F_j for $j := j(t) < i$, contradicting our assumption that $(T_j, \mathcal{V}_j)$ displays all the ends of G that do not live at points in F_j.

We terminate the construction at the first ordinal κ with $T_{\kappa+1} = T_\kappa$. Then $\kappa \leqslant \omega_1$ because each $(T_\alpha, \mathcal{V}_\alpha)$ defines a T_α-graph $G/\mathcal{V}_\alpha$ of finite adhesion that has a final level F_α of height at least α, and these T_α-graphs cannot have uncountable branches by Corollary 4.6. By assumption, $(T_\kappa, \mathcal{V}_\kappa)$ is a sequentially faithful partition tree of G that displays all the ends of G which do not live at points in F_κ. We claim that $(T_\kappa, \mathcal{V}_\kappa)$ displays all the ends of G. For this, it suffices to show that no end ε of G lives at a point in F_κ. And indeed, no end of G can live at a point in F_κ, because otherwise the construction would not have terminated. Therefore, $(T_\kappa, \mathcal{V}_\kappa)$ is the desired partition tree. □

8. Building a suitable T-graph

In this section, we prove the main result of this paper:

Theorem 1. *Every end space is homeomorphic to the end space of a special order tree.*

Proof. By Theorem 4.5 and Proposition 5.4 it suffices to show: Every end space is homeomorphic to the end space of a uniform graph G' on some order tree T'.

Let $\Omega(G)$ be any end space and recall that we may assume G to be connected. By Theorem 7.3 we find a sequentially faithful partition tree $(T, \mathcal{V})$ of G that displays all the ends of G. Without loss of generality all non-limits $t \in T$ are named so that $V_t = \{t\}$.

Construction of T' and G'. Intuitively, we obtain T' from T by splitting up limit nodes of T in a careful manner. Formally, we define an order tree T' and an epimorphism $\varphi\colon T' \to T$ as follows. Let $L \subseteq T$ consist of all the limits of T that have at least one successor in T, and for every $\ell \in L$ let the set $S(\ell)$ consist of all the successors of ℓ in T. For a non-limit $t \in T$ we write N_t for the finite neighbourhood of $V_{\lfloor t \rfloor}$ in G. For each limit $\ell \in L$ we put $\mathcal{N}_\ell := \{\, N_t \colon t \in S(\ell)\,\}$. We obtain the order tree T' from T as follows. First, add for each $\ell \in L$ and $X \in \mathcal{N}_\ell$ a new node $v(\ell, X)$ that we declare to be a successor of ℓ and a predecessor of all $t \in S(\ell)$ with $N_t = X$. Then delete L.

We let the epimorphism $\varphi\colon T' \to T$ be the identity on $T \setminus L$ and we let it send each $v(\ell, X)$ to ℓ. Then φ is onto. Further, it is a homomorphism (i.e. $t' < s'$ in T' implies $\varphi(t') < \varphi(s')$ in T) that is non-injective only at limits, i.e. if $t' \neq s'$ but $\varphi(t') = \varphi(s')$, then t' and s' are limits of T' (with $\lceil \mathring{t}' \rceil = \lceil \mathring{s}' \rceil$, by the homomorphism property). In particular, φ defines a bijection $\varrho \mapsto \varphi[\varrho]$ between the high-rays of T' and T, which we denote by Φ.

Finally, let G' be the graph with $V(G') := T'$ and

$$E(G') := \Big\{ tt' \colon t < t' \in T' \;\wedge\; \varphi(t)\varphi(t') \in E(\dot{G}) \Big\}.$$

G' is a uniform T'-graph. First, we verify that G' is a T'-graph. From the definition of $E(G')$ it is clear that the end vertices of any edge in $E(G')$ are comparable in T'. Hence, it remains to show that the set of lower neighbours of any point $t \in T'$ is cofinal in $\lceil \mathring{t} \rceil$. Towards this end, let $t' < t$ in T' be arbitrary. We need to find some $x \in T'$ with $t' \leqslant x < t$ and $xt \in E(G')$. Since φ is a homomorphism, $\varphi(t') < \varphi(t)$. Since $\dot{G}$ is a T-graph, there is $y \in T$ with $\varphi(t') \leqslant y < \varphi(t)$ and $y\varphi(t) \in E(\dot{G})$. Since φ is onto and level-preserving, there is a unique $x \in T'$ with $t' \leqslant x < t$ and $\varphi(x) = y$. Then $xt \in E(G')$ as desired.

