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Path decompositions of tournaments

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Abstract

In 1976, Alspach, Mason, and Pullman conjectured that any tournament T of even order can be decomposed into exactly $\mathrm{ex}(T)$ paths, where $\mathrm{ex}(T) := \frac{1}{2}\sum_{v\in V(T)} |d_T^+(v) - d_T^-(v)|$. We prove this conjecture for all sufficiently large tournaments. We also prove an asymptotically optimal result for tournaments of odd order.

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1 | INTRODUCTION

Path and cycle decomposition problems have a long history. For example, the Walecki construction [17], which goes back to the 19th century, gives a decomposition of the complete graph of odd order into Hamilton cycles (see also [2]). A version of this for (regular) directed tournaments was conjectured by Kelly in 1968 and proved for large tournaments by Kühn and Osthus [12]. Beautiful open problems in the area include the Erdős–Gallai conjecture which asks for a decomposition of any graph into linearly many cycles and edges. The best bounds for this are due to Conlon, Fox, and Sudakov [6]. Another famous example is the linear arboricity conjecture, which asks for a decomposition of a d-regular graph into $\lceil \frac{d+1}{2} \rceil$ linear forests. The latter was resolved asymptotically by Alon [1] and the best current bounds are due to Lang and Postle [15].

1.1 Background

The problem of decomposing digraphs into paths was first explored by Alspach and Pullman [4], who provided bounds for the minimum number of paths needed in path decompositions of digraphs. (Throughout this paper, in a digraph, for any two vertices $u \neq v$, we allow a directed edge uv from u to v as well as a directed edge vu from v to u, whereas in an oriented graph we allow at most one directed edge between any two distinct vertices.) Given a digraph D, define the *path number of* D, denoted by $\mathrm{pn}(D)$, as the minimum integer k such that D can be decomposed into k paths. Alspach and Pullman [4] proved that, for any oriented graph D on n vertices, $\mathrm{pn}(D) \leqslant \frac{n^2}{4}$, with equality holding for transitive tournaments. O'Brien [18] showed that the same bound holds for digraphs on at least 4 vertices.

The path number of digraphs can be bounded below by the following quantity. Let D be a digraph and $v \in V(D)$. Define the *excess* at v as $\mathrm{ex}_D(v) := d_D^+(v) - d_D^-(v)$. Let $\mathrm{ex}_D^+(v) := \max\{0, \mathrm{ex}_D(v)\}$ and $\mathrm{ex}_D^-(v) := \max\{0, -\mathrm{ex}_D(v)\}$ be the *positive excess* and *negative excess* at v, respectively. Then, as observed in [4], if $d_D^+(v) > d_D^-(v)$, a path decomposition of D contains at most $d_D^-(v)$ paths which have v as an internal vertex, and thus at least $d_D^+(v) - d_D^-(v) = \mathrm{ex}_D^+(v)$ paths starting at v. Similarly, a path decomposition will contain at least $\mathrm{ex}_D^-(v)$ paths ending at v. Thus, the *excess* of D, defined as

$$\mathrm{ex}(D) := \sum_{v \in V(D)} \mathrm{ex}_D^+(v) = \sum_{v \in V(D)} \mathrm{ex}_D^-(v) = \frac{1}{2} \sum_{v \in V(D)} |\,\mathrm{ex}_D(v)|, \tag{1.1}$$

provides a natural lower bound for the path number of D, that is, any digraph D satisfies

$$\mathrm{pn}(D) \geqslant \mathrm{ex}(D). \tag{1.2}$$

It was shown in [4] that equality is satisfied for acyclic digraphs. A digraph satisfying equality in (1.2) is called *consistent*. Clearly, not all digraphs are consistent (for example, regular digraphs have excess 0). However, Alspach, Mason, and Pullman [3] conjectured in 1976 that tournaments of even order are consistent.

Conjecture 1.1 (Alspach, Mason, and Pullman [3]). *Let $n \in \mathbb{N}$ be even. Then, any tournament T on n vertices satisfies* $\mathrm{pn}(T) = \mathrm{ex}(T)$.

This conjecture is discussed also, for example, in the Handbook of Combinatorics [5].

Note that the results of Alspach and Pullman [4] mentioned above imply that Conjecture 1.1 holds for tournaments of excess $\frac{n^2}{4}$. Moreover, as observed by Lo, Patel, Skokan, and Talbot [16], Conjecture 1.1 for tournaments of excess $\frac{n}{2}$ is equivalent to Kelly's conjecture on Hamilton decompositions of regular tournaments. Recently, Conjecture 1.1 was verified in [16] for sufficiently large tournaments of sufficiently large excess. Moreover, they extended this result to tournaments of odd order n whose excess is at least $n^{2-\frac{1}{18}}$.

Theorem 1.2 [16]. *The following hold.*

(a) *There exists $C \in \mathbb{N}$ such that, for any tournament T of even order n, if* $\mathrm{ex}(T) \geqslant Cn$, *then* $\mathrm{pn}(T) = \mathrm{ex}(T)$.

(b) *There exists $n_0 \in \mathbb{N}$ such that, for any $n \geqslant n_0$, if T is a tournament on n vertices satisfying* $\mathrm{ex}(T) \geqslant n^{2-\frac{1}{18}}$, *then* $\mathrm{pn}(T) = \mathrm{ex}(T)$.

1.2 | New results

Building on the results and methods of [12, 16], we prove Conjecture 1.1 for large tournaments.

Theorem 1.3. *There exists $n_0 \in \mathbb{N}$ such that, for any even $n \geqslant n_0$, any tournament T on n vertices satisfies* $\mathrm{pn}(T) = \mathrm{ex}(T)$.

In fact, our methods are more general and allow us to determine the path number of most tournaments of odd order, whose behaviour turns out to be more complex. As mentioned above, not every digraph is consistent.

Let D be a digraph. Let $\Delta^0(D)$ denote the largest semidegree of D, that is $\Delta^0(D) := \max\{d^+(v), d^-(v) \mid v \in V(D)\}$. Note that $\Delta^0(D)$ is a natural lower bound for $\mathrm{pn}(D)$ as every vertex $v \in V(D)$ must be in at least $\max\{d^+(v), d^-(v)\}$ paths. This leads to the notion of the *modified excess* of a digraph D, which is defined as

$$\widetilde{\mathrm{ex}}(D) := \max\{\mathrm{ex}(D), \Delta^0(D)\}.$$

This provides a natural lower bound for the path number of any digraph D.

Fact 1.4. *Any digraph D satisfies* $\mathrm{pn}(D) \geqslant \widetilde{\mathrm{ex}}(D)$.

(Note that one can easily verify that any tournament T of even order satisfies $\widetilde{\mathrm{ex}}(T) = \mathrm{ex}(T)$ (see, for example, Proposition 6.1), so Fact 1.4 is consistent with Conjecture 1.1.)

Observe that, by Theorem 1.2(b), equality holds for large tournaments of excess at least $n^{2-\frac{1}{18}}$. However, note that equality does not hold for regular digraphs. (Here a digraph is r-regular if for every vertex, both its in- and outdegree equal r.) Indeed, by considering the number of edges, one can show that any path decomposition of an r-regular digraph will contain at least $r+1$ paths. Thus, any r-regular digraph satisfies

$$\mathrm{pn}(D) \geqslant r + 1 = \widetilde{\mathrm{ex}}(D) + 1. \tag{1.3}$$

Alspach, Mason, and Pullman [3] conjectured that equality holds in (1.3) whenever D is a regular tournament. We verify this conjecture for sufficiently large tournaments.

Theorem 1.5. *There exists $n_0 \in \mathbb{N}$ such that any regular tournament T on $n \geqslant n_0$ vertices satisfies* $\mathrm{pn}(T) = \frac{n+1}{2} = \widetilde{\mathrm{ex}}(T) + 1$.

In fact, our argument also applies to regular oriented graphs of large enough degree.

Theorem 1.6. *For any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, if D is an r-regular oriented graph on $n \geqslant n_0$ vertices satisfying $r \geqslant (\frac{3}{8} + \varepsilon)n$, then* $\mathrm{pn}(D) = r + 1 = \widetilde{\mathrm{ex}}(D) + 1$.

More generally, we will see that Theorem 1.6 can be extended to regular digraphs of linear degree which are 'robust outexpanders' (see Theorem 5.2).

There also exist non-regular tournaments for which equality does not hold in Fact 1.4. Indeed, let $\mathcal{T}_{\text{apex}}$ be the set of tournaments T on $n \geqslant 5$ vertices for which there exists a partition $V(T) = V_0 \cup \{v_+\} \cup \{v_-\}$ such that $T[V_0]$ is a regular tournament on $n-2$ vertices (and so n is odd), $N_T^+(v_+) = V_0 = N_T^-(v_-)$, $N_T^-(v_+) = \{v_-\}$, and $N_T^+(v_-) = \{v_+\}$. The tournaments in $\mathcal{T}_{\text{apex}}$ are called *apex tournaments*. We show that any sufficiently large tournament $T \in \mathcal{T}_{\text{apex}}$ satisfies $\text{pn}(T) = \widetilde{\text{ex}}(T) + 1$ (see Theorem 5.1). Denote by $\mathcal{T}_{\text{reg}}$ the class of regular tournaments and let $\mathcal{T}_{\text{excep}} := \mathcal{T}_{\text{apex}} \cup \mathcal{T}_{\text{reg}}$. The tournaments in $\mathcal{T}_{\text{excep}}$ are called *exceptional*. We conjecture that the tournaments in $\mathcal{T}_{\text{excep}}$ are the only ones which do not satisfy equality in Fact 1.4.

Conjecture 1.7. *There exists $n_0 \in \mathbb{N}$ such that any tournament $T \notin \mathcal{T}_{\text{excep}}$ on $n \geqslant n_0$ vertices satisfies* $\text{pn}(T) = \widetilde{\text{ex}}(T)$.

We prove an approximate version of this conjecture (see Corollary 1.9). Moreover, in Theorem 1.8, we prove Conjecture 1.7 exactly unless n is odd and T is extremely close to being a regular tournament.

Theorem 1.8. *For all $\beta > 0$, there exists $n_0 \in \mathbb{N}$ such that the following holds. If T is a tournament on $n \geqslant n_0$ vertices such that $T \notin \mathcal{T}_{\text{excep}}$ and*

(a) $\widetilde{\text{ex}}(T) \geqslant \frac{n}{2} + \beta n$*; or*
(b) $N^+(T), N^-(T) \geqslant \beta n$, *where* $N^+(T) := |\{v \in V(T) \mid \text{ex}_T^+(v) > 0\}| + \widetilde{\text{ex}}(T) - \text{ex}(T)$ *and* $N^-(T) := |\{v \in V(T) \mid \text{ex}_T^-(v) > 0\}| + \widetilde{\text{ex}}(T) - \text{ex}(T)$*;*

then $\text{pn}(T) = \widetilde{\text{ex}}(T)$.

In Section 6, we will derive Theorem 1.3 (that is, the exact solution when n is even) from Theorem 1.8. This will make use of the fact that $\widetilde{\text{ex}}(T) = \text{ex}(T)$ for n even (see Proposition 6.1). We will also derive an approximate version of Conjecture 1.7 from Theorem 1.8.

Corollary 1.9. *For all $\beta > 0$, there exists $n_0 \in \mathbb{N}$ such that, for any tournament T on $n \geqslant n_0$ vertices,* $\text{pn}(T) \leqslant \widetilde{\text{ex}}(T) + \beta n$.

Note that Theorem 1.8(b) corresponds to the case where linearly many different vertices can be used as endpoints of paths in a path decomposition of size $\widetilde{\text{ex}}(T)$. Indeed, let T be a tournament and $\mathcal{P}$ be a path decomposition of T. Then, as mentioned above, each $v \in V(T)$ must be the starting point of at least $\text{ex}_T^+(v)$ paths in $\mathcal{P}$. Thus, for any tournament T, $N^+(T)$ is the maximum number of distinct vertices which can be a starting point of a path in a path decomposition of T of size $\widetilde{\text{ex}}(T)$ and similarly for $N^-(T)$ and the ending points of paths.

One can show that almost all large tournaments satisfy $\text{ex}(T) = n^{\frac{3}{2}+o(1)}$. Indeed, consider a tournament T on n vertices, where the orientation of each edge is chosen uniformly at random, independently of all other orientations. For the upper bound on $\text{ex}(T)$, one can simply apply a Chernoff bound to show that for a given vertex v and $\varepsilon > 0$, we have $\text{ex}_T^+(v) \leqslant n^{\frac{1}{2}+\varepsilon}$ with probability $1 - o(\frac{1}{n})$. The result follows by a union bound over all vertices. For the lower bound, let X denote the number of vertices v with $d_T^-(v) \in [\frac{n}{2} - 2\sqrt{n}, \frac{n}{2} - \sqrt{n}]$. Then it is easy to see that, for large enough n, we have $\mathbb{E}[X] \geqslant \frac{n}{10^4}$, say. Moreover, Chebyshev's inequality can be used to show that, with probability $1 - o(1)$, we have $X \geqslant \frac{n}{2\cdot 10^4}$, again with room to spare. Thus, Theorem 1.8 implies the following.

Corollary 1.10. *As $n \to \infty$, the proportion of tournaments T on n vertices satisfying* $\mathrm{pn}(T) = \widetilde{\mathrm{ex}}(T)$ *tends to* 1.

Note that the case when n is even already follows from Theorem 1.2(a). Corollary 1.10 is an analogue of a result of Kühn and Osthus [13], which states that almost all sufficiently large tournaments T contain $\delta^0(T) := \min\{d_T^+(v), d_T^-(v) \mid v \in V(T)\}$ edge-disjoint Hamilton cycles and which proved a conjecture of Erdős (see [20]).

Rather than random tournaments, it is also natural to consider the following related question: for which densities p is the random binomial digraph $D_{n,p}$ likely to be consistent? Very recently, significant partial results towards this question were obtained by Espuny Díaz, Patel, and Stroh [7].

Finally, we will see in Section 14 that our methods give a short proof of (a stronger version of) a result of Osthus and Staden [19], which guarantees an approximate decomposition of regular 'robust outexpanders' of linear degree into Hamilton cycles and which was used as a tool in the proof of Kelly's conjecture [12].

1.3 | Organisation of the paper

This paper is organised as follows. In Section 2, we give a proof overview of Theorem 1.8. Notation will be introduced in Section 3, while tools and preliminary results will be collected in Section 4. We consider exceptional tournaments in Section 5 and derive Theorem 1.3 and Corollary 1.9 from Theorem 1.8 in Section 6. Then, Sections 7–13 are devoted to proving Theorem 1.8. In particular, the approximate decomposition step is carried out in Section 7 and Theorem 1.8 is derived in Section 10. Finally, in Section 14, we discuss Hamilton decompositions of robust outexpanders and conclude with a remark about Conjecture 1.7.

2 | PROOF OVERVIEW

2.1 | Robust outexpanders

Our proof of Theorem 1.8 will be based on the concept of robust outexpanders. Roughly speaking, a digraph D is called a *robust outexpander* if, for any set $S \subseteq V(D)$ which is neither too small nor too large, there exist significantly more than $|S|$ vertices with many inneighbours in S. (Robust outexpanders will be defined formally in Section 4.1.) Any (almost) regular tournament is a robust outexpander and we will use that this property is inherited by random subdigraphs. The main result of [12] states that any regular robust outexpander of linear degree has a Hamilton decomposition (see Theorem 4.9). We can apply this to obtain an optimal path decomposition in the following setting. Let D be a digraph on n vertices, $0 < \eta < 1$, and suppose that $X^+ \cup X^- \cup X^0$ is a partition of $V(D)$ such that $|X^+| = |X^-| = \eta n$ and the following hold.

$$\begin{aligned} &\textit{Each } v \in X^0 \textit{ satisfies } d_D^+(v) = \eta n = d_D^-(v).\\ &\textit{Each } v \in X^+ \textit{ satisfies } d_D^+(v) = \eta n \textit{ and } d_D^-(v) = \eta n - 1.\\ &\textit{Each } v \in X^- \textit{ satisfies } d_D^+(v) = \eta n - 1 \textit{ and } d_D^-(v) = \eta n.\end{aligned} \tag{†}$$

Then, the digraph D' obtained from D by adding a new vertex v with $N^+_{D'}(v) = X^+$ and $N^-_{D'}(v) = X^-$ is ηn-regular. Thus, if D is a robust outexpander, then so is D' and there exists a decomposition of D' into Hamilton cycles. This induces a decomposition $\mathcal{P}$ of D into ηn Hamilton paths, where each vertex in X^+ is the starting point of exactly one path in $\mathcal{P}$ and each vertex in X^- is the ending point of exactly one path in $\mathcal{P}$. This is formalised in Corollary 4.10. (A similar observation was already made and used in [16].) Our main strategy will be to reduce our tournament to a digraph of the above form. This will be achieved as follows.

