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Duality theorems for stars and combs IV: Undominating stars

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Abstract

In a series of four papers we determine structures whose existence is dual, in the sense of complementary, to the existence of stars or combs from the well-known star—comb lemma for infinite graphs. Call a set U of vertices in a graph G *tough* in G if only finitely many components of $G - X$ meet U for every finite vertex set $X \subseteq V(G)$. In this fourth and final paper of the series, we structurally characterise the connected graphs G in which a given vertex set $U \subseteq V(G)$ is tough. Our characterisations are phrased in terms of tree-decompositions, tangle-distinguishing separators and tough subgraphs (a graph G is *tough* if its vertex set is tough in G). From the perspective of stars and combs, we thereby find structures whose existence is complementary to the existence of so-called undominating stars.

KEYWORDS

complementary, critical vertex set, dual, duality, infinite graph, star comb lemma, star decomposition, tough vertex set, tree set, undominating star

1 | INTRODUCTION

Two properties of infinite graphs are *complementary* in a class of infinite graphs if they partition the class. In a series of four papers we determine structures whose existence is complementary to the existence of two substructures that are particularly fundamental to the study of connectedness in infinite graphs: stars and combs. See [2] for a comprehensive introduction, and a brief overview of results, for the entire series of four papers ([2,3,1] and this paper).

A *comb* is the union of a ray R (the comb's *spine*) with infinitely many disjoint finite paths, possibly trivial, that have precisely their first vertex on R. The last vertices of those paths are the *teeth* of this comb. Given a vertex set U, a *comb attached to* U is a comb with all its teeth in U, and a *star attached to* U is a subdivided infinite star with all its leaves in U. Then the set of teeth is the *attachment set* of the comb, and the set of leaves is the *attachment set* of the star. In the first paper [2] of this series we found structures whose existence is complementary to the existence of a star or a comb attached to a given set U of vertices.

As stars and combs can interact with each other, this is not the end of the story. For example, a given vertex set U might be connected in a graph G by both a star and a comb, even with infinitely intersecting sets of leaves and teeth. To formalise this, let us say that a subdivided star S *dominates* a comb C if infinitely many of the leaves of S are also teeth of C. A *dominating star* in a graph G then is a subdivided star $S \subseteq G$ that dominates some comb $C \subseteq G$; and a *dominated comb* in G is a comb $C \subseteq G$ that is dominated by some subdivided star $S \subseteq G$. Thus, a star $S \subseteq G$ is undominating in G if it is not dominating in G; and a comb $C \subseteq G$ is undominated in G if it is not dominated in G. In the second and third paper of the series we determined structures whose existence is complementary to the existence of dominating stars, dominated combs or undominated combs [1,3].

Here, in the fourth and final paper of the series, we determine structures whose existence is complementary to the existence of undominating stars.

As a first step, we will show that the existence of an undominating star attached to a given set U of vertices in a connected graph G is complementary to U being tough in G. Here, U is *tough* in G if only finitely many components of $G - X$ meet U for every finite vertex set $X \subseteq V(G)$. Hence, to find structures whose existence is complementary to undominated stars attached to U, it suffices to structurally characterise the connected graphs G in which $U \subseteq V(G)$ is tough.

In the first two papers of the series, many of the structural characterisations are phrased in terms of normal trees. Given a graph G, a rooted tree $T \subseteq G$ is *normal* in G if the endvertices of every T-path in G are comparable in the tree-order of T, compare [5]. Indeed, we showed that a connected graph G contains no star or no comb attached to a given vertex set $U \subseteq V(G)$ if and only if there is a normal tree $T \subseteq G$ that contains U and satisfies some properties which depend on whether we are considering a star, a comb, or whatever. Normal trees that contain U, however, do not always exist in G when U is tough in G. For instance, if G is an uncountable complete graph and $U = V(G)$, then U is tough in G but G has no normal spanning tree.

Instead of normal trees, we will consider tough subgraphs of G. Call a graph H *tough* if its vertex set is tough in H, that is, if $H - X$ has only finitely many components for every finite vertex set $X \subseteq V(H)$. It is well known that the tough graphs are precisely the graphs that are compactified by their ends [7]. Clearly, if U is contained in a tough subgraph of G, then U is tough in G. We will show that the converse holds as well, and thereby obtain a characterisation.

Tree-decompositions play a role in all other three papers of the series, and here they will also play a role: We show that U being tough in G is characterised by G admitting a particular star-decomposition with U contained in the central part. For the definition of tree-decompositions, we refer to [5].

Our main result reads as follows. Missing definitions follow.

Theorem 1. *Let G be any connected graph and let $U \subseteq V(G)$ be any vertex set. Then the following assertions are equivalent:*

(A1) *U is tough in G;*
(A2) *G contains no undominating star attached to U;*
(A3) *G has no critical vertex set that lies in the closure of U;*
(A4) *there is a tough subgraph $H \subseteq G$ that contains U;*
(A5) *G has a tame star-decomposition such that U is contained in the central part and every critical vertex set of G lives in a leaf's part.*

Moreover, if U is normally spanned in G, then we may add

(A6) *G contains a locally finite normal tree that contains U cofinally.*

Here, a finite vertex set $X \subseteq V(G)$ is *critical* if the collection

$$\breve{\mathscr{C}}_X := \{C \in \mathscr{C}_X \mid N(C) = X\}$$

is infinite, where $\mathscr{C}_X$ is the collection of all components of $G - X$. The collection of all critical vertex sets of G is denoted by $\operatorname{crit}(G)$. A critical vertex set X of G lies *in the closure* of M, where M is either a subgraph of G or a set of vertices of G, if infinitely many components in $\breve{\mathscr{C}}_X$ meet M. Critical vertex sets were introduced in [10]. As tangle-distinguishing separators, they have a surprising background involving the Stone–Čech compactification of G, Robertson and Seymour's tangles from their graph-minor series, and Diestel's tangle compactification, compare [11,12,6].

For the definitions of 'tame' and 'live' in (A5), see Section 4. Tame tree-decompositions have finite adhesion sets.

A vertex set $U \subseteq V(G)$ is *normally spanned* in G if there is a normal tree $T \subseteq G$ which contains U. Recall that a subset X of a poset $P = (P, \leq)$ is *cofinal* in P, and $\leq$, if for every $p \in P$ there is an $x \in X$ with $x \geq p$. We say that a rooted tree $T \subseteq G$ contains a set U of vertices *cofinally* if $U \subseteq V(T)$ and U is cofinal in the tree-order of T.

The proof of Theorem 1 is completely different from all the proofs presented in the other three papers of the series. In fact, a whole new strategy is needed to prove Theorem 1. The starting point of our strategy will be a very recent generalisation [8] of Robertson and Seymour's tree-of-tangles theorem from their graph-minor series [12].

This paper is organised as follows. In Section 2, we prove Theorem 1 except (A4) and (A5). In Section 3, we extend the proof to include (A4). In Section 4, we further extend the proof to include (A5). In Section 5, we summarise the duality theorems of the complete series. Section 6 gives an outlook.

Throughout this paper, $G = (V, E)$ is an arbitrary graph. We use the graph-theoretic notation of Diestel's book [5], and we assume familiarity with the tools and terminology described in the first paper of this series [2, Section 2].

2 | PROOF OF THEOREM 1 EXCEPT (A4) AND (A5)

Proof. ¬(A2)→¬(A3). If G contains an undominating star attached to U, then the attachment set $U' \subseteq U$ of this star has no end of G in its closure. We know by [2, Lemma 2.9] that every infinite set of vertices in a connected graph has an end or a critical vertex set in its closure. Hence, some critical vertex set of G must lie in the closure of $U' \subseteq U$.

¬(A3)→¬(A2). Let $X \subseteq V(G)$ be a critical vertex set of G in the closure of U. Then infinitely many components in $\breve{\mathscr{C}}_X$ meet U. Pick $x \in X$ arbitrarily (note that X is nonempty since G is connected). For each component $C \in \breve{\mathscr{C}}_X$ that meets U, pick an $x-U$ path in $G[x + C]$. Then the union of these paths is a star S attached to U. Denote its attachment set by U'. The finite separator X obstructs the existence of a comb in G attached to U'. Hence S must be undominating.

(A1)↔(A3) is immediate from the pigeonhole principle.

Now assume that U is normally spanned in G.

(A6)→(A1) holds because locally finite graphs are tough.

¬(A6)→¬(A1). Suppose that $T \subseteq G$ is a normal tree which contains U cofinally, and suppose that T has a vertex t of infinite degree. Using that T contains U cofinally, we find infinitely many vertices $u_0, u_1, \ldots$ in U that lie strictly above t and are pairwise incomparable in the tree-order of T. Since T is normal in G, the down-closure $\lceil t \rceil$ of t in T pairwise separates the vertices u_n with $n \in \mathbb{N}$, compare [2, Lemma 2.10]. In particular, infinitely many components of $G - \lceil t \rceil$ meet U, so U is not tough. □

In the remainder of this paper, we extend this proof to include (A4) and (A5). We prove (A3)→(A4)→(A1) in Section 3, and we prove (A3)↔(A5) in Section 4.

3 | INCLUDING TOUGH SUBGRAPHS (A4)

If U is any set of vertices in a graph G and $H \subseteq G$ is a tough subgraph that contains U, then U is tough in G. Hence (A4)→(A1). The remainder of this section is dedicated to the proof of the implication (A3)→(A4).

We will need the following theorem by Gollin and Kneip. A chain $\mathcal{C}$ in a given poset is said to have *order-type* α for an ordinal α if $\mathcal{C}$ with the induced linear order is order-isomorphic to α. The chain $\mathcal{C}$ is then said to be an *α-chain*.

