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Duality theorems for stars and combs III: Undominated combs

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Abstract

In a series of four papers we determine structures whose existence is dual, in the sense of complementary, to the existence of stars or combs. Here, in the third paper of the series, we present duality theorems for a combination of stars and combs: undominated combs. We describe their complementary structures in terms of rayless trees and of tree-decompositions. Applications include a complete characterisation, in terms of normal spanning trees, of the graphs whose rays are dominated but which have no rayless spanning tree. Only two such graphs had so far been constructed, by Seymour and Thomas and by Thomassen. As a corollary, we show that graphs with a normal spanning tree have a rayless spanning tree if and only if all their rays are dominated.

KEYWORDS

complementary, dual, duality, infinite graph, normal tree, rayless spanning tree, star–comb lemma, star-decomposition, tree-decomposition, undominated comb, undominated ends

MATHEMATICAL SUBJECT CLASSIFICATION

05C63, 05C40, 05C75, 05C05

1 | INTRODUCTION

Two properties of infinite graphs are *complementary* in a class of infinite graphs if they partition the class. In a series of four papers we determine structures whose existence is complementary to the existence of two substructures that are particularly fundamental to the study of connectedness in infinite graphs: stars and combs. See [3] for a comprehensive introduction, and a brief overview of results, for the entire series of four papers ([2,1,3] and this paper).

In the first paper [3] of this series we found structures whose existence is complementary to the existence of a star or a comb attached to a given set U of vertices, and two types of these structures turned out to be relevant for both stars and combs: normal trees and tree-decompositions. A *comb* is the union of a ray R (the comb's *spine*) with infinitely many disjoint finite paths, possibly trivial, that have precisely their first vertex on R. The last vertices of those paths are the *teeth* of this comb. Given a vertex set U, a *comb attached to* U is a comb with all its teeth in U, and a *star attached to* U is a subdivided infinite star with all its leaves in U. Then the set of teeth is the *attachment set* of the comb, and the set of leaves is the *attachment set* of the star. Given a graph G, a rooted tree $T \subseteq G$ is *normal* in G if the endvertices of every T-path in G are comparable in the tree-order of T, compare [4]. For the definition of tree-decompositions see [4].

As stars and combs can interact with each other, this is not the end of the story. For example, a given vertex set U might be connected in a graph G by both a star and a comb, even with infinitely intersecting sets of leaves and teeth. To formalise this, let us say that a subdivided star S *dominates* a comb C if infinitely many of the leaves of S are also teeth of C. A *dominating star* in a graph G then is a subdivided star $S \subseteq G$ that dominates some comb $C \subseteq G$; and a *dominated comb* in G is a comb $C \subseteq G$ that is dominated by some subdivided star $S \subseteq G$. Thus, a comb $C \subseteq G$ is undominated in G if it is not dominated in G. Recall that a vertex v of G *dominates* a ray $R \subseteq G$ if there is an infinite v–$(R - v)$ fan in G, see [4]. A ray $R \subseteq G$ is *dominated* if some vertex of G dominates it. Rays not dominated by any vertex of G are *undominated*. Dominated combs are related to dominated rays in that a comb is dominated in G if and only if its spine is dominated in G.

In the second paper [2] of our series we determined structures whose existence is complementary to the existence of dominating stars or dominated combs—again in terms of normal trees or tree-decompositions.

Here, in the third paper of the series, we determine structures whose existence is complementary to the existence of undominated combs. A candidate for a normal tree that is complementary to an undominated comb in G attached to a given set U of vertices is a normal tree $T \subseteq G$ that contains U and all whose rays are dominated in G, for if $U = V(G)$ then T is spanning and hence its (dominated) rooted rays are in a natural one-to-one correspondence to the ends of G. Such normal trees T are easily seen to be complementary structures for undominated combs whenever G happens to contain some normal tree that contains U. But in general, normal trees $T \subseteq G$ containing U all whose rays are dominated in G are not complementary to undominated combs, because the absence of an undominated comb does not imply the existence of such a normal tree: for example, if G is an uncountable complete graph and $U = V(G)$, then every normal tree in G containing U must be spanning but G does not have any normal spanning tree.

As our first main result, we show that if U is contained in any normal tree $T \subseteq G$, there is a more elementary structure that is complementary to undominated combs attached to U and which obstructs undominated combs attached to U immediately: a rayless tree containing U.

Call a set $U \subseteq V(G)$ of vertices of a graph G *normally spanned* in G if U is contained in a tree $T \subseteq G$ that is normal in G. The graph G is *normally spanned* if $V(G)$ is normally spanned in G, that is, if G has a normal spanning tree.

