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*Citation for published version (Harvard):*

Sergeev, S & Kennedy-Cochran-Patrick, A 2023, 'Extending CSR decomposition to tropical inhomogenous matrix products', *Electronic Journal of Linear Algebra*, vol. 38, pp. 820-851.  
<<https://journals.uwo.edu/index.php/ela/article/view/7169/6095>>

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# EXTENDING CSR DECOMPOSITION TO TROPICAL INHOMOGENEOUS MATRIX PRODUCTS\*

ARTHUR KENNEDY-COCHRAN-PATRICK<sup>†</sup> AND SERGEI SERGEEV<sup>‡</sup>

**Abstract.** This article presents an attempt to extend the CSR decomposition, previously introduced for tropical matrix powers, to tropical inhomogeneous matrix products. The CSR terms for inhomogeneous matrix products are introduced, then a case is described where an inhomogeneous product admits such CSR decomposition after some length and give a bound on this length. In the last part of the paper a number of counterexamples are presented to show that inhomogeneous products do not admit CSR decomposition under more general conditions.

**Key words.** max-plus algebra, matrix product, factor-rank, walk, matrix decompositions

**AMS subject classifications.** 15A80, 68R99, 16Y60, 05C20, 05C22, 05C25

**1. Introduction.** Tropical (max-plus) linear algebra is the linear algebra developed over the set  $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$  equipped with the additive operator  $\oplus : a \oplus b = \max(a, b)$  and the multiplicative operator  $\otimes : a \otimes b = a + b$ . For brevity we denote  $\varepsilon = -\infty$ : this element of the semiring is neutral with respect to addition, thus playing the role of semiring zero. In turn, the usual zero 0 plays the role of semiring unity, being neutral with respect to multiplication. Note that for any  $a \in \mathbb{R}$  there is a multiplicative inverse: element  $a^- = a$  such that  $a^- \otimes a = a \otimes a^- = 0$ .

We will be working with the max-plus multiplication of matrices  $A \otimes B$  defined by the operation

$$(A \otimes B)_{i,j} = \bigoplus_{1 \leq k \leq n} a_{i,k} \otimes b_{k,j} = \max_{1 \leq k \leq n} (a_{i,k} + b_{k,j})$$

using two matrices  $A = (a_{i,j})$  and  $B = (b_{i,j})$  of appropriate sizes.

Consider the tropical dynamical system of equations given by

$$\begin{aligned} x(0) &= x_0 \\ x(k) &= x(k-1) \otimes A_k \quad \text{for } k \geq 1 \\ x(k) &= x_0 \otimes A_1 \otimes \dots \otimes A_k = x_0 \otimes \Gamma(k). \end{aligned}$$

Here the matrices  $A_i$  are taken in some unspecified order from a possibly infinite set of matrices  $\mathcal{A}$ . In practical terms, this represents a dynamical system where some accidental changes may occur over time. This has useful applications in modelling scheduling systems that are subject to change.

Much work has been done for the case where the matrix  $A_i$  is the same at each step. Cohen et al. [8, 7] were the first to observe that, under some mild conditions, the tropical powers  $\{A^t\}_{t \geq 1}$  become periodic after a big enough time. A number of bounds on the transient of such periodicity were then obtained, in particular, by Hartmann and Arguelles [9], Akian et al. [2], and Merlet et al. [17, 16]. In particular, Merlet et al. [17] offer an approach based on the CSR decompositions and CSR expansions of tropical matrix powers introduced by Sergeev and Schneider [20, 22]. Let us note that a preliminary version of such decompositions was introduced and studied before by Nachtigall [19] and Molnárová [18], and that similar decompositions appear in Akian et al. [2].

It is difficult to speak of ultimate periodicity in the case of inhomogeneous products. However, one can observe that CSR decompositions are an algebraic expression of turnpike phenomena occurring in

\*Submitted to the editors on June 7, 2022.

**Funding:** This work was supported by EPSRC Grant EP/P019676/1.

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tropical dynamical systems driven by one matrix. Namely, they express the fact that in such systems there are optimal trajectories (or walks) with a special structure: after a finite number of steps they arrive to a well-defined group of nodes called critical nodes, then dwell within that group of nodes, and then use a finite number of steps to reach the destination. The same phenomena will likely occur in inhomogeneous products as well, but only under certain restrictive conditions. In particular, we can agree that all matrices constituting these inhomogeneous products have the same sets of critical nodes, and for a starter, we can consider the case where all these matrices have just one critical node. Under this and some other assumptions, Shue et al. [24] found that products  $\Gamma(k)$  become tropical rank-1 matrices (i.e., tropical outer products) when  $k$  is sufficiently big. Kennedy-Cochran-Patrick et al. [13] improved this result by giving a lower bound for  $k$  to guarantee that  $\Gamma(k)$  becomes a rank-1 matrix (i.e., a tropical outer product). In the present paper we show that the above results of [13, 24] can be generalised further by introducing the *factor rank transient*: the length of the product after which the product is guaranteed to have a tropical factor rank not exceeding certain number. Rather than directly proving the factor rank property from an inhomogeneous product, a CSR analogue is used, which changes the aim to develop bounds on CSR transients rather than factor rank transients. Upon showing that the analogue definition of CSR exhibits similar properties to the original CSR (see the apper by Sergeev and Schneider [22]) then we can use similar proof methods and results from Merlet, Nowak, Schneider and Sergeev [16] as well as Brualdi and Ryser [5] to develop the key result, which is Theorem 5.8, together with Corollary 5.9, which gives an explicit bound on the length of the product after which it becomes CSR. However there are limitations to this approach, namely, where it can be shown for other cases that no bound exists for the CSR transient, and then we cannot guarantee a factor rank property. Three cases where CSR does not work are given along with the counterexamples that demonstrate this. In all these counterexamples we present families of words of infinite length, in which the product made using such a word is not CSR.

Recall that tropical factor rank of a matrix  $A$ , studied together with many other concepts of rank in Akian et al. [1], can be defined as follows: for a matrix  $A \in \mathbb{R}_{\max}^{n \times m}$ , the *tropical factor rank*  $r$  of  $A$  is the smallest  $r \in \mathbb{N}$  such that  $A = U \otimes L$  where  $U \in \mathbb{R}_{\max}^{n \times r}$  and  $L \in \mathbb{R}_{\max}^{r \times m}$  for some  $n, m \in \mathbb{N}$ . Note that the factor rank of  $A$  is also equal to the minimum number of factor rank-1 matrices whose sum is equal to  $A$ , see [1][Definition 7.1].

For wider reading, Hook [11] shows that, by approximating the rank of the product in a min-plus setting, one can find and express the predominant structure in the associated digraph of the matrices forming the product. Hook has also looked at turnpike theory with respect to the max-plus linear systems in [12]. In this paper he studies infinite length products, then uses a turnpike property to develop a factorisation of said matrix product. In terms of turnpikes, many results were obtained for them in the context of dynamic programming, in both discrete and continuous settings. Specifically, Kontorer and Yakovenko [15] used turnpike theory and Bellman equations to work with discrete optimal control problems. Following his work, Kolokoltsov and Maslov [14] developed turnpike theory for discrete optimal control problems in the context of idempotent analysis and tropical mathematics.

The paper will proceed as follows. The first section will cover the necessary definitions and notation as well as a brief overview of [13] to give a more concrete background for the ensuing work. In section 5 we generalise the work from [13] to a general case. For section 6 we look at the cases where no bound can exist using counterexamples.

## 2. Definitions and Notation.

**2.1. Weighted digraphs and tropical matrices.** This subsection presents some concepts and notation expressing the connection between tropical matrices and weighted digraphs. Monographs [6, 10] are our basic references for such definitions.

**DEFINITION 2.1** (Weighted digraphs). *A directed graph (digraph) is a pair  $(N, E)$  where  $N$  is*

a finite set of nodes and  $E \subseteq N \times N = \{(i, j) : i, j \in N\}$  is the set of edges, where  $(i, j)$  is a directed edge from node  $i$  to node  $j$ .

A weighted digraph is a digraph with associated weights  $w_{i,j} \in \mathbb{R}_{\max}$  for each edge  $(i, j)$  in the digraph.

A digraph associated with a square matrix  $A$  is a weighted digraph  $\mathcal{D}(A) = (N_A, E_A)$  where the set  $N_A$  has the same number of elements as the number of rows or columns in the matrix  $A$ . The set  $E_A \subseteq N_A \times N_A$  is the set of edges in  $\mathcal{D}(A)$ , where  $(i, j)$  is an edge if and only if  $a_{i,j} \neq \varepsilon$ , and in this case the weight of  $(i, j)$  equals the corresponding entry in the matrix  $A$ , i. e.  $w_{i,j} = a_{i,j} \in \mathbb{R}_{\max}$ .

**DEFINITION 2.2** (Walks, paths and weights). A sequence of nodes  $W = (i_0, \dots, i_l)$  is called a walk on a weighted digraph  $\mathcal{D} = (N, E)$  if  $(i_{s-1}, i_s) \in E$  for each  $s : 1 \leq s \leq l$ . This walk is a cycle if the start node  $i_0$  and the end node  $i_l$  are the same. It is a path if no two nodes in  $i_0, \dots, i_l$  are the same. The length of  $W$  is  $l(W) = l$ .

The weight of  $W$  is defined as the max-plus product (i. e., the usual arithmetic sum) of the weights of each edge  $(i_{s-1}, i_s)$  traversed throughout the walk, and it is denoted by  $p_{\mathcal{D}}(W)$ . Note that a sequence  $W = (i_0)$  is also a walk (without edges), and we assume that it has weight and length 0.

The mean weight of  $W$  is defined as the ratio  $p_{\mathcal{D}}(W)/l(W)$ .

For a digraph, being strongly connected is a particularly useful property.

**DEFINITION 2.3** (Strongly connected, irreducible, completely reducible). A digraph is strongly connected, if for any two nodes  $i$  and  $j$  there exists a walk connecting  $i$  to  $j$ . A square matrix is irreducible if the graph associated with it in the sense of Definition 2.1 is strongly connected.

A digraph is called completely reducible, if it consists of a number of strongly connected components, such that no two nodes of any two different components can be connected to each other by a walk.

Note that, trivially, any strongly connected digraph is completely reducible.

The following more refined notions are crucial in the study of ultimate periodicity of tropical matrix powers, and also for the present paper.

**DEFINITION 2.4** (Cyclicity and cyclic classes). Suppose that a digraph is completely reducible. Then the cyclicity of that digraph is the lowest common multiple of the greatest common divisors of the lengths of cycles within each strongly connected component. It will be denoted by  $\gamma$ .

Suppose now that a digraph with set of nodes  $N$  and cyclicity  $\gamma$  is strongly connected. For two nodes  $i, j \in N$  we say that  $i$  and  $j$  are in the same cyclic class if there exists a walk of length modulo  $\gamma$  connecting  $i$  to  $j$  or  $j$  to  $i$ . This splits the set of nodes into  $\gamma$  cyclic classes:  $C_0, \dots, C_{\gamma-1}$ . The notation  $C_l \rightarrow_k C_m$  means that some (and hence all) walks connecting nodes of  $C_l$  to nodes of  $C_m$  have lengths congruent to  $k$  modulo  $\gamma$ . The cyclic class containing  $i$  will be also denoted by  $[i]$ .

The correctness of the above definition of cyclic classes follows, for example, from [5, Lemma 3.4.1]: in fact, every walk from  $i$  to  $j$  on  $\mathcal{D}$  has the same length modulo  $\gamma$ .

In tropical algebra, we often have to deal with two digraphs: 1) the digraph associated with  $A$  and 2) the critical digraph of  $A$ . The latter digraph (being a subdigraph of the first) is defined below.

**DEFINITION 2.5** (Maximum cycle mean and critical digraph). For a square matrix  $A$ , the maximum cycle mean of  $\mathcal{D}(A)$  denoted as  $\lambda(A)$  (equivalently, the maximum cycle mean of  $A$ ) is the biggest mean weight of all cycles of  $\mathcal{D}(A)$ .

A cycle in  $\mathcal{D}(A)$  is called critical if its mean weight is equal to the maximum cycle mean (i. e., is maximal).

The critical digraph of  $A$ , denoted by  $\mathbf{C}(A)$ , is the subdigraph of  $\mathcal{D}(A)$  whose node set  $N_c$  and edge set  $E_c$  consist of all nodes and edges that belong to the critical cycles (i. e., that are critical).

Note that any critical digraph is completely reducible. As shown already in [8, 7], the cyclicity of critical digraph of  $A$  is the ultimate period of the tropical matrix powers sequence  $\{A^t\}_{t \geq 1}$ , provided that  $A$  is irreducible and  $\lambda(A) = 0$ . See also Butkovič [6] and Sergeev [20] for more detailed analysis of the ultimate periodicity of this sequence.

Below we will use notation for walk sets and their maximal weights that is similar to that of Merlet et al. [17].

**DEFINITION 2.6** (Sets of walks). *Let  $\mathcal{D} = (N, E)$  be a weighted digraph and let  $i, j \in N$ . The three sets  $\mathcal{W}_{\mathcal{D}}(i \rightarrow j)$ ,  $\mathcal{W}_{\mathcal{D}}^k(i \rightarrow j)$  and  $\mathcal{W}_{\mathcal{D}}(i \xrightarrow{\mathcal{N}} j)$ , where  $\mathcal{N} \subseteq N$  is a subset of nodes, are defined as follows:*

$\mathcal{W}_{\mathcal{D}}(i \rightarrow j)$  is the set of walks over  $\mathcal{D}$  connecting  $i$  to  $j$ ;

$\mathcal{W}_{\mathcal{D}}^k(i \rightarrow j)$  is the set of walks over  $\mathcal{D}$  of length  $k$  connecting  $i$  to  $j$ ;

$\mathcal{W}_{\mathcal{D}}(i \xrightarrow{\mathcal{N}} j)$  is the set of walks over  $\mathcal{D}$  connecting  $i$  to  $j$  that traverse at least one node of  $\mathcal{N}$ .

The supremum of the weights of walks in these sets will be denoted by  $p(\mathcal{W})$ .

**2.2. Main assumptions.** In this subsection, we set out the main assumptions about  $\mathcal{X}$  and the matrices  $A_{\alpha}$  that are drawn from this set and give some relevant definitions.

**DEFINITION 2.7** (Geometrical equivalence). *Let the matrices  $A$  and  $B$  have their respective digraphs  $\mathcal{D}(A) = (N_A, E_A)$  and  $\mathcal{D}(B) = (N_B, E_B)$ . We say that  $A$  and  $B$  are weakly geometrically equivalent if  $N_A = N_B$  and  $E_A = E_B$ , and they are strongly geometrically equivalent if they are weakly geometrically equivalent and  $\mathbf{C}(A) = \mathbf{C}(B)$ .*

We cannot assume that the maximum cycle mean of each  $A_{\alpha} \in \mathcal{X}$  is zero therefore we normalise each matrix to give the new set of matrices  $\mathcal{Y}$ , where

$$\mathcal{Y} = \{A'_{\alpha} : A'_{\alpha} = \lambda^{-}(A_{\alpha}) \otimes A_{\alpha} \forall A_{\alpha} \in \mathcal{X}\}.$$

Here  $\lambda^{-}(A_{\alpha}) = -\lambda(A_{\alpha})$ . From Assumption  $\mathcal{A}$  stated below it follows that  $\lambda(A_{\alpha}) \in \mathbb{R}$ , thus the inverse  $\lambda^{-}(A_{\alpha})$  is well defined.

**NOTATION 2.8** ( $A^{\sup}$  and  $A^{\inf}$ ).

$A^{\sup}$ : entrywise supremum of all matrices in  $\mathcal{Y}$ . In formula,  $A^{\sup} = \bigoplus_{\alpha: A_{\alpha} \in \mathcal{Y}} A_{\alpha}$ .

$A^{\inf}$ : entrywise infimum of all matrices in  $\mathcal{Y}$ .

Note that the concept of  $A^{\sup}$  has been used before for various purposes. In [4], Gursoy, Mason and Sergeev use the same definition to develop a common subeigenvector for the entire semigroup of matrices used to create  $A^{\sup}$ , which is a technique we will use later on. In [3], Gursoy and Mason use  $A^{\sup}$ , and  $\lambda(A^{\sup})$  to develop bounds for the max-eigenvalues over a set of matrices.

We now state the main assumptions to be used in the paper.

**ASSUMPTION  $\mathcal{A}$ .** Any matrix  $A_{\alpha} \in \mathcal{X}$  is irreducible.

**ASSUMPTION  $\mathcal{B}$ .** Any two matrices  $A_{\alpha}, A_{\beta} \in \mathcal{X}$  are strongly geometrically equivalent to each other and to  $A^{\sup}$ , which has all entries in  $\mathbb{R}_{\max}$ .

The following notation is defined under assumptions  $\mathcal{A}$  and  $\mathcal{B}$ .

