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# A random version of Sperner's theorem

József Balogh,<sup>\*</sup> Richard Mycroft<sup>†</sup> and Andrew Treglown<sup>‡</sup>

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## Abstract

Let  $\mathcal{P}(n)$  denote the power set of  $[n]$ , ordered by inclusion, and let  $\mathcal{P}(n, p)$  be obtained from  $\mathcal{P}(n)$  by selecting elements from  $\mathcal{P}(n)$  independently at random with probability  $p$ . A classical result of Sperner [12] asserts that every antichain in  $\mathcal{P}(n)$  has size at most that of the middle layer,  $\binom{n}{\lfloor n/2 \rfloor}$ . In this note we prove an analogous result for  $\mathcal{P}(n, p)$ : If  $pn \rightarrow \infty$  then, with high probability, the size of the largest antichain in  $\mathcal{P}(n, p)$  is at most  $(1 + o(1))p \binom{n}{\lfloor n/2 \rfloor}$ . This solves a conjecture of Osthus [9] who proved the result in the case when  $pn/\log n \rightarrow \infty$ . Our condition on  $p$  is best-possible. In fact, we prove a more general result giving an upper bound on the size of the largest antichain for a wider range of values of  $p$ .

We write  $[n]$  for the set of natural numbers up to  $n$ , and  $\mathcal{P}(n)$  for the power set of  $[n]$ . Also, for any  $0 \leq k \leq n$  we write  $\binom{[n]}{k}$  for the subset of  $\mathcal{P}(n)$  consisting of all sets of size  $k$ . A subset  $\mathcal{A} \subseteq \mathcal{P}(n)$  is an *antichain* if for any  $A, B \in \mathcal{A}$  with  $A \subseteq B$  we have  $A = B$ . So  $\binom{[n]}{k}$  is an antichain for any  $0 \leq k \leq n$ ; Sperner's theorem [12] states that in fact no antichain in  $\mathcal{P}(n)$  has size larger than  $\binom{n}{\lfloor n/2 \rfloor}$ . Our main theorem is a random version of Sperner's theorem. For this, let  $\mathcal{P}(n, p)$  be the set obtained from  $\mathcal{P}(n)$  by selecting elements randomly with probability  $p$  and independently of all other choices. Write  $m := \binom{n}{\lfloor n/2 \rfloor}$ . Roughly speaking, our main result asserts that if  $p > C/n$  for some constant  $C$ , then with high probability, the largest antichain in  $\mathcal{P}(n, p)$  is approximately the same size as the 'middle layer' in  $\mathcal{P}(n, p)$ .

**Theorem 1.** *For any  $\varepsilon > 0$  there exists a constant  $C$  such that if  $p > C/n$  then with high probability the largest antichain in  $\mathcal{P}(n, p)$  has size at most  $(1 + \varepsilon)pm$ .*

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(Here, by ‘with high probability’ we mean with probability tending to 1 as  $n$  tends to infinity.)

The model  $\mathcal{P}(n, p)$  was first investigated by Rényi [10] who determined the probability threshold for the property that  $\mathcal{P}(n, p)$  is not itself an antichain, thereby answering a question of Erdős. The size of the largest antichain in  $\mathcal{P}(n, p)$  for  $p$  above this threshold was first studied by Kohayakawa and Kreuter [6]. In [6] they raised the question of which values of  $p$  does the conclusion of Theorem 1 hold. Osthus [9] proved Theorem 1 in the case when  $pn/\log n \rightarrow \infty$  and conjectured that this can be replaced by  $pn \rightarrow \infty$ . (So Theorem 1 resolves this conjecture.) Moreover, Osthus showed that, for a fixed  $c > 0$ , if  $p = c/n$  then with high probability the largest antichain in  $\mathcal{P}(n, p)$  has size at least  $(1 + o(1))(1 + e^{-c/2})p \binom{n}{\lfloor n/2 \rfloor}$ . So the bound on  $p$  in Theorem 1 is best-possible up to the constant  $C$ . There have also been a number of results concerning the length of (the longest) chains in  $\mathcal{P}(n, p)$  and related models of random posets (see for example, [2, 7, 8]).