Then we verify that the T'-graph G' is uniform. For this, let $t = v(\ell, X)$ be any limit of T'. Recall that X is a finite set of vertices in G, and we write $\dot{X} \subseteq T$ for the finitely many nodes in T whose parts intersect X non-trivially. We claim that $S_t := \varphi^{-1}(\dot{X}) \cap \lceil \mathring{t} \rceil$ is as desired. First of all, since φ is a homomorphism, S_t is finite. Now we argue that any $t' > t$ in T' has all its down-neighbours below t inside S_t. So consider some $x < t < t'$ with $xt' \in E(G')$. Then $\varphi(x) < \varphi(t) < \varphi(t')$ and $\varphi(x)\varphi(t') \in E(\dot{G})$. Now $t' > v(\ell, X)$ in T' implies by construction that $\varphi(x) \in \dot{X}$, and hence $x \in S_t$ as desired.

The end spaces are homeomorphic. Since $(T, \mathcal{V})$ is a sequentially faithful partition tree for G, we have a natural bijection $\tau\colon \Omega(G) \to \mathcal{R}(T)$ from the ends of G to the high-rays of T such that Lemma 7.1 provides a combinatorial description of convergence of a sequence of ends $\varepsilon_n \to \varepsilon$ in terms of their associated high-rays $\tau(\varepsilon_n)$ and $\tau(\varepsilon)$.

Similarly, since G' is a uniform T'-graph, we have a natural bijection $\sigma\colon \Omega(G') \to \mathcal{R}(T')$ from the ends of G' to the high-rays of T' such that Lemma 5.3 provides a combinatorial description of convergence of a sequence of ends $\varepsilon'_n \to \varepsilon'$ in terms of their associated high-rays $\sigma(\varepsilon'_n)$ and $\sigma(\varepsilon')$.

To complete the proof, we show that the bijection $\Phi\colon \mathcal{R}(T') \to \mathcal{R}(T)$ lifts to a homeomorphism

$$f := \tau^{-1} \circ \Phi \circ \sigma\colon \Omega(G') \to \Omega(G)$$

as in the following diagram:

$$\begin{array}{ccc} \Omega(G') \ni \varepsilon' & \xmapsto{\ \sigma\ } & \sigma(\varepsilon') \in \mathcal{R}(T') \\ {\scriptstyle f}\,\big\downarrow & & \big\downarrow\,{\scriptstyle \Phi} \\ \Omega(G) \ni \varepsilon & \xleftarrow[\tau^{-1}]{} & \varphi[\sigma(\varepsilon')] \in \mathcal{R}(T) \end{array}$$

Towards this aim, consider ends ε'_n (for $n \in \mathbb{N}$) and $\varepsilon'_\star$ in $\Omega(G')$, with images $\varepsilon_s := f(\varepsilon'_s)$ in $\Omega(G)$ for $s \in \mathbb{N} \cup \{\star\}$. We show that $\varepsilon'_n \to \varepsilon'_\star$ in $\Omega(G')$ if and only if $\varepsilon_n \to \varepsilon_\star$ in $\Omega(G)$. Write $\varrho'_s := \sigma(\varepsilon'_s)$ $\mathcal{R}(T')$ and $\varrho_s := \tau(\varepsilon_s)$ in $\mathcal{R}(T)$ for $s \in \mathbb{N} \cup \{\star\}$ for the associated high-rays. By definition of f, we have $\varrho_s := \Phi(\varrho'_s)$ for all $s \in \mathbb{N} \cup \{\star\}$. By definition of Φ, we have

$$\{n \in \mathbb{N} \colon \varrho_\star \subsetneq \varrho_n\} = A = \{n \in \mathbb{N} \colon \varrho'_\star \subsetneq \varrho'_n\}.$$