2.2 | Simplified approach for well-behaved tournaments

Let $\beta > 0$ and fix additional constants such that $0 < \frac{1}{n_0} \ll \varepsilon \ll \gamma \ll \eta \ll \beta$. Let T be a tournament on $n \geqslant n_0$ vertices. Note that by Theorem 1.2, we may assume that $\widetilde{\mathrm{ex}}(T) \leqslant \varepsilon^2 n^2$. Moreover, for simplicity, we first also assume that each $v \in V(T)$ satisfies $|\,\mathrm{ex}_T(v)| \leqslant \varepsilon n$ (that is, T is almost regular), $\widetilde{\mathrm{ex}}(T) = \mathrm{ex}(T)$, and both $|\{v \in V(T) \mid \mathrm{ex}^+_T(v) > 0\}|, |\{v \in V(T) \mid \mathrm{ex}^-_T(v) > 0\}| \geqslant \eta n$. In Section 2.3, we will briefly explain how the argument can be generalised if any of these conditions is not satisfied. (An in-depth discussion of these modifications can be found in Sections 8 and 9.)

Since T is almost regular, it is a robust outexpander. Let Γ be obtained by including each edge of T with probability γ. Using Chernoff bounds, we may assume that Γ is a robust outexpander of density almost γ and $D := T \setminus \Gamma$ is almost regular. The digraph Γ will serve two purposes. Firstly, its robust outexpansion properties will be used to construct an approximate path decomposition of T. Secondly, provided few edges of Γ are used throughout this approximate decomposition, it will guarantee that the leftover (consisting of all of those edges of T not covered by the approximate path decomposition) is still a robust outexpander, as required to complete our decomposition of T in the way described in Section 2.1.

Fix $X^+ \subseteq \{v \in V(T) \mid \mathrm{ex}^+_T(v) > 0\}$ and $X^- \subseteq \{v \in V(T) \mid \mathrm{ex}^-_T(v) > 0\}$, both of size ηn and denote $X^0 := V(T) \setminus (X^+ \cup X^-)$. Our goal is then to find an approximate path decomposition $\mathcal{P}$ of T such that $|\mathcal{P}| = \widetilde{\mathrm{ex}}(T) - \eta n$ and such that the leftover $T \setminus E(\mathcal{P})$ satisfies the degree conditions in (†). Thus, it suffices to show that $\mathcal{P}$ satisfies the following.

(i) Each $v \in X^+$ is the starting point of exactly $\mathrm{ex}^+_T(v) - 1$ paths in $\mathcal{P}$, while each $v \in V(T) \setminus X^+$ is the starting point of exactly $\mathrm{ex}^+_T(v)$ paths in $\mathcal{P}$. Similarly, each $v \in X^-$ is the ending point of exactly $\mathrm{ex}^-_T(v) - 1$ paths in $\mathcal{P}$, while each $v \in V(T) \setminus X^-$ is the ending point of exactly $\mathrm{ex}^-_T(v)$ paths in $\mathcal{P}$.

(ii) Each $v \in V(T) \setminus (X^+ \cup X^-)$ is the internal vertex of exactly $\frac{(n-1)-|\,\mathrm{ex}_T(v)|}{2} - \eta n$ paths in $\mathcal{P}$, while each $v \in X^+ \cup X^-$ is the internal vertex of exactly $\frac{(n-1)-|\,\mathrm{ex}_T(v)|}{2} - \eta n + 1$ paths in $\mathcal{P}$.

Indeed, (i) ensures that $|\mathcal{P}| = \mathrm{ex}(T) - \eta n$ and each vertex has the desired excess in $T \setminus E(\mathcal{P})$, namely $\mathrm{ex}_{T\setminus E(\mathcal{P})}(v) = +1$ if $v \in X^+$, $\mathrm{ex}_{T\setminus E(\mathcal{P})}(v) = -1$ if $v \in X^-$, and $\mathrm{ex}_{T\setminus E(\mathcal{P})}(v) = 0$ otherwise. In addition, (ii) ensures that the degrees in $T \setminus E(\mathcal{P})$ satisfy (†).

Recall that, by assumption, T is almost regular. Thus, in a nutshell, (i) and (ii) state that we need to construct edge-disjoint paths with specific endpoints and such that each vertex is covered by about $(\frac{1}{2} - \eta)n$ paths. To ensure the latter, we will in fact approximately decompose T into about $(\frac{1}{2} - \eta)n$ spanning sets of internally vertex-disjoint paths. To ensure the former, we will start by constructing $(\frac{1}{2} - \eta)n$ auxiliary digraphs on $V(T)$ such that, for each $v \in V(T)$, the total number of edges starting (and ending) at v is the number of paths that we want to start (and end, respectively) at v. These auxiliary digraphs will be called *layouts*. These layouts are constructed in Section 13. Then, it will be enough to construct, for each layout L, a spanning set $\mathcal{P}_L$ of paths, called a *spanning configuration of shape* L, such that each path $P \in \mathcal{P}_L$ corresponds to some edge $e \in E(L)$ and such that the starting and ending points of P equal those of e. Roughly speaking, a spanning configuration $\mathcal{P}_L$ is a set of internally vertex-disjoint paths and L indicates the starting and ending points of these paths. (See Section 7 for further motivation of layouts.)

These spanning configurations will be constructed one by one as follows. (See also Figure 1.) At each stage, given a layout L, fix an edge $yz \in E(L)$. Then, using the robust outexpansion properties of (the remainder of) Γ, find short internally vertex-disjoint paths with endpoints corresponding to the endpoints of the edges in $L \setminus \{yz\}$. Denote by $\mathcal{P}'_L$ the set containing these paths. Then, it only remains to construct a path from y to z spanning $V(T) \setminus V(\mathcal{P}'_L)$. We achieve this as follows.

Let D' and Γ' be obtained from (the remainders of) $D - V(\mathcal{P}'_L)$ and $\Gamma - V(\mathcal{P}'_L)$ by merging the vertices y and z into a new vertex v_{yz} whose outneighbourhood is the outneighbourhood of y and whose inneighbourhood is the inneighbourhood of z. Then, observe that a Hamilton cycle of $D' \cup \Gamma'$ corresponds to a path from y to z of T which spans $V(T) \setminus V(\mathcal{P}'_L)$. To construct such a Hamilton cycle of $D' \cup \Gamma'$, one can simply use the fact that Γ' is a robust outexpander to find a

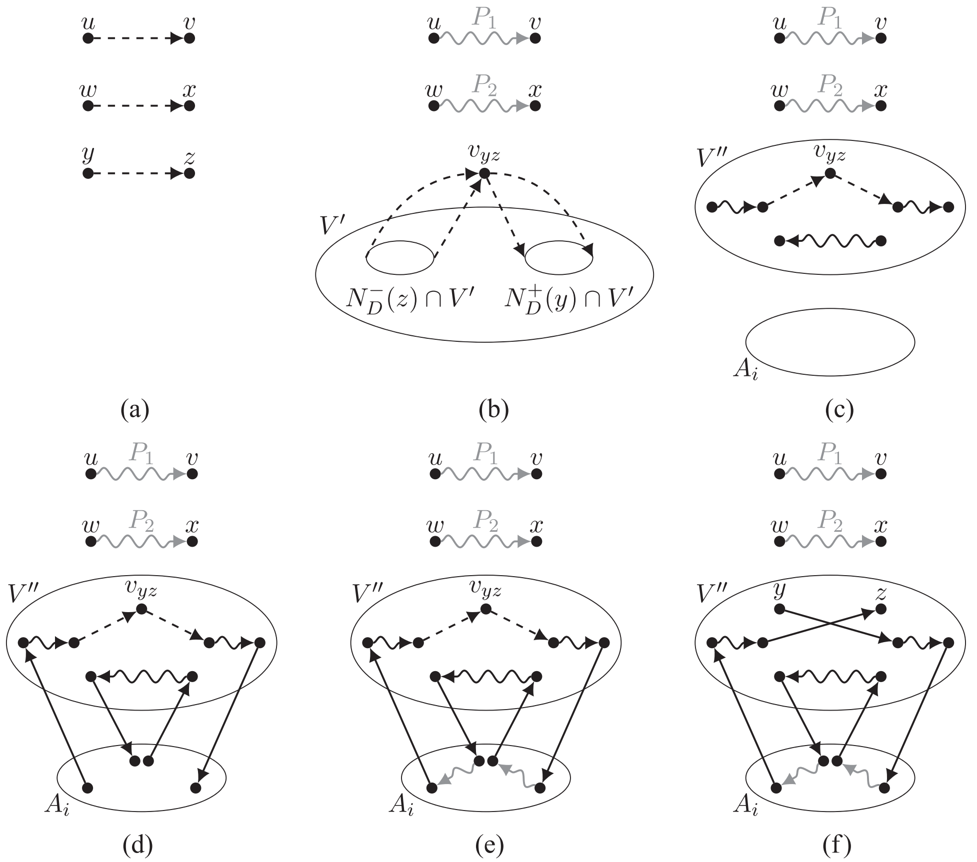


FIGURE 1 Constructing a spanning set of vertex-disjoint paths in $D \cup \Gamma$ with prescribed endpoints and few edges of Γ. Dashed edges represent auxiliary edges, full black edges represent edges of D, and grey edges represent edges of Γ. Wavy black edges represent paths in D and wavy grey edges represent paths in Γ.

Hamilton cycle. However, if we proceed in this way, then the robust outexpansion property of Γ' might be destroyed before constructing all the desired spanning configurations.

So instead we construct a Hamilton cycle of $D' \cup \Gamma'$ with only few edges in Γ' as follows. As a preparatory step in advance of choosing the spanning configurations, we consider a random partition of $V(T)$ into $A_1, \dots, A_a$ each of size $\frac{n}{a}$. We choose one A_i for the current layout. We restrict ourselves to use Γ' inside A_i only. Note that $\Gamma'[A_i]$ is a robust outexpander and $D' - A_i$ is a dense almost regular digraph. The latter means that we can find a spanning linear forest F in $D' - A_i$ which has few components. Since F has few components, we can then greedily extend the components of F to obtain a linear forest $F' \subseteq D'$ which covers all the vertices in $V(D') \setminus A_i$ and whose endpoints are all in A_i. Finally, we use the robust outexpansion properties of $\Gamma'[A_i]$ to close F' in to a Hamilton cycle of $D' \cup \Gamma'$. None of the A_i will be used too often when constructing the spanning configurations, which will mean that $\Gamma'[A_i]$ is always a robust outexpander. When the desired spanning configuration is a Hamilton cycle, this approach of finding many edge-disjoint spanning configurations by first finding a suitable linear forest F, and then tying F together via some small set A_i (with varying A_i in order to avoid over-using a particular set of vertices) has been used successfully in several earlier papers (for example, [8, 12]). This construction of spanning configurations is carried out in Section 7.

We illustrate this argument with the following example. Suppose that L is a layout consisting of three edges uv, wx, and yz (Figure 1(a)). We want to construct a spanning configuration of shape L, that is, a set of paths which consists of a path from u to v, a path from w to x, and a path from y to z such that these three paths are vertex-disjoint and altogether cover all the vertices of T. First, we use robust outexpansion to construct a short path P_1 from u to v and a short path P_2 from w to x in Γ (Figure 1(b)). Denote $V' := V(T) \setminus (V(P_1) \cup V(P_2) \cup \{y, z\})$. The goal is now to construct a path from y to z which covers all the vertices in V'. To do so, we replace y and z by an auxiliary vertex v_{yz} whose outneighbourhood is $N^+(v_{yz}) := N^+_D(y) \cap V'$ and whose inneighbourhood is $N^-(v_{yz}) := N^-_D(z) \cap V'$ (Figure 1(b)) and we consider a small preselected random set of vertices $A_i \subseteq V'$. It is then enough to find a cycle on $V' \cup \{v_{yz}\}$ which uses Γ inside A_i only. Denote $V'' := (V' \cup \{v_{yz}\}) \setminus A_i$. Firstly, we use almost regularity of D to find a spanning linear forest on V'' which consists of few components (Figure 1(c)). Secondly, we use the large degree of D to extend the endpoints of the linear forest to A_i (Figure 1(d)). Finally, we use the robust outexpansion of Γ to close the linear forest into a cycle which covers all the vertices in A_i (Figure 1(e)). This gives a cycle on $V' \cup \{v_{yz}\}$. Replacing the auxiliary vertex v_{yz} by the original vertices y and z, we obtain a path from y to z which covers all the vertices in V', as desired (Figure 1(f)).

2.3 | General tournaments

For a general tournament T, we adapt the above argument as follows. Let W be the set of vertices $v \in V(T)$ such that $|\operatorname{ex}_T(v)| > \varepsilon n$. If $W \neq \emptyset$, then T is no longer almost regular and we cannot proceed as above. However, since $\operatorname{ex}(T) \leqslant \varepsilon^2 n^2$, $|W|$ is small. Thus, we can start with a cleaning procedure which efficiently decreases the excess and degree at W by taking out a few edge-disjoint paths. The corresponding proof is deferred until Section 12, as it is quite involved and carrying out the other steps first helps to give a better picture of the overall argument. Then, we apply the above argument to (the remainder of) $T - W$. We incorporate all remaining edges at W in the approximate decomposition by generalising the concept of a layout introduced above. This is discussed in more detail in Section 9.

If $|\{v \in V(T) \mid \mathrm{ex}_T^+(v) > 0\}| < \eta n$ but $\widetilde{\mathrm{ex}}(T) = \mathrm{ex}(T)$, say, then we cannot choose $X^+ \subseteq \{v \in V(T) \mid \mathrm{ex}_T^+(v) > 0\}$ of size ηn. We circumvent this problem as follows. Select a small set of vertices W_A such that $\sum_{v \in W_A} \mathrm{ex}_T^+(v) \geqslant \eta n$ and let A be a set of ηn edges such that the following hold. Each edge in A starts in W_A and ends in $V(T) \setminus W_A$. Moreover, each $v \in W_A$ is the starting point of at most $\mathrm{ex}_T^+(v)$ edges in A and each $v \in V(T) \setminus W_A$ is the ending point of at most one edge in A. We will call A an *absorbing set of starting edges*. Let V_A be the set of ending points of the edges in A. Then, $V_A \subseteq V(T) \setminus W_A$. Observe that any path which starts in V_A and is disjoint from W_A can be extended to a path starting in W_A using an edge from A. Thus, we can let the ending points of the edges in A play the role of X^+ and add the vertices in W_A to W so that, at the end of the approximate decomposition, the only remaining edges at W_A are the edges in A. Thus, in the final decomposition step, we can use the edges in A to extend the paths starting at X^+ into paths starting in W_A. (See Section 8.2 for details.) If $|\{v \in V(T) \mid \mathrm{ex}_T^-(v) > 0\}| < \eta n$, then we proceed analogously.

If $\widetilde{\mathrm{ex}}(T) > \mathrm{ex}(T)$, then not all paths will 'correspond' to some excess. To be able to adopt a unified approach, we will choose which additional endpoints to use at the beginning and artificially add excess to those vertices. This then enables us to proceed as if $\mathrm{ex}(T) = \widetilde{\mathrm{ex}}(T)$. More precisely, we will choose a set $U^* \subseteq \{v \in V(T) \mid \mathrm{ex}_T(v) = 0\}$ of size $\widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T)$ and we will treat the vertices in U^* in the same way as we treat those with $\mathrm{ex}_T^+(v) = 1$ and $\mathrm{ex}_T^-(v) = 1$. Note that selecting additional endpoints in this way maximises the number of distinct endpoints, which will enable us to choose $X^+ \subseteq \{v \in V(T) \mid \mathrm{ex}_T^+(v) > 0\} \cup U^*$ and/or $X^- \subseteq \{v \in V(T) \mid \mathrm{ex}_T^-(v) > 0\} \cup U^*$ when $N^+(T) = |\{v \in V(T) \mid \mathrm{ex}_T^+(v) > 0\}| + \widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T) \geqslant \eta n$ and/or $N^-(T) = |\{v \in V(T) \mid \mathrm{ex}_T^-(v) > 0\}| + \widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T) \geqslant \eta n$, and use absorbing edges otherwise, that is, if condition (b) fails in Theorem 1.8. More details of this approach are given in Section 8.2.

3 | NOTATION

In this section, we collect the notation that will be used throughout this paper. The non-standard pieces of notation will be recalled to the reader when first needed.

