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van der Giessen, Iris

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Admissible rules for six intuitionistic modal logics

Iris van der Giessen

School of Computer Science, University of Birmingham, Edgbaston, Birmingham, B15 2TT, United Kingdom

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This paper characterizes the admissible rules for six interesting intuitionistic modal logics: iCK4, iCS4 ≡ IPC, strong Löb logic iSL, modalized Heyting calculus mHC, Kuznetsov-Muravitsky logic KM, and propositional lax logic PLL. Admissible rules are rules that can be added to a logic without changing the set of theorems of the logic. We provide a Gentzen-style proof theory for admissibility that combines methods known for intuitionistic propositional logic and classical modal logic. From this proof theory, we extract bases for the admissible rules, i.e., sets of admissible rules that derive all other admissible rules. In addition, we show that admissibility is decidable for these logics.

0. Introduction

Proof-theoretic research of logical systems is usually concerned with axiomatisation and derivation. The aim is to find a minimal set of axioms and rules that define the logic. An interesting question is: what is the maximal set of inference rules? Or in other words, given a logic, which rules can be added without changing the set of theorems of the logic? These rules are called the admissible rules of the logic. For so-called structurally complete logics, for example classical propositional logic CPC, all admissible rules are derivable from its axiomatization. Interestingly, for many other logics there are admissible rules invisible from its axiomatization. Even more interestingly, the admissible rules form an invariant for the logic independent from the chosen axiomatisation. Thereby they give insight in the structure of all possible inferences in a logic.

E-mail address: i.vandergiessen@bham.ac.uk.

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The research on admissible rules got a boost in 1975 with one of Friedman's problems [12]: Is admissibility in intuitionistic propositional logic IPC decidable? The question was positively answered by Rybakov, who published a series of papers showing that admissibility in IPC, many intermediate logics, and many modal logics above K4 is decidable, see [32]. Later, full descriptions of the admissible rules have been established in terms of a basis for many of these logics. A basis is a set of admissible rules that derive all other admissible rules in the logic. The so-called Visser rules form a basis for the admissible rules for IPC, independently shown by Rozière [31] and Iemhoff [19]. Jeřábek [23] constructed modal Visser rules in the setting of classical modal logic. After that multiple papers appeared about descriptions, bases, and complexity of the admissible rules for modal logic and Łukasiewics logic, see e.g. [8,24,33].

This paper is motivated by the question: can the methods and results for IPC and classical modal logic be combined to obtain admissibility results for intuitionistic modal logics? This is a broad question, because intuitionistic modal logics can be defined in different ways. We focus on intuitionistic modal logics only containing the $\Box$, and without a $\Diamond$. Using machinery similar to [22], we show that a natural combination of the admissible rules for IPC and classical modal logics form the admissible rules for six interesting intuitionistic modal logics.

The six logics are: iCK4, iCS4 $\equiv$ IPC, strong Löb logic iSL, modalized Heyting calculus mHC, Kuznetsov-Muravitsky logic KM, and propositional lax logic PLL (see Section 1 for the precise definitions of the logics). The crucial axiom of all these logics is the *coreflection principle*, $A \to \Box A$. In Kripke semantics, this corresponds to frames equipped with a partial order and a modal relation with the strong condition that the modal relation is contained in the partial order. The reason to require the coreflection principle is that it enables us to use existing methods in the research of admissible rules. It imposes a very strong condition on the logics, so that for classical modal logics the box collapses, i.e. $(\Box A \equiv \top) \vee (\Box A \equiv A)$. However, in the intuitionistic setting, these logics have interesting interpretations and applications which we discuss here.

Logic iCK4 is the smallest intuitionistic modal logic with coreflection. It is also known as logic IEL^- that features as an intuitionistic epistemic logic of belief [3]. It is based on the idea that intuitionistic knowledge is the result of verification. The axiom $A \to \Box A$ is interpreted as the fact that a proof of A is a verification of A implying the knowledge of A. In [3], also the intuitionistic epistemic logic of knowledge IEL is discussed. This is not part of our investigation, but we think that the methods presented here can be easily adjusted to this logic.

Logic iCS4 contains the reflexivity axiom $\Box A \to A$ and is thus equivalent to IPC, since together with the coreflection principle we have $\Box A \equiv A$. The admissible rules that we derive for iCS4 are equivalent to the admissible rules known from the literature [19,31,22].

The three logics iSL, mHC, and KM have close connections to provability logic in which the coreflection principle is also called the completeness principle. Logic iSL is an important logic in the study of provability of Heyting Arithmetic. In the classical case, GL is the provability logic of Peano Arithmetic [34]. However, it is an open problem what the provability of Heyting Arithmetic is. In [35], it is shown that iSL is the provability logic of an extension of Heyting Arithmetic with the completeness principle with respect to what is called 'slow provability'. In addition, iSL plays an important role in the Σ_1-provability logic for Heyting Arithmetic [2]. Cut-free sequent calculi and countermodel constructions for iSL are studied in [17].

Muravitsky has written a joyful overview about the birth of logic KM [29]. Logic KM is an extension of iSL and its interest lies in the close relationship between intuitionistic propositional logic and provability logic. Concerning the former one, any modal-free extension of KM is conservative with respect to the corresponding extension of intuitionistic logic, known as *Kuznetsov's Theorem*. Concerning the latter one, the lattices of the extensions of KM and those of GL are isomorphic. The *modalized Heyting calculus* mHC is weaker than KM and is extensively studied by Esakia [10]. He provides different semantics for it. Compared to the work on KM, the lattice of extensions of mHC is isomorphic to the lattice of normal extensions of K4.Grz (announced in [10], and proven in [30]).

Logic PLL is different from the other five logics, containing a modality with flavors from both $\Box$ and $\Diamond$. Therefore its modality is typically denoted by $\bigcirc$. This logic is interesting for different reasons. Lax logic naturally arises from the algebraic study of nuclei and subframe logics [6,7,5]. Goldblatt frames provide a semantics involving topology [18]. It also has practical applications using a Curry-Howard isomorphism [4] and for software verification using constraint models (that we also use in this paper) [11]. From a proof-theoretical point of view it has a lot of similarities to other modal logics, despite the non-standard properties of the modality [21].

The main contribution of this paper is a full description of the admissible rules of these logics using the same strategy from Iemhoff and Metcalfe [22]. They provide Gentzen-style proof systems for admissibility for IPC and several modal logics above K4. We combine these systems into a system for admissibility of the intuitionistic modal logics that we study. The admissibility proof systems have three nice properties. First, in contrast to well-known proof systems for logics that reason about formulas or sequents, these admissibility proof systems reason about rules. In other words, they contain rules about rules. Second, the shape of the rules in these proof systems is independent of the proof theory of the logics. Third, we can immediately conclude that the admissible rules for the logics are decidable, based on the decidability of the logic.

The paper is structured as follows. Section 1 provides definitions of the logics, Kripke semantics, and admissibility. Sections 2 to 4 treat logics iCK4, iSL, iCS4, mHC, and KM. Section 2 is a technical discussion about projective formulas and the extension property, which play an important role [13]. For readers interested in the main story, we advise to only read the definitions and skip the proofs. The main results are presented in Section 3 which provides the Gentzen-style proof systems for admissibility. Section 4 discusses the bases for the five logics. Logic PLL is treated separately in Section 5 along the same line of reasoning.

1. Preliminaries

We consider the modal language with constant $\bot$, (propositional) atoms $p, q, \ldots$, connectives $\wedge, \vee, \to$ and modal operator $\Box$ (no $\Diamond$). We denote Prop for the set of all atoms and Form denotes the set of all well-formed formulas in this language. We use the notation $\overline{p}$ to represent a finite list of atoms.

Given formula A, $\mathsf{Var}(A)$ is the set of atoms occurring in A. Similarly for set of formulas Γ, $\mathsf{Var}(\Gamma)$ is the set of atoms occurring in formulas from Γ. And given formulas A and B, the following formulas are defined as usual:

$$\neg A := A \to \bot \qquad A \equiv B := (A \to B) \wedge (B \to A)$$
$$\top := \neg\bot \qquad \boxdot A := A \wedge \Box A.$$

We call formulas of the form $p \to q$ and $\Box p$, *atom implications* and *boxed atoms*, respectively. Given a finite set of formulas Γ, we define

$$\Box\Gamma := \{\Box A \mid A \in \Gamma\},$$
$$\boxdot\Gamma := \Gamma \cup \Box\Gamma,$$
$$\Box\Gamma \to \Gamma := \{\Box A \to A \mid A \in \Gamma\}.$$

We write $\bigwedge\Gamma$ and $\bigvee\Gamma$ for the iterated conjunction and disjunction of Γ, where $\bigwedge\emptyset := \top$ and $\bigvee\emptyset := \bot$, by definition. A sequent is an expression between finite sets of formulas, denoted $\Gamma \Rightarrow \Delta$. Its formula interpretation is defined as usual as $I(\Gamma \Rightarrow \Delta) = \bigwedge\Gamma \to \bigvee\Delta$. We often write, Γ, Δ for $\Gamma \cup \Delta$ and Γ, A for $\Gamma \cup \{A\}$. We let S range over sequents and we let $\mathcal{G}, \mathcal{H}$ range over finite sets of sequents. Similarly, we often write $\mathcal{G}, \mathcal{H}$ for $\mathcal{G} \cup \mathcal{H}$ and $\mathcal{G}, S$ for $\mathcal{G} \cup \{S\}$. We write $I(\mathcal{G})$ to mean $\{I(S) \mid S \in \mathcal{G}\}$.

(k)	$\Box(A \to B) \to \Box A \to \Box B$	(t)	$\Box A \to A$	(derv)	$\Box A \to ((B \to A) \vee B)$
(c)	$A \to \Box A$	(wlöb)	$\Box(\Box A \to A) \to \Box A$	(bind)	$\bigcirc\bigcirc A \to \bigcirc A$
(4)	$\Box A \to \Box\Box A$	(slöb)	$(\Box A \to A) \to A$		

Fig. 1. Modal axioms.

$\mathsf{iCK4} := \mathsf{iK} + (\mathsf{c})$,	$\mathsf{mHC} := \mathsf{iCK4} + (\mathsf{derv})$,
$\mathsf{iCGL} := \mathsf{iCK4} + (\mathsf{wlöb}) \equiv \mathsf{iSL}$,	$\mathsf{KM} := \mathsf{iCK4} + (\mathsf{wlöb}) + (\mathsf{derv})$,
$\mathsf{iCS4} := \mathsf{iCK4} + (\mathsf{t}) \equiv \mathsf{IPC}$,	$\mathsf{PLL} := \mathsf{iCK4} + (\mathsf{bind})$.

Fig. 2. Intuitionistic modal logic with coreflection.

We consider *intuitionistic normal modal logics* L, which is a set of formulas containing all *intuitionistic* tautologies, the (k)-axiom (see Fig. 1), and is closed under modus ponens (if A in L and $A \to B$ in L, then B in L), uniform substitution and necessitation (if A in L then $\Box A$ in L). We call the minimal logic with these axioms and rules iK.

For all logics that we consider in this paper, we require the *coreflection axiom* (c). Note that (c) is denoted by (R) and (r) in [11] and [27], respectively. We work with the logics as defined in Fig. 2 with axioms from Fig. 1. See [27] for a nice schema of intuitionistic modal logics including these.

We mention a few things. Axiom (k) is redundant for the axiomatization of KM, see [26]. The weak Löb axiom (wlöb) is also known as the Löb axiom (löb). And logic iCGL is equivalent to intuitionistic strong Löb logic iSL [27]:

$$\mathsf{iSL} := \mathsf{iK} + (\mathsf{slöb}).$$

If Γ is a set of formulas, $\Gamma \vdash_{\mathsf{L}} A$ means that A is provable from Γ using axioms and rules of L. Note that there is no distinction between the local and global consequence relation, in particular, we also have $p \vdash_{\mathsf{L}} \Box p$ if we read $\vdash_{\mathsf{L}}$ in terms of the local consequence relation. This is due to the axiom (c) and modus ponens. Moreover, we have the deduction theorem which can be shown by induction on the derivations in $\vdash_{\mathsf{L}}$, cf. Theorem 3.51 from [9].

Theorem 1.1 *(Deduction theorem). Let* $\mathsf{L} \in \{\mathsf{iCK4}, \mathsf{iSL}, \mathsf{iCS4}, \mathsf{mHC}, \mathsf{KM}, \mathsf{PLL}\}$. *For all finite sets of formulas* Γ *and formulas* A *and* B *it holds that*

$$\Gamma, A \vdash_{\mathsf{L}} B \textit{ if and only if } \Gamma \vdash_{\mathsf{L}} A \to B.$$

The reason that we require the coreflection axiom is that it enables us to use existing methods in the research of admissible rules. It imposes a very strong condition on the logics. It immediately implies that $\boxdot A \equiv A$. It also implies transitivity of the Kripke frames, which is the reason why we include $\mathsf{4}$ in the name of base logic $\mathsf{iCK4}$. For classical modal logic it leads to trivial logics, i.e. $\Box A \equiv \top \vee \Box A \equiv A$. To be more specific, for classical Gödel-Löb logic GL plus coreflection, we obtain $\Box A \equiv \top$ and for the reflexive logic T plus coreflection we have $\Box A \equiv A$. Although it seems a very strong condition, the six intuitionistic logics in this paper have interesting applications as explained in the Introduction.

In this paper, we consider *strong intuitionistic modal Kripke frames* which are structures $(W, \leq, R)$, where W is a finite set of worlds with partial order $\leq$ and binary relation R, that satisfy the following requirements:

iCK4	finite strong Kripke frames	[27]
iSL	finite strong irreflexive Kripke frames	[17]
iCS4	finite strong reflexive Kripke frames, i.e. $R = \leq$	[27]
mHC	finite strong Kripke frames with $< \subseteq R$	[10]
KM	finite strong irreflexive Kripke frames with $< \subseteq R$, i.e. $R = <$	[28]
PLL	finite constraint frames	[11]

Fig. 3. Finite model property for logics with coreflection.

1. R is closed under prefixing with $\leq$, i.e. $x \leq yRz$ implies xRz,
2. *strongness*: xRy implies $x \leq y$.

The first requirement is standard for intuitionistic modal logics. The strongness condition on frames corresponds to the coreflection axiom (c). The two conditions imply that R is also transitive.

A *strong intuitionistic modal Kripke model* is a structure $K = (W, \leq, R, V)$ where $(W, \leq, R)$ is a strong intuitionistic modal Kripke frame and $V : W \times \mathsf{Prop} \to \{0, 1\}$ a valuation that is monotonic in $\leq$, i.e., $w \leq w' \Rightarrow V(w, p) \leq V(w', p)$. We simply call these models strong Kripke models.

We consider rooted frames with root ρ satisfying $\rho \leq w$ for all w. For model $\mathsf{K} = (W, \leq, R, V)$ we write $w \in K$ to mean $w \in W$. The forcing relation $\Vdash$ is defined as usual as follows:

$$
\begin{array}{ll}
K, w \Vdash p & \text{iff } V(w, p) = 1, \\
K, w \Vdash \bot & \text{never}, \\
K, w \Vdash A \wedge B & \text{iff } K, w \Vdash A \text{ and } K, w \Vdash B, \\
K, w \Vdash A \vee B & \text{iff } K, w \Vdash A \text{ or } K, w \Vdash B, \\
K, w \Vdash A \to B & \text{iff for all } w' \geq w,\ K, w' \Vdash A \text{ implies } K, w' \Vdash B, \\
K, w \Vdash \Box A & \text{iff for all } v \text{ such that } wRv, \text{ we have } K, v \Vdash A.
\end{array}
$$

We write $K \models A$ to mean $K, w \Vdash A$ for every $w \in K$ and say that K *satisfies* A. Note that this is equivalent to $K, \rho \Vdash A$. We write $K(w) := \{p \mid K, w \Vdash p\}$ for the set of atoms forced in w. We denote K_v for the submodel of K generated by v. By definition, v is the root of K_v. We say that model K *almost satisfies* A if $K_w \models A$ for all w except for possibly $w = \rho$.

Fig. 3 presents the completeness with respect to Kripke frames shown in corresponding cited papers [10,11,17,27,28]. As validity is preserved in generated submodels, the completeness result also holds for the corresponding rooted frames. We call these rooted frames L-frames and models based on these L-models. The finite model property is not explicitly shown for iCK4 and mHC in the mentioned references, but can be obtained by a filtration argument. Note that the iCS4 models correspond to IPC-models, because $R = \leq$. We write $Mod_{\mathsf{L}}(A)$ to be the set of all L-models that satisfy A.

For PLL, we work with constraint models after Fairtlough and Mendler [11], which we will define in Section 5. Although PLL is sound and complete with respect to finite strong dense Kripke frames due to Goldblatt [18], we cannot use these models for our purpose. We want to be able to add a new root to each finite set of L-models to create a new L-model (used in Theorem 3.6). Logics with this property are called *extensible* logics. For logics $\mathsf{L} \in \{\mathsf{iCK4}, \mathsf{iSL}, \mathsf{iCS4}, \mathsf{mHC}, \mathsf{KM}\}$, this is the case. However, for the Goldblatt frames it is not guaranteed that the new created model is indeed dense. The constraint models avoid the problem.

All our logics $\mathsf{L} \in \{\mathsf{iCK4}, \mathsf{iSL}, \mathsf{iCS4}, \mathsf{mHC}, \mathsf{KM}, \mathsf{PLL}\}$ are finitely axiomatizable and have the finite model property, this means that these logics are decidable.

Theorem 1.2 *(decidability). Logics* iCK4, iSL, iCS4, mHC, KM, *and* PLL *are decidable.*

1.1. Admissibility

The main purpose of this paper is to construct a proof theory for admissibility for iCK4, iCK4, iSL, mHC, KM, and PLL following strategies from [22]. This proof theory deals with *generalized rules*

$$\frac{A_1, \ldots, A_k}{B_1, \ldots, B_l}$$

with multiple formulas in its conclusion. Generalized rules are interesting to study and is extensively done so since Jeřábek [23]. They capture the disjunction property, which is an important admissible rule in logics like S4, GL, and IPC, and also for the intuitionistic modal logics with coreflection that we study in this paper (see Example 3.4). See also [1] for a study of the disjunction property in other intuitionistic modal logics.

A *substitution* is a function $\sigma : \mathsf{Form} \to \mathsf{Form}$ that commutes with $\bot$, the connectives, and the modality $\Box$. An L-*unifier* for A is a substitution σ such that $\vdash_{\mathsf{L}} \sigma(A)$.

Definition 1.3. A generalized rule Γ/Δ is admissible in L, written $\Gamma \mid\!\sim_{\mathsf{L}} \Delta$, if every substitution σ that is an L-unifier of all formulas in Γ is an L-unifier of some formula in Δ.

The set of all admissible rules can be described by a basis. A basis is a set of admissible rules that derive all other admissible rules of the logic. For this we introduce multi-conclusion consequence relations. We introduce the definitions here and characterize specific bases in Section 4. We write $\sigma(\Gamma)$ for the set $\{\sigma(A) \mid A \in \Gamma\}$.

Definition 1.4. A *finitary multi-conclusion consequence relation* or *m-consequence relation*, denoted $\vdash^m$ or simply $\vdash$, is a relation between finite sets of formulas which satisfies the following properties where A is a formula and $\Gamma, \Pi, \Delta, \Sigma$ are finite sets of formulas:

reflexivity:	$A \vdash A$;
monotonicity:	if $\Gamma \vdash \Delta$ then $\Gamma, \Pi \vdash \Delta, \Sigma$;
transitivity:	if $\Gamma \vdash \Delta, A$ and $A, \Pi \vdash \Sigma$ then $\Gamma, \Pi \vdash \Delta, \Sigma$;
structurality:	if $\Gamma \vdash \Delta$ then $\sigma(\Gamma) \vdash \sigma(\Delta)$.

Formula A is said to be a *theorem* of this m-consequence relation if $\vdash A$. The set of all theorems of m-consequence relation $\vdash$ is denoted by $Th(\vdash) := \{A \mid \vdash A\}$.

We define m-consequence relation based on logic L as follows, denoted $\vdash^m_{\mathsf{L}}$,

$$\Gamma \vdash^m_{\mathsf{L}} \Delta \text{ if and only if } \bigwedge \Gamma \to A \in \mathsf{L} \text{ for some } A \in \Delta.$$

It is easy to see that $\Gamma \vdash^m_{\mathsf{L}} \Delta$ implies $\Gamma \mid\!\sim_{\mathsf{L}} \Delta$, so derivability implies admissibility. Note that with the deduction theorem in L we have $\Gamma \vdash^m_{\mathsf{L}} A$ if and only if $\Gamma \vdash_{\mathsf{L}} A$ for each formula A.