Theorem 3.1 ([9, Theorem 1]). *A tree set is isomorphic to the edge tree set of a tree if and only if it is regular and contains no $(\omega + 1)$-chain.*

We also remind the reader that we use two different notations for separations, see [2, Section 2.2]. In the event of ambiguity, we will explicitly say which notation we are using: either the standard notation $\{A, B\}$ with $A, B \subseteq V(G)$, or the nonstandard notation $\{X, \mathscr{C}\}$ with $X \subseteq V(G)$ and $\mathscr{C} \subseteq \mathscr{C}_X$ (where $\mathscr{C}_X$ is the set of components of $G - X$). From now on, we will refer to the nonstandard notation as *component-notation*.

3.1 | Motivation of the proof

Suppose that no critical vertex set of G lies in the closure of U. Our aim is to find a tough subgraph of G that contains U.

For every critical vertex set X of G, the number of components in $\breve{\mathscr{C}}_X$ which meet U is finite, whereas $\breve{\mathscr{C}}_X$ is infinite. So one could say that the number of components in $\breve{\mathscr{C}}_X$ which meet U is negligible. Let us assume for the sake of convenience that no component in $\breve{\mathscr{C}}_X$ meets U for every critical vertex set X of G.

To find a tough subgraph of G that contains U, it might be helpful to first learn more about the structure of G around U from the perspective of the critical vertex sets. Every critical vertex set X of G defines a separation of G, namely $\{X, \breve{\mathscr{C}}_X\}$. For each X, the vertex set U uniquely orients the separation $\{X, \breve{\mathscr{C}}_X\}$ away from the component collection $\breve{\mathscr{C}}_X$, that is, U induces the orientation $O := \{(\breve{\mathscr{C}}_X, X) \mid X \in \text{crit}(G)\}$ of $S := \{\{X, \breve{\mathscr{C}}_X\} \mid X \in \text{crit}(G)\}$. For an illustration, see Figure 1 without the caption and read $\mathscr{K}(X)$ as $\breve{\mathscr{C}}_X$ et cetera (the missing definitions in the figure will be introduced in Section 3.3). Now, let us suppose that S somehow happens to be the set of induced separations of some tree-decomposition of G. Then O uniquely determines a part of this tree-decomposition which contains U on the one hand, and is tough in G on the other hand! (Here we use (A3)→(A1) to see that the part is tough in G).

We will see in Example 3.2 that the part need not induce a tough subgraph of G, so we are not done yet. If we want to find a tough subgraph of G that contains the part, we have to add connectivity to the part. The adhesion sets of the part are critical vertex sets, and any two vertices in a critical vertex X are linked by infinitely many internally disjoint paths which are internally disjoint from the part (use the infinitely many components in $\breve{\mathscr{C}}_X$ which avoid the part to find these paths). Hence, there is a chance that adding such paths to the part could result in the tough subgraph of G that we seek, and indeed we will see in the proof that this can be done.

In this motivation, we assumed that S is the set of induced separations of some tree-decomposition. This assumption is too optimistic, but we shall find a viable compromise. We will discuss this in Section 3.2.

Now, we present an example where the part does not induce a tough subgraph.

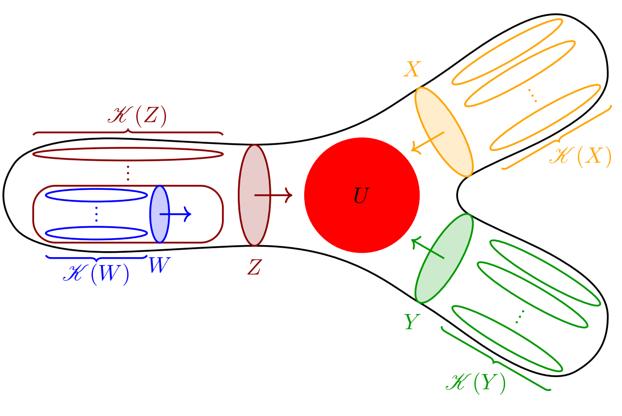

FIGURE 1 A principal set $\mathcal{Y} = \{W, X, Y, Z\}$ with admissable function $\mathscr{K}$. (The figure is based on [8, Figure 5])

Example 3.2. For every $n \in \mathbb{N}$ let A_n be some countably infinite set, such that A_n is disjoint from every A_m with $m \neq n$ and also disjoint from $\mathbb{N}$. Pick a vertex $a_n \in A_n$ for each n. Let G be the graph on $\mathbb{N} \sqcup \bigsqcup_{n\in\mathbb{N}} A_n$ where every vertex in $A_n \setminus \{a_n\}$ is joined completely to $\{0, ..., n\} \sqcup \{a_n\}$. Then the critical vertex sets of G are precisely the vertex sets of the form $X_n := \{0, ..., n\} \sqcup \{a_n\}$ where $n \in \mathbb{N}$. For every critical vertex set X_n the collection of components $\breve{\mathscr{C}}_{X_n}$ consists of the singletons in $A_n \setminus \{a_n\}$. Let $U := \mathbb{N} \cup \{a_n \mid n \in \mathbb{N}\}$. Then U has no critical vertex set in its closure. The set $\{(\breve{\mathscr{C}}_X, X) \mid X \in \mathrm{crit}(G)\}$ is an infinite star. In particular, the separations $\{X, \breve{\mathscr{C}}_X\}$ with $X \in \mathrm{crit}(G)$ are the induced separations of a star-decomposition of G. The central part of this star-decomposition is the intersection $\bigcap_{n\in\mathbb{N}} (V(G) \setminus \bigcup \breve{\mathscr{C}}_{X_n})$ which is equal to U. Since $G[U]$ has no edges, the central part induces a subgraph of G which is not tough.

3.2 | Motivation of admissable functions

In the motivation given in Section 3.1, it was useful to assume that the separations $\{X, \breve{\mathscr{C}}_X\}$ of a graph G defined by its critical vertex sets X are the induced separations of some tree-decomposition of G. In general, however, these tree-decompositions do not exist, for two reasons.

First, the separations $\{X, \breve{\mathscr{C}}_X\}$ need not be nested with each other. For an example, see [8, Example 3.7].

Second, even if the separations $\{X, \breve{\mathscr{C}}_X\}$ are nested, they can form a tree set which contains $(\omega + 1)$-chains (see [8, Example 5.12]), and then no corresponding tree-decomposition exists by Theorem 3.1.

To overcome the latter problem, we will relax tree-decompositions to tree sets. Tree sets generalise tree-decompositions, and our proof will work for tree sets as well.

To overcome the former problem, we will adjust the separations $\{X, \breve{\mathscr{C}}_X\}$. For each X, let $\mathscr{K}(X) := \breve{\mathscr{C}}_X$. Perhaps surprisingly, it will suffice to delete at most one component from each set $\mathscr{K}(X)$ to achieve that $\{\{X, \mathscr{K}(X)\} \mid X \in \mathrm{crit}(G)\}$ is a tree set. Functions $X \mapsto \mathscr{K}(X)$ which achieve this will be called strongly admissable. They exist for the collections of critical vertex sets of all graphs. In fact, the construction of these functions relies only on a single property of the collections of critical vertex sets. This property will be captured by the so-called principal collections of vertex sets. Hence, whenever we say principal collection, we may think of the collection of critical vertex sets of a graph. And whenever we say that $\mathscr{K}$ is strongly principal, we may think that $\mathscr{K}(X) = \breve{\mathscr{C}}_X$ for the sake of convenience.

The precise definition of an 'admissable' function that we give in Section 3.3 will be technical. Hence we remark that we will not actually use this technical definition in our proofs. Instead, we will work with Theorems 3.4 and 3.5, which will mostly hide this technical definition.

3.3 | Admissable functions for principal collections

In this section, we formally introduce admissable functions for principal collections of vertex sets.

Given a collection $\mathcal{Y}$ of vertex sets of G we say that a vertex set X of G is *$\mathcal{Y}$-principal* if X meets for every $Y \in \mathcal{Y}$ at most one component of $G - Y$. And we say that $\mathcal{Y}$ is *principal* if all its elements are $\mathcal{Y}$-principal.

If $X \subseteq V(G)$ meets precisely one component of $G - Y$ for some $Y \subseteq V(G)$, then we denote this component by $C_Y(X)$.

Every critical vertex set of a graph G is $\mathcal{X}$-principal for the collection $\mathcal{X}$ of all finite vertex sets of G: since every two vertices in a critical vertex set X are linked by infinitely many independent paths (these exist as $\breve{\mathscr{C}}_X$ is infinite), no two vertices in X are separated by a finite vertex set. The vertex sets of cliques in G are $\mathcal{X}$-principal as well.

Definition 3.3 ([8, Definition 5.9]). Suppose that $\mathcal{Y}$ is a principal collection of vertex sets of a graph G. A function that assigns to every $X \in \mathcal{Y}$ a subset $\mathscr{K}(X) \subseteq \breve{\mathscr{C}}_X$ is called *admissable* for $\mathcal{Y}$ if for every two $X, Y \in \mathcal{Y}$ that are incomparable as sets we have either $C_X(Y) \notin \mathscr{K}(X)$ or $C_Y(X) \notin \mathscr{K}(Y)$. If additionally $|\breve{\mathscr{C}}_X \setminus \mathscr{K}(X)| \leq 1$ for all $X \in \mathcal{Y}$, then $\mathscr{K}$ is *strongly* admissable for $\mathcal{Y}$.

Theorem 3.4 ([8, Theorem 5.10]). *For every principal collection of vertex sets of a connected graph there is a strongly admissable function.*

Theorem 3.5 ([8, Theorem 5.11]). *Let G be any connected graph, let $\mathcal{Y}$ be any principal collection of vertex sets of G and let $\mathscr{K}$ be any admissable function for $\mathcal{Y}$. Then for every distinct two $X, Y \in \mathcal{Y}$, after possibly swapping X and Y,*

$$\text{either } (\mathscr{K}(X), X) \leq (Y, \mathscr{K}(Y)) \text{ or } (\mathscr{K}(X), X) \leq (C_Y(X), Y) \leq (\mathscr{K}(Y), Y).$$

In particular, if $\emptyset \subsetneq \mathscr{K}(X) \subsetneq \mathscr{C}_X$ for all $X \in \mathcal{Y}$, then the separations $\{X, \mathscr{K}(X)\}$ form a regular tree set for which the separations $(\mathscr{K}(X), X)$ form a consistent orientation.

