Edge-connectivity and tree-structure in finite and infinite graphs

Elbracht, Christian; Kurkofka, Jan; Teegen, Maximilian

EDGE-CONNECTIVITY AND TREE-STRUCTURE IN FINITE AND INFINITE GRAPHS

CHRISTIAN ELBRACHT, JAN KURKOFKA, AND MAXIMILIAN TEEGEN

Abstract. We show that every graph admits a canonical tree-like decomposition into its k-edge-connected pieces for all $k \in \mathbb{N} \cup \{\infty\}$ simultaneously.

1. Introduction

Finding a tree-like decomposition of any finite graph into its 'k-vertex-connected pieces', for just one given $k \in \mathbb{N}$ or all $k \in \mathbb{N}$ at once, has been a longstanding quest in graph theory until recently, when it was solved comprehensively by Diestel, Hundertmark and Lemanczyk [20]. One of the complications was that there are many competing notions of what a 'k-vertex-connected piece' of a graph should be. Instead of providing a dozen independent solutions for the dozen different notions of 'k-vertex-connected pieces' that are in use, the solution in [20] deals with all these notions at once. Related results can be found in [5–10, 15–28, 30, 32, 34, 36].

If we consider edge-connectivity instead of vertex-connectivity, however, there does exist a single notion of 'k-edge-connected pieces' that undeniably is the most natural one. Let $k \in \mathbb{N} \cup \{\infty\}$ and let G be any connected graph, possibly infinite. We say that two vertices or ends are $({<}k)$-*inseparable* in G if they cannot be separated in G by fewer than k edges. This defines an equivalence relation on $\hat{V}(G) := V(G) \cup \Omega(G)$ where $\Omega(G)$ denotes the set of ends of G (which is empty if G is finite). Its equivalence classes are the 'k-edge-connected pieces' of G, its *k-edge-blocks*. A subset of $\hat{V}(G)$ is an *edge-block* if it is a k-edge-block for some k. Note that any two edge-blocks are either disjoint or one contains the other. In this paper we find a canonical tree-like decomposition of any connected graph, finite or infinite, into its k-edge-blocks—for all $k \in \mathbb{N} \cup \{\infty\}$ simultaneously. To state our result, we only need a few intuitive definitions.

A subset $X \subseteq \hat{V}(G)$ *lives in* a subgraph $C \subseteq G$ or vertex set $C \subseteq V(G)$ if all the vertices of X lie in C and all the rays of ends in X have tails in C or $G[C]$, respectively. If G is finite, saying that X lives in C simply means that $X \subseteq C$. An edge set $F \subseteq E(G)$ *distinguishes* two edge-blocks of G, not necessarily k-edge-blocks for the same k, if they live in distinct components of $G - F$. It distinguishes them *efficiently* if they are not distinguished by any edge set of smaller size. Note that if F distinguishes two edge-blocks efficiently, then F must be a *bond*, a cut with connected sides. A set B of bonds *distinguishes* some set of edge-blocks of G *efficiently* if every two disjoint edge-blocks in this set are distinguished efficiently by a bond in B. Two cuts F_1, F_2 of G are *nested* if F_1 has a side V_1 and F_2 has a side V_2 such that $V_1 \subseteq V_2$. Note that this is symmetric. The fundamental cuts of a spanning tree, for example, are (pairwise) nested. Our main result reads as follows:

Theorem 1. *Every connected graph G has a nested set of bonds that efficiently distinguishes all the edge-blocks of G.*

The nested sets $N = N(G)$ that we construct, one for every G, have two strong additional properties:

(i) They are canonical in that they are invariant under isomorphisms: if $\phi \colon G \to G'$ is a graph-isomorphism, then $\phi(N(G)) = N(\phi(G))$.

(ii) For every $k \in \mathbb{N}$, the subset $N_k \subseteq N$ formed by the bonds of size less than k is equal to the set of fundamental cuts of a tree-cut decomposition of G that decomposes G into its k-edge-blocks.

Tree-cut decompositions are decompositions of graphs similar to tree-decompositions but based on edge-cuts rather than vertex-separators. They were introduced by Wollan [37], and they are more general than the 'tree-partitions' introduced by Seese [35] and by Halin [31]; see Section 4.

2020 *Mathematics Subject Classification.* 05C40, 05C05, 05C69, 05C70, 05C83, 05C63.

Key words and phrases. edge-connected; tree; edge-block; inseparable; distinguish efficiently; infinite.

The second additional property above is best possible in the sense that N_k cannot be replaced with N: there exists a graph G (see Example 4.4) that has no nested set of cuts which, on the one hand, distinguishes all the edge-blocks of G efficiently, and on the other hand, is the set of fundamental cuts of some tree-cut decomposition. (This is because the 'tree-structure' defined by a nested set of cuts may have limit points, and hence not be representable by a graph-theoretical tree.)

It turns out that the nested sets of bonds which make Theorem 1 true can be characterised in terms of generating bonds (for the definition of *generate* see Section 5):

Theorem 2. *Let G be any connected graph and let M be any nested set of bonds of G. Then the following assertions are equivalent:*

(i) *M efficiently distinguishes all the edge-blocks of G;*
(ii) *For every $k \in \mathbb{N}$, the $\leq k$-sized bonds in M generate all the k-sized cuts of G.*

Nested sets of bonds which are canonical and satisfy assertion (ii) of Theorem 2 have been constructed by Dicks and Dunwoody using their algebraic theory of graph symmetries. This is one of the main results of their monograph [11, II 2.20f]. Since the implication (ii)→(i) of Theorem 2 is straightforward, Theorem 1 can be deduced from their theory, but it is not stated in [11] explicitly. Our Theorem 2 itself, in particular its highly non-trivial forward implication (i)→(ii), does not follow from material in [11]. Since our proofs are purely combinatorial, we can combine Theorem 1 and the forward implication (i)→(ii) of Theorem 2 to obtain a purely combinatorial proof of the main result of Dicks and Dunwoody. Together, our proofs of Theorem 1 and Theorem 2 take just over 7 pages in total.

This paper is organised as follows. In Section 2 we introduce the tools and terminology that we need. In Section 3 we prove our main result, Theorem 1, and we show that we obtain a canonical set N. In Section 4 we relate each N_k to a tree-cut decomposition. In Section 5 we prove Theorem 2. In Section 6 we recall a theorem about spanning trees that distinguish all ∞-edge-blocks.

2. Tools and terminology

We use the graph-theoretic notation of Diestel's book [13]. Throughout this paper, $G = (V, E)$ denotes any connected graph, finite or infinite. When we say ends we mean vertex-ends as usual, not edge-ends. If a subset $X \subseteq \hat{V}(G)$, usually an edge-block, lives in a subgraph $C \subseteq G$ or vertex set $C \subseteq V(G)$, we denote this by $X \sqsubseteq C$ for short. Recall that $X \sqsubseteq C$ defaults to $X \subseteq C$ if G is finite.

The following lemma is well known [13, Exercise 8.12]; we provide a proof for the reader's convenience.