Given a set of rules $\mathcal{R}$ and an m-consequence relation $\vdash$ we denote the smallest m-consequence relation extending $\vdash$ and $\mathcal{R}$ as $\vdash_{\mathcal{R}}$.

Definition 1.5. A set of rules $\mathcal{R}$ is called a *basis* for $\mid\!\sim_{\mathsf{L}}$ if $\mid\!\sim_{\mathsf{L}} = \vdash^m_{\mathsf{L},\mathcal{R}}$.

So far we introduced generalized rules and m-consequence relations with multiple conclusions. In Section 4, we will also shortly discuss bases for single-conclusion admissible rules. A *single-conclusion rule* is a special generalized rule, having exactly one formula B_1 in its conclusion. A *single-conclusion* consequence relation is similarly defined as an m-consequence relation, but then reading $\Gamma \vdash \Delta$ with $|\Delta| = 1$. A basis for the single-conclusion rules is defined analogously to the multi-conclusion setting. Finally, note that in the single-conclusion setting, admissibility is equivalent to the statement that $\mathrel{|\!\sim}_{\mathsf{L}}$ is the maximal single-conclusion consequence relation $\vdash$ containing $\vdash_{\mathsf{L}}$ such that $Th(\vdash) = \mathsf{L}$, see [20].

2. Projective formulas and the extension property

Projective formulas are useful in the study of admissible rules, because for these formulas admissibility reduces to derivability as shown in Lemma 2.2. Moreover, in many decidable intermediate and modal logics, projectivity is decidable as shown by Ghilardi [13,14]. It results in a new proof of the decidability of the admissible rules in these logics, replacing the method by Rybakov, see e.g. [32]. The decidability of projective formulas follows from a semantic characterization in terms of the extension property. This section provides the characterization for the intuitionistic modal logics with coreflection considered in this paper.

Definition 2.1. A formula A is *projective* in L if there is an L-unifier σ for it such that

$$A \vdash_{\mathsf{L}} p \leftrightarrow \sigma(p) \text{ for all atoms } p. \tag{1}$$

We call σ an L-*projective unifier* for A.

By induction it is easily checked that also $A \vdash_{\mathsf{L}} B \leftrightarrow \sigma(B)$, for each formula B.

Lemma 2.2. *If A is projective in L, then $A \mathrel{|\!\sim}_{\mathsf{L}} \Delta$ if and only if $A \vdash_{\mathsf{L}} B$ for some $B \in \Delta$.*

Proof. From right to left follows immediately. For the other direction suppose $A \mathrel{|\!\sim}_{\mathsf{L}} \Delta$. This means that each unifier for A is a unifier for some $B \in \Delta$. Let σ be a projective unifier for A, then $\vdash_{\mathsf{L}} \sigma(B)$ for some $B \in \Delta$. Also $A \vdash_{\mathsf{L}} B \leftrightarrow \sigma(B)$, therefore $A \vdash_{\mathsf{L}} B$. □

Now we turn to the extension property. Here we provide the characterization for all logics except PLL. Although the method presented here directly applies to Goldblatt frames for PLL, we are interested in the characterization for constraint models for PLL as presented in Section 5. So for the rest of this section, let $\mathsf{L} \in \{\mathsf{iCK4}, \mathsf{iSL}, \mathsf{iCS4}, \mathsf{mHC}, \mathsf{KM}\}$.

Definition 2.3. A *variant* of an L-model K is an L-model K', such that they have the same frame and their valuation agree on all worlds except for possibly the root. A class $\mathcal{K}$ of L-models is said to have the *extension property* if for every model K, if $K_w \in \mathcal{K}$ for each $w \neq \rho$, then there is a variant K' of K such that $K' \in \mathcal{K}$.

Theorem 2.4. *Formula A is projective in L if and only if $Mod_{\mathsf{L}}(A)$ has the extension property.*

We show the two directions separately in Theorem 2.6 and Theorem 2.11. Given substitution σ, the semantic operator σ^* on models is defined as follows. For model K, model $\sigma^*(K)$ has the same frame as K and its valuation is defined according to:

$$\sigma^*(K), w \Vdash p \Leftrightarrow K, w \Vdash \sigma(p) \text{ for all } p.$$

Note that model $\sigma^*(K)$ is well defined, in particular, its valuation map is monotone. The following lemma is shown by induction on A, see the proof of Proposition 1.3 in [14].

Lemma 2.5. *Let A be a formula and let σ be a substitution. For every Kripke L-model K, we have*

(i) $\sigma^*(K) \models A$ *iff* $K \models \sigma(A)$,
(ii) *and for every substitution* τ, $(\tau\sigma)^*(K) = \sigma^*(\tau^*(K))$.

Theorem 2.6. *If formula* A *is projective in* L, *then* $Mod_{\mathsf{L}}(A)$ *has the extension property.*

Proof. Let K be an L-model almost satisfying A. Since A is projective, we can take a substitution σ such that $\vdash_{\mathsf{L}} \sigma(A)$ and $A \vdash_{\mathsf{L}} p \leftrightarrow \sigma(p)$ for all atoms p. We will show that $\sigma^*(K)$ is a variant of K that satisfies A. $\sigma^*(K)$ is a variant, because for all $w \neq \rho$ we have $\sigma^*(K)(w) = K(w)$, since $K_w \models A$ and $A \vdash_{\mathsf{L}} p \leftrightarrow \sigma(p)$ for each atom p. And $\sigma^*(K) \models A$ follows from $\vdash_{\mathsf{L}} \sigma(A)$ by Kripke completeness and Lemma 2.5. □

Now we focus on the other direction of Theorem 2.4. In [14], Ghilardi treats classical modal logics using a bisimulation argument. See also [16] for an extensive analysis of the role of bisimulation in this work. For IPC, there is an easier method [13]. This will be the method that we follow here.

Analogous to [13], we define simple substitutions σ_a that satisfy $A \vdash_{\mathsf{L}} p \leftrightarrow \sigma(p)$ for all p. Let $\overline{p}$ be the atoms occurring in A, that is, $\mathsf{Var}(A) = \overline{p}$. Let $a \subseteq \overline{p}$; the substitution σ_a is defined as follows for all atoms p:

$$\sigma_a^A(p) = \begin{cases} A \to p & \text{if } p \in a, \\ A \wedge p & \text{if } p \notin a. \end{cases}$$

Note that in [13,14], substitutions are restricted to the subset of formulas only containing atoms from $\overline{p}$. But here we assume substitutions defined on all formulas Form. This makes no difference in the applied method. We simply write σ_a when A is clear from the context. The following lemma is similarly proved as Lemma 2.2 in [13].

Lemma 2.7. *Let* A *be a formula with* $\mathsf{Var}(A) = \overline{p}$ *and let* K *be an* L*-model. Suppose* $a \subseteq \overline{p}$. *For all* $w \in K$, *we have*
(i) $(\sigma_a^*(K))(w) = K(w)$, *if* $K_w \models A$,
(ii) $(\sigma_a^*(K))(w) \subseteq a$, *if* $K_w \not\models A$, *more precisely,*

$$(\sigma_a^*(K))(w) = \{p \in a \mid \text{ for all } w' > w \text{ if } K_{w'} \models A, \text{ then } K_{w'} \models p\},$$

(iii) $\sigma_a^* \sigma_a^* = \sigma_a^*$.

Lemma 2.8. *Let* A *be a formula with* $\mathsf{Var}(A) = \overline{p}$ *and suppose that* $Mod_{\mathsf{L}}(A)$ *has the extension property. Let* K *be an* L*-model that almost satisfies* A. *Then there is a subset* $a \subseteq \overline{p}$ *such that* $\sigma_a^*(K) \models A$ *and* $\sigma_a^*(K)(\rho) = a$.

Proof. $Mod_{\mathsf{L}}(A)$ has the extension property, so there is a variant K' of K that satisfies A. Since the validity of A only depends on atoms from $\overline{p}$, we may assume that the only atoms forced in the root are among these atoms. Take $a = K'(\rho)$. For all w such that $w \neq \rho$ we have $\sigma_a^*(K_w) = K_w$ by Lemma 2.7 (i) and so $\sigma_a^*(K_w) \models A$. For the root ρ we have $\sigma_a^*(K)(\rho) = a$, because of Lemma 2.7 (ii) and monotonicity of $\leq$. Moreover, we have proved that $\sigma_a^*(K) = K'$ and so $\sigma_a^*(K) \models A$. □

We use the following notions, introduced in [14,16], where we explicitly define frontier points in contrast to [13]. We write $|a|$ for the cardinality of set a. Let A be a formula with $\mathsf{Var}(A) = \overline{p}$ and suppose that $Mod_{\mathsf{L}}(A)$ has the extension property. Let K be an L-model.

- The set of *frontier points* in K is

$$\{w \in K \mid K_w \not\models A \text{ and } \forall v (w < v \Rightarrow K_v \models A)\}.$$

- Let w be a frontier point in K and let σ_a^* be defined as in Lemma 2.8 for K_w such that for all b that also satisfy the properties of Lemma 2.8 we have $|b| \leq |a|$. We fix such a and call it the *corresponding substitution* for K_w.
- Let w be a frontier point in K. We define the *rank* of K_w to be $r(K_w) = |a|$, where σ_a^* is the corresponding substitution of K_w.
- We call a frontier point w *maximal* in K, if $r(K_w) \geq r(K_v)$, for all other frontier points v in K.

Frontier points are worlds w such that K_w almost satisfies A. Now we define θ as in [13], that is, $\theta := \sigma_{a_1} \dots \sigma_{a_s}$, where the a_i are subsets of $\overline{p}$ ordered according to $a_i \supseteq a_j$ implies $i \leq j$. We will show that θ is a projective unifier for A. We can divide θ in parts as follows:

$$\theta = \tau_m \cdots \tau_1 \tau_0, \tag{2}$$

where τ_j contains the σ_a's for which $|a| = j$ and m is the number of atoms in $\overline{p}$.

Lemma 2.9. *Let A be a formula with $\mathsf{Var}(A) = \overline{p}$ and suppose that $Mod_{\mathsf{L}}(A)$ has the extension property. Let K be an L-model and let k be the rank of its maximal frontier points. Then, for any frontier point v in $\tau_k^*(K)$, we have $r(\tau_k^*(K_v)) < k$.*

Proof. The result follows from induction on the number of simple substitutions occurring in τ_k^*. To see this, let us show that for any L-model M with maximal frontier points of rank k, for any σ_a^* from τ_k^*, for any frontier point v in $\sigma_a^*(M)$, and for any frontier point w in M such that $v \leq w$ we have,

$$r(\sigma_a^*(M_v)) \leq r(M_w). \tag{3}$$

We distinguish two cases. If $\sigma_a^*(M_w) \not\models A$, then $v = w$ and $r(\sigma_a^*(M_w)) = r(M_w) \leq k$ since the corresponding substitution of M_w only depends on the submodels above w, which remains the same after application of σ_a^* by Lemma 2.7 (i). Now suppose $\sigma_a^*(M_w) \models A$. Then by Lemma 2.7 (i) and (ii), $a' := (\sigma_a^*(M))(w)) \subseteq a$ also satisfies $\sigma_{a'}^*(M_w) \models A$. Let σ_b be the corresponding substitution of M_w. In particular, $|a'| \leq |b|$. Suppose σ_c^* is the corresponding substitution of $\sigma_a^*(M_v)$. So $\sigma_c^* \sigma_a^*(M_v) \models A$, and $(\sigma_c^* \sigma_a^*(M))(v) = c$. By Lemma 2.7 (i) and the monotonicity in $\leq$, we know that $c \subseteq a'$. Therefore $r(\sigma_a^*(M_v)) = |c| \leq |a'| \leq |b| = r(M_w)$.

Now, if σ_a is the corresponding substitution of M_w, i.e. $r(M_w) = k$, we show that

$$r(\sigma_a^*(M_v)) < r(M_w) = k. \tag{4}$$

This is sufficient to show the desired result in the lemma, because (3) takes care of frontier points w with rank strictly lower than k and (4) takes care of frontier points with rank k, since for these there is a corresponding substitution among τ_k^* reducing the rank of the new frontier points below k. We show (4) by contradiction by supposing that $r(\sigma_a^*(M_v)) = r(M_w)$. We will show that $\sigma_a^*(M_v) \models A$ and so v cannot be a frontier point in $\sigma_a^*(M)$ after all. σ_a is the corresponding substitution of M_w. So using the observations from above we see that $a' = a$ and so $c \subseteq a$. But $|c| = |a|$ by assumption, so $a = c$. Thus $\sigma_a^* \sigma_a^*(M_v) \models A$ and by Lemma 2.7 (iii) we have $\sigma_a^*(M_v) \models A$. □

Theorem 2.10. *Let A be a formula with $\mathsf{Var}(A) = \overline{p}$ and suppose that $Mod_{\mathsf{L}}(A)$ has the extension property. Then $\theta^*(K) \models A$ for all L-models K.*

Proof. Let $\overline{p} = \{p_1, \dots, p_m\}$, so the rank is at most m and $\theta = \tau_m \cdots \tau_1 \tau_0$ as defined before. From Lemma 2.9 it follows that for each $m \geq k \geq 0$ the rank of maximal frontier points in $(\tau_k)^* \cdots (\tau_m)^*(K)$ is smaller than k. So for $k = 0$, we have that $(\tau_0)^* \cdots (\tau_m)^*(K)$ does not contain any frontier point. Therefore $(\tau_0)^* \cdots (\tau_m)^*(K) \models A$. So $\theta^*(K) \models A$. □

Theorem 2.11. *If $Mod_{\mathsf{L}}(A)$ has the extension property, then formula A is projective in* L.

Proof. This is an immediate consequence from Theorem 2.10 and the observation that θ satisfies property (1) making it a projective unifier for A. $\square$

Corollary 2.12. *Projectivity is decidable in* $\mathsf{L} \in \{\mathsf{iCK4}, \mathsf{iSL}, \mathsf{iCS4}, \mathsf{mHC}, \mathsf{KM}\}$.

Proof. From Theorem 2.4 and Theorem 2.11 it follows that A is projective iff θ is a unifier for A iff $Mod_{\mathsf{L}}(A)$ has the extension property. Using Theorem 1.2, it is decidable to check whether θ is a unifier for A or not. $\square$

3. Proof system for admissible rules

We determine the admissible rules via a sequent system for admissibility following [22]. Therefore we introduce sequent versions of generalized rules.

Definition 3.1. A *generalized sequent rule (gs-rule)* $\mathtt{R}$ is an ordered pair of finite sets of sequents, written $\mathcal{G} \rhd \mathcal{H}$.

- $\mathtt{R}$ is *admissible (in* L*)*, written $\mathrel{|\!\sim}_{\mathsf{L}} \mathtt{R}$, iff $I(\mathcal{G}) \mathrel{|\!\sim}_{\mathsf{L}} I(\mathcal{H})$. This means that each unifier for all $I(S)$ with $S \in \mathcal{G}$ is a unifier for $I(S')$ for some $S' \in \mathcal{H}$.
- $\mathtt{R}$ is *derivable (in* L*)*, written $\vdash_{\mathsf{L}} \mathtt{R}$, iff $I(\mathcal{G}) \vdash^m_{\mathsf{L}} I(\mathcal{H}), \bot$. This means that $I(\mathcal{G}) \vdash_{\mathsf{L}} I(S)$ for some $S \in \mathcal{H}$ or $I(\mathcal{G}) \vdash_{\mathsf{L}} \bot$.

The reason that we include $\bot$ in the definition of derivability is that it allows us to speak about derivability in case $\mathcal{H}$ is empty. The rules that we use in the proof system of admissibility consist of inferences between gs-rules. Therefore we consider rules that reason about rules in the form

$$\frac{\mathtt{R}_1 \qquad \dots \qquad \mathtt{R}_n}{\mathtt{R}}\ .$$

Definition 3.2. Such a rule is called

- L-*sound* if whenever $\mathrel{|\!\sim}_{\mathsf{L}} \mathtt{R}_i$ for all i, then $\mathrel{|\!\sim}_{\mathsf{L}} \mathtt{R}$,
- L-*invertible* if whenever $\mathrel{|\!\sim}_{\mathsf{L}} \mathtt{R}$, then $\mathrel{|\!\sim}_{\mathsf{L}} \mathtt{R}_i$ for all i.

Note that this definition of invertibility is with regard to admissibility in L and is not invertible in the usual sense with regard to a certain proof system.

Now we define the proof systems for admissibility for five of our logics, denoted by GAL for logic L. Logic PLL is treated in Section 5, so for the rest of this section, let $\mathsf{L} \in \{\mathsf{iCK4}, \mathsf{iCS4}, \mathsf{iSL}, \mathsf{KM}, \mathsf{mHC}\}$. The rules from Fig. 4 combine the intuitionistic propositional rules and the modal rules from [22]. All proof systems contain the rules from Fig. 4 except that $\mathsf{GAiCS4}$, GAiSL, and GAKM only contain (AC) and not ($\mathrm{AC}_\Box$). In addition, each proof system has logic specific modal Visser rules from Fig. 5 defined as follows:

$$\begin{aligned} (\mathrm{V}^{\bullet,i,1}), (\mathrm{V}^{\circ,i}) &\in \mathsf{GAiCK4}, & (\mathrm{V}^{\bullet,i,2}), (\mathrm{V}^{\circ,i}) &\in \mathsf{GAmHC},\\ (\mathrm{V}^{\circ,i}) &\in \mathsf{GAiCS4}, & (\mathrm{V}^{\bullet,i,2}) &\in \mathsf{GAKM}.\\ (\mathrm{V}^{\bullet,i,1}) &\in \mathsf{GAiSL}, \end{aligned}$$

Remark 3.3. In contrast to [22], we choose to leave out the right logical rules as presented there and replace it by the one right rule $\rhd()$ that reflects the truth of a sequent in logic L. This gives us the freedom to use the

Right rule

$$\frac{}{\mathcal{G} \rhd S, \mathcal{H}} \rhd(), I(S) \in \mathsf{L}$$

Left logical rules

$$\frac{\mathcal{G} \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, \bot \Rightarrow \Delta) \rhd \mathcal{H}} (\bot \Rightarrow)\rhd \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \bot, \Delta) \rhd \mathcal{H}} (\Rightarrow \bot)\rhd$$

$$\frac{\mathcal{G}, (\Gamma, A, B \Rightarrow \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, A \wedge B \Rightarrow \Delta) \rhd \mathcal{H}} (\wedge \Rightarrow)\rhd \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow A, \Delta), (\Gamma \Rightarrow B, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \wedge B, \Delta) \rhd \mathcal{H}} (\Rightarrow \wedge)\rhd$$

$$\frac{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Gamma, B \Rightarrow \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, A \vee B \Rightarrow \Delta) \rhd \mathcal{H}} (\vee \Rightarrow)\rhd \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow A, B, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \vee B, \Delta) \rhd \mathcal{H}} (\Rightarrow \vee)\rhd$$

$$\frac{\mathcal{G}, (\Gamma, A \to B \Rightarrow A, \Delta), (\Gamma, B \Rightarrow \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, A \to B \Rightarrow \Delta) \rhd \mathcal{H}} (\to)\rhd$$

$$\frac{\mathcal{G}, (\Gamma, p \to q \Rightarrow \Delta), (p \Rightarrow A), (B \Rightarrow q) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, A \to B \Rightarrow \Delta) \rhd \mathcal{H}} (\to\Rightarrow)\rhd^i \quad \frac{\mathcal{G}, (\Gamma \Rightarrow p, \Delta), (p, A \Rightarrow B) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \to B, \Delta) \rhd \mathcal{H}} (\Rightarrow\to)\rhd^i$$

$$\frac{\mathcal{G}, (\Gamma, \Box p \Rightarrow \Delta), (A \Rightarrow p) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, \Box A \Rightarrow \Delta) \rhd \mathcal{H}} (\Box \Rightarrow)\rhd \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow \Box p, \Delta), (p \Rightarrow A) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \Box A, \Delta) \rhd \mathcal{H}} (\Rightarrow \Box)\rhd$$

p, q do not occur in $\mathcal{G}, \mathcal{H}, \Gamma, \Delta, A, B$ in $(\to\Rightarrow)\rhd^i, (\Rightarrow\to)\rhd^i, (\Box \Rightarrow)\rhd$ and $(\Rightarrow \Box)\rhd$.