**NOTATION 2.9.** The common associated digraph of the matrices from  $\mathcal{X}$  will be denoted by  $\mathcal{D}(\mathcal{X}) = (N, E)$ , and the common critical digraph by  $\mathbf{C}(\mathcal{X}) = (N_c, \mathcal{E}_c)$ . In general, this critical digraph has  $m \geq 1$  strongly connected components, denoted by  $\mathbf{C}_{\nu}$ , for  $\nu = 1, \dots, m$ .

**ASSUMPTION  $\mathcal{C}$ .** Any matrix  $A_{\alpha} \in \mathcal{X}$  is weakly geometrically equivalent to  $A^{\inf}$ . In other words, for each  $(i, j) \in E$ , we have  $(A^{\inf})_{ij} \neq -\infty$ .

**ASSUMPTION  $\mathcal{D}1$ .** For the matrix  $A^{\sup}$ , we have  $\lambda(A^{\sup}) = 0$ .

The first three assumptions come from the previous works by Shue et al. [24] and Kennedy-Cochran-Patrick et al. [13]: however, we will no longer assume that the critical graph consists just of one loop.

The final assumption below is inspired by the visualisation scaling studied in Sergeev et al [23], see also [21] and references therein for more background on this scaling.

**DEFINITION 2.10** (Visualisation). *Matrix  $B$  is called a visualisation of  $A$  if there exists a diagonal matrix  $X = \text{diag}(x)$ , with entries  $X_{ii} = x_i$  on the diagonal and  $X_{ij} = \varepsilon$  off the diagonal (i.e., if  $i \neq j$ ), such that  $B = X^{-1}AX$  and  $B$  satisfies the following conditions:  $B_{ij} = \lambda(B)$  for  $(i, j) \in \mathcal{E}_c(B)$  and  $B_{ij} \leq \lambda(B)$  for  $(i, j) \notin \mathcal{E}_c(B)$ .*

Once  $\lambda(A) \neq \varepsilon$ , a visualisation of  $A$  always exists and, moreover, vectors  $x$  providing a visualisation by means of diagonal matrix scaling  $A \mapsto X^{-1}AX$  are precisely the tropical subeigenvectors of  $A$ , i.e., vectors satisfying  $Ax \leq \lambda(A)x$ . Using this information we have the following lemma.

**LEMMA 2.11.** *Suppose that the vector  $x$  satisfies  $A^{\sup}x \leq x$ . Then  $x$  provides a simultaneous visualisation for all matrices of  $\mathcal{X}$  (and  $\mathcal{Y}$ ).*

*Proof.* Let  $x$  be the vector that satisfies  $A^{\sup}x \leq x$ . By construction,  $A^{\sup}$  is the supremum matrix of all the normalised generators in  $\mathcal{X}$ . Therefore for these normalised generators  $A_\alpha$ ,  $A_\alpha \leq A^{\sup}$ . Hence the vector  $x$  also satisfies  $A_\alpha x \leq x$  and it can be used to visualise  $A_\alpha$ . As this applies for all  $\alpha$  then they can be simultaneously visualised. As  $\mathcal{Y}$  is the set of normalised matrices from  $\mathcal{X}$  then the same applies to any matrix from  $\mathcal{Y}$  as well.  $\square$

This is referred to as the set of matrices having a *common visualisation*, therefore, in what follows we assume that we have performed this common visualisation on all of the matrices in  $\mathcal{X}$  (and  $\mathcal{Y}$ ) to give the final core assumption.

**ASSUMPTION D2.** *For all  $A_\alpha \in \mathcal{Y}$ , we have  $(A_\alpha)_{ij} = 0$  and  $(A^{\sup})_{ij} = 0$  for  $(i, j) \in \mathcal{E}_c$ , and  $(A_\alpha)_{ij} \leq 0$  and  $(A^{\sup})_{ij} \leq 0$  for  $(i, j) \notin \mathcal{E}_c$ .*

From now on we will use Assumption D2 instead of Assumption D1. Note however, if the theory developed in this paper is applied to a set of matrices satisfying Assumption D1, then the parameters appearing in the bounds are computed using the entries of their visualised counterparts.

**2.3. Extension to inhomogeneous products.** Recall now that we have a set of matrices  $\mathcal{Y}$ , from which we can select matrices in arbitrary sequence.

**DEFINITION 2.12.** *The word associated with the matrix product  $\Gamma(k)$  is the string of characters (subscript)  $i$  from  $A_i \in \mathcal{Y}$  that make up said  $\Gamma(k)$ .*

Let us also introduce the trellis digraph associated with a matrix product  $\Gamma(k) = A_1 \otimes A_2 \otimes \dots \otimes A_k$  (as in [13], inspired by Viterbi algorithm).

**DEFINITION 2.13.** *The trellis digraph  $\mathcal{T}(P) = (\mathcal{N}, \mathcal{E})$  associated with the product  $\Gamma(k) = A_1 \otimes A_2 \otimes \dots \otimes A_k$  made from the word  $P$  is the digraph with the set of nodes  $\mathcal{N}$  and the set of edges  $\mathcal{E}$ , where:*

(1)  $\mathcal{N}$  consists of  $k+1$  copies of  $N$  which are denoted  $N_0, \dots, N_k$ , and the nodes in  $N_l$  for each  $0 \leq l \leq k$  are denoted by  $1:l, \dots, n:l$ ;

(2)  $\mathcal{E}$  is defined by the following rules:

a) there are edges only between  $N_l$  and  $N_{l+1}$  for each  $l$ ,

b) we have  $(i:(l-1), j:l) \in \mathcal{E}$  if and only if  $(i, j)$  is an edge of  $\mathcal{D}(\mathcal{Y})$ , and the weight of that edge is  $(A_l)_{i,j}$ .

The weight of a walk  $W$  on  $\mathcal{T}(P)$  is denoted by  $p_{\mathcal{T}}(W)$ .



Below we will need to use 1) walks that start at one side of the trellis and end at an intermediate node, 2) walks that start at an intermediate node and end at the other side of the trellis, 3) walks that connect one side of the trellis to the other. More formally, we give the following definition.

DEFINITION 2.14. Consider a trellis digraph  $\mathcal{T}(P)$ .

By an initial walk connecting  $i$  to  $j$  on  $\mathcal{T}(P)$  we mean a walk on  $\mathcal{T}(P)$  connecting node  $i : 0$  to  $j : m$ , where  $0 \leq m \leq k$ .

By a final walk connecting  $i$  to  $j$  on  $\mathcal{T}(P)$  we mean a walk on  $\mathcal{T}(P)$  connecting node  $i : l$  to  $j : k$ , where  $0 \leq l \leq k$ .

A full walk connecting  $i$  to  $j$  on  $\mathcal{T}(P)$  is a walk on  $\mathcal{T}(P)$  connecting node  $i : 0$  to  $j : k$ .

We will mostly work with the following sets of walks on  $\mathcal{T}$ .

NOTATION 2.15 (Walk sets on  $\mathcal{T}(P)$ ).

$\mathcal{W}_{\mathcal{T},\text{full}}^k(i \rightarrow j)$ ,  $\mathcal{W}_{\mathcal{T},\text{init}}^l(i \rightarrow j)$  and  $\mathcal{W}_{\mathcal{T},\text{final}}^l(i \rightarrow j)$  : set of full walks (of length  $k$ ), and sets of initial and final walks of length  $l$  on  $\mathcal{T}$  connecting  $i$  to  $j$ .

$\mathcal{W}_{\mathcal{T},\text{full}}^k(i \xrightarrow{\mathcal{N}_c} j)$ ,  $\mathcal{W}_{\mathcal{T},\text{init}}^l(i \xrightarrow{\mathcal{N}_c} j)$  and  $\mathcal{W}_{\mathcal{T},\text{final}}^l(i \xrightarrow{\mathcal{N}_c} j)$  : set of full walks (of length  $k$ ), and sets of initial and final walks of length  $l$  on  $\mathcal{T}$  traversing a critical node and connecting  $i$  to  $j$ ;

$\mathcal{W}_{\mathcal{T},\text{init}}(i \rightarrow \mathcal{N}_c)$  : set of initial walks connecting  $i$  to a node in  $\mathcal{N}_c$  so that this node of  $\mathcal{N}_c$  is the only node of  $\mathcal{N}_c$  that is visited by the walk and it is visited only once;

$\mathcal{W}_{\mathcal{T},\text{final}}(\|\mathcal{N}_c \rightarrow j)$  : set of final walks connecting a node in  $\mathcal{N}_c$  to  $j$  so that this node of  $\mathcal{N}_c$  is the only node of  $\mathcal{N}_c$  that is visited by the walk and it is visited only once.

$i \rightarrow_{\mathcal{T}} j$  : this denotes the situation where  $i : 0$  can be connected to  $j : k$  on  $\mathcal{T}$  by a full walk.

Recall that  $p(\mathcal{W})$  denotes the optimal weight of a walk in a set of walks  $\mathcal{W}$ . The optimal walk interpretation of entries of  $\Gamma(k)$  in terms of walks on  $\mathcal{T} = \mathcal{T}(P)$  is now apparent:

$$(1) \quad \Gamma(k)_{i,j} = p(\mathcal{W}_{\mathcal{T},\text{full}}^k(i \rightarrow j)).$$

We will also need special notation for the optimal weights of walks in the sets  $\mathcal{W}_{\mathcal{T},\text{init}}(i \rightarrow \mathcal{N}_c)$  and  $\mathcal{W}_{\mathcal{T},\text{final}}(\|\mathcal{N}_c \rightarrow j)$  introduced above.

NOTATION 2.16 (Optimal weights of walks on  $\mathcal{T}(P)$ ).

$w_{i,\mathcal{N}_c}^* = p(\mathcal{W}_{\mathcal{T},\text{init}}(i \rightarrow \mathcal{N}_c))$  : the maximal weight of walks in  $\mathcal{W}_{\mathcal{T},\text{init}}(i \rightarrow \mathcal{N}_c)$ ,

$v_{\mathcal{N}_c,j}^* = p(\mathcal{W}_{\mathcal{T},\text{final}}(\|\mathcal{N}_c \rightarrow j))$  : the maximal weight of walks in  $\mathcal{W}_{\mathcal{T},\text{final}}(\|\mathcal{N}_c \rightarrow j)$ .

The following notation is for optimal values of various optimisation problems involving paths and walks on  $\mathcal{D}(A^{\text{sup}})$ ,  $\mathcal{D}(A^{\text{inf}})$ , which will be used in our factor rank bounds.

NOTATION 2.17 (Optimal weights of walks on  $\mathcal{D}(A^{\text{sup}})$  and  $\mathcal{D}(A^{\text{inf}})$ ).

$\alpha_{i,\mathcal{N}_c}$  : the weight of an optimal path on  $\mathcal{D}(A^{\text{sup}})$  connecting node  $i$  to a node in  $\mathcal{N}_c$ ;

$\beta_{\mathcal{N}_c,j}$  : the weight of an optimal path on  $\mathcal{D}(A^{\text{sup}})$  connecting a node in  $\mathcal{N}_c$  to node  $j$ ;

$\gamma_{i,j}$  : the weight of an optimal path on  $\mathcal{D}(A^{\text{sup}})$  connecting node  $i$  to node  $j$  without traversing any node in  $\mathcal{N}_c$ .

$w_{i,\mathcal{N}_c}$  : the weight of an optimal path on  $\mathcal{D}(A^{\text{inf}})$  connecting node  $i$  to a node in  $\mathcal{N}_c$ ;

$v_{\mathcal{N}_c,j}$  : the weight of an optimal path on  $\mathcal{D}(A^{\text{inf}})$  connecting a node in  $\mathcal{N}_c$  to node  $j$ ;

$u_{i,j}^k$  : the weight of an optimal walk on  $\mathcal{D}(A^{\text{inf}})$  of length  $k$  connecting node  $i$  to node  $j$ .

We remark by saying that the Kleene star, which is explored in [6] and is defined as  $(A)^* = I \oplus A \oplus A^2 \oplus \dots$ , of  $A^{\text{sup}}$  can be used to find the values of  $\alpha_{i,\mathcal{N}_c}$  and  $\beta_{\mathcal{N}_c,j}$ . Similarly the Kleene star of  $A^{\text{inf}}$  can be used to find  $w_{i,\mathcal{N}_c}$  and  $v_{\mathcal{N}_c,j}$ . Let us end this section with the following observation, which follows from the geometric equivalence (Assumptions **B** and **C**)

LEMMA 2.18. The following are equivalent: (i)  $i \rightarrow_{\mathcal{T}} j$ ; (ii)  $(\Gamma(k))_{i,j} > \varepsilon$ ; (iii)  $u_{i,j}^k > \varepsilon$ .

**3. CSR products.** In this section we introduce CSR decomposition of inhomogeneous products and study its properties. It should be noted that in this section we will use Assumptions  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{D}2$  for every proof presented. We will give the two definitions of the CSR decomposition of  $\Gamma(k)$  and prove their equivalence. However in order to do that we require another definition.

**DEFINITION 3.1.** *Let the matrix  $A$  have cyclicity  $\gamma$ . The threshold of ultimate periodicity of powers of  $A$ , is a bound  $T(A)$  such that  $\forall k \geq T(A)$ ,  $A^k = A^{k+\gamma}$ .*

This threshold is required to develop the CSR decomposition for  $\Gamma(k)$  as seen in the following definitions.

**DEFINITION 3.2 (CSR-1).** *Let  $\Gamma(k) = A_1 \otimes \dots \otimes A_k$  be a matrix product of length  $k$  made using the word  $P$ . Define  $C$ ,  $S$  and  $R$  as follows:*

*$S$  is the matrix associated with the critical graph, i.e.*

$$(2) \quad S = (s_{i,j}) = \begin{cases} 0 & \text{if } (i,j) \in \mathcal{E}_c \\ \varepsilon & \text{otherwise.} \end{cases}$$

*Let  $\gamma$  be the cyclicity of critical graph, and  $t$  be a big enough number, such that  $t\gamma \geq T(S)$ , where  $T(S)$  is the threshold of ultimate periodicity of (the powers of)  $S$ .  $C$  and  $R$  are defined by the following formulae:*

$$C = \Gamma(k) \otimes S^{(t+1)\gamma - k(\bmod \gamma)}, \quad R = S^{(t+1)\gamma - k(\bmod \gamma)} \otimes \Gamma(k).$$

*The product of  $C$ ,  $S^{k(\bmod \gamma)}$  and  $R$  will be denoted by  $CS^{k(\bmod \gamma)}R[\Gamma(k)]$ . We say that  $\Gamma(k)$  is CSR if  $CS^{k(\bmod \gamma)}R[\Gamma(k)]$  is equal to  $\Gamma(k)$ .*

For completeness we must also state that for any matrix in  $A \in \mathbb{R}_{\max}^{n \times n}$ ,  $A^0 = I$ , where  $I$  is the tropical identity matrix, i.e.  $I = \text{diag}(0)$ . In the next definition, we prefer to define CSR terms corresponding to the components of the critical graph.