Lemma 2.1. *Every edge of a graph lies in only finitely many bonds of size k of that graph, for any $k \in \mathbb{N}$.*

Proof. Let e be any edge of a graph G, and suppose for a contradiction that e lies in infinitely many distinct bonds $B_0, B_1, \dots$ of size k, say. Let F be an inclusionwise maximal set of edges of G such that F is included in B_n for infinitely many n (all n, without loss of generality). Then $|F| < k$ because the bonds are distinct, and any bond $B_n \supsetneq F$ gives rise to a path P in $G - F$ that links the endvertices of e. Now all the infinitely many bonds B_n must contain an edge of the finite path P. But by the choice of F, each edge of P lies in only finitely many B_n, a contradiction. □

Corollary 2.2. *Let G be any connected graph, $k \in \mathbb{N}$, and let $F_0, F_1, \dots$ be infinitely many distinct bonds of G of size at most k such that each bond F_n has a side A_n with $A_n \subsetneq A_m$ for all $n < m$. Then $\bigcup_{n \in \mathbb{N}} A_n = V$.*

Proof. If $\bigcup_n A_n$ is a proper subset of V, then any A_0–$(V \setminus \bigcup_n A_n)$ path in G admits an edge that lies in infinitely many F_n, contradicting Lemma 2.1. □

2.1. **Cuts, bonds and separations.** The *order* of a cut is its size. A *cut-separation* of a graph G is a bipartition $\{A, B\}$ of the vertex set of G, and it *induces* the cut $E(A, B)$. Then the order of the cut $E(A, B)$ is also the *order* of $\{A, B\}$. Recall that in a connected graph, every cut is induced by a unique cut-separation in this way, to which it *corresponds.* A *bond-separation* of G is a cut-separation that induces a bond of G, a cut with connected sides. We say that a cut-separation *distinguishes* two edge-blocks (*efficiently*) if its corresponding cut does, and we call two cut-separations *nested* if their corresponding cuts are nested. Thus, two cut-separations $\{A, B\}$ and $\{C, D\}$ are nested if one of the four inclusions $A \subseteq C$, $A \subseteq D$, $B \subseteq C$ or $B \subseteq D$ holds.

2.2. **Key tool.** The proof of our main result relies on a result from [24]. To state it, we shall need the following definitions. Let $\mathcal{A}$ be some set and $\sim$ a reflexive and symmetric binary relation on $\mathcal{A}$. We say that two elements a and b of $\mathcal{A}$ are *nested* if $a \sim b$ and two elements of $\mathcal{A}$ which are not nested *cross*. A subset of $\mathcal{A}$ is called nested if its elements are pairwise nested. In our setting, $\mathcal{A}$ will be the set of all the bond-separations of a connected graph G that efficiently distinguish some edge-blocks of G, and $\sim$ will encode 'being nested' for bond-separations.

Given $a, b \in \mathcal{A}$, we call $c \in \mathcal{A}$ a *corner* of a and b if every element of $\mathcal{A}$ which is nested with both a and b is also nested with c. When $a = \{A, B\}$ and $b = \{C, D\}$ are two bond-separations, then c will usually be one of the following four possible corners: either $\{A \cap C, B \cup D\}$, $\{A \cap D, B \cup C\}$, $\{B \cap D, A \cup C\}$ or $\{B \cap C, A \cup D\}$. These are the four possibilities of how a new cut-separation can be built from $\{A, B\}$ and $\{C, D\}$ using just '$\cup$' and '$\cap$'. Note that sometimes an intersection may be empty so some of the four possibilities may not be valid cut-separations; and sometimes a possibility is a cut-separation but not an element of $\mathcal{A}$. We will see in Lemma 3.2 that every possibility that happens to lie in $\mathcal{A}$ is already a corner of $\{A, B\}$ and $\{C, D\}$, provided that $\{A, B\}$ and $\{C, D\}$ cross.

Consider a family $(\, \mathcal{A}_i \mid i \in I\,)$ of non-empty subsets of $\mathcal{A}$ and some function $|\cdot|\colon I \to \mathbb{N}$, where I is a possibly infinite index set. We call $|i|$ the *order* of the elements of $\mathcal{A}_i$. We will consider I to be the collection of all the unordered pairs formed by two disjoint edge-blocks of G, and each $\mathcal{A}_i$ will consist of all the bond-separations of G that efficiently distinguish the two edge-blocks forming the pair i. Then every $\mathcal{A}_i$ will be non-empty because the edge-blocks forming i are disjoint. Our choice for $|i|$ will be the unique natural number that is the order of all the bond-separations in $\mathcal{A}_i$. Note that each of the two edge-blocks forming i will be a k-edge-block for some $k > |i|$.

When we wish to prove Theorem 1 without its additional properties, then it suffices to find a subset $N \subseteq \mathcal{A}$ that meets each $\mathcal{A}_i$ and that is nested. One of the main results of [24] states that we can find N if the setup of the sets $\mathcal{A}_i$ and their order function $|\cdot|$ satisfies a number of properties. The result can be applied even when I is infinite, and moreover it ensures that N is 'canonical' for the given setup. To state the properties and the result, we need one more definition.

The *k-crossing number* of a, for an $a \in \mathcal{A}$ and $k \in \mathbb{N}$, is the number of elements of $\mathcal{A}$ that cross a and lie in some $\mathcal{A}_i$ with $|i| = k$. Note that in our case, every bond-separation of order k can only possibly lie in sets $\mathcal{A}_i$ with $|i| = k$. Thus, the k-crossing number of a bond-separation of arbitrary finite order will be the number of efficiently distinguishing bond-separations of order k crossing it.

We say that the family $(\, \mathcal{A}_i \mid i \in I\,)$ *thinly splinters* if it satisfies the following three properties:

(i) For every $i \in I$ all elements of $\mathcal{A}_i$ have finite k-crossing number for all $k \leq |i|$.
(ii) If $a_i \in \mathcal{A}_i$ and $a_j \in \mathcal{A}_j$ cross with $|i| < |j|$, then $\mathcal{A}_j$ contains some corner of a_i and a_j that is nested with a_i.
(iii) If $a_i \in \mathcal{A}_i$ and $a_j \in \mathcal{A}_j$ cross with $|i| = |j| = k \in \mathbb{N}$, then either $\mathcal{A}_i$ contains a corner of a_i and a_j with strictly lower k-crossing number than a_i, or else $\mathcal{A}_j$ contains a corner of a_i and a_j with strictly lower k-crossing number than a_j.

The following theorem from [24], whose proof takes little more than half a page, will be the key ingredient for our proof of Theorem 1:

Theorem 2.3 ([24, Theorem 1.2]). *If $(\, \mathcal{A}_i \mid i \in I\,)$ thinly splinters with respect to some reflexive symmetric relation $\sim$ on $\mathcal{A} := \bigcup_{i \in I} \mathcal{A}_i$, then there is a set $N = N((\, \mathcal{A}_i \mid i \in I\,)) \subseteq \mathcal{A}$ which meets every $\mathcal{A}_i$ and is nested, i.e., $n_1 \sim n_2$ for all $n_1, n_2 \in N$. Moreover, this set N can be chosen invariant under isomorphisms: if ϕ is an isomorphism between $(\mathcal{A}, \sim)$ and $(\mathcal{A}', \sim')$, then we have $N((\, \phi(\mathcal{A}_i) \mid i \in I\,)) = \phi(N((\, \mathcal{A}_i \mid i \in I\,)))$.*

3. Proof of Theorem 1

Let G be any connected graph, possibly infinite, and consider the set $\mathcal{A}$ with the relation $\sim$ of 'being nested', the family $(\,\mathcal{A}_i \mid i \in I\,)$ and the function $|\cdot|$, all defined with regard to the efficiently distinguishing bond-separations of G like in Section 2.2. Our aim is to employ Theorem 2.3 to deduce Theorem 1. In order to do that, we first have to verify that $(\,\mathcal{A}_i \mid i \in I\,)$ thinly splinters. To this end, we verify all the three properties (i)–(iii) below. The following lemma clearly implies property (i):

Lemma 3.1. *Every finite-order bond-separation of a graph G is crossed by only finitely many bond-separations of G of order at most k, for any given $k \in \mathbb{N}$.*

Proof. Our proof starts with an observation. If two bond-separations $\{A, B\}$ and $\{A', B'\}$ cross, then A' contains a vertex from A and a vertex from B. Let $v \in A' \cap A$ and $w \in A' \cap B$. Since $G[A']$ is connected, there exists a path from v to w in $G[A']$. This path, and thus $G[A']$, must contain an edge from A to B. Similarly, $G[B']$ must contain an edge from A to B.