Anti-cut, anti-cut for boxed formulas, and projection rule

$$\frac{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Pi \Rightarrow A, \Sigma), (\Gamma, \Pi \Rightarrow \Sigma, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Pi \Rightarrow A, \Sigma) \rhd \mathcal{H}} (\mathrm{AC})$$

$$\frac{\mathcal{G}, (\Gamma, \Theta \Rightarrow \Delta), (\Pi \Rightarrow \Psi, \Sigma), (\Gamma, \Pi, \Box A \to A \Rightarrow \Sigma, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, \Theta \Rightarrow \Delta), (\Pi \Rightarrow \Psi, \Sigma) \rhd \mathcal{H}} (\mathrm{AC}_\Box)$$

where A is box-free, $(\Theta \cup \Psi) \subseteq \{A, \Box A\}$ and $\Theta, \Psi \neq \emptyset$.

$$\frac{\mathcal{G}, S \rhd (\Gamma, I(S) \Rightarrow \Delta), \mathcal{H}}{\mathcal{G}, S \rhd \mathcal{H}} (\mathrm{PJ}), \Gamma \Rightarrow \Delta \in \mathcal{H} \cup \{\Rightarrow\}$$

Fig. 4. Rules for admissibility.

semantic notions of the logic instead of searching for a sequent calculus that reflects the derivability of the logic. In a sense, this shows that the shape of the rules in the proof systems for admissibility is independent of the proof theory of the logics.

So, the right-hand side of a gs-rule $\mathcal{G} \rhd \mathcal{H}$ reflects derivability/truth in logic L. The left-hand side of a gs-rule $\mathcal{G} \rhd \mathcal{H}$ captures the admissibility. For each logic, we have logic specific Visser rules depending on their Kripke semantics. We have two rules that reflects irreflexive extensions of models and one rule that reflects reflexive extensions. Informally speaking, these rules are fusions from the modal Visser rules and

Irreflexive

$$\frac{\begin{array}{c}[\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Box\Sigma, \Gamma \Rightarrow D) \rhd \mathcal{H}]_{D\in\Delta} \\ [\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Sigma, \Gamma \Rightarrow O) \rhd \mathcal{H}]_{O\in\Omega} \\ [\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \rhd (\Box\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta), \mathcal{H}]_{\emptyset\neq\Pi\subseteq\Gamma_{\Box\Omega,\Delta}}\end{array}}{\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \rhd \mathcal{H}} \ (\text{V}^{\bullet,i,1})$$

$$\frac{\begin{array}{c}[\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Sigma, \Gamma \Rightarrow D) \rhd \mathcal{H}]_{D\in\Delta} \\ [\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Sigma, \Gamma \Rightarrow O) \rhd \mathcal{H}]_{O\in\Omega} \\ [\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \rhd (\Box\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta), \mathcal{H}]_{\emptyset\neq\Pi\subseteq\Gamma_{\Box\Omega,\Delta}}\end{array}}{\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \rhd \mathcal{H}} \ (\text{V}^{\bullet,i,2})$$

Reflexive

$$\frac{\begin{array}{c}[\mathcal{G}, (\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Sigma, \Gamma \Rightarrow D) \rhd \mathcal{H}]_{D\in\Delta} \\ [\mathcal{G}, (\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Sigma, \Gamma \Rightarrow O) \rhd \mathcal{H}]_{O\in\Omega} \\ [\mathcal{G}, (\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \rhd (\Box\Sigma \to \Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta), \mathcal{H}]_{\emptyset\neq\Pi\subseteq\Gamma_{\Box\Omega,\Delta}}\end{array}}{\mathcal{G}, (\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \rhd \mathcal{H}} \ (\text{V}^{\circ,i})$$

where for all these rules it holds that Γ contains only implications,
$\Gamma^{\Pi} := \{A \to B \in \Gamma \mid A \notin \Pi\}$ and $\Gamma_{\Box\Omega,\Delta} := \{A \notin \Box\Omega \cup \Delta \mid \exists B(A \to B \in \Gamma)\}$

Fig. 5. Intuitionistic modal Visser rules.

intuitionistic Visser rule from [22]. The only rule that connects the left-hand side to the right-hand side is the projection rule (PJ). This rule corresponds to the fact that derivability implies admissibility (see [22]).

The semantics we have in mind for GAL is admissibility in L. We write $\vdash_{\mathsf{GAL}} \mathcal{G} \rhd \mathcal{H}$ denoting that there is a tree using the rules from GAL that ends in gs-rule $\mathcal{G} \rhd \mathcal{H}$ and its leaves are instances of the right rule or one of the Visser rules from GAL with no premises. Note that it is decidable whether a gs-rule is a conclusion of the right rule, because of the decidability of logic L (Theorem 1.2).

Example 3.4. The *disjunction property* can be expressed as $A \vee B \mathrel{|\!\sim}_{\mathsf{L}} A, B$. Once we show soundness and completeness of GAL we will know that the disjunction property holds for all L using the following derivation in GAL, where (V) can be any Visser rule:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(\Rightarrow A, B), (\Rightarrow A) \rhd (A \Rightarrow A), (\Rightarrow B)}\rhd()}{(\Rightarrow A, B), (\Rightarrow A) \rhd (\Rightarrow A), (\Rightarrow B)}(\text{PJ}) \quad \dfrac{\dfrac{}{(\Rightarrow A, B), (\Rightarrow B) \rhd (\Rightarrow A), (B \Rightarrow B)}\rhd()}{(\Rightarrow A, B), (\Rightarrow B) \rhd (\Rightarrow A), (\Rightarrow B)}(\text{PJ})}{(\Rightarrow A, B) \rhd (\Rightarrow A), (\Rightarrow B)}(\text{V})}{(\Rightarrow A \vee B) \rhd (\Rightarrow A), (\Rightarrow B)}(\Rightarrow \vee)\rhd$$

In the realm of admissible rules, it is interesting to think about admissible rules in the proof system GAL. In [22], (W)$\rhd$ and $\rhd$(W) are part of system GAL, but we show that they are admissible in GAL:

Lemma 3.5. *The following weakening rules are admissible in* GAL*.*

$$\frac{\mathcal{G} \rhd \mathcal{H}}{\mathcal{G}, S \rhd \mathcal{H}} \ (\text{W})\rhd \qquad \frac{\mathcal{G} \rhd \mathcal{H}}{\mathcal{G} \rhd S, \mathcal{H}} \ \rhd(\text{W})$$

Proof. It can be shown by a standard induction on the height of proofs in GAL. For rules $(\to\Rightarrow)\triangleright^i$, $(\Rightarrow\to)\triangleright^i$, $(\Box\Rightarrow)\triangleright$, and $(\Rightarrow\Box)\triangleright$, we have to be careful when sequent S contains atoms p and q present in the rules. For these cases we change all p and q in the proof of the premise of these rules into fresh variables p' and q' not occurring in its proof, and also not in S. Application of the induction hypothesis implies the desired result. □

3.1. Soundness

In this section we show the soundness of proof system GAL. The next section treats the completeness theorem.

Theorem 3.6 *(soundness). If* $\vdash_{\mathsf{GAL}} \mathcal{G} \triangleright \mathcal{H}$*, then* $\mid\!\sim_{\mathsf{L}} \mathcal{G} \triangleright \mathcal{H}$*.*

Proof. We show that each rule in GAL is L-sound. The weakening rules and right rule are clearly L-sound. Also the soundness of the left logical rules follow easily, see [22]. Here we treat rule $(\Rightarrow\to)\triangleright^i$. Suppose that the premises of the rule are admissible and suppose that σ is an L-unifier for $I(S)$ for all $S \in \mathcal{G}$ and for $I(\Gamma \Rightarrow A \to B, \Delta)$. Atom p does not occur in $\mathcal{G}, \mathcal{H}, \Gamma, \Delta, A, B$, so we can extend σ to define a new substitution σ_1 with $\sigma_1(p) = A \to B$. It follows immediately that σ_1 is an L-unifier for $I(\Gamma \Rightarrow p, \Delta)$ and $I(p, A \Rightarrow B)$. Since the premise of the rule is admissible, we conclude that σ_1 is an L-unifier for $I(S)$ for some $S \in \mathcal{H}$. Since p does not occur in $\mathcal{H}$, also σ is an L-unifier for this $I(S)$.

For rule $(\text{AC}_\Box)$, let σ be an L-unifier for $I(S)$ for all $S \in \mathcal{G}$, for $I(\Gamma, \Theta \Rightarrow \Delta)$, and for $I(\Pi \Rightarrow \Psi, \Sigma)$ where $(\Theta \cup \Psi) \subseteq \{A, \Box A\}$, A is box-free, and $\Theta, \Psi \neq \emptyset$. Using Kripke models, we show that σ is also an L-unifier for $I(\Gamma, \Pi, \Box A \to A \Rightarrow \Sigma, \Delta)$ which immediately implies the desired result. Let K be a strong intuitionistic Kripke model with world w. Let $w' \geq w$ such that $K, w' \models \sigma(\bigwedge \Gamma \wedge \bigwedge \Pi \wedge \Box A \to A)$. There are two cases: $K, w' \models \sigma(A)$ or $K, w' \not\models \sigma(A)$. In the first case we have $K, w' \models \sigma(A \wedge \Box A)$ and by the assumption on $(\Gamma, \Theta \Rightarrow \Delta)$ we obtain $K, w' \models \sigma(\bigvee \Delta)$. In the second case we have $K, w' \not\models \sigma(A)$ and $K, w' \not\models \sigma(\Box A)$. By the assumption on $(\Pi \Rightarrow \Psi, \Sigma)$ we obtain $K, w' \models \sigma(\bigvee \Sigma)$. The soundness of (AC) can be verified in a similar way.

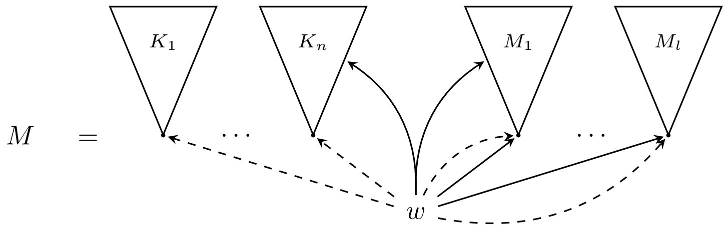
For the projection rule (PJ), suppose that σ is an L-unifier for $I(S')$ for all $S' \in \mathcal{G}$ and for $I(S)$. By the admissibility of the premise, σ is an L-unifier for $I(\Gamma, I(S) \Rightarrow \Delta)$ or for $I(S')$ for some $S' \in \mathcal{H}$. In the latter case we are done. In the former case, since σ is an L-unifier for $I(S)$ and $I(\Gamma, I(S) \Rightarrow \Delta)$, it is also a unifier for $\Gamma \Rightarrow \Delta$. The empty sequent can never be unified, therefore $\Gamma \Rightarrow \Delta \in \mathcal{H}$ by the condition of rule (PJ). Therefore σ is an L-unifier for some sequent in $\mathcal{H}$.

Now we turn to the soundness of the intuitionistic modal Visser rules, which are all shown in a similar way. We start with $(\text{V}^{\bullet,i,1})$ which is a rule in GAiCK4 and GAiSL, so let $\mathsf{L} \in \{\mathsf{iCK4}, \mathsf{iSL}\}$. Suppose that σ is an L-unifier for $I(S)$ for all $S \in \mathcal{G}$ and for $I(\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta)$. Write $\Delta = \{D_1, \ldots, D_n\}$ and $\Omega = \{O_1, \ldots, O_l\}$ (including the cases where the sets are empty). Using the third set of premises, we have for all $\emptyset \neq \Pi \subseteq \Gamma_{\Box\Omega,\Delta}$ that σ is either an L-unifier for some $S \in \mathcal{H}$ or for $I(\Box\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta)$. If there is such a Π for which the first case holds we are done. If for all such Π we have the second case (or in case there is no such Π at all), we will show that σ is an L-unifier for $I(\Box\Sigma, \Gamma \Rightarrow D_i)$ for some i, or for $I(\Sigma, \Gamma \Rightarrow O_j)$ for some j. This is sufficient, because that implies that σ is an L-unifier for some $S \in \mathcal{H}$ by the first or second set of premises of $(\text{V}^{\bullet,i,1})$. Suppose for a contradiction that this is not the case. Then there exist L-countermodels $K_1, \ldots, K_n$ and $M_1, \ldots, M_l$ such that

$$K_i \models \sigma(\bigwedge \Box\Sigma \wedge \bigwedge \Gamma) \text{ and } K_i \not\models \sigma(D_i),$$

$$M_j \models \sigma(\bigwedge \Sigma \wedge \bigwedge \Gamma) \text{ and } M_j \not\models \sigma(O_j).$$

Consider the following L-model M with irreflexive world w, where $\leq$ is drawn by a dashed line, and R by a straight line. R should be closed under prefixing with $\leq$ as indicated by the lines into model K_n and M_1. Model M is a one-node model if $\Delta = \Omega = \emptyset$.



Let us first note that $M \not\models \sigma(A)$ for all $A \in \Box\Omega \cup \Delta$. We also note that $M \models \sigma(\bigwedge(\Box\Sigma))$. We let $\Pi = \{A \in \Gamma_{\Box\Omega,\Delta} \mid M \models \sigma(A)\}$. Thus $M \models \sigma(\bigwedge \Pi)$. We also claim that $M \models \sigma(\bigwedge \Gamma^{\Pi})$. Let $A \to B \in \Gamma^{\Pi}$. Observe that either $A \in \Delta \cup \Box\Omega$ or $M \not\models \sigma(A)$. The first implies the second, so $M \not\models \sigma(A)$. And since $K_i \models \sigma(A \to B)$ for all i and $M_j \models \sigma(A \to B)$ for all j, we have $M \models \sigma(A \to B)$. So far we have shown that $M \models \sigma(\bigwedge(\Box\Sigma \cup \Gamma^{\Pi} \cup \Pi))$. If $\Pi = \emptyset$, then $\Gamma^{\Pi} = \Gamma$ and so $M \models \sigma(\bigwedge(\Box\Sigma \cup \Gamma))$. But σ is an L-unifier for $I(\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta)$. If $\Pi \neq \emptyset$, then σ is an L-unifier for $I(\Box\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta)$ by assumption. In both cases we have $M \models \sigma(\bigvee(\Box\Omega \cup \Delta))$, which is a contradiction with our first observation about model M.

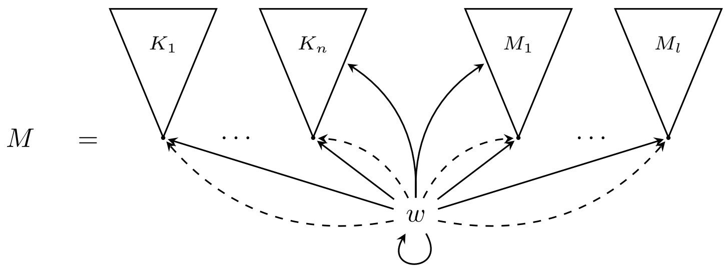
For rule ($\vee^{\bullet,i,2}$) for logics mHC and KM, the proof is completely analogous, where we construct model M as in the previous picture but now by extending each model with both relations $\leq$ and R.

Rule ($\vee^{\circ,i}$) is present in GAiCK4, GAiCS4, and GAmHC, so let L $\in$ {iCK4, iCS4, mHC}. Suppose that σ is an L-unifier for $I(S)$ for all $S \in \mathcal{G}$ and for $I(\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta)$. Again, write $\Delta = \{D_1, \ldots, D_n\}$ and $\Omega = \{O_1, \ldots, O_l\}$. By a similar argument as above, it is sufficient to show that σ is an L-unifier for $I(\Sigma, \Gamma \Rightarrow D_i)$ for some i, or for $I(\Sigma, \Gamma \Rightarrow O_j)$ for some j. Suppose this is not the case. Then there exist L-countermodels $K_1, \ldots, K_n$ and $M_1, \ldots M_l$ such that

$$K_i \models \sigma(\bigwedge \Sigma \wedge \bigwedge \Gamma) \text{ and } K_i \not\models \sigma(D_i),$$

$$M_j \models \sigma(\bigwedge \Sigma \wedge \bigwedge \Gamma) \text{ and } M_j \not\models \sigma(O_j).$$

Consider the following L-model M with reflexive root w, drawn in a similar way as above.



By a similar argument as above, this leads to a contradiction with σ being an L-unifier for formula $I(\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta)$ and for formulas $I(\Box\Sigma \to \Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta)$ with $\Pi \neq \emptyset$. $\Box$

Remark 3.7. Rule ($\vee^{\circ,i}$) can be considered as the reflexive version of ($\vee^{\bullet,i,2}$). One might ask why we do not use a reflexive version of ($\vee^{\bullet,i,1}$) for logics iCK4 and iCS4 which is the rule

$$\frac{\begin{array}{c}[\mathcal{G}, (\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Box\Sigma \to \Sigma, \Gamma \Rightarrow D) \rhd \mathcal{H}]_{D \in \Delta} \\ [\mathcal{G}, (\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Sigma, \Gamma \Rightarrow O) \rhd \mathcal{H}]_{O \in \Omega} \\ [\mathcal{G}, (\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \rhd (\Box\Sigma \to \Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta), \mathcal{H}]_{\emptyset \neq \Pi \subseteq \Gamma_{\Box\Omega,\Delta}}\end{array}}{\mathcal{G}, (\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \rhd \mathcal{H}} \ (\mathrm{V}^{\circ,i,1})$$

The reason is that it can be shown that adding this rule is equivalent to adding $(\mathrm{V}^{\circ,i})$. In Section 4 we will give an example of this equivalence for the corresponding bases (Example 4.3).

3.2. Completeness

This section shows the completeness of proof system GAL.

Theorem 3.8 *(completeness). If* $\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$*, then* $\vdash_{\mathsf{GAL}} \mathcal{G} \rhd \mathcal{H}$*.*

The theorem is shown in several steps in the same line of reasoning from [22]. We first show that derivability is captured by the proof system GAL as shown in the following lemma. Note the difference between Lemma 19 from [22] due to the small difference between the definition of derivability. After that we present some lemmas that show that each gs-rule is derivable in GAL from a certain class of irreducible gs-rules that have special properties.

Lemma 3.9. *If* $\vdash_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$*, then* $\vdash_{\mathsf{GAL}} \mathcal{G} \rhd \mathcal{H}$

Proof. Suppose $\vdash_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$. By definition, $I(\mathcal{G}) \vdash_{\mathsf{L}} I(\Gamma \Rightarrow \Delta)$ for some $\Gamma \Rightarrow \Delta \in \mathcal{H}$ or $I(\mathcal{G}) \vdash_{\mathsf{L}} \bot$. Consider $\Gamma = \Delta = \emptyset$, then the latter is equivalent to $I(\mathcal{G}) \vdash_{\mathsf{L}} \bot \equiv I(\Gamma \Rightarrow \Delta)$. So assume $I(\mathcal{G}) \vdash_{\mathsf{L}} I(\Gamma \Rightarrow \Delta)$. The deduction theorem (Theorem 1.1) of L implies $\vdash_{\mathsf{L}} \bigwedge I(\mathcal{G}) \to I(\Gamma \Rightarrow \Delta)$. By the formula interpretation, we have $\vdash_{\mathsf{L}} I(\Gamma, I(\mathcal{G}) \Rightarrow \Delta)$. Now apply the right rule $\rhd()$ to obtain $\vdash_{\mathsf{GAL}} \mathcal{G} \rhd (\Gamma, I(\mathcal{G}) \Rightarrow \Delta), \mathcal{H}$. Repeated applications of (PJ) give us $\vdash_{\mathsf{GAL}} \mathcal{G} \rhd \mathcal{H}$. $\Box$

Lemma 3.10. *All rules in* GAL *are* L*-invertible.*

Proof. Consider for example left rule $(\vee \Rightarrow)\rhd$. Suppose that the conclusion $\mathcal{G}, (\Gamma, A \vee B \Rightarrow \Delta) \rhd \mathcal{H}$ is admissible in L. Let σ be an L-unifier for $I(S)$ for all $S \in \mathcal{G}$, for $I(\Gamma, A \Rightarrow \Delta)$, and for $I(\Gamma, B \Rightarrow \Delta)$. By intuitionistic reasoning, σ is also an L-unifier for $I(\Gamma, A \vee B \Rightarrow \Delta)$, hence σ is an L-unifier for $I(S)$ for some $S \in \mathcal{H}$.