Instead of proving Theorem 1 directly we prove the following more general result.

**Theorem 2.** *Let  $n \in \mathbb{N}$  and  $m := \binom{n}{\lfloor n/2 \rfloor}$ . For any  $\varepsilon > 0$  and  $t \in \mathbb{N}$ , there exists a constant  $C$  such that if  $p > C/n^t$  then with high probability the largest antichain in  $\mathcal{P}(n, p)$  has size at most  $(1 + \varepsilon)pmt$ .*

Osthus [9] proved this result in the case when  $p(n/t)^t/\log n \rightarrow \infty$ . (In fact, Osthus’s result allows for  $t$  to be an integer function, see [9] for the precise statement.) Moreover, Osthus showed that, for  $1/n^t \ll p \ll 1/n^{t-1}$ , with high probability,  $\mathcal{P}(n, p)$  has an antichain of size at least  $(1 + o(1))pmt$  (so Theorem 2 is ‘tight’ in this window of  $p$ ).

The method of proof of Theorem 2 also allows us to estimate the number of antichains in  $\mathcal{P}(n)$  of certain fixed sizes.

**Proposition 3.** *Fix any  $t \in \mathbb{N}$ , and suppose that  $m/n^t \ll s \ll m/n^{t-1}$ . Then the number of antichains of size  $s$  in  $\mathcal{P}(n)$  is  $\binom{(t+o(1))m}{s}$ .*

To prove Theorem 2, let  $G$  be the graph with vertex set  $\mathcal{P}(n)$  in which distinct sets  $A$  and  $B$  are adjacent if  $A \subseteq B$  or  $B \subseteq A$ . Then an antichain in  $\mathcal{P}(n)$  is precisely an independent set in  $G$ . We follow the ‘hypergraph container’ approach (see, for example, [1, 11]): indeed, we show that all independent sets in  $G$  are contained within a fairly small number of low-density sets in  $G$ . Crucially, for this method to work, we have to construct our ‘containers’ in two phases (see Lemma 6). For this we use a result of Kleitman [5] on the minimum number of edges induced by a subset of  $G$  with a given fixed size. Define the *centrality order* on the vertices of  $\mathcal{P}(n)$  as follows: we begin with the elements of  $\binom{[n]}{\lfloor n/2 \rfloor}$ , ordered arbitrarily, then the elements of  $\binom{[n]}{\lfloor n/2 \rfloor + 1}$ , then the elements of  $\binom{[n]}{\lfloor n/2 \rfloor - 1}$ , then the elements of  $\binom{[n]}{\lfloor n/2 \rfloor + 2}$ , and so forth until all vertices of  $\mathcal{P}(n)$  have been ordered. For any  $r \in \mathbb{N}$  let  $I_r$  denote the initial segment of this order of length  $r$ ; Kleitman [5] proved that  $I_r$  minimises the number of induced edges over all sets of size  $r$  (see also [4], which characterises all the sets  $U$  of size  $r$  for which  $e(G[U])$  is minimised).

**Theorem 4** (Kleitman [5]). *For any  $r \leq 2^n$  and any  $U \subseteq V(G)$  of size  $r$  we have  $e(G[U]) \geq e(G[I_r])$ .*

We apply this theorem in the form of the following corollary.

**Corollary 5.** *Let  $U \subseteq V(G)$ , and suppose that  $0 < \varepsilon \leq 1/2$  and  $t \in \mathbb{N}$ . If  $|U| \geq (t + \varepsilon)m$ , then  $e(G[U]) > \varepsilon n^t |U| / (2t)^{t+1}$ .*

*Proof.* Let  $r := |U|$ . We have  $r \geq (t + \varepsilon)m$ , so in particular  $r - mt \geq r(1 - t/(t + \varepsilon)) \geq 2\varepsilon r / (1 + 2t)$  since  $\varepsilon \leq 1/2$ . Observe that  $I_r$  contains all of the at most  $mt$  elements of the  $t$  ‘middle layers’,  $\binom{[n]}{\lfloor n/2 \rfloor}$ ,  $\binom{[n]}{\lfloor n/2 \rfloor + 1}$ , and so forth. Further,  $I_r$  contains at least  $r - mt$  elements from outside these layers, each of which has at least  $\binom{[n/2]}{t} \geq (n/2t)^t$  neighbours in the  $t$  middle layers. So by Theorem 4 we have

$$e(G[U]) \geq e(G[I_r]) \geq \frac{2\varepsilon r}{1 + 2t} \cdot \left(\frac{n}{2t}\right)^t \geq \frac{\varepsilon n^t r}{(2t)^{t+1}}. \quad \square$$