Suppose now that $\mathcal{Y}$ is a principal collection of vertex sets of a graph G and that $\mathscr{K}$ is an admissable function for $\mathcal{Y}$ satisfying $\emptyset \subsetneq \mathscr{K}(X) \subsetneq \mathscr{C}_X$ for all $X \in \mathcal{Y}$. If T is the regular tree set $\{\{X, \mathscr{K}(X)\} \mid X \in \mathcal{Y}\}$ provided by Theorem 3.5, then we call T a *principal* tree set of G. By a slight abuse of notation, we also call the triple $(T, \mathcal{Y}, \mathscr{K})$ a principal tree set. In this context, we write $O_{\mathscr{K}}$ for the consistent orientation $\{(\mathscr{K}(X), X) \mid X \in \mathcal{Y}\}$ of T.

For an illustration of the following corollary, see Figure 1 in Section 3.1.

Corollary 3.6. *Let G be any connected graph and let $U \subseteq V(G)$ be any vertex set. If no critical vertex of G lies in the closure of U, then there is a principal tree set $(T, \mathrm{crit}(G), \mathscr{K})$ of G satisfying the following two conditions:*

(i) *no element of $\mathscr{K}(X)$ meets U for any critical vertex set X;*
(ii) *$\mathscr{K}(X)$ is a cofinite subset of $\breve{\mathscr{C}}_X$ for every critical vertex set X.*

Proof. For every critical vertex set X of G, only finitely many components in $\breve{\mathscr{C}}_X$ meet U; we write $\mathscr{F}_X$ for this finite collection. Theorem 3.4 yields a strongly admissable function $\mathscr{K}$ for the collection $\mathrm{crit}(G)$ of all the critical vertex sets of G. We alter this function by removing $\mathscr{F}_X$ from $\mathscr{K}(X)$ for all X. Then $\mathscr{K}$ is still admissable for $\mathrm{crit}(G)$, and $\mathscr{K}(X)$ is a cofinite subcollection of $\breve{\mathscr{C}}_X \setminus \mathscr{F}_X$ for all X. Now Theorem 3.5 says that the separations $\{X, \mathscr{K}(X)\}$ with X critical form a tree set, and that the oriented separations $(\mathscr{K}(X), X)$ form a consistent orientation of this tree set. $\square$

3.4 | The proof

Let S be any tree set consisting of finite-order separations of G. A *part* of S is a vertex set of the form $\bigcap\{B \mid (A, B) \in O\}$ where O is a consistent orientation of S. Thus, if O is any consistent orientation of S, then it defines a part, which in turn induces a subgraph of G. The graph obtained from this subgraph by adding an edge xy whenever x and y are two vertices of the part that lie together in the separator of some separation in O is called the *torso* of O (or of the part, if O is clear from context). Thus, torsos usually will not be subgraphs of G. We need the following standard lemma:

Lemma 3.7 ([8, Corollary 2.11]). *Let G be any graph and let $W \subseteq V(G)$ be any connected vertex set. If B is a part of a tree set of separations of G, then $W \cap B$ is connected in the torso of B.*

Proof of (A3)→(A4). Let G be any connected graph and let $U \subseteq V(G)$ be any vertex set. Suppose that G has no critical vertex set that lies in the closure of U. Our aim is to find a tough subgraph of G that contains U. By Corollary 3.6 we find a principal tree set $(T, \mathrm{crit}(G), \mathscr{K})$ so that, for every critical vertex set X, no element of $\mathscr{K}(X)$ meets U and $\mathscr{K}(X)$ is a cofinite subset of $\breve{\mathscr{C}}_X$. We write B for the part of T that is defined by $O_{\mathscr{K}}$. Note that U is included in B.

First we claim that the torso of the part B is tough. To see this, consider any finite vertex set $X \subseteq B$. Only finitely many components of $G - X$ meet B: Indeed, if infinitely many components of $G - X$ meet B, then by the pigeonhole principle we deduce that a subset X' of X is critical in G with infinitely many components in $\breve{\mathscr{C}}_{X'}$ meeting B. But then $\bigcup\mathscr{K}(X')$ must meet B, contradicting that B is the part of T that is defined by $O_{\mathscr{K}} = \{(\mathscr{K}(X), X) \mid X \in \mathrm{crit}(G)\}$. Thus $G - X$ has only finitely many components meeting B. By Lemma 3.7 each of these components induces a component of the torso minus X, and so deleting X from the torso results in at most finitely many components.

The tough torso of the part B, however, need not be a subgraph of G. That is why as our next step, we construct a subgraph H of G that imitates the torso of B to inherit its toughness. More precisely, we obtain H from $G[B]$ by adding a subgraph L of G that has the following three properties:

(L1) Every vertex of $L - B$ has finite degree in L.
(L2) For every finite $X \subseteq B$ only finitely many components of $L - X$ avoid B.
(L3) If x and y are distinct vertices in B that lie together in a critical vertex set of G, then L contains a B-path between x and y.

Before we begin the construction of L, let us verify that any L satisfying these three properties really gives rise to a tough subgraph $H = G[B] \cup L$. For this, consider any finite vertex set $X \subseteq V(H)$. By (L1) every vertex of $H - B$ has finite degree in H, and hence deleting it produces only finitely many new components. Therefore we may assume that X is included in B entirely. Every component of $H - X$ avoiding B is a component of $L - X$ avoiding B, and there are only finitely many such components by (L2). Hence it remains to show that there are only finitely many components of $H - X$ that meet B. We already know that the torso of B is tough, so deleting X from it results in at most finitely many components. Then property (L3) ensures that each of these finitely many components has its vertex set included in a component of $H - X$, as follows. Let u and v be any two vertices in the torso of B that lie in the same component after we delete X from the torso. Let P be a u–v path in the torso of B that avoids X. Every edge of P that is not an edge of G joins two vertices that lie together in a critical vertex of G, by the definition of the torso of B. Hence we may use (L3) to replace all of these edges with B-paths in L. In this way, we obtain a connected subgraph of $L - X$ that contains u and v. Hence u and v lie in the same component of $L - X$. Hence every component of the torso of B minus X has its vertex set included in a component of $H - X$. It follows that there can only be finitely many components of $H - X$ that meet B.

Finally, we construct a subgraph $L \subseteq G$ satisfying the three properties (L1), (L2) and (L3). Choose $(\{x_\alpha, y_\alpha\})_{\alpha<\kappa}$ to be a transfinite enumeration of the collection of all unordered pairs $\{x, y\}$ where x and y are distinct vertices in B that lie together in a critical vertex set of G. Then we recursively construct L as a union $L = \bigcup_{\alpha<\kappa} P_\alpha$ where at step α we choose P_α from among all B-paths P in G between x_α and y_α so as to minimise the number $|E(P)\setminus E(\bigcup_{\xi<\alpha} P_\xi)|$ of new edges. (There is a B-path in G between x_α and y_α since x_α and y_α lie together in some critical vertex set X of G and $\mathscr{K}(X) \subseteq \breve{\mathscr{C}}_X$ is nonempty).

We verify that our construction yields an L satisfying (L1), (L2) and (L3). (L1) For this, fix any vertex $\ell \in L - B$. It suffices to show that the edges of L at ℓ simultaneously extend to an ℓ–B fan in L. To see that this really suffices, use that ℓ is not contained in B to find some critical vertex set X of G with $\ell \in \bigcup \mathscr{K}(X)$. Then the ℓ–B fan at ℓ extending the edges of L at ℓ must have all its ℓ–B paths pass through the finite X, and so there can be only finitely many such paths, meaning that ℓ has finite degree in L.

Now to find the ℓ–B fan we proceed as follows. For every edge e of L at ℓ we write $\alpha(e)$ for the minimal ordinal α with $e \in E(P_\alpha)$. Then we write P_e for $P_{\alpha(e)}$, and we write Q_e for the ℓ–B subpath of P_e containing e. The paths Q_e form an ℓ–B fan, as we verify now. For this, we show that, if $e \neq e'$ are two distinct edges of L at ℓ, then Q_e and $Q_{e'}$ meet precisely in ℓ. Let e and e' be given. We abbreviate $\alpha(e) = \alpha$ and $\alpha(e') = \alpha'$. If $\alpha = \alpha'$ then $Q_e \cup Q_{e'} = P_\alpha$ and we are done. Otherwise $\alpha < \alpha'$, say. Then we assume for a contradiction that $\mathring{\ell} Q_{e'}$ does meet $\mathring{\ell} Q_e$. Without loss of generality we may assume that $Q_{e'}$ starts in ℓ and ends in $y_{\alpha'}$. We let t be the last vertex of $Q_{e'}$ in $\mathring{\ell} Q_e$. But then the graph $x_{\alpha'} P_{e'} \ell \cup \ell Q_e t P_{e'} y_{\alpha'}$ is connected and meets B precisely in the two vertices $x_{\alpha'}$ and $y_{\alpha'}$. Consequently, it contains a B-path P between $x_{\alpha'}$ and $y_{\alpha'}$. But then P avoids the edge e', so the inclusion $E(P)\setminus E\left(\bigcup_{\xi<\alpha'} P_\xi\right) \subseteq E(P_{e'})\setminus E\left(\bigcup_{\xi<\alpha'} P_\xi\right)$ must be proper. Therefore, P contradicts the choice of $P_{\alpha'}$ as desired.