Theorem 1. *Let G be any graph and let $U \subseteq V(G)$ be normally spanned in G. Then the following assertions are complementary*:

(i) *G contains an undominated comb attached to U*;
(ii) *there is a rayless tree $T \subseteq G$ that contains U.*

This extends results of Polat [9,10] and Širáň [13], who proved the case $U = V(G)$ for countable G: *A countable connected graph has a rayless spanning tree if and only if all its rays are dominated.*

There are uncountable graphs G for which this duality fails, even for $U = V(G)$. By Theorem 1, such graphs G cannot have a normal spanning tree. There are two known constructions of such graphs, by Seymour and Thomas [12] and by Thomassen [14]. Both these constructions are involved.

As a corollary of Theorem 1 we obtain a full characterisation of the graphs that contain a rayless tree containing a given set U of vertices: they are precisely the graphs G that have a subgraph H in which U is normally spanned and all whose rays are dominated in H. In particular, we obtain the following corollary:

Corollary 2. *Graphs with a normal spanning tree have a rayless spanning tree if and only if all their rays are dominated.*

The graphs with a normal spanning tree are well studied and are quite well known: see [5,6,8].

While it is not always possible to find normal trees or rayless trees that are complementary to undominated combs, it turns out that suitable tree-decompositions still serve as complementary structures:

Theorem 3. *Let G be any connected graph and let $U \subseteq V(G)$ be any vertex set. Then the following assertions are complementary*:

(i) *G contains an undominated comb attached to U*;
(ii) *G has a star-decomposition with finite adhesion sets such that U is contained in the central part and all undominated ends of G live in the leaves' parts.*

Moreover, we may assume that the adhesion sets of the tree-decomposition in (ii) *are connected.*

As discussed above, rayless trees are in general too strong to serve as complementary structures for undominated combs. It turns out that less specific structures than rayless trees, subgraphs all of whose rays are dominated, yield another complementary structure for undominated combs:

Theorem 4. *Let G be any connected graph and let $U \subseteq V(G)$ be any vertex set. Then the following assertions are complementary*:

(i) *G contains an undominated comb attached to U;*
(ii) *G has a connected subgraph that contains U and all whose rays are dominated in it.*

This paper is organised as follows. In Section 2, we prove our duality theorem for undominated combs in terms of rayless trees, Theorem 1. In Section 3, we provide our two full duality theorems for undominated combs: Theorems 3 and 4.

Throughout this paper, $G = (V, E)$ is an arbitrary graph. We use the graph theoretic notation of Diestel's book [4], and we assume familiarity with the tools and terminology described in the first paper of this series [3, Section 2].

2 | UNDOMINATED COMBS AND RAYLESS TREES

In this section, we will consider rayless trees as structures that are complementary to undominated combs. As usual, let G be any connected graph and let $U \subseteq V(G)$ be any vertex set. There are three reasons why rayless trees containing U are good candidates. First, an undominated comb attached to U is more specific than a comb attached to U and in [3, Theorem 1] we proved that rayless normal trees $T \subseteq G$ that contain U are complementary to combs. Therefore, structures that are complementary to undominated combs should be less specific than such normal trees.

Second, by the star–comb lemma, G containing no undominated comb attached to U can be rephrased as follows: for every infinite subset $U' \subseteq U$ the graph G contains a star attached to U'. So combining such stars in a clever way might lead to a rayless tree containing U.

Finally, a graph cannot contain both an undominated comb attached to U and a rayless tree containing U at the same time:

Lemma 2.1 (Bürger and Kurkofka [3, Lemma 2.4])**.** *If U is an infinite set of vertices in a rayless rooted tree T, then T contains a star attached to U which is contained in the up-closure of its central vertex in the tree-order of T.*

For $U = V(G)$, Širáň [13] conjectured that G having a rayless spanning tree is complementary to G containing an undominated comb attached to U. Surprisingly, his conjecture has turned out to be false, as shown by Seymour and Thomas [12]. The counterexample they have found is also a big surprise. Recall that T_κ for a cardinal κ denotes the tree all whose vertices have degree κ.

Theorem 2.2 (Seymour and Thomas [12, Theorem 1.6])**.** *There is an infinitely connected, in particular one-ended, graph G of order $2^{\aleph_0}$ which does not contain a subdivided $K^{\aleph_1}$, such that every spanning tree of G contains a subdivision of $T_{\aleph_1}$.*

Indeed, the end of a graph G as in Theorem 2.2 is dominated as G is infinitely connected, but for $U = V(G)$ the graph does not contain a rayless tree containing U.