3.1 | Hierarchies

We denote by $\mathbb{N}$ the set of natural numbers (including 0). Let $a, b, c \in \mathbb{R}$. We write $a = b \pm c$ if $b - c \leqslant a \leqslant b + c$. For simplicity, we use hierarchies instead of explicitly calculating the values of constants for which our statements hold. More precisely, if we write $0 < a \ll b \ll c \leqslant 1$ in a statement, we mean that there exist non-decreasing functions $f : (0,1] \longrightarrow (0,1]$ and $g : (0,1] \longrightarrow (0,1]$ such that the statement holds for all $0 < a, b, c \leqslant 1$ satisfying $b \leqslant f(c)$ and $a \leqslant g(b)$. Hierarchies with more constants are defined in a similar way. We assume large numbers to be integers and omit floors and ceilings, provided this does not affect the argument.

3.2 | $\pm$-notation

In general, a statement $C^\pm$ will mean that both statements C^+ and C^- hold simultaneously. If used in the form that $C^\pm$ is the statement '$\mathcal{A}^\pm$ implies $\mathcal{B}^\pm$', the convention means that '$\mathcal{A}^+$ implies $\mathcal{B}^+$'

and '$\mathcal{A}^-$ implies $\mathcal{B}^-$'. Similarly, the statement '$\mathcal{A}^\pm$ implies $\mathcal{B}^\mp$' means that '$\mathcal{A}^+$ implies $\mathcal{B}^-$' and '$\mathcal{A}^-$ implies $\mathcal{B}^+$'.

3.3 | Graphs and digraphs

A *digraph* D is a directed graph without loops which contains, for any distinct vertices u and v of D, at most two edges between u and v, at most one in each direction. A digraph D is called an *oriented graph* if it contains, for any distinct vertices u and v of D, at most one edge between u and v; that is, D can be obtained by orienting the edges of an undirected graph.

Let G be a (di)graph. We denote by $V(G)$ and $E(G)$ the vertex and edge sets of G, respectively. We say G is *non-empty* if $E(G) \neq \emptyset$. Let $u, v \in V(G)$ be distinct. If G is undirected, then we write uv for an edge between u and v. If G is directed, then we write uv for an edge directed from u to v, where u and v are called the *starting* and *ending points* of the edge uv, respectively. Let $A, B \subseteq V(G)$ be disjoint. Denote $E_A(G) := \{e \in E(G) \mid V(e) \cap A \neq \emptyset\}$. Moreover, we write $G[A, B]$ for the undirected graph with vertex set $A \cup B$ and edge set $\{ab \in E(G) \mid a \in A, b \in B\}$ and $e(A, B) := |E(G[A, B])|$.

Given $S \subseteq V(G)$, we write $G[S]$ for the sub(di)graph of G induced on S and $G - S$ for the (di)graph obtained from G by deleting all vertices in S. Given $E \subseteq E(G)$, we write $G \setminus E$ for the (di)graph obtained from G by deleting all edges in E. Similarly, given a sub(di)graph $H \subseteq G$, we write $G \setminus H := G \setminus E(H)$. If F is a set of non-edges of G, then we write $G \cup F$ for the (di)graph obtained by adding all edges in F. Given a (di)graph H, if G and H are edge-disjoint, then we write $G \cup H$ for the (di)graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

3.4 | Degrees and neighbourhood

Assume G is an undirected graph. For any $v \in V(G)$, we write $N_G(v)$ for the *neighbourhood* of v in G and $d_G(v)$ for the *degree* of v in G. Given $S \subseteq V(G)$, we denote $N_G(S) := \bigcup_{v \in S} N_G(v)$.

Let D be a digraph and $v \in V(D)$. We write $N_D^+(v)$ and $N_D^-(v)$ for the *outneighbourhood* and *inneighbourhood* of v in D, respectively, and define the *neighbourhood* of v in D as $N_D(v) := N_D^+(v) \cup N_D^-(v)$. We denote by $d_D^+(v)$ and $d_D^-(v)$ the *outdegree* and *indegree* of v in D, respectively, and define the *degree* of v in D as $d_D(v) := d_D^+(v) + d_D^-(v)$. Denote $d_D^{\min}(v) := \min\{d_D^+(v), d_D^-(v)\}$ and $d_D^{\max}(v) := \max\{d_D^+(v), d_D^-(v)\}$. If $d_D^+(v) \neq d_D^-(v)$, then define

$$N_D^{\min}(v) := \begin{cases} N_D^+(v) & \text{if } d_D^{\min} = d_D^+(v), \\ N_D^-(v) & \text{if } d_D^{\min} = d_D^-(v), \end{cases} \quad \text{and} \quad N_D^{\max}(v) := \begin{cases} N_D^+(v) & \text{if } d_D^{\max} = d_D^+(v), \\ N_D^-(v) & \text{if } d_D^{\max} = d_D^-(v). \end{cases} \tag{3.1}$$

The *minimum semidegree* of D is defined as $\delta^0(D) := \min\{d_D^{\min}(v) \mid v \in V(D)\}$ and, similarly, $\Delta^0(D) := \max\{d_D^{\max}(v) \mid v \in V(D)\}$ is called the *maximum semidegree* of D. Define the *minimum degree* and *maximum degree* of D by $\delta(D) := \min\{d_D(v) \mid v \in V(D)\}$ and $\Delta(D) := \max\{d_D(v) \mid v \in V(D)\}$, respectively. Given $S \subseteq V(D)$, we denote $N_D^\pm(S) := \bigcup_{v \in S} N_D^\pm(v)$ and $N_D(S) := \bigcup_{v \in S} N_D(v)$.

Let D be a digraph on n vertices. We say D is *r-regular* if, for any $v \in V(D)$, $d_D^+(v) = d_D^-(v) = r$. We say D is *regular* if it is r-regular for some $r \in \mathbb{N}$. Let $\varepsilon, \delta > 0$. We say D is *(δ, ε)-almost regular* if, for each $v \in V(D)$, both $d_D^+(v) = (\delta \pm \varepsilon)n$ and $d_D^-(v) = (\delta \pm \varepsilon)n$.

3.5 | Multidigraphs

Let A and B be multisets. The *support* of A is the set $S(A) := \{a \mid a \in A\}$. For each $a \in S(A)$, we denote by $\mu_A(a)$ the *multiplicity* of a in A. For any $a \notin S(A)$, we define $\mu_A(a) := 0$. We write $A \cup B$ for the multiset with support $S(A \cup B) := S(A) \cup S(B)$ and such that, for each $a \in S(A \cup B)$, $\mu_{A\cup B}(a) := \mu_A(a) + \mu_B(b)$. We denote by $A \setminus B$ the multiset with support $S(A \setminus B) := \{a \in S(A) \mid \mu_A(a) > \mu_B(a)\}$ and such that, for each $a \in S(A \setminus B)$, $\mu_{A\setminus B}(a) := \mu_A(a) - \mu_B(a)$. We say A is a *submultiset* of B, denoted $A \subseteq B$, if $S(A) \subseteq S(B)$ and, for each $a \in S(A)$, $\mu_A(a) \leqslant \mu_B(a)$.

By a *multidigraph*, we mean a directed graph where we allow multiple edges but no loops. All the notation and definitions introduced thus far extend naturally to multidigraphs, with unions/differences of edge sets now interpreted as multiset unions/differences. In a multidigraph, two instances of an edge are considered to be distinct. In particular, given a multidigraph D, we say $D_1, D_2 \subseteq D$ are *edge-disjoint* submultidigraphs of D if, for any $e \in E(D)$, $\mu_{E(D_1)}(e) + \mu_{E(D_2)}(e) \leqslant \mu_{E(D)}$.

3.6 | Paths

In this paper, all paths and cycles are directed, with edges consistently oriented. The *length* of a path P, denoted by $e(P)$, is the number of edges it contains. A path on one vertex, that is, a path of length 0 is called *trivial*. Let $P = v_1v_2 \dots v_\ell$ be a path. We say v_1 is the *starting point* of P and v_ℓ is the *ending point* of P. We say v is an *endpoint* of a path P if v is the starting or ending point of P. We say $v_2, \dots, v_{\ell-1}$ are *internal vertices* of P. We write $V^+(P) = \{v_1\}$, $V^-(P) = \{v_\ell\}$, and $V^0(P) = \{v_2, \dots, v_{\ell-1}\}$. We say that a path P is a (u, v)-*path* if $V^+(P) = \{u\}$ and $V^-(P) = \{v\}$. Given $1 \leqslant i < j \leqslant \ell$, we denote $v_iPv_j := v_iv_{i+1} \dots v_j$. A *linear forest* is a set of pairwise vertex-disjoint paths.

Similarly, given a (multi)set $\mathcal{P}$ of paths, we write $V^+(\mathcal{P})$ for the set of vertices which are the starting point of a path in $\mathcal{P}$. Similarly, we write $V^-(\mathcal{P})$ for the set of vertices which are the ending point of a path in $\mathcal{P}$ and $V^0(\mathcal{P})$ for the set of vertices which are an internal vertex of a path in $\mathcal{P}$. (Note that $V^\pm(\mathcal{P})$ and $V^0(\mathcal{P})$ are sets and not multisets.)

Given a directed edge xy and a path P, we say P has *shape xy* if P is an (x, y)-path. Similarly, let E be a (multi)set of (auxiliary) directed edges and $\mathcal{P}$ be a (multi)set of paths. We say $\mathcal{P}$ has *shape E* if there exists a bijection $\phi : E \longrightarrow \mathcal{P}$ such that, for each $xy \in E$, $\phi(xy)$ is an (x, y)-path.

For convenience, a (multi)set $\mathcal{P}$ of paths will sometimes be viewed as the (multi)digraph consisting of their union. In particular, given a (multi)set $\mathcal{P}$ of paths, we write $V(\mathcal{P})$ for the set of vertices of $\mathcal{P}$ and $E(\mathcal{P})$ for the (multi)set of edges of $\mathcal{P}$, that is, $V(\mathcal{P})$ is the set $\bigcup_{P\in\mathcal{P}} V(P)$ and $E(\mathcal{P})$ is the (multi)set $\bigcup_{P\in\mathcal{P}} E(P)$. (Note that $V(\mathcal{P})$ is a set and not a multiset.) For any $v \in V(\mathcal{P})$, we write $d^\pm_{\mathcal{P}}(v)$ and $\mathrm{ex}^\pm_{\mathcal{P}}(v)$ for the in/outdegree and positive/negative excess of v in $\mathcal{P}$ when viewed as a multidigraph, that is, $d^\pm_{\mathcal{P}}(v) := d^\pm_{\bigcup \mathcal{P}}(v)$ and $\mathrm{ex}^\pm_{\mathcal{P}}(v) := \mathrm{ex}^\pm_{\bigcup \mathcal{P}}(v)$. We define $d_{\mathcal{P}}(v)$ and $\mathrm{ex}_{\mathcal{P}}(v)$ similarly. For any digraph D, we denote $D \setminus \mathcal{P} := D \setminus E(\mathcal{P})$.

3.7 | Subdivisions and contractions

Let D and D' be digraphs and $uv \in E(D)$. We say D' is obtained from D by *subdividing* uv, if $V(D') = V(D) \cup \{w\}$, for some $w \notin V(D)$, and $E(D') = (E(D) \setminus \{uv\}) \cup \{uw, wv\}$. We say D' is a *subdivision* of D if D' is obtained by successively subdividing some edges of D. Let P be a (u, v)-path satisfying $V^0(P) \cap V(D) = \emptyset$. We say D' is obtained from D by *subdividing uv into P*, if $V(D') = V(D) \cup V^0(P)$ and $E(D') = (E(D) \setminus \{uv\}) \cup E(P)$. Similarly, given an induced (u, v)-path $P \subseteq D$, we say D' is obtained from D by *contracting the path P into an edge uv* if $V(D') = V(D) \setminus V^0(P)$ and $E(D') = (E(D) \setminus E(P)) \cup \{uv\}$.

3.8 | Decompositions

Let D be a digraph. A *decomposition* of D is set $\mathcal{D}$ of non-empty edge-disjoint subdigraphs of D such that every edge of D is in one of these subdigraphs. A *(Hamilton) path decomposition* of D is a decomposition $\mathcal{P}$ of D such that each subdigraph $P \in \mathcal{P}$ is a (Hamilton) path of D. Similarly, a *(Hamilton) cycle decomposition* of D is a decomposition $\mathcal{C}$ of D such that each subdigraph $C \in \mathcal{C}$ is a (Hamilton) cycle of D. By a *Hamilton decomposition* of D, we mean a Hamilton cycle decomposition of D.

4 | PRELIMINARIES

In this section, we introduce some tools which will be used throughout the rest of the paper.

4.1 | Robust outexpanders

Let D be a digraph on n vertices. Given $S \subseteq V(D)$, the *ν-robust outneighbourhood of S* is the set $RN^+_{\nu,D}(S) := \{v \in V(D) \mid |N^-_D(v) \cap S| \geqslant \nu n\}$. We say that D is a *robust (ν, τ)-outexpander* if, for any $S \subseteq V(D)$ satisfying $\tau n \leqslant |S| \leqslant (1-\tau)n$, $|RN^+_{\nu,D}(S)| \geqslant |S| + \nu n$.

In this section, we state some useful properties of robust outexpanders. First, observe that the next fact follows immediately from the definition.

Fact 4.1. *Let D be a robust (ν, τ)-outexpander. Then, for any $\nu' \leqslant \nu$ and $\tau' \geqslant \tau$, D is a robust (ν', τ')-outexpander.*

The following lemma states that robust outexpansion is preserved when few edges are removed and/or few vertices are removed and/or added. This follows immediately from the definition and so we omit details. (Note that a similar result was already observed, for example, in [11, Lemma 4.8].)

Lemma 4.2. *Let $0 < \varepsilon \leqslant \nu \leqslant \tau \leqslant 1$. Let D be a robust (ν, τ)-outexpander on n vertices.*

(a) *If D' is obtained from D by removing at most εn inedges and at most εn outedges at each vertex, then D' is a robust $(\nu - \varepsilon, \tau)$-outexpander.*
(b) *Suppose that $\tau \geqslant (1 + 2\tau)\varepsilon$. If D' is obtained from D by adding or removing at most εn vertices, then D' is a robust $(\nu - \varepsilon, 2\tau)$-outexpander.*

One can easily show that the τ-parameter of robust outexpansion can be decreased when the minimum semidegree is large. This will enable us to state some results of [11, 12] with slightly adjusted parameters.

Lemma 4.3. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leqslant \frac{\delta}{2} \leqslant 1$. Assume D is a robust $(\nu, \frac{\delta}{2})$-outexpander on n vertices satisfying $\delta^0(D) \geqslant \delta n$. Then, D is a robust (ν, τ)-outexpander.*

Proof. Let $S \subseteq V(D)$ satisfy $\tau n \leqslant |S| \leqslant (1-\tau)n$ and denote $T := RN^+_{\nu,D}(S)$. We need to show that $|T| \geqslant |S| + \nu n$. If $\frac{\delta n}{2} \leqslant |S| \leqslant (1-\frac{\delta}{2})n$, then we are done by assumption. If $|S| > (1-\frac{\delta}{2})n$, then each $v \in V(D)$ satisfies $|N^{\pm}_D(v) \cap S| \geqslant \delta n - |V(D) \setminus S| \geqslant \frac{\delta n}{2} \geqslant \nu n$ and so $T = V(D)$.

We may therefore assume that $|S| < \frac{\delta n}{2}$. Note that the number of edges of D which start in S is $\sum_{v \in S} d^+_D(v)$. By definition of T, we have

$$|S|\delta n \overset{\delta^0(D) \geqslant \delta n}{\leqslant} \sum_{v \in S} d^+_D(v) \leqslant |T||S| + (n-|T|)\nu n \leqslant |T||S| + \nu n^2.$$

Therefore, $|T| \geqslant \frac{|S|\delta n - \nu n^2}{|S|} \geqslant \delta n - \frac{\nu n}{\tau} \geqslant \frac{\delta n}{2} + \nu n \geqslant |S| + \nu n$, as desired. □

The next result states that oriented graphs of sufficiently large minimum semidegree are robust outexpanders.

Lemma 4.4 [12, Lemma 13.1]. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leqslant \varepsilon \leqslant 1$. Let D be an oriented graph on n vertices with $\delta^0(D) \geqslant (\frac{3}{8} + \varepsilon)n$. Then, D is a robust (ν, τ)-outexpander.*

The next lemma follows easily from the definition of robust outexpansion and states that robust outexpanders of linear minimum semidegree have small diameter.

Lemma 4.5 [11, Lemma 6.6]. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leqslant \frac{\delta}{2} \leqslant 1$. Let D be a robust (ν, τ)-outexpander on n vertices with $\delta^0(D) \geqslant \delta n$. Then, for any $x, y \in V(D)$, D contains an (x, y)-path of length at most ν^{-1}.*

One can iteratively apply Lemma 4.5 to obtain a small set of short internally vertex-disjoint paths with prescribed endpoints. After each application of Lemma 4.5, one can check that the remaining digraph is still a robust outexpander by applying Lemma 4.2(b).