Now suppose for a contradiction that there are infinitely many bond-separations of order at most a given $k \in \mathbb{N}$, which all cross some finite-order bond-separation $\{A, B\}$. Without loss of generality, all the crossing bond-separations have order k. Using our observation, the pigeon-hole principle and the finite order of $\{A, B\}$, we find two edges $e, f \in E(A, B)$ and infinitely many bond-separations $\{A_0, B_0\}, \{A_1, B_1\}, \ldots$ that all cross $\{A, B\}$ so that $e \in G[A_n]$ and $f \in G[B_n]$ for all $n \in \mathbb{N}$. Let P be a path in G that links an endvertex v of e to an endvertex w of f. Now v is contained in all the A_n and w is contained in all the B_n, thus for every $\{A_n, B_n\}$ there exists an edge of P with one end in A_n and the other in B_n. However, every $\{A_n, B_n\}$ corresponds to a bond of size k of G and, again by the pigeon-hole principle, infinitely many of theses bonds must contain the same edge of P. This contradicts Lemma 2.1. □

Next, to show the second property, we need the following lemma:

Lemma 3.2. *If two cut-separations $\{A_1, B_1\}$ and $\{A_2, B_2\}$ cross, and a third cut-separation $\{X, Y\}$ is nested with both $\{A_1, B_1\}$ and $\{A_2, B_2\}$, then $\{X, Y\}$ is nested with $\{A_1 \cap A_2, B_1 \cup B_2\}$ (provided that this is a cut-separation).*

Proof. As $\{X, Y\}$ is a cut-separation that is nested with $\{A_1, B_1\}$ and $\{A_2, B_2\}$, either X or Y is a subset of B_1 or B_2, in which case it is immediate that $\{X, Y\}$ is nested with $\{A_1 \cap A_2, B_1 \cup B_2\}$ as desired, or, one of X and Y is a subset of A_1 and one of X and Y is a subset of A_2. However, since $A_1 \cup A_2 \neq V(G)$ (as $\{A_1, B_1\}$ and $\{A_2, B_2\}$ cross) it needs to be the case that either $X \subseteq A_1 \cap A_2$ or $Y \subseteq A_1 \cap A_2$, so in either case $\{X, Y\}$ is nested with $\{A_1 \cap A_2, B_1 \cup B_2\}$ as desired. □

Using this lemma, we can now show property (ii):

Lemma 3.3. *If $\{A, B\} \in \mathcal{A}_i$ and $\{C, D\} \in \mathcal{A}_j$ cross with $|i| < |j|$, then $\mathcal{A}_j$ contains some corner of $\{A, B\}$ and $\{C, D\}$ that is nested with $\{A, B\}$.*

Proof. Let us denote the two edge-blocks in j as β and β' so that $\beta \sqsubseteq C$ and $\beta' \sqsubseteq D$. Since the order of $\{A, B\}$ is less than $|j|$, we may assume without loss of generality that $\beta, \beta' \sqsubseteq A$. We claim that either $\{A \cap C, B \cup D\}$ or $\{A \cap D, B \cup C\}$ is the desired corner in $\mathcal{A}_j$, and we refer to them as *corner candidates*. Both are cut-separations that distinguish β and β', and both are nested with $\{A, B\}$. Furthermore, by Lemma 3.2, every cut-separation that is nested with both $\{A, B\}$ and $\{C, D\}$ is also nested with both corner candidates. It remains to show that at least one of the two corner candidates has order at most $|j|$, because then it lies in $\mathcal{A}_j$ as desired.

Let us assume for a contradiction that both corner candidates have order greater than $|j|$. Then the two inequalities

$$|E(A \cap C, B \cup D)| + |E(B \cap D, A \cup C)| \leq |E(A, B)| + |E(C, D)|$$
$$\text{and}\quad |E(A \cap D, B \cup C)| + |E(B \cap C, A \cup D)| \leq |E(A, B)| + |E(C, D)|$$

imply

$$|E(B \cap D, A \cup C)| < |i| \quad \text{and} \quad |E(B \cap C, A \cup D)| < |i|.$$

Recall that the edge-blocks forming the pair i are k-edge-blocks for some values k greater than $|i|$. One of the edge-blocks of the pair i lives in B, and due to the latter two inequalities, this edge-block must live either in $B \cap D$ or in $B \cap C$. But then either $\{B \cap D, A \cup C\}$ or $\{B \cap C, A \cup D\}$ is a cut-separation of order less than $|i|$ that distinguishes the two edge-blocks forming the pair i, contradicting the fact that an order of at least $|i|$ is required for that. □

Finally, to show the third property, we need the following lemma:

Lemma 3.4. *Let $\{A_1, B_1\}$ and $\{A_2, B_2\}$ be crossing cut-separations such that both $\{A_1 \cap A_2, B_1 \cup B_2\}$ and $\{A_1 \cup A_2, B_1 \cap B_2\}$ are cut-separations as well. Then every cut-separation that crosses both $\{A_1 \cap A_2, B_1 \cup B_2\}$ and $\{A_1 \cup A_2, B_1 \cap B_2\}$ must also cross both $\{A_1, B_1\}$ and $\{A_2, B_2\}$.*

Proof. Consider any cut-separation $\{X, Y\}$ that crosses both $\{A_1 \cap A_2, B_1 \cup B_2\}$ and $\{A_1 \cup A_2, B_1 \cap B_2\}$. Since $\{X, Y\}$ crosses $\{A_1 \cap A_2, B_1 \cup B_2\}$, both X and Y contain a vertex from $A_1 \cap A_2$. Since $\{X, Y\}$ crosses $\{A_1 \cup A_2, B_1 \cap B_2\}$, both X and Y contain a vertex from $B_1 \cap B_2$. Hence $\{X, Y\}$ crosses both $\{A_1, B_1\}$ and $\{A_2, B_2\}$. □

Let us now show property (iii):

Lemma 3.5. *If $\{A, B\} \in \mathcal{A}_i$ and $\{C, D\} \in \mathcal{A}_j$ cross with $|i| = |j| = k \in \mathbb{N}$, then either $\mathcal{A}_i$ contains a corner of $\{A, B\}$ and $\{C, D\}$ with strictly lower k-crossing number than $\{A, B\}$, or else $\mathcal{A}_j$ contains a corner of $\{A, B\}$ and $\{C, D\}$ with strictly lower k-crossing number than $\{C, D\}$.*

Proof. Let us assume without loss of generality that the k-crossing number of $\{A, B\}$ is less than or equal to the k-crossing number of $\{C, D\}$, and let us denote the edge-blocks in j as β and β' so that $\beta \sqsubseteq C$ and $\beta' \sqsubseteq D$. We consider two cases.