A similar counterexample has been obtained independently by Thomassen [14]. Set-theoretic points of view are presented in both [12] and Komjáth's [7]. Komjáth even gives a positive consistency result under Martin's axiom for graphs G with $<2^{\aleph_0}$ many vertices: *If $\kappa < 2^{\aleph_0}$ is a cardinal, $MA(\kappa)$ holds, and G is infinitely connected with $|V(G)| \leq \kappa$, then G has a rayless spanning tree.*

Nevertheless, it is known that requiring G to be countable does suffice to ensure the existence of a rayless spanning tree when G is connected and every end is dominated, giving the following duality:

Theorem 2.3. *Let G be any connected countable graph. Then the following assertions are complementary*:

(i) *G contains an undominated comb attached to $V(G)$*;
(ii) *G has a rayless spanning tree.*

Proofs are due to Polat [9,10] and Širáň [13]. Our main result in this section extends Theorem 2.3:

Theorem 1. *Let G be any graph and let $U \subseteq V(G)$ be normally spanned in G. Then the following assertions are complementary*:

(i) *G contains an undominated comb attached to U*;
(ii) *there is a rayless tree $T \subseteq G$ that contains U.*

Note that this extends Theorem 2.3 twofold: On the one hand, we localise the statement to an arbitrary vertex set $U \subseteq V(G)$. On the other hand, we extend the statement to the class of all graphs in which U is normally spanned.

While our focus in this paper is to find duality theorems for undominated combs, Polat and Širáň were rather interested in a characterisation of those graphs that have rayless spanning trees. The strongest sufficient condition for the existence of a rayless spanning tree, other than Theorem 1 (to the knowledge of the authors), is due to Polat [11]: *If every end of a connected graph G is dominated and G contains no subdivided $T_{\aleph_1}$, then G has a rayless spanning tree*. His result does not imply our Theorem 1, for example, consider G to be the graph obtained from $T_{\aleph_1}$ by completely joining an arbitrarily chosen root to all other nodes, and $U = V(G)$. However, as a corollary of Theorem 1, we obtain a full characterisation of the graphs that have rayless spanning trees. Our characterisation even takes an arbitrary vertex set $U \subseteq V(G)$ into account:

Corollary 2.4. *Let G be any graph. Then the following assertions are equivalent*:

(i) *There is a rayless tree $T \subseteq G$ that contains U*;
(ii) *G has a subgraph H in which $U \subseteq V(H)$ is normally spanned and all whose rays are dominated in H.*

If the graph G itself has a normal spanning tree, then our characterisation simplifies as follows:

Corollary 2. *Graphs with a normal spanning tree have a rayless spanning tree if and only if all their rays are dominated.* □

This section is organised as follows. In Section 2.1 we will prove Theorem 1 for normally spanned graphs. Then, in Section 2.2, we will deduce Theorem 1.

2.1 | Proof for normally spanned graphs

As a first approximation to Theorem 1 we prove the following:

Theorem 2.5. *Let G be any normally spanned graph and let $U \subseteq V(G)$ be any vertex set. Then the following assertions are complementary*:

(i) *G contains an undominated comb attached to U*;
(ii) *G contains a rayless tree that contains U*.

Our proof consists of three key ideas, organised in three lemmas: Lemmas 2.6, 2.7 and 2.9. Recall that a subset X of a poset $P = (P, \leq)$ is *cofinal* in P, and $\leq$, if for every $p \in P$ there is an $x \in X$ with $x \geq p$. We say that a rooted tree $T \subseteq G$ contains a set W *cofinally* if $W \subseteq V(T)$ and W is cofinal in the tree-order of T.

Lemma 2.6 (Bürger and Kurkofka [3, Lemma 2.13])**.** *Let G be any graph. If $T \subseteq G$ is a rooted tree that contains a vertex set W cofinally, then $\partial_\Omega T = \partial_\Omega W$.*

Lemma 2.7. *Let G be any graph and let $U \subseteq V(G)$ be any vertex set. If $\hat{U}$ is the union of U and the set of vertices dominating an end in the closure of U, then $\partial_\Omega \hat{U} = \partial_\Omega U$. In particular, $\partial_\Omega U' = \partial_\Omega U$ for all vertex sets U' with $U \subseteq U' \subseteq \hat{U}$ and $\hat{U}$ contains all the vertices dominating an end in the closure of $\hat{U}$.*

Proof. Every end in the closure of U is contained in the closure of $\hat{U}$ because $\hat{U}$ contains U. For the other inclusion consider any end ω in the closure of $\hat{U}$. Given a finite vertex set $X \subseteq V(G)$ we show that $C(X, \omega)$ contains a vertex from U. Fix a comb attached to $\hat{U}$ and with spine in ω, and pick any tooth v of the comb in the component $C(X, \omega)$ of $G - X$. Then either v is contained in U, or v dominates an end ω' in the closure of U in which case U must meet $C(X, \omega') = C(X, \omega)$. Therefore, $C(X, \omega)$ meets U for all finite vertex sets $X \subseteq V(G)$, and so ω lies in the closure of U. □

For our last key lemma, we shall need the following result of Jung (cf. [3, Theorem 3.5]):