Corollary 4.6. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \nu \ll \tau \leqslant \frac{\delta}{2} \leqslant 1$. Let D be a robust (ν, τ)-outexpander on n vertices. Suppose that $\delta^0(D) \geqslant \delta n$ and let $S \subseteq V(D)$ be such that $|S| \leqslant \varepsilon n$. Let $k \leqslant \nu^3 n$ and $x_1, \dots, x_k, x'_1, \dots, x'_k$ be (not necessarily distinct) vertices of D. Let $X := \{x_1, \dots, x_k, x'_1, \dots, x'_k\}$. Then, there exist internally vertex-disjoint paths $P_1, \dots, P_k \subseteq D$ such that, for each $i \in [k]$, P_i is an (x_i, x'_i)-path of length at most $2\nu^{-1}$ and $V^0(P_i) \subseteq V(D) \setminus (X \cup S)$.*

We will use the fact that robust outexpanders of linear minimum degree contain Hamilton paths from any fixed vertex x to any vertex $y \neq x$. This immediately follows by identifying x and y to a single vertex z whose outneighbourhood is that of x and whose inneighbourhood is that of y. The resulting digraph is a robust outexpander which contains a Hamilton cycle. The Hamiltonicity of such digraphs was first proven in [10, 14].

Lemma 4.7 [11, Corollary 6.9]. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leqslant \frac{\delta}{2} \leqslant 1$. Let D be a robust (ν, τ)-outexpander on n vertices with $\delta^0(D) \geqslant \delta n$. Then, for any distinct $x, y \in V(D)$, D contains a Hamilton (x, y)-path.*

Using similar arguments as in Corollary 4.6, we can iteratively apply Lemma 4.5 to tie together a small set of paths into a short path (Corollary 4.8(a)). By replacing the last iteration of Lemma 4.5 with an application of Lemma 4.7, we can tie together a small set of paths into a Hamilton path (Corollary 4.8(b)). Similarly, we can tie together a small set of paths into a Hamilton cycle (Corollary 4.8(c)) by first using Lemma 4.5 to tie these paths into a short path P and then tie together the endpoints of P using Lemma 4.7.

Corollary 4.8. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leqslant \frac{\delta}{2} \leqslant 1$ and $k \leqslant \nu^3 n$. Let D be a digraph and $P_1, \dots, P_k \subseteq D$ be vertex-disjoint paths. For each $i \in [k]$, denote by v_i^+ and v_i^- the starting and ending points of P_i, respectively. Let $V' := V(D) \setminus \bigcup_{i \in [k]} V(P_i)$ and $S \subseteq V'$. Suppose that $D' := D[V' \setminus S]$ is a robust (ν, τ)-outexpander on n vertices satisfying $\delta^0(D') \geqslant \delta n$. Assume that for each $i \in [k-1]$, $|N_D^+(v_i^-) \cap (V' \setminus S)| \geqslant 2k$ and $|N_D^-(v_{i+1}^+) \cap (V' \setminus S)| \geqslant 2k$. Then, the following hold.*

(a) *There exists a (v_1^+, v_k^-)-path $Q \subseteq D - S$ of length at most $2\nu^{-1}k + \sum_{i \in [k]} e(P_i)$ such that Q contains $\bigcup_{i \in [k]} P_i$.*
(b) *There exists a (v_1^+, v_k^-)-Hamilton path Q' of $D - S$ which contains $\bigcup_{i \in [k]} P_i$.*
(c) *There exists a Hamilton cycle C of $D - S$ which contains $\bigcup_{i \in [k]} P_i$.*

Proof. By assumption, there exist distinct $w_1^+, \dots, w_k^+, w_1^-, \dots, w_k^- \in V' \setminus S$ such that, for each $i \in [k]$, $w_i^+ \in N_D^-(v_i^+)$ and $w_i^- \in N_D^+(v_i^-)$. In particular, observe that $w_1^+ v_1^+ P_1 v_1^- w_1^-, \dots, w_k^+ v_k^+ P_k v_k^- w_k^-$ are vertex-disjoint paths of $D - S$. Apply Corollary 4.6 with $(x_i, x_i') = (w_i^-, w_{i+1}^+)$ for each $i \in [k-1]$ to obtain vertex-disjoint paths $P_1', \dots, P_{k-1}'$ such that for each $i \in [k-1]$, P_i' is a (w_i^-, w_{i+1}^+)-path of length at most $2\nu^{-1}$. Note that (a) holds by setting

$$Q := v_1^+ P_1 v_1^- w_1^- P_1' w_2^+ v_2^+ P_2 v_2^- w_2^- \dots w_{k-1}^- P_{k-1}' w_k^+ v_k^+ P_k v_k^-.$$

For (b), let $D'' := D' - \bigcup_{i \in [k-2]} V(P_i')$. By Lemma 4.2(b) and Lemma 4.7, D'' contains a Hamilton (w_{k-1}^-, w_k^+)-path P_{k-1}''. Let

$$Q' := v_1^+ P_1 v_1^- w_1^- P_1' w_2^+ v_2^+ P_2 v_2^- w_2^- \dots w_{k-2}^- P_{k-2}' w_{k-1}^+ v_{k-1}^+ P_{k-1} v_{k-1}^- w_{k-1}^- P_{k-1}'' w_k^+ v_k^+ P_k v_k^-.$$

To prove (c), a similar argument shows that there exists a Hamilton (w_k^-, w_1^+)-path P_k' in $D' - \bigcup_{i \in [k-1]} V(P_i')$. Let $C := w_1^+ v_1^+ Q v_k^- w_k^- P_k' w_1^+$. $\square$

The main result of [12] states that regular robust outexpanders of linear degree can be decomposed into Hamilton cycles. Note that this implies Kelly's conjecture on Hamilton decompositions of regular tournaments. Indeed, any regular tournament T on n vertices satisfies $\delta^0(T) = \frac{n-1}{2}$. Thus, Lemma 4.4 implies that any regular tournament is in fact a robust outexpander (of linear degree).

Theorem 4.9 [12, Theorem 1.2]. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leqslant \frac{\delta}{2} \leqslant 1$ and $r \geqslant \delta n$. Suppose D is an r-regular robust (ν, τ)-outexpander on n vertices. Then, D has a Hamilton decomposition.*

The following result is a consequence of Theorem 4.9 and will be used to complete our path decompositions (as described in the proof overview). In particular, this implies that any digraph D satisfying (†) from Section 2.1 is consistent, that is, $\mathrm{pn}(D) = \mathrm{ex}(D)$. Note that Corollary 4.10 is slightly more general than the argument described in Section 2.1. Indeed, since Theorem 4.9 holds for digraphs (rather than just oriented graphs), we can allow X^+ and X^- from Section 2.1 to 'intersect': we may have some vertices with one less inedge and one less outedge and then join these vertices to the auxiliary vertex by an inedge and an outedge. The set X^* in Corollary 4.10 consists of these vertices.

Corollary 4.10. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leqslant \frac{\delta}{2} \leqslant 1$ and $r \geqslant \delta n$. Suppose D is a robust (ν, τ)-outexpander on n vertices with a vertex partition $V(D) = X^+ \cup X^- \cup X^* \cup X^0$ such that $|X^+ \cup X^*| = |X^- \cup X^*| = r$ and, for all $v \in V(D)$, the following hold.*

$$\mathrm{ex}_D(v) = \begin{cases} \pm 1 & \text{if } v \in X^\pm, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad d_D(v) = \begin{cases} 2r-1 & \text{if } v \in X^\pm, \\ 2r-2 & \text{if } v \in X^*, \\ 2r & \text{otherwise.} \end{cases} \tag{4.1}$$

Then, $\mathrm{pn}(D) = r$*. Moreover, D has a path decomposition which consists of precisely r Hamilton paths with distinct starting points in $X^+ \cup X^*$ and distinct ending points in $X^- \cup X^*$.*

The proof is very similar to [16, Theorem 4.7], but we include it here for completeness.

Proof. By Fact 4.1 and Lemma 4.3, we may assume that $\tau \ll \delta$.

Note that $\mathrm{pn}(D) \geqslant r$. Indeed, if $X^* = V(D)$, then D is $(r-1)$-regular and so (1.3) implies that $\mathrm{pn}(D) \geqslant r$. Otherwise, $\Delta^0(D) = r$ and so, by Fact 1.4, $\mathrm{pn}(D) \geqslant \widetilde{\mathrm{ex}}(D) \geqslant r$. Thus, it is enough to decompose D into r Hamilton paths with distinct starting points in $X^+ \cup X^*$ and distinct ending points in $X^- \cup X^*$.

Let D' be obtained from D by adding a new vertex v with $N^\pm_{D'}(v) := X^\pm \cup X^*$. Then, by Lemma 4.2(b), D' is an r-regular robust $(\frac{\nu}{2}, 2\tau)$-outexpander. Applying Theorem 4.9 with $D', \frac{\delta}{2}, \frac{\nu}{2}$, and 2τ playing the roles of D, δ, ν, and τ yields a Hamilton decomposition of D' into r Hamilton cycles. Removing v, we obtain a path decomposition of D which consists of precisely r Hamilton paths with distinct starting points in $X^+ \cup X^*$ and distinct ending points in $X^- \cup X^*$, as desired. □

4.2 | Probabilistic estimates

In this section, we introduce a Chernoff-type bound and derive several easy probabilistic lemmas which will be used in the approximate decomposition step.

Let X be a random variable. We write $X \sim \mathrm{Bin}(n, p)$ if X follows a binomial distribution with parameters n and p. Let $N, n, m \in \mathbb{N}$ be such that $\max\{n, m\} \leqslant N$. Let Γ be a set of size N and $\Gamma' \subseteq \Gamma$ be of size m. Recall that X has a *hypergeometric distribution with parameters N, n, and m* if $X = |\Gamma_n \cap \Gamma'|$, where Γ_n is a random subset of Γ with $|\Gamma_n| = n$ (that is, Γ_n is obtained by drawing n elements of Γ without replacement). We will denote this by $X \sim \mathrm{Hyp}(N, n, m)$.

We will use the following Chernoff-type bound.

Lemma 4.11 (see, for example, [9, Theorems 2.1 and 2.10]). *Assume that $X \sim \mathrm{Bin}(n, p)$ or $X \sim \mathrm{Hyp}(N, n, m)$. Then, for any $0 < \varepsilon \leqslant 1$, the following hold.*

(a) $\mathbb{P}[X \leqslant (1-\varepsilon)\mathbb{E}[X]] \leqslant \exp(-\frac{\varepsilon^2}{3}\mathbb{E}[X])$.

(b) $\mathbb{P}[X \geqslant (1+\varepsilon)\mathbb{E}[X]] \leqslant \exp(-\frac{\varepsilon^2}{3}\mathbb{E}[X])$.

Using Lemma 4.11, it is easy to see that robust outexpansion is preserved with high probability when taking random edge-slices.

Lemma 4.12 [13, Lemma 3.2(ii)]. *Let $0 < \frac{1}{n} \ll \nu \ll \tau, \gamma \leqslant 1$. Let D be a robust (ν, τ)-outexpander on n vertices. Suppose Γ is obtained from D by taking each edge independently with probability γ. Then, with probability at least $1 - \exp(-\nu^3 n^2)$, Γ is a robust $(\frac{\gamma\nu}{2}, \tau)$-outexpander.*

The bound on the probability is not part of the statement in [13] but follows immediately from the proof. (The latter considers each $S \subseteq V(D)$ of size $\tau n \leqslant |S| \leqslant (1-\tau)n$ and uses Lemma 4.11 to show that for all $v \in RN^+_{\nu,D}(S)$, the probability that $|N^-_\Gamma(v) \cap S|$ is small is exponentially small in n.)

Similarly, using Lemma 4.11, it is easy to see that the property of being almost regular is preserved when a random edge-slice is taken.

Lemma 4.13. *Let $0 < \frac{1}{n} \ll \varepsilon, \gamma \ll \delta \leqslant 1$. Let D be a (δ, ε)-almost regular digraph on n vertices. Let Γ be obtained from D by taking each edge independently with probability $\frac{\gamma}{\delta}$. Then, with probability at least $1 - \frac{1}{n}$, Γ is (γ, ε)-almost regular and $D \setminus \Gamma$ is $(\delta - \gamma, \varepsilon)$-almost regular.*

Let D be a digraph on n vertices. We say D is an *(ε, p)-robust (ν, τ)-outexpander* if D is a robust (ν, τ)-outexpander and, for any integer $k \geqslant \varepsilon n$, if $S \subseteq V(D)$ is a random subset of size k, then $D[S]$ is a robust (ν, τ)-outexpander with probability at least $1 - p$. Note that the following analogue of Fact 4.1 holds for this new notion of robust outexpansion.

Fact 4.14. *Let D be an (ε, p)-robust (ν, τ)-outexpander. Then, for any $\varepsilon' \geqslant \varepsilon$, $p' \geqslant p$, $\nu' \leqslant \nu$, and $\tau' \geqslant \tau$, D is an (ε', p')-robust (ν', τ')-outexpander.*

Moreover, by Lemma 4.2(a), the following holds.

Lemma 4.15. *Let $0 < \varepsilon \leqslant \nu \leqslant \tau \leqslant 1$. Let D be an (ε, p)-robust (ν, τ)-outexpander on n vertices. If D' is obtained from D by removing at most εn inedges and at most εn outedges at each vertex, then D' is a $(\sqrt{\varepsilon}, p)$-robust $(\nu - \sqrt{\varepsilon}, \tau)$-outexpander.*

Proof. Let $S \subseteq V(D)$ satisfy $|S| \geqslant \sqrt{\varepsilon} n$ and suppose that $D[S]$ is a robust (ν, τ)-outexpander. By assumption, $D'[S]$ is obtained from $D[S]$ by removing at most $\varepsilon n \leqslant \sqrt{\varepsilon}|S|$ inedges and at most $\sqrt{\varepsilon}|S|$ outedges at each vertex. Thus, Lemma 4.2(a) implies that $D'[S]$ is a robust $(\nu - \sqrt{\varepsilon}, \tau)$-outexpander. □

We will see in the concluding remarks that any robust outexpander is in fact (ε, p)-robust (for some suitable parameters). However, our method for showing this requires the regularity lemma and so, for brevity, we will not prove this result. In this paper, we work with almost regular tournaments. Thus, it will be enough to use the next lemma, which shows that (ε, p)-robustness is easily inherited from almost regular robust outexpanders of sufficiently large minimum semidegree.

Lemma 4.16. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \nu \ll \tau \ll \gamma \ll \frac{3}{7} \leqslant \delta \leqslant 1$. Let D be a (δ, ε)-almost regular oriented graph on n vertices. Then, there exists a (γ, ε)-almost regular spanning subdigraph Γ of D which is an (ε, n^{-3})-robust (ν, τ)-outexpander and such that $D \setminus \Gamma$ is $(\delta - \gamma, \varepsilon)$-almost regular.*

Proof. Let Γ be obtained from D by taking each edge independently with probability $\frac{\gamma}{\delta}$. By Lemma 4.13, with probability at least $1-\frac{1}{n}$, Γ is (γ,ε)-almost regular and $D\setminus\Gamma$ is $(\delta-\gamma,\varepsilon)$-almost regular.

By Lemma 4.4, D is a robust $(2\gamma^{-1}\nu,\tau)$-outexpander. Therefore, by Lemma 4.12, Γ is a robust (ν,τ)-outexpander with probability at least $1-\exp(-8\gamma^{-3}\nu^3n^2)$.

Assume that $S\subseteq V(D)$ is such that $|S|\geqslant\varepsilon n$ and $D[S]$ is a robust $(2\nu^{-1}\gamma,\tau)$-outexpander. Then, Lemma 4.12 implies that $\Gamma[S]$ is a robust (ν,τ)-outexpander with probability at least $1-\exp(-8\gamma^{-3}\nu^3\varepsilon^2n^2)$. Therefore, the probability that $\Gamma[S]$ is a robust (ν,τ)-outexpander for each such S is at least $1-2^n\exp(-8\gamma^{-3}\nu^3\varepsilon^2n^2)$.

Thus, by a union bound, there exists a (γ,ε)-almost regular $\Gamma\subseteq D$ which is a robust (ν,τ)-outexpander and such that $D\setminus\Gamma$ is $(\delta-\gamma,\varepsilon)$-almost regular and, for each $S\subseteq V(D)$ with $|S|\geqslant\varepsilon n$, if $D[S]$ is a robust $(2\gamma^{-1}\nu,\tau)$-outexpander, then $\Gamma[S]$ is also a robust (ν,τ)-outexpander.