In the first case, $\{A, B\}$ distinguishes the two edge-blocks β and β'. Hence $\beta \sqsubseteq A \cap C$ and $\beta' \sqsubseteq B \cap D$, say. Then both $\{A \cap C, B \cup D\}$ and $\{B \cap D, A \cup C\}$ distinguish the two edge-blocks β and β' that form the pair j, and so they have order at least $|j| = k$. Furthermore, we have

$$|E(A \cap C, B \cup D)| + |E(B \cap D, A \cup C)| \leq |E(A, B)| + |E(C, D)| = 2k, \tag{1}$$

so both $\{A \cap C, B \cup D\}$ and $\{B \cap D, A \cup C\}$ must have order exactly k. In particular, both are contained in $\mathcal{A}_j$, and they are corners of $\{A, B\}$ and $\{C, D\}$ by Lemma 3.2. Next, we assert that the k-crossing numbers of $\{A \cap C, B \cup D\}$ and $\{B \cap D, A \cup C\}$ in sum are less than the sum of the k-crossing numbers of $\{A, B\}$ and $\{C, D\}$. Indeed, all the k-crossing numbers involved are finite by property (i), and the two cut-separations $\{A, B\}$ and $\{C, D\}$ cross which allows us to deduce the desired inequality between the sums by Lemmas 3.2 and 3.4, as follows:

- by Lemma 3.2, every $\{X, Y\} \in \mathcal{A}$ of order k that crosses at least one of $\{A \cap C, B \cup D\}$ and $\{B \cap D, A \cup C\}$ must cross at least one of $\{A, B\}$ and $\{C, D\}$; and
- by Lemma 3.4, every $\{X, Y\} \in \mathcal{A}$ of order k that crosses both $\{A \cap C, B \cup D\}$ and $\{B \cap D, A \cup C\}$ must cross both $\{A, B\}$ and $\{C, D\}$.

But then the strict inequality between the sums, plus our initial assumption that the k-crossing number of $\{A, B\}$ is less than or equal to that of $\{C, D\}$, implies that one of $\{A \cap C, B \cup D\}$ and $\{B \cap D, A \cup C\}$ must have a k-crossing number less than the one of $\{C, D\}$, as desired.

In the second case, $\{A, B\}$ does not distinguish the two edge-blocks β and β'. Recall that all the edge-blocks in the two pairs i and j are ℓ-edge-blocks for some values $\ell > k$. Hence $\beta \cup \beta' \sqsubseteq A$, say. Let us denote by β'' the edge-block in i that lives in B. Then either $\beta'' \sqsubseteq B \cap C$ or $\beta'' \sqsubseteq B \cap D$, say $\beta'' \sqsubseteq B \cap D$. In total:

$$\beta \sqsubseteq A \cap C, \;\; \beta' \sqsubseteq A \cap D \text{ and } \beta'' \sqsubseteq B \cap D.$$

Therefore, $\{A\cap C, B\cup D\}$ distinguishes the two edge-blocks β and β' forming the pair j which imposes an order of at least k, and $\{B\cap D, A\cup C\}$ distinguishes the two edge-blocks forming the pair i which imposes an order of at least k as well. Combining these lower bounds with (1) we deduce that both $\{A\cap C, B\cup D\}$ and $\{B\cap D, A\cup C\}$ have order exactly k. In particular, they are contained in $\mathcal{A}_j$ and $\mathcal{A}_i$ respectively, and they are corners of $\{A, B\}$ and $\{C, D\}$ by Lemma 3.2. Repeating the final argument of the first case, we deduce from the strict inequality between the sums of the k-crossing numbers that either $\{A\cap C, B\cup D\} \in \mathcal{A}_j$ has strictly lower k-crossing number than $\{C, D\}$, or else $\{B\cap D, A\cup C\} \in \mathcal{A}_i$ has strictly lower k-crossing number than $\{A, B\}$, completing the proof. □

We can now prove our main result:

Proof of Theorem 1. Let G be any connected graph. By Lemma 3.1, Lemma 3.3 and Lemma 3.5 we may apply Theorem 2.3 to the family $(\,\mathcal{A}_i \mid i \in I\,)$ defined at the beginning of the section. This results in the desired nested set $N(G) \subseteq \mathcal{A}$. To see that it is canonical, note that any isomorphism $\phi\colon G \to G'$ induces an isomorphism between $(\mathcal{A}, \sim)$ and $(\mathcal{A}', \sim')$, where the latter is defined like the former but with regard to G'. Thus, by the 'moreover' part of Theorem 2.3, we indeed obtain that $\phi(N(G)) = N(\phi(G))$. □

4. Nested sets of bonds and tree-cut decompositions

Recall that, given a connected graph G, we denote by $N = N(G)$ the canonical set of nested bonds from Theorem 1 that efficiently distinguishes all the edge-blocks of G. Furthermore, recall that the subset $N_k \subseteq N$ is formed by the bonds in N of order less than k. In this section, we show that:

- For every $k \in \mathbb{N}$, the subset $N_k \subseteq N$ is equal to the set of fundamental cuts of a tree-cut decomposition of G that decomposes G into its k-edge-blocks.

To this end, we first introduce the notion of a tree-cut decomposition. Recall that a *near-partition* of a set M is a family of pairwise disjoint subsets $M_\xi \subseteq M$, possibly empty, such that $\bigcup_\xi M_\xi = M$.

Let G be a graph, T a tree, and let $\mathcal{X} = (X_t)_{t\in T}$ be a family of vertex sets $X_t \subseteq V(G)$ indexed by the nodes t of T. The pair $(T, \mathcal{X})$ is called a *tree-cut decomposition* of G if $\mathcal{X}$ is a near-partition of $V(G)$. The vertex sets X_t are the *parts* or *bags* of the tree-cut decomposition $(T, \mathcal{X})$. When we say that $(T, \mathcal{X})$ *decomposes* G into its k-edge-blocks for a given k, we mean that the non-empty parts of $(T, \mathcal{X})$ are the sets of vertices of the k-edge-blocks of G. In this paper, we require the nodes with non-empty parts to be *dense* in T in that every edge of T lies on a path in T that links up two nodes with non-empty parts.

If $(T, \mathcal{X})$ is a tree-cut decomposition, then every edge t_1t_2 of its *decomposition tree* T induces a cut $E(\,\bigcup_{t\in T_1} X_t\,,\,\bigcup_{t\in T_2} X_t\,)$ of G where T_1 and T_2 are the two components of $T - t_1t_2$ with $t_1 \in T_1$ and $t_2 \in T_2$. Here, the nodes with non-empty parts densely lying in T ensures that both unions are non-empty, which is required of the sides of a cut. We call these induced cuts the *fundamental cuts* of the tree-cut decomposition $(T, \mathcal{X})$. Note that, unlike the fundamental cuts of a spanning tree, the fundamental cuts of a tree-cut decomposition need not be bonds.

It is important that parts of a tree-cut decomposition are allowed to be empty, as the following example demonstrates.

Example 4.1. Let the graph G arise from the disjoint union of three copies G_1, G_2 and G_3 of K^4 by selecting one vertex $v_i \in G_i$ for all $i \in [3]$ and adding all edges v_iv_j $(i \neq j \in [3])$. Then the 3-edge-blocks of G are the three vertex sets $V(G_1)$, $V(G_2)$ and $V(G_3)$. Since $N(G)$ is canonical, we have $N_3(G) = \{\,F_1, F_2, F_3\,\}$ where $F_i := \{\,v_iv_j \mid j \neq i\,\}$. However, we cannot find a tree-cut decomposition $(T, \mathcal{X})$ of G such that, on the one hand, T is a tree on three nodes t_1, t_2, t_3 and $X_{t_i} = V(G_i)$ for all $i \in [3]$, and on the other hand, the fundamental cuts of $(T, \mathcal{X})$ are precisely the bonds in $N_3(G)$: the decomposition tree T would then be a path of length two, and hence would induce two fundamental cuts, but $N_3(G)$ consists of three bonds.