**DEFINITION 3.3 (CSR-2).** *Let  $\Gamma(k) = A_1 \otimes \dots \otimes A_k$  be a matrix product of length  $k$ , and let  $\mathbf{C}_\nu$ , for  $\nu = 1, \dots, m$  be the components of  $\mathbf{C}(\mathcal{Y})$ . For each  $\nu = 1, \dots, m$  define  $C_\nu$ ,  $S_\nu$  and  $R_\nu$  as follows:*

*$S_\nu \in \mathbb{R}_{\max}^{n \times n}$  is the matrix associated with the s.c.c.  $\mathbf{C}_\nu$  of the critical graph, i.e.,*

$$(3) \quad S_\nu = (s_{i,j}) = \begin{cases} 0 & \text{if } (i,j) \in \mathbf{C}_\nu, \\ \varepsilon & \text{otherwise.} \end{cases}$$

*Let  $\gamma_\nu$  be the cyclicity of critical component, and  $t_\nu$  be a big enough number, such that  $t_\nu \gamma_\nu \geq T(S_\nu)$ , where  $T(S_\nu)$  is the threshold of ultimate periodicity of (the powers of)  $S_\nu$ .  $C_\nu$  and  $R_\nu$  are defined by the following formulae:*

$$C_\nu = \Gamma(k) \otimes S_\nu^{(t_\nu+1)\gamma_\nu - k(\bmod \gamma_\nu)}, \quad R_\nu = S_\nu^{(t_\nu+1)\gamma_\nu - k(\bmod \gamma_\nu)} \otimes \Gamma(k).$$

*The product of  $C_\nu$ ,  $S_\nu^{k(\bmod \gamma_\nu)}$  and  $R_\nu$  will be denoted by  $C_\nu S_\nu^{k(\bmod \gamma_\nu)} R_\nu[\Gamma(k)]$ . We say that  $\Gamma(k)$  is CSR if*

$$\Gamma(k) = \bigoplus_{\nu=1}^m C_\nu S_\nu^{k(\bmod \gamma_\nu)} R_\nu[\Gamma(k)].$$



Using the definitions given above, we can write out the CSR terms more explicitly:

$$\begin{aligned}
CS^{k(\bmod \gamma)}R[\Gamma(k)] &= \Gamma(k) \otimes S^{(t+1)\gamma-k(\bmod \gamma)} \otimes S^{k(\bmod \gamma)} \otimes S^{(t+1)\gamma-k(\bmod \gamma)} \otimes \Gamma(k) \\
&= \Gamma(k) \otimes S^{2(t+1)\gamma-k(\bmod \gamma)} \otimes \Gamma(k), \\
C_\nu S_\nu^{k(\bmod \gamma_\nu)} R_\nu[\Gamma(k)] &= \Gamma(k) \otimes S_\nu^{2(t_\nu+1)\gamma_\nu-k(\bmod \gamma_\nu)} \otimes \Gamma(k),
\end{aligned}$$

Since the powers of  $S$  are ultimately periodic with period  $\gamma$  and the powers of  $S_\nu$  are ultimately periodic with period  $\gamma_\nu$ , and since also we have  $t\gamma \geq T(S)$  and  $t_\nu\gamma_\nu \geq T(S_\nu)$ , we can reduce the exponents of  $S$  and  $S_\nu$  to  $(t+1)\gamma - k(\bmod \gamma)$  and  $(t_\nu+1)\gamma_\nu - k(\bmod \gamma_\nu)$ , respectively, and thus

$$\begin{aligned}
(4) \quad CS^{k(\bmod \gamma)}R[\Gamma(k)] &= \Gamma(k) \otimes S^v \otimes \Gamma(k), \quad C_\nu S_\nu^{k(\bmod \gamma_\nu)} R_\nu[\Gamma(k)] = \Gamma(k) \otimes S_\nu^{v_\nu} \otimes \Gamma(k), \\
\text{for } v &= (t+1)\gamma - k(\bmod \gamma), \quad v_\nu = (t_\nu+1)\gamma_\nu - k(\bmod \gamma_\nu), \quad t\gamma \geq T(S), \quad t_\nu\gamma_\nu \geq T(S_\nu).
\end{aligned}$$

Below we will also need the following elementary observation.

LEMMA 3.4. *Let  $v = (t+1)\gamma - k(\bmod \gamma)$ , where  $t\gamma \geq T(S)$ . Then, for any  $\nu$ , we can find  $t_\nu$  such that  $v = (t_\nu+1)\gamma_\nu - k(\bmod \gamma_\nu)$  and  $t_\nu\gamma_\nu \geq T(S_\nu)$ .*

*Proof.* The existence of  $t_\nu$  such that  $v = (t_\nu+1)\gamma_\nu - k(\bmod \gamma_\nu)$  follows since  $\gamma$  is a multiple of  $\gamma_\nu$ , and then we also have  $t_\nu\gamma_\nu \geq t\gamma \geq T(S) \geq T(S_\nu)$ .  $\square$

This lemma allows us to also write

$$(5) \quad C_\nu S_\nu^{k(\bmod \gamma_\nu)} R_\nu[\Gamma(k)] = \Gamma(k) \otimes S_\nu^v \otimes \Gamma(k),$$

with  $v$  as in (4).

PROPOSITION 3.5.  $\Gamma(k)$  is CSR by Definition 3.2 if and only if it is CSR by Definition 3.3.

*Proof.* We need to show that

$$(6) \quad CS^{k(\bmod \gamma)}R[\Gamma(k)] = \bigoplus_{\nu=1}^m C_\nu S_\nu^{k(\bmod \gamma_\nu)} R_\nu[\Gamma(k)]$$

for arbitrary  $k$ . Using (4) and (5) we can rewrite this equivalently as

$$(7) \quad \Gamma(k) \otimes S^{(t+1)\gamma-k(\bmod \gamma)} \otimes \Gamma(k) = \Gamma(k) \otimes \left( \bigoplus_{\nu=1}^m S_\nu^{(t+1)\gamma-k(\bmod \gamma)} \right) \otimes \Gamma(k) \quad \square$$

with  $t\gamma \geq T(S)$ . To obtain this equality, observe that  $S = \bigoplus_{\nu=1}^m S_\nu$ , and as  $S_{\nu_1} \otimes S_{\nu_2} = -\infty$  for any  $\nu_1$  and  $\nu_2$  we can raise both sides to the same power to give us  $S^t = \bigoplus_{\nu=1}^m S_\nu^t$  for any  $t$ . This shows (7), and the claim follows.

For a similar reason, we also have the following identities:

$$\begin{aligned}
(8) \quad C &= \bigoplus_{\nu=1}^m C_\nu, \quad R = \bigoplus_{\nu=1}^m R_\nu, \\
C \otimes S^{k(\bmod \gamma)} &= \bigoplus_{\nu=1}^m C_\nu \otimes S_\nu^{k(\bmod \gamma_\nu)}, \quad S^{k(\bmod \gamma)} \otimes R = \bigoplus_{\nu=1}^m S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu.
\end{aligned}$$

To give an optimal walk interpretation of CSR, we will need to define the trellis graph corresponding to these terms, by modifying Definition 2.13.

DEFINITION 3.6 (Symmetric extension of the trellis graph). Let  $v = (t+1)\gamma - k(\bmod \gamma)$ , where  $t$  is a large enough number such that  $t\gamma \geq T(S)$ .

Define  $\mathcal{T}'(\Gamma(k))$  as the digraph  $\mathcal{T}' = (\mathcal{N}', \mathcal{E}')$  with the set of nodes  $\mathcal{N}'$  and edges  $\mathcal{E}'$ , such that:

- (1)  $\mathcal{N}'$  consists of  $2k + v + 1$  copies of  $N$  which are denoted  $N_0, \dots, N_{2k+v}$  and the nodes for  $N_l$  for each  $0 \leq l \leq 2k + v$  are denoted by  $1 : l, \dots, n : l$ ;
- (2)  $\mathcal{E}'$  is defined by the following rules:
  - a) there are edges only between  $N_l$  and  $N_{l+1}$ ,
  - b) for  $1 \leq l \leq k$  we have  $(i : l - 1, j : l) \in \mathcal{E}'$  if and only if  $(i, j) \in E(\mathcal{Y})$  and the weight of the edge is  $(A_l)_{i,j}$ ,
  - c) for  $k + v + 1 \leq l \leq 2k + v$  we have  $(i : l - 1, j : l) \in \mathcal{E}'$  if and only if  $(i, j) \in E(\mathcal{Y})$  and the weight of the edge is  $(A_{l-k-v})_{i,j}$ ,
  - d) for  $k < l < k + v + 1$  we have  $(i : l - 1, j : l) \in \mathcal{E}'$  if and only if  $(i, j) \in \mathbf{C}(\mathcal{Y})$  and the weight of the edge is 0.

The weight of a walk on  $\mathcal{T}'(\Gamma(k))$  is denoted by  $p_{\mathcal{T}'}(W)$ .

If we consider the walks in  $\mathcal{W}_{\mathcal{T}', \text{full}}^{2k+v}(i \rightarrow j)$  then, in the middle of the walk for  $l$  satisfying  $k < l < k + v + 1$ , the walk is confined in one of the components of  $\mathbf{C}(\mathcal{Y})$ . The set of walks confined in the  $\nu^{\text{th}}$  component of  $\mathbf{C}(\mathcal{Y})$  in the middle of the walk for  $l$  satisfying  $k < l < k + v + 1$ , is denoted by  $\mathcal{W}_{\mathcal{T}', \text{full}}^{2k+v}(i \xrightarrow{[\mathcal{N}_c^\nu]} j)$ . The following optimal walk interpretation of CSR terms on  $\mathcal{T}'$  is now obvious.

LEMMA 3.7 (CSR and optimal walks). The following identities hold for all  $i, j$

$$(9) \quad \begin{aligned} (CS^{k(\bmod \gamma)} R[\Gamma(k)])_{i,j} &= p\left(\mathcal{W}_{\mathcal{T}', \text{full}}^{2k+v}(i \rightarrow j)\right), \\ (C_\nu S_\nu^{k(\bmod \gamma_\nu)} R_\nu[\Gamma(k)])_{i,j} &= p\left(\mathcal{W}_{\mathcal{T}', \text{full}}^{2k+v}(i \xrightarrow{[\mathcal{N}_c^\nu]} j)\right), \end{aligned}$$

where  $v = (t+1)\gamma - k(\bmod \gamma)$ , with  $t\gamma \geq T(S)$ .

*Proof.* With (4), the first identity follows from the optimal walk interpretation of  $\Gamma(k) \otimes S^v \otimes \Gamma(k)$ , and the second identity follows from (5) and the optimal walk interpretation of  $\Gamma(k) \otimes S_\nu^v \otimes \Gamma(k)$ .  $\square$

In what follows, we mostly work with Definition 3.3, but we can switch between the equivalent definitions if we find it convenient.

We now present a useful lemma that shows equality for columns of  $C_\nu$  and rows of  $R_\nu$  with indices in the same cyclic class.

LEMMA 3.8. For any  $i$  and for any two nodes  $x$  and  $y$  in the same cyclic class of the critical component  $\mathbf{C}_\nu$  we have

$$(10) \quad (C_\nu)_{i,x} = (C_\nu)_{i,y} \quad \text{and} \quad (R_\nu)_{x,i} = (R_\nu)_{y,i}$$

*Proof.* We prove the lemma for columns, as the case of the rows is similar.

For any  $i, x$ , denote  $(C_\nu)_{i,x}$  by  $c_{i,x}$ . From the definition of  $C_\nu$ , it follows that  $c_{i,x}$  is the weight of an optimal walk in  $\mathcal{W}_{\mathcal{T}', \text{init}}^{k+(t_\nu+1)\gamma_\nu-k(\bmod \gamma_\nu)}(i \xrightarrow{\mathcal{N}_c^\nu} j)$  where  $t_\nu\gamma_\nu \geq T(S_\nu)$ , and such walk consists of two parts. The first part is a full walk on  $\mathcal{T}$  connecting  $i$  to the critical subgraph at some node  $s$ . The second part is a walk over the critical subgraph of length  $(t_\nu + 1)\gamma_\nu - k(\bmod \gamma_\nu)$  connecting  $s$  to  $x$  with weight zero. As the length of the second walk is greater than  $T(S_\nu)$ , a walk connecting  $s$  to  $x$  exists if and only if  $[s] \rightarrow_{-k(\bmod \gamma_\nu)} [x]$ . If a full walk connecting  $i$  to  $[s]$  on  $\mathcal{T}$  exists then, for arbitrary  $x, y$  in the same cyclic class,  $c_{i,x}$  and  $c_{i,y}$  are both equal to the optimal weight of all walks connecting  $i$  to  $[s]$  on  $\mathcal{T}$ , where  $[s] \rightarrow_{-k(\bmod \gamma_\nu)} [x]$ , otherwise both  $c_{i,x}$  and  $c_{i,y}$  are equal to  $-\infty$ . This shows that  $c_{i,x} = c_{i,y}$ .

356 The case of rows of  $R_\nu$  is considered similarly, but instead of initial walks one has to use final  
 357 walks on  $\mathcal{T}'$ .  $\square$

358 We can use this to prove the same property for  $C$  and  $R$  of Definition 3.2.

359 COROLLARY 3.9. *For any  $i$  and for any two nodes  $x$  and  $y$  in the same critical component and*  
 360 *the same cyclic class of said critical component, we have*

$$361 \quad (11) \quad C_{i,x} = C_{i,y} \quad \text{and} \quad R_{x,i} = R_{y,i}$$

362 *Proof.* We will prove only the first identity, as the proof of the second identity is similar. Let  
 363  $x, y$  belong to the same component  $\mathbf{C}_\mu$  of  $\mathbf{C}(\mathcal{Y})$ , and let them belong to the same cyclic class of that  
 364 component. By Lemma 3.8 we have  $(C_\mu)_{i,x} = (C_\mu)_{i,y}$ , and we also have  $(C_\nu)_{i,x} = (C_\nu)_{i,y} = \varepsilon$  for  
 365 any  $\nu \neq \mu$ . Using these identities and (8), we have

$$366 \quad C_{i,x} = \left( \bigoplus_{\nu=1}^m C_\nu \right)_{i,x} = (C_\mu)_{i,x} = (C_\mu)_{i,y} = \left( \bigoplus_{\nu=1}^m C_\nu \right)_{i,y} = C_{i,y}. \quad \square$$

368 The next theorem explains why CSR is useful for inhomogeneous products. Note that in the  
 369 proof of it we use the CSR structure rather than the  $\Gamma(k) \otimes S^v \otimes \Gamma(k)$  representation that was used  
 370 above.

371 THEOREM 3.10. *The factor rank of each  $C_\nu S_\nu^{k(\bmod \gamma_\nu)} R_\nu[\Gamma(k)]$  is no more than  $\gamma_\nu$ , for  $\nu =$   
 372  $1, \dots, m$ , and the factor rank of  $CS^{k(\bmod \gamma)} R[\Gamma(k)]$  is no more than  $\sum_{\nu=1}^m \gamma_\nu$ .*

373 *Proof.* For each  $\nu = 1, \dots, m$ , take all the nodes from  $\mathcal{G}_\nu$  and order them into cyclic classes  
 374  $\mathcal{C}_0^\nu, \dots, \mathcal{C}_{\gamma_\nu-1}^\nu$ . Take two columns with indices  $x, y \in \mathcal{C}_i^\nu$  from the matrix  $C_\nu$ . As they are in the same  
 375 cyclic class, by Lemma 3.8 the columns are equal to each other. This means that we can take a  
 376 column representing a single node from each cyclic class and since there are  $\gamma_\nu$  distinct classes then  
 377 there will be  $\gamma_\nu$  distinct columns of  $C_\nu$ . The same also holds for any two rows of  $R_\nu$ : if the row  
 378 indices are in the same cyclic class, then the rows are equal, so that we have  $\gamma_\nu$  distinct rows.

Let us now check that the same holds for  $S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu$ . By the construction of  $S_\nu^{k(\bmod \gamma_\nu)}$  we  
 know that if  $(S_\nu^{k(\bmod \gamma_\nu)})_{ij} \neq 0$  then  $[i] \rightarrow_{k(\bmod \gamma_\nu)} [j]$ . Therefore

$$(S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu)_{i,\cdot} = \bigoplus_{j \in N_c} (S_\nu^{k(\bmod \gamma_\nu)})_{ij} \otimes (R_\nu)_{j,\cdot} = \bigoplus_{j: [i] \rightarrow_{k(\bmod \gamma_\nu)} [j]} (S_\nu^{k(\bmod \gamma_\nu)})_{ij} \otimes (R_\nu)_{j,\cdot} = (R_\nu)_{j,\cdot}.$$

379 This means that for a row  $i$  such that  $[i] \rightarrow_{k(\bmod \gamma_\nu)} [j]$  we have  $(S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu)_{i,\cdot} = (R_\nu)_{j,\cdot}$  and all  
 380 such rows of  $S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu$  are equal to each other.

381 Our next aim is to define, for each  $\nu$ , matrices  $C'_\nu$  and  $R'_\nu$  with  $\gamma_\nu$  rows and  $\gamma_\nu$  columns, such  
 382 that  $C_\nu S_\nu^{k(\bmod \gamma_\nu)} R_\nu[\Gamma(k)] = C'_\nu \otimes R'_\nu$ . To form matrix  $C'_\nu$ , we select a node of  $\mathbf{C}_\nu$  from each cyclic  
 383 class  $\mathcal{C}_0^\nu, \dots, \mathcal{C}_{\gamma_\nu-1}^\nu$  and define the column of  $C'_\nu$  whose index is the number of this node to be the  
 384 column of  $C_\nu$  with the same index. The rest of the columns of  $C'_\nu$  are set to  $-\infty$ . To form matrix  
 385  $R'_\nu$ , we use the same selected nodes, but this time (instead of taking columns of  $C_\nu$  and making them  
 386 columns of  $C'_\nu$ ) we take the rows from  $S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu$  whose indices are the numbers of selected  
 387 nodes and make them rows of  $R'_\nu$ . The rest of the rows of  $R'_\nu$  are set to  $-\infty$ . Since the rows of  $C_\nu$   
 388 with indices in the same cyclic class are equal to each other and the same is true about the rows  
 389 of  $S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu$ , we have  $C_\nu S_\nu^{k(\bmod \gamma_\nu)} R_\nu[\Gamma(k)] = C'_\nu \otimes R'_\nu$ , thus the factor rank of any of these  
 390 terms is no more than  $\gamma_\nu$ .