It now suffices to check that for any integer $k\geqslant\varepsilon n$, if $S\subseteq V(D)$ is chosen uniformly at random among the subsets of $V(D)$ of size k, then $D[S]$ is a robust $(2\gamma^{-1}\nu,\tau)$-outexpander with probability at least $1-n^{-3}$. Fix an integer $k\geqslant\varepsilon n$ and let $S\subseteq V(D)$ satisfy $|S|=k$. Then, for any $v\in V(D)$, $\mathbb{E}[d^{\pm}_{D[S]}(v)]=(\delta\pm\varepsilon)|S|$ and, by Lemma 4.11,

$$\mathbb{P}\left[d^{\pm}_{D[S]}(v)<\left(\frac{3}{8}+\gamma\right)|S|\right]\leqslant\mathbb{P}\left[d^{\pm}_{D[S]}(v)<\frac{9}{10}\mathbb{E}[d^{\pm}_{D[S]}(v)]\right]\leqslant\exp\left(-\varepsilon^2n\right).$$

Therefore, by Lemma 4.4, $D[S]$ is a robust $(2\gamma^{-1}\nu,\tau)$-outexpander with probability at least $1-n\exp(-\varepsilon^2n)\geqslant1-n^{-3}$. This completes the proof. □

The following result is an easy and well-known consequence of Lemma 4.11.

Lemma 4.17. *Let $0<\frac{1}{n}\ll\frac{1}{k},\varepsilon,\delta\ll1$. Let D be a (δ,ε)-almost regular digraph on n vertices. Let $n_1,\dots,n_k\in\mathbb{N}$ be such that $\sum_{i\in[k]}n_i=n$ and, for each $i\in[k]$, $n_i=\frac{n}{k}\pm1$. Assume $V_1,\dots,V_k$ is a random partition of $V(D)$ such that, for each $i\in[k]$, $|V_i|=n_i$. Then, with probability at least $1-n^{-1}$, the following holds. For each $i\in[k]$ and $v\in V(D)$, $|N^{\pm}_D(v)\cap V_i|=(\delta\pm2\varepsilon)\frac{n}{k}$.*

4.3 | Some tools for finding matchings

In this subsection, we record two easy consequences of Hall's theorem which will enable us to construct matchings.

Proposition 4.18. *Let G be a bipartite graph on vertex classes A and B with $|A|\leqslant|B|$. Suppose that, for each $a\in A$, $d_G(a)\geqslant\frac{|B|}{2}$ and, for each $b\in B$, $d_G(b)\geqslant|A|-\frac{|B|}{2}$. Then, G contains a matching covering A.*

Proposition 4.19. *Let $0<\frac{1}{n}\ll\varepsilon\ll\delta\leqslant1$. Let G be a bipartite graph on vertex classes A and B such that $|A|,|B|=(1\pm\varepsilon)n$. Suppose that, for each $v\in V(G)$, $d_G(v)=(\delta\pm\varepsilon)n$. Then, G contains a matching of size at least $(1-\frac{3\varepsilon}{\delta})n$.*

4.4 | Some properties of the excess function

We will need the following inequalities, which hold by definition of the excess function.

Fact 4.20. *Let D be a digraph and $v \in V(D)$.*

(a) $\widetilde{\mathrm{ex}}(D) \geqslant \Delta^0(D) \geqslant \frac{\Delta(D)}{2} \geqslant \frac{\delta(D)}{2}$.
(b) $d_D^{\min}(v) = \min\{d_D^+(v), d_D^-(v)\} = \frac{d_D(v) - |\mathrm{ex}_D(v)|}{2}$.
(c) $d_D^{\max}(v) = \max\{d_D^+(v), d_D^-(v)\} = \frac{d_D(v) + |\mathrm{ex}_D(v)|}{2}$.
(d) $\widetilde{\mathrm{ex}}(D) \geqslant \Delta^0(D) \geqslant d_D^{\max}(v) = d_D^{\min}(v) + |\mathrm{ex}_D(v)|$.

Given a digraph D and $S \subseteq V(D)$, define $\mathrm{ex}_D^{\pm}(S) := \sum_{v \in S} \mathrm{ex}_D^{\pm}(v)$. The next fact follows immediately from (1.1).

Fact 4.21. *Let D be a digraph and $S \subseteq V$. Then,* $\mathrm{ex}(D) = \mathrm{ex}_D^{\pm}(V(D)) = \mathrm{ex}_D^{\pm}(S) + \mathrm{ex}_D^{\pm}(V(D) \setminus S)$.

Fact 4.22. *Any tournament T on n vertices which is not regular satisfies* $\widetilde{\mathrm{ex}}(T) \geqslant \Delta^0(T) \geqslant \lceil \frac{n}{2} \rceil$.

5 | EXCEPTIONAL TOURNAMENTS

Recall the definition of the class $\mathcal{T}_{\mathrm{excep}} = \mathcal{T}_{\mathrm{reg}} \cup \mathcal{T}_{\mathrm{apex}}$ of exceptional tournaments from Section 1. The main purpose of this section is to prove Theorem 1.6 as well as the following result.

Theorem 5.1. *There exists $n_0 \in \mathbb{N}$ such that any tournament $T \in \mathcal{T}_{\mathrm{excep}}$ on $n \geqslant n_0$ vertices satisfies* $\mathrm{pn}(T) = \widetilde{\mathrm{ex}}(T) + 1$.

By Lemma 4.4, Theorem 1.6 (and thus also Theorem 5.1 in the case when $T \in \mathcal{T}_{\mathrm{reg}}$) is an immediate corollary of the following result.

Theorem 5.2. *Let $0 < \frac{1}{n} \ll \nu \ll \tau \leqslant \frac{\delta}{2} \leqslant 1$ and $r \geqslant \delta n$. Let D be an r-regular digraph on n vertices. Assume D is a robust (ν, τ)-outexpander. Then,* $\mathrm{pn}(D) = \widetilde{\mathrm{ex}}(D) + 1 = r + 1$.

Proof. By (1.3), we have $\widetilde{\mathrm{ex}}(D) = r$ and $\mathrm{pn}(D) \geqslant r + 1$. Let $P := v_1 \dots v_{r+1}$ be a path of D. Then, by Lemma 4.2(a), $D \setminus P$ is a robust $(\frac{\nu}{2}, \tau)$-outexpander. Let $X^+ := \{v_{r+1}\}$, $X^- := \{v_1\}$, $X^* := \{v_2, \dots, v_r\}$, and $X^0 := V(D) \setminus (X^+ \cup X^- \cup X^*)$. Applying Corollary 4.10 with $D \setminus P$ and $\frac{\nu}{2}$ playing the roles of D and ν completes the proof. □

Denote $U^{\pm}(D) := \{v \in V(D) \mid \mathrm{ex}_D^{\pm}(v) > 0\}$ and $U^0(D) := \{v \in V(D) \mid \mathrm{ex}_D(v) = 0\}$. In order to prove Theorem 5.1 for $T \in \mathcal{T}_{\mathrm{apex}}$, we need the following result.

Proposition 5.3. *Any $T \in \mathcal{T}_{\mathrm{apex}}$ on n vertices satisfies $\mathrm{ex}(T) = n - 3$ and* $\mathrm{pn}(T) \geqslant \widetilde{\mathrm{ex}}(T) + 1 = n - 1$.

Proof. Denote by $v_{\pm} \in V(T)$ the unique vertices such that $v_{\pm} \in U^{\pm}(T)$ and $V_0 := V(T) \setminus \{v_+, v_-\} = U^0(T)$. Thus, $v^- v^+ \in E(T)$.

Claim 1. $\mathrm{ex}(T) = n - 3$ and $\widetilde{\mathrm{ex}}(T) = n - 2$.

Proof of Claim. By definition of $\mathcal{T}_{\mathrm{apex}}$, we have $d_T^+(v_+) = n - 2 = d_T^-(v_-)$ and $d_T^-(v_+) = 1 = d_T^+(v_-)$. Moreover, each $v \in V_0$ satisfies $d_T^+(v) = \frac{n-1}{2} = d_T^-(v)$. Therefore, $\Delta^0(T) = n - 2$ and $\mathrm{ex}(T) = \frac{1}{2} \sum_{v \in V(T)} |d_T^+(v) - d_T^-(v)| = n - 3$. □

It remains to show that $\mathrm{pn}(T) \geqslant n-1$. Let $P \subseteq T$ be a path containing the edge v_-v_+. It suffices to show that $\mathrm{pn}(T \setminus P) \geqslant n-2$.

Let v be the starting point of P. Observe that, since $v_-v_+ \in E(P)$, we have $v \neq v_+$. If $v = v_-$, then $\mathrm{ex}(D \setminus P) \geqslant \mathrm{ex}^-_{D\setminus P}(v_-) = n-2$; otherwise, $v \in U^0(T)$ and so $\mathrm{ex}(D \setminus P) \geqslant \mathrm{ex}^-_{D\setminus P}(v_-) + \mathrm{ex}^-_{D\setminus P}(v) = (n-3)+1 = n-2$. Thus, we have shown that $\mathrm{ex}(T \setminus P) \geqslant n-2$. By (1.2), $T \setminus P$ cannot be decomposed into fewer than $\widetilde{\mathrm{ex}}(T \setminus \mathcal{P}) \geqslant n-2$ paths. Therefore, $\mathrm{pn}(T) \geqslant 1 + (n-2) = \widetilde{\mathrm{ex}}(T) + 1$. □

Proof of Theorem 5.1. By Lemma 4.4 and Theorem 5.2, we may assume that $T \in T_{\mathrm{apex}}$. Fix additional constants such that $0 < \frac{1}{n_0} \ll \nu \ll \tau \ll 1$. Let $T \in \mathcal{T}_{\mathrm{apex}}$ be a tournament on $n \geqslant n_0$ vertices. By Proposition 5.3, $\mathrm{pn}(T) \geqslant \widetilde{\mathrm{ex}}(T) + 1 = n-1$. Thus, it suffices to find a path decomposition of T of size $n-1$.

Let $v_\pm \in V(T)$ denote the unique vertices such that $v_\pm \in U^\pm(T)$. Let $V' := U^0(T)$. Let $v_1, \dots, v_{n-2}$ be an enumeration of V' and $r := \frac{n-3}{2}$. Since $T[V']$ is a regular tournament on $n-2$ vertices, Lemma 4.4 implies that $T[V']$ is a robust (ν, τ)-outexpander. Thus, by Lemma 4.7, we may assume without loss of generality that $v_1 \dots v_{r+1}$ is a path in $T[V']$.

Define a set of $r+2$ paths in T by

$$\mathcal{P} := \{v_-v_+, v_+v_1 \dots v_{r+1}v_-, v_+v_{r+2}v_-, \dots, v_+v_{n-2}v_-\}.$$

We now decompose $T \setminus \mathcal{P}$ into $(n-1)-(r+2) = r$ paths. Note that $d^\pm_{T\setminus\mathcal{P}}(v_\pm) = d_{T\setminus\mathcal{P}}(v_\pm) = r$. Thus, each path must start at v_+ and end at v_-. Let $A^+ := \{v_+v_i \mid 2 \leqslant i \leqslant r+1\}$ and $A^- := \{v_iv_- \mid i \in [r]\}$. Denote $D := T \setminus (A^+ \cup A^- \cup \mathcal{P})$. Then, $d^\pm_D(v_+) = 0 = d^\pm_D(v_-)$. Moreover, each $i \in [r]$ satisfies $d^+_D(v_i) = \frac{n-1}{2} - 2 = r-1$ and each $j \in [n-2] \setminus [r]$ satisfies $d^+_D(v_j) = \frac{n-1}{2} - 1 = r$. Similarly, each $i \in \{2, \dots, r+1\}$ satisfies $d^+_D(v_i) = \frac{n-1}{2} - 2 = r-1$ and each $j \in [n-2] \setminus \{2, \dots, r+1\}$ satisfies $d^+_D(v_j) = \frac{n-1}{2} - 1 = r$. Let $X^+ := \{v_{r+1}\}$, $X^- := \{v_1\}$, $X^* := \{v_2, \dots, v_r\}$, and $X^0 := \{v_i \mid i \in [n-2] \setminus [r+1]\}$. Then, $|X^+ \cup X^*| = r = |X^- \cup X^*|$ and (4.1) holds with $D - \{v_+, v_-\}$ playing the role of D. By Corollary 4.10, there exists a path decomposition $\mathcal{P}' = \{P_1, \dots, P_r\}$ of $D - \{v_+, v_-\}$. For each $i \in [r]$, let w_i^+ and w_i^- denote the starting and ending points of P_i. By the 'moreover part' of Corollary 4.10, we may assume that $w_1^+, \dots, w_r^+$ are distinct and $\{w_i^+ \mid i \in [r]\} = X^+ \cup X^*$. We may also assume that $w_1^-, \dots, w_r^-$ are distinct and $\{w_i^- \mid i \in [r]\} = X^- \cup X^*$. Thus, $\mathcal{P}'' := \{v_+w_i^+P_iw_i^-v_- \mid i \in [r]\}$ is a path decomposition of $D \cup A^+ \cup A^- = T \setminus \mathcal{P}$. Therefore, $\mathcal{P} \cup \mathcal{P}''$ is a path decomposition of T of size $2r+2 = n-1$. That is, $\mathrm{pn}(T) \leqslant n-1$, as desired. □

In Section 8.2, we will introduce the concept of 'absorbing edges' which plays a similar role as the edge sets A^+ and A^- above.

We will need the following observation about tournaments in $\mathcal{T}_{\mathrm{apex}}$ for later.

Proposition 5.4. *A tournament T satisfies $|U^+(T)| = |U^-(T)| = 1$, $e(U^-(T), U^+(T)) = 1$ and $\widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T) < 2$ if and only if $T \in \mathcal{T}_{\mathrm{apex}}$.*

Proof. Suppose that $T \in \mathcal{T}_{\mathrm{apex}}$. By definition and Proposition 5.3, $|U^+(T)| = |U^-(T)| = 1$, $e(U^-(T), U^+(T)) = 1$, $\widetilde{\mathrm{ex}}(T) = n-2$, and $\mathrm{ex}(T) = n-3$.

Suppose $|U^+(T)| = |U^-(T)| = 1$ and $e(U^-(T), U^+(T)) = 1$ and $\widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T) < 2$. Let $v_\pm \in U^\pm(T)$. By (1.1), we have $\mathrm{ex}^+_T(v_+) = \mathrm{ex}^-_T(v_-)$ and so, as T is a tournament, $d^+_T(v_+) = d^-_T(v_-)$. Since

$e(U^-(T), U^+(T)) = 1$, $d_T^{\mp}(v_\pm) \geqslant 1$. On the other hand,

$$2 > \widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T) \geqslant \Delta^0(T) - \mathrm{ex}(T) \geqslant d_T^{\pm}(v_\pm) - \mathrm{ex}_T^{\pm}(v_\pm) \overset{\text{Fact 4.20(d)}}{=} d_T^{\mp}(v_\pm).$$

Therefore, $N^{\mp}(v_\pm) = \{v_\mp\}$ and $N^{\pm}(v_\pm) = U^0(T)$. In particular, $T - \{v_+, v_-\} = T[U^0(T)]$ is regular. Hence, $T \in \mathcal{T}_{\mathrm{apex}}$, as required. □

6 | DERIVING THEOREM 1.3 AND COROLLARY 1.9 FROM THEOREM 1.8

In this section, we assume that Theorem 1.8 holds and derive Theorem 1.3 and Corollary 1.9. For Theorem 1.3, we first observe that if n is even, then $\widetilde{\mathrm{ex}}(T) = \mathrm{ex}(T)$.

Proposition 6.1. *Let T be a tournament of even order n. Then, $\widetilde{\mathrm{ex}}(T) = \mathrm{ex}(T)$ and $U^0(T) = \emptyset$.*

Proof. It is easy to see that each $v \in V(T)$ satisfies $\mathrm{ex}_T(v) \neq 0$ and so $U^0(T) = \emptyset$. Let $v \in V(T)$ be such that $d_T^{\max}(v) = \Delta^0(T)$. Thus,

$$\mathrm{ex}(T) = \frac{1}{2} \sum_{u \in V(T)} |\,\mathrm{ex}_T(u)| \geqslant \frac{n - 1 + |\,\mathrm{ex}_T(v)|}{2} \overset{\text{Fact 4.20(c)}}{=} d_T^{\max}(v) = \Delta^0(T),$$

so $\widetilde{\mathrm{ex}}(T) = \mathrm{ex}(T)$, as desired. □

Proof of Theorem 1.3. Let $0 < \frac{1}{n_0} \ll \beta \ll 1$. Let $n \geqslant n_0$ be even and T be a tournament on n vertices. It is easy to see that $T \notin \mathcal{T}_{\mathrm{excep}}$. We show that one of Theorem 1.8(a) and (b) holds. Suppose that Theorem 1.8(b) does not hold. Without loss of generality, we may assume that $N^+(T) \leqslant \beta n$. Thus, $|U^+(T)| \leqslant \beta n$. Since n is even, each $v \in V(T)$ satisfies $\mathrm{ex}_T(v) \neq 0$. Thus, $\widetilde{\mathrm{ex}}(T) \geqslant \mathrm{ex}(T) \geqslant |U^-(T)| = n - |U^+(T)| \geqslant n - \beta n \geqslant \frac{n}{2} + \beta n$ and so Theorem 1.8(a) holds. Therefore, by Theorem 1.8 and Proposition 6.1, $\mathrm{pn}(T) = \widetilde{\mathrm{ex}}(T) = \mathrm{ex}(T)$. □

Finally, we derive Corollary 1.9 from Theorem 1.8. The idea is that if none of Theorems 1.3, 1.8, and 5.1 apply to T, then we can transform T into a tournament T' which satisfies the conditions of Theorem 1.8 by flipping a small number of edges, and so that $\mathrm{pn}(T) \sim \mathrm{pn}(T')$ and $\widetilde{\mathrm{ex}}(T) \sim \widetilde{\mathrm{ex}}(T')$.