To relate N_k to a tree-cut decomposition, we will use a theorem by Gollin and Kneip. In order to state their theorem, we need to introduce separation systems and S-trees first.

4.1. **Separation systems and S-trees.** Separation systems and S-trees are two fundamental tools in graph minor theory. In this section we briefly introduce the definitions from [12–14] that we need.

A *separation of a set* V is an unordered pair $\{A,B\}$ such that $A\cup B=V$. The ordered pairs (A,B) and (B,A) are its *orientations*. Then the *oriented separations* of V are the orientations of its separations. The map that sends every oriented separation (A,B) to its *inverse* (B,A) is an involution that reverses the partial ordering

$$(A,B)\le(C,D)\;:\Leftrightarrow\; A\subseteq C \text{ and } B\supseteq D$$

since $(A,B)\le(C,D)$ is equivalent to $(D,C)\le(B,A)$.

More generally, a *separation system* is a triple $(\vec{S},\le,{}^*)$ where $(\vec{S},\le)$ is a partially ordered set and ${}^*\colon\vec{S}\to\vec{S}$ is an order-reversing involution. We refer to the elements of $\vec{S}$ as *oriented separations*. If an oriented separation is denoted by $\vec{s}$, then we denote its *inverse* $\vec{s}^{\,*}$ as $\overleftarrow{s}$, and vice versa. That * is *order-reversing* means $\vec{r}\le\vec{s}\Leftrightarrow\overleftarrow{r}\ge\overleftarrow{s}$ for all $\vec{r},\vec{s}\in\vec{S}$.

A *separation* is an unordered pair of the form $\{\vec{s},\overleftarrow{s}\}$, and then denoted by s. Its elements $\vec{s}$ and $\overleftarrow{s}$ are the *orientations* of s. The set of all separations $\{\vec{s},\overleftarrow{s}\}\subseteq\vec{S}$ is denoted by S. When a separation is introduced as s without specifying its elements first, we use $\vec{s}$ and $\overleftarrow{s}$ (arbitrarily) to refer to these elements.

Separations of sets, and their orientations, are an instance of this abstract setup if we identify $\{A,B\}$ with $\{\,(A,B),(B,A)\,\}$. Hence the cut-separations of a graph define a separation system. Here is another example: The set $\vec{E}(T):=\{\,(x,y)\mid xy\in E(T)\,\}$ of all *orientations* (x,y) of the edges $xy=\{x,y\}$ of a tree T forms a separation system with the involution $(x,y)\mapsto(y,x)$ and the natural partial ordering on $\vec{E}(T)$ in which $(x,y)<(u,v)$ if and only if $xy\ne uv$ and the unique $\{x,y\}$–$\{u,v\}$ path in T is $\mathring{x}yTu\mathring{v}=yTu$.

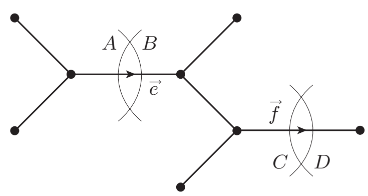

FIGURE 1. An S-tree with $\alpha(\vec{e})=(A,B)\le(C,D)=\alpha(\vec{f})$. [13]

An *S-tree* is a pair (T,α) such that T is a tree and $\alpha\colon\vec{E}(T)\to\vec{S}$ propagates the ordering on $\vec{E}(T)$ and commutes with inversion: that $\alpha(\vec{e})\le\alpha(\vec{f})$ if $\vec{e}\le\vec{f}\in\vec{E}(T)$ and $(\alpha(\overleftarrow{e}))^*=\alpha(\vec{e})$ for all $\vec{e}\in\vec{E}(T)$; see Figure 1. A tree-decomposition $(T,\mathcal{V})$, for example, makes T into an S-tree for the set of separations it induces [13, §12.5]. Similarly, a tree-cut decomposition $(T,\mathcal{X})$ makes T into an S-tree for the set of cut-separations which correspond to its fundamental cuts.

An *isomorphism* between two separation systems is a bijection between their underlying sets that respects both their partial orderings and their involutions. We need the following fragment of [29, Theorem 1] by Gollin and Kneip:

Theorem 4.2. *Let G be any connected graph, and let $\vec{S}$ be any nested separation system formed by oriented cut-separations of G. Then the following assertions are equivalent:*

(i) *There exists an S-tree (T,α) such that $\alpha\colon\vec{E}(T)\to\vec{S}$ is an isomorphism between separation systems;*
(ii) *$\vec{S}$ contains no chain of order-type $\omega+1$.*

4.2. **N_k is a set of fundamental cuts.** The following theorem clearly implies that N_k is the set of fundamental cuts of a tree-cut decomposition of G that decomposes G into its k-edge-blocks:

Theorem 4.3. *Let G be any connected graph and $k\in\mathbb{N}$. Every nested set of bonds of G of order less than k is the set of fundamental cuts of some tree-cut decomposition of G.*

Proof. Let G be any connected graph, $k \in \mathbb{N}$, and let B be any nested set of bonds of G of order less than k. We write S for the set of bond-separations which correspond to the bonds in B.

First, we wish to use Theorem 4.2 to find an S-tree (T, α) such that $\alpha\colon \vec{E}(T) \to \vec{S}$ is an isomorphism. For this, it suffices to show that B cannot contain pairwise distinct bonds $F_0, F_1, \dots, F_\omega$ such that each bond F_α has a side A_α with $A_\alpha \subsetneq A_\beta$ for all $\alpha < \beta \le \omega$. This is immediate from Corollary 2.2.

Second, we wish to find a tree-cut decomposition $(T, \mathcal{X})$ whose fundamental cuts are precisely equal to the bonds in B. We define the parts X_t of $(T, \mathcal{X})$ by letting

$$X_t := \bigcap \{\, D \mid (C, D) = \alpha(x, t) \text{ where } xt \in E(T) \,\}.$$

Then clearly the parts X_t are pairwise disjoint. To see that $\bigcup_t X_t$ includes the whole vertex set of G, consider any vertex $v \in V(G)$. We orient each edge $t_1t_2 \in T$ towards the t_i with $v \in D$ for $(C, D) = \alpha(t_{3-i}, t_i)$. By Corollary 2.2 we may let t be the last node of a maximal directed path in T; then all the edges of T at t are oriented towards t, and $v \in X_t$ follows. Therefore, $\mathcal{X}$ is a near-partition of $V(G)$. It is straightforward to see that B is the set of fundamental cuts of $(T, \mathcal{X})$. □

4.3. ***N* is not a set of fundamental cuts.** Finally, we show that there exists a graph G that has no nested set of cuts which, on the one hand, distinguishes all the edge-blocks of G efficiently, and on the other hand, is the set of fundamental cuts of some tree-cut decomposition.

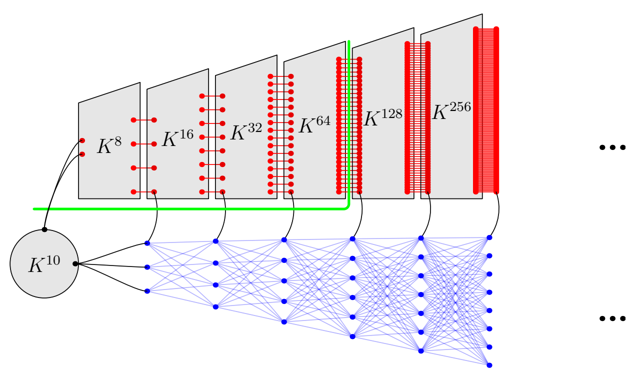

FIGURE 2. The only cut that efficiently distinguishes the two edge-blocks defined by K^{64} and by K^{128} is drawn in green.