(L2) For this, fix any finite vertex set $X \subseteq B$. Let $\mathscr{C}$ be the set consisting of all the components of $L - X$ that avoid B. And let F consist of all the edges inside components from $\mathscr{C}$ and all the edges of L between components from $\mathscr{C}$ and X, that is, $F = E(\bigcup \mathscr{C}) \cup E_L(\bigcup \mathscr{C}, X)$. As every component from $\mathscr{C}$ meets some edge from F it suffices to show that F is finite, a fact that we verify as follows. Every edge in F lies on a path P_α, and since P_α is a B-path between x_α and y_α we deduce $\{x_\alpha, y_\alpha\} \in [X]^2$. Thus the finite edge sets of the paths P_α with $\{x_\alpha, y_\alpha\} \in [X]^2$ cover F. Since X is finite so is $[X]^2$, and hence there are only finitely many such paths, meaning that F is finite.

(L3) This property holds by construction.

As (L1), (L2) and (L3) are now verified we conclude that L is as desired, which completes the proof. □

4 | INCLUDING STAR-DECOMPOSITIONS (A5)

In this section we prove that the two assertions (A3) and (A5) are equivalent. Before we restate the assertions, let us recall the following definitions from [2, Section 3.5]. A finite-order separation $\{X, \mathscr{C}\}$ of a graph G in component-notation is *tame* if for no $Y \subseteq X$ both $\mathscr{C}$ and $\mathscr{C}_X \setminus \mathscr{C}$ contain infinitely many components whose neighbourhoods are precisely equal to Y. The tame separations of G are precisely the finite-order separations of G that respect the critical vertex sets:

Lemma 4.1 ([2, Lemma 3.16]). *A finite-order separation $\{A, B\}$ of a graph G in standard-notation is tame if and only if every critical vertex set X of G together with all but finitely many components from $\breve{\mathscr{C}}_X$ is contained in one side of $\{A, B\}$.*

Recall that the set of all finite-order separations of a graph G is denoted by $S_{\aleph_0} = S_{\aleph_0}(G)$. An $S_{\aleph_0}$-tree (T, α) is *tame* if all the separations in the image of α are tame. As a consequence of Lemma 4.1, if X is a critical vertex set of G and (T, α) is a tame $S_{\aleph_0}$-tree, then X induces a consistent orientation of the image of α by orienting every tame finite-order separation $\{A, B\}$ towards the side that contains X and all but finitely many of the components from $\breve{\mathscr{C}}_X$. This consistent orientation, via α, also induces a consistent orientation of $\vec{E}(T)$. Then, just like for ends, the critical vertex set X either *lives* at a unique node $t \in T$ or *corresponds* to a unique end of T. As usual, these definitions for $S_{\aleph_0}$-trees carry over to tree-decompositions.

Now here are the statements of (A3) and (A5):

(A3) G has no critical vertex set that lies in the closure of U;
(A5) G has a tame star-decomposition such that U is contained in the central part and every critical vertex set of G lives in a leaf's part.

The proof of the backward implication is short:

Proof of (A5)→(A3). For this, consider any critical vertex set X of G. Then X lives in a leaf's part of the star-decomposition provided by (A5), while U is contained in the central part. It follows that only finitely many components in $\breve{\mathscr{C}}_X$ meet U. Hence X does not lie in the closure of U. □

The remainder of this section is dedicated to the proof of the forward implication (A3)→(A5). For the motivation of the proof, we need one more definition.

The *supremum* sup$L$ of a set L of oriented separations of a graph is the oriented separation (A, B) with $A = \bigcup \{C \mid (C, D) \in L\}$ and $B = \bigcap \{D \mid (C, D) \in L\}$. The *infimum* of L is defined analogously.

4.1 | Motivation of the proof

Suppose that no critical vertex set of G lies in the closure of U. Our aim is to find a star-decomposition of G as described in (A5). As our first step, we consider a principal tree set $(T_{\mathscr{K}}, \text{crit}(G), \mathscr{K})$ as provided by Corollary 3.6, with the consistent orientation $O_{\mathscr{K}} = \{(\mathscr{K}(X), X) \mid X \in \text{crit}(G)\}$ of $T_{\mathscr{K}}$ just like in the construction of the tough subgraph in Section 3.

If each element of $O_{\mathscr{K}}$ lies below some maximal element of $O_{\mathscr{K}}$, then the set of all maximal elements of $O_{\mathscr{K}}$ forms a star which defines a star-decomposition with the properties described in (A5). Therefore, it is the case where some elements of $O_{\mathscr{K}}$ do not lie below any maximal elements of $O_{\mathscr{K}}$ that we have to deal with.

If some elements of $O_{\mathscr{K}}$ do not lie below any maximal elements of $O_{\mathscr{K}}$, there is a natural way to obtain a star from $O_{\mathscr{K}}$ such that every element of $O_{\mathscr{K}}$ lies below some element of the star. This star can be obtained in two steps, as follows. First, we consider the 'corridors' of $O_{\mathscr{K}}$. These 'corridors' are the classes of a natural equivalence relation on $O_{\mathscr{K}}$, in which two separations are equivalent precisely if we would expect them to lie below the same maximal element of $O_{\mathscr{K}}$. That is, two elements $\vec{s_1}, \vec{s_2} \in O_{\mathscr{K}}$ will be equivalent if and only if there is $\vec{r} \in O_{\mathscr{K}}$ with $\vec{s_1}, \vec{s_2} \le \vec{r}$. Second, we define the star: It is the set of the suprema of the corridors of $O_{\mathscr{K}}$. If a corridor already has a maximal element in $O_{\mathscr{K}}$, then its supremum will coincide with this maximal element.

Every element of $O_{\mathscr{K}}$ will lie below the supremum of the corridor it belongs to. Hence we obtain a star-decomposition which is almost as desired, but there is a problem: The adhesion sets of this star-decomposition could be infinite in size. In fact, $O_{\mathscr{K}}$ might contain separations $\vec{s_0} < \vec{s_1} < \cdots$ of orders 1, 2, ... and then all these separations will belong to the same corridor whose supremum will have infinite order (by Lemma 4.5, which we will state and prove in Section 4.2).

We will see in Lemma 4.5 that this problem does not occur if for each corridor of $O_{\mathscr{K}}$ there exists a natural number which bounds the sizes of all the separations in that corridor. This leads us to the following idea. For each $n \in \mathbb{N}$, let $O_{\mathscr{K}}^{\le n}$ consist only of the separations in $O_{\mathscr{K}}$ with order at most n. Then, for each n, let π_n be the star which consists of the suprema of the corridors of $O_{\mathscr{K}}^{\le n}$. Each separation in π_n will have size at most n by Lemma 4.5. However, instead of just one star which defines a star-decomposition, we now have a sequence of stars $\pi_0, \pi_1, \ldots$ and this sequence does not define a star-decomposition in an obvious way.

In the formal proof, we will refer to the union $O \coloneqq \bigcup_{n \in \mathbb{N}} \pi_n$ as the 'parliament' of $O_{\mathscr{K}}$, and this will be a consistent orientation. See Figure 2 on the next page for an illustration. In the figure, separations with the same shade of blue belong to the same star π_n. The parliament will have the following useful property: For each critical X, there is $\vec{s_X} \in O$ with $(\breve{\mathscr{C}}_X, X) \le \vec{s_X}$. Unfortunately though, $\vec{s_X}$ usually will not be maximal in O.

While the parliament O will face the same problem as $O_{\mathscr{K}}$ does, namely that not every element of O lies below some maximal element of O, the parliament has a great advantage over $O_{\mathscr{K}}$: Since O is the union of the sequence of stars π_n, the corridors of O will be much easier to control. More precisely, we will see that each corridor of O translates to a tree-decomposition of G; compare Lemma 4.7. This plus of control will allow us to obtain the desired star-decomposition from O in a final step, using a trick that has been discovered by Carmesin [4]. Roughly, here is what we will do. For each corridor γ of O, we will define a star σ_γ such that $\sigma \coloneqq \bigcup_\gamma \sigma_\gamma$ is a star which induces a star-decomposition as in (A5). So consider any corridor γ of O; we shall define σ_γ. It will be important that σ_γ satisfies the following three conditions:

(i) the separations in σ_γ have finite order;

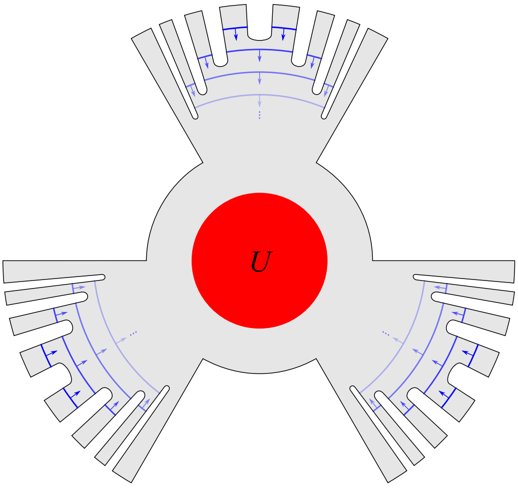

FIGURE 2 The blue separations form a parliament in the grey graph. This parliament has three corridors

(ii) for each critical X with $\vec{s}_X \in \gamma$ there is $\vec{r} \in \sigma_\gamma$ such that $(\mathscr{C}, X) \le \vec{r}$ for some cofinite subset $\mathscr{C} \subseteq \breve{\mathscr{C}}_X$ (this property will ensure that each critical vertex set of G lives in a leaf's part of the decomposition defined by σ);
(iii) U is contained in the part of σ_γ.

If γ has a maximal element, then we will let σ_γ be the singleton star that consists of this maximal element, so σ_γ obviously satisfies (i)–(iii).