Proof of Corollary 1.9. We may assume without loss of generality that $\beta \ll 1$. Let $0 < \frac{1}{n_0} \ll \beta \ll 1$. Let T be a tournament on $n \geqslant n_0$ vertices. By Theorems 1.3 and 5.1, we may assume that $T \notin \mathcal{T}_{\mathrm{excep}}$ and that n is odd. If $\Delta^0(T) \geqslant \frac{n}{2} + \frac{\beta n}{5}$, then, by Theorem 1.8 applied with $\frac{\beta}{5}$ playing the role of β, $\mathrm{pn}(T) = \widetilde{\mathrm{ex}}(T)$ and we are done. We may therefore assume that $\Delta^0(T) < \frac{n}{2} + \frac{\beta n}{5}$. Let $v \in V(T)$. Since T is not regular, we may assume without loss of generality that $v \in U^+(T)$. Then, note that $d_T^+(v) \geqslant \frac{n+1}{2}$. Let $S \subseteq N_T^-(v)$ satisfy $|S| = \lceil \frac{n}{2} + \frac{\beta n}{5} \rceil - d_T^+(v)$ (this is possible since $d_T^-(v) = (n - 1) - d_T^+(v)$). Note that $|S| \leqslant \lceil \frac{n}{2} + \frac{\beta n}{5} \rceil - \frac{n+1}{2} \leqslant \frac{\beta n}{4}$.

Let T' be obtained from T by flipping the direction of all edges between v and S. Then, observe that $\widetilde{\mathrm{ex}}(T') \geqslant \Delta^0(T') \geqslant d_{T'}^+(v) = \lceil \frac{n}{2} + \frac{\beta n}{5} \rceil$ and, in particular, $T' \notin \mathcal{T}_{\mathrm{excep}}$. Moreover, we claim that $\widetilde{\mathrm{ex}}(T') \leqslant \widetilde{\mathrm{ex}}(T) + 2|S|$. Since $\Delta^0(T') \leqslant \Delta^0(T) + |S|$, it suffices to show that $\mathrm{ex}(T') \leqslant \mathrm{ex}(T) + 2|S|$.

Note that, by Fact 4.20(c), $\mathrm{ex}^+_{T'}(v) - \mathrm{ex}^+_T(v) = 2(d^+_{T'}(v) - d^+_T(v)) = 2|S|$. For each $\diamond \in \{+, -, 0\}$, denote $S^\diamond := S \cup U^\diamond(T)$. Then, by Fact 4.20(c), for each $u \in S^+$, $\mathrm{ex}^+_{T'}(u) - \mathrm{ex}^+_T(u) = -2$ and, for each $u \in S^- \cup S^0$, $\mathrm{ex}^+_{T'}(u) = 0 = \mathrm{ex}^+_T(v)$. Thus, $\mathrm{ex}(T') - \mathrm{ex}(T) = 2|S| - 2|S^+| \leqslant 2|S|$, as desired.

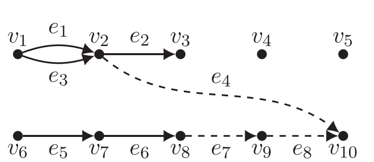
By Theorem 1.8, $\mathrm{pn}(T') = \widetilde{\mathrm{ex}}(T') \leqslant \widetilde{\mathrm{ex}}(T) + 2|S|$ and thus, since $|S| \leqslant \frac{\beta n}{4}$, it suffices to show that $\mathrm{pn}(T) \leqslant \mathrm{pn}(T') + 2|S|$. Let $\mathcal{P}'$ be a path decomposition of T' of size $\mathrm{pn}(T')$. Let $\mathcal{P}'_1$ consist of all the paths $P \in \mathcal{P}'$ such that $E(P) \subseteq E(T)$. Let $\mathcal{P}'_2 := \mathcal{P}' \setminus \mathcal{P}'_1$. Let $\mathcal{P}_2$ be set of paths obtained from $\mathcal{P}'_2$ by deleting all the edges in $E(T') \setminus E(T)$. Then, $\mathcal{P} := \mathcal{P}'_1 \cup \mathcal{P}_2 \cup (E(T) \setminus E(T'))$ is a path decomposition of T. Moreover, by construction, all the edges in $E(T') \setminus E(T)$ are incident to v. Thus, each path in $\mathcal{P}'_2$ contains exactly one edge of $E(T') \setminus E(T)$ and so $|\mathcal{P}_2 \cup (E(T) \setminus E(T'))| \leqslant 3|\mathcal{P}'_2| = |\mathcal{P}'_2| + 2|E(T') \setminus E(T)| = |\mathcal{P}'_2| + 2|S|$. Therefore, $\mathrm{pn}(T) \leqslant |\mathcal{P}| \leqslant |\mathcal{P}'_1| + |\mathcal{P}'_2| + 2|S| = \mathrm{pn}(T') + 2|S|$. This completes the proof. $\square$

7 | APPROXIMATE DECOMPOSITION OF ROBUST OUTEXPANDERS

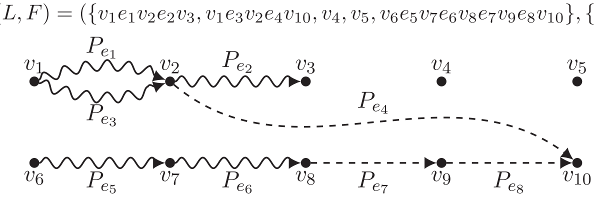
The rest of the paper is devoted to the proof of Theorem 1.8. We start by discussing and proving the approximate decomposition step, which is achieved via Lemma 7.3.

As mentioned in the proof overview, in order to reduce the excess and the vertex degrees at the correct rate, we will approximately decompose our digraphs into sets of paths. To do so, we will start by constructing auxiliary multidigraphs called *layouts* which will prescribe the 'shape' of the structures in our approximate decomposition.

Suppose that we would like to find a Hamilton (v_+, v_-)-path which contains a fixed edge $f = u_+u_-$. We can view this as the task of finding two paths of shapes v_+u_+ and u_-v_-, respectively, that are vertex-disjoint and cover all remaining vertices. (Recall from Section 3 that, given an (auxiliary) edge uv, we say that a (u, v)-path has shape uv.) We now generalise this approach to *layouts*, which will tell us the shapes of paths required, the set F of fixed edges to be included, and the vertices to be avoided by these paths. The 'spanning' extension of a layout will be called a *spanning configuration*. To ensure that the spanning configuration has a suitable path decomposition, we will define a layout to consist of a (multi)set of paths rather than a multiset of edges. The concepts of layout and spanning configuration are also illustrated in Figure 2.



(a) A layout $(L, F) = (\{v_1e_1v_2e_2v_3, v_1e_3v_2e_4v_{10}, v_4, v_5, v_6e_5v_7e_6v_8e_7v_9e_8v_{10}\}, \{e_4, e_7, e_8\})$ on V.



(b) A spanning configuration $\mathcal{H}$ of shape (L, F) on V. The wavy edges represent internally vertex-disjoint paths on V which altogether cover all the vertices in $V \setminus V(L)$.

FIGURE 2 A layout (L, F) and a spanning configuration of shape (L, F). Dashed edges represent fixed edges (that is, the edges of F).

We will be working with multidigraphs. Let V be a vertex set. We say (L, F) is a *layout* if the following hold.

(L1) L is a multiset consisting of paths on V and isolated vertices.
(L2) $F \subseteq E(L)$.
(L3) $E(L) \setminus F \neq \emptyset$.

Conditions (L1)–(L3) can be motivated as follows. Suppose that (L, F) is a layout on V. As described above, our goal is to construct a spanning set of paths whose shapes correspond to those of the paths in L and which contain the edges in F. This will be achieved by replacing the edges in $E(L) \setminus F$ by internally vertex-disjoint paths which cover all the vertices in $V \setminus V(L)$. This motivates (L3): if $E(L) \setminus F$ was empty, then there would be no edge to replace by a path and so we would not be able cover the vertices in $V \setminus V(L)$, that is, our set of paths would not be spanning. Moreover, as we will be constructing the paths according to the shapes of the paths in L, we need to make sure that the fixed edge set F is covered by L. This explains (L2). Finally, since we will have already covered some edges before the approximate decomposition (recall from the proof overview that we will need a cleaning step) and since not all vertices have the same excess, we do not actually want our structures to be completely spanning, but want them to avoid a suitable small set of vertices. This is why we allow paths of length 0 in (L1).

Let (L, F) be a layout on V. A multidigraph $\mathcal{H}$ on V is a *spanning configuration of shape* (L, F) if $\mathcal{H}$ can be decomposed into internally vertex-disjoint paths $\{P_e \mid e \in E(L)\}$ such that each P_e has shape e; $P_f = f$ for all $f \in F$; and $\bigcup_{e \in E(L)} V^0(P_e) = V \setminus V(L)$ (recall that given a path P, $V^0(P)$ denotes the set of internal vertices of P). (Note that the last equality implies that the isolated vertices of L remain isolated in $\mathcal{H}$.) See Figure 2(b) for an example of a spanning configuration.

Let (L, F) be a layout on V and $\mathcal{H}$ be a spanning configuration of shape (L, F) on V. There is a natural bijection between a path Q in L and the path $P_Q := \bigcup\{P_e \mid e \in E(Q)\}$ in $\mathcal{H}$. (For example, in the example presented in Figure 2, the path $v_1 e_1 v_2 e_2 v_3$ in L corresponds to the path $v_1 P_{e_1} v_2 P_{e_2} v_3$ in $\mathcal{H}$.) Note that this bijection is not necessarily unique since if e has multiplicity more than 1 in L, then there are different ways to define P_e. (For example, in the example presented in Figure 2, we could have exchanged P_{e_1} and P_{e_3}.) A path decomposition $\mathcal{P}$ of $\mathcal{H}$ consisting of all such P_Q for all the paths $Q \in L$ is said to be *induced* by (L, F). (For example, in the example presented in Figure 2, $\{v_1 P_{e_1} v_2 P_{e_2} v_3, v_1 P_{e_3} v_2 P_{e_4} v_{10}, v_6 P_{e_5} v_7 P_{e_6} v_8 P_{e_7} v_9 P_{e_8} v_{10}\}$ is a path decomposition of $\mathcal{H}$ induced by (L, F).) Note that if $\mathcal{P}$ is a path decomposition $\mathcal{H}$ induced by (L, F), then the paths in $\mathcal{P}$ are non-trivial and have the same endpoints as their corresponding path in L.

Fact 7.1. *Let (L, F) be a layout on V and $\mathcal{H}$ be a spanning configuration of shape (L, F) on V. Let L' denote the set of (non-trivial) paths contained in L (that is, L' is obtained by deleting all the isolated vertices in L). For any path Q in L, the corresponding path $P_Q := \bigcup\{P_e \mid e \in E(Q)\}$ in $\mathcal{H}$ satisfies $V^{\pm}(P_Q) = V^{\pm}(Q)$ and $V^0(Q) \subseteq V^0(P_Q) \subseteq V^0(Q) \cup (V \setminus V(L))$. Thus, if $\mathcal{P}$ is a path decomposition of $\mathcal{H}$ which is induced by (L, F), then $V^{\pm}(\mathcal{P}) = V^{\pm}(L')$ and $V^0(\mathcal{P}) = V^0(L) \cup (V \setminus V(L))$.*

Let V be a vertex set. Let (L, F) be a layout on V and $\mathcal{H}$ be a spanning configuration of shape (L, F) on V. By Fact 7.1, the degree of each $v \in V$ in $\mathcal{H}$ is entirely determined by the degree of v in L. Thus, the following holds.

Fact 7.2. *Let D be a digraph on a vertex set V and $(L_1, F_1), \dots, (L_\ell, F_\ell)$ be layouts on V. For each $i \in [\ell]$, let $\mathcal{H}_i$ be a spanning configuration of shape (L_i, F_i). Suppose that $\mathcal{H}_1, \dots, \mathcal{H}_\ell$ are pairwise edge-*

disjoint. Then, for all $v \in V$,

$$d^{\pm}_{\mathcal{H}}(v) = \sum_{i\in[\ell]} d^{\pm}_{\mathcal{H}_i}(v) = \sum_{i\in[\ell]} (d^{\pm}_{L_i}(v) + \mathbb{1}_{v\notin V(L_i)}) = d^{\pm}_{L}(v) + |\{i \in [\ell] \mid v \in V \setminus V(L_i)\}|.$$

Roughly speaking, the approximate decomposition lemma says that given a dense almost regular digraph D and a sparse almost regular robust outexpander Γ, we can transform a suitable set of small layouts into edge-disjoint spanning configurations of corresponding shape in $D \cup \Gamma$. Moreover, the number of layouts that we are allowed to prescribe is close to $\delta^0(D)$, in which case the configurations form an approximate decomposition of $D \cup \Gamma$.

Lemma 7.3 (Approximate decomposition lemma for robust outexpanders). *Let $0 < \frac{1}{n} \ll \varepsilon \ll \nu \ll \tau \ll \gamma \ll \eta, \delta \leqslant 1$. Suppose $\ell \in \mathbb{N}$ satisfies $\ell \leqslant (\delta - \eta)n$. If $\ell \leqslant \varepsilon^2 n$, then let $p \leqslant n^{-1}$; otherwise, let $p \leqslant n^{-2}$. Let D and Γ be edge-disjoint digraphs on a common vertex set V of size n. Suppose that D is (δ, ε)-almost regular and Γ is (γ, ε)-almost regular. Suppose further that Γ is an (ε, p)-robust (ν, τ)-outexpander. Let $\mathcal{F}$ be a multiset of directed edges on V. Any edge in $\mathcal{F}$ is considered to be distinct from the edges of $D \cup \Gamma$, even if the starting and ending points are the same (recall Section 3). Let $F_1, \dots, F_\ell$ be a partition of $\mathcal{F}$. Assume that $(L_1, F_1), \dots, (L_\ell, F_\ell)$ are layouts such that $V(L_i) \subseteq V$ for each $i \in [\ell]$ and the following hold, where $L := \bigcup_{i\in[\ell]} L_i$.*

(a) *For each $i \in [\ell]$, $|V(L_i)| \leqslant \varepsilon^2 n$ and $|E(L_i)| \leqslant \varepsilon^4 n$.*
(b) *Moreover, for each $v \in V$, $d_L(v) \leqslant \varepsilon^3 n$ and there exist at most $\varepsilon^2 n$ indices $i \in [\ell]$ such that $v \in V(L_i)$.*

Then, there exist edge-disjoint submultidigraphs $\mathcal{H}_1, \dots, \mathcal{H}_\ell \subseteq D \cup \Gamma \cup \mathcal{F}$ such that, for each $i \in [\ell]$, $\mathcal{H}_i$ is a spanning configuration of shape (L_i, F_i) and the following hold, where $\mathcal{H} := \bigcup_{i\in[\ell]} \mathcal{H}_i$, $D' := D \setminus \mathcal{H}$, and $\Gamma' := \Gamma \setminus \mathcal{H}$.