We next form the matrices  $C' = \bigoplus_{\nu=1}^m C'_\nu$  and  $R' = \bigoplus_{\nu=1}^m R'_\nu$ . Obviously,  $C'_{\nu_1} \otimes R'_{\nu_2} = -\infty$  for  $\nu_1 \neq \nu_2$  and therefore

$$C' \otimes R' = \bigoplus_{\nu=1}^m C'_\nu \otimes R'_\nu = \bigoplus_{\nu=1}^m C_\nu S_\nu^{k(\bmod \gamma_\nu)} R_\nu[\Gamma(k)] = CS^{k(\bmod \gamma)} R[\Gamma(k)]. \quad \square$$

Finally, as  $C'$  and, respectively,  $R'$  have  $\sum_{\nu=1}^m \gamma_\nu$  columns with finite entries and, respectively, rows with finite entries with the same indices,  $CS^{k(\bmod \gamma)} R[\Gamma(k)] = C' \otimes R'$  has factor rank at most  $\sum_{\nu=1}^m \gamma_\nu$ .

**COROLLARY 3.11.** *If  $\Gamma(k)$  is CSR, then its rank is no more than  $\sum_{\nu=1}^m \gamma_\nu$ .*

Let us also prove the following results that are similar to [22, Corollary 3.7].

**PROPOSITION 3.12.** *For each  $\nu = 1, \dots, m$*

$$\begin{aligned} (C_\nu \otimes S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu)_{\cdot, j} &= (C_\nu \otimes S_\nu^{k(\bmod \gamma_\nu)})_{\cdot, j} \quad \text{for } j \in \mathcal{N}_c^\nu \\ (C_\nu \otimes S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu)_{i, \cdot} &= (S_\nu^{k(\bmod \gamma_\nu)} \otimes R_\nu)_{i, \cdot} \quad \text{for } i \in \mathcal{N}_c^\nu. \end{aligned}$$

*Proof.* As the proofs are very similar for both statements we will only prove the first and omit the proof for the second statement. We begin by observing that

$$(C_\nu \otimes S_\nu^{k(\bmod \gamma_\nu)})_{i, j} = p\left(\mathcal{W}_{\mathcal{T}', \text{init}}^{k+t_\nu \gamma_\nu}(i \rightarrow j)\right),$$

where we used the definitions of  $C_\nu$  and  $S_\nu$  and the identity  $S_\nu^{(t_\nu+1)\gamma_\nu} = S_\nu^{t_\nu \gamma_\nu}$  (since  $t_\nu \gamma_\nu \geq T(S_\nu)$ ). Here it is convenient to choose  $t_\nu$  that satisfies  $(t_\nu + 1)\gamma_\nu - k(\bmod \gamma_\nu) = (t + 1)\gamma - k(\bmod \gamma)$ , with  $t$  used in the definition of  $\mathcal{T}'$ . With this choice  $t_\nu \gamma_\nu \leq t\gamma$ .

Using (9), all we need to show is that  $p\left(\mathcal{W}_{\mathcal{T}', \text{full}}^{2k+v}(i \xrightarrow{[\mathcal{N}_c^\nu]} j)\right) = p\left(\mathcal{W}_{\mathcal{T}', \text{init}}^{k+t_\nu \gamma_\nu}(i \rightarrow j)\right)$ , where  $v = (t + 1)\gamma - k(\bmod \gamma)$ . We will achieve this by proving these two inequalities:

$$\begin{aligned} p\left(\mathcal{W}_{\mathcal{T}', \text{full}}^{2k+v}(i \xrightarrow{[\mathcal{N}_c^\nu]} j)\right) &\geq p\left(\mathcal{W}_{\mathcal{T}', \text{init}}^{k+t_\nu \gamma_\nu}(i \rightarrow j)\right), \\ p\left(\mathcal{W}_{\mathcal{T}', \text{full}}^{2k+v}(i \xrightarrow{[\mathcal{N}_c^\nu]} j)\right) &\leq p\left(\mathcal{W}_{\mathcal{T}', \text{init}}^{k+t_\nu \gamma_\nu}(i \rightarrow j)\right) \end{aligned} \quad (12)$$

To prove the first inequality of (12) we first consider  $\mathcal{W}_{\mathcal{T}', \text{init}}^{k+t_\nu \gamma_\nu}(i \rightarrow j')$ , where  $j' \in [j]$ . Optimal walk in any of these sets can be decomposed into 1) an optimal full walk on  $\mathcal{T}$  connecting  $i$  to a node of  $[j]$ , and 2) a walk of weight 0 and length  $t_\nu \gamma_\nu$  on  $\mathbf{C}_\nu$  connecting that node of  $[j]$  to  $j'$ , whose existence follows since  $t_\nu \gamma_\nu \geq T(S_\nu)$ . This decomposition implies that the weights of all these optimal walks are equal. One of them, denote it by  $W_1$  can be concatenated with a walk  $W_2$  on  $\mathbf{C}_\nu$  of length  $k - k(\bmod \gamma_\nu) + \gamma$  and ending in  $j$ . We see that  $p(W_1 W_2) = p(W_1)$  and  $W_1 W_2 \in \mathcal{W}_{\mathcal{T}', \text{full}}^{2k+v}(i \xrightarrow{[\mathcal{N}_c^\nu]} j)$ .

To prove the second inequality of (12) we take a walk in  $\mathcal{W}_{\mathcal{T}', \text{full}}^{2k+v}(i \xrightarrow{[\mathcal{N}_c^\nu]} j)$  and decompose it into 1) a walk in  $\mathcal{W}_{\mathcal{T}', \text{init}}^{k+t_\nu \gamma_\nu}(i \rightarrow j')$ , where  $j' \in [j]$ , 2) a walk in  $\mathcal{W}_{\mathcal{T}', \text{final}}^{k-k(\bmod \gamma_\nu)+\gamma_\nu}(j' \rightarrow j)$ . The weight of the first walk is bounded by  $p\left(\mathcal{W}_{\mathcal{T}', \text{init}}^{k+t_\nu \gamma_\nu}(i \rightarrow j)\right)$ , and the weight of the second walk is bounded by 0, thus the second inequality also holds.  $\square$

**COROLLARY 3.13.** *For CSR as defined in Definition 3.2 we have,*

$$\begin{aligned} (C \otimes S^{k(\bmod \gamma)} \otimes R)_{\cdot, j} &= (C \otimes S^{k(\bmod \gamma)})_{\cdot, j} \quad \text{for } j \in \mathcal{N}_c \\ (C \otimes S^{k(\bmod \gamma)} \otimes R)_{i, \cdot} &= (S^{k(\bmod \gamma)} \otimes R)_{i, \cdot} \quad \text{for } i \in \mathcal{N}_c. \end{aligned}$$

424 *Proof.* The proofs for both statements are similar so we will only prove the first one.  
 425 Let  $j \in \mathcal{N}_c$ . As all nodes from  $\mathcal{N}_c$  can be sorted into  $\mathcal{N}_c^\nu$  for some  $\nu = 1, \dots, m$ , assume without  
 426 loss of generality that  $j \in \mathcal{N}_c^\mu$ .  
 427 Taking the right-hand side of the first statement and using (8), we have

$$428 \quad (C \otimes S^{k(\text{mod } \gamma)})_{\cdot, j} = \left( \bigoplus_{\nu=1}^m C_\nu \otimes S_\nu^{k(\text{mod } \gamma_\nu)} \right)_{\cdot, j}.$$

429 By Definition 3.3, if  $j \in \mathcal{N}_c^\mu$  then for all  $\nu \neq \mu$ ,  $(C_\nu \otimes S_\nu^{k(\text{mod } \gamma_\nu)})_{\cdot, j} = -\infty$ . Therefore, for every  $\nu$ ,  
 430  $(C_\nu \otimes S_\nu^{k(\text{mod } \gamma_\nu)})_{\cdot, j}$  will be dominated by  $(C_\mu \otimes S_\mu^{k(\text{mod } \gamma_\mu)})_{\cdot, j}$ . Hence,

$$431 \quad (13) \quad \left( \bigoplus_{\nu=1}^m C_\nu \otimes S_\nu^{k(\text{mod } \gamma_\nu)} \right)_{\cdot, j} = (C_\mu \otimes S_\mu^{k(\text{mod } \gamma_\mu)})_{\cdot, j}.$$

432 Turning our attention to the left-hand side of the first statement, by (8) we get

$$433 \quad (C \otimes S^{k(\text{mod } \gamma)} \otimes R)_{\cdot, j} = \left( \bigoplus_{\nu=1}^m C_\nu \otimes S_\nu^{k(\text{mod } \gamma_\nu)} \otimes R_\nu \right)_{\cdot, j}.$$

434 Now we must show that, for  $j \in \mathcal{N}_c^\mu$  and for all  $\nu$ ,  $(C_\nu \otimes S_\nu^{k(\text{mod } \gamma_\nu)} \otimes R_\nu)_{\cdot, j} \leq (C_\mu \otimes S_\mu^{k(\text{mod } \gamma_\mu)} \otimes R_\mu)_{\cdot, j}$ .  
 435 By (9) this is the same as saying

$$436 \quad p \left( \mathcal{W}_{\mathcal{T}', \text{full}}^{2k+v} (i \xrightarrow{[\mathcal{N}_c^\nu]} j) \right) \leq p \left( \mathcal{W}_{\mathcal{T}', \text{full}}^{2k+v} (i \xrightarrow{\mathcal{N}_c^\mu} j) \right)$$

437  
 438 for some arbitrary node  $i$ . Let  $W$  be the walk of length  $2k + v$  connecting  $i$  to  $j$  that traverses  
 439  $\mathcal{N}_c^\nu$ , such that  $p(W) = p \left( \mathcal{W}_{\mathcal{T}', \text{full}}^{2k+v} (i \xrightarrow{[\mathcal{N}_c^\nu]} j) \right)$ . As  $j \in \mathcal{N}_c^\mu$  then  $W$  is also a walk of length  $2k + v$   
 440 connecting  $i$  to  $j$  that traverses  $\mathcal{N}_c^\mu$ , hence  $W \in \mathcal{W}_{\mathcal{T}', \text{full}}^{2k+v} (i \xrightarrow{\mathcal{N}_c^\mu} j)$  and the inequality holds.  
 441 Therefore, as with the right-hand side, we have

$$442 \quad (14) \quad \left( \bigoplus_{\nu=1}^m C_\nu \otimes S_\nu^{k(\text{mod } \gamma_\nu)} \otimes R_\nu \right)_{\cdot, j} = (C_\mu \otimes S_\mu^{k(\text{mod } \gamma_\mu)} \otimes R_\mu)_{\cdot, j}.$$

443 Finally the first statement of Proposition 3.12 gives us equality between (13) and (14). As  $j$  was  
 444 chosen arbitrarily, this holds for any  $j \in \mathcal{N}_c$  and the result follows.  $\square$

445 **4. General results.** This section presents some results that hold for general inhomogeneous  
 446 products satisfying Assumptions **A**, **B** and **D2**. Before we proceed, let us introduce the following  
 447 piece of notation, inspired by the weak CSR expansion of Merlet et al. [17]:

448 NOTATION 4.1 ( $B^{\text{sup}}$  and  $\lambda_*$ ). Denote

$$449 \quad (B^{\text{sup}})_{i, j} = \begin{cases} \varepsilon, & \text{if } i \in \mathcal{N}_c \text{ or } j \in \mathcal{N}_c, \\ (A^{\text{sup}})_{i, j}, & \text{otherwise} \end{cases}$$

450 and by  $\lambda_*$  the maximum cycle mean of  $B^{\text{sup}}$ .

We remark that the the metric matrix, given in [6] and defined as  $A^+ = A \oplus A^2 \oplus \dots$ , of  $B^{\text{sup}}$  is useful in calculating all the entries of  $\gamma_{i,j}$  simultaneously.

NOTATION 4.2 ( $q$ ). We will denote by  $q$  the number of critical nodes, i.e.,  $q = |\mathcal{N}_c|$ .

The following results generalize [13, Lemmas 3.1-3.2] for initial and final walks to the case of a general critical subgraph. Observe that, under Assumptions **B** and **D2**, we have  $\lambda_* < 0$ , so that the bounds in the following lemmas make sense. Recall the sets of walks  $\mathcal{W}_{\mathcal{T},\text{init}}(i \rightarrow \mathcal{N}_c)$  and  $\mathcal{W}_{\mathcal{T},\text{final}}(\|\mathcal{N}_c \rightarrow j)$  introduced in Notation 2.15.

LEMMA 4.3. Let  $W_{i,\mathcal{N}_c}$  be an optimal walk in  $\mathcal{W}_{\mathcal{T},\text{init}}(i \rightarrow \mathcal{N}_c)$ , so that  $p(W_{i,\mathcal{N}_c}) = w_{i,\mathcal{N}_c}^*$ . Then we have the following bound on the length of  $W_{i,\mathcal{N}_c}$ :

$$(15) \quad l(W_{i,\mathcal{N}_c}) \leq \begin{cases} n - q, & \text{if } \lambda_* = \varepsilon, \\ \frac{w_{i,\mathcal{N}_c}^* - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (n - q), & \text{if } \lambda_* > \varepsilon \end{cases}$$

*Proof.* If  $\lambda_* = \varepsilon$ , then any walk in  $\mathcal{W}_{\mathcal{T},\text{init}}(i \rightarrow \mathcal{N}_c)$  has to be a path, and its length is bounded by  $n - q$ . Now let  $\lambda_* > \varepsilon$ . As  $\lambda_* < 0$ , the weight of the walk  $W_{i,\mathcal{N}_c}$  connecting  $i$  to a node in  $\mathcal{N}_c$  is less than or equal to that of a path  $P_{i,\mathcal{N}_c}$  on  $\mathcal{D}(A^{\text{sup}})$  connecting  $i$  to a node in  $\mathcal{N}_c$  plus the remaining length multiplied by  $\lambda_*$ . The remaining length is bounded from above by  $n - q$ , since all intermediate nodes in  $W_{i,\mathcal{N}_c}$  are non-critical. Hence

$$p_{\mathcal{T}}(W_{i,\mathcal{N}_c}) \leq p_{\text{sup}}(P_{i,\mathcal{N}_c}) + (l(W_{i,\mathcal{N}_c}) - (n - q))\lambda_*.$$

We can bound  $p_{\text{sup}}(P_{i,\mathcal{N}_c}) \leq \alpha_{i,\mathcal{N}_c}$ , so

$$(16) \quad p_{\mathcal{T}}(W_{i,\mathcal{N}_c}) \leq \alpha_{i,\mathcal{N}_c} + (l(W_{i,\mathcal{N}_c}) - (n - q))\lambda_*.$$

Now assuming for contradiction that  $l(W_{i,\mathcal{N}_c}) > \frac{w_{i,\mathcal{N}_c}^* - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (n - q)$ . This is equivalent to

$$(17) \quad \alpha_{i,\mathcal{N}_c} + (l(W_{i,\mathcal{N}_c}) - (n - q))\lambda_* < w_{i,\mathcal{N}_c}^*.$$

In combining (16) and (17) we get  $p_{\mathcal{T}}(W_{i,\mathcal{N}_c}) < w_{i,\mathcal{N}_c}^*$  meaning that  $W_{i,\mathcal{N}_c}$  is not optimal, a contradiction. So we know that for any  $l \in \mathcal{N}_c$

$$l(W_{i,\mathcal{N}_c}) \leq \frac{w_{i,\mathcal{N}_c}^* - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (n - q).$$

The proof is complete.  $\square$

LEMMA 4.4. Let  $W_{\mathcal{N}_c,j}$  be an optimal walk in  $\mathcal{W}_{\mathcal{T},\text{final}}(\|\mathcal{N}_c \rightarrow j)$ , so that  $p(W_{\mathcal{N}_c,j}) = v_{\mathcal{N}_c,j}^*$ . Then we have the following bound on the length of  $W_{\mathcal{N}_c,j}$ :

$$(18) \quad l(W_{\mathcal{N}_c,j}) \leq \begin{cases} n - q, & \text{if } \lambda_* = \varepsilon, \\ \frac{v_{\mathcal{N}_c,j}^* - \beta_{\mathcal{N}_c,j}}{\lambda_*} + (n - q), & \text{if } \lambda_* > \varepsilon. \end{cases}$$

As the proof of this lemma is analogous to the proof of Lemma 4.3 it is omitted. Also, we can observe that  $n - q$  is the limit of the expressions on the right-hand side of (15) and (18) as  $\lambda_* \rightarrow \varepsilon$ , hence we will not consider this case separately in the rest of the paper.

The following result is a generalised form of [13, Lemma 3.4] which uses a nominal weight  $\omega$ .