For $(\Rightarrow \Box)\rhd$, suppose the conclusion $\mathcal{G}, (\Gamma \Rightarrow \Box A, \Delta) \rhd \mathcal{H}$ is admissible in L. Let σ be an L-unifier for $I(S)$ for all $S \in \mathcal{G}$, for $I(\Gamma \Rightarrow \Box p, \Delta)$, and for $I(p \Rightarrow A)$. Since we work with normal modal logics, σ is also an L-unifier for $I(\Box p \Rightarrow \Box A)$. Using intuitionistic reasoning we obtain that σ is an L-unifier for $I(\Gamma \Rightarrow \Box A, \Delta)$. Hence, σ is an L-unifier for $I(S)$ for some $S \in \mathcal{H}$.

For (AC), $(\mathrm{AC}_\Box)$, (PJ), and all Visser rules we have that all the sequents in the conclusion appear in the premises, which immediately implies the invertibility of the rules. $\Box$

Recall that atom implications and boxed atoms are of the form $p \to q$ and $\Box p$, respectively.

Definition 3.11. A sequent $\Lambda \Rightarrow \Phi$ is called

- *semi-modal-implication-irreducible*, if Λ contains only atoms, boxed atoms, atom implications, and implications of the form $\Box p \to p$, and if Φ contains only atoms and boxed atoms.
- *modal-implication-irreducible* if it is semi-modal-implication-irreducible without the implications of the form $\Box p \to p$.

A gs-rule $\mathcal{G} \rhd \mathcal{H}$ is called *(semi-)modal-implication-irreducible* if all sequents in $\mathcal{G}$ are *(semi-)modal-implication-irreducible.*

Lemma 3.12. *Every (admissible) gs-rule is derivable in* GAL *from an (admissible) modal-implication-irreducible gs-rule only using the left logical rules without using* $(\to)\rhd$.

Proof. The fact that every gs-rule can be derived from one modal-implication-irreducible gs-rule follows from the fact that applications of left logical rules (except $(\to)\rhd$) terminate in one modal-implication-irreducible gs-rule. This is seen as follows reading the rules bottom-up. Every left logical rule (except $(\to)\rhd$) replaces a sequent on the left in the conclusion with sequents on the left in the premise that have fewer connectives. For rules $(\Box \Rightarrow)\rhd$ and $(\Rightarrow \Box)\rhd$ we stop if A is an atom. Similarly so for $(\to\Rightarrow)\rhd^i$ if $A \to B$ is of the form $p \to q$. Admissibility follows from the invertibility of the left logical rules shown in Lemma 3.10. □

Definition 3.13. A gs-rule $\mathcal{G} \rhd \mathcal{H}$ is *full* with respect to a set of gs-rules X if whenever

$$\frac{\mathcal{G}_1 \rhd \mathcal{H}_1 \qquad \dots \qquad \mathcal{G}_n \rhd \mathcal{H}_n}{\mathcal{G} \rhd \mathcal{H}}$$

is an instance of a rule in X, then there exists an i such that $\mathcal{G}_i \subseteq \mathcal{G}$ and $\mathcal{H}_i \subseteq \mathcal{H}$.

Lemma 3.14. *Let* $X \subset \{(\to)\rhd, (\mathrm{AC}), (\mathrm{AC}_\Box), (\mathrm{V}^{\bullet,i,1}), (\mathrm{V}^{\bullet,i,2}), (\mathrm{V}^{\circ,i})\}$ *be rules of* GAL. *Every (admissible) gs-rule is derivable in* GAL *from (admissible) semi-modal-implication-irreducible gs-rules that are full with respect to* X.

Proof. First apply Lemma 3.12 to backwards reach a modal-implication-irreducible gs-rule. Note that bottom-up applications of rules in X to a semi-modal-implication-irreducible conclusion yield semi-modal-implication-irreducible premises. In particular, only $(\mathrm{AC}_\Box)$ can introduce implications of the form $\Box p \to p$ in the premise, so only $(\mathrm{AC}_\Box)$ can introduce semi-modal-implication-irreducible sequents. Note that in rule $(\mathrm{AC}_\Box)$, A can only be an atom since A is assumed to be box-free. All rules in X do not add new atoms into the premises compared to the conclusion and the premises contain more sequents than the conclusion. Since there are only finitely many different semi-modal-implication-irreducible sequents for a fixed set of atoms, applying these rules exhaustively backwards terminates with a set of semi-modal-implication-irreducible gs-rules full with respect to these rules. Again, admissibility follows from Lemma 3.10. □

We provide some technical definitions and lemmas that we use in the completeness proof of Theorem 3.8 on page 18. We advise the reader to first skip the technical definitions and lemmas and to go immediately to the proof. The lemmas play a role in a resolution refutation argument of the kind also present in the completeness proof in [22]. Informally, the lemmas show that a 'cut' on p in sequents of the form $S_1 = (\Lambda_1, p \Rightarrow \Phi_1)$ and $S_2 = (\Lambda_2 \Rightarrow \Phi_2, p)$ resulting in $S = (\Lambda_1, \Lambda_2 \Rightarrow \Phi_1, \Phi_2)$ preserves some desirable technical properties.

Definition 3.15. Define the following property $\bullet$ on pairs consisting of a semi-modal-implication-irreducible sequent $(\Lambda \Rightarrow \Phi)$ and a set of sequents $\mathcal{G}$:

$$\bullet((\Lambda \Rightarrow \Phi), \mathcal{G}) \Leftrightarrow \forall \Pi \subseteq \Lambda_\Phi,\ \exists \Lambda' \subseteq \Pi \cup \Lambda^\Pi \cup \Lambda^a \cup \Lambda^b,\ \exists \Phi' \subseteq \Phi \text{ s.t. } \Lambda' \Rightarrow \Phi' \in \mathcal{G},$$

where $\Lambda_\Phi = \{A \notin \Phi \mid \exists B(A \to B \in \Lambda)\}$, $\Lambda^\Pi = \{A \to B \in \Lambda \mid A \notin \Pi\}$, Λ^a is the set of all atoms in Λ and Λ^b is the set of all boxed atoms in Λ.

Note that the property automatically holds for $\Pi = \emptyset$, by taking $(\Lambda \Rightarrow \Phi)$ for $(\Lambda' \Rightarrow \Phi')$. Also note that Π may contain atoms and boxed atoms, since all implications in Λ are atom implications or have the form $\Box p \to p$.

Lemma 3.16. *Let $S_1 = (\Lambda_1, p \Rightarrow \Phi_1)$, $S_2 = (\Lambda_2 \Rightarrow \Phi_2, p)$, and $S = (\Lambda_1, \Lambda_2 \Rightarrow \Phi_1, \Phi_2)$ be semi-modal-implication-irreducible sequents. Let $\mathcal{G}, S_1, S_2 \rhd \mathcal{H}$ be a gs-rule full with respect to* (AC). *Then $\bullet(S_1, \mathcal{G})$ and $\bullet(S_2, \mathcal{G})$ imply $\bullet(S, \mathcal{G})$.*

Proof. Suppose $\Pi \subseteq (\Lambda_1 \cup \Lambda_2)_{\Phi_1 \cup \Phi_2}$. Then also $\Pi \subseteq (\Lambda_1)_{\Phi_1} \cup (\Lambda_2)_{\Phi_2}$. Write $\Pi = \Pi_1 \cup \Pi_2$ such that $\Pi_i = \Pi \cap (\Lambda_i)_{\Phi_i}$ for $i = 1, 2$. Note that in this way $\Lambda_i^{\Pi} = \Lambda_i^{\Pi_i}$ for $i = 1, 2$. So we want to find sets

$$\Lambda' \subseteq \Pi_1 \cup \Pi_2 \cup \Lambda_1^{\Pi_1} \cup \Lambda_2^{\Pi_2} \cup \Lambda_1^a \cup \Lambda_2^a \cup \Lambda_1^b \cup \Lambda_2^b \quad \text{and} \quad \Phi' \subseteq \Phi_1 \cup \Phi_2$$

such that $(\Lambda' \Rightarrow \Phi') \in \mathcal{G}$.

First assume $p \in \Pi_2$. (Here we cannot use the assumption for S_2, because $\Pi_2 \not\subseteq (\Lambda_2)_{\Phi_2 \cup \{p\}}$). For this case we only have to use the assumption for S_1. Note that $\Pi_1 \subseteq (\Lambda_1)_{\Phi_1}$, so

$$\exists \Lambda_1' \subseteq \Pi_1 \cup \Lambda_1^{\Pi_1} \cup \Lambda_1^a \cup \Lambda_1^b \cup \{p\},\ \exists \Phi_1' \subseteq \Phi_1 \text{ such that } (\Lambda_1' \Rightarrow \Phi_1') \in \mathcal{G}.$$

Define $\Lambda' = \Lambda_1'$ and $\Phi' = \Phi_1'$. Since $p \in \Pi_2$, we have indeed $\Lambda' \subseteq \Pi_1 \cup \Pi_2 \cup \Lambda_1^{\Pi_1} \cup \Lambda_2^{\Pi_2} \cup \Lambda_1^a \cup \Lambda_2^a \cup \Lambda_1^b \cup \Lambda_2^b$.

If $p \notin \Pi_2$, then we have $\Pi_1 \subseteq (\Lambda_1)_{\Phi_1}$ and $\Pi_2 \subseteq (\Lambda_2)_{\Phi_2 \cup \{p\}}$, so by assumption for S_1 and S_2 we have

$$\exists \Lambda_1' \subseteq \Pi_1 \cup \Lambda_1^{\Pi_1} \cup \Lambda_1^a \cup \Lambda_1^b \cup \{p\},\ \exists \Phi_1' \subseteq \Phi_1 \text{ such that } (\Lambda_1' \Rightarrow \Phi_1') \in \mathcal{G},$$
$$\exists \Lambda_2' \subseteq \Pi_2 \cup \Lambda_2^{\Pi_2} \cup \Lambda_2^a \cup \Lambda_2^b,\ \exists \Phi_2' \subseteq \Phi_2 \cup \{p\} \text{ such that } (\Lambda_2' \Rightarrow \Phi_2') \in \mathcal{G}.$$

We distinguish 3 cases. If $p \notin \Lambda_1'$, we can take $\Lambda' = \Lambda_1'$ and $\Phi' = \Phi_1'$. If $p \notin \Phi_2'$, we can take $\Lambda' = \Lambda_2'$ and $\Phi' = \Phi_2'$. Otherwise, they have the form $\Lambda_1' = \Lambda_1'' \cup \{p\}$ and $\Phi_2' = \Phi_2'' \cup \{p\}$. Then $S' = (\Lambda_1'', \Lambda_2' \Rightarrow \Phi_1', \Phi_2'') \in \mathcal{G}$ by fullness of (AC). Take $\Lambda' = \Lambda_1'' \cup \Lambda_2'$ and $\Phi' = \Phi_1' \cup \Phi_2''$. □

Definition 3.17. Define the following property $\circ$ on pairs consisting of a semi-modal-implication-irreducible sequent $(\Lambda \Rightarrow \Phi)$ and a set of sequents $\mathcal{G}$:

$$\circ((\Lambda \Rightarrow \Phi), \mathcal{G}) \Leftrightarrow \forall \Pi \subseteq \Lambda_{\Phi}^a,\ \exists \Lambda' \subseteq \boxdot\Pi \cup \Lambda^{\Pi} \cup \Lambda^a \cup \Lambda^b,\ \exists \Phi' \subseteq \Phi \text{ s.t. } \Lambda' \Rightarrow \Phi' \in \mathcal{G},$$

where $\Lambda_{\Phi}^a = \{p \notin \Phi \mid \exists q(p \to q \in \Lambda)\}$, $\Lambda^{\Pi} = \{A \to B \in \Lambda \mid A \notin \Pi\}$, Λ^a is the set of all atoms in Λ, and Λ^b is the set of all boxed atoms in Λ.

Note that Λ_{Φ}^a only contains atoms by definition, and so does Π. This means that Λ^{Π} contains all implications from Λ of the form $\Box p \to p$. Again, the property holds automatically for $\Pi = \emptyset$, by taking $(\Lambda \Rightarrow \Phi)$ for $(\Lambda' \Rightarrow \Phi')$.

Lemma 3.18. *Let $S_1 = (\Lambda_1, \Theta \Rightarrow \Phi_1)$, $S_2 = (\Lambda_2 \Rightarrow \Phi_2, \Psi)$, and $S = (\Lambda_1, \Lambda_2, \Box p \to p \Rightarrow \Phi_1, \Phi_2)$ be semi-modal-implication-irreducible sequents with non-empty subsets $\Theta, \Psi \subseteq \{p, \Box p\}$. Let $\mathcal{G}, S_1, S_2 \rhd \mathcal{H}$ be a gs-rule full with respect to* ($\text{AC}_\Box$). *Then $\circ(S_1, \mathcal{G})$ and $\circ(S_2, \mathcal{G})$ imply $\circ(S, \mathcal{G})$.*

Proof. Suppose that $\Pi \subseteq (\Lambda_1 \cup \Lambda_2 \cup \{\Box p \to p\})_{\Phi_1 \cup \Phi_2}^a$. Note that Π only contains atoms. Then also $\Pi \subseteq (\Lambda_1)_{\Phi_1}^a \cup (\Lambda_2)_{\Phi_2}^a$. Write $\Pi = \Pi_1 \cup \Pi_2$ such that $\Pi_i = \Pi \cap (\Lambda_i)_{\Phi_i}^a$ for $i = 1, 2$. Note that in this way $\Lambda_i^{\Pi} = \Lambda_i^{\Pi_i}$ for $i = 1, 2$. So we want to find sets

$$\Lambda' \subseteq \boxdot\Pi_1 \cup \boxdot\Pi_2 \cup \Lambda_1^{\Pi_1} \cup \Lambda_2^{\Pi_2} \cup \{\Box p \to p\} \cup \Lambda_1^a \cup \Lambda_2^a \cup \Lambda_1^b \cup \Lambda_2^b \quad \text{and} \quad \Phi' \subseteq \Phi_1 \cup \Phi_2$$

such that $(\Lambda' \Rightarrow \Phi') \in \mathcal{G}$.

We distinguish between $p \in \Pi_2$ and $p \notin \Pi_2$. First assume $p \in \Pi_2$. (Here we cannot always use the assumption for S_2, because Π_2 may not be a subset of $(\Lambda_2)^a_{\Phi_2 \cup \Psi}$). For this case we only have to use the assumption for S_1. Note that $\Pi_1 \subseteq (\Lambda_1)^a_{\Phi_1}$, so

$$\exists \Lambda_1' \subseteq \boxdot\Pi_1 \cup \Lambda_1^{\Pi_1} \cup \Lambda_1^a \cup \Lambda_1^b \cup \Theta,\ \exists \Phi_1' \subseteq \Phi_1 \text{ such that } (\Lambda_1' \Rightarrow \Phi_1') \in \mathcal{G}.$$

Define $\Lambda' = \Lambda_1'$ and $\Phi' = \Phi_1'$. Since $\Theta \subseteq \boxdot\Pi_2$, we have $\Lambda' \subseteq \boxdot\Pi_1 \cup \boxdot\Pi_2 \cup \Lambda_1^{\Pi_1} \cup \Lambda_1^a \cup \Lambda_1^b$ and we are done.

If $p \notin \Pi_2$, then we have $\Pi_1 \subseteq (\Lambda_1)^a_{\Phi_1}$ and $\Pi_2 \subseteq (\Lambda_2)^a_{\Phi_2 \cup \Psi}$, so by assumption for S_1 and S_2 we have

$$\exists \Lambda_1' \subseteq \boxdot\Pi_1 \cup \Lambda_1^{\Pi_1} \cup \Lambda_1^a \cup \Lambda_1^b \cup \Theta,\ \exists \Phi_1' \subseteq \Phi_1 \text{ such that } (\Lambda_1' \Rightarrow \Phi_1') \in \mathcal{G},$$

$$\exists \Lambda_2' \subseteq \boxdot\Pi_2 \cup \Lambda_2^{\Pi_2} \cup \Lambda_2^a \cup \Lambda_2^b,\ \exists \Phi_2' \subseteq \Phi_2 \cup \Psi \text{ such that } (\Lambda_2' \Rightarrow \Phi_2') \in \mathcal{G}.$$

We distinguish 3 cases. If $p, \Box p \notin \Lambda_1'$, we can take $\Lambda' = \Lambda_1'$ and $\Phi' = \Phi_1'$. If $p, \Box p \notin \Phi_2'$, we can take $\Lambda' = \Lambda_2'$ and $\Phi' = \Phi_2'$. Otherwise Λ_1' is of the form $\Lambda_1'' \cup \Theta_1$ and Φ_2' is of the form $\Phi_2'' \cup \Psi_2$ with non-empty sets $\Theta_1, \Psi_2 \subseteq \{p, \Box p\}$. Then $S' = (\Lambda_1'', \Lambda_2', \Box p \to p \Rightarrow \Phi_1', \Phi_2'') \in \mathcal{G}$ by fullness of (AC$_\Box$). Take $\Lambda' = \Lambda_1'' \cup \Lambda_2' \cup \{\Box p \to p\}$ and $\Phi' = \Phi_1' \cup \Phi_2''$. $\Box$

Now we give the proof of completeness from Theorem 3.8, i.e. if $\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$, then $\vdash_{\mathsf{GAL}} \mathcal{G} \rhd \mathcal{H}$. The idea is that the derivation of $\mathcal{G} \rhd \mathcal{H}$ in GAL starts with bottom-up applications of left logical rules and rules from the set $X = \mathsf{GAL} \cap \{(\to)\rhd, (\text{AC}), (\text{AC}_\Box), (\text{V}^{\bullet,i,1}), (\text{V}^{\bullet,i,2}), (\text{V}^{\circ,i})\}$ resulting in a set of semi-modal-implication-irreducible gs-rules full with respect to that set by Lemma 3.14. For each such gs-rule $\mathcal{G}' \rhd \mathcal{H}'$, we show that admissibility $\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{G}' \rhd \mathcal{H}'$ reduces to derivability $\vdash_{\mathsf{L}} \mathcal{G}' \rhd \mathcal{H}'$. This results in a derivation of $\vdash_{\mathsf{GAL}} \mathcal{G}' \rhd \mathcal{H}'$ using the rules (PJ) and $\rhd()$ as shown in Lemma 3.9. The proof is based on a distinction between $C := \bigwedge I(\mathcal{G})$ being inconsistent, consistent and projective, and consistent and not projective. The latter case is very difficult and relies on the extension property discussed in Section 2. The proof includes a resolution refutation argument using the technical lemmas presented before.

Proof of Theorem 3.8. Suppose $\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$. Using Lemma 3.14, it is sufficient to assume that $\mathcal{G} \rhd \mathcal{H}$ is a semi-modal-implication-irreducible gs-rule that is full with respect to the set of rules

$$X = \mathsf{GAL} \cap \{(\to)\rhd, (\text{AC}), (\text{AC}_\Box), (\text{V}^{\bullet,i,1}), (\text{V}^{\bullet,i,2}), (\text{V}^{\circ,i})\}.$$

Define formula

$$C := \bigwedge I(\mathcal{G}).$$

We consider three cases. Only for one case we use the assumption $\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$ to prove $\vdash_{\mathsf{GAL}} \mathcal{G} \rhd \mathcal{H}$. For the other cases $\vdash_{\mathsf{GAL}} \mathcal{G} \rhd \mathcal{H}$ follows immediately. If C is inconsistent, then $I(\mathcal{G}) \vdash_{\mathsf{L}} \bot$ and so $\vdash_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$ by definition. By Lemma 3.9 we have $\vdash_{\mathsf{GAL}} \mathcal{G} \rhd \mathcal{H}$. Now assume C is consistent. For the case that C is projective, we use the assumption $\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$ and Lemma 2.2, to conclude $C \vdash_{\mathsf{L}} I(S)$ for some $S \in \mathcal{H}$. So $I(\mathcal{G}) \vdash_{\mathsf{L}} I(S)$ for some $S \in \mathcal{H}$ and so $\vdash_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$ by definition. Again by Lemma 3.9, we obtain $\vdash_{\mathsf{GAL}} \mathcal{G} \rhd \mathcal{H}$.

The case remains that C is consistent and not projective. By Theorem 2.4, there is an L-model K with root ρ such that $K_w \models C$ for each $w \neq \rho$ and for each variant K' of K we have $K' \not\models C$. Formula C holds in at least one 1-point model, because it is consistent. Therefore there exists at least one $w \neq \rho$ in K.