Let  $s \in \mathbb{N}$ ,  $t > 0$  and let  $S$  be a set of size  $|S| = s$ . Define  $\binom{S}{\leq t}$  to be the set of all subsets of  $S$  of size at most  $t$  and  $\binom{s}{\leq t} := \left| \binom{S}{\leq t} \right|$ .

**Lemma 6.** *Suppose that  $t \in \mathbb{N}$ ,  $0 < \varepsilon \leq 1/(2t)^{t+1}$  and  $n$  is sufficiently large. Then there exist functions  $f : \binom{V(G)}{\leq n^{-(t+0.9)2n}} \rightarrow \binom{V(G)}{\leq (t+1+\varepsilon)m}$  and  $g : \binom{V(G)}{\leq (t+2)m/(\varepsilon^2 n^t)} \rightarrow \binom{V(G)}{\leq (t+\varepsilon)m}$  such that, for any independent set  $I$  in  $G$ , there are disjoint subsets  $S_1, S_2 \subseteq I$  with  $S_1 \in \binom{V(G)}{\leq n^{-(t+0.9)2n}}$ ,  $S_2 \in \binom{V(G)}{\leq (t+2)m/(\varepsilon^2 n^t)}$  such that  $S_1 \cup S_2$  and  $g(S_1 \cup S_2)$  are disjoint,  $S_2 \subseteq f(S_1)$ , and  $I \subseteq S_1 \cup S_2 \cup g(S_1 \cup S_2)$ .*

Roughly speaking, Lemma 6 ensures that every independent set  $I$  in  $G$  lies in some (not too big) sparse ‘container’ set  $S_1 \cup S_2 \cup g(S_1 \cup S_2)$ , and in total we do not have ‘too many’ containers. Indeed, since  $S_1$  and  $S_2$  are small sets, there are not too many possibilities for the set  $S_1 \cup S_2$ , which in turn means there are not too many containers  $S_1 \cup S_2 \cup g(S_1 \cup S_2)$  to consider. This property is crucial to the proof of Theorem 2, as it enables us to take a union bound to show that it is unlikely that the number of vertices randomly selected from any container is significantly higher than expected.

*Proof of Lemma 6.* Fix an arbitrary total order  $v_1, \dots, v_n$  on the vertices of  $V(G)$ . Given any independent set  $I$  in  $G$ , define  $G_0 := G$ , and take  $S_1$  and  $S_2$  to be initially empty. We add vertices to  $S_1$  and  $S_2$  through the following iterative process, beginning at Step 1 in Phase 1.

*Phase 1:* At Step  $i$ , let  $u$  be the maximum degree vertex of  $G_{i-1}$  (with ties broken by our fixed total order). If  $u \notin I$  then define  $G_i := G_{i-1} \setminus \{u\}$ , and proceed to Step  $i + 1$  (still in Phase 1). Alternatively, if  $u \in I$  and  $\deg_{G_{i-1}}(u) \geq n^{t+0.9}$  then add  $u$  to  $S_1$ , define  $G_i := G_{i-1} \setminus (\{u\} \cup N_G(u))$ , and proceed to Step  $i + 1$  (still in Phase 1). Finally, if  $u \in I$  and  $\deg_{G_{i-1}}(u) < n^{t+0.9}$ , then add  $u$  to  $S_1$ , define  $G_i := G_{i-1} \setminus \{u\}$  and  $f(S_1) := V(G_i)$ , and proceed to Step  $i + 1$  of Phase 2.