LEMMA 4.5. If  $\gamma_{i,j} = \varepsilon$ , then any full walk connecting  $i$  to  $j$  on  $\mathcal{T}(P)$  traverses a node in  $\mathcal{N}_c$ .  
If  $\gamma_{i,j} > \varepsilon$ , let

$$(19) \quad k > \frac{\omega - \gamma_{i,j}}{\lambda_*} + (n - q)$$

for some  $\omega \in \mathbb{R}$ . Then any full walk  $W$  connecting  $i$  to  $j$  on  $\mathcal{T}(P)$  that does not go through any node  $l \in \mathcal{N}_c$  has weight smaller than  $\omega$ .

*Proof.* In the case when  $\gamma_{i,j} = \varepsilon$ , the claim follows by the definition of  $\gamma_{i,j}$  and by the geometric equivalence between  $A^{\text{sup}}$  and the matrices from  $\mathcal{Y}$ . So we assume that  $\gamma_{i,j} > \varepsilon$ . Any walk  $W$  that does not traverse any node in  $\mathcal{N}_c$  can be decomposed into a path  $P$  connecting  $i$  to  $j$  avoiding  $\mathcal{N}_c$  and a number of cycles. Hence we have the following bound:

$$(20) \quad p_{\mathcal{T}}(W) \leq p_{\text{sup}}(P) + (k - (n - q))\lambda_*.$$

We can further bound  $p_{\text{sup}}(P) \leq \gamma_{i,j}$  so

$$(21) \quad p_{\mathcal{T}}(W) \leq \gamma_{i,j} + (k - (n - q))\lambda_*.$$

Now (19) can be rewritten as

$$(22) \quad \gamma_{i,j} + (k - (n - q))\lambda_* < \omega.$$

By combining (20) with (21) we have  $p_{\mathcal{T}}(W) < \omega$ , which completes the proof.  $\square$

Using this bound we can obtain a condition under which the CSR term is (non-strictly) above  $\Gamma(k)$ .

THEOREM 4.6. If  $\gamma_{i,j} = \varepsilon$  then  $\Gamma(k) \leq CS^{k(\text{mod } \gamma)} R[\Gamma(k)]$ .  
If  $\gamma_{i,j} > \varepsilon$ , let

$$(23) \quad k > \max_{i,j: i \rightarrow_{\mathcal{T}} j, \gamma_{i,j} > \varepsilon} \left( \frac{\Gamma(k)_{i,j} - \gamma_{i,j}}{\lambda_*} + (n - q) \right).$$

Then  $\Gamma(k) \leq CS^{k(\text{mod } \gamma)} R[\Gamma(k)]$ .

*Proof.* If  $i \not\rightarrow_{\mathcal{T}} j$ , then  $(\Gamma(k))_{i,j} = -\infty$ . In this case, obviously,  $\Gamma(k)_{i,j} \leq (CS^{k(\text{mod } \gamma)} R[\Gamma(k)])_{i,j}$ .  
If  $i \rightarrow_{\mathcal{T}} j$ , then  $(\Gamma(k))_{i,j} \neq \varepsilon$ . Let  $W^*$  be the optimal walk of length  $k$  on  $\mathcal{T}(P)$  connecting  $i$  to  $j$  with weight  $\Gamma(k)_{i,j}$ . If  $k$  is greater than the bound (23) then, by Lemma 4.5, for the walk to have weight equal to  $\Gamma(k)_{i,j}$ , it must traverse at least one node in  $\mathcal{N}_c$ , and the same is true when  $\gamma_{i,j} = \varepsilon$ . Hence this walk belongs to the set  $\mathcal{W}_{\mathcal{T}}^k(i \xrightarrow{\mathcal{N}_c} j)$  and further  $\Gamma(k)_{i,j} = p(W^*) \leq p(\mathcal{W}_{\mathcal{T}}^k(i \xrightarrow{\mathcal{N}_c} j))$ .

Let  $f \in \mathcal{N}_c$  be the first critical node in the first critical s.c.c  $\mathbf{C}_\nu$ , with cyclicity  $\gamma_\nu$ , that  $W^*$  traverses. We can split the walk into  $W^* = W_1 W_3$  where  $W_1$  is a walk connecting  $i$  to  $f$  of length  $r$  and  $W_3$  is a walk connecting  $f$  to  $j$  of length  $k - r$ . We have  $p(W^*) = p(W_1) + p(W_3)$ .

Let  $\mathcal{T}'$  be the trellis extension for the matrix product  $CS^{k(\text{mod } \gamma)} R[\Gamma(k)]$  with length  $2k + v$  where  $v = (t + 1)\gamma - k(\text{mod } \gamma)$  as described in Definition 3.6.

We now introduce the new walk  $W' = W_1 W_2 W_3$  on  $\mathcal{T}'$ . Here  $W_1$  and  $W_3$  are the subwalks from  $W^*$  introduced before, where  $W_1$  is viewed as an initial walk on  $\mathcal{T}'$  and  $W_3$  as a final walk on  $\mathcal{T}'$ , and  $W_2$  is a closed walk of length  $k + v$  that starts and ends at  $f$ . Since  $k + v \equiv 0(\text{mod } \gamma_\nu)$  and  $k + v \geq T(S) \geq T(S_\nu)$ , this closed walk exists and can be entirely made up of edges from  $\mathbf{C}_\nu$ . This means the walk  $W'$  is of length  $2k + v$  and it traverses the set of nodes  $\mathcal{N}_c^\nu$  therefore  $W' \in \mathcal{W}_{\mathcal{T}'}^{2k+v}(i \xrightarrow{\mathcal{N}_c^\nu} j)$ .

As  $W_2$  is made entirely from critical edges, we have  $p(W_2) = 0$  and  $p(W^*) = p(W') \leq p\left(\mathcal{W}_{\mathcal{T}'}^{2k+v}(i \xrightarrow{\mathcal{N}_c^\nu} j)\right)$ , and using (31) gives us

$$\Gamma(k)_{i,j} = p(W^*) \leq (C_\nu S_\nu^{k(\bmod \gamma)} R_\nu[\Gamma(k)])_{i,j} \leq (CS^{k(\bmod \gamma)} R[\Gamma(k)])_{i,j},$$

where the last inequality is due to Proposition 3.5. The claim follows.  $\square$

This condition looks like a bound for  $\Gamma(k)$  to become equal to the corresponding CSR product, but it is implicit since it requires  $\Gamma(k)$  to be calculated in order to generate the bound. However, we can develop a condition that does not depend on  $\Gamma(k)$ . This following result requires Assumption C.

COROLLARY 4.7. *Let*

$$(23) \quad k > \max_{i,j: i \rightarrow_{\mathcal{T}} j, \gamma_{i,j} > \varepsilon} \left( \frac{u_{i,j}^k - \gamma_{i,j}}{\lambda_*} + (n - q) \right).$$

Then  $\Gamma(k) \leq CS^{k(\bmod \gamma)} R[\Gamma(k)]$ .

*Proof.* By Lemma 2.18,  $i \rightarrow_{\mathcal{T}} j$  is equivalent to  $u_{i,j}^k > \varepsilon$ , so maximum in (23) is taken over  $i, j$  for which  $u_{i,j}^k$  and  $\gamma_{i,j}$  are finite. We also have  $u_{i,j}^k \leq (\Gamma(k))_{i,j}$  by the definition of  $A^{\text{inf}}$ .

Further, as  $\lambda_* < 0$ , then any  $k$  that satisfies (23) will also satisfy (22). The claim now follows from Theorem 4.6.  $\square$

**5. The case where CSR works.** In the case when  $\mathbf{C}(\mathcal{X})$  is just one loop, Kennedy-Cochran-Patrick et al. [13] established a bound on the lengths of inhomogeneous products, after which these products are of tropical factor rank 1. In this section we extend this result to the case when  $\mathcal{D}(\mathcal{X})$  and  $\mathbf{C}(\mathcal{X})$  satisfy the following assumption, in addition to Assumptions A, B and D2.

ASSUMPTION P0.  $\mathbf{C}(\mathcal{X})$  is strongly connected and its cyclicity  $\gamma$  is equal to the cyclicity of  $\mathcal{D}(\mathcal{X})$ .

The equality between cyclicities means that the associated digraph  $\mathcal{D}(\mathcal{X})$  has the same number of cyclic classes  $\gamma$  as  $\mathbf{C}(\mathcal{X})$ .

NOTATION 5.1. The cyclic classes of  $\mathcal{D}(\mathcal{X})$  are denoted by  $\mathcal{C}'_0, \dots, \mathcal{C}'_{\gamma-1}$ .

For a node  $i \in N$ , the cyclic class of this node with respect to  $\mathcal{D}(\mathcal{X})$  will be denoted by  $[i]'$ .

For a node  $i \in \mathcal{N}_c$ , we will use both  $[i]$  (the cyclic class with respect to  $\mathbf{C}(\mathcal{X})$ ) and  $[i]'$  (the cyclic class with respect to  $\mathcal{D}(\mathcal{X})$ ), and an obvious inclusion relation between them:  $[i] \subseteq [i]'$ .

One of the ideas is to combine Lemmas 4.3 and 4.4 together with Schwarz's bound. To define this bound, following [17], we first introduce *Wielandt's number*

$$\text{Wi}(n) = \begin{cases} (n-1)^2 + 1 & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \end{cases}$$

and then *Schwarz's number*

$$\text{Sch}(\gamma, n) = \gamma \text{Wi}\left(\left\lfloor \frac{n}{\gamma} \right\rfloor\right) + n(\bmod \gamma).$$

Let us now prove the following lemma.

LEMMA 5.2. *Let*

$$(24) \quad k \geq \frac{w_{i,\mathcal{N}_c}^* - \alpha_{i,\mathcal{N}_c}}{\lambda_{q*}} + (n - q) + \text{Sch}(\gamma, q) + \frac{v_{\mathcal{N}_c,j}^* - \beta_{\mathcal{N}_c,j}}{\lambda_{q*}} + (n - q).$$

*Then*

- (i) *If  $[i]' \not\rightarrow_k [j]'$  then there are no full walks connecting  $i$  to  $j$  on  $\mathcal{T}(P)$  (i.e.,  $i \not\rightarrow_{\mathcal{T}} j$ ).*
- (ii) *If  $[i]' \rightarrow_k [j]'$ , then there is a full walk  $W$  connecting  $i$  to  $j$  on  $\mathcal{T}(P)$  and going through a critical node, and we have  $p_{\mathcal{T}}(W) = w_{i,\mathcal{N}_c}^* + v_{\mathcal{N}_c,j}^*$  if  $W$  is optimal.*

*Proof.* The property  $[i]' \not\rightarrow_k [j]'$  implies that there is no full walk  $W$  connecting  $i$  to  $j$  on  $\mathcal{T}(P)$ . In the case  $[i]' \rightarrow_k [j]'$ , we construct a walk  $W' = W_{i,\mathcal{N}_c} W_c W_{\mathcal{N}_c,j}$  of length  $k$ , where  $W_{i,\mathcal{N}_c}$  be an optimal walk in  $\mathcal{W}_{\mathcal{T},\text{init}}(i \rightarrow \mathcal{N}_c)$  (see Lemma 4.3),  $W_{\mathcal{N}_c,j}$  be an optimal walk in  $\mathcal{W}_{\mathcal{T},\text{final}}(\|\mathcal{N}_c \rightarrow j)$  (see Lemma 4.4), and  $W_c$  is a walk that connects the end of  $W_{i,\mathcal{N}_c}$  to the beginning of  $W_{\mathcal{N}_c,j}$  and such that all edges of  $W_c$  are critical (the existence of such  $W_c$  is yet to be proved). Without loss of generality set  $[i]' = \mathcal{C}'_0$  and  $[j]' = \mathcal{C}'_{p_3}$ : the cyclic classes of  $\mathcal{D}(\mathcal{X})$  to which  $i$  and  $j$  belong. Let  $x$  be the final node of  $W_{i,\mathcal{N}_c}$  and let  $y$  be the first node of  $W_{\mathcal{N}_c,j}$ . Set  $[x]' = \mathcal{C}'_{p_1}$  and  $[y]' = \mathcal{C}'_{p_2}$ .

By [5, Lemma 3.4.1.iv]  $l(W_{i,\mathcal{N}_c}) \equiv p_1 \pmod{\gamma}$ ,  $l(W_{\mathcal{N}_c,j}) \equiv (p_3 - p_2) \pmod{\gamma}$ . Hence the congruence of the walk  $W_c$  to be inserted is  $(p_3 - p_1 - (p_3 - p_2)) \pmod{\gamma} \equiv (p_2 - p_1) \pmod{\gamma}$ . As the cyclicity of the critical subgraph is the same as that of the digraph, the cyclic classes of the critical subgraph are  $\mathcal{C}_0, \dots, \mathcal{C}_{\gamma-1}$  and we can assume that the numbering is such that  $\mathcal{C}_0 \subseteq \mathcal{C}'_0, \dots, \mathcal{C}_{\gamma-1} \subseteq \mathcal{C}'_{\gamma-1}$ . Then  $x \in \mathcal{C}_{p_1}$  and  $y \in \mathcal{C}_{p_2}$  and by [5, Lemma 3.4.1.iv] there exists a walk on the critical subgraph of length congruent to  $(p_2 - p_1) \pmod{\gamma}$ . Moreover, all walks connecting  $x$  to  $y$  have such length and by Schwarz's bound if  $k - l(W_{i,\mathcal{N}_c}) - l(W_{\mathcal{N}_c,j}) \geq \text{Sch}(\gamma, q)$  then there is a walk of length equal to  $l(W') - l(W_{i,\mathcal{N}_c}) - l(W_{\mathcal{N}_c,j})$ . According to Lemmas 4.3 and 4.4  $l(W_{i,\mathcal{N}_c}) \leq \frac{w_{i,\mathcal{N}_c}^* - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (n - q)$ ,  $l(W_{\mathcal{N}_c,j}) \leq \frac{v_{\mathcal{N}_c,j}^* - \beta_{\mathcal{N}_c,j}}{\lambda_*} + (n - q)$ , therefore  $k$  is a sufficient length for  $k - l(W_{i,\mathcal{N}_c}) - l(W_{\mathcal{N}_c,j})$  to satisfy Schwarz's bound, so a walk of the form  $W' = W_{i,\mathcal{N}_c} W_c W_{\mathcal{N}_c,j}$  exists and  $p(W') = w_{i,\mathcal{N}_c}^* + v_{\mathcal{N}_c,j}^*$ .

Let now  $W$  be an optimal full walk connecting  $i$  to  $j$  on  $\mathcal{T}$  that passes through  $\mathcal{N}_c$  at least once. As it passes through the critical nodes then the walk can be decomposed into  $W = \tilde{W}_{i,\mathcal{N}_c} \tilde{W}_c \tilde{W}_{\mathcal{N}_c,j}$  where  $\tilde{W}_{i,\mathcal{N}_c}$  is a walk in  $\mathcal{W}_{\mathcal{T},\text{init}}(i \rightarrow \mathcal{N}_c)$ , and  $\tilde{W}_{\mathcal{N}_c,j}$  is a walk in  $\mathcal{W}_{\mathcal{T},\text{final}}(\|\mathcal{N}_c \rightarrow j)$ , and  $\tilde{W}_c$  connects the end of  $\tilde{W}_{i,\mathcal{N}_c}$  to the beginning of  $\tilde{W}_{\mathcal{N}_c,j}$  on  $\mathcal{T}(P)$ . We then have  $p_{\mathcal{T}}(\tilde{W}_{i,\mathcal{N}_c}) \leq p_{\mathcal{T}}(W_{i,\mathcal{N}_c})$  and  $p_{\mathcal{T}}(\tilde{W}_{\mathcal{N}_c,j}) \leq p_{\mathcal{T}}(W_{\mathcal{N}_c,j})$  and also  $p_{\mathcal{T}}(\tilde{W}_c) \leq p(W_c) = 0$ . Since  $W$  is optimal then all of these inequalities hold with equality, and  $p_{\mathcal{T}}(W) = w_{i,\mathcal{N}_c}^* + v_{\mathcal{N}_c,j}^*$ , as claimed.  $\square$

REMARK 5.3. *It follows from the proof that, under the conditions of this lemma and in the case  $[i] \rightarrow_k [j]$ , there is an optimal full walk connecting  $i$  to  $j$  on  $\mathcal{T}_{\Gamma(k)}$  and traversing a critical node that can be decomposed as  $W = W_{i,\mathcal{N}_c} W_c W_{\mathcal{N}_c,j}$ , where  $W_{i,\mathcal{N}_c}$  is an optimal walk in  $\mathcal{W}_{\mathcal{T},\text{init}}(i \rightarrow \mathcal{N}_c)$  and  $W_{\mathcal{N}_c,j}$  is an optimal walk in  $\mathcal{W}_{\mathcal{T},\text{final}}(\|\mathcal{N}_c \rightarrow j)$ , and  $W_c$  consists of edges solely in the critical subgraph. If the elements of  $\mathcal{Y}$  are also strictly visualised in the sense of [23], then any such optimal full walk has to be of this form.*

Lemma 5.2 gives us the first part of the final bound for the case. In order to be able to use this lemma we must ensure that the walk must traverse  $\mathcal{N}_c$  hence we can use Lemma 4.5 in conjunction with Lemma 5.2 to give us the following theorem.