Example 4.4. This example is a variation of [24, Example 4.9]. Consider the locally finite graph displayed in Figure 2. This graph G is constructed as follows. For every $n \in \mathbb{N}_{\ge 1}$ we pick a copy of $K^{2^{n+2}}$ together with $n+2$ additional vertices $w^n_1, \dots, w^n_{n+2}$. Then we select 2^n vertices of the $K^{2^{n+2}}$ and call them $u^n_1, \dots, u^n_{2^n}$. Furthermore, we select 2^{n+1} vertices of the $K^{2^{n+2}}$, other than the previously chosen u^n_i, and call them $v^n_1, \dots, v^n_{2^{n+1}}$. Now we add all the red edges $v^n_i u^{n+1}_i$, all the blue edges $w^n_i w^{n+1}_j$, and if $n \ge 2$ we also add the black edge $u^n_1 w^n_1$. Finally, we disjointly add one copy of K^{10} and join one vertex v^0_1 of this K^{10} to u^1_1 and u^1_2; and we select another vertex $w^0_1 \in K^{10}$ distinct from v^0_1 and add all edges $w^0_1 w^1_i$. This completes the construction.

Now the vertex sets of the chosen $K^{2^{n+2}}$ are $(2^{n+2}-1)$-edge-blocks B_n. The only cut-separation that efficiently distinguishes B_n and B_{n+1} is $F_n := \{\bigcup_{k=1}^n B_n\,,\, V \setminus \bigcup_{k=1}^n B_n\,\}$. Additionally, the vertex set of the K^{10} is a 9-edge-block B_0. The only cut-separation that efficiently distinguishes B_0 and B_1 is $F_0 := \{B_0, V \setminus B_0\}$. Therefore, $N(G)$ must contain all the cuts corresponding to the cut-separations F_n $(n \in \mathbb{N})$. But the cut-separations F_n define an $(\omega+1)$-chain

$$(B_1, V \setminus B_1) < (B_1 \cup B_2, V \setminus (B_1 \cup B_2)) < \cdots < (V \setminus B_0, B_0),$$

so $N(G)$ cannot be equal to the set of fundamental cuts of a tree cut-decomposition of G by Theorem 4.2.

5. Generating all bonds

A set S of cut-separations *generates* a cut $\{X,Y\}$ if there exists a finite subset $\{\,\{A_k,B_k\}\mid k<n\,\}\subseteq S$ such that

$$\{X,Y\}=\{\,\bigcup_{k<n}A_k\,,\,\bigcap_{k<n}B_k\,\}.$$

If S generates $\{X,Y\}$, then the cuts corresponding to the cut-separations in S *generate* the cut corresponding to $\{X,Y\}$. Note that S generates $\{X,Y\}$ if and only if both (X,Y) and (Y,X) can be obtained from finitely many oriented cut-separations in $\vec{S}$ by taking suprema and infima, where

- $(A,B)\vee(A',B'):=(A\cup A',B\cap B')$ is the *supremum* and
- $(A,B)\wedge(A',B'):=(A\cap A',B\cup B')$ is the *infimum*

of two cut-separations (A,B) and (A',B'). In this section we prove our second main result:

Theorem 2. *Let G be any connected graph and let M be any nested set of bonds of G. Then the following assertions are equivalent:*

(i) *M efficiently distinguishes all the edge-blocks of G;*
(ii) *For every $k\in\mathbb{N}$, the $\le k$-sized bonds in M generate all the k-sized cuts of G.*

For the proof, we need a generalised version of the star-comb lemma [13, Lemma 8.2.2]. A *comb* in a given graph G means one of the following two substructures of G:

(i) The union of a ray R (the comb's *spine*) with infinitely many disjoint finite paths, possibly trivial, that have precisely their first vertex on R. The last vertices of those paths are the *teeth* of this comb.
(ii) The union of a ray R (the comb's *spine*) with infinitely many disjoint pairwise inequivalent rays $R_0,R_1,\dots$ that have precisely their first vertex on R. The ends to which the rays $R_0,R_1,\dots$ belong are the *teeth* of this comb.

Given a set $U\subseteq V(G)\cup\Omega(G)$, a *comb attached to* U is a comb with all its teeth in U. A *star attached to* U is either a subdivided infinite star with all its leaves in U, or a union of infinitely many rays that meet precisely in their first vertex and belong to distinct ends in U.

Lemma 5.1 (Generalised star-comb lemma)**.** *Let $U\subseteq V(G)\cup\Omega(G)$ be an infinite set for a connected graph G. Then G contains either a comb attached to U or a star attached to U.*

Proof. If U contains infinitely many vertices of G, then we are done by the standard star-comb lemma [13, Lemma 8.2.2]. Hence we may assume that U consists of ends and, say, is countable. Inductively, we choose for each end $\omega\in U$ a ray $R_\omega\in\omega$ so that R_ω is disjoint from all previously chosen rays, ensuring that all chosen rays are pairwise disjoint, and we let U' consist of the first vertices of these rays. Then we consider an inclusionwise minimal tree $T\subseteq G$ that extends all the rays R_ω with $\omega\in U$. Let $T'\subseteq T$ be the inclusionwise minimal subtree that contains U'. Then, by the standard star-comb lemma, T' contains either a star or a comb attached to U', and either extends to a star or comb attached to U. □

For more on stars and combs, see the series [1–4].

Proof of Theorem 2. (ii)→(i) Let M be any nested set of bonds of G such that, for every $k\in\mathbb{N}$, the $\le k$-sized bonds in M generate all the k-sized cuts of G, and suppose for a contradiction that there are two edge-blocks β_1,β_2 which are not efficiently distinguished by any bond in M. Let $\{X,Y\}$ be some bond-separation which efficiently distinguishes β_1 and β_2, and let k be its order. Let $\{\,\{A_\ell,B_\ell\}\mid \ell<n\,\}$ be a finite set of $\le k$-sized bonds in M which generate $\{X,Y\}$ so that $\{X,Y\}=\{\,\bigcup_{\ell<n}A_\ell\,,\,\bigcap_{\ell<n}B_\ell\,\}$. Since M does not efficiently distinguish β_1 from β_2, for every $\ell<n$ we either have that both β_1 and β_2 live in A_ℓ, or that both of them live in B_ℓ. However, this implies that either both β_1 and β_2 live in X, or that both of them live in Y, contradicting the fact that $\{X,Y\}$ distinguishes β_1 and β_2.

(i)→(ii) We assume (i). It suffices to prove (ii) for finite bonds. Let $B = E(V_1, V_2)$ be any bond of G of size k, say. By Theorem 4.3, the set formed by the $\le k$-sized bonds in M is the set of fundamental cuts of a tree-cut decomposition $(T, \mathcal{X})$ of G. Write (T, α) for the S-tree that arises from $(T, \mathcal{X})$.

Since B is finite, only finitely many parts of $(T, \mathcal{X})$ contain endvertices of edges in B. We let H be the minimal subtree of T which contains all the nodes corresponding to these parts. Note that H is finite. Then we let H' be the subtree of T which is induced by the nodes of H and all their neighbours in T. The subtree H' might be infinite, but it is rayless. Let $\mathcal{H}$ be the tree-cut decomposition of G which corresponds to the S-tree $(H', \alpha \restriction \vec{E}(H'))$.