*Phase 2:* At Step  $i$ , let  $u$  be the maximum degree vertex of  $G_{i-1}$ . If  $u \notin I$  then define  $G_i := G_{i-1} \setminus \{u\}$ , and proceed to Step  $i + 1$  (still in Phase 2). Alternatively, if  $u \in I$  and  $\deg_{G_{i-1}}(u) \geq \varepsilon^2 n^t$  then add  $u$  to  $S_2$ , define  $G_i := G_{i-1} \setminus (\{u\} \cup N_G(u))$ , and proceed to Step

$i + 1$  (still in Phase 2). Finally, if  $u \in I$  and  $\deg_{G_{i-1}}(u) < \varepsilon^2 n^t$ , then add  $u$  to  $S_2$ , define  $G_i := G_{i-1} \setminus \{u\}$  and  $g(S_1 \cup S_2) := V(G_i)$ , and terminate.

Observe first that for any independent set  $I$  in  $G$  the process defined ensures that  $S_1$  and  $S_2$  are disjoint subsets of  $I$ , that  $S_1 \cup S_2$  is disjoint from  $g(S_1 \cup S_2)$ , that  $S_2 \subseteq f(S_1)$  and that  $I \subseteq S_1 \cup S_2 \cup g(S_1 \cup S_2)$ .

Next, note that for any independent set  $I$ , if a vertex  $u$  is added to  $S_1$  at step  $i$ ,  $u$  and at least  $n^{t+0.9}$  neighbours of  $u$  are deleted from  $G_{i-1}$  in forming  $G_i$ , with a single exception (when  $u$  is the final vertex added to  $S_1$ ). So we must have  $|S_1| \leq 1 + |V(G)|/(n^{t+0.9} + 1) \leq n^{-(t+0.9)} 2^n$ . Furthermore, at the end of Phase 1 we know that every vertex  $v$  of  $G_i$  has  $\deg_{G_i}(v) \leq n^{t+0.9}$ , and so Corollary 5 implies that  $f(S_1)$ , the set of all vertices not deleted up to this point, must have size  $|f(S_1)| < (t + 1 + \varepsilon)m$ . Then, in Phase 2, if a vertex  $u$  is added to  $S_2$  at step  $i$ , at least  $\varepsilon^2 n^t$  neighbours of  $u$  are deleted from  $G_{i-1}$  in forming  $G_i$ , again with the single exception of the final vertex added to  $S_2$ . So we must have  $|S_2| \leq 1 + |f(S_1)|/(\varepsilon^2 n^t)$  and thus

$$|S_1 \cup S_2| \leq 1 + (t + 1 + \varepsilon)m/(\varepsilon^2 n^t) + n^{-(t+0.9)} 2^n \leq (t + 2)m/(\varepsilon^2 n^t).$$

Moreover, at the end of Phase 2 every vertex  $v$  of the final  $G_i$  has  $\deg_{G_i}(v) \leq \varepsilon^2 n^t$  and so  $e(G_i) \leq \varepsilon^2 n^t |G_i| \leq \varepsilon n^t |G_i|/(2t)^{t+1}$ . Thus, Corollary 5 implies that  $|g(S_1 \cup S_2)| \leq (t + \varepsilon)m$ .

So it is sufficient to check that the functions  $f$  and  $g$  are well-defined. That is, we must check that if the process described above yields the same set  $S_1$  when applied to independent sets  $I$  and  $I'$ , then it should also yield the same set  $f(S_1)$ , and if additionally the same set  $S_2$  is returned then the sets  $g(S_1 \cup S_2)$  should be identical. However, this is a consequence of the fact that we always chose  $u$  to be the vertex of  $I$  of maximum degree in  $G_{i-1}$ . Moreover, if our algorithm produces sets  $S_1, S_2$  for an independent set  $I$  and sets  $S'_1, S'_2$  for an independent set  $I'$  such that  $S_1 \cup S_2 = S'_1 \cup S'_2$  then  $S_1 = S'_1$  (and  $S_2 = S'_2$ ). Thus, indeed  $f$  and  $g$  are well-defined.  $\square$

The reason for using a two-phase algorithm in the proof of Lemma 6 is that the structure of the hypercube graph is locally highly asymmetric; even worse, the size of the targeted independent set  $I$  is very small compared to the number of vertices in the graph. Roughly speaking, the main objective of Phase 1 (where in each step many vertices are removed) is to decrease the number of potential vertices of  $I$  sufficiently for the standard ‘hypergraph container’ approach of Phase 2 to be successful.