THEOREM 5.4. *Denote  $u_{i,\mathcal{N}_c,j}^* = w_{i,\mathcal{N}_c}^* + v_{\mathcal{N}_c,j}^*$ . Let*

$$(25) \quad k \geq \max \left( \frac{u_{i,\mathcal{N}_c,j}^* - \alpha_{i,\mathcal{N}_c} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + 2(n - q) + \text{Sch}(\gamma, q), \frac{u_{i,\mathcal{N}_c,j}^* - \gamma_{i,j}}{\lambda_*} + (n - q + 1) \right)$$

593 if  $\gamma_{i,j} > \varepsilon$  or just

$$594 \quad (26) \quad k \geq \frac{u_{i,\mathcal{N}_c,j}^* - \alpha_{i,\mathcal{N}_c} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + 2(n - q) + \text{Sch}(\gamma, q),$$

596 if  $\gamma_{i,j} = \varepsilon$ , for some  $i, j \in N$ . Then

- 597 (i) If  $[i]' \not\rightarrow_k [j]'$  then  $\Gamma(k)_{i,j} = -\infty$ ,  
 598 (ii) If  $[i]' \rightarrow_k [j]'$  then  $\Gamma(k)_{i,j} = u_{i,\mathcal{N}_c,j}^* = w_{i,\mathcal{N}_c}^* + v_{\mathcal{N}_c,j}^*$ .

599 *Proof.* We only need to prove the second part. By Lemma 4.5 and taking  $\omega = w_{i,\mathcal{N}_c}^* + v_{\mathcal{N}_c,j}^*$ , if

$$600 \quad k > \frac{w_{i,\mathcal{N}_c}^* + v_{\mathcal{N}_c,j}^* - \gamma_{i,j}}{\lambda_{q*}} + (n - q)$$

601 then any walk on  $\mathcal{T}(P)$  that does not traverse the nodes in  $\mathcal{N}_c$  will have weight smaller than  
 602  $w_{i,\mathcal{N}_c}^* + v_{\mathcal{N}_c,j}^*$ , or such walk will not exist if  $\gamma_{i,j} = \varepsilon$ . Using Lemma 5.2, if

$$603 \quad k \geq \frac{w_{i,\mathcal{N}_c}^* - \alpha_{i,\mathcal{N}_c}}{\lambda_{q*}} + (n - q) + \text{Sch}(\gamma, q) + \frac{v_{\mathcal{N}_c,j}^* - \beta_{\mathcal{N}_c,j}}{\lambda_{q*}} + (n - q)$$

605 and  $[i]' \rightarrow_k [j]'$  then the weight of any optimal full walk on  $\mathcal{T}(P)$  connecting  $i$  to  $j$  and traversing a  
 606 critical node will be equal to  $w_{i,\mathcal{N}_c}^* + v_{\mathcal{N}_c,j}^*$ . If  $\gamma_{i,j} = \varepsilon$ ,  $[i]' \rightarrow_k [j]'$  and the above inequality holds, or  
 607 if  $\gamma_{i,j} > \varepsilon$ ,  $k$  satisfies both inequalities and  $[i] \rightarrow_k [j]$ , then any optimal full walk traverses nodes in  
 608  $\mathcal{N}_c$  and has weight

$$609 \quad \Gamma(k)_{i,j} = w_{i,\mathcal{N}_c}^* + v_{\mathcal{N}_c,j}^*. \quad \square$$

611 Our next aim is to rewrite Theorem 5.4 in a CSR form, and we first want to look at the optimal  
 612 walk representation of  $w_{i,\mathcal{N}_c}^*$  and  $v_{\mathcal{N}_c,j}^*$ . This leads to the following lemma.

613 LEMMA 5.5. *We have*

$$614 \quad (27) \quad w_{i,\mathcal{N}_c}^* = p(\mathcal{W}_{\mathcal{T},\text{full}}^k(i \rightarrow \mathcal{N}_c)), \quad v_{\mathcal{N}_c,j}^* = p(\mathcal{W}_{\mathcal{T},\text{full}}^k(\mathcal{N}_c \rightarrow j)).$$

615 *Proof.* We will prove only the first of these two equalities, as the second one can be proved in a  
 616 similar way.

617 Let  $W_{i,\mathcal{N}_c}$  be an optimal walk in  $\mathcal{W}_{\mathcal{T},\text{init}}(i \rightarrow \mathcal{N}_c\|)$ , with weight  $w_{i,\mathcal{N}_c}^*$ . We are required to prove  
 618 that

$$619 \quad (28) \quad p(\mathcal{W}_{\mathcal{T},\text{init}}(i \rightarrow \mathcal{N}_c\|)) = p(\mathcal{W}_{\mathcal{T},\text{full}}^k(i \rightarrow \mathcal{N}_c)),$$

620 where on the right we have the set of full walks connecting  $i$  to a critical node on  $\mathcal{T}(P)$ . We split (28)  
 621 into two inequalities,

$$622 \quad (29) \quad p(\mathcal{W}_{\mathcal{T},\text{init}}(i \rightarrow \mathcal{N}_c\|)) \leq p(\mathcal{W}_{\mathcal{T},\text{full}}^k(i \rightarrow \mathcal{N}_c)), \quad p(\mathcal{W}_{\mathcal{T},\text{init}}(i \rightarrow \mathcal{N}_c\|)) \geq p(\mathcal{W}_{\mathcal{T},\text{full}}^k(i \rightarrow \mathcal{N}_c))$$

623 For the first inequality in (29), observe that we can concatenate  $W_{i,\mathcal{N}_c}$  with a walk  $V$  on  
 624 the critical graph which has length  $l(V) = k - l(W_{i,\mathcal{N}_c})$ . The resulting walk  $W_{i,\mathcal{N}_c}V$  belongs to  
 625  $\mathcal{W}_{\mathcal{T},\text{full}}^k(i \rightarrow \mathcal{N}_c)$  and has weight  $w_{i,\mathcal{N}_c}^*$ , which proves the first inequality. For the second inequality,  
 626 take an optimal walk  $W^* \in \mathcal{W}_{\mathcal{T},\text{full}}^k(i \rightarrow \mathcal{N}_c)$ , whose weight is  $p(\mathcal{W}_{\mathcal{T},\text{full}}^k(i \rightarrow \mathcal{N}_c))$ . By observing the  
 627 first occurrence of a critical node in this walk, we represent  $W^* = WV$ , where  $W \in \mathcal{W}_{\mathcal{T},\text{init}}(i \rightarrow \mathcal{N}_c\|)$ .  
 628 We then have  $p(W^*) = p(W) + p(V) \leq p(W) \leq w_{i,\mathcal{N}_c}^*$  proving the second inequality. Combining  
 629 both inequalities gives the equality (28) and finishes the proof of  $w_{i,\mathcal{N}_c}^* = p(\mathcal{W}_{\mathcal{T},\text{full}}^k(i \rightarrow \mathcal{N}_c))$ . The  
 630 second part of the claim is proved similarly.  $\square$

REMARK 5.6. In the previous lemma, the length of the walks on the right-hand side does not have to be restricted to  $k$ . We can obtain the following results:

$$(30) \quad \begin{aligned} w_{i, \mathcal{N}_c}^* &= p(\mathcal{W}_{\mathcal{T}, \text{init}}^l(i \rightarrow \mathcal{N}_c)) \quad \text{for any } l \geq \min \left( \frac{w_{i, \mathcal{N}_c}^* - \alpha_{i, \mathcal{N}_c}}{\lambda_{q*}} + (n - q), k \right) \\ v_{\mathcal{N}_c, j}^* &= p(\mathcal{W}_{\mathcal{T}, \text{final}}^m(\mathcal{N}_c \rightarrow j)) \quad \text{for any } m \geq \min \left( \frac{v_{\mathcal{N}_c, j}^* - \beta_{\mathcal{N}_c, j}}{\lambda_{q*}} + (n - q), k \right). \end{aligned}$$

We now establish the connection between the previous Lemma and CSR.

LEMMA 5.7. We have one of the following cases:

- (i)  $(CS^{k(\text{mod } \gamma)} R[\Gamma(k)])_{i, j} = \varepsilon$  if  $[i]' \not\rightarrow_k [j]'$ ,
- (ii)  $(CS^{k(\text{mod } \gamma)} R[\Gamma(k)])_{i, j} = w_{i, \mathcal{N}_c}^* + v_{\mathcal{N}_c, j}^*$  if  $[i]' \rightarrow_k [j]'$ .

*Proof.* By Lemma 3.7 we have  $p(\mathcal{W}_{\mathcal{T}', \text{full}}^{2k+v}(i \rightarrow j)) = (CS^{k(\text{mod } \gamma)} R[\Gamma(k)])_{i, j}$ , where  $v = (t + 1)\gamma - k(\text{mod } \gamma)$  and  $t\gamma \geq T(S)$ , and let  $W \in \mathcal{W}_{\mathcal{T}', \text{full}}^{2k+v}(i \rightarrow j)$  be optimal.  $W$  can be decomposed as  $W_1 W_2 W_3$  where  $W_1$  is a full walk (of length  $k$ ) connecting  $i$  to some  $l \in \mathcal{N}_c$  on  $\mathcal{T}$ ,  $W_3$  is a (full) walk of length  $k$  connecting some  $m \in \mathcal{N}_c$  to  $j$  and  $W_2$  is a walk on the critical graph of length  $v$  connecting the end of  $W_1$  to the beginning of  $W_3$ . In formula,

$$(31) \quad \begin{aligned} (CS^{k(\text{mod } \gamma)} R[\Gamma(k)])_{i, j} &= \max\{p(W_1) + p(W_2) + p(W_3) : \\ &W_1 \in \mathcal{W}_{\mathcal{T}, \text{full}}^k(i \rightarrow l), W_2 \in \mathcal{W}_{\mathcal{C}}^v(l \rightarrow m), W_3 \in \mathcal{W}_{\mathcal{T}, \text{full}}^k(m \rightarrow j), l, m \in \mathcal{N}_c\} \end{aligned}$$

If the weights of  $W_1$ ,  $W_2$  and  $W_3$  in (31) are finite then  $[i]' \rightarrow_k [l]'$ ,  $[l]' \rightarrow_v [m]'$  and  $[m]' \rightarrow_k [j]'$ , hence  $[i]' \rightarrow_k [j]'$ . Thus  $(CS^t R[\Gamma(k)])_{i, j} > \varepsilon$  implies  $[i]' \rightarrow_k [j]'$  proving (i).

As the cyclicity of the associated graph is the same as the cyclicity of the critical graph, Lemma 5.5 implies that

$$(32) \quad w_{i, \mathcal{N}_c}^* = p(\mathcal{W}_{\mathcal{T}}^k(i \rightarrow \mathcal{C}_{i, k})), \quad v_{\mathcal{N}_c, j}^* = p(\mathcal{W}_{\mathcal{T}}^k(\mathcal{C}_{k, j} \rightarrow j)),$$

where  $\mathcal{C}_{i, k} = \mathcal{C}'_{i, k} \cap \mathcal{N}_c$  is the cyclic class of  $\mathbf{C}(\mathcal{X})$  that can be found by intersecting with critical nodes  $\mathcal{N}_c$  the cyclic class  $\mathcal{C}'_{i, k}$  of  $\mathcal{D}$  defined by  $[i]' \rightarrow_k \mathcal{C}'_{i, k}$ . Similarly,  $\mathcal{C}_{k, j} = \mathcal{C}'_{k, j} \cap \mathcal{N}_c$  is the cyclic class of  $\mathbf{C}(\mathcal{X})$  that can be found by intersecting with critical nodes  $\mathcal{N}_c$  the cyclic class  $\mathcal{C}'_{k, j}$  of  $\mathcal{D}$  defined by  $\mathcal{C}'_{k, j} \rightarrow_k [j]'$ .

Now note that in (31) we can similarly restrict  $l$  to  $\mathcal{C}_{i, k}$  and  $m$  to  $\mathcal{C}_{k, j}$ , which transforms it to

$$(33) \quad \begin{aligned} (CS^{k(\text{mod } \gamma)} R[\Gamma(k)])_{i, j} &= \max\{p(W_1) + p(W_2) + p(W_3) : \\ &W_1 \in \mathcal{W}_{\mathcal{T}}^k(i \rightarrow l), W_2 \in \mathcal{W}_{\mathcal{C}}^v(l \rightarrow m), W_3 \in \mathcal{W}_{\mathcal{T}}^k(m \rightarrow j), l \in \mathcal{C}_{i, k}, m \in \mathcal{C}_{k, j}\} \end{aligned} \quad \square$$

Note that if a walk  $W_2$  exists between any  $l \in \mathcal{C}_{i, k}$  and  $m \in \mathcal{C}_{k, j}$  then using (32) we immediately obtain  $(CS^{k(\text{mod } \gamma)} R[\Gamma(k)])_{i, j} = w_{i, \mathcal{N}_c}^* + v_{\mathcal{N}_c, j}^*$ . Thus it remains to show existence of  $W_2 \in \mathcal{W}_{\mathcal{C}}^v(l \rightarrow m)$  between any  $l \in \mathcal{C}_{i, k}$  and  $m \in \mathcal{C}_{k, j}$ . For this note that since  $v = (t + 1)\gamma - k(\text{mod } \gamma) \geq T(S)$ , either  $\mathcal{C}_{i, k} \rightarrow_{(\gamma - k(\text{mod } \gamma))} \mathcal{C}_{k, j}$  and a walk on  $\mathbf{C}(\mathcal{X})$  of length  $v$  exists between each pair of nodes in  $\mathcal{C}_{i, k}$  and  $\mathcal{C}_{k, j}$ , or  $\mathcal{C}_{i, k} \not\rightarrow_{(\gamma - k(\text{mod } \gamma))} \mathcal{C}_{k, j}$  and then no such walk exists. We thus have to check that  $\mathcal{C}_{i, k} \rightarrow_{(\gamma - k(\text{mod } \gamma))} \mathcal{C}_{k, j}$  on  $\mathcal{D}$ . But this follows since we have  $[i]' \rightarrow_k [j]'$ , and since in the sequence  $[i]' \rightarrow_k \mathcal{C}'_{i, k} \rightarrow_l \mathcal{C}'_{k, j} \rightarrow_k [j]'$  we then must have  $l \equiv_{\gamma} \gamma - k(\text{mod } \gamma)$ .

Combining Theorem 5.4 and Lemma 5.7 we obtain the following result.

THEOREM 5.8. Denote  $u_{i,\mathcal{N}_c,j}^* = w_{i,\mathcal{N}_c}^* + v_{\mathcal{N}_c,j}^*$ . Let  $k$  be greater than or equal to

$$\max \left( \max_{i,j} \frac{u_{i,\mathcal{N}_c,j}^* - \alpha_{i,\mathcal{N}_c} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + 2(n - q) + \text{Sch}(\gamma, q), \max_{i,j: \gamma_{i,j} > \varepsilon} \frac{u_{i,\mathcal{N}_c,j}^* - \gamma_{i,j}}{\lambda_*} + n - q + 1 \right)$$

Then  $\Gamma(k) = CS^{k(\bmod \gamma)} R[\Gamma(k)]$ .

As with Theorem 4.6 this bound requires  $\Gamma(k)$  in order to calculate the bound, which makes it implicit, but as with Corollary 4.7 we can use  $w_{i,\mathcal{N}_c} \leq w_{i,\mathcal{N}_c}^*$  and  $v_{\mathcal{N}_c,j} \leq v_{\mathcal{N}_c,j}^*$  to give us an explicit bound. The following result requires Assumption C on  $A^{\text{inf}}$ .

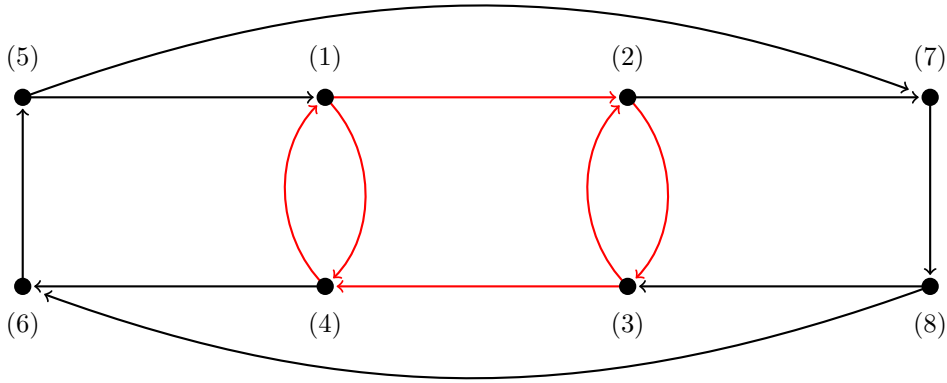
COROLLARY 5.9. Denote  $u_{i,\mathcal{N}_c,j} = w_{i,\mathcal{N}_c} + v_{\mathcal{N}_c,j}$ . Let  $k$  be greater than or equal to

$$\max \left( \max_{i,j} \frac{u_{i,\mathcal{N}_c,j} - \alpha_{i,\mathcal{N}_c} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + 2(n - q) + \text{Sch}(\gamma, q), \max_{i,j: \gamma_{i,j} > \varepsilon} \frac{u_{i,\mathcal{N}_c,j} - \gamma_{i,j}}{\lambda_*} + n - q + 1 \right)$$

Then  $\Gamma(k) = CS^{k(\bmod \gamma)} R[\Gamma(k)]$ .