Theorem 2.8 (Jung)**.** *Let G be any graph. A vertex set $W \subseteq V(G)$ is normally spanned in G if and only if it is a countable union of dispersed sets. In particular, G is normally spanned if and only if $V(G)$ is a countable union of dispersed sets.*

Lemma 2.9. *Let G be any graph and let $U \subseteq V(G)$ be normally spanned. If every end in the closure of U is dominated by some vertex in U, then there is a rayless tree $T \subseteq G$ containing U.*

Normal trees follow the concept of depth-first search trees. Speaking informally, all ends of G are 'far away' from the perspective of any fixed vertex. This is why normal spanning trees grow towards the ends of the underlying graph in the sense that they contain (precisely) one normal ray from every end. We, however, seek to avoid having any rays in our tree. This is why our construction of a rayless tree containing U will follow the opposite concept to depth-first search trees, namely that of breadth-first search trees.

Proof of Lemma 2.9. First we choose a well-ordering of U all whose proper initial segments are dispersed: By Theorem 2.8, we have that U is a countable union $\bigcup_{n\in\mathbb{N}} U_n$ of, say pairwise disjoint, dispersed sets U_n. Choose a well-ordering $\preccurlyeq_n$ of every vertex set U_n. Given $u, u' \in U$ with $u \in U_m$ and $u' \in U_n$, we write $u \preccurlyeq u'$ if either $m < n$ or $m = n$ with $u \preccurlyeq_m u'$ holds. It is straightforward to show that $\preccurlyeq$ defines a well-ordering of U that is as desired. From now on we view U as well-ordered set $(U, \preccurlyeq)$.

We recursively construct an ascending sequence $(T_\alpha)_{\alpha<\kappa}$ of rooted trees T_α sharing their root and satisfying that the overall union of the T_α is a rayless tree containing U. Let T_0 be the tree consisting of and rooted in the smallest vertex of U. In a limit step $\beta > 0$ we let T_β be the tree $\bigcup\{T_\alpha | \alpha < \beta\}$. In a successor step $\beta = \alpha + 1$ we terminate and set $\kappa = \beta$ if U is included in T_α. Otherwise we let u be the smallest vertex in $U \setminus V(T_\alpha)$. Following the concept of a breadth-first search tree, among all u–T_α paths fix one P_β whose endvertex in T_α has minimal height in T_α. We obtain T_β from T_α by adding the path P_β.

Let T be the overall union of the trees T_α, that is, $T := \bigcup\{T_\alpha | \alpha < \kappa\}$. Then T is a rooted tree that contains U cofinally. It remains to check that T is rayless. Suppose for a contradiction that R is a ray in T starting in the root, say. By Lemma 2.6 the end of the ray R is contained in the closure of U. As all ends in $\partial_\Omega U$ are dominated by vertices in U, we find a vertex $u^* \in U$ dominating R. Let P_{α^*} be the path from the construction of T that added u^*.

We claim that every tree T_α meets R in a finite initial subpath. This can be seen as follows. Since all proper initial segments of U are dispersed, by Lemma 2.6 it suffices to show that every T_α with $\alpha > 0$ contains a subset of such a segment cofinally. A transfinite induction on α shows that for T_α this subset may be chosen as the set of starting vertices of the paths P_ξ with $\xi \leq \alpha$ a successor ordinal while the proper initial segment may be chosen as the down-closure in U of the starting vertex of $P_{\alpha+1}$. Here we remark that $\alpha + 1 < \kappa$ for all $\alpha < \kappa$ (i.e., κ is a limit ordinal): Indeed, by our assumption that $R \subseteq T$ we know that the vertex set U is not dispersed and, therefore, meets infinitely many U_n.

Finally, we derive the desired contradiction. Fix $\beta > \alpha^*$ so that the endvertex x of $P_{\beta+1}$ in T_β has larger height than u^* has in T_β and so that $P_{\beta+1}$ contains an edge of R. Let u be the first vertex of $P_{\beta+1}$, that is, the smallest vertex in $U \setminus V(T_\beta)$. Note that the first vertex w of $P_{\beta+1}$ that is contained in R is distinct from x. (Also see Figure 1). As u^* dominates R we