Let $\overline{p}$ such that all the atoms occurring in $\mathcal{G} \rhd \mathcal{H}$ are among $\overline{p}$. There are finitely many variants of K that only force atoms among $\overline{p}$ in the root. Let $M_1, \dots M_k$ be all such variants of K. We have $M_i \not\models C$ for all i, hence $M_i \not\models I(S_i)$ for some semi-modal-implication-irreducible sequent $S_i \in \mathcal{G}$. Denote $S_i = (\Lambda_i \Rightarrow \Phi_i)$, so $M_i \not\models I(\Lambda_i \Rightarrow \Phi_i)$. Since $M_i, w \Vdash I(\Lambda_i \Rightarrow \Phi_i)$ for each $w \neq \rho$, we have $M_i \models \bigwedge \Lambda_i$ and $M_i \not\models \bigvee \Phi_i$. We assume that

$$(p \to q) \in \Lambda_i \implies p \in \Phi_i, \tag{5}$$
$$(\Box p \to p) \in \Lambda_i \implies \Box p \in \Phi_i. \tag{6}$$

This is possible because suppose that $(\Box p \to p) \in \Lambda_i$ and $\Box p \notin \Phi_i$. We show that $\Lambda_i \Rightarrow \Phi_i$ can be replaced by another sequent $S \in \mathcal{G}$ that has property (6) and $M_i \not\models I(S)$. Since $M_i \models \bigwedge \Lambda_i$ and $M_i \not\models \bigvee \Phi_i$ it follows that $M_i \models \Box p \to p$, which means $M_i, \rho \nVdash \Box p$ or $M_i, \rho \Vdash p$. So either $M_i \not\models I(\Lambda_i \Rightarrow \Box p, \Phi_i)$ or $M_i \not\models I(\Lambda_i \setminus \{\Box p \to p\}, p \Rightarrow \Phi_i)$. Since $\mathcal{G} \rhd \mathcal{H}$ is full with respect to $(\to)\rhd$, both of these sequents belong to $\mathcal{G}$ and can replace $\Lambda_i \Rightarrow \Phi_i$. Similar for property (5).

Define the set of atoms:

$$P := \{p \mid p \text{ occurs in } C \text{ and } K_w \models p \text{ for all } w \neq \rho\}.$$

There are several possibilities depending on the specific logic.

1. For logics iCK4 and iSL, root ρ can be irreflexive (in case of iSL, ρ must be irreflexive).
2. For logics mHC and KM, root ρ can be irreflexive and model K satisfies $< \subseteq R$ (in case of KM, ρ must be irreflexive).
3. For logics iCK4, iCS4, and mHC, root ρ can be reflexive (in case of iCS4, ρ must be reflexive).

We treat each case separately using the Visser rules $(\mathrm{V}^{\bullet,i,1})$, $(\mathrm{V}^{\bullet,i,2})$, and $(\mathrm{V}^{\circ,i})$, respectively.

Case 1:
Let $\mathsf{L} \in \{\mathsf{iCK4}, \mathsf{iSL}\}$ and suppose that ρ is irreflexive. We define formulas for $i = 1, \ldots, k$ as follows:

$$A_i := \bigwedge_{p \in \Lambda_i} \neg p \wedge \bigwedge_{p \in \Phi_i \cap P} p \quad \text{and} \quad A := \bigvee_{i=1}^{k} A_i.$$

Note that $\mathsf{Var}(\Lambda_i) \subseteq P$, because $M_i \models \bigwedge \Lambda_i$ and so for variant K we have $K_w \models p$ for all $w \neq \rho$ by monotonicity. We show that A is a classical tautology. If the conjuncts in A_i are empty for some i, then A is equivalent to $\top$. Otherwise, let v be a classical valuation on P. This valuation corresponds to a variant M of K for which

$$M \models p \Leftrightarrow v(p) = 0.$$

Note that this correspondence is well-defined because $M_w \models p$ for each $w \neq \rho$ and $p \in P$. Let $M = M_i$. We have $M \models p$ for all atoms $p \in \Lambda_i$ and $M \not\models p$ for all atoms $p \in \Phi_i$. Hence $v(p) = 0$ if $p \in \Lambda_i$ and $v(p) = 1$ if $p \in \Phi_i$. Thus $v(A_i) = 1$, hence A is a classical tautology. So $\neg A$ is classically inconsistent and by DeMorgan laws, $\neg A$ is classically equivalent to:

$$\neg A \equiv \bigwedge_{i=1}^{k} (\bigvee_{p \in \Lambda_i} p \vee \bigvee_{p \in \Phi_i \cap P} \neg p).$$

Therefore there exists a resolution refutation starting with the clauses

$$\{p \mid p \in \Lambda_i\} \cup \{\neg p \mid p \in \Phi_i \cap P\} \text{ for } i = 1, \ldots, k,$$

that ends in the empty clause $\emptyset$. In case the conjuncts of A_i are empty for some i, the empty clause is already among the starting clauses. Each clause in the resolution refutation is of the form $\Theta \cup \Psi'$ where Θ contains only atoms and Ψ' contains only negated atoms. Define $\Psi := \{p \mid \neg p \in \Psi'\}$.

The resolution refutation can be mimicked by applications of the rule (AC), such that each class $\Theta \cup \Psi'$ corresponds to a semi-modal-implication-irreducible sequent of the form $\Box\Sigma, \Gamma, \Theta \Rightarrow \Box\Omega, \Delta, \Psi$, where Γ only contains atom implications or implications of the form $\Box p \to p$, sets Σ and Ω only contain atoms, and Δ only contains atoms with $\Delta \cap P = \emptyset$. To see this, first note that the starting classes are of this form. Further, each resolution on $\Theta_1 \cup \{p\} \cup \Psi_1'$ and $\Theta_2 \cup \Psi_2' \cup \{\neg p\}$ can be mimicked by (AC) from sequents

$$\Box\Sigma_1, \Gamma_1, \Theta_1, p \Rightarrow \Box\Omega_1, \Delta_1, \Psi_1 \quad \text{and} \quad \Box\Sigma_2, \Gamma_2, \Theta_2 \Rightarrow \Box\Omega_2, \Delta_2, \Psi_2, p$$

by a 'cut' on p, resulting in sequent

$$\Box\Sigma_1, \Box\Sigma_2, \Gamma_1, \Gamma_2, \Theta_1, \Theta_2 \Rightarrow \Box\Omega_1, \Box\Omega_2, \Delta_1, \Delta_2, \Psi_1, \Psi_2,$$

also of the right form. Moreover, since $\mathcal{G}$ is full with respect to (AC), it is guaranteed that all such sequents $\Box\Sigma, \Gamma, \Theta \Rightarrow \Box\Omega, \Delta, \Psi$ are in $\mathcal{G}$.

In addition we have the following properties for each clause $\Theta \cup \Psi'$ in the refutation and its corresponding sequent $\Box\Sigma, \Gamma, \Theta \Rightarrow \Box\Omega, \Delta, \Psi \in \mathcal{G}$:

1. $K_w \models \bigwedge \Box\Sigma \wedge \bigwedge \Gamma$ for all $w \neq \rho$,
2. $K_v \models \bigwedge \Sigma \wedge \bigwedge \Gamma$ for all v such that $\rho R v$,
3. $\Box q \in \Box\Omega$ implies $K_v \not\models q$ for some v such that $\rho R v$,
4. $\bullet((\Box\Sigma, \Gamma, \Theta \Rightarrow \Box\Omega, \Delta, \Psi), \mathcal{G})$.

We show these properties inductively following the resolution refutation. Properties 1 and 2 are true for the initial clauses $\{p \mid p \in \Lambda_i\} \cup \{\neg p \mid p \in \Phi_i \cap P\}$ with corresponding sequents $\Lambda_i \Rightarrow \Phi_i$, because $M_i \models \bigwedge \Lambda_i$. So $K_w \models \Lambda_i$ for each $w \neq \rho$ since K is a variant of M_i and by monotonicity of $\leq$ in L-models. Moreover, for $\Box q \in \Lambda_i$ we have $K_v \models q$ for all v such that $\rho R v$. For all other corresponding sequents in the refutation it follows immediately from backwards applications of (AC).

Property 3 holds for initial sequents $\Lambda_i \Rightarrow \Phi_i$, because suppose $\Box q \in \Phi_i$. We know $M_i \not\models \Phi_i$, so there must be a v such that $M_v \not\models q$. We assumed ρ to be irreflexive, so $v \neq \rho$. Since M_i is a variant of K we have $K_v \not\models q$. Again, for all other corresponding sequents in the refutation it follows immediately from backwards applications of (AC).

For property 4 observe that (5) implies $\bullet((\Lambda_i \Rightarrow \Phi_i), \mathcal{G})$, because $(\Lambda_i)_{\Phi_i} = \emptyset$ (and note that $\Lambda_i \Rightarrow \Phi_i$ is indeed semi-modal-implication-irreducible by assumption). For the other corresponding sequents in the refutation we use Lemma 3.16 to prove the property.

Now we use all those facts for the empty clause $\emptyset$. There is a corresponding semi-modal-implication-irreducible sequent for the empty clause, $\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta$, where Γ only contains atom implications or implications of the form $\Box p \to p$, sets Σ and Ω only contain atoms, and Δ only contains atoms such that $\Delta \cap P = \emptyset$.

Gs-rule $\mathcal{G} \triangleright \mathcal{H}$ is full with respect to ($\mathrm{V}^{\bullet,i,1}$), so we have at least one of the following:

(i) $(\Box\Sigma, \Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Delta$,
(ii) $(\Sigma, \Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Omega$,
(iii) $(\Box\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta) \in \mathcal{H}$ for some $\emptyset \neq \Pi \subseteq \Gamma_{\Box\Omega\cup\Delta}$.

We will show that the first two lead to a contradiction. For (i), we use the fact that $K_w \models C$ for all $w \neq \rho$, so $K_w \models I(\Box\Sigma, \Gamma \Rightarrow q)$ for all $w \neq \rho$. Since $K_w \models \bigwedge \Box\Sigma \wedge \bigwedge \Gamma$ by property 1, we have $K_w \models q$ for all $w \neq \rho$. But then $q \in P$. This is a contradiction, because $\Delta \cap P = \emptyset$.

For (ii), we also use the fact that $K_w \models C$ for all $w \neq \rho$, so $K_w \models I(\Sigma, \Gamma \Rightarrow q)$ for all $w \neq \rho$. Since ρ is irreflexive, this also holds for all v such that $\rho R v$. By property 2 we have that $K_v \models \bigwedge \Sigma \wedge \bigwedge \Gamma$ for all v such that $\rho R v$. Hence for all these v's, $K_v \models q$. But this contradicts property 3.

For case (iii), we use property 4 saying that $\bullet((\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), \mathcal{G})$. So

$$\exists \Lambda' \subseteq \Pi \cup \Gamma^{\Pi} \cup \Box\Sigma,\ \exists \Phi' \subseteq \Box\Omega \cup \Delta \text{ such that } \Lambda' \Rightarrow \Phi' \in \mathcal{G}.$$

Clearly $\vdash_{\mathsf{L}} (\Lambda' \Rightarrow \Phi') \triangleright (\Box\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta)$ by intuitionistic reasoning using Kripke models (or via weakening in a multi-succedent sequent system if available). Since the left sequent is in $\mathcal{G}$ and the right sequent is in $\mathcal{H}$ we have $\vdash_{\mathsf{L}} \mathcal{G} \triangleright \mathcal{H}$. Now we apply Lemma 3.9 to conclude $\vdash_{\mathsf{GAL}} \mathcal{G} \triangleright \mathcal{H}$.

Case 2:
Let $\mathsf{L} \in \{\mathsf{mHC}, \mathsf{KM}\}$ and suppose that ρ is irreflexive. The proof proceeds in a similar way as for Case 1. The only difference is that we replace property 1 with the following property:

1. $K_v \models \bigwedge \Sigma \wedge \bigwedge \Gamma$ for all $v \neq \rho$.

This property is shown using the fact that $<\subseteq R$ in mHC-models and KM-models. The rest of the proof proceeds the same using Visser rule $(\mathrm{V}^{\bullet,i,2})$ instead of $(\mathrm{V}^{\bullet,i,1})$.

Case 3:
Let $\mathsf{L} \in \{\mathsf{iCK4}, \mathsf{iCS4}, \mathsf{mHC}\}$ and suppose that ρ is reflexive. Now define formulas for $i = 1, \ldots, k$ as follows, where l_p are new introduced atoms:

$$A_i := \bigwedge_{p \in \Lambda_i} \neg l_p \wedge \bigwedge_{\Box p \in \Lambda_i} \neg l_p \wedge \bigwedge_{p \in P, p \in \Phi_i} l_p \wedge \bigwedge_{p \in P, \Box p \in \Phi_i} l_p \quad \text{and} \quad A := \bigvee_{i=1}^{k} A_i.$$

Note that $\mathsf{Var}(\Lambda_i) \subseteq P$, because $M_i \models \bigwedge \Lambda_i$ and so for variant K we have $K_w \models p$ for all $w \neq \rho$ by monotonicity. In addition, if $\Box p \in \Lambda_i$, then $p \in P$ by reflexivity of the root ρ.

We show that A is a classical tautology. If the conjuncts in A_i are empty for some i, then A is equivalent to $\top$. Otherwise, let v be a classical valuation on the l_p's for $p \in P$. This valuation corresponds to a variant M of K so that

$$M \models p \Leftrightarrow v(l_p) = 0.$$

Note that this correspondence is well-defined because $M_w \models p$ for each $w \neq \rho$ and $p \in P$. Let $M = M_i$. We have $M \models p$ for all atoms $p \in \Lambda_i$ and $M \not\models p$ for all atoms $p \in \Phi_i \cap P$. Since ρ is reflexive we also have $M \models p$ for all $\Box p \in \Lambda_i$ and $M \not\models p$ for all p such that $p \in P, \Box p \in \Phi_i$. Hence, for all $p \in P$, $v(l_p) = 0$ if $p \in \Lambda_i$ or $\Box p \in \Lambda_1$, and $v(l_p) = 1$ if $p \in \Phi_i$ or $\Box p \in \Phi_i$. Thus $v(A_i) = 1$, hence A is a classical tautology. So $\neg A$ is classically inconsistent and by DeMorgan laws, $\neg A$ is classically equivalent to:

$$\neg A \equiv \bigwedge_{i=1}^{k} \Big(\bigvee_{p \in \Lambda_i} l_p \vee \bigvee_{\Box p \in \Lambda_i} l_p \vee \bigvee_{p \in \Phi_i \cap P} \neg l_p \vee \bigvee_{p \in P, \Box p \in \Phi_i} \neg l_p \Big).$$

Therefore there exists a resolution refutation starting with the clauses

$$\{l_p \mid p \in \Lambda_i \text{ or } \Box p \in \Lambda_i\} \cup \{\neg l_p \mid p \in P, \text{ and } p \in \Phi_i \text{ or } \Box p \in \Phi_i\} \text{ for } i = 1, \ldots, k,$$

that ends in the empty clause $\emptyset$. In case the conjuncts of A_i are empty for some i, the empty clause is already among the starting clauses. Each clause in the resolution refutation is of the form $\Theta' \cup \Psi'$ where Θ' contains only atoms l_p and Ψ' contains only negated atoms $\neg l_p$.

Now, the resolution refutation can be mimicked by applications of (AC$_\Box$), such that each class $\Theta' \cup \Psi'$ corresponds to a semi-modal-implication-irreducible sequent of the form $\Box\Sigma \to \Sigma, \Box\Sigma' \to \Sigma', \Gamma, \Theta \Rightarrow \Box\Omega, \Delta, \Psi$, where Γ only contains atom implications, Σ only atoms in P, and Ω, Δ and Σ' atoms not in P. And where Θ and Ψ are sets of atoms and boxed atoms satisfying

$$\Theta = \bigcup_{l_p \in \Theta'} \Theta_p, \text{ with } \emptyset \neq \Theta_p \subseteq \{p, \Box p\} \quad \text{and} \quad \Psi = \bigcup_{\neg l_p \in \Psi'} \Psi_p, \text{ with } \emptyset \neq \Psi_p \subseteq \{p, \Box p\}.$$

To see this, first note that the starting classes are of this form. Further, each resolution on $\Theta'_1 \cup \{l_p\} \cup \Psi'_1$ and $\Theta'_2 \cup \Psi'_2 \cup \{\neg l_p\}$ can be mimicked by (AC$_\Box$) from sequents with non-empty sets $\Theta_p, \Psi_p \subseteq \{p, \Box p\}$,

$$\Box\Sigma_1 \to \Sigma_1, \Box\Sigma'_1 \to \Sigma'_1, \Gamma_1, \Theta_1, \Theta_p \Rightarrow \Box\Omega_1, \Delta_1, \Psi_1,$$
$$\Box\Sigma_2 \to \Sigma_2, \Box\Sigma'_2 \to \Sigma'_2, \Gamma_2, \Theta_2 \Rightarrow \Box\Omega_2, \Delta_2, \Psi_2, \Psi_p,$$

by a 'cut' on $p, \Box p$, resulting in sequent

$$\Box p \to p, \Box\Sigma_1 \to \Sigma_1, \Box\Sigma_2 \to \Sigma_2, \Box\Sigma'_1 \to \Sigma'_1, \Box\Sigma'_2 \to \Sigma'_2, \Gamma_1, \Gamma_2, \Theta_1, \Theta_2 \Rightarrow \Box\Omega_1, \Box\Omega_2, \Delta_1, \Delta_2, \Psi_1, \Psi_2.$$

Moreover, since $\mathcal{G}$ is full with respect to (AC$_\Box$), we know that all such sequents are in $\mathcal{G}$.

In addition we have the following properties for each clause $\Theta' \cup \Psi'$ in the refutation and its corresponding sequent $(\Box\Sigma \to \Sigma, \Box\Sigma' \to \Sigma', \Gamma, \Theta \Rightarrow \Box\Omega, \Delta, \Psi) \in \mathcal{G}$:

1. $K_v \models \bigwedge \Sigma \wedge \bigwedge(\Box\Sigma' \to \Sigma') \wedge \bigwedge \Gamma$ for all $v \neq \rho$,
2. $q \in \Sigma'$ implies $\Box q \in \Box\Omega$,
3. $\Box q \in \Box\Omega$ implies $K_v \not\models q$ for some $v \neq \rho$,
4. $\circ((\Box\Sigma \to \Sigma, \Box\Sigma' \to \Sigma', \Gamma, \Theta \Rightarrow \Box\Omega, \Delta, \Psi), \mathcal{G})$.

We show these properties inductively following the resolution refutation. Property 1 is true for the initial clauses $\{l_p \mid p \in \Lambda_i \text{ or } \Box p \in \Lambda_i\} \cup \{\neg l_p \mid p \in P, \text{and } p \in \Phi_i \text{ or } \Box p \in \Phi_i\}$ with corresponding sequents $\Lambda_i \Rightarrow \Phi_i$, because $M_i \models \bigwedge \Lambda_i$. So $K_v \models \bigwedge \Lambda_i$ for each $v \neq \rho$ since K is a variant of M_i and by monotonicity of $\leq$ in L-models. Moreover, for $q \in P$ we have $K_v \models q$ for all $v \neq \rho$, so if $\Box q \to q \in \Lambda_i$ for $q \in P$, then $K_v \models q$ (corresponding to the Σ of Λ_i). For all other corresponding sequents in the refutation it follows immediately from backwards applications of (AC$_\Box$). In particular, if the 'cut' is applied to subsets of $\{p, \Box p\}$, we have $p \in P$ so that $\Box p \to p \in \Box\Sigma \to \Sigma$ and $K_v \models p$ for all $v \neq \rho$. In addition, it shows that each $\Box q \to q \in \Box\Sigma' \to \Sigma'$ was already present in the sequents to which the 'cut' was applied, because we do not cut on $\Box q, q$ when $q \notin P$. So indeed the property holds.

Property 2 holds for initial sequents $\Lambda_i \Rightarrow \Phi_i$ because of assumption (6). For the other corresponding sequents it follows from the fact that we do not cut on $\Box q, q$ when $q \notin P$.

Property 3 follows immediately from the fact that $q \notin P$.

For property 4 observe that (5) implies $\circ((\Lambda_i \Rightarrow \Phi_i), \mathcal{G})$, because $(\Lambda_i)^q_{\Phi_i} = \emptyset$ (and note that $\Lambda_i \Rightarrow \Phi_i$ is indeed semi-modal-implication-irreducible by assumption). For the other corresponding sequents in the refutation we use Lemma 3.18 to prove the property.