We will now present an example of this bound in action.

Let  $\mathcal{D}(G)$  be the eight node digraph with the following structure:



along with the associated weight matrix.

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & A_{2,7} & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{4,6} & \varepsilon & \varepsilon \\ A_{5,1} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{5,7} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{6,5} & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{7,8} \\ \varepsilon & \varepsilon & A_{8,3} & \varepsilon & \varepsilon & A_{8,6} & \varepsilon & \varepsilon \end{pmatrix}$$

There are three critical cycles in this digraph, one cycle of length 4 traversing  $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$ , and two cycles of length 2 traversing  $1 \rightarrow 4 \rightarrow 1$  and  $2 \rightarrow 3 \rightarrow 2$  respectively. There are also cycles of length 4, 6 and 8 which means that the cyclicity of the whole digraph is 2, which is the same cyclicity of the critical subgraph. Therefore Assumption P0 is satisfied and we can continue.



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$$w_{i,N_c}^T = (0 \ 0 \ 0 \ 0 \ -19 \ -37 \ -39 \ -19), v_{N_c,j} = (0 \ 0 \ 0 \ 0 \ -34 \ -16 \ -19 \ -39).$$

696 With all the pieces ready we can now form the bound of Corollary 5.9,

$$697 \quad k \geq \max \left( \begin{pmatrix} 12 & 12 & 12 & 12 & 16.4 & 14.2 & 15.6 & 18.9 \\ 12 & 12 & 12 & 12 & 16.4 & 14.2 & 15.6 & 18.9 \\ 12 & 12 & 12 & 12 & 16.4 & 14.2 & 15.6 & 18.9 \\ 12 & 12 & 12 & 12 & 16.4 & 14.2 & 15.6 & 18.9 \\ 14.2 & 14.2 & 14.2 & 14.2 & 18.7 & 16.4 & 17.8 & 21.1 \\ 16.4 & 16.4 & 16.4 & 16.4 & 20.9 & 18.7 & 20 & 23.3 \\ 19.3 & 19.3 & 19.3 & 19.3 & 23.8 & 21.6 & 22.9 & 26.2 \\ 16 & 16 & 16 & 16 & 20.4 & 18.22 & 19.6 & 22.9 \end{pmatrix}, \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 12.8 & 10.6 & 12.8 & 16.1 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 19 & 12.8 & 15 & 18.3 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 17.9 & 15.7 & 13.9 & 21.2 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 14.6 & 12.3 & 10.6 & 13.9 \end{pmatrix} \right)$$

698  $\Rightarrow k \geq 23.8.$

699 Therefore by Corollary 5.9 if the length of a product using the matrices from  $\mathcal{X}$  is greater than or  
700 equal to 24 then the resulting product will be CSR. We will show such a product. Let  $\Gamma(24)$  be the  
701 inhomogeneous matrix product made using the word  $P = 551541235515535135454155$  which gives us:

$$702 \quad \Gamma(24) = \begin{pmatrix} 0 & \varepsilon & 0 & \varepsilon & \varepsilon & -16 & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & -28 & \varepsilon & \varepsilon & -21 \\ 0 & \varepsilon & 0 & \varepsilon & \varepsilon & -16 & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & -28 & \varepsilon & \varepsilon & -21 \\ \varepsilon & -19 & \varepsilon & -19 & -47 & \varepsilon & \varepsilon & -40 \\ -31 & \varepsilon & -31 & \varepsilon & \varepsilon & -47 & -42 & \varepsilon \\ -11 & \varepsilon & -11 & \varepsilon & \varepsilon & -27 & -22 & \varepsilon \\ \varepsilon & -1 & \varepsilon & -1 & -29 & \varepsilon & \varepsilon & -22 \end{pmatrix}.$$

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704 This matrix product is indeed CSR and by Definition 3.2 we have,

$$705 \quad \Gamma(24) = \begin{pmatrix} 0 & \varepsilon & 0 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ 0 & \varepsilon & 0 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & -19 & \varepsilon & -19 \\ -31 & \varepsilon & -31 & \varepsilon \\ -11 & \varepsilon & -11 & \varepsilon \\ \varepsilon & -1 & \varepsilon & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \varepsilon & 0 & \varepsilon & \varepsilon & -16 & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & -28 & \varepsilon & \varepsilon & -21 \\ 0 & \varepsilon & 0 & \varepsilon & \varepsilon & -16 & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & -28 & \varepsilon & \varepsilon & -21 \end{pmatrix}$$

$$706 \quad \Gamma(24) = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \\ 0 & \varepsilon \\ \varepsilon & 0 \\ \varepsilon & -19 \\ -31 & \varepsilon \\ -11 & \varepsilon \\ \varepsilon & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \varepsilon & 0 & \varepsilon & \varepsilon & -16 & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & -28 & \varepsilon & \varepsilon & -21 \end{pmatrix}.$$

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708 We can see that, for the  $C$  matrix, columns 3 and 4 are copies of columns 1 and 2 respectively. The  
709 same is also true for the rows of the  $R$  matrix so they can be deleted. As  $24 \pmod 2 = 0$  we replace  
710 the  $S$  matrix with the tropical identity matrix which shows us that the matrix product  $\Gamma(24)$  using  
711 the word  $P$  is indeed CSR and it has factor rank-2.

712 **6. Counterexamples.** Here we present a number of counterexamples for the different cases of  
713 digraph structure. These counterexamples present families of products which are not CSR, and we  
714 construct them in such a way that they have no upper bound on their length.

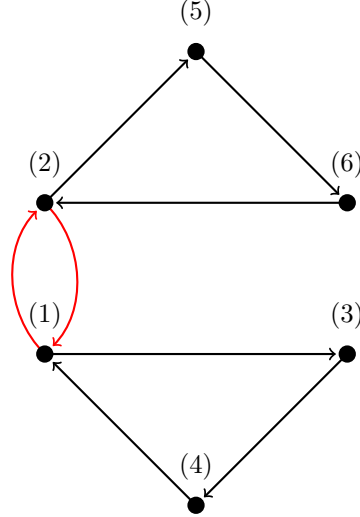
715 **6.1. The ambient graph is primitive but the critical graph is not.** We will now look at  
716 two cases where we are unable to create a bound for matrix products to become CSR. For the first

case we will be looking at digraphs that are primitive but have a critical subgraph with a non-trivial cyclicity. Therefore we have the following assumption:

ASSUMPTION  $\mathcal{P}1$ .  $\mathcal{D}(\mathcal{X})$  is primitive (i.e.,  $\gamma(\mathcal{D}(\mathcal{X})) = 1$ ) and the critical subgraph  $\mathbf{C}(\mathcal{X})$ , which is a single strongly connected component, has cyclicity  $\gamma(\mathbf{C}(\mathcal{X})) = \gamma > 1$ .

We now present a counterexample which shows that under this assumption, in general, no bound for  $k$  in terms of  $A^{\sup}$  and  $A^{\inf}$  can exist that ensures that  $\Gamma(k)$  is equal to the corresponding CSR product.

Let  $\mathcal{D}(G)$  be the five node digraph with the following structure:



This digraph will have the following associated weight matrix.

$$A = \begin{pmatrix} \varepsilon & 0 & A_{1,3} & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & A_{2,5} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & A_{3,4} & \varepsilon & A_{3,6} \\ A_{4,1} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{5,6} \\ \varepsilon & A_{6,2} & A_{6,3} & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

There is a critical subgraph consisting of the cycle between nodes 1 and 2. There also exist two cycles,  $1 \rightarrow 3 \rightarrow 4 \rightarrow 1$  and  $2 \rightarrow 5 \rightarrow 6 \rightarrow 2$ , both of length 3 which makes  $\mathcal{D}(A)$  primitive. We aim to present a family of words with infinite length such that the products made up using these words are not CSR. Since the cyclicity of the critical subgraph is 2 then we will have to create two classes of words, one of even length and one of odd length to define the family.

The semigroup of matrices we will use is generated by the two matrices:

$$A_1 = \begin{pmatrix} \varepsilon & 0 & -100 & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & -100 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & -100 & \varepsilon & \varepsilon \\ -100 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -100 \\ \varepsilon & -100 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, A_2 = \begin{pmatrix} \varepsilon & 0 & -100 & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & -1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & -100 & \varepsilon & \varepsilon \\ -1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -100 \\ \varepsilon & -100 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

Let us first consider the class of words  $(1)^{2t}2$  where  $t \geq 2$ , and let  $U = (A_1)^{2t}A_2$  for arbitrary such  $t$ . We will first examine entries  $U_{6,1}$ ,  $U_{2,5}$ ,  $U_{6,2}$  and  $U_{1,5}$ .

The entry  $U_{6,1}$  can be obtained as the weight of the walk  $6 \underbrace{(21)(21) \dots (21)}_{t-1} 341$ , which is  $-301$ .

For this observe that the walk  $621$  has an even length and therefore we need to use one of the three-cycles to make it odd, and using the southern three-cycle in the end of the walk is the most profitable way to do so. The entry  $U_{25}$  is equal to  $-1$ , as there is a walk that mostly rests on the critical cycle and only in the end jumps to node 5. We also have  $U_{6,2} = -100$  (go to node 2 and remain on the critical cycle) and  $U_{1,5} = -301$  (use the southern triangle once, then dwell on the critical cycle and in the end jump to node 5). Note that in the case of  $U_{1,5}$  we again need to use one of the triangles to create a walk of an odd length.

We then compute

$$(CSR)[U]_{6,5} = (US^3U)_{6,5} = \max(U_{6,1} + U_{2,5}, U_{6,2} + U_{1,5}) = -301 - 1 = -302.$$

However,  $U_{6,5}$  results from the walk  $6 \underbrace{(21)(21) \dots (21)}_{t-1} 2562$ , with weight  $-401$ , needing to use the northern triangle to make a walk of odd length.

The following an example of  $U$  and  $CS^{2t+1}R[U]$  for  $t = 10$ :

$$U = \begin{pmatrix} -201 & 0 & -100 & -500 & -301 & -200 \\ 0 & -300 & -400 & -200 & -1 & -500 \\ -401 & -200 & -300 & -700 & -501 & -400 \\ -100 & -400 & -500 & -300 & -101 & -600 \\ -200 & -500 & -600 & -400 & -201 & -700 \\ -301 & -100 & -200 & -600 & -401 & -300 \end{pmatrix}$$

$$CS^{21(\bmod 2)}R[U] = \begin{pmatrix} -201 & 0 & -100 & -401 & -202 & -200 \\ 0 & -300 & -400 & -200 & -1 & -500 \\ -401 & -200 & -300 & -601 & -402 & -400 \\ -100 & -400 & -500 & -300 & -101 & -600 \\ -200 & -500 & -600 & -400 & -201 & -700 \\ -301 & -100 & -200 & -501 & -302 & -300 \end{pmatrix}$$

We now consider the class of words  $(1)^{2t+1}2$  where  $t \geq 1$ , and let  $V = (A_1)^{2t+1}A_2$  for arbitrary such  $t$ . We will first examine entries  $V_{2,1}$ ,  $V_{1,5}$ ,  $V_{2,2}$  and  $V_{2,5}$ .

The entry  $V_{2,1} = -201$  is obtained as the weight of the walk  $2 \underbrace{(12)(12) \dots (12)}_{t-1} 341$ : it is necessary

to use one of the triangles to create a walk of even length, and using the southern triangle once in the end of the walk is the most profitable way to do so. The walk  $125$  already has an even length, and we only have to augment it with enough copies of the critical cycle and use the arc  $2 \rightarrow 5$  in the end of the walk, thus getting  $V_{1,5} = -1$ . Obviously,  $V_{2,2} = 0$ : we just stay on the critical cycle. The entry  $V_{2,5} = -301$  is obtained as the weight of the walk  $\underbrace{(21)(21) \dots (21)}_{t-1} 5625$ , where we have to use

the northern triangle in the end of the walk to create a walk of even walk and minimise the loss.

We then find

$$(CS^2R[V])_{2,5} = (VS^2V)_{2,5} = \max(V_{2,1} + V_{1,5}, V_{2,2} + V_{2,5}) = V_{2,1} + V_{1,5} = -202,$$

which is bigger than  $V_{2,5} = -301$ .

The case for  $V_{2,5}$  is one for connecting a critical node to a non critical node. For completeness we should also look at a walk connecting two non critical nodes, namely the walk representing  $V_{4,5}$ . To

do this we will need to also look at the entries  $V_{4,1}$  and  $V_{4,2}$ . For  $V_{4,1} = -301$  the entry is obtained as the weight of the walk  $4 \underbrace{(12)(12) \dots (12)}_{t-1} 341$ . As the walk 41 has odd length, one of the triangles

is required to make the walk even so choosing the southern triangle is the most profitable way to achieve an even length walk. The walk 412 already has an even length so we can augment it with enough copies of the critical cycle to give us the desired length for the walk representing the entry  $V_{4,2} = -100$ . Using  $V_{1,5}$  and  $V_{2,5}$  discussed earlier we calculate

$$(CS^2R[V])_{4,5} = (VS^2V)_{4,5} = \max(V_{4,1} + V_{1,5}, V_{4,2} + V_{2,5}) = V_{4,1} + V_{1,5} = -302,$$

763 which is bigger than  $V_{4,5} = -401$ .

764 We now show an example of  $V$  for  $t = 10$ :

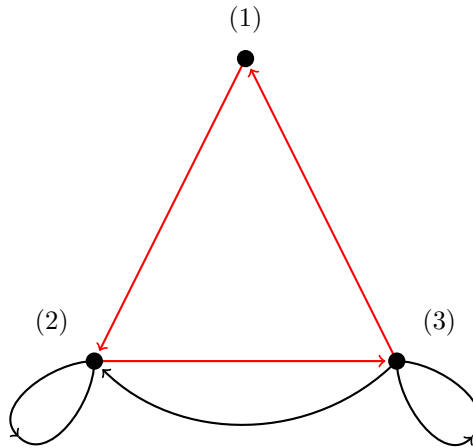
$$765 \quad V = \begin{pmatrix} 0 & -300 & -400 & -200 & -1 & -500 \\ -201 & 0 & -100 & -500 & -301 & -200 \\ -200 & -500 & -600 & -400 & -201 & -700 \\ -301 & -100 & -200 & -600 & -401 & -300 \\ -401 & -200 & -300 & -700 & -501 & -400 \\ -100 & -400 & -500 & -300 & -101 & -600 \end{pmatrix}$$

$$766 \quad CS^{22(\bmod 2)}R[V] = \begin{pmatrix} 0 & -300 & -400 & -200 & -1 & -500 \\ -201 & 0 & -100 & -401 & -202 & -200 \\ -200 & -500 & -600 & -400 & -201 & -700 \\ -301 & -100 & -200 & -501 & -302 & -300 \\ -401 & -200 & -300 & -601 & -402 & -400 \\ -100 & -400 & -500 & -300 & -101 & -600 \end{pmatrix}$$

767

768 Combining both classes we have a family of words covering all lengths greater than 29 such that  
 769 any product made using these words will not be equal to the corresponding CSR product. Therefore  
 770 there cannot be a transient for this case as there is no upper limit to the lengths of these words.

771 We now also construct a counterexample where all nodes of  $\mathcal{D}(G)$  are critical. Let  $\mathcal{D}(G)$  be the  
 772 three node digraph with the following structure:



773

774 The digraph has the following associated weight matrix.

$$775 \quad A = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & A_{2,2} & 0 \\ 0 & A_{3,2} & A_{3,3} \end{pmatrix}.$$

776 For this example there is a single critical cycle of length 3 traversing all of the nodes. There also  
 777 exists two loops  $2 \rightarrow 2$  and  $3 \rightarrow 3$  and a cycle  $2 \rightarrow 3 \rightarrow 2$  of length 2. Like the previous example this  
 778 digraph is primitive but the critical subgraph has cyclicity 3. As the cyclicity is greater than one we  
 779 need to present three different classes of words making up a family of words such that any product  
 780  $\Gamma(k)$  made using these words will not be CSR.