*Proof of Theorem 2.* Fix  $\varepsilon > 0$  and  $t \in \mathbb{N}$ ; we may assume that  $\varepsilon < 1/(2t)^{t+1}$ . Define  $C := 10^{10} \varepsilon^{-5}$  and  $\varepsilon_1 := \varepsilon/4$ . Let  $G_p$  be the graph formed from  $G$  by selecting vertices independently at random with probability  $p > C/n^t$ . Then we must show that, with high probability,  $G_p$  has no independent set of size greater than  $(1 + \varepsilon)pmt$ . Apply Lemma 6 with  $\varepsilon_1$  playing the role of  $\varepsilon$ . Suppose for a contradiction that  $G_p$  does contain some independent set  $I$  with  $|I| > (1 + \varepsilon)pmt$ . Then all vertices of the sets  $S_1$  and  $S_2$  given by Lemma 6 for this  $I$  must have been selected for  $G_p$ , along with at least  $|I| - |S_1 \cup S_2| \geq (1 + \varepsilon)pmt - (t + 2)m/(\varepsilon_1^2 n^t) \geq (1 + \varepsilon/2)pmt$  vertices of  $g(S_1 \cup S_2)$  (the second inequality follows from  $C = 10^{10} \varepsilon^{-5}$ ).

However, the number of possibilities for  $S_1$  is  $\binom{2^n}{\leq n^{-(t+0.9)} 2^n}$ , and for each possibility the probability that  $S_1 \subseteq V(G_p)$  is  $p^{|S_1|}$ . For any fixed  $S_1$  we have  $|f(S_1)| \leq (t + 2)m$  and

$S_2 \subseteq f(S_1)$ , so the number of possibilities for  $S_2$  is at most  $\binom{(t+2)m}{\leq (t+2)m/(\varepsilon_1^2 n^t)}$ , and for each possibility the probability that  $S_2 \subseteq V(G_p)$  is  $p^{|S_2|}$ . Finally, for any fixed  $S_1$  and  $S_2$  we have  $g(S_1 \cup S_2) \leq (t + \varepsilon_1)m \leq (1 + \varepsilon/4)mt$ , so the expected number of vertices of  $g(S_1 \cup S_2)$  selected for  $G_p$  is at most  $(1 + \varepsilon/4)pmt$ . By a standard Chernoff bound the probability that at least  $(1 + \varepsilon/2)pmt$  vertices of  $g(S_1 \cup S_2)$  are selected for  $G_p$  is therefore at most  $e^{-\varepsilon^2 pmt/100}$ . Taking a union bound, we conclude that the probability that  $G_p$  contains an independent set  $I$  of size greater than  $(1 + \varepsilon)pmt$  is at most

$$\begin{aligned} \Pi &:= \sum_{0 \leq a \leq n^{-(t+0.9)}2^n} \sum_{0 \leq b \leq (t+2)m/(\varepsilon_1^2 n^t)} \binom{2^n}{a} \cdot p^a \cdot \binom{(t+2)m}{b} \cdot p^b \cdot e^{-\varepsilon^2 pmt/100} \\ &\leq (n^{-(t+0.9)}2^n + 1)((t+2)m/(\varepsilon_1^2 n^t) + 1) \binom{2^n}{n^{-(t+0.9)}2^n} \cdot p^{n^{-(t+0.9)}2^n} \binom{(t+2)m}{(t+2)m/(\varepsilon_1^2 n^t)} \\ &\quad \cdot p^{(t+2)m/(\varepsilon_1^2 n^t)} \cdot e^{-\varepsilon^2 pmt/100}. \end{aligned}$$

Note that for large  $n$ , with plenty of room to spare we have

$$(n^{-(t+0.9)}2^n + 1)((t+2)m/(\varepsilon_1^2 n^t) + 1) \leq e^{\varepsilon^2 pmt/400}$$

and

$$\binom{2^n}{n^{-(t+0.9)}2^n} \cdot p^{n^{-(t+0.9)}2^n} \leq e^{\varepsilon^2 pmt/400}.$$