We claim that every two edge-blocks of G that are distinguished by B are also distinguished by some fundamental cut of $\mathcal{H}$. For this, let $\beta_1 \sqsubseteq V_1$ and $\beta_2 \sqsubseteq V_2$ be any two edge-blocks of G that are distinguished by B. Then β_1 and β_2 are also distinguished by a $\le k$-sized bond in M, and hence some fundamental cut of $(T, \mathcal{X})$ distinguishes β_1 and β_2 as well. Let st be an edge of T whose induced fundamental cut distinguishes β_1 and β_2, chosen at minimal distance to H' in T. Then β_1 lives in C and β_2 lives in D for $(C, D) = \alpha(s, t)$, say. We claim that st is also an edge of H', and assume for a contradiction that it is not. Then s, say, is not a vertex of H' and t lies on the s–H' path in T. Since $\{C, D\}$ is an element of M, it is a bond and in particular $G[C]$ is connected. Moreover, C avoids the endvertices of the edges in B, because t separates s from H. Therefore, C is included in one of the two sides of B, say in V_1, so β_1 lives in V_1. The node t, however, cannot lie in H because this would imply $s \in H'$, so t has a neighbour u in T which separates t (and s) from H. Let $(C', D') := \alpha(t, u)$. Since s and u are distinct neighbours of t, we have $(C, D) \le (C', D')$. As argued for (C, D), we find that C' must be included in one of the two sides of B, and this side must be V_1 since C is included in both V_1 and C'. By the choice of st at minimal distance to H', the edge-block β_2 must live in C' (or we could replace st with tu, contradicting the choice of st). But then both β_1 and β_2 live in V_1, the desired contradiction.

We replace $(T, \mathcal{X})$ with $\mathcal{H}$. Then:

> Every two edge-blocks of G that are distinguished by B are also distinguished by some fundamental cut of $(T, \mathcal{X})$. $(*)$

Given a node $t \in T$, we denote by $\hat{X}_t$ the subset of $\hat{V}(G)$ which is the union of all the $(k+1)$-edge-blocks of G that live in D for all cut-separations $(C, D) = \alpha(s, t)$ with $(s, t) \in \vec{E}(T)$. Then $\hat{X}_t \cap V(G) = X_t$ and we call $\hat{X}_t$ the *extended part* of t. Note that extended parts of distinct nodes are disjoint. Since T is rayless, the extended parts near-partition $\hat{V}(G)$. As an immediate consequence of $(*)$, every extended part of $(T, \mathcal{X})$ lives either in V_1 or V_2.

We colour the nodes of T using red and blue, as follows. We colour a node $t \in T$ red if $\hat{X}_t$ is non-empty and $\hat{X}_t \sqsubseteq V_1$. Similarly, we colour a node $t \in T$ blue if $\hat{X}_t$ is non-empty and $\hat{X}_t \sqsubseteq V_2$. Finally, we consider all the nodes $t \in \hat{T}$ with $\hat{X}_t = \emptyset$. These induce a forest in T. We colour all the nodes in a component of this forest red if the component has a red neighbour, and blue otherwise.

We let $T_1 \subseteq T$ be the forest induced by the red nodes, and we let $T_2 \subseteq T$ be the forest induced by the blue nodes. The way in which we coloured the nodes with empty extended parts ensures that, for every connected component C of T_1 or of T_2, some node $t \in C$ has a non-empty extended part $\hat{X}_t$. Note that $B = E(\bigcup_{t \in T_1} X_t, \bigcup_{t \in T_2} X_t)$ by the definition of T_1 and T_2. We claim that we are done if T contains only finitely many T_1–T_2 edges. Indeed, if $s_0 t_0, \dots, s_n t_n$ are the finitely many T_1–T_2 edges with $s_\ell \in T_1$ and $t_\ell \in T_2$, then

$$(V_1, V_2) = \bigwedge_{C\colon \text{ a component of } T_2} \; \bigvee_{\ell\colon\, t_\ell \in C} \alpha(s_\ell, t_\ell) \,.$$

Thus, it remains to show that T contains only finitely many T_1–T_2 edges. For this, we consider the tree $\tilde{T}$ that arises from T by contracting every component of T_1 and every component of T_2 to a single node. Since T is rayless, so is $\tilde{T}$. By Kőnig's lemma, it remains to show that $\tilde{T}$ is locally finite.

Suppose for a contradiction that $d \in \tilde{T}$ is a vertex that has some infinitely many neighbours c_n ($n \in \mathbb{N}$). Recall that all the sets $Y_c := \bigcup \{ \hat{X}_t \mid t \in c \}$ where c is a node of $\tilde{T}$ are non-empty. We choose one point $u_n \in Y_{c_n}$ for every $n \in \mathbb{N}$, and we apply the star-comb lemma in the connected side $G[V_i]$ of B where all sets Y_{c_n} live to the infinite set $U := \{ u_n \mid n \in \mathbb{N} \}$. Then we cannot get a star, because the finite fundamental cuts of $(T, \mathcal{X})$ induced by its T_i–d edges would force the centre vertex to lie in Y_d, contradicting the fact that Y_d lives in V_{3-i}. Therefore, the star-comb lemma must return a comb contained in $G[V_i]$ and attached to U. Without loss of generality, each u_n is a tooth of this comb.

Let us consider the end of G that contains the spine of the comb. This end is contained in a $(k+1)$-edge-block $\beta \sqsubseteq V_i$. And β in turn is included in a set Y_c where c is a component of T_i. Hence $c \neq d$. But then the fundamental cut of $(T, \mathcal{X})$ which corresponds to the T_i–d edge on the c–d path in T separates a tail of the comb from infinitely many u_n, a contradiction. □

6. Finitely separating spanning trees and ∞-edge-blocks

By the second property of our nested set $N(G)$, we find a tree-cut decomposition of any connected graph G into its k-edge-blocks, one for every $k \in \mathbb{N}$. But for $k = \infty$, such a decomposition does not in general exist, e.g., consider Example 4.4 with each K^n of the graph replaced by $K^{\aleph_0}$ (or any other infinitely edge-connected graph). The reason why, however, is not that there are no meaningful tree-cut decompositions of G into its ∞-edge-blocks, but that we considered only those decompositions whose sets of fundamental cuts are equal to $N(G)$. If we drop this requirement, then we find tree-cut decompositions of G into its ∞-edge-blocks, meaningful in the sense that all their fundamental cuts are finite. Let us call a graph *finitely separable* if any two of its vertices can be separated by finitely many edges. And let us call a spanning tree, respectively a tree-cut decomposition, *finitely separating* if all its fundamental cuts are finite. The following theorem has been introduced in [3] as Theorem 3.9, and it is Theorem 5.1 in [33]:

Theorem 6.1 ([3])**.** *Every finitely separable connected graph has a finitely separating spanning tree.*

If G is any connected graph, then the graph $\tilde{G}$ obtained from G by collapsing every ∞-edge-block to a single vertex is finitely separable and connected. Hence $\tilde{G}$ has a finitely separating spanning tree by the theorem, and this tree is easily translated to a finitely separating tree-cut decomposition of G, even with all parts non-empty:

Theorem 6.2 ([33])**.** *Every connected graph has a finitely separating tree-cut decomposition into its ∞-edge-blocks.* □

This result, phrased in terms of S-trees, is extensively used in [33] to study infinite edge-connectivity.

References

[1] C. Bürger and J. Kurkofka, *Duality theorems for stars and combs I: Arbitrary stars and combs* (2020), available at arXiv:2004.00594. To appear in Journal of Graph Theory.