Otherwise, we will use Lemma 4.7 to find an $S_{\aleph_0}$-tree $(T_\gamma, \alpha_\gamma)$ such that α_γ is an isomorphism between $\vec{E}(T_\gamma)$ and $\gamma \cup \gamma^* \subseteq \vec{S}_{\aleph_0}$. The corridor γ induces a consistent orientation of $\vec{E}(T_\gamma)$ which directs all the edges of T_γ. This consistent orientation of the edges of T_γ will be determined by a forward-directed ray in T_γ. Then this ray determines a cofinal sequence $\vec{s}_\gamma^{\,0} < \vec{s}_\gamma^{\,1} < \cdots$ in γ. Here, Carmesin's trick is to obtain a star σ_γ from this sequence by letting

$$\sigma_\gamma := \left\{\vec{s}_\gamma^{\,0}\right\} \cup \left\{\vec{s}_\gamma^{\,n} \wedge \overleftarrow{s}_\gamma^{\,n-1} : n \ge 1\right\}.$$

An illustration will follow in the formal proof, see Figure 3. Carmesin came up with this trick for a different purpose, but with a little luck it turns out that this suffices to ensure that σ_γ satisfies (i)–(iii).

4.2 | The proof

The proof is organised as follows. First, we state without proof a technical theorem, Theorem 4.2 below, and then we use it to derive the implication (A3)→(A5). In a last step we prepare and provide the proof of the technical theorem.

Note that the part of a star σ of separations of a graph G is $\bigcap\{B \mid (A,B) \in \sigma\}$. Given two oriented separations $\vec{s_1}, \vec{s_2}$ of G we write $\vec{s_1} \lesssim \vec{s_2}$ if either $\vec{s_1} \le \vec{s_2}$ or there is a component $C \in \mathscr{C}$ for $(\mathscr{C}, X) = \vec{s_1}$ such that $(\mathscr{C} \setminus \{C\}, X) \le \vec{s_2}$. Here is the technical theorem:

Theorem 4.2. *Let G be any graph, and let $(T, \mathcal{Y}, \mathscr{K})$ be any principal tree set such that $O_{\mathscr{K}}$ defines an infinite part. Then G admits a star σ of finite-order separations such that the following two conditions hold:*

(i) *the part defined by $O_{\mathscr{K}}$ is included in the part of σ;*
(ii) *for every $\vec{s} \in O_{\mathscr{K}}$ there is some $\vec{r} \in \sigma$ with $\vec{s} \lesssim \vec{r}$.*

We derive the implication (A3)→(A5) from Theorem 4.2, as follows:

Proof of (A3)→(A5). If U is finite, then the star

$$\{(\mathscr{C}_U(Y), Y) \mid Y \in 2^U \setminus \{\emptyset\}\}$$

gives the desired star-decomposition with central part equal to U, where $\mathscr{C}_U(Y)$ is the collection of all components $C \in \mathscr{C}_U$ with $N(C) = Y$.

Otherwise U is infinite. Then, by Corollary 3.6, we find a principal tree set $(T, \mathrm{crit}(G), \mathscr{K})$ such that, for every critical vertex set X, no element of $\mathscr{K}(X)$ meets U and the inclusion $\mathscr{K}(X) \subseteq \breve{\mathscr{C}}_X$ is cofinite. We claim that the star provided by Theorem 4.2 gives a star-decomposition of G meeting the requirements of (ii), a fact that can be verified as follows: First, the separations of the form $(\mathscr{K}(X), X)$ with X critical and $\mathscr{K}(X)$ a cofinite subset of $\breve{\mathscr{C}}_X$ are tame and thus our star-decomposition is tame. Next, by Theorem 4.2 (i), we have that U is contained in the central part of the star-decomposition. Finally, by Theorem 4.2 (ii), every critical vertex set of G lives in a leaf's part, which can be verified as follows. Let X be any critical vertex set of G, and let σ be the star provided by Theorem 4.2. By (ii) there is some $\vec{r} \in \sigma$ with $(\mathscr{K}(X), X) \lesssim \vec{r}$. Let us write $\vec{r} = (A, B)$ in standard notation. Since $(\mathscr{K}(X), X) \lesssim \vec{r}$ we either have $(\mathscr{K}(X), X) \le \vec{r}$ or there is a component $C \in \mathscr{K}(X)$ such that $(\mathscr{K}(X) \setminus \{C\}, X) \le \vec{r}$. In either case, since $\mathscr{K}(X)$ is a cofinite subset of $\breve{\mathscr{C}}_X$, it follows that X and all but finitely many of the components in $\breve{\mathscr{C}}_X$ are contained in $G[A]$. Thus, the critical vertex set X orients the separation $\{A, B\}$ towards A, while the central part is contained in B. Hence X lives in a leaf's part. □

Next, we prepare the proof of our technical theorem, Theorem 4.2. We will need the following concept of a corridor from [8]. Suppose that $(\vec{T}, \le, ^*)$ is a tree set, and that O is a consistent orientation of $\vec{T}$. A *corridor* of O is an equivalence class of separations in O, where two separations $\vec{s_1}, \vec{s_2} \in O$ are considered equivalent if there is $\vec{r} \in O$ with $\vec{s_1}, \vec{s_2} \le \vec{r}$, compare [8, Lemma 7.1 and Definition 7.2]. Note that if $\vec{r} \in O$ witnesses that $\vec{s_1}, \vec{s_2} \in O$ are equivalent, then $\vec{r}$ is contained in the equivalence class of $\vec{s_1}$ and $\vec{s_2}$ as well. As corridors are consistent partial orientations of tree sets on the one hand, and directed posets on the other hand, they come with a number of useful properties.

Lemma 4.3. *Let T be any regular tree set of separations of any graph G, let O be any consistent partial orientation of T and let γ be any corridor of O. Then the supremum of γ is nested with $\vec{T}$.*

Proof. Consider any unoriented separation $r \in T$. If there is a separation $\vec{s} \in \gamma$ such that r has an orientation $\vec{r}$ with $\vec{r} \le \vec{s}$, then $\vec{r} \le \vec{s} \le \sup\gamma$ as desired. As T is nested, r has for every separation $\vec{s} \in \gamma$ an orientation $\vec{r}(\vec{s})$ such that either $\vec{r}(\vec{s}) \le \vec{s}$ or $\vec{s} \le \vec{r}(\vec{s})$. By our first observation, we may assume that $\vec{s} \le \vec{r}(\vec{s})$ for all $\vec{s} \in \gamma$. It suffices to show that $\vec{r}(\vec{s_1}) = \vec{r}(\vec{s_2})$ for all $\vec{s_1}, \vec{s_2} \in \gamma$, since then r has one orientation that lies above all elements of γ and, in particular, above the supremum of γ. Given $\vec{s_1}, \vec{s_2} \in \gamma$, consider any $\vec{s_3} \in O$ with $\vec{s_1}, \vec{s_2} \le \vec{s_3}$. Then $\vec{s_3} \in \gamma$ and $\vec{s_1}, \vec{s_2} \le \vec{s_3} \le \vec{r}(\vec{s_3})$. We claim that $\vec{r}(\vec{s_i}) = \vec{r}(\vec{s_3})$ for both $i = 1, 2$. Assume for a contradiction that $\vec{r}(\vec{s_i}) \ne \vec{r}(\vec{s_3})$ for some $i \in \{1, 2\}$. Then

$$\vec{s_i} \le \vec{s_3} \le \vec{r}(\vec{s_3}) = (\vec{r}(\vec{s_i}))^* \le \overleftarrow{s_i}$$

follows. Hence $\vec{s_i}$ is small, contradicting that T is regular. □

Lemma 4.4. *Let T be any tree set of separations of any graph G and let O be any consistent orientation of T. Then the suprema of the corridors of O form a star.*

Proof. We have to show that for every two distinct corridors γ and δ of O the supremum (A, B) of γ and the supremum (C, D) of δ satisfy $(A, B) \le (D, C)$. Let us write $\gamma = \{(A_i, B_i) \mid i \in I\}$ and $\delta = \{(C_j, D_j) \mid j \in J\}$. As γ is distinct from δ we have $(A_i, B_i) \le (D_j, C_j)$ for all $i \in I$ and $j \in J$. Hence $(A, B) = (\bigcup_i A_i, \bigcap_i B_i) \le (\bigcap_j D_j, \bigcup_j C_j) = (D, C)$. □

Lemma 4.5. *Suppose that T is any tree set of separations of any graph G, that O is any consistent orientation of T, and that γ is any corridor of O. Then every finite subset of the separator of the supremum of γ is contained in the separator of some separation in γ.*

In particular, if the order of the separations in γ is bounded by some natural number n, then the supremum of γ has order at most n.

Proof. Let us write (A, B) for the supremum of γ and let Y be any finite subset of its separator $X := A \cap B$. For every vertex $y \in Y \subseteq A$ there is a separation $(C_y, D_y) \in \gamma$ with $y \in C_y$. Since γ is a corridor we find a separation $(C, D) \in \gamma$ lying above all (C_y, D_y). Then $Y \subseteq C$ as C includes all C_y, and $Y \subseteq D$ because $(C, D) \le (A, B)$ gives $Y \subseteq X \subseteq B \subseteq D$. □

Before we start with the proof of Theorem 4.2 we need two final ingredients: induced separation systems and parliaments. If $\vec{S} = (\vec{S}, \le, ^*)$ is a separation system and $O \subseteq \vec{S}$ is any subset (usually a partial orientation of S), then O induces a separation system $O \cup O^*$ that is a subsystem of $\vec{S}$ with the partial ordering and involution induced by $\le$ and *. We denote this subsystem by $\vec{S}[O]$.

Next, we define parliaments. Suppose that G is any graph, that $\vec{T} = (\vec{T}, \le, ^*)$ is any regular tree set of finite-order separations of G, and that O is any consistent orientation of $\vec{T}$. For every number $n \in \mathbb{N}$ let $O_{\le n}$ be the subset of O formed by the oriented separations in O whose separators have size at most n. Then, by Lemma 4.5, every corridor of $O_{\leq n}$ has a supremum of order at most n, and these suprema form a star for fixed n (cf. Lemma 4.4) which we denote by $\pi_n(O)$. The *parliament* of O, denoted by $\pi(O)$, is the union $\bigcup_{n\in\mathbb{N}} \pi_n(O)$. See Figure 2 for an illustration. In the figure, separations with the same shade of blue belong to $\pi_n(O)$ for the same n. Notably, the parliament of O is a cofinal subset of $O \cup \pi(O)$: Each $\vec{s} \in O$ is contained in $O_{\leq n}$ for some $n \in \mathbb{N}$, so $\vec{s} \in \gamma$ for a corridor γ of $O_{\leq n}$ and hence $\vec{s} \leq \sup\gamma \in \pi(O)$. The parliament of O induces a separation system $\vec{S}_{\aleph_0}[\pi(O)]$ that is a subsystem of $\vec{S}_{\aleph_0}$ whose separations are all nested with each other. Furthermore, $\vec{S}_{\aleph_0}[\pi(O)]$ and $\vec{T}$ are nested with each other in $\vec{S}_{\aleph_0}$ by Lemma 4.3. Also, the parliament of O is a consistent orientation of $\vec{S}_{\aleph_0}[\pi(O)]$ where it defines the same part as O does for $\vec{T}$.