For each $n \in A$ let s_n denote the successor in ϱ_n of the top of ϱ in ϱ_n. Then

$$\begin{aligned} &\varepsilon'_n \to \varepsilon'_\star \in \Omega(G') \text{ as } n \to \infty \text{ for } n \in A \\ \Leftrightarrow\ & |A| = \infty \text{ and for every top } t = v(\ell, X) \text{ of } \varrho'_\star \text{ there are only finitely many } n \in A \text{ with } t \in \varrho'_n \\ \Leftrightarrow\ & |A| = \infty \text{ and for every finite } X \subseteq V(G) \text{ there are only finitely many } n \in A \text{ with } X = N_{s_n} \\ \Leftrightarrow\ & \varepsilon_n \to \varepsilon_\star \in \Omega(G) \text{ as } n \to \infty \text{ for } n \in A, \end{aligned}$$

where the first equivalence is Lemma 5.3 (i) and the third is Lemma 7.1 (i). To see the backward implication of the second equivalence, consider any top $t = v(\ell, X)$ of $\varrho'_\star$. For each ϱ'_n with $t \in \varrho'_n$, we have that (ℓ is the predecessor of s_n and) $N_{s_n} = X$ by definition of Φ. Since there are only finitely many $n \in A$ with $X = N_{s_n}$ by assumption, the latter implies that there are only finitely many $n \in A$ with $t \in \varrho'_n$. To see the forward implication of the second equivalence, consider any finite $X \subseteq V(G)$. For each ϱ_n we have that N_{s_n} meets V_{ℓ_n} where ℓ_n is the top of ϱ in ϱ_n, but N_{s_n} avoids V_ℓ for all other tops ℓ of ϱ, because $\dot{G}$ is a T-graph. Hence $N_{s_n} = X$ for some $n \in A$ implies that only those ϱ_m with $\ell_m = \ell_n$ can possibly satisfy $N_{s_m} = X$. Therefore, we have $N_{s_m} = X$ if and only if $v(\ell_n, X) \in \varrho'_m$, and by assumption there are only finitely many such m as desired.

Similarly, setting $B = \mathbb{N} \setminus A$, we get

$$\begin{aligned} &\varepsilon'_n \to \varepsilon'_\star \in \Omega(G') \text{ as } n \to \infty \text{ for } n \in B \\ \Leftrightarrow\ & |B| = \infty \text{ and for every successor } t \in \varrho'_\star \text{ there are only finitely many } n \in B \text{ with } \varrho'_\star \cap \varrho'_n \subseteq \lceil \mathring{t} \rceil_{T'} \\ \Leftrightarrow\ & |B| = \infty \text{ and for every successor } t \in \varrho_\star \text{ there are only finitely many } n \in B \text{ with } \varrho_\star \cap \varrho_n \subseteq \lceil \mathring{t} \rceil_{T} \\ \Leftrightarrow\ & \varepsilon_n \to \varepsilon_\star \in \Omega(G) \text{ as } n \to \infty \text{ for } n \in B, \end{aligned}$$

where the first equivalence is Lemma 5.3 (ii), the second is evident by the properties of Φ, and the third is Lemma 7.1 (ii). Together, we have $\varepsilon'_n \to \varepsilon'_\star$ in $\Omega(G')$ if and only if $\varepsilon_n \to \varepsilon_\star$ in $\Omega(G)$ as desired. □

9. Applications

9.1. **Nested sets of clopen bipartitions.** Two bipartitions $\{A, B\}$ and $\{C, D\}$ of the same ground set are *nested* if there are choices $X \in \{A, B\}$ and $Y \in \{C, D\}$ with $X \subseteq Y$. The bipartitions of $V(G)$ induced by the fundamental cuts of a spanning tree of G, for example, are (pairwise) nested. A set of bipartitions of the same ground set is *nested* if its elements are pairwise nested. A bipartition $\{A, B\}$ *distinguishes* two points p and q if $p \in A$ and $q \in B$ or vice versa. A set S of bipartitions *distinguishes* a set P of points if every pair of two points in P is distinguished by a bipartition in S. The following theorem is a direct corollary of a deep result by Carmesin [9, 5.17] which he proved on thirty pages.