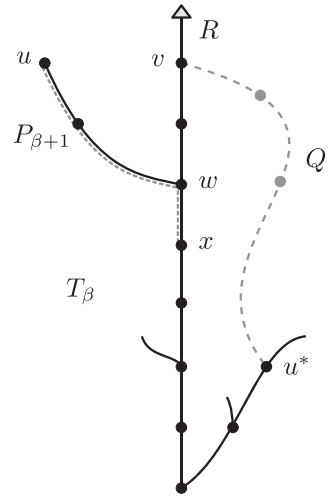

FIGURE 1 The situation in the last paragraph of the proof of Lemma 2.9

find an infinite set $\mathcal{Q}$ of u^*–R paths in G such that distinct paths in $\mathcal{Q}$ only meet in u^*. All but finitely many paths in $\mathcal{Q}$ meet $T_{\beta+1}$ precisely in u^*: Otherwise the end of R is contained in the closure of $T_{\beta+1}$ contradicting that the vertex set of $T_{\beta+1}$ is dispersed. Fix a path $Q \in \mathcal{Q}$ meeting $T_{\beta+1}$ precisely in u^* and having its endvertex v in $\mathring{w}R$. We conclude that $uP_{\beta+1}wRvQu^*$ would have been a better choice than $P_{\beta+1}$ in the construction of $T_{\beta+1}$ (contradiction). □

Proof of Theorem 2.5. By Lemma 2.1 at most one of (i) and (ii) holds at a time. To verify that at least one of (i) and (ii) holds, we show ¬(i)→(ii). By Lemma 2.7 we may assume that U contains all vertices dominating an end in the closure of U, and by Lemma 2.9 there is a rayless tree $T \subseteq G$ that contains U. □

2.2 | Deducing our duality theorem in terms of rayless trees

Let us analyse why the proof of our duality theorem for undominated combs in terms of rayless trees for normally spanned graphs, Theorem 2.5, does not immediately give a proof for arbitrary graphs. For this, consider any graph G and let $U \subseteq V(G)$ be any vertex set. Furthermore, suppose that there is a normal tree $T \subseteq G$ that contains U and that G contains no undominated comb attached to U. In the proof of Theorem 2.5 we assume without loss of generality that U contains all the vertices dominating an end in the closure of U. This is possible because, by Lemma 2.7, adding all the vertices to U that dominate an end in the closure of U does not change the set $\partial_\Omega U$ of ends in the closure of U. However, after adding all these vertices it may happen—in contrast to the situation in the proof of Theorem 2.5 where G has a normal spanning tree—that U is no longer normally spanned in G (e.g., consider any countably infinite set U of vertices in an uncountable complete graph). And U being normally spanned in G is a crucial requirement of the lemma that yields the desired rayless tree, Lemma 2.9.

But maybe adding all the vertices that dominate an end in the closure of U and maintaining that U is normally spanned was too much to ask. Indeed, Lemma 2.9 only requires that U contains for every end $\omega \in \partial_\Omega U$ at least one vertex dominating ω, and adding just one dominating vertex for every end ω might preserve the property of U being normally spanned in G. The following example shows that this is in general false:

Example 2.10. Let G be a *binary tree with tops*, that is, let G be obtained from the rooted infinite binary tree T_2 by adding for every normal ray R of T_2 a new vertex v_R, its *top*, that is joined completely to R (cf. Diestel and Leader's [5]). Let U be the vertex set of T_2. Then $\partial_\Omega U = \Omega(G)$ and every end ω is dominated precisely by the top that was added for the unique normal ray of T_2 that is contained in ω. Hence adding for every end in $\partial_\Omega U$ a vertex dominating it to U results in the whole vertex set of G. However, as pointed out in [5], the graph G does not have a normal spanning tree.

Our way out is to work in a suitable contraction minor, which requires some preparation: Let H and G be any two graphs. We say that H is a contraction minor of G with *fixed branch sets* if an indexed collection of branch sets $\{V_x | x \in V(H)\}$ is fixed to witness that G is an IH. In this case, we write $[v] = [v]_H$ for the branch set V_x containing a vertex v of G and also refer to x by $[v]$. Similarly, we write $[U] = [U]_H := \{[u] | u \in U\}$ for vertex sets $U \subseteq V(G)$.

Lemma 2.11. *Let G be any graph and let H be any contraction minor of G with fixed branch sets that induce subgraphs of G with rayless spanning trees. Furthermore, let $U \subseteq V(G)$ be any vertex set. If H contains a rayless tree that contains $[U]$, then G contains a rayless tree that contains U.*

Proof. Let $T \subseteq H$ be a rayless tree that contains $[U]$. Fix for every branch set $W \in [V(T)]$ a rayless spanning tree T_W in the subgraph that G induces on W. Furthermore, select one edge $e_f \in E_G(t_1, t_2)$ for every edge $f = t_1t_2 \in T$. It is straightforward to show that the union of all the trees T_W plus all the edges e_f is a rayless tree in G that contains U. □

Let H be a contraction minor of a graph G with fixed branch sets. A subgraph $G' = (V', E')$ of G can be passed on to H as follows. Take as vertex set the set $[V']$ and declare W_1W_2 to be an edge whenever E' contains an edge between W_1 and W_2. We write $[G'] = [G']_H$ for the resulting subgraph of H and call it the graph that is obtained by *passing on* G' to H. If every vertex $W \in [V']$ meets V' in precisely one vertex, then we say that G' is *properly passed on* to H. Note that if G' is properly passed on to H, then $[G']$ and G' are isomorphic.