[2] ———, *Duality theorems for stars and combs II: Dominating stars and dominated combs* (2020), available at arXiv:2004.00593.

[3] ———, *Duality theorems for stars and combs III: Undominated combs* (2020), available at arXiv:2004.00592.

[4] ———, *Duality theorems for stars and combs IV: Undominating stars* (2020), available at arXiv:2004.00591.

[5] J. Carmesin, *A short proof that every finite graph has a tree-decomposition displaying its tangles*, European Journal of Combinatorics **58** (2016), 61–65, DOI 10.1016/j.ejc.2016.04.007, available at arXiv:1511.02734. MR3530620

[6] J. Carmesin, R. Diestel, M. Hamann and F. Hundertmark, *Canonical tree-decompositions of finite graphs I. Existence and algorithms*, J. Combin. Theory (Series B) **116** (2016), 1–24, DOI 10.1016/j.jctb.2014.04.001, available at arXiv:1305.4668. MR3425235

[7] ———, *Canonical tree-decompositions of finite graphs II. Essential parts*, J. Combin. Theory (Series B) **118** (2016), 268–283, DOI 10.1016/j.jctb.2014.12.009, available at arXiv:1305.4909. MR3471852

[8] ———, *k-Blocks: A Connectivity Invariant for Graphs*, SIAM Journal on Discrete Mathematics **28** (2014), no. 4, 1876-1891, DOI 10.1137/130923646, available at arXiv:1305.4557.

[9] J. Carmesin, R. Diestel, F. Hundertmark and M. Stein, *Connectivity and tree structure in finite graphs*, Combinatorica **34** (2014), no. 1, 11–46, DOI 10.1007/s00493-014-2898-5, available at arXiv:1105.1611. MR3213840

[10] J. Carmesin, M. Hamann and B. Miraftab, *Canonical trees of tree-decompositions* (2020), available at arXiv:2002.12030.

[11] W. Dicks and M.J. Dunwoody, *Groups Acting on Graphs*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1989. MR1001965

[12] R. Diestel, *Abstract Separation Systems*, Order **35** (2018), no. 1, 157–170, DOI 10.1007/s11083-017-9424-5, available at arXiv:1406.3797v6. MR3774512

[13] ______, *Graph Theory*, 5th, Springer, 2016.

[14] ______, *Tree Sets*, Order **35** (2018), no. 1, 171–192, DOI 10.1007/s11083-017-9425-4, available at arXiv:1512.03781.

[15] R. Diestel, P. Eberenz and J. Erde, *Duality Theorems for Blocks and Tangles in Graphs*, SIAM Journal on Discrete Mathematics **31** (2017), 1514–1528, DOI 10.1137/16M1077763, available at arXiv:1605.09139. MR3670715

[16] R. Diestel, J. Erde and D. Weißauer, *Structural submodularity and tangles in abstract separation systems*, J. Combin. Theory (Series A) **167** (2019), 155–180, DOI 10.1016/j.jcta.2019.05.001, available at arXiv:1805.01439. MR3950083

[17] R. Diestel and S. Oum, *Tangle-tree duality in abstract separation systems*, Advances in Mathematics **377** (2021), 107470, DOI 10.1016/j.aim.2020.107470, available at arXiv:1701.02509.

[18] R. Diestel and S. Oum, *Tangle-Tree Duality: In Graphs, Matroids And Beyond*, Combinatorica **39** (2019), 879—910, DOI 10.1007/s00493-019-3798-5, available at arXiv:1701.02651. MR4015355

[19] R. Diestel and G. Whittle, *Tangles and the Mona Lisa* (2016), available at arXiv:1603.06652.

[20] R. Diestel, F. Hundertmark and S. Lemanczyk, *Profiles of separations: in graphs, matroids, and beyond*, Combinatorica **39** (2019), no. 1, 37–75, DOI 10.1007/s00493-017-3595-y, available at arXiv:1110.6207. MR3936191

[21] M. Dunwoody and B. Krön, *Vertex Cuts*, Journal of Graph Theory **80** (2014), no. 2, 136—171, DOI 10.1002/jgt.21844. MR3385727

[22] C. Elbracht and J. Kneip, *A canonical tree-of-tangles theorem for submodular separation systems* (2020), available at arXiv:2009.02091. To appear in Combinatorial Theory.

[23] C. Elbracht, J. Kneip and M. Teegen, *Trees of tangles in abstract separation systems*, J. Combin. Theory (Series A) **180** (2021), 105425, DOI 10.1016/j.jcta.2021.105425, available at arXiv:1909.09030.

[24] ______, *Trees of tangles in infinite separation systems* (2020), available at arXiv:2005.12122.

[25] A.K. Elm and J. Kurkofka, *A tree-of-tangles theorem for infinite tangles* (2020), available at arXiv:2003.02535.

[26] J. Erde, *Refining a Tree-Decomposition which Distinguishes Tangles*, SIAM Journal on Discrete Mathematics **31** (2017), no. 3, 1529–1551, DOI 10.1137/16M1059539, available at arXiv:1512.02499.

[27] J. Geelen, B. Gerards, N. Robertson and G. Whittle, *Obstructions to branch-decomposition of matroids*, J. Combin. Theory (Series B) **96** (2006), no. 4, 560–570, DOI 10.1016/j.jctb.2005.11.001. MR2232391

[28] J. Geelen, B. Gerards and G. Whittle, *Tangles, tree-decompositions and grids in matroids*, J. Combin. Theory (Series B) **99** (2009), no. 4, 657–667, DOI 10.1016/j.jctb.2007.10.008.

[29] J.P. Gollin and J. Kneip, *Representations of Infinite Tree Sets*, Order **38** (2021), 79–96, DOI 10.1007/s11083-020-09529-0, available at arXiv:1908.10327.

[30] M. Grohe, *Quasi-4-Connected Components*, 43rd International Colloquium on Automata, Languages, and Programming (ICALP 2016), 2016, pp. 8:1–8:13, DOI 10.4230/LIPIcs.ICALP.2016.8, available at arXiv:1602.04505.

[31] R. Halin, *Tree-partitions of infinite graphs*, Disc. Math. **97** (1991), no. 1, 203–217, DOI 10.1016/0012-365X(91)90436-6. MR1140802

[32] V. Hofmaier, *Kantenblöcke und Strukturbäume*, Bachelor's Thesis, Universität Hamburg, 2016.

[33] J. Kurkofka, *Every infinitely edge-connected graph contains the Farey graph or $T_{\aleph_0} * t$ as a minor* (2020), available at arXiv:2004.06710.

[34] N. Robertson and P.D. Seymour, *Graph minors. X. Obstructions to tree-decomposition*, J. Combin. Theory (Series B) **52** (1991), no. 2, 153–190, DOI 10.1016/0095-8956(91)90061-N. MR1110468

[35] D. Seese, *Tree-partite graphs and the complexity of algorithms*, Fundamentals of Computation Theory, 1985, pp. 412–421, DOI 10.1007/BFb0028825.

[36] W. Tutte, *Graph Theory*, Addison-Wesley, 1984.

[37] P. Wollan, *The structure of graphs not admitting a fixed immersion*, J. Combin. Theory (Series B) **110** (2015), 47–66, DOI 10.1016/j.jctb.2014.07.003.

Universität Hamburg, Department of Mathematics, Bundesstrasse 55 (Geomatikum), 20146 Hamburg, Germany
Email address: {christian.elbracht, jan.kurkofka, maximilian.teegen}@uni-hamburg.de