Further, since  $C = 10^{10}\varepsilon^{-5}$ , for large  $n$  we have that

$$\binom{(t+2)m}{(t+2)m/(\varepsilon_1^2 n^t)} \cdot p^{(t+2)m/(\varepsilon_1^2 n^t)} \leq e^{\varepsilon^2 pmt/400}.$$

Thus, the upper bound  $\Pi$  on the probability is  $o(1)$ .  $\square$

We conclude with a sketch of the proof of Proposition 3, on the number of antichains of given fixed sizes in  $\mathcal{P}(n)$ .

*Proof sketch of Proposition 3.* The lower bound can be obtained by greedily choosing vertices from within the  $t$  middle layers of  $\mathcal{P}(n)$  to form an antichain of size  $s$ , and counting the number of ways to make these choices. For the upper bound, fix any  $\varepsilon > 0$  and apply Lemma 6 with this  $\varepsilon$  and  $t$ . Then any independent set in  $G$  of size  $s$  is uniquely determined by the choice of

1. a set  $S_1$  of size  $s_1 \leq \ell_1 := 2^n/n^{t+0.9}$ , for which there are at most  $\binom{2^n}{\leq \ell_1}$  choices,
2. a set  $S_2 \subseteq f(S_1)$  of size  $s_2 \leq \ell_2 := (t+2)m/(\varepsilon^2 n^t)$ , for which there are at most  $\binom{(t+1+\varepsilon)m}{\leq \ell_2}$  choices, and
3. a set  $S \subseteq g(S_1 \cup S_2)$  of size  $s - s_1 - s_2$ , for which there are at most  $\binom{(t+\varepsilon)m}{s-s_1-s_2}$  choices.

Summing over all these choices by a similar calculation as in the proof of Theorem 2, we find that (for large  $n$ ) there are at most  $\binom{t+2\epsilon)m}{s}$  independent sets of size  $s$  in  $G$ .  $\square$

When we completed the project, we were informed that Collares Neto and Morris [3] independently proved Theorem 1. Their method is however different. We used the proof technique of [1], and they followed the method of [11]. In particular, when we constructed containers, we aimed at having few vertices, whilst they aimed at having only few edges.

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## References

- [1] J. Balogh, R. Morris and W. Samotij, Independent sets in hypergraphs, *J. Amer. Math. Soc.*, to appear. Preprint available at arXiv:1204.6530.
- [2] B. Bollobás and G. Brightwell, The height of a random partial order: Concentration of measure, *Ann. Appl. Probab.* **2** (1992), 1009–1018.
- [3] M. Collares Neto and R. Morris, Maximum antichains in random set, preprint.
- [4] S. Das, W. Gan and B. Sudakov, Sperner’s Theorem and a Problem of Erdős, Katona and Kleitman, *Combin. Probab. Comput.*, to appear. Preprint available at arXiv:1302.5210.
- [5] D. Kleitman, A conjecture of Erdős–Katona on commensurable pairs among subsets of an  $n$ -set, *Theory of Graphs, Proc. Colloq., Tihany* (1966), 215–218.
- [6] Y. Kohayakawa, B. Kreuter, The width of random subsets of Boolean lattices, *J. Combin. Theory A* **100** (2002), 376–386.
- [7] Y. Kohayakawa, B. Kreuter, D. Osthus, The length of random subsets of Boolean lattices, *Random Struct. Algorithms* **16** (2000), 177–194.
- [8] B. Kreuter, Small sublattices in random subsets of Boolean lattices, *Random Struct. Algorithms* **13** (1998), 383–407.
- [9] D. Osthus, Maximum antichains in random subsets of a finite set, *J. Combin. Theory A* **90** (2000), 336–346.

- [10] A. Rényi, On random subsets of a finite set, *Mathematica (Cluj)* **3**, No. 26 (1961), 355–362.
- [11] D. Saxton and A. Thomason, Hypergraph containers, arXiv:1204.6595.
- [12] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, *Math. Zeit.* **27** (1928) 544–548.