Lemma 2.12. *Let H be a contraction minor of a graph G with fixed branch sets and let $T \subseteq G$ be a tree that is normal in G. If T is properly passed on to H, then $[T] \subseteq H$ is a tree that is normal in H.*

Proof. Since T is properly passed on to G we have that T and $[T]$ are isomorphic as witnessed by the bijection φ that maps every vertex $t \in T$ to $[t]$. To see that $[T]$ is normal in H when it is rooted in $[r]$ for the root r of T, consider any $[T]$-path $W_0 \dots W_k$ in $[H]$. Using that branch sets are connected, it is straightforward to show that there is T-path in G between the two vertices $\varphi^{-1}(W_0)$ and $\varphi^{-1}(W_k)$ of T. Hence W_0 and W_k must be comparable in $[T]$. □

We need two more lemmas for the proof of Theorem 1. Recall that the *generalised up-closure* $\lfloor\!\lfloor x \rfloor\!\rfloor$ of a vertex $x \in T$ is the union of $\lfloor x \rfloor$ with the vertex set of $\bigcup \mathscr{C}(x)$, where the set $\mathscr{C}(x)$ consists of those components of $G - T$ whose neighbourhoods meet $\lfloor x \rfloor$.

Lemma 2.13 (Bürger and Kurkofka [3, Lemma 2.10]). *Let G be any graph and $T \subseteq G$ any normal tree.*

(i) *Any two vertices $x, y \in T$ are separated in G by the vertex set $\lceil x \rceil \cap \lceil y \rceil$.*
(ii) *Let $W \subseteq V(T)$ be down-closed. Then the components of $G - W$ come in two types: the components that avoid T; and the components that meet T, which are spanned by the sets $\lfloor\!\lfloor x \rfloor\!\rfloor$ with x minimal in $T - W$.*

Lemma 2.14 (Bürger and Kurkofka [3, Lemma 2.11]). *If G is any graph and $T \subseteq G$ is any normal tree, then every end of G in the closure of T contains exactly one normal ray of T. Moreover, sending these ends to the normal rays they contain defines a bijection between $\partial_\Omega T$ and the normal rays of T.*

Proof of Theorem 1. Given a normally spanned vertex set $U \subseteq V(G)$ we have to show that the following assertions are complementary:

(i) G contains an undominated comb attached to U;
(ii) G contains a rayless tree that contains U.

By Lemma 2.1 at most one of (i) and (ii) holds at a time. To verify that at least one of (i) and (ii) holds, we show ¬(i)→(ii). For this, we may assume by Lemma 2.6 that U is the vertex set of a normal tree $T \subseteq G$. In the following we will find a contraction minor H of G with fixed branch sets V_x such that:

- all $G[V_x]$ have rayless spanning trees;
- T is properly passed on to H;
- and every end of H in the closure of $[T] \subseteq H$ is dominated in H by some vertex of $[T]$.

Before we prove that such H exists, let us see how to complete the proof once H is found. By Lemma 2.12, the tree $[T]$ is normal in H, and it has vertex set $[U]$ because $V(T) = U$. So, by Lemma 2.9, the graph H contains a rayless tree that contains $[U]$. Finally, by Lemma 2.11, this rayless tree in H containing $[U]$ gives rise to a rayless tree in G containing U as desired.

To construct H, fix for every normal ray R of T a vertex v_R dominating R in G. Let $\mathcal{R}$ be the set of all normal rays R of T for which v_R is contained in a component C_R of $G - T$. Note that the down-closure of the neighbourhood of each C_R is $V(R)$ due to the separation properties of normal trees (Lemma 2.13). Thus, we have $C_R \neq C_{R'}$ for distinct normal rays $R, R' \in \mathcal{R}$. Fix a v_R–R path P_R for every $R \in \mathcal{R}$. Then the overall union of the paths P_R is a forest of subdivided stars, each having its centre on T. Let us refer by S_R to the subdivided star that contains v_R for $R \in \mathcal{R}$, that is, S_R is the union of all the paths $P_{R'}$ that contain the last vertex of P_R and this last vertex is the centre of S_R. Let H be the contraction minor of G with fixed branch sets defined as follows: If v is contained on a path P_R, then put $[v] := S_R$; otherwise let $[v] := \{v\}$. Then, in particular, every branch set of H induces a subgraph of G that has a rayless spanning tree.

As every star S_R meets T precisely in its centre, the tree T is properly passed on to H. By Lemma 2.12, the tree $[T] \subseteq H$ is normal in H and $V([T]) = [U]$ since $V(T) = U$. And by Lemma 2.14 it remains to show that every normal ray of $[T]$ is dominated in H by some vertex of $[T]$. For this, we consider three cases. In all three cases, fix any normal ray $R \subseteq T$ and some collection $\mathcal{P}$ of infinitely many v_R–R paths in G meeting precisely in v_R.