781 The semigroup of matrices that we will use is again generated only by two matrices:

$$782 \quad A_1 = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & -100 & 0 \\ 0 & -100 & -100 \end{pmatrix} \quad A_2 = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & -1 & 0 \\ 0 & -100 & -1 \end{pmatrix}$$

783 Let the first class of words be  $(1)^{3t+2}2$  for  $t \geq 0$ , and let  $M = (A_1)^{3t+2}A_2$  for any arbitrary  $t$ .  
 784 We will now examine the entries  $M_{1,1}$ ,  $M_{1,2}$ ,  $M_{2,2}$ ,  $M_{1,3}$  and  $M_{3,2}$ .

785 Since all the walks are of length 0 modulo 3 then any walk connecting  $i$  to  $i$  will have weight  
 786 zero as we can simply use the critical cycle. This gives  $M_{1,1} = M_{2,2} = 0$ . The entry  $M_{1,2}$  can be  
 787 obtained as the weight of the walk  $(123)^{t+1}2$  which is  $-100$ . In this entry observe that the walk 12  
 788 is of length 1 modulo 3 therefore we need to use the two cycle  $2 \rightarrow 3 \rightarrow 2$  to give us a walk of the  
 789 desired length. The entry  $M_{1,3}$  is equal to the weight of the walk  $(123)^{t+1}3$  and the entry  $M_{3,2}$  is  
 790 equal to the weight of the walk  $(312)^{t+1}2$ . For these entries observe that the walks 123 and 312 are  
 791 both of length 2 modulo 3 therefore we require a loop for both walks to give us the required length.  
 792 The most profitable time to use these loops are right at the end of the walk.

We then compute

$$(CSR)[M]_{1,2} = (MS^3M)_{1,2} = \max(M_{1,1} + M_{1,2}, M_{1,2} + M_{2,2}, M_{1,3} + M_{3,2}) = -1 - 1 = -2.$$

793 However, as seen earlier the entry  $M_{12}$  has weight  $-100$  which is less than the CSR suggestion.  
 794 The following is an example of  $M$  and  $CS^{3t+3}R[M]$  for  $t = 10$ :

$$795 \quad M = \begin{pmatrix} 0 & -100 & -1 \\ -100 & 0 & -100 \\ -100 & -1 & 0 \end{pmatrix} \quad CS^{33(\bmod 3)}R[M] = \begin{pmatrix} 0 & -2 & -1 \\ -100 & 0 & -100 \\ -100 & -1 & 0 \end{pmatrix}$$

796 For efficiency we will simply present the final two classes and omit the in-depth analysis of them:

797 For walks of length 1 modulo 3 we have the class of words  $(1)^{3t+3}2$  for  $t \geq 0$ .

798 For walks of length 2 modulo 3 we have the class of words  $(1)^{3t+4}2$  for  $t \geq 0$ .

799 We will also present examples of products and their CSR counterparts made using these words for  
 800  $t = 10$  where  $N = (A_1)^{3t+3}A_2$  and  $P = (A_1)^{3t+4}A_2$ .

$$801 \quad N = \begin{pmatrix} -100 & 0 & -100 \\ -100 & -1 & 0 \\ 0 & -100 & -1 \end{pmatrix} \quad CS^{34(\bmod 3)}R[N] = \begin{pmatrix} -100 & 0 & -100 \\ -100 & -1 & 0 \\ 0 & -2 & -1 \end{pmatrix}$$

$$802 \quad P = \begin{pmatrix} -100 & -1 & 0 \\ 0 & -100 & -1 \\ -100 & 0 & -100 \end{pmatrix} \quad CS^{35(\bmod 3)}R[P] = \begin{pmatrix} -100 & -1 & 0 \\ 0 & -2 & -1 \\ -100 & 0 & -100 \end{pmatrix}.$$

803



The combination of these three classes create a family of words such that any product  $\Gamma(k)$  made using these words is not equal to the corresponding CSR product.

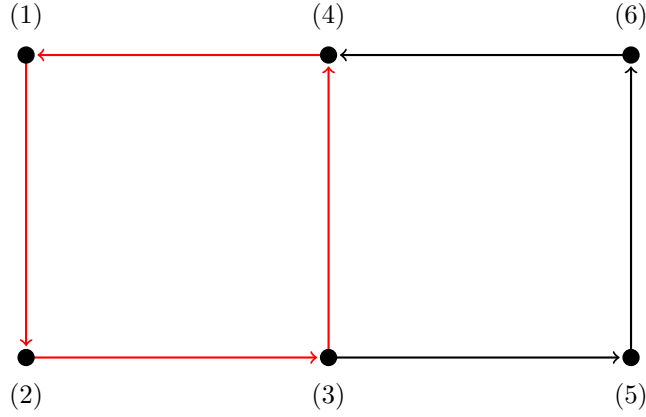
We now extend these counterexamples to a more general form where we consider digraphs with non-trivial cyclicity  $r$  along with critical subgraphs with cyclicity  $\gamma$  which is greater than  $r$ . This leads to the following assumptions.

### 6.2. More general case.

ASSUMPTION  $\mathcal{P}2$ .  $\mathcal{D}(\mathcal{X})$  has cyclicity  $r$  and the critical subgraph  $\mathbf{C}(\mathcal{X})$ , which is strongly connected, has cyclicity  $\gamma > r$ .

In a similar method to the primitive example above, using the new assumptions, we can now describe a counterexample that shows that no bound for  $k$  in terms of  $A^{\sup}$  and  $A^{\inf}$  can exist that ensures  $\Gamma(k)$  is equal to the corresponding CSR product.

Let  $\mathcal{D}(\mathcal{X})$  be a six node digraph with the following structure:



along with the following associated weight matrix,

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & A_{3,5} & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{5,6} \\ \varepsilon & \varepsilon & \varepsilon & A_{6,4} & \varepsilon & \varepsilon \end{pmatrix}$$

Here the critical cycle traverses nodes  $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$  however there also exists another non-critical cycle of length six traversing  $1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 6 \rightarrow 4 \rightarrow 1$ . This means that while the cyclicity of the critical subgraph is 4 the cyclicity of  $\mathcal{D}(G)$  is 2. Therefore the digraph structure satisfies the assumptions and we can develop a family of words with infinite length such that any  $\Gamma(k)$  made using these words will not be equal to the corresponding CSR product. As the cyclicity of the critical subgraph is 4 then we will require four classes of words to fully define the family.

The semigroup of matrices that will be used is generated by two matrices:

$$A_1 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & -100 & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -100 \\ \varepsilon & \varepsilon & \varepsilon & -100 & \varepsilon & \varepsilon \end{pmatrix} \quad A_2 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -100 \\ \varepsilon & \varepsilon & \varepsilon & -1 & \varepsilon & \varepsilon \end{pmatrix}$$

Let us begin with the first class of words  $(1)^{4t}2$  where  $t \geq 2$ , and let  $L = (A_1)^{4t}A_2$  for arbitrary such  $t$ . We will begin by examining the entries  $L_{1,2}$ ,  $L_{1,5}$ ,  $L_{1,4}$  and  $L_{3,5}$ .

The entry  $L_{1,2}$  can be obtained as the weight of the walk  $\underbrace{(1234)}_t 12$ , which is 0. As the walk 12

has length congruent to  $1 \pmod{4}$  then a walk exists on the critical cycle connecting these nodes. The entry  $L_{1,5}$  is obtained from the weight of the walk  $\underbrace{(1234)}_{t-2} 1235641235$ , which is  $-301$ . As the walk

1235 has length congruent to  $3 \pmod{4}$  then we need to add on the six cycle with weight  $-300$  to

give us a walk of length congruent to  $1 \pmod{4}$  and finally the last step of the walk is to go from 3 to

5 with weight  $-1$ . For the entry  $L_{1,4} = -201$  which is the weight of the walk  $\underbrace{(1234)}_{t-1} 123564$  and the

entry  $L_{35} = -1$  comes from the weight of the walk  $\underbrace{(3412)}_t 35$ . Note that in the case of  $L_{1,4}$  we used

the six cycle to give us the desired length of walk.

We then compute

$$(CSR)[L]_{1,5} = (L \otimes S^3 \otimes L)_{1,5} = \max(L_{1,2} + L_{1,5}, L_{1,4} + L_{3,5}) = -201 - 1 = -202.$$

However  $L_{15}$ , as explained earlier, results from a walk with weight  $-301$ .

The following is an example of  $L$  and  $CS^{4t+1}R[L]$  for  $t = 10$

$$L = \begin{pmatrix} \varepsilon & 0 & \varepsilon & -201 & -301 & \varepsilon \\ -300 & \varepsilon & 0 & \varepsilon & \varepsilon & -401 \\ \varepsilon & -300 & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & -300 & \varepsilon & \varepsilon & -101 \\ -500 & \varepsilon & -200 & \varepsilon & \varepsilon & -601 \\ \varepsilon & -400 & \varepsilon & -100 & -101 & \varepsilon \end{pmatrix}$$

$$CS^{41 \pmod{4}}R[L] = \begin{pmatrix} \varepsilon & 0 & \varepsilon & -201 & -202 & \varepsilon \\ -300 & \varepsilon & 0 & \varepsilon & \varepsilon & -401 \\ \varepsilon & -300 & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & -300 & \varepsilon & \varepsilon & -101 \\ -500 & \varepsilon & -200 & \varepsilon & \varepsilon & -601 \\ \varepsilon & -400 & \varepsilon & -100 & -101 & \varepsilon \end{pmatrix}$$

The other classes behave in a similar way so we omit the in depth explanation of them. We present the words used for each class:

For walks of length congruent to  $2 \pmod{4}$  we have the words  $(1)^{4t+1}2$  for  $t \geq 2$ ;

For walks of length congruent to  $3 \pmod{4}$  we have the words  $(1)^{4t+2}2$  for  $t \geq 2$ ;

For walks of length congruent to  $0 \pmod{4}$  we have the words  $(1)^{4t+3}2$  for  $t \geq 2$ .

850 For example, if  $t = 10$  then for the first of these classes

$$\begin{aligned}
 851 \quad F &= (A_1)^{41} \otimes A_2 = \begin{pmatrix} -300 & \varepsilon & 0 & \varepsilon & \varepsilon & -401 \\ \varepsilon & -300 & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & -300 & \varepsilon & \varepsilon & -101 \\ \varepsilon & 0 & \varepsilon & -201 & -301 & \varepsilon \\ \varepsilon & -500 & \varepsilon & -200 & -201 & \varepsilon \\ -100 & \varepsilon & -400 & \varepsilon & \varepsilon & -201 \end{pmatrix}, \\
 852 \quad CS^{42(\bmod 4)}R[F] &= \begin{pmatrix} -300 & \varepsilon & 0 & \varepsilon & \varepsilon & -401 \\ \varepsilon & -300 & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & -300 & \varepsilon & \varepsilon & -101 \\ \varepsilon & 0 & \varepsilon & -201 & -202 & \varepsilon \\ \varepsilon & -500 & \varepsilon & -200 & -201 & \varepsilon \\ -100 & \varepsilon & -400 & \varepsilon & \varepsilon & -201 \end{pmatrix} \\
 853
 \end{aligned}$$

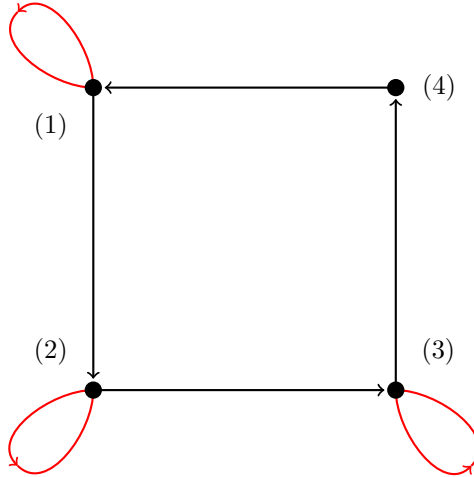
854 Combining all classes gives us a family of words covering all lengths greater than 9 such that any  
 855 product made using these words will not be equal to the corresponding CSR product.

856 **6.3. Critical graph is not connected.** For this counterexample we now consider a digraph  
 857 with multiple critical components  $\mathbf{C}_1, \dots, \mathbf{C}_m$  which are each strongly connected components with  
 858 respective cyclicities  $\gamma_1, \dots, \gamma_m$ .

859 ASSUMPTION  $\mathcal{P}3$ .  $\mathbf{C}(\mathcal{X})$  is composed of multiple strongly connected components  $\mathbf{C}_1, \dots, \mathbf{C}_m$   
 860 where the component  $\mathbf{C}_i$  has cyclicity  $\gamma_i$ . The cyclicity of  $\mathcal{D}(\mathcal{X})$  is  $\text{lcm}_i(\gamma_i)$ , which is the same as the  
 861 cyclicity of  $\mathbf{C}(\mathcal{X})$ .

862 Let us now show a counterexample, which demonstrates that, for the case of several critical  
 863 components, we cannot have any bounds after which the product becomes CSR in terms of  $A^{\text{sup}}$  and  
 864  $A^{\text{inf}}$ . The reason is that the non-critical parts of optimal walks whose weights are the entries of  $C$   
 865 and  $R$  cannot be separated in time: in general, they will use the same letters, and such walks on the  
 866 symmetric extension of  $\mathcal{T}(P)$  cannot be transformed back to the walks on  $\mathcal{T}(P)$ .

867 Let  $\mathcal{D}(\mathcal{X})$  be the four node digraph with the following structure:



868

869 along with the following associated weight matrix

$$870 \quad A = \begin{pmatrix} 0 & A_{12} & \varepsilon & \varepsilon \\ \varepsilon & 0 & A_{23} & \varepsilon \\ \varepsilon & \varepsilon & 0 & A_{34} \\ A_{41} & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

871 For this digraph we have a the critical subgraph comprised of three separate loops at nodes 1,2  
872 and 3. There is also a cycle of length 4 which means the cyclicity of the digraph is 1. We are going  
873 to present a class of words of infinite length such that the matrix generated by this class of words is  
874 not CSR.

875 We introduce a semigroup of tropical matrices with two generators  $\mathcal{X} = \{A_1, A_2\}$  where  $A_1$  to  
876  $A_2$  are

$$877 \quad A_1 = \begin{pmatrix} 0 & -100 & \varepsilon & \varepsilon \\ \varepsilon & 0 & -100 & \varepsilon \\ \varepsilon & \varepsilon & 0 & -100 \\ -100 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 & \varepsilon & \varepsilon \\ \varepsilon & 0 & -1 & \varepsilon \\ \varepsilon & \varepsilon & 0 & -100 \\ -100 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

878 and the class of the words that we will consider is  $(1)^t 2$ , where  $t \geq 2$ . In other words we will consider  
879 a set of matrices of the form  $U = (A_1)^t A_2$  (the actual value of  $t \geq 2$  will not matter to us).

880 We have:  $U_{1,2} = -1$  (as the weight of the walk  $\underbrace{11 \dots 1}_{t+1} 2$ ),  $U_{2,3} = -1$  (as the weight of the walk  
881  $\underbrace{22 \dots 2}_{t+1} 3$ ), and therefore  $(CS^{t+1}R[U])_{1,3} = U_{1,3}^2 = U_{1,2} \otimes U_{2,3} = -2$ , but  $U_{1,3} = -101$  (as the weight  
882 of the walk  $1 \underbrace{22 \dots 2}_t 3$ ).

883 Similarly, we can also look at the entry  $U_{4,3}$ . Then we have  $U_{4,2} = -101$  (as the weight of  
884 the walk  $4 \underbrace{11 \dots 1}_t 2$ ),  $U_{2,3} = -1$  and hence  $(CS^{t+1}R)_{4,3} = (USU)_{4,3} = U_{4,2} \otimes U_{2,3} = -102$ , but  
885  $U_{4,3} = -201$  (as the weight of the walk  $41 \underbrace{22 \dots 2}_{t-1} 3$ ).

886 Here is an example of the word from the class for  $t = 10$  and the corresponding CSR

$$887 \quad W = \begin{pmatrix} 0 & -1 & -101 & -300 \\ -300 & 0 & -1 & -200 \\ -200 & -201 & 0 & -100 \\ -100 & -101 & -201 & -400 \end{pmatrix}, \quad CS^{11(\bmod 1)}R[W] = \begin{pmatrix} 0 & -1 & -2 & -201 \\ -201 & 0 & -1 & -101 \\ -200 & -201 & 0 & -100 \\ -100 & -101 & -102 & -301 \end{pmatrix}.$$

888 Therefore any matrix product of length greater than 3 which has been made following this word  
889 will not be CSR. Hence there can be no upper bound to guarantee the CSR decomposition in this  
890 case.

891 **Acknowledgments.** The authors are grateful to Oliver Mason, Glenn Merlet, Thomas Nowak  
892 and Stephane Gaubert with whom the ideas of this paper were discussed.

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