(i) *If $\ell \leqslant \varepsilon^2 n$, then Γ' is obtained from Γ by removing at most $3\varepsilon^3\nu^{-4}n$ edges incident to each vertex, that is, $\Delta(\Gamma \setminus \Gamma') \leqslant 3\varepsilon^3\nu^{-4}n$.*
(ii) *If $\ell \leqslant \nu^5 n$, then D' is $(\delta - \frac{\ell}{n}, 2\varepsilon)$-almost regular and Γ' is $(\gamma, 2\varepsilon)$-almost regular. Moreover, Γ' is a $(\sqrt{\varepsilon}, p)$-robust $(\nu - \sqrt{\varepsilon}, \tau)$-outexpander.*
(iii) *$D' \cup \Gamma'$ is a robust $(\frac{\nu}{2}, \tau)$-outexpander.*

The approximate decomposition guaranteed by Lemma 7.3 is constructed in stages. The core of the approximate decomposition occurs in Lemma 7.3(i), where a small set of layouts is converted into spanning configurations one by one (see Section 2.2). Repeated applications of Lemma 7.3(i) will then enable us to transform larger sets of layouts into spanning configurations (Lemma 7.3(ii)). Then, one can obtain the final approximate decomposition (Lemma 7.3(iii)) by repeatedly applying Lemma 7.3(ii), adjusting the parameters in each iteration. This can be seen as a semirandom 'nibble' process, where the applications of Lemma 7.3(i) are the 'nibbles' (which are chosen via a probabilistic argument) and the applications of Lemma 7.3(ii) correspond to 'bites' consisting of several 'nibbles'. We prove (ii), (iii), and (i) in this order.

Proof of Lemma 7.3(ii). Let $\ell' := \lfloor \varepsilon^2 n \rfloor$ and $k := \lceil \frac{\ell}{\ell'} \rceil$. Note that $k \leqslant 2\nu^5\varepsilon^{-2}$. We now group $(L_1, F_1), \dots, (L_\ell, F_\ell)$ into k batches, each of size at most ℓ'. For each $m \in [k]$, the mth batch will consist of $(L_i, \mathcal{F}_i)$ with $(m-1)\ell' \leqslant i \leqslant \min\{m\ell', \ell\}$. We aim to apply Lemma 7.3(i) to each batch in turn.

Assume that we have done m batches for some $0 \leqslant m \leqslant k$. This means that we have constructed edge-disjoint $\mathcal{H}_1, \dots, \mathcal{H}_{\min\{m\ell', \ell\}} \subseteq D \cup \Gamma \cup \mathcal{F}$ such that, for each $i \in [\min\{m\ell', \ell\}]$, $\mathcal{H}_i$ is a spanning configuration of shape (L_i, F_i) satisfying $E(\mathcal{H}_i) \cap E(\mathcal{F}) = E(F_i)$ and the following holds. Let $\Gamma_m := \Gamma \setminus \bigcup_{i \in [\min\{m\ell', \ell\}]} \mathcal{H}_i$. Then, for each $v \in V$,

$$|N_{\Gamma \setminus \Gamma_m}(v)| \leqslant m \cdot 25\varepsilon^3 \nu^{-4} n \leqslant 50\varepsilon\nu n \leqslant \frac{\varepsilon n}{2}. \tag{7.1}$$

Let $D_m := D \setminus \bigcup_{i \in [\min\{m\ell', \ell\}]} \mathcal{H}_i$. Observe that, by Fact 7.2 and (b), $\bigcup_{i \in [\min\{m\ell', \ell\}]} \mathcal{H}_i \setminus F_i$ is $(\frac{\min\{m\ell', \ell\}}{n}, \varepsilon^2 + \varepsilon^3)$-almost regular. Together with (7.1), this implies that D_m is $(\delta - \frac{\min\{m\ell', \ell\}}{n}, 2\varepsilon)$-almost regular and Γ_m is $(\gamma, 2\varepsilon)$-almost regular.

Moreover, by Lemma 4.15, Γ_m is a $(\sqrt{\varepsilon}, p)$-robust $(\nu - \sqrt{\varepsilon}, \tau)$-outexpander. Thus, if $m = k$, we are done.

Suppose $m < k$. We show that Γ_m is a $(2\varepsilon, n^{-1})$-robust $(\nu - \varepsilon, \tau)$-outexpander. If $m = 0$, then $\Gamma_m = \Gamma$ and we are done. We may therefore assume that $m \geqslant 1$. Then, note that $k \geqslant 2$ so $\ell > \ell' = \lfloor \varepsilon^2 n \rfloor$ and, thus, $p \leqslant n^{-2}$. Fix an integer $k' \geqslant 2\varepsilon n$. Suppose $S \subseteq V$ is a random subset of size k'. We show that $\Gamma_m[S]$ is a robust $(\nu - \varepsilon, \tau)$-robust outexpander with probability at least $1 - n^{-1}$. Let $v \in V$. If $|N_{\Gamma \setminus \Gamma_m}(v)| \leqslant \varepsilon^2 n$, then $|N_{\Gamma \setminus \Gamma_m}(v) \cap S| \leqslant \varepsilon^2 n \leqslant \varepsilon k'$. Suppose $|N_{\Gamma \setminus \Gamma_m}(v)| \geqslant \varepsilon^2 n$. Then, by (7.1), $\mathbb{E}[|N_{\Gamma \setminus \Gamma_m}(v) \cap S|] = \frac{k'}{n} |N_{\Gamma \setminus \Gamma_m}(v)| \leqslant \frac{\varepsilon k'}{2}$. Thus, Lemma 4.11 implies that

$$\mathbb{P}\left[|N_{\Gamma \setminus \Gamma_m}(v) \cap S| > \varepsilon k'\right] \leqslant \mathbb{P}\left[|N_{\Gamma \setminus \Gamma_m}(v) \cap S| > 2\mathbb{E}[|N_{\Gamma \setminus \Gamma_m}(v) \cap S|]\right] \leqslant \exp\left(-\frac{2\varepsilon^3 n}{3}\right).$$

Therefore, by a union bound, with probability at least $1 - n\exp(-\frac{2\varepsilon^3 n}{3})$, the digraph $\Gamma_m[S]$ is obtained from $\Gamma[S]$ by removing at most $\varepsilon k'$ edges incident to each vertex. Our assumption on Γ implies that $\Gamma[S]$ is a robust (ν, τ)-outexpander with probability at least $1 - p \geqslant 1 - n^{-2}$. Therefore, by Lemma 4.2(a), we conclude that $\Gamma_m[S]$ is a robust $(\nu - \varepsilon, \tau)$-outexpander with probability at least $1 - p - n\exp(-\frac{2\varepsilon^3 n}{3}) \geqslant 1 - n^{-1}$. Thus, Γ_m is a $(2\varepsilon, n^{-1})$-robust $(\nu - \varepsilon, \tau)$-outexpander.

Let $\ell'' := \min\{\ell - m\ell', \ell'\}$ and $\mathcal{F}' := \bigcup_{i \in [\ell'']} \mathcal{F}_{m\ell' + i}$. Apply Lemma 7.3(i) with D_m, Γ_m, $\mathcal{F}'$, n^{-1}, $\delta - \frac{m\ell'}{n}$, $\nu - \varepsilon$, 2ε, ℓ'', $L_{m\ell'+1}, \dots, L_{m\ell'+\ell''}$, and $F_{m\ell'+1}, \dots, F_{m\ell'+\ell''}$ playing the roles of D, Γ, $\mathcal{F}$, p, δ, ν, ε, ℓ, $L_1, \dots, L_\ell$, and $F_1, \dots, F_\ell$ to obtain edge-disjoint $\mathcal{H}_{m\ell'+1}, \dots, \mathcal{H}_{m\ell'+\ell''} \subseteq D_m \cup \Gamma_m \cup \mathcal{F}'$ such that, for each $i \in [\ell'']$, $\mathcal{H}_{m\ell'+i}$ is a spanning configuration of shape $(L_{m\ell'+i}, F_{m\ell'+i})$ and, for each $v \in V$, $|N_{\Gamma_m \setminus \Gamma_{m+1}}(v)| \leqslant 3(2\varepsilon)^3(\nu - \varepsilon)^{-4} n \leqslant 25\varepsilon^3\nu^{-4} n$, where $\Gamma_{m+1} := \Gamma_m \setminus \bigcup_{i \in [\ell'']} \mathcal{H}_{m\ell'+i}$. In particular, (7.1) holds. This completes the proof. □

Proof of Lemma 7.3(iii). Let $\ell' := \lfloor \nu^5 n \rfloor$ and $k := \lceil \frac{\ell}{\ell'} \rceil$. Note that $k \leqslant \nu^{-5}$. For each $i \in \mathbb{N}$, denote $\varepsilon_i := 2^i \varepsilon^{\frac{1}{2^i}}$. Assume inductively that, for some $0 \leqslant m \leqslant k$, we have constructed edge-disjoint $\mathcal{H}_1, \dots, \mathcal{H}_{\min\{m\ell', \ell\}} \subseteq D \cup \Gamma \cup \mathcal{F}$ such that:

- for each $i \in [\min\{m\ell', \ell\}]$, $\mathcal{H}_i$ is a spanning configuration of shape (L_i, F_i) satisfying $E(\mathcal{H}_i) \cap E(\mathcal{F}) = E(F_i)$;
- for each $i \in [m]$, $D_i := D \setminus \bigcup_{j \in [\min\{i\ell', \ell\}]} \mathcal{H}_j$ is $(\delta - \frac{\min\{i\ell', \ell\}}{n}, \varepsilon_i)$-almost regular; and
- for each $i \in [m]$, $\Gamma_i := \Gamma \setminus \bigcup_{j \in [\min\{i\ell', \ell\}]} \mathcal{H}_j$ is a (γ, ε_i)-almost regular (ε_i, p)-robust $(\nu - \varepsilon_i, \tau)$-outexpander.

If $m = k$, then, since $k \leqslant \nu^{-5}$ and $\varepsilon \ll \nu$, Γ_m is a robust $(\frac{\nu}{2}, \tau)$-outexpander and so is $D_m \cup \Gamma_m$, as desired. Assume $m < k$. Let $\ell'' := \min\{\ell - m\ell', \ell'\}$ and $\mathcal{F}' := \bigcup_{i\in[\ell'']} \mathcal{F}_{m\ell'+i}$. Then, apply Lemma 7.3(ii) with $D_m, \Gamma_m, \mathcal{F}', \delta - \frac{m\ell'}{n}, \nu - \varepsilon_m, \varepsilon_m, \ell''$, $L_{m\ell'+1}, \dots, L_{m\ell'+\ell''}$, and $F_{m\ell'+1}, \dots, F_{m\ell'+\ell''}$ playing the roles of $D, \Gamma, \mathcal{F}, \delta, \nu, \varepsilon, \ell, L_1, \dots, L_\ell$, and $F_1, \dots, F_\ell$ to obtain edge-disjoint $\mathcal{H}_{m\ell'+1}, \dots, \mathcal{H}_{m\ell'+\ell''} \subseteq D_m \cup \Gamma_m \cup \mathcal{F}'$ such that the following hold. For each $i \in [\ell'']$, $\mathcal{H}_{m\ell'+i}$ is a spanning configuration of shape $(L_{m\ell'+i}, F_{m\ell'+i})$. Moreover, $D_{m+1} := D_m \setminus \bigcup_{i\in[\ell'']} \mathcal{H}_{m\ell'+i}$ is $(\delta - \frac{\min\{(m+1)\ell', \ell\}}{n}, \varepsilon_{m+1})$-almost regular and $\Gamma_{m+1} := \Gamma_m \setminus \bigcup_{i\in[\ell'']} \mathcal{H}_{m\ell'+i}$ is a $(\gamma, \varepsilon_{m+1})$-almost regular (ε_{m+1}, p)-robust $(\nu - \varepsilon_{m+1}, \tau)$-outexpander , as desired. □

As discussed in Section 2, the key idea in the proof of Lemma 7.3(i) is how to use the robust outexpander Γ efficiently, that is, to find the required number of spanning configurations $\mathcal{H}_i$ without using too many edges of Γ. We achieve this by considering a random partition $A_1, \dots, A_a$ of V. To build $\mathcal{H}_i$, we find an almost cover of V in D with few long paths (which exists since D is almost regular) and tie them together into a single spanning path using only $\Gamma[A_j]$ for a suitable $j \in [a]$. The remainder of $\mathcal{H}_i$ is comparatively small and its construction does not affect Γ significantly. (See also Figure 1.)

Proof of Lemma 7.3(i). Let $a := \lceil \varepsilon^{-1}\nu^4 \rceil$. By Lemma 4.17 (successively applied to D and Γ) and since Γ is an (ε, p)-robust (ν, τ)-outexpander, we can fix a partition $A_1, \dots, A_a$ of V such that, for each $i \in [a]$, the following hold.

(α) $|A_i| = \frac{n}{a} \pm 1 = \varepsilon(\nu^{-4} \pm 1)n$.
(β) $\Gamma[A_i]$ is a robust (ν, τ)-outexpander.
(γ) For each $v \in V$, $|N_\Gamma^\pm(v) \cap A_i| = (\gamma \pm 2\varepsilon)\frac{n}{a}$.
(δ) For each $v \in V$, $|N_D^\pm(v) \cap A_i| = (\delta \pm 2\varepsilon)\frac{n}{a}$.

For each $i \in [\ell]$, let $j \in [a]$ be such that $i \equiv j \mod a$ and define $A_i' := A_j \setminus V(L_i)$. Using (a) and Lemma 4.2(b), it is easy to check that, for each $i \in [\ell]$, the following hold.

(α') $|A_i'| = \varepsilon(\nu^{-4} \pm 2)n$.
(β') $\Gamma[A_i']$ and $\Gamma - A_i'$ are both robust $(\frac{\nu}{2}, 2\tau)$-outexpanders.
(γ') $\Gamma[A_i']$ and $\Gamma - A_i'$ are both $(\gamma, 3\varepsilon)$-almost regular.
(δ') $D - A_i'$ is $(\delta, 3\varepsilon)$-almost regular.
(ε') For each $v \in V \setminus A_i'$, $|N_D^\pm(v) \cap A_i'| \geqslant \frac{\varepsilon\delta n}{2\nu^4}$.

For each $i \in [\ell]$, fix $e_i \in E(L_i) \setminus F_i$ (this is possible by (L3)). Assume inductively that for some $0 \leqslant m \leqslant \ell$ we have constructed, for each $i \in [m]$, a set of paths $\mathcal{P}_i = \{P_e^i \mid e \in E(L_i) \setminus F_i\}$ in $D \cup \Gamma$ such that $\mathcal{P}_1, \dots, \mathcal{P}_m$ are edge-disjoint and the following hold.

(A) Let $i \in [m]$. For each $e \in E(L_i) \setminus F_i$, P_e^i is a path of shape e. Moreover, the paths in $\mathcal{P}_i$ are internally vertex-disjoint and $V^0(\mathcal{P}_i) = V \setminus V(L_i)$. In particular, $\mathcal{P}_i \cup F_i$ is a spanning configuration of shape (L_i, F_i).
(B) For each $i \in [m]$ and $e \in E(L_i) \setminus (F_i \cup \{e_i\})$, $P_e^i \subseteq \Gamma - A_i'$ and $e(P_e^i) \leqslant 8\nu^{-1}$. Moreover, for each $v \in V$, there exist at most $\varepsilon^3 n$ indices $i \in [m]$ such that $v \in V^0(\mathcal{P}_i \setminus \{P_{e_i}^i\})$.
(C) For each $i \in [m]$, $E(P_{e_i}^i) \cap E(\Gamma) \subseteq E(\Gamma[A_i'])$.

Denote $D_m := D \setminus \bigcup_{i\in[m]} E(\mathcal{P}_i)$ and $\Gamma_m := \Gamma \setminus \bigcup_{i\in[m]} E(\mathcal{P}_i)$. For each $i \in [m]$, define $\mathcal{H}_i := \mathcal{P}_i \cup F_i$. Denote $\mathcal{H}^m := \bigcup_{i\in[m]} \mathcal{H}_i$. Then, note that, for each $v \in V$, since $d_L(v) \leqslant \varepsilon^3 n$, there are at most $\varepsilon^3 n$ indices $i \in [m]$ such that $v \in V^+(\mathcal{P}_i \setminus \{P^i_{e_i}\}) \cup V^-(\mathcal{P}_i \setminus \{P^i_{e_i}\})$ and, by (B), there are at most $\varepsilon^3 n$ indices $i \in [m]$ such that $v \in V^0(\mathcal{P}_i \setminus \{P^i_{e_i}\})$. Moreover, by (C) and construction of the A'_i, there are, for each $v \in V$, at most $\lceil \frac{\ell}{a} \rceil$ indices $i \in [m]$ such that $v \in V(E(P^i_{e_i}) \cap E(\Gamma))$. Hence, each $v \in V$ satisfies

$$|N_{\mathcal{H}^m \cap \Gamma}(v)| \leqslant \varepsilon^3 n + 2\varepsilon^3 n + 2\left\lceil \frac{\ell}{a} \right\rceil \leqslant 3\varepsilon^3 n + \frac{2\varepsilon^2 n}{\varepsilon^{-1}\nu^4} + 2 \leqslant 3\varepsilon^3 \nu^{-4} n. \tag{7.2}$$

Assume $m = \ell$. Then, by (A), $\mathcal{H}_i$ is a spanning configuration of shape (L_i, F_i) for each $i \in [\ell]$. Moreover, (i) holds by (7.2) and we are done.