First assume that $R \in \mathcal{R}$. Note that only finitely many of the paths in $\mathcal{P}$ meet $\mathring{v}_R P_R$, without loss of generality none. Then all graphs $[P] \subseteq H$ with $P \in \mathcal{P}$ are $[v_R]$–$[R]$ paths that meet only in $[v_R]$. This shows that $[v_R] \in [T]$ dominates $[R]$ in H.

Second, suppose that $R \notin \mathcal{R}$ and that every branch set of H other than $[v_R]$ meets only finitely many of the paths in $\mathcal{P}$. By thinning out $\mathcal{P}$ we may assume that every branch set other than $[v_R]$ meets at most one of the paths in $\mathcal{P}$. Then the connected graphs $[P]$ with $P \in \mathcal{P}$ pairwise meet in $[v_R]$ but nowhere else and all contain a vertex of $[R]$ other than $[v_R]$. Taking one $[v_R]$–$([R] - [v_R])$ path inside each $[P]$ yields a fan witnessing that $[v_R] \in [T]$ dominates $[R]$ in H.

Finally, suppose that $R \notin \mathcal{R}$ and that some branch set $S \neq [v_R]$ of H meets infinitely many of the paths in $\mathcal{P}$, say all of them. We write c for the centre of S. Without loss of generality none of the paths in $\mathcal{P}$ contains c. Also note that c is contained in $V(R)$ as otherwise all the paths in $\mathcal{P}$ need to pass through the finite down-closure of c in T in vertices other than v_R. Let $\mathcal{R}'$ be the collection of normal rays of T that satisfies

$S = \bigcup\{V(P_{R'}) | R' \in \mathcal{R}'\}$. For every v_R–R path $P \in \mathcal{P}$ let v_P be the last vertex on P that is contained in S, let w_P be the first vertex on P after v_P in which P meets T and let Q_P be the unique w_P–R path in T. (See Figure 2.) For every path $P \in \mathcal{P}$ let $P' = P'(P) := v_P P w_P Q_P$, and let $\mathcal{P}' = \mathcal{P}'(\mathcal{P}) := \{P' | P \in \mathcal{P}\}$.

Each path $P_{R'} \mathring{c} \subseteq S$ with $R' \in \mathcal{R}'$ meets only finitely many paths from $\mathcal{P}'$, and these latter paths are precisely the paths in $\mathcal{P}'$ that meet $C_{R'}$: This is because every path in $\mathcal{P}'$ that meets $C_{R'}$ starts in a vertex $v_P \in C_{R'}$ and after leaving $C_{R'}$ only traverses through vertices of T. Therefore, by replacing $\mathcal{P}$ with an infinite subset of $\mathcal{P}$, we can see to it that every component $C_{R'}$ with $R' \in \mathcal{R}'$ meets at most one of the paths in the then smaller set $\mathcal{P}' = \mathcal{P}'(\mathcal{P})$. In countably many steps we fix paths $P'_1, P'_2, \ldots$ in $\mathcal{P}'$ so that their last vertices are pairwise distinct: To see that this is possible suppose for a contradiction that $t \in R$ is maximal in the tree-order of T so that t is the last vertex of a path in $\mathcal{P}'$. Note that R together with the paths $v_P P$ with $P \in \mathcal{P}$ forms a comb in G. Hence infinitely many of the paths $v_P P$ are contained in the same component of $G - \lceil t \rceil$ as some tail of R. By Lemma 2.13, this component is of the form $\lVert t' \rVert$ for the successor t' of t on R. In particular, we find some $P \in \mathcal{P}$ so that w_P lies above t' in the tree-order of T. But then the endvertex of Q_P in R lies above t' and, in particular, above t, contradicting the choice of t.

So let $P'_1, P'_2, \ldots$ be paths in $\mathcal{P}'$ with pairwise distinct last vertices. We show that the paths P'_i give rise to S–$[R]$ paths $[P'_i]$ in H that form an infinite S–$[R]$ fan witnessing that S dominates $[R]$ in H. Every path P'_i is an S–R path because every path in $\mathcal{P}'$ is an S–R path by the choice of the vertices v_P. Moreover, the paths P'_i are pairwise disjoint: Every path P'_i starts in a component $C_{R'}$. Using the choice of the vertices v_P with $P \in \mathcal{P}$ as the last vertex on P that is contained in S we have that the $[P'_i]$ are S–$[R]$ paths of H that only share their first vertex S. Hence the $[P'_i]$ form an infinite S–R fan in H and we conclude that $S \in [T]$ dominates $[R]$ in H. □