Theorem 9.1. *Every end space admits a nested set of clopen bipartitions that distinguishes all its ends.*

This theorem follows directly from our main result, Theorem 1, resulting in a significantly shorter proof.

Proof. Let Ω be any end space. By Theorem 1 we may assume that $\Omega = \Omega(G)$ for a uniform T-graph G. For every non-limit $t \in T$ we let

$$\Omega_t := \Omega\big(N(\lfloor t \rfloor),\, G[\lfloor t \rfloor]\big).$$

We claim that

$$\big\{\, \{\Omega_t, \Omega \setminus \Omega_t\} \;\big|\; t \in T \text{ is a non-limit} \,\big\}$$

is the desired nested set. As discussed at the end of Section 6, we can obtain a partition tree $(T, \mathcal{V})$ of G from T, plus a canonical bijection $\sigma\colon \Omega \to \mathcal{R}(T)$. Thus, each set Ω_t consists precisely of those ends ε of G whose corresponding high-ray $\sigma(\varepsilon) \subseteq T$ includes $\lceil t \rceil$ as an initial segment. Now to see that the bipartitions $\{\Omega_t, \Omega \setminus \Omega_t\}$ are pairwise nested, note that $t < t'$ implies $\Omega_t \supseteq \Omega_{t'}$, and that incomparability of t and t' implies $\Omega_t \cap \Omega_{t'} = \emptyset$. To see that any two ends ε_1 and ε_2 of G are distinguished by a bipartition $\{\Omega_t, \Omega \setminus \Omega_t\}$, we assume without loss of generality that $\sigma(\varepsilon_1)$ is not included in $\sigma(\varepsilon_2)$, let t' be the least element of $\sigma(\varepsilon_1) \setminus \sigma(\varepsilon_2)$, and consider $\{\Omega_t, \Omega \setminus \Omega_t\}$ for the successor t of t' in the high-ray $\sigma(\varepsilon_1)$. □

9.2. **Discrete expansions.** A *discrete expansion of length* σ of a topological space X is an increasing sequence $(X_i : i < \sigma)$ of non-empty closed subsets of X such that

- $X = \bigcup_{i<\sigma} X_i$,
- X_0 and $X_{i+1} \setminus X_i$ are discrete for all $i + 1 < \sigma$, and
- $X_\ell = \overline{\bigcup_{i<\ell} X_i}$ for all limits $\ell < \sigma$.

It is trivial that every Hausdorff space X has a discrete expansion of length $|X|$, see [43, Remark 7.3]. The following remarkable theorem that end spaces have "short" expansions is a deep result by Polat [43, Theorem 8.4] which he proved on twenty pages. Once again, this theorem is now a direct consequence of our main result, Theorem 1.

Theorem 9.2. *Every end space admits a discrete expansion of length at most* ω_1.

Proof. Let Ω be any end space. By Theorem 1 we may assume that $\Omega = \Omega(G)$ for a uniform T-graph G. For every limit $\ell < \omega_1$ write $\Omega_\ell \subseteq \Omega(G)$ for the set of ends whose high-rays belong to $T^{<\ell}$. Then $\Omega = \bigcup_\ell \Omega_\ell$ is an increasing cover (Corollary 4.6) of closed sets (by finite adhesion of G). To get a discrete development, for every limit $\ell < \omega_1$ and every $n \in \mathbb{N}$ we now define a set $\Omega_{\ell+n}$ with $\Omega_\ell \subseteq \Omega_{\ell+n} \subseteq \Omega_{\ell+\omega}$ as follows: For every component C of $T - T^{\leqslant \ell+n}$ choose, if possible, one end $\varepsilon_C \in \Omega_{\ell+\omega}$ whose highray eventually belongs to C. Let $\Omega_{\ell+n}$ consists of all ends in the previous set $\Omega_{\ell+n-1}$ together with all the chosen ε_C.

Then the increasing sequence $(\Omega_i : i < \omega_1)$ is the desired discrete development. Indeed, all sets are closed by construction, $\Omega_{i+1} \setminus \Omega_i$ is discrete by construction, and $\Omega_\ell = \overline{\bigcup_{i<\ell} \Omega_i}$ for all limits $\ell < \omega_1$. □

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