The inverses of corridors of parliaments have no ω-chains:

Lemma 4.6. *Let G be any graph, let $\vec{T}$ be any regular tree set of finite-order separations of G, and let O be any consistent orientation of $\vec{T}$. Then the inverse γ^* of any corridor γ of $\pi(O)$ has no ω-chain.*

Proof. Suppose for a contradiction that there is a sequence $\overleftarrow{s_0} < \overleftarrow{s_1} < \cdots$ of separations $\overleftarrow{s_n} \in \gamma^*$. Note that $\vec{s} < \vec{r}$ with $\vec{s} \in \pi_m(O)$ and $\vec{r} \in \pi_n(O)$ implies $m < n$. Hence the function $g : \omega \to \omega$ assigning to each $n < \omega$ the least $k < \omega$ with $\vec{s_n} \in \pi_k(O)$ is strictly decreasing in that $g(m) > g(n)$ for all $m < n$, contradicting that there are only finitely many natural numbers $<g(0)$. □

The corridors of a parliament usually stem from $S_{\aleph_0}$-trees:

Theorem 4.7. *Let G be any graph, let $\vec{T}$ be any regular tree set of finite-order separations of G, and let O be any consistent orientation of $\vec{T}$ such that $\vec{S}_{\aleph_0}[\pi(O)]$ is regular. Then for every corridor γ of the parliament of O the corresponding regular tree set $\vec{S}_{\aleph_0}[\gamma]$ is isomorphic to the edge tree set of a tree.*

Proof. Let γ be any corridor of the parliament of O. By Theorem 3.1, it suffices to show that $\vec{S}_{\aleph_0}[\gamma]$ has no $(\omega+1)$-chain. For this, suppose for a contradiction that $\vec{s_0} < \vec{s_1} < \cdots < \vec{s_\omega}$ is an $(\omega+1)$-chain in $\vec{S}_{\aleph_0}[\gamma]$.

If $\vec{s_\omega}$ lies in γ, then so do all the other $\vec{s_n}$ as γ is consistent. Note that $\vec{s} < \vec{r}$ with $\vec{s} \in \pi_m(O)$ and $\vec{r} \in \pi_n(O)$ implies $m < n$. Hence the function $f : \omega + 1 \to \omega$ assigning to each $\alpha \leq \omega$ the least $n < \omega$ with $\vec{s_\alpha} \in \pi_n(O)$ is strictly increasing in that $f(\alpha) < f(\beta)$ for all $\alpha < \beta$, contradicting $f(\omega) < \omega$.

Otherwise $\vec{s_\omega}$ lies in γ^*. By Lemma 4.6 there is no ω-chain in γ^*, so $\vec{s_n} \in \gamma$ for all but finitely many n. Hence $\vec{s_n} \in \gamma$ for all $n < \omega$ by consistency. Using that γ is a corridor we find a separation $\vec{r} \in \gamma$ with $\overleftarrow{s_\omega} \leq \vec{r}$ and $\vec{s_0} \leq \vec{r}$. For every $n < \omega$, either $\vec{s_n} \leq \vec{r}$ or $\vec{s_n} \leq \overleftarrow{r}$ or $\overleftarrow{s_n} \leq \vec{r}$ or $\overleftarrow{s_n} \leq \overleftarrow{r}$.

We cannot have $\vec{s_n} \leq \overleftarrow{r}$ for any n, since this would imply $\vec{s_0} \leq \vec{s_n} \leq \overleftarrow{r} \leq \overleftarrow{s_0}$ contradicting that $\vec{S}_{\aleph_0}[\pi(O)]$ is regular. In particular $\vec{s_n} \neq \overleftarrow{r}$ for all n, which implies

$\overleftarrow{s_n} \neq \overrightarrow{r}$. We cannot have $\overleftarrow{s_n} < \overrightarrow{r}$ for any n because γ is consistent. And we cannot have $\overleftarrow{s_n} \leq \overleftarrow{r}$, because then $\overleftarrow{s_\omega} \leq \overrightarrow{r} \leq \overrightarrow{s_n} < \overrightarrow{s_\omega}$ contradicts that $\vec{S}_{\aleph_0}[\pi(O)]$ is regular. Hence $\overrightarrow{s_n} \leq \overrightarrow{r}$ for all n. As γ contains no $(\omega + 1)$-chains by the first case, there must be an $\ell < \omega$ with $\overrightarrow{s_\ell} = \overrightarrow{r}$. But this then contradicts $\overrightarrow{r} = \overrightarrow{s_\ell} < \overrightarrow{s_{\ell+1}} \leq \overrightarrow{r}$, completing the proof that $\vec{S}_{\aleph_0}[\gamma]$ has no $(\omega + 1)$-chains. □

Finally, we prove our technical theorem:

Proof of Theorem 4.2. Let $(T_{\mathscr{K}}, \mathcal{Y}, \mathscr{K})$ be any principal tree set of a connected graph G so that $O_{\mathscr{K}}$ defines an infinite part. We let O be the parliament of $O_{\mathscr{K}}$. Then the tree set $\vec{S}_{\aleph_0}[O]$ is regular: For every $n \in \mathbb{N}$ and every $(A, B) \in \pi_n(O_{\mathscr{K}}) \subseteq O$ we have that $A \backslash B$ contains the nonempty vertex set of the graph $\bigcup \mathscr{K}(X)$ for some $X \in \mathcal{Y}$, and $B \backslash A$ contains all but at most $|A \cap B| \leq n$ of the infinitely many vertices of the infinite part defined by O. Therefore, by Theorem 4.7 we find for every corridor γ of O an $S_{\aleph_0}$-tree $(T_\gamma, \alpha_\gamma)$ such that α_γ is an isomorphism between the edge tree set $\vec{E}(T_\gamma)$ of T_γ and $\vec{S}_{\aleph_0}[\gamma]$.

In a first step, we will use the $S_{\aleph_0}$-trees $(T_\gamma, \alpha_\gamma)$ to define stars σ_γ, one for every corridor γ of O, such that their union $\sigma = \bigcup_\gamma \sigma_\gamma$ is a candidate for the star that we seek. Then, in a second step, we will verify that σ is indeed as desired, completing the proof.

First step. For this, consider any corridor γ of O. Then γ, as it orients the image of α_γ consistently, defines either a node or an end of T_γ (see Section 2.8 of [2]).

If γ defines a node t of T_γ, then t has precisely one neighbour in T_γ. Indeed, γ is the down-closure in $\vec{S}_{\aleph_0}[\gamma]$ of the star $\alpha_\gamma(\vec{F_t})$ where $\vec{F_t} = \{(s, t) \mid st \in E(T_\gamma)\}$. Note that all separations in $\alpha_\gamma(\vec{F_t})$ are maximal in γ. Hence, if t has two distinct neighbours k_1 and k_2 in T_γ, then γ contains a separation $\vec{r}$ that lies above both $\alpha_\gamma(k_1, t)$ and $\alpha_\gamma(k_2, t)$, contradicting the maximality in the corridor γ of at least one of these two separations (here we also use that $\alpha_\gamma(k_1, t)$ and $\alpha_\gamma(k_2, t)$ are distinct for distinct neighbours k_1 and k_2 of t because α_γ is injective). Therefore, t is a leaf of T_γ. Call its neighbour k. Then $\alpha_\gamma(k, t)$ is the maximal element of the corridor γ, and we let $\sigma_\gamma := \{\alpha_\gamma(k, t)\}$.

Otherwise γ defines an end of T_γ from which we pick a ray $R_\gamma = v_\gamma^0 v_\gamma^1 \ldots$ all whose edges are oriented forward by γ in that $\vec{s}_\gamma^{\,n} := \alpha_\gamma(v_\gamma^n, v_\gamma^{n+1})$ lies in γ for all $n \in \mathbb{N}$. Then we let

$$\sigma_\gamma := \left\{ \vec{s}_\gamma^{\,0} \right\} \cup \left\{ \vec{s}_\gamma^{\,n} \wedge \overleftarrow{s}_\gamma^{\,n-1} : \; n \geq 1 \right\}. \tag{1}$$

(See Figure 3).