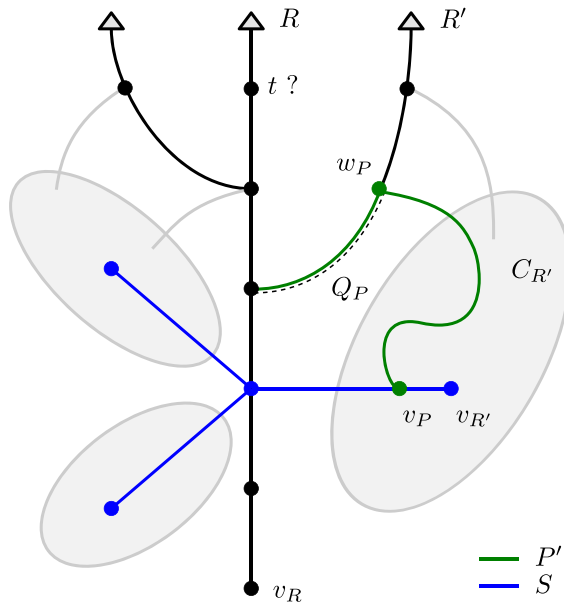

FIGURE 2 The final case in the proof of our duality theorem for undominated combs in terms of rayless trees

3 | DUALITY THEOREMS FOR UNDOMINATED COMBS

In this section we prove our two duality theorems for undominated combs in full generality. The first theorem is phrased in terms of star-decompositions:

Theorem 3. *Let G be any connected graph and let $U \subseteq V(G)$ be any vertex set. Then the following assertions are complementary*:

(i) *G contains an undominated comb attached to U*;
(ii) *G has a star-decomposition with finite separators such that U is contained in the central part and all undominated ends of G live in the leaves' parts.*

Moreover, we may assume that the separators of the tree-decomposition in (ii) are connected.

For the proof we need Carmesin's theorem:

Theorem 3.1 (Carmesin; Bürger and Kurkofka [3, Theorem 2.17])**.** *Every connected graph G has a rooted tree-decomposition with upwards disjoint finite connected separators that displays the undominated ends of G.*

Proof of Theorem 3. Clearly, at most one of (i) and (ii) can hold.

To establish that at least one of (i) and (ii) holds, we show ¬(i)→(ii). By Theorem 3.1 we find a rooted tree-decomposition $(T, \mathcal{V})$ of G with upwards disjoint finite connected separators that display the undominated ends of G. We let $W \subseteq V(T)$ consist of those nodes $t \in T$ whose parts V_t meet U. Then we root T arbitrarily and let T' be the subtree $\lceil W \rceil$ of T. Since U does not have any undominated end of G in its closure, it follows that T' must be rayless. We obtain the star S from T by contracting T' and all of the components of $T - T'$. Then we let (T, α) be the $S_{\aleph_0}$-tree corresponding to $(T, \mathcal{V})$, so $(S, \alpha \restriction \vec{E}(S))$ is an $S_{\aleph_0}$-tree that induces the desired star-decomposition which even satisfies the 'moreover' part. □

The central part of the star-decomposition in Theorem 3(ii) induces a subgraph of G that seems to carry the information that there is no undominated comb attached to U. Our second duality theorem for undominated combs confirms this suspicion:

Theorem 4. *Let G be any connected graph and let $U \subseteq V(G)$ be any vertex set. Then the following assertions are complementary*:

(i) *G contains an undominated comb attached to U*;
(ii) *G has a connected subgraph that contains U and all whose rays are dominated in it.*

Proof. To see that at most one of (i) and (ii) holds, consider any connected subgraph $H \subseteq G$ containing U such that every ray of H is dominated in H. We show that H obstructs the existence of an undominated comb in G attached to U. Assume for a contradiction that such a comb exists. Then the undominated end $\omega \in \Omega(G)$ of that comb's spine lies in the closure of U, and so applying the star–comb lemma in H to the

attachment set $U' \subseteq U$ of that comb must yield another comb attached to U'. But this latter comb is dominated in H by assumption, and at the same time its spine is equivalent in G to the first comb's spine, contradicting that ω is undominated in G.

To establish that at least one of (i) and (ii) holds, we show ¬(i)→(ii). Let $(T, \mathcal{V})$ be the star-decomposition from Theorem 3(ii) also satisfying the 'moreover' part of the theorem. We claim that the graph $H = G[V_c]$ that is induced by the central part V_c of $(T, \mathcal{V})$ is as desired. Clearly, H contains U. And H is connected because the separators of $(T, \mathcal{V})$ are connected. Now if R is any ray in H, it is dominated in G by some vertex $v \in V_c$. This vertex v also dominates R in H because every infinite v–$(R - v)$ fan in G can be greedily turned into an infinite v–$(R - v)$ fan in H by employing the connectedness of the finite separators of the star-decomposition. □

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