Now we use all those facts for the empty clause $\emptyset$. There is a corresponding sequent for the empty clause, $\Box\Sigma \to \Sigma, \Box\Sigma' \to \Sigma', \Gamma \Rightarrow \Box\Omega, \Delta$, where Γ only contains atom implications, Σ only atoms in P, and Ω, Δ and Σ' atoms not in P.

Gs-rule $\mathcal{G} \rhd \mathcal{H}$ is full with respect to (V$^{\circ,i}$), so we have at least one of the following:

(i) $(\Sigma, \Box\Sigma' \to \Sigma', \Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Delta$,
(ii) $(\Sigma, \Box\Sigma' \to \Sigma', \Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Omega$,
(iii) $(\Box\Sigma \to \Sigma, (\Box\Sigma' \to \Sigma')^{\Pi}, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta) \in \mathcal{H}$ for some $\emptyset \neq \Pi \subseteq (\Gamma \cup (\Box\Sigma' \to \Sigma'))_{\Box\Omega\cup\Delta}$.

We will show that the first two lead to a contradiction. For (i), we use the fact that $K_v \models C$ for all $v \neq \rho$, so $K_v \models I(\Sigma, \Box\Sigma' \to \Sigma', \Gamma \Rightarrow q)$ for all $v \neq \rho$. Since $K_v \models \bigwedge\Sigma \wedge \Box\Sigma' \to \Sigma' \wedge \bigwedge\Gamma$ by property 1, we have $K_v \models q$ for all $v \neq \rho$. But then $q \in P$. This is a contradiction, because $\Delta \cap P = \emptyset$.

For (ii), we also use the fact that $K_v \models C$ for all $v \neq \rho$, so $K_v \models I(\Sigma, \Box\Sigma' \to \Sigma', \Gamma \Rightarrow q)$ for all $v \neq \rho$. By property 1 we have that $K_v \models \bigwedge\Sigma \wedge \Box\Sigma' \to \Sigma' \wedge \bigwedge\Gamma$ for all $v \neq \rho$, hence $K_v \models q$ for all $v \neq \rho$. But this contradicts property 3.

For case (iii), we use property 4 saying that $\circ((\Box\Sigma \to \Sigma, \Box\Sigma' \to \Sigma', \Gamma \Rightarrow \Box\Omega, \Delta), \mathcal{G})$. Note that Π is a subset of $(\Gamma \cup (\Box\Sigma' \to \Sigma'))_{\Box\Omega\cup\Delta}$. By property 2 we know that Π does not contain boxed atoms, and so $\Pi \subseteq (\Gamma \cup (\Box\Sigma' \to \Sigma'))^a_{\Box\Omega\cup\Delta}$ and $(\Box\Sigma' \to \Sigma')^{\Pi} = \Box\Sigma' \to \Sigma'$. So

$$\exists \Lambda' \subseteq \boxdot\Pi \cup \Gamma^{\Pi} \cup (\Box\Sigma \to \Sigma) \cup (\Box\Sigma' \to \Sigma'),\ \exists \Phi' \subseteq \Box\Omega \cup \Delta \text{ such that } \Lambda' \Rightarrow \Phi' \in \mathcal{G}.$$

We have $\vdash_{\mathsf{L}} (\Lambda' \Rightarrow \Phi') \rhd (\Box\Sigma \to \Sigma, (\Box\Sigma' \to \Sigma')^{\Pi}, \Gamma^{\Pi}, \boxdot\Pi \Rightarrow \Box\Omega, \Delta)$ by intuitionistic reasoning (i.e. using Kripke models). Recall that $\boxdot A \equiv A$ for all formulas A in intuitionistic modal logics with coreflection. So we can conclude that $\vdash_{\mathsf{L}} (\Lambda' \Rightarrow \Phi') \rhd (\Box\Sigma \to \Sigma, (\Box\Sigma' \to \Sigma')^{\Pi}, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta)$. Since the left sequent is in $\mathcal{G}$ and the right sequent is in $\mathcal{H}$ we have $\vdash_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$. Now we apply Lemma 3.9 to conclude $\vdash_{\mathsf{GAL}} \mathcal{G} \rhd \mathcal{H}$.

This finishes the proof of Theorem 3.8. □

Corollary 3.19. *Admissibility is decidable for* L*.*

Proof. We obtain a terminating procedure to decide the admissibility of a gs-rule $\mathcal{G} \rhd \mathcal{H}$. First apply the left logical rules to reach a modal-implication-irreducible gs-rule (Lemma 3.10). After that apply the logic specific rules from the set $X := \mathsf{GAL} \cap \{(\to)\rhd, (\mathrm{AC}), (\mathrm{AC}_{\Box}), (\mathrm{V}^{\bullet,i,1}), (\mathrm{V}^{\bullet,i,2}), (\mathrm{V}^{\circ,i})\}$ to obtain leaves $\mathcal{G} \rhd \mathcal{H}$ of the proof search tree that are full with respect to X. Similarly to the proof of Theorem 3.8, for each such a leaf $\mathcal{G} \rhd \mathcal{H}$ there are three cases: $C := \bigwedge I(\mathcal{G})$ is inconsistent, consistent and projective, or consistent and not projective. By inspection of the proof, in all cases admissibility $\mid\!\sim_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$ reduces to derivability $\vdash_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$ which is decidable by Theorem 1.2. □

4. Bases for admissible rules

Bases give a nice description of all admissible rules in a logic. A basis is a set of admissible rules that derive all other admissible rules of the logic. Recall Definition 1.5: A set of rules $\mathcal{R}$ is called a *basis* for $\mid\!\sim_{\mathsf{L}}$ if $\mid\!\sim_{\mathsf{L}} = \vdash^m_{\mathsf{L},\mathcal{R}}$. In Fig. 6 we define sets of intuitionistic modal Visser rules that we use to define the bases of the six logics, where we denote the rules without brackets in contrast to the Visser rules from GAL defined in Fig. 5. Each line should be understood as a set of rules with natural numbers n, m, k, l (that may be 0). We define bases $\mathcal{B}_{\mathsf{L}}$ for logic L as follows:

$$\mathcal{B}_{\mathsf{iCK4}} = \mathrm{V}^{\bullet,i,1} \cup \mathrm{V}^{\circ,i}, \qquad \mathcal{B}_{\mathsf{mHC}} = \mathrm{V}^{\bullet,i,2} \cup \mathrm{V}^{\circ,i},$$
$$\mathcal{B}_{\mathsf{iCS4}} = \mathrm{V}^{\circ,i}, \qquad \mathcal{B}_{\mathsf{KM}} = \mathrm{V}^{\bullet,i,2}.$$
$$\mathcal{B}_{\mathsf{iSL}} = \mathrm{V}^{\bullet,i,1},$$

We use the proof systems for admissibility from the previous section to show that these form indeed a basis. This is not done in [22], since the bases for the logics discussed there were already known in the literature [19,23]. Here we show that the general sequent Visser rules from the proof theories GAL derive the formula Visser rules for the bases $\mathcal{B}_{\mathsf{L}}$.

Let $Y = \bigwedge_{i<n}(A_i \to B_i) \wedge \bigwedge_{i<m}(\Box E_i \to F_i)$.

Irreflexive

$$\frac{\Box C \wedge Y \to \bigvee_{j<k} \Box O_j \vee \bigvee_{j<l} D_j}{\{\Box C \wedge Y \to D_j\}_{j<l}\ ,\ \{C \wedge Y \to O_j\}_{j<k}\ ,\ \{\Box C \wedge Y \to A_j\}_{j<n}\ ,\ \{C \wedge Y \to E_j\}_{j<m}}\ \mathrm{V}^{\bullet,i,1}$$

$$\frac{\Box C \wedge Y \to \bigvee_{j<k} \Box O_j \vee \bigvee_{j<l} D_j}{\{C \wedge Y \to D_j\}_{j<l}\ ,\ \{C \wedge Y \to O_j\}_{j<k}\ ,\ \{C \wedge Y \to A_j\}_{j<n}\ ,\ \{C \wedge Y \to E_j\}_{j<m}}\ \mathrm{V}^{\bullet,i,2}$$

Reflexive

$$\frac{\bigwedge_{i<h}(\Box C_i \to C_i) \wedge Y \to \bigvee_{j<k} \Box O_j \vee \bigvee_{j\leq l} D_j}{\begin{array}{c}\{\bigwedge_{i<h} C_i \wedge Y \to D_j\}_{j<l}\ ,\ \{\bigwedge_{i<h} C_i \wedge Y \to O_j\}_{j<k} \\ \{\bigwedge_{i<h} C_i \wedge Y \to A_j\}_{j<n}\ ,\ \{\bigwedge_{i<h} C_i \wedge Y \to E_j\}_{j<m}\end{array}}\ \mathrm{V}^{\circ,i}$$

Fig. 6. Intuitionistic modal Visser rules for bases.

Theorem 4.1. *Let* $\mathsf{L} \in \{\mathsf{iCK4}, \mathsf{iCS4}, \mathsf{iSL}, \mathsf{mHC}, \mathsf{KM}\}$*, then* $\mathcal{B}_\mathsf{L}$ *forms a basis for the admissible rules of* L.

Proof. We have to show that $\mathrel{|\!\sim}_\mathsf{L} = \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}}$. It is easy to show that the Visser rules from $\mathcal{B}_\mathsf{L}$ are admissible in L using a similar strategy as the proof of Theorem 3.6. Therefore, for all rules Γ/Δ, $\Gamma \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}} \Delta$ implies $\Gamma \mathrel{|\!\sim}_\mathsf{L} \Delta$. For the other direction we prove the following theorem based on the sequent system for admissibility. Note that Theorem 3.8 implies the desired result. □

Theorem 4.2. *If* $\vdash_{\mathsf{GAL}} \mathcal{G} \rhd \mathcal{H}$ *then* $I(\mathcal{G}) \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}} I(\mathcal{H})$.

Proof. We proceed by induction on the height of the derivation of $\vdash_{\mathsf{GAL}} \mathcal{G} \rhd \mathcal{H}$. Almost all rules can be easily checked using intuitionistic reasoning from L. Let us check $(\Rightarrow \wedge)\rhd$, $(\Box \Rightarrow)\rhd$, and (PJ). After that we discuss the difficult cases of the modal Visser rules.

For $(\Rightarrow \wedge)\rhd$ we have

$$\frac{\mathcal{G}, (\Gamma \Rightarrow A, \Delta), (\Gamma \Rightarrow B, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \wedge B, \Delta) \rhd \mathcal{H}}\ (\Rightarrow \wedge)\rhd\ .$$

Using the induction hypothesis we know $I(\mathcal{G}), \bigwedge \Gamma \to A \vee \bigvee \Delta, \bigwedge \Gamma \to B \vee \bigvee \Delta \ \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}} I(\mathcal{H})$. By intuitionistic reasoning we have $\bigwedge \Gamma \to (A \wedge B) \vee \bigvee \Delta \vdash_\mathsf{L} \bigwedge \Gamma \to A \vee \bigvee \Delta$, and similarly with formula B. Using transitivity of $\vdash^m_\mathsf{L}$ we obtain $I(\mathcal{G}), I(\Gamma \Rightarrow A \wedge B, \Delta) \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}} I(\mathcal{H})$.

For $(\Box \Rightarrow)\rhd$ consider

$$\frac{\mathcal{G}, (\Gamma, \Box p \Rightarrow \Delta), (A \Rightarrow p) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, \Box A \Rightarrow \Delta) \rhd \mathcal{H}}\ (\Box \Rightarrow)\rhd\ ,$$

where p does not occur in any sequent of the conclusion. By the induction hypothesis, it follows that $I(\mathcal{G}), \bigwedge \Gamma \wedge \Box p \to \bigvee \Delta, A \to p \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}} I(\mathcal{H})$. Since p is not present in $\mathcal{G}, \mathcal{H}, \Gamma, \Delta, A$, we can substitute A for p and use the structurality of the consequence relation to show that $I(\mathcal{G}), \bigwedge \Gamma \wedge \Box A \to \bigvee \Delta, A \to A \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}} I(\mathcal{H})$. Of course, formula $A \to A$ is valid, and therefore $I(\mathcal{G}), I(\Gamma, \Box A \Rightarrow \Delta) \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}} I(\mathcal{H})$.

The rule (PJ) is

$$\frac{\mathcal{G}, S \rhd (\Gamma, I(S) \Rightarrow \Delta), \mathcal{H}}{\mathcal{G}, S \rhd \mathcal{H}}\ (\mathrm{PJ})$$

where $(\Gamma \Rightarrow \Delta) \in \mathcal{H} \cup \{\Rightarrow\}$. By induction we have $I(\mathcal{G}), I(S) \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}} \bigwedge \Gamma \wedge I(S) \to \bigvee \Delta, I(\mathcal{H})$. By the fact that $I(S), \bigwedge \Gamma \wedge I(S) \to \bigvee \Delta \vdash_\mathsf{L} \bigwedge \Gamma \to \bigvee \Delta$ together with transitivity of the consequence relation, we have $I(\mathcal{G}), I(S) \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}} \bigwedge \Gamma \to \bigvee \Delta, I(\mathcal{H})$. Since $(\Gamma \Rightarrow \Delta) \in \mathcal{H} \cup \{\Rightarrow\}$ we conclude $I(\mathcal{G}), I(S) \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}} I(\mathcal{H})$.

Now we turn to the Visser rules in GAL. We treat rule $(\mathrm{V}^{\bullet,i,1})$. The other Visser rules can be handled in the same way and are left to the reader. So consider rule $(\mathrm{V}^{\bullet,i,1})$ and let $\mathsf{L} = \{\mathsf{iCK4}, \mathsf{iSL}\}$.

$$\frac{\begin{array}{c}[\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Box\Sigma, \Gamma \Rightarrow D) \triangleright \mathcal{H}]_{D\in\Delta} \\ [\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Sigma, \Gamma \Rightarrow O) \triangleright \mathcal{H}]_{O\in\Omega} \\ [\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \triangleright (\Box\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta), \mathcal{H}]_{\emptyset\neq\Pi\subseteq\Gamma_{\Box\Omega,\Delta}}\end{array}}{\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \triangleright \mathcal{H}} \; (\mathrm{V}^{\bullet,i,1})$$

Define sets of formulas

$$\Theta_\Pi := \{D \to (A \to B) \mid A \in \Pi, A \to B \in \Gamma, D \in \Delta\}, \quad \Theta := \Theta_{\Gamma_{\Box\Omega,\Delta}}$$
$$\Psi_\Pi := \{\Box O \to (A \to B) \mid A \in \Pi, A \to B \in \Gamma, O \in \Omega\}, \quad \Psi := \Psi_{\Gamma_{\Box\Omega,\Delta}}.$$

Note that $C \to (A \to B)$ is equivalent to $(C \wedge A) \to B$ for all formulas A, B, C. So Θ_Π can be considered as the set of implications from Γ not in Γ^Π whose antecedents are enriched with a formula from Δ. Similarly for Ψ_Π and $\Box\Omega$.

We need the following statement which we will show at the end of this proof.

$$\{I(\Box\Sigma, \Gamma^\Pi, \Pi \Rightarrow \Box\Omega, \Delta)\}_{\Pi\subseteq\Gamma_{\Box\Omega,\Delta}} \vdash_\mathsf{L} I(\Box\Sigma, \Gamma^{\Gamma_{\Box\Omega,\Delta}}, \Theta, \Psi \Rightarrow \Box\Omega, \Delta). \tag{7}$$

First we use it to show the desired result. Use the induction hypothesis of the third set of premises in $(\mathrm{V}^{\bullet,i,1})$ and apply transitivity together with (7) to obtain

$$I(\mathcal{G}), I(\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}} I(\Box\Sigma, \Gamma^{\Gamma_{\Box\Omega,\Delta}}, \Theta, \Psi \Rightarrow \Box\Omega, \Delta), I(\mathcal{H}). \tag{8}$$

Note that $\Gamma^{\Gamma_{\Box\Omega,\Delta}}$ only contains implications of the form $D \to B$ or $\Box O \to B$ with $D \in \Delta$ and $O \in \Omega$. Also all antecedents of implications from Θ and Ψ are D and $\Box O$, respectively. At this point we apply the basis Visser rule $\mathrm{V}^{\bullet,i,1}$ to $I(\Box\Sigma, \Gamma^{\Gamma_{\Box\Omega,\Delta}}, \Theta, \Psi \Rightarrow \Box\Omega, \Delta)$ to obtain

$$I(\mathcal{G}), I(\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}} \{I(\Box\Sigma, \Gamma^{\Gamma_{\Box\Omega,\Delta}}, \Theta, \Psi \Rightarrow D)\}_{D\in\Delta}, \{I(\Sigma, \Gamma^{\Gamma_{\Box\Omega,\Delta}}, \Theta, \Psi \Rightarrow O)\}_{O\in\Omega}, I(\mathcal{H}). \tag{9}$$

Note that $\Gamma \vdash^m_\mathsf{L} A$ for all $A \in \Gamma^{\Gamma_{\Box\Omega,\Delta}} \cup \Theta \cup \Psi$. This follows from the fact that $A \to B \vdash_\mathsf{L} C \to (A \to B)$ for all formulas A, B, C. Therefore

$$I(\mathcal{G}), I(\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}} \{I(\Box\Sigma, \Gamma \Rightarrow D)\}_{D\in\Delta}, \{I(\Sigma, \Gamma \Rightarrow O)\}_{O\in\Omega}, I(\mathcal{H}) \tag{10}$$

Now we use the induction hypothesis of the first two sets of premises in $(\mathrm{V}^{\bullet,i,1})$ to conclude the desired result, $I(\mathcal{G}), I(\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}} I(\mathcal{H})$.