Let us check that σ_γ really is a star. On the one hand, it follows from $\vec{s}_\gamma^{\,0} \leq \vec{s}_\gamma^{\,n-1}$ that $\vec{s}_\gamma^{\,0} \leq \overleftarrow{s}_\gamma^{\,n} \vee \vec{s}_\gamma^{\,n-1} = (\vec{s}_\gamma^{\,n} \wedge \overleftarrow{s}_\gamma^{\,n-1})^*$ for all $n \geq 1$. And on the other hand, for $1 \leq n < m$, we infer from $\vec{s}_\gamma^{\,n-1} \leq \vec{s}_\gamma^{\,n} \leq \vec{s}_\gamma^{\,m-1} \leq \vec{s}_\gamma^{\,m}$ that

$$\vec{s}_\gamma^{\,m} \wedge \overleftarrow{s}_\gamma^{\,m-1} \leq \overleftarrow{s}_\gamma^{\,m-1} \leq \overleftarrow{s}_\gamma^{\,n} \leq \overleftarrow{s}_\gamma^{\,n} \vee \vec{s}_\gamma^{\,n-1} = \left(\vec{s}_\gamma^{\,n} \wedge \overleftarrow{s}_\gamma^{\,n-1} \right)^*.$$

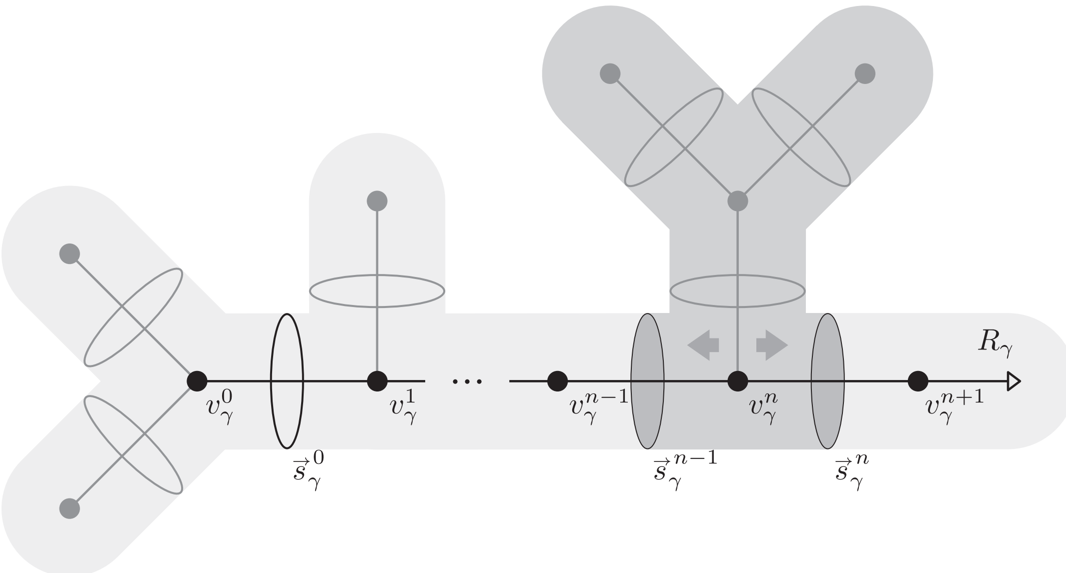

FIGURE 3 The light grey area depicts $B\backslash A$, the grey area depicts $A\backslash B$ and the dark grey area depicts $A \cap B$ of the separation $(A, B) := \overrightarrow{s}_\gamma^{\,n} \wedge \overleftarrow{s}_\gamma^{\,n-1}$ from the proof of Theorem 4.2

Since all $\overrightarrow{s}_\gamma^{\,n}$ have finite order, so do the infima of which σ_γ is composed. This technique of turning a ray into a star of separations has been introduced by Carmesin [4] in his 'Proof that Lemma 6.8 implies Lemma 6.7.'

Second step. We prove that σ is as desired. First, we show condition (i), which states that the part defined by $O_{\mathscr{K}}$ is included in the part of σ. For every separation $\vec{s} \in \sigma$ there is some separation $\vec{r} \in O$ satisfying $\vec{s} \leq \vec{r}$. Hence the part of σ includes the part of O, which in turn includes the part of $O_{\mathscr{K}}$ because O is the parliament of $O_{\mathscr{K}}$.

It remains to verify condition (ii), which states that for every $(\mathscr{K}(X), X) \in O_{\mathscr{K}}$ there is some $\vec{s} \in \sigma$ with $(\mathscr{K}(X), X) \lesssim \vec{s}$. For this, let any vertex set $X \in \mathcal{Y}$ be given. As O is cofinal in $O_{\mathscr{K}} \cup O$, there is a separation $\vec{s}_X \in O$ above $(\mathscr{K}(X), X)$. Let γ be the corridor of O containing $\vec{s}_X$. We check the following two cases.

In the first case, σ_γ is a singleton, formed by the maximal element $\vec{s}$ of γ, giving

$$(\mathscr{K}(X), X) \leq \vec{s}_X \leq \vec{s} \in \sigma.$$

In the second case, σ_γ is of the form (1). Then, as O is nested with $T_{\mathscr{K}}$, the separation $(\mathscr{K}(X), X)$ induces a consistent orientation of the image of α_γ, as follows. The orientation consists of all $\vec{r} \in \vec{S}_{\aleph_0}[\gamma]$ that satisfy either $\vec{r} \leq (\mathscr{K}(X), X)$ or $(\mathscr{K}(X), X) < \overleftarrow{r}$. Now this consistent orientation defines either a node or an end of T_γ. Since $\vec{s}_X \in \gamma$ lies above $(\mathscr{K}(X), X)$ and since γ^* contains no ω-chains by Lemma 4.6, it must be a node t of T_γ. Let $P = t_0 \ldots t_k$ be the t–R_γ path in T_γ and let $n \in \mathbb{N}$ be the number with $v_\gamma^n = t_k$, see Figure 4 (the ray $R_\gamma = v_\gamma^0 v_\gamma^1 \ldots$ was defined right above Equation 1).

Assume first that $n = 0$, which implies that $\overleftarrow{s}_\gamma^0$ lies in the orientation that defines t. In this case we have either $\overleftarrow{s}_\gamma^0 \le (\mathscr{K}(X), X)$ or $(\mathscr{K}(X), X) < \overrightarrow{s}_\gamma^0$. But actually, we cannot have $\overleftarrow{s}_\gamma^0 \le (\mathscr{K}(X), X)$ because otherwise $(\mathscr{K}(X), X) \le \overrightarrow{s_X}$ would imply that $\overleftarrow{s}_\gamma^0 \le \overrightarrow{s_X}$ meaning that $\overleftarrow{s}_\gamma^0$ and $\overrightarrow{s_X}$ violate the consistency of γ. Therefore, we must have $(\mathscr{K}(X), X) < \overrightarrow{s}_\gamma^0$, and then we are done because $\overrightarrow{s}_\gamma^0$ is an element of σ_γ. Thus, we may assume $n > 0$.

If the path P is nontrivial, that is, if $t_0 = t$ is distinct from $t_k = v_\gamma^n$, then we consider the separation $\overrightarrow{r_P} = \alpha_\gamma(t_{k-1}, t_k) \in \gamma$ associated with the last edge $t_{k-1}t_k$ of P. By the definition of P, the separation $\overleftarrow{r_P}$ satisfies either $\overleftarrow{r_P} \le (\mathscr{K}(X), X)$ or $(\mathscr{K}(X), X) < \overrightarrow{r_P}$. The former inequality would violate the consistency of γ as $\overleftarrow{r_P} \le (\mathscr{K}(X), X) \le \overrightarrow{s_X}$ would follow (here we use that $\vec{S}_{\aleph_0}[\gamma] \subseteq \vec{S}_{\aleph_0}[O]$ is regular to ensure $\overrightarrow{r_P} \neq \overleftarrow{s_X}$). Hence $(\mathscr{K}(X), X) < \overrightarrow{r_P}$. As t_{k-1} is distinct from v_γ^{n-1}, and both vertices have v_γ^n as a neighbour in T_γ, we obtain the inequalities $\overrightarrow{r_P} \le \overrightarrow{s}_\gamma^n$ and $\overrightarrow{r_P} \le \overleftarrow{s}_\gamma^{n-1}$. Thus,

$$(\mathscr{K}(X), X) \le \overrightarrow{r_P} \le \overrightarrow{s}_\gamma^n \wedge \overleftarrow{s}_\gamma^{n-1} \in \sigma.$$

Otherwise the path P is trivial, that is, $t_0 = t_k$ where $t_0 = t$ and $t_k = v_\gamma^n$. By the definition of t we have either $\overrightarrow{s}_\gamma^{n-1} \le (\mathscr{K}(X), X)$ or $(\mathscr{K}(X), X) < \overleftarrow{s}_\gamma^{n-1}$, and we have either $\overleftarrow{s}_\gamma^n \le (\mathscr{K}(X), X)$ or $(\mathscr{K}(X), X) < \overrightarrow{s}_\gamma^n$. The case $\overleftarrow{s}_\gamma^n \le (\mathscr{K}(X), X)$ is impossible since otherwise $(\mathscr{K}(X), X) \le \overrightarrow{s_X} \in \gamma$ would imply that $\overleftarrow{s}_\gamma^n \le \overrightarrow{s_X}$ meaning that $\overrightarrow{s}_\gamma^n$ and $\overrightarrow{s_X}$ violate the consistency of γ. Therefore, we have either $(\mathscr{K}(X), X) \le \overrightarrow{s}_\gamma^n \wedge \overleftarrow{s}_\gamma^{n-1} \in \sigma$ as desired, or we have $\overrightarrow{s}_\gamma^{n-1} \le (\mathscr{K}(X), X) < \overrightarrow{s}_\gamma^n$. For this latter case, we show that there is a component $C \in \mathscr{K}(X)$ such that $\overrightarrow{s}_\gamma^{n-1} \le (C, X)$ holds. This suffices to complete the proof,