It remains us to show statement (7). To do so, we show by induction on the cardinality of set $\Pi' \subseteq \Gamma_{\Box\Omega,\Delta}$ that the following statement holds for all sets of formulas Ξ and Υ:

$$\{I(\Upsilon, \Gamma^{\Pi,\Xi}, \Pi \Rightarrow \Box\Omega, \Delta)\}_{\Pi\subseteq\Pi'} \vdash_\mathsf{L} I(\Upsilon, \Gamma^{\Pi',\Xi}, \Theta_{\Pi'}, \Psi_{\Pi'} \Rightarrow \Box\Omega, \Delta). \tag{11}$$

When Π' is empty the result follows immediately. Now suppose $|\Pi'| = n + 1$. This means that we have 2^{n+1} formulas on the left-hand side of (11). Let $A \in \Pi'$ be a formula and consider the following 2^n derivations for $\Pi \subseteq \Pi' \setminus \{A\}$,

$$I(\Upsilon, \Gamma^{\Pi,\Xi}, \Pi \Rightarrow \Box\Omega, \Delta), I(\Upsilon, \Gamma^{\Pi,\Xi,\{A\}}, \Pi, A \Rightarrow \Box\Omega, \Delta) \vdash_{\mathsf{L}} I(\Upsilon, \Gamma^{\Pi,\Xi,\{A\}}, \Theta_A, \Psi_A, \Pi \Rightarrow \Box\Omega, \Delta). \tag{12}$$

This follows from the fact that $\Upsilon, \Gamma^{\Pi,\Xi,\{A\}}, \Pi \vdash_{\mathsf{L}} A \leftrightarrow (\bigvee \Delta \vee \bigvee \Box\Omega) \wedge A$ using the second premise in (12). So this means that $\Theta_A, \Psi_A \vdash_{\mathsf{L}} A \to B$ for each B such that $A \to B \in \Gamma$. Using the first premise in (12) completes the desired result. Now we can apply the induction hypothesis to $|\Pi' \setminus \{A\}| = n$ stating that:

$$\{I(\Upsilon, \Theta_A, \Psi_A, \Gamma^{\Pi,\Xi,\{A\}}, \Pi \Rightarrow \Box\Omega, \Delta)\}_{\Pi\subseteq\Pi'\setminus\{A\}} \vdash_{\mathsf{L}} I(\Upsilon, \Theta_A, \Psi_A, \Gamma^{\Pi'\setminus\{A\},\Xi,\{A\}}, \Theta_{\Pi'\setminus\{A\}}, \Psi_{\Pi'\setminus\{A\}} \Rightarrow \Box\Omega, \Delta). \tag{13}$$

Since $\Theta_{\Pi'} = \Theta_A \cup \Theta_{\Pi'\setminus\{A\}}$ and $\Psi_{\Pi'} = \Psi_A \cup \Psi_{\Pi'\setminus\{A\}}$, this is the same as:

$$\{I(\Upsilon, \Theta_A, \Psi_A, \Gamma^{\Pi,\Xi,\{A\}}, \Pi \Rightarrow \Box\Omega, \Delta)\}_{\Pi\subseteq\Pi'\setminus\{A\}} \vdash_{\mathsf{L}} I(\Upsilon, \Gamma^{\Pi',\Xi}, \Theta_{\Pi'}, \Psi_{\Pi'} \Rightarrow \Box\Omega, \Delta). \tag{14}$$

Now 2^n applications of transitivity on (12) and (14) results in equation (11) for $|\Pi'| = n + 1$, concluding the induction proof for (11). To conclude (7), take $\Xi = \emptyset$, $\Upsilon = \Box\Sigma$, and $\Pi' = \Gamma_{\Box\Omega,\Delta}$. $\Box$

Example 4.3. Basis Visser rules $\mathrm{V}^{\circ,i}$ can be considered as the reflexive version of $\mathrm{V}^{\bullet,i,2}$ and the following rule as the reflexive version of $\mathrm{V}^{\bullet,i,1}$, where $Y = \bigwedge_{i<n}(A_i \to B_i) \wedge \bigwedge_{i<m}(\Box E_i \to F_i)$.

$$\frac{\bigwedge_{i<h}(\Box C_i \to C_i) \wedge Y \to \bigvee_{j<k} \Box O_j \vee \bigvee_{j<l} D_j}{\begin{array}{c}\{\bigwedge_{i<h}(\Box C_i \to C_i) \wedge Y \to D_j\}_{j<l}\ ,\ \{\bigwedge_{i<h} C_i \wedge Y \to O_j\}_{j<k} \\ \{\bigwedge_{i<h}(\Box C_i \to C_i) \wedge Y \to A_j\}_{j<n}\ ,\ \{\bigwedge_{i<h} C_i \wedge Y \to E_j\}_{j<m}\end{array}} \mathrm{V}^{\circ,i,1}$$

This rule is equivalent to $\mathrm{V}^{\circ,i}$. That $\mathrm{V}^{\circ,i}$ follows from $\mathrm{V}^{\circ,i,1}$ is due to the fact that $C \vdash_{\mathsf{L}} \Box C \to C$. For the other direction, we let $C_i' := \Box C_i \to C_i$ for each $i < h$. It can be checked that $\vdash_{\mathsf{iK}} (\Box C_i' \to C_i') \to C_i'$ for each $i < h$. Let us write $\Gamma = \{C_i \mid i < h\}$, $\Gamma' = \{C_i' \mid i < h\}$, $\Omega = \{O_j \mid j < k\}$, and $\Delta = \{O_j \mid j < l\}$. So we have

$$I(\Box\Gamma \to \Gamma, Y \Rightarrow \Box\Omega, \Delta)\ \vdash_{\mathsf{L}}\ I(\Box\Gamma' \to \Gamma', Y \Rightarrow \Box\Omega, \Delta). \tag{15}$$

An application of $\mathrm{V}^{\circ,i}$ to the right-hand side and transitivity of $\vdash^m_{\mathsf{L},\mathrm{V}^{\circ,i}}$ gives us

$$I(\Box\Gamma \to \Gamma, Y \Rightarrow \Box\Omega, \Delta) \vdash^m_{\mathsf{L},\mathrm{V}^{\circ,i}} \begin{array}{c}\{I(\Box\Gamma \to \Gamma, Y \Rightarrow D_j)\}_{j<l}\ ,\ \{I(\Box\Gamma \to \Gamma, Y \Rightarrow O_j)\}_{j<k}, \\ \{I(\Box\Gamma \to \Gamma, Y \Rightarrow A_j)\}_{j<n}\ ,\ \{I(\Box\Gamma \to \Gamma, Y \Rightarrow E_j)\}_{j<m}.\end{array} \tag{16}$$

Since $C \vdash_{\mathsf{L}} \Box C \to C$ for all formulas C we have the following application of $\mathrm{V}^{\circ,i,1}$ as desired.

$$I(\Box\Gamma \to \Gamma, Y \Rightarrow \Box\Omega, \Delta) \vdash^m_{\mathsf{L},\mathrm{V}^{\circ,i}} \begin{array}{c}\{I(\Box\Gamma \to \Gamma, Y \Rightarrow D_j)\}_{j<l}\ ,\ \{I(\Gamma, Y \Rightarrow O_j)\}_{j<k}, \\ \{I(\Box\Gamma \to \Gamma, Y \Rightarrow A_j)\}_{j<n}\ ,\ \{I(\Gamma, Y \Rightarrow E_j)\}_{j<m}.\end{array} \tag{17}$$

This shows that $\mathrm{V}^{\circ,i}$ and $\mathrm{V}^{\circ,i,1}$ are equivalent.

We now turn to single-conclusion admissible rules of the form Γ/A for finite set of formulas Γ and formula A. We denote $\mathrel|\!\sim^s_{\mathsf{L}}$ for the single-conclusion admissible rules in L. Analogous to [23], we are able to extract the single-conclusion admissible rules from the multi-conclusion admissible rules via the disjunction property (see Example 3.4). For bases $\mathscr{B}_{\mathsf{L}}$ of the multi-conclusion rules we define

$$\widehat{\mathscr{B}}_{\mathsf{L}} := \{(\bigwedge \Gamma \vee D)/(\bigvee \Delta \vee D) \mid \Gamma/\Delta \in \mathscr{B}_{\mathsf{L}}\}.$$

Lemma 4.4. *If* $\Gamma \vdash_{\mathsf{L},\widehat{\mathscr{B}}_{\mathsf{L}}} A$*, then* $\bigwedge \Gamma \vee B \vdash_{\mathsf{L},\widehat{\mathscr{B}}_{\mathsf{L}}} A \vee B$ *for any formula* B.

Proof. By induction on the structure of the single-conclusion consequence relation $\vdash_{\mathsf{L},\widehat{\mathcal{B}}_\mathsf{L}}$. If Γ/A is derivable in L, i.e. $\Gamma \vdash_\mathsf{L} A$, then also $\bigwedge\Gamma \vee B \vdash_\mathsf{L} A \vee B$ by intuitionistic reasoning. If Γ/A is in $\widehat{\mathcal{B}}_\mathsf{L}$, it is of the form $(\bigwedge\Gamma \vee D)/(\bigwedge\Delta \vee D)$ for some $(\Gamma/\Delta) \in \mathcal{B}_\mathsf{L}$. We have $(\bigwedge\Gamma \vee D) \vee B \vdash_\mathsf{L} \bigwedge\Gamma \vee (D \vee B) \vdash_{\mathsf{L},\widehat{\mathcal{B}}_\mathsf{L}} \bigvee\Delta \vee (D \vee B) \vdash_\mathsf{L} (\bigvee\Delta \vee D) \vee B$. The result for the closure properties of the consequence relation follows easily by induction. □

Theorem 4.5. *Let* $\mathsf{L} \in \{\mathsf{iCK4}, \mathsf{iCS4}, \mathsf{iSL}, \mathsf{mHC}, \mathsf{KM}\}$*, then* $\widehat{\mathcal{B}}_\mathsf{L}$ *forms a basis for the single-conclusion admissible rules of* L*.*

Proof. We show that $\mid\!\sim^s_\mathsf{L} = \vdash_{\mathsf{L},\widehat{\mathcal{B}}_\mathsf{L}}$. Note that $\Gamma \mid\!\sim^s_\mathsf{L} A$ iff $\Gamma \mid\!\sim_\mathsf{L} A$ iff $\Gamma \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}} A$. The fact that $\vdash_{\mathsf{L},\widehat{\mathcal{B}}_\mathsf{L}} \subseteq\ \mid\!\sim^s_\mathsf{L}$ follows from the fact that each rule in $\widehat{\mathcal{B}}_\mathsf{L}$ is admissible using the disjunction property. For the other inclusion, it is sufficient to prove that $\Gamma \vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}} \Delta$ implies $\Gamma \vdash_{\mathsf{L},\widehat{\mathcal{B}}_\mathsf{L}} \bigvee\Delta$. We do so by induction on the structure of $\vdash^m_{\mathsf{L},\mathcal{B}_\mathsf{L}}$. We only treat two cases, the other cases are left to the reader. If Γ/Δ is in $\mathcal{B}_\mathsf{L}$, we have $\Gamma \vdash_\mathsf{L} \bigwedge\Gamma \vee \bot \vdash_{\mathsf{L},\widehat{\mathcal{B}}_\mathsf{L}} \bigvee\Delta \vee \bot \vdash_\mathsf{L} \bigvee\Delta$. The induction step for transitivity is as follows. By induction hypothesis, we have $\Gamma_1 \vdash_{\mathsf{L},\widehat{\mathcal{B}}_\mathsf{L}} \bigvee\Delta_1 \vee C$ and $\Gamma_2, C \vdash_{\mathsf{L},\widehat{\mathcal{B}}_\mathsf{L}} \bigvee\Delta_2$. Note that in general we have $F, G \vdash_{\mathsf{L},\widehat{\mathcal{B}}_\mathsf{L}} F \wedge G$ and together with $\Gamma \vdash_{\mathsf{L},\widehat{\mathcal{B}}_\mathsf{L}} F$ we have $\Gamma, G \vdash_{\mathsf{L},\widehat{\mathcal{B}}_\mathsf{L}} F \wedge G$. So for the first we obtain $\Gamma_1, \Gamma_2 \vdash_{\mathsf{L},\widehat{\mathcal{B}}_\mathsf{L}} \bigvee\Delta_1 \vee (\bigwedge\Gamma_2 \wedge C)$. Lemma 4.4 applied to the second yields $(\Gamma_2 \wedge C) \vee \bigvee\Delta_1 \vdash_{\mathsf{L},\widehat{\mathcal{B}}_\mathsf{L}} \bigvee\Delta_2 \vee \bigvee\Delta_1$. Using transitivity we have $\Gamma_1, \Gamma_2 \vdash_{\mathsf{L},\widehat{\mathcal{B}}_\mathsf{L}} \bigvee\Delta_1 \vee \bigvee\Delta_2$. □

5. Propositional lax logic

Recall that $\mathsf{PLL} := \mathsf{iCK4} + \bigcirc\bigcirc A \to \bigcirc A$. We work with constraint models defined by Fairtlough and Mendler [11]. We would like to stress again that the presented method to characterize the admissible rules of PLL does not work when using Goldblatt frames [18]. The reason is that these models are not *extensible*, meaning that extending Goldblatt frames with a new root does not necessarily result in a Goldblatt frame.

A PLL-*frame* is a quadruple $(W, \leq, R, F)$, where W is a non-empty set with partial order $\leq$, binary relation R, and $F \subseteq W$ the set of *fallible* worlds, with: $w \leq w', w \in F \Rightarrow w' \in F$. A PLL-*model* is a quintuple $K = (W, \leq, R, V, F)$, where $(W, \leq, R, F)$ is a PLL-frame and V is the valuation map which is monotonic in $\leq$, which means that $w \leq w'$ implies $V(w,p) \leq V(w',p)$. In addition we have the following conditions:

1. *strongness*: xRy implies $x \leq y$,
2. R is reflexive and transitive,
3. V is *full* on F: for all atoms p and $w \in F$, $V(w,p) = 1$.

The forcing relation $\Vdash$ changes as follows, where the forcing relation of $\bigcirc$ contains a universal quantifier and an existential quantifier that covers both aspects of $\Box$ and $\Diamond$:

$$\begin{array}{ll} K, w \Vdash \bot & \text{iff } w \in F \\ K, w \Vdash \bigcirc A & \text{iff for all } w' \geq w \text{ there exists a } v \text{ such that } w'Rv \text{ and } K, v \Vdash A. \end{array}$$

By induction on A it can be shown that if $K, w \models A$ and $w \leq v$, then $K, v \models A$. Similarly, if $K, w \models A$ and wRv, then $K, v \models A$. We also have that $K, w \models A$ for all $w \in F$. We assume the frames and models to be rooted.

We will show that the constraint models also satisfy the correspondence between projective formulas and the extension property in PLL-models. A similar characterization is shown with respect to the so-called subframe semantics for PLL in [15]. We use the same definitions of projectivity and the extension property from Section 2. Note for example that $Mod_{\mathsf{PLL}}(\bot)$ does not have the extension property, because world ρ in K does not have to be a fallible world. And, indeed, there is no unifier for $\bot$.

Theorem 5.1. *Formula A is projective in* PLL *if and only if $Mod_{\mathsf{PLL}}(A)$ has the extension property.*

The theorem is similarly shown as Theorem 2.4, where semantic operator σ^* is now defined as follows on PLL-models. $\sigma^*(K)$ has the same frame as K, in particular they have the same set of fallible worlds, and the valuation on $\sigma^*(K)$ is defined as:

$$\sigma^*(K), w \Vdash p \Leftrightarrow K, w \Vdash \sigma(p).$$

This definition is well defined. Indeed, it is easy to check that the valuation is full with respect to the set of fallible worlds of $\sigma^*(K)$. We denote the set of fallible worlds of $\sigma^*(K)$ by $\sigma^*(F)$.

Lemma 5.2. *Let A be a formula and let σ be a substitution. For every* PLL*-model K, we have*

(i) $\sigma^*(K) \models A$ *iff* $K \models \sigma(A)$,
(ii) and for every substitution τ, $(\tau\sigma)^*(K) = \sigma^*(\tau^*(K))$.

Proof. (ii) easily follows from (i). For (i) we have to take into account the fallible worlds. It is shown with induction to the structure of formula A. We only treat $A = \bot$, where we use the fact that $\sigma(\bot) = \bot$. We have $\sigma^*(K), w \Vdash \bot$ iff $w \in \sigma^*(F)$ iff $w \in F$ iff $K, w \Vdash \bot$ iff $K, w \Vdash \sigma(\bot)$. □

Now we show Theorem 5.1. The direction from left to right has the same proof as Theorem 2.6. The other direction proceeds analogously to the method presented in Section 2. We use the same simple substitutions σ_a as defined in Section 2. Lemma 2.7 and Lemma 2.8 also hold for PLL. The proofs are completely identical, because the interpretation of the connectives $\to$ and $\wedge$ in PLL-models remain standard and only these connectives play a role in the substitution σ_a^*. We take the same definitions of frontier points, corresponding substitution, rank and maximal frontier points as in Section 2. Note that fallible worlds can never be frontier points. We take the same definitions of θ and τ's as from (2). Lemma 2.9 and Lemma 2.11 are shown exactly the same for the setting of PLL showing that θ is a projective unifier for A. So we have Theorem 5.1. We conclude with the equivalent of Corollary 2.12.

Corollary 5.3. *Projectivity in* PLL *is decidable.*

5.1. Proof theory for admissibility

Let the proof system for admissibility for PLL, written GAPLL, contain all rules from Fig. 4, where each $\Box$ is replaced by $\bigcirc$, and the modal Visser rules ($\mathrm{V}^{\leq,R}$) and ($\mathrm{V}^{\leq}$) from Fig. 7. Such as for the other logics, the Visser rules reflect extensions in PLL-models. Rule ($\mathrm{V}^{\leq,R}$) reflects extensions with at least one R relation, whereas ($\mathrm{V}^{\leq}$) reflects extensions with no R relations. Note that ($\mathrm{V}^{\leq}$) is identical to ($\mathrm{V}^{\circ,i}$) and can correspond to the reflexive extensions in the Goldblatt models for PLL.

Note that the weakening rules

$$\frac{\mathcal{G} \rhd \mathcal{H}}{\mathcal{G}, S \rhd \mathcal{H}}\,(\mathrm{W})\rhd \qquad \frac{\mathcal{G} \rhd \mathcal{H}}{\mathcal{G} \rhd S, \mathcal{H}}\,\rhd(\mathrm{W})$$

are also admissible in GAPLL.

Theorem 5.4 *(soundness). If $\vdash_{\mathsf{GAPLL}} \mathcal{G} \rhd \mathcal{H}$, then $\mathrel{|\!\sim}_{\mathsf{PLL}} \mathcal{G} \rhd \mathcal{H}$.*

Proof. Similar to the proof of Theorem 3.6. Here we consider the rules ($\mathrm{V}^{\leq,R}$) and ($\mathrm{V}^{\leq}$).

$$\frac{\begin{array}{c}[\mathcal{G}, (\bigcirc\Sigma, \Gamma \Rightarrow \Delta), (\bigcirc\Sigma, \Gamma \Rightarrow D) \triangleright \mathcal{H}]_{D\in\Delta} \\ [\mathcal{G}, (\bigcirc\Sigma, \Gamma \Rightarrow \Delta) \triangleright (\bigcirc\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Delta), \mathcal{H}]_{\emptyset\neq\Pi\subseteq\Gamma_{\Delta}}\end{array}}{\mathcal{G}, (\bigcirc\Sigma, \Gamma \Rightarrow \Delta) \triangleright \mathcal{H}} \ (\mathrm{V}^{\leq,R})$$

$$\frac{\begin{array}{c}[\mathcal{G}, (\bigcirc\Sigma \to \Sigma, \Gamma \Rightarrow \bigcirc\Omega, \Delta), (\Sigma, \Gamma \Rightarrow D) \triangleright \mathcal{H}]_{D\in\Delta} \\ [\mathcal{G}, (\bigcirc\Sigma \to \Sigma, \Gamma \Rightarrow \bigcirc\Omega, \Delta), (\Sigma, \Gamma \Rightarrow O) \triangleright \mathcal{H}]_{O\in\Omega} \\ [\mathcal{G}, (\bigcirc\Sigma \to \Sigma, \Gamma \Rightarrow \bigcirc\Omega, \Delta) \triangleright (\bigcirc\Sigma \to \Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \bigcirc\Omega, \Delta), \mathcal{H}]_{\emptyset\neq\Pi\subseteq\Gamma_{\bigcirc\Omega,\Delta}}\end{array}}{\mathcal{G}, (\bigcirc\Sigma \to \Sigma, \Gamma \Rightarrow \bigcirc\Omega, \Delta) \triangleright \mathcal{H}} \ (\mathrm{V}^{\leq})$$

where for all these rules it holds that Γ contains only implications,
$\Gamma^{\Pi} := \{A \to B \in \Gamma \mid A \notin \Pi\}$ and $\Gamma_{\bigcirc\Omega,\Delta} := \{A \notin \bigcirc\Omega \cup \Delta \mid \exists B(A \to B \in \Gamma)\}$

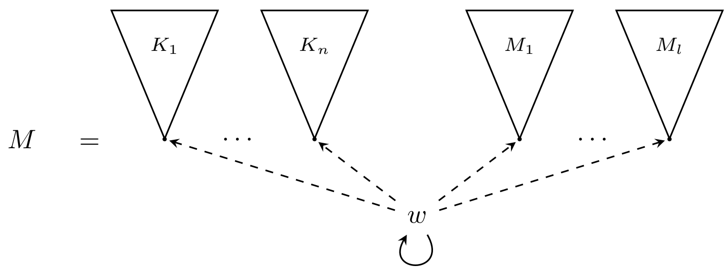
Fig. 7. Visser rules for PLL.

For rule ($\mathrm{V}^{\leq}$), suppose that σ is a unifier for $I(S)$ for all $S \in \mathcal{G}$ and $I(\bigcirc\Sigma \to \Sigma, \Gamma \Rightarrow \bigcirc\Omega, \Delta)$. We write $\Delta = \{D_1, \dots, D_n\}$ and $\Omega = \{O_1, \dots, O_l\}$ (including the cases where the sets are empty). Using the third set of premises, we have for all $\emptyset \neq \Pi \subseteq \Gamma_{\bigcirc\Omega,\Delta}$ that σ is either a unifier for some $S \in \mathcal{H}$ or for $I(\bigcirc\Sigma \to \Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \bigcirc\Omega, \Delta)$. If there is such a Π for which the first case holds we are done. If for all such Π we have the second case (or in case there is no such Π at all), we will show that σ is a unifier for $I(\Sigma, \Gamma \Rightarrow D_i)$ for some i or for $I(\Sigma, \Gamma \Rightarrow O_j)$ for some j. This is sufficient, because it implies that σ is a unifier for some $S \in \mathcal{H}$ by the first or second set of premises of ($\mathrm{V}^{\leq}$). Suppose for a contradiction that this is not the case. Then there exist PLL-countermodels $K_1, \dots, K_n$ and $M_1, \dots M_l$ such that

$$K_i \models \sigma(\bigwedge \Sigma \wedge \bigwedge \Gamma) \text{ and } K_i \not\models \sigma(D_i),$$

$$M_j \models \sigma(\bigwedge \Sigma \wedge \bigwedge \Gamma) \text{ and } M_j \not\models \sigma(O_j).$$

Consider the following PLL-model M with reflexive root w. (M is a one-node model if $\Delta = \Omega = \emptyset$.) Relation $\leq$ is drawn by dashed lines, and R by a straight line. We take R to be reflexive and transitive by definition of PLL-models. Root w is not a fallible world in the model.