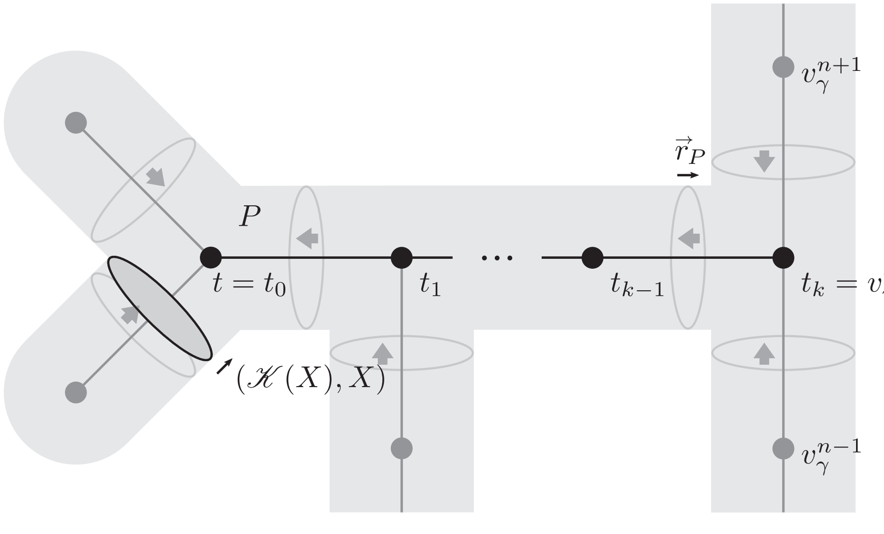

FIGURE 4 The orientation of the image $\vec{S}_{\aleph_0}[\gamma]$ of α_γ and the path P in the second step of the proof of Theorem 4.2

because then the inequalities $(\mathscr{K}(X)\setminus\{C\}, X) \le (X, C) \le \overleftarrow{s}_\gamma^{n-1}$ and $(\mathscr{K}(X)\setminus\{C\}, X) \le (\mathscr{K}(X), X) < \overrightarrow{s}_\gamma^{n}$ give

$$(\mathscr{K}(X)\setminus\{C\}, X) \le \overrightarrow{s}_\gamma^{n} \wedge \overleftarrow{s}_\gamma^{n-1} \in \sigma.$$

The separation $\overrightarrow{s}_\gamma^{n-1} \in O$ is, by definition, the supremum of some corridor δ of $\{(A, B) \in O_{\mathscr{K}} : |A \cap B| \le \ell\}$ for some number $\ell \in \mathbb{N}$. Then every separation $(\mathscr{K}(Y), Y) \in \delta$ satisfies $(\mathscr{K}(Y), Y) \le \overrightarrow{s}_\gamma^{n-1} \le (\mathscr{K}(X), X)$. In particular, as the principal tree set $T_{\mathscr{K}}$ satisfies the conclusions of Theorem 3.5 and $(\mathscr{K}(X), X)$ is nonsmall, every separation $(\mathscr{K}(Y), Y) \in \delta$ satisfies $(\mathscr{K}(Y), Y) \le (C_X(Y), X)$. Hence to show that $\overrightarrow{s}_\gamma^{n-1} \le (C, X)$ for some component $C \in \mathscr{K}(X)$, it suffices to show that $C_X(Y) = C_X(Y')$ for every two separations $(\mathscr{K}(Y), Y)$ and $(\mathscr{K}(Y'), Y')$ in δ. Given $(\mathscr{K}(Y), Y)$ and $(\mathscr{K}(Y'), Y')$, consider any separation $(\mathscr{K}(Z), Z) \in \delta$ above the two. Then $(\mathscr{K}(Z), Z) \le (C_X(Z), X)$ implies that both $C_X(Y)$ and $C_X(Y')$ are contained in $C_X(Z)$, giving $C_X(Y) = C_X(Y')$ as desired. □

5 | OVERVIEW OF ALL DUALITY RESULTS

In this section we summarise all duality theorems of this series. A very brief overview of the complementary structures is given by the following table:

	Normal tree	Tree-decomposition	Other
Combs	✓	✓	✓
Stars	✓	✓	
Dominated combs	✓	✓	
Dominating stars	✓	✓	
Undominated comb	✗	✓	✓
Undominating star	✗	✓	✓

Here, a check mark means, for example, that we proved a duality theorem for combs in terms of normal trees, whereas the two crosses mean that normal trees cannot serve as complementary structures for undominated combs or undominating stars.

Finally, we summarise our duality theorem for combs, stars and combinations of the two explicitly in five theorems:

Theorem (Combs). *Let G be any connected graph and let $U \subseteq V(G)$ be any vertex set. Then the following assertions are equivalent:*

(i) *G does not contain a comb attached to U;*
(ii) *there is a rayless normal tree $T \subseteq G$ that contains U (moreover, T can be chosen such that it contains U cofinally);*

(iii) *G has a rayless tree-decomposition into parts each containing at most finitely many vertices from U and whose parts at nonleaves of the decomposition tree are all finite* (*moreover, the tree-decomposition displays $\partial_\Omega U$ and may be chosen with connected separators*);
(iv) *for every infinite $U' \subseteq U$ there is a critical vertex set $X \subseteq V(G)$ such that infinitely many of the components in $\breve{\mathscr{C}}_X$ meet U'*;
(v) *G has a U-rank;*
(vi) *G has a rooted tame tree-decomposition $(T, \mathcal{V})$ that covers U cofinally and satisfies the following four assertions*:
- *$(T, \mathcal{V})$ is the squeezed expansion of a normal tree of G that contains the vertex set U cofinally*;
- *every part of $(T, \mathcal{V})$ meets U finitely and parts at nonleaves are finite*;
- *$(T, \mathcal{V})$ displays $\partial_\Gamma U \subseteq \mathrm{crit}(G)$*;
- *the rank of T is equal to the U-rank of G.*

Theorem (Stars). *Let G be any connected graph and let $U \subseteq V(G)$ be any vertex set. Then the following assertions are equivalent:*

(i) *G does not contain a star attached to U*;
(ii) *there is a locally finite normal tree $T \subseteq G$ that contains U and all whose rays are undominated in G* (*moreover, T can be chosen such that it contains U cofinally and every component of $G - T$ has finite neighbourhood*);
(iii) *G has a locally finite tree-decomposition with finite and pairwise disjoint separators such that each part contains at most finitely many vertices of U* (*moreover, the tree-decomposition can be chosen with connected separators and such that it displays $\partial_\Gamma U \subseteq \Omega(G)$*);

Theorem (Dominating stars and dominated combs). *Let G be any connected graph and let $U \subseteq V(G)$ be any vertex set. Then the following assertions are equivalent*:

(i) *G does not contain a dominating star attached to U*;
(ii) *G does not contain a dominated comb attached to U*;
(iii) *there is a normal tree $T \subseteq G$ that contains U and all whose rays are undominated in G* (*moreover, the normal tree T can be chosen such that it contains U cofinally and every component of $G - T$ has finite neighbourhood*);
(iv) *G has a rooted tree-decomposition $(T, \mathcal{V})$ such that*
- *each part contains at most finitely many vertices from U;*
- *all parts at nonleaves of T are finite;*
- *$(T, \mathcal{V})$ has essentially disjoint connected separators;*
- *$(T, \mathcal{V})$ displays $\partial_\Omega U$.*

Theorem (Undominated combs). *Let G be any connected graph and let $U \subseteq V(G)$ be any vertex set. Then the following assertions are equivalent:*

(i) *G does not contain an undominated comb attached to U;*

(ii) *G has a star-decomposition with finite separators such that U is contained in the central part and all undominated ends of G live in the leaves' parts (moreover, the star-decomposition can be chosen with connected separators);*
(iii) *G has a connected subgraph that contains U and all whose rays are dominated in it.*

Moreover, if U is normally spanned in G, we may add

(iv) *there is a rayless tree $T \subseteq G$ that contains U.*

Theorem (Undominating stars). *Let G be any connected graph and let $U \subseteq V(G)$ be any vertex set. Then the following assertions are equivalent:*

(i) *U is tough in G;*
(ii) *G contains no undominating star attached to U;*
(iii) *G has no critical vertex set that lies in the closure of U;*
(iv) *there is a tough subgraph $H \subseteq G$ that contains U;*
(v) *G has a tame star-decomposition such that U is contained in the central part and every critical vertex set of G lives in a leaf's part.*

Moreover, if U is normally spanned in G, then we may add

(i) *G contains a locally finite normal tree that contains U cofinally.*

6 | OUTLOOK

There are a number of possible future research directions and unsolved problems related to this series; here we give a few examples. We have raised the following problem in the first paper [2] of the series (see the paper for definitions):

Problem 6.1 ([2, Problem 3.22]). Characterise, for all connected graphs G, the subsets $\Psi \subseteq \Omega(G)$ or $\Psi \subseteq \Gamma(G)$ for which G admits a rooted tame tree-decomposition displaying Ψ.

6.1 | Finite graphs

A possible version of the star–comb lemma for finite graphs is the following. A *comb of order n* is the union of a finite path P (the comb's *spine*) with n disjoint finite paths, possibly trivial, that have precisely their first vertex on P. The last vertices of those paths are the *teeth* of this comb. Given a vertex set U, a *comb of order n attached to U* is a comb of order n with all its teeth in U, and a *star of order n attached to U* is a subdivision of $K_{1,n}$ with all its leaves in U. Then the set of teeth is the *attachment set* of the comb, and the set of leaves is the *attachment set* of the star.

Lemma 6.2. *For every $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that the following assertion holds: If U is a set of at least n vertices in a connected graph G, then G either contains a star or a comb of order m attached to U.*

Proof. Given $m \in \mathbb{N}$, let $n \in \mathbb{N}$ be large enough such that every tree with at least n vertices either has a vertex of degree at least $m + 1$ or contains a path with at least $m + 1$ vertices (this is an easy instance of [5, Proposition 9.4.1]). Next, let U be any set of at least n vertices in a connected graph G. Pick any rooted tree $T \subseteq G$ that contains U cofinally. Obtain the tree T' from T by suppressing all vertices of degree two that are not contained in U. Then T' has at least $|U| \geq n$ vertices. If T' has a vertex of degree at least $m + 1$, we find a star of order m attached to U in T. Otherwise T' contains a path P with at least $m + 1$ vertices, from which we obtain a comb of order m attached to U in T. □

Problem 6.3. Are there variants of the main results of this series for finite stars and combs in finite graphs?

These variants could build on the definition of stars and combs of order n given above, but other definitions of finite stars and combs could be of interest as well.

6.2 | Beyond graphs

There are a number of objects that generalise graphs. Classical examples are matroids, graph-like spaces, or even topological spaces.

Problem 6.4. Can the main results of this series be generalised to these objects, possibly under some mild assumptions?

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