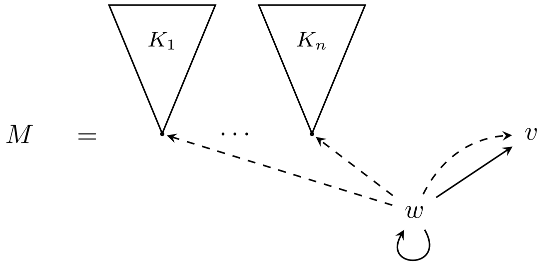


First note that $M \not\models \sigma(D_i)$ for all i. We also have $M \not\models \sigma(\bigcirc O_j)$ for all j, because for all worlds v such that wRv we have $v \not\models \sigma(O_j)$ (indeed v can only be w). Also note that $M \models \sigma(\bigwedge(\bigcirc\Sigma \to \Sigma))$ because the only world that is modal related to w is w itself. Let $\Pi := \{A \in \Gamma_{\bigcirc\Omega,\Delta} \mid M \models \sigma(A)\}$. Thus $M \models \sigma(\bigwedge \Pi)$. We also claim that $M \models \sigma(\bigwedge \Gamma^{\Pi})$. Let $A \to B \in \Gamma^{\Pi}$. Observe that either $A \in \Delta \cup \bigcirc\Omega$ or $M \not\models \sigma(A)$. We already saw that the first implies the second, so $M \not\models \sigma(A)$. And since $K_i \models \sigma(A \to B)$ for all i and $M_j \models \sigma(A \to B)$ for all j, we have $M \models \sigma(A \to B)$. So far we have shown that $M \models \sigma(\bigwedge(\bigcirc\Sigma \to \Sigma \cup \Gamma^{\Pi} \cup \Pi))$. If $\Pi = \emptyset$, then $\Gamma^{\Pi} = \Gamma$ and so $M \models \sigma(\bigwedge(\bigcirc\Sigma \to \Sigma \cup \Gamma))$. But σ is a unifier for $I(\bigcirc\Sigma \to \Sigma, \Gamma \Rightarrow \bigcirc\Omega, \Delta)$. If $\Pi \neq \emptyset$, then σ is a unifier for $I(\bigcirc\Sigma \to \Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \bigcirc\Omega, \Delta)$ by assumption. In both cases we have $M \models \sigma(\bigvee(\bigcirc\Omega \cup \Delta))$, which is a contradiction with our first observation about model M.

For rule ($\mathrm{V}^{\leq,R}$), suppose that σ is a unifier for $I(S)$ for all $S \in \mathcal{G}$ and for $I(\bigcirc\Sigma, \Gamma \Rightarrow \Delta)$. Again, we write $\Delta = \{D_1, \ldots, D_n\}$. By a similar argument as above, it is sufficient to show that σ is a unifier for $I(\bigcirc\Sigma, \Gamma \Rightarrow D_i)$ for some i. Suppose for a contradiction that this is not the case. Then there exist PLL-countermodels $K_1, \ldots, K_n$ such that

$$K_i \models \sigma(\bigwedge \bigcirc\Sigma \wedge \bigwedge \Gamma) \text{ and } K_i \not\models \sigma(D_i).$$

Consider the following PLL-model M with reflexive root w and $v \in F$ is a fallible world. Again, relation $\leq$ is drawn by dashed lines, and R by a straight line. We take R to be reflexive and transitive by definition of PLL-models. Strictly speaking, the fallible world v is only necessary in case $\Delta = \emptyset$, otherwise one R-relation to one of K_i will suffice.



First note that $M \not\models \sigma(D_i)$ for all i. Also note that $M \models \sigma(\bigwedge(\bigcirc\Sigma))$ because $K_i \models \sigma(\bigwedge(\bigcirc\Sigma))$ for all i, and for w we have fallible world v such that wRv and $v \models \bigwedge \Sigma$ (note that this also holds for $\Delta = \emptyset$). Let $\Pi = \{A \in \Gamma_\Delta \mid M \models \sigma(A)\}$. Thus $M \models \sigma(\bigwedge \Pi)$. Again, $M \models \sigma(\bigwedge \Gamma^\Pi)$, so $M \models \sigma(\bigwedge(\bigcirc\Sigma \cup \Gamma^\Pi \cup \Pi))$. If $\Pi = \emptyset$, then $\Gamma^\Pi = \Gamma$ and so $M \models \sigma(\bigwedge(\bigcirc\Sigma \cup \Gamma))$. But σ is a unifier for $I(\bigcirc\Sigma, \Gamma \Rightarrow \Delta)$. If $\Pi \neq \emptyset$, then σ is a unifier for $I(\bigcirc\Sigma, \Gamma^\Pi, \Pi \Rightarrow \Delta)$ by assumption. In both cases we have $M \models \sigma(\bigvee(\Delta))$, which is a contradiction with our first observation about model M. □

All results from Lemma 3.9 to Lemma 3.18 are similarly proved for PLL. So all rules in GAPLL are invertible and each admissible gs-rule is derivable in GAPLL from admissible semi-modal-implication-irreducible gs-rules that are full with respect to $\{(\to)\triangleright, (\mathrm{AC}), (\mathrm{AC}_\bigcirc), (\mathrm{V}^{\leq,R}), (\mathrm{V}^{\leq})\}$. This enables us to show completeness.

Theorem 5.5 *(completeness). If* $\mathrel{|\!\sim}_{\mathsf{PLL}} \mathcal{G} \triangleright \mathcal{H}$*, then* $\vdash_{\mathsf{GAPLL}} \mathcal{G} \triangleright \mathcal{H}$*.*

Proof. Same strategy as the proof of Theorem 3.8. Suppose $\mathrel{|\!\sim}_{\mathsf{PLL}} \mathcal{G} \triangleright \mathcal{H}$. It is sufficient to assume that $\mathcal{G} \triangleright \mathcal{H}$ is a semi-modal-implication-irreducible gs-rule that is full with respect to $(\to)\triangleright$, $(\mathrm{V}^{\leq,R})$, $(\mathrm{V}^{\leq})$, (AC) and $(\mathrm{AC}_\bigcirc)$.

Define formula $C := \bigwedge I(\mathcal{G})$. We consider three cases. The two cases for which C is inconsistent, or consistent and projective, are shown similarly to the proof for Theorem 3.8.

So suppose that C is consistent and not projective. By Theorem 5.1, there is a PLL-model K with root ρ such that $K_w \models C$ for each $w \neq \rho$ and for each variant K' of K we have $K' \not\models C$. Since formula C is consistent it holds in at least one non-fallible world of some PLL-model. Therefore, it holds in a model in which all the nodes are fallible worlds except for the root. For model K this means that there exists at least one non-fallible world $w \neq \rho$ in K.

Let $M_1, \ldots M_k$ be all the variants of K that only force atoms among the atoms occurring in $\mathcal{G}$ and $\mathcal{H}$. So $M_i \not\models C$ for all i, hence $M_i \not\models I(S_i)$ for some $S_i \in \mathcal{G}$. Write $S_i = (\Lambda_i \Rightarrow \Phi_i)$, then $M_i \not\models I(\Lambda_i \Rightarrow \Phi_i)$. Since $M_i, w \Vdash I(\Lambda_i \Rightarrow \Phi_i)$ for each $w \neq \rho$, we have $M_i \models \bigwedge \Lambda_i$ and $M_i \not\models \bigvee \Phi_i$. Such as for Theorem 3.8, since $\mathcal{G} \triangleright \mathcal{H}$ is full with respect to $(\to)\triangleright$, we can assume that

$$(p \to q) \in \Lambda_i \implies p \in \Phi_i, \tag{18}$$

$$(\bigcirc p \to p) \in \Lambda_i \implies \bigcirc p \in \Phi_i. \tag{19}$$

Define the set of atoms:

$$P := \{p \mid p \text{ occurs in } C \text{ and } K_w \models p \text{ for all } w \neq \rho\}.$$

We consider two possibilities for the frame of model K (note that each M_i has the same frame): there is some world $w \neq \rho$ such that $\rho R w$ or there is not. In the first case we use the fact that $\mathcal{G} \rhd \mathcal{H}$ is full with respect to ($\mathrm{V}^{\leq,R}$) and (AC). For the second case we use fullness with respect to ($\mathrm{V}^{\leq}$) and ($\mathrm{AC}_{\bigcirc}$). Case 1 can be compared to the completeness proof for logics with rule ($\mathrm{V}^{\bullet,i,1}$) and Case 2 to logics with rule ($\mathrm{V}^{\circ,i}$).

Case 1:
Define formulas for $i = 1, \ldots, k$ as follows:

$$A_i := \bigwedge_{p \in \Lambda_i} \neg p \wedge \bigwedge_{p \in \Phi_i \cap P} p \quad \text{and} \quad A := \bigvee_{i=1}^{k} A_i.$$

Note again that $p \in \Lambda_i$ implies $p \in P$. Using the same proof for Case 1 in Theorem 3.8, we have that A is a classical tautology and that

$$\neg A \equiv \bigwedge_{i=1}^{k} \Big(\bigvee_{p \in \Lambda_i} p \vee \bigvee_{p \in \Phi_i \cap P} \neg p \Big)$$

is classically inconsistent. Therefore there exists a resolution refutation starting with the clauses

$$\{p \mid p \in \Lambda_i\} \cup \{\neg p \mid p \in \Phi_i \cap P\} \text{ for } i = 1, \ldots, k,$$

that ends in the empty clause $\emptyset$. Again, the resolution refutation can be mimicked by rule (AC), where each clause $\Theta \cup \Psi'$ corresponds to a semi-modal-implication-irreducible sequent $\bigcirc\Sigma, \Gamma, \Theta \Rightarrow \bigcirc\Omega, \Delta, \Psi \in \mathcal{G}$, where Γ only contains atom implications or implications of the form $\bigcirc p \to p$, sets Σ and Ω only contain atoms, and Δ only contains atoms with $\Delta \cap P = \emptyset$ (and Θ and Ψ as in Theorem 3.8). We have the following properties for each clause $\Theta \cup \Psi'$:

1. $K_v \models \bigwedge \bigcirc\Sigma \wedge \bigwedge \Gamma$ for all $v \neq \rho$,
2. $\bigcirc q \in \bigcirc\Omega$ implies $K_v \not\models \bigcirc q$ for some $v \neq \rho$,
3. $\bullet((\bigcirc\Sigma, \Gamma, \Theta \Rightarrow \bigcirc\Omega, \Delta, \Psi), \mathcal{G})$.

Here we only present the proof for property 2. It holds for initial sequents $\Lambda_i \Rightarrow \Phi_i$, because suppose $\bigcirc q \in \Phi_i$. We know $M_i \not\models \Phi_i$, so there must be a $w \geq \rho$ such that for all v with wRv we have $M_v \not\models q$. If $w \neq \rho$ we are done, since K is a variant of M_i. Suppose $w = \rho$, then for all v with $\rho R v$ we have $M_v \not\models q$. We assumed that there is at least one $v \neq \rho$ so that $\rho R v$. For this particular v we know that if vRy then $\rho R y$ by transitivity of relation R. Hence $M_v \not\models \bigcirc q$. For all other corresponding sequents in the refutation it follows immediately from backwards applications of (AC), since we do not 'cut' on boxed formulas.

Now we use all those facts for the empty clause $\emptyset$. There is a corresponding sequent for the empty clause, $\bigcirc\Sigma, \Gamma \Rightarrow \bigcirc\Omega, \Delta$, where Γ contains atom implications or implications of the form $\bigcirc p \to p$, Σ and Ω only contain atoms, and Δ atoms not in P.

Gs-rule $\mathcal{G} \rhd \mathcal{H}$ is full with respect to ($\mathrm{V}^{\leq,R}$), so we have at least one of the following:

(i) $(\bigcirc\Sigma, \Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Delta$,

(ii) $(\bigcirc\Sigma, \Gamma \Rightarrow \bigcirc q) \in \mathcal{G}$ for some $q \in \Omega$,
(iii) $(\bigcirc\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \bigcirc\Omega, \Delta) \in \mathcal{H}$ for some $\emptyset \neq \Pi \subseteq \Gamma_{\bigcirc\Omega\cup\Delta}$.

The proof proceeds analogous to Case 1 in Theorem 3.8, showing that (i) and (ii) lead to a contradiction and that case (iii) implies $\vdash_{\mathsf{GAPLL}} \mathcal{G} \triangleright \mathcal{H}$.

Case 2:
Now we define formulas for $i = 1, \dots, k$ as follows, where l_p are new introduced atoms:

$$A_i := \bigwedge_{p\in\Lambda_i} \neg l_p \wedge \bigwedge_{\bigcirc p\in\Lambda_i} \neg l_p \wedge \bigwedge_{p\in P, p\in\Phi_i} l_p \wedge \bigwedge_{p\in P, \bigcirc p\in\Phi_i} l_p \quad \text{and} \quad A := \bigvee_{i=1}^{k} A_i.$$

Note that $p \in \Lambda_i$ implies $p \in P$, because $M_i \models \bigwedge \Lambda_i$ and so for variant K we have $K_w \models p$ for all $w \neq \rho$ by monotonicity. We also have that $\bigcirc p \in \Lambda_i$ implies $p \in P$. This follows from the form of model M_i as follows. $M_i \models \bigcirc p$, which means that for all $w \geq \rho$ there exists a v such that wRv and $M_i, v \models p$. For $w = \rho$ we have that ρ itself is the only world v such that ρRv, hence $M_i \models p$ and so $p \in P$. Using a similar argument we have $M_i \not\models p$ for all p such that $p \in P$ and $\bigcirc p \in \Phi_i$.

Again, A is a classical tautology. So

$$\neg A \equiv \bigwedge_{i=1}^{k} (\bigvee_{p\in\Lambda_i} l_p \vee \bigvee_{\bigcirc p\in\Lambda_i} l_p \vee \bigvee_{p\in\Phi_i\cap P} \neg l_p \vee \bigvee_{p\in P, \bigcirc p\in\Phi_i} \neg l_p)$$

is classically inconsistent. Therefore there exists a resolution refutation starting with the clauses

$$\{l_p \mid p \in \Lambda_i \text{ or } \bigcirc p \in \Lambda_i\} \cup \{\neg l_p \mid p \in P, \text{ and } p \in \Phi_i \text{ or } \bigcirc p \in \Phi_i\} \text{ for } i = 1, \dots, k,$$

that ends in the empty clause $\emptyset$. Define $\Theta', \Psi', \Theta, \Psi$ as in Case 3 of Theorem 3.8. The resolution refutation can be mimicked by $(\mathrm{AC}_{\bigcirc})$ where each clause $\Theta' \cup \Psi'$ corresponds to a semi-modal-implication-irreducible sequent $\bigcirc\Sigma \to \Sigma, \bigcirc\Sigma' \to \Sigma', \Gamma, \Theta \Rightarrow \bigcirc\Omega, \Delta, \Psi \in \mathcal{G}$ where Γ only contains atom implications, Σ only atoms in P, and Ω, Δ and Σ' atoms not in P. We have the following properties for each clause $\Theta' \cup \Psi'$ and its corresponding sequent $\bigcirc\Sigma \to \Sigma, \bigcirc\Sigma' \to \Sigma', \Gamma, \Theta \Rightarrow \bigcirc\Omega, \Delta, \Psi \in \mathcal{G}$:

1. $K_v \models \bigwedge\Sigma \wedge \bigcirc\Sigma' \to \Sigma' \wedge \bigwedge\Gamma$ for all $v \neq \rho$,
2. $q \in \Sigma'$ implies $\bigcirc q \in \bigcirc\Omega$,
3. $\circ((\bigcirc\Sigma \to \Sigma, \bigcirc\Sigma' \to \Sigma', \Gamma, \Theta \Rightarrow \bigcirc\Omega, \Delta, \Psi), \mathcal{G})$.

The properties are similarly shown as in Case 3 of Theorem 3.8. Gs-rule $\mathcal{G} \triangleright \mathcal{H}$ is full with respect to $(\mathrm{V}^{\leq})$, so we have at least one of the following:

(i) $(\Sigma, \bigcirc\Sigma' \to \Sigma', \Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Delta$,
(ii) $(\Sigma, \bigcirc\Sigma' \to \Sigma', \Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Omega$,
(iii) $(\bigcirc\Sigma \to \Sigma, (\bigcirc\Sigma' \to \Sigma')^{\Pi}, \Gamma^{\Pi}, \Pi \Rightarrow \bigcirc\Omega, \Delta) \in \mathcal{H}$ for some $\emptyset \neq \Pi \subseteq (\Gamma \cup (\bigcirc\Sigma' \to \Sigma'))_{\bigcirc\Omega\cup\Delta}$.

The proof proceeds analogously to Case 3 in Theorem 3.8, showing that (i) and (ii) lead to a contradiction and that case (iii) implies $\vdash_{\mathsf{GAPLL}} \mathcal{G} \triangleright \mathcal{H}$. □

Corollary 5.6. *Admissibility is decidable for* PLL*.*

Let $Y = \bigwedge_{i<n}(A_i \to B_i) \wedge \bigwedge_{i<m}(\bigcirc E_i \to F_i)$.

$$\frac{\bigcirc C \wedge \bigwedge_{i<n}(A_i \to B_i) \to \bigvee_{j<l} D_j}{\{\bigcirc C \wedge \bigwedge_{i<n}(A_i \to B_i) \to D_j\}_{j<l}\ ,\ \{\bigcirc C \wedge \bigwedge_{i<n}(A_i \to B_i) \to A_j\}_{j<n}}\ \mathrm{V}^{\leq,R}$$

$$\frac{\bigwedge_{i<h}(\bigcirc C_i \to C_i) \wedge Y \to \bigvee_{j<k} \bigcirc O_j \vee \bigvee_{j<l} D_j}{\begin{array}{c}\{\bigwedge_{i<h} C_i \wedge Y \to D_j\}_{j<l}\ ,\ \{\bigwedge_{i<h} C_i \wedge Y \to O_j\}_{j<k}\\ \{\bigwedge_{i<h} C_i \wedge Y \to A_j\}_{j<n}\ ,\ \{\bigwedge_{i<h} C_i \wedge Y \to E_j\}_{j<m}\end{array}}\ \mathrm{V}^{\leq}$$

Fig. 8. Basis for the admissible rules in PLL.

5.2. Basis

See Fig. 8 for the multi-conclusion Visser rules for the basis of PLL. Define $\mathcal{B}_{\mathsf{PLL}} = \mathrm{V}^{\leq,R} \cup \mathrm{V}^{\leq}$. Single-conclusion Visser rules for PLL can be defined in the same way as done in Section 4.

Theorem 5.7. *$\mathcal{B}_{\mathsf{PLL}}$ forms a basis for the admissible rules for* PLL*.*

Proof. Analogous to Theorem 4.1. The fact that $\vdash_{\mathsf{GAPLL}} \mathcal{G} \rhd \mathcal{H}$ implies $I(\mathcal{G}) \vdash^m_{\mathsf{PLL},\mathcal{B}_{\mathsf{PLL}}} I(\mathcal{H})$ is shown exactly the same as Theorem 4.2, using intuitionistic reasoning. □

6. Conclusion

The paper provides Gentzen-style proof systems for the admissible rules for the six intuitionistic modal logics iCK4, iCS4 ≡ IPC, iSL, mHC, KM, and PLL. From these systems, we extract bases for the admissible rules and prove that admissibility in these logics is decidable. Our machinery uses the characterization of projective formulas in terms of the extension property that we prove in Section 2 based on methods by Ghilardi [13]. Our proof relies on the strongness condition in the Kripke models imposed by the coreflection axiom $A \to \Box A$ of the logics. It is an open question whether a similar characterization can be established for other intuitionistic modal logics. It might not be surprising if it turns out to be impossible, because the methods studied in the literature so far do not work for all classical modal logics. Much is known for logics extending K4, but for modal logic K it is an open problem how to determine all the admissible rules. Methods so far do not work for it as implied in [25]. Therefore, research to the admissible rules for intuitionistic modal logic in general could demand for some essentially new ideas.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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