Assume $m < \ell$. Using (α')–(ε'), (7.2), (b), and Lemma 4.2(a), it is easy to check that the following hold.

(I) $\Gamma_m[A'_{m+1}]$ and $\Gamma_m - A'_{m+1}$ are robust $(\frac{\nu}{4}, 2\tau)$-outexpanders.
(II) $\Gamma_m[A'_{m+1}]$ and $\Gamma_m - A'_{m+1}$ are both $(\gamma, 4\varepsilon)$-almost regular.
(III) $D_m - A'_{m+1}$ is $(\delta - \frac{m}{n}, 4\varepsilon)$-almost regular.
(IV) For each $v \in V \setminus A'_{m+1}$, $|N^{\pm}_{D_m}(v) \cap A'_{m+1}| \geqslant \frac{\varepsilon\delta n}{3\nu^4}$.

We first construct P^{m+1}_e for each $e \in E(L_{m+1}) \setminus (F_{m+1} \cup \{e_{m+1}\})$ in the following way. Let S be the set of vertices $v \in V$ for which there exist $\lfloor \varepsilon^3 n \rfloor$ indices $i \in [m]$ such that $v \in V^0(\mathcal{P}_i \setminus \{P^i_{e_i}\})$. Observe that, by (a) and (B), $|S| \leqslant \frac{8\nu^{-1} \cdot \ell \cdot \varepsilon^4 n}{\lfloor \varepsilon^3 n \rfloor} \leqslant \varepsilon |V \setminus A'_{m+1}|$. Denote $E(L_{m+1}) \setminus (F_{m+1} \cup \{e_{m+1}\}) =: \{x_1 x'_1, \dots, x_k x'_k\}$. Apply Corollary 4.6 with $\Gamma_m - A'_{m+1}, \frac{\nu}{4}, 2\tau, \gamma - 4\varepsilon$, and $S \cup V(L_{m+1})$ playing the roles of D, ν, τ, δ, and S to obtain internally vertex-disjoint paths $P^{m+1}_{x_1 x'_1}, \dots, P^{m+1}_{x_k x'_k} \subseteq \Gamma_m - A'_{m+1}$ such that, for each $i \in [k]$, $P^{m+1}_{x_i x'_i}$ is an (x_i, x'_i)-path of length at most $8\nu^{-1}$ with $V^0(P^{m+1}_{x_i x'_i}) \subseteq V \setminus (A'_{m+1} \cup S \cup V(L_{m+1}))$. Let $\mathcal{P}'_{m+1} := \{P^{m+1}_{x_i x'_i} \mid i \in [k]\}$.

Let $z \notin V$ be a new vertex. Let H be the digraph on vertex set $V(H) := V \setminus (V(L_{m+1}) \cup V(\mathcal{P}'_{m+1})) \cup \{z\}$ defined as follows. Denote $v^+ v^- := e_{m+1}$ and recall that, by construction, $v^{\pm} \notin A'_{m+1}$. Then, let $N^{\pm}_H(z) := N^{\pm}_{D_m}(v^{\pm}) \cap V(H)$, $H[A'_{m+1}] := \Gamma_m[A'_{m+1}]$, and, for each $v \in V(H) \setminus (A'_{m+1} \cup \{z\})$, $N^{\pm}_{H-\{z\}}(v) := N^{\pm}_{D_m}(v) \cap V(H)$. Note that, by (I)–(IV), the following hold.

(I′) $H[A'_{m+1}]$ is a robust $(\frac{\nu}{4}, 2\tau)$-outexpander.
(II′) $H[A'_{m+1}]$ is $(\gamma, 4\varepsilon)$-almost regular.
(III′) $H - A'_{m+1}$ is $(\delta - \frac{m}{n}, 5\varepsilon)$-almost regular.
(IV′) For each $v \in V(H) \setminus A'_{m+1}$, $|N^{\pm}_H(v) \cap A'_{m+1}| \geqslant \frac{\varepsilon\delta n}{3\nu^4}$.

Indeed, to check (III′), note that, by (a), $H - A'_{m+1}$ is obtained from $D_m - A'_{m+1}$ by adding z and deleting $|V(L_{m+1}) \cup V(\mathcal{P}'_{m+1})| \leqslant \varepsilon^2 n + \varepsilon^4 n \cdot 8\nu^{-1} \leqslant 2\varepsilon^2 n$ vertices.

Our aim is to find a Hamilton cycle of H which contains few edges of $\Gamma[A'_{m+1}]$. First, we cover $V(H) \setminus A'_{m+1}$ with a small number of paths as follows. Let $k' := \lfloor \frac{|V(H) \setminus A'_{m+1}|}{\varepsilon n} \rfloor$. Apply Lemma 4.17 with $H - A'_{m+1}, |V(H) \setminus A'_{m+1}|, \delta - \frac{m}{n}$, and 5ε playing the roles of D, n, δ, and ε to obtain a partition $V_1, \dots, V_{k'}$ of $V(H) \setminus A'_{m+1}$ such that, for each $i \in [k']$, $|V_i| = (1 \pm 2\varepsilon)\varepsilon n$ and, for each $i \in [k']$ and $v \in V_i$, $|N^-_H(v) \cap V_{i-1}| = (\delta - \frac{m}{n} \pm 10\varepsilon)\varepsilon n$ if $i > 1$ and $|N^+_H(v) \cap V_{i+1}| = (\delta - \frac{m}{n} \pm 10\varepsilon)\varepsilon n$ if $i < k'$.

Then, for each $i \in [k' - 1]$, apply Proposition 4.19 with $H[V_i, V_{i+1}]$, V_i, V_{i+1}, εn, $\delta - \frac{m}{n}$, and 10ε playing the roles of G, A, B, n, δ, and ε to obtain a matching M_i of $H[V_i, V_{i+1}]$ of size at least

$(1-\frac{31\varepsilon}{\delta})\varepsilon n$. For each $i \in [k'-1]$, denote by $\overrightarrow{M}_i$ the directed matching obtained from M_i by directing all edges from V_i to V_{i+1}. Note that, by construction, $\overrightarrow{M}_i \subseteq H$. Define $F \subseteq H$ by letting $V(F) := V(H) \setminus A'_{m+1}$ and $E(F) := \bigcup_{i\in[k'-1]} \overrightarrow{M}_i$. Observe that F is a linear forest which spans $V(H) \setminus A'_{m+1}$ and has $f \leqslant \frac{33\varepsilon n}{\delta}$ components. Indeed, one can count the number of paths in F by counting the number of ending points as follows. (An isolated vertex is considered as the ending point of a trivial path of length 0.) Note that, for each $i \in [k'-1]$, $v \in V_i$ is the ending point of a path in F if and only if $v \notin V(M_i)$, while every $v \in V_{k'}$ is the ending point of a path in F. Moreover, for each $i \in [k'-1]$, we have $|V_i \setminus V(M_i)| \leqslant |V_i| - |M_i| \leqslant \varepsilon n + 2\varepsilon^2 n - (1-\frac{31\varepsilon}{\delta})\varepsilon n \leqslant \frac{32\varepsilon^2 n}{\delta}$. Thus, since $k'-1 \leqslant \varepsilon^{-1}-1$, we have $f \leqslant \frac{32\varepsilon^2 n}{\delta}(\varepsilon^{-1}-1) + |V_k| \leqslant \frac{33\varepsilon n}{\delta}$, as desired.

Denote the components of F by $P_1, \dots, P_f$. We now join $P_1, \dots, P_f$ into a Hamilton cycle as follows. Note that, by (α'), $f \leqslant (\frac{\nu}{4})^3 |A'_{m+1}|$. For each $i \in [f]$, denote by v_i^+ and v_i^- the starting and ending points of P_i. By (IV$'$), for each $i \in [f]$, we have $|N_H^{\mp}(v_i^{\pm}) \cap A'_{m+1}| \geqslant 2f$. Apply Corollary 4.8(c) with $H, A'_{m+1}, \emptyset, f, \frac{\nu}{4}, 2\tau$, and $\gamma - \nu$ playing the roles of D, V', S, k, ν, τ, and δ to obtain a Hamilton cycle C of H such that $F \subseteq C$. Denote by $u^{\pm}$ the (unique) vertices such that $u^{\pm} \in N_C^{\pm}(z)$, respectively. Let $P_{e_{m+1}}^{m+1} := (C - \{z\}) \cup \{v^+u^+, u^-v^-\}$. By construction, $P_{e_{m+1}}^{m+1}$ is a path of shape e_{m+1} such that $P_{e_{m+1}}^{m+1} \subseteq (D_m \cup \Gamma_m) - V(\mathcal{P}'_{m+1})$ and $V^0(P_{e_{m+1}}^{m+1}) = V \setminus (V(\mathcal{P}'_{m+1}) \cup V(L_{m+1}))$. Moreover, $P_{e_{m+1}}^{m+1}[A'_{m+1}] \subseteq \Gamma_m$ and $P_{e_{m+1}}^{m+1} \setminus P_{e_{m+1}}^{m+1}[A'_{m+1}] \subseteq D_m$. Let $\mathcal{P}_{m+1} := \mathcal{P}'_{m+1} \cup \{P_{e_{m+1}}^{m+1}\}$. Thus, (A)–(C) hold. This completes the induction step. □

8 | GOOD PARTIAL PATH DECOMPOSITIONS AND ABSORBING EDGES

Lemma 7.3 only covers most of the edges. Moreover, we will see that we also need an extra cleaning step before being able to apply Lemma 7.3. This means that our path decomposition will be constructed in several stages.

Suppose that we have already constructed an intermediate set of paths $\mathcal{P}$ and that we want to extend $\mathcal{P}$ to a path decomposition of T. Then, $\widetilde{\mathrm{ex}}(T \setminus \mathcal{P})$ must not be too large (for otherwise we will not have any hope of extending $\mathcal{P}$ to a path decomposition of the desired size $\widetilde{\mathrm{ex}}(T)$). This is encapsulated in the concept of a good partial path decomposition, which is defined and discussed in Section 8.1.

Moreover, we will need to make sure that, in the last stage, the remaining digraph D has a nice structure (for otherwise we may not know how to decompose D). In Corollary 4.10, we saw an example of a digraph that we can decompose efficiently. Unfortunately, it will not always be possible to get a leftover of that form, so, in Section 8.2, we will generalise Corollary 4.10 using the concept of absorbing edges.

8.1 | Partial path decompositions

Recall that $U^{\pm}(D) := \{v \in V(D) \mid \mathrm{ex}_D^{\pm}(v) > 0\}$ and $U^0(D) := \{v \in V(D) \mid \mathrm{ex}_D(v) = 0\}$.

Proposition 8.1. *Any oriented graph D satisfies $|U^0(D)| \geqslant \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$.*

Proof. Assume for a contradiction that there exists an oriented graph D such that $|U^0(D)| < \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$. Then, note that $\widetilde{\mathrm{ex}}(D) = \Delta^0(D)$ and let $v \in V$ be such that $d_D^{\max}(v) = \Delta^0(D)$. Assume without loss of generality that $v \in U^+(D)$. Then, $d_D^+(v) = \widetilde{\mathrm{ex}}(D) > \mathrm{ex}(D)$. By Fact 4.21, $\mathrm{ex}(D) \geqslant \mathrm{ex}_D^+(v) + |U^+(D)| - 1$ and so $|U^+(D)| \leqslant \mathrm{ex}(D) - \mathrm{ex}_D^+(v) + 1$. Moreover, by assumption, we have $|U^0(D)| < d_D^+(v) - \mathrm{ex}(D)$. Therefore, by Facts 4.20(b) and 4.20(c), we have

$$\begin{aligned}\mathrm{ex}(D) \geqslant |U^-(D)| &= n - |U^+(D)| - |U^0(D)| > n - (\mathrm{ex}(D) - \mathrm{ex}_D^+(v) + 1) - (d_D^+(v) - \mathrm{ex}(D))\\ &= n - 1 - d_D^-(v) \geqslant d_D^+(v),\end{aligned}$$

a contradiction. □

Let D be an oriented graph. Recall that in a path decomposition $\mathcal{P}$ of D of size $\widetilde{\mathrm{ex}}(D)$, each $v \in V(D)$ will be the starting point of at least $\mathrm{ex}_D^+(v)$ paths in $\mathcal{P}$ and the ending point of at least $\mathrm{ex}_D^-(v)$ paths in $\mathcal{P}$. When $\widetilde{\mathrm{ex}}(D) > \mathrm{ex}(D)$, there are $\widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$ starting (and ending) points unaccounted for. By Proposition 8.1, we can choose these endpoints (to be distinct vertices) in $U^0(D)$. Thus, our path decomposition $\mathcal{P}$ will also maximise the number of distinct vertices that are an endpoint of some path in $\mathcal{P}$. This motivates the following definition.

Definition 8.2 (Partial path decomposition). Let D be a digraph. A set $\mathcal{P}$ of edge-disjoint paths of D is called a *partial path decomposition of D* if the following hold.

(P1) Any vertex $v \in V(D) \setminus U^0(D)$ is the starting point of at most $\mathrm{ex}_D^+(v)$ paths in $\mathcal{P}$ and the ending point of at most $\mathrm{ex}_D^-(v)$ paths in $\mathcal{P}$.
(P2) Any vertex $v \in U^0(D)$ is the starting point of at most one path in $\mathcal{P}$ and the ending point of at most one path in $\mathcal{P}$.
(P3) There are at most $\widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$ vertices $v \in U^0(D)$ such that v is an endpoint of a path in $\mathcal{P}$, that is, $|U^0(D) \cap (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))| \leqslant \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$.

By (P3), we will need to construct sets of edge-disjoint paths which do not contain too many paths which start and/or end at vertices of zero excess. It will turn out to be convenient to fix in advance which zero-excess vertices will be used as endpoints. This motivates the following definition. Let D be a digraph and suppose that $U^* \subseteq U^0(D)$ satisfies $|U^*| \leqslant \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$. We say that $\mathcal{P}$ is a *U^*-partial path decomposition of D* if $\mathcal{P}$ is a partial path decomposition where $(V^+(\mathcal{P}) \cup V^-(\mathcal{P})) \cap U^0(D) \subseteq U^*$, that is, no path in $\mathcal{P}$ has an endpoint in $U^0(D) \setminus U^*$.

Let D be a digraph. Recall that in Theorem 1.8, we defined

$$N^{\pm}(D) = |U^{\pm}(D)| + \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D). \tag{8.1}$$

Note that (P1) and (P3) imply that if $\mathcal{P}$ is a partial path decomposition of D, then there are at most $N^+(D)$ distinct vertices which are the starting point of a path in $\mathcal{P}$ ((P1) implies that the vertices in $U^-(D)$ cannot be used as starting points and (P3) implies that at most $\widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$ vertices in $U^0(D)$ may be used as starting points). Similarly, there are at most $N^-(D)$ distinct vertices which are the ending point of a path in $\mathcal{P}$.

Proposition 8.3. *Let D be a digraph and $\mathcal{P}$ be a partial path decomposition of D. Then,* $\mathrm{ex}(D \setminus \mathcal{P}) = \mathrm{ex}(D) - |\mathcal{P}| + |U^0(D) \cap (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))| \leqslant \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$.

Proof. By (P3), $|U^0(D) \cap (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))| \leqslant \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$ and so it is enough to show that $\mathrm{ex}(D \setminus \mathcal{P}) = \mathrm{ex}(D) - |\mathcal{P}| + |U^0(D) \cap (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))|$. For each $v \in V(D)$, denote by $n_{\mathcal{P}}^+(v)$ and

$n^-_{\mathcal{P}}(v)$ the number of paths on $\mathcal{P}$ which start and end at v, respectively. Let $v \in V(D)$ and note that

$$\begin{aligned}\mathrm{ex}_{D\setminus\mathcal{P}}(v) &= d^+_{D\setminus\mathcal{P}}(v) - d^-_{D\setminus\mathcal{P}}(v) = (d^+_D(v) - d^-_D(v)) - (d^+_{\mathcal{P}}(v) - d^-_{\mathcal{P}}(v)) \\ &= \mathrm{ex}_D(v) - n^+_{\mathcal{P}}(v) + n^-_{\mathcal{P}}(v). \end{aligned} \tag{8.2}$$

Let n^+ be the number of paths in $\mathcal{P}$ which start in $U^+(D)$. Since $\mathcal{P}$ is a partial path decomposition, we have

$$\begin{aligned}\mathrm{ex}(D\setminus\mathcal{P}) &\overset{(1.1)}{=} \sum_{v\in V(D)} \mathrm{ex}^+_{D\setminus\mathcal{P}}(v) \overset{(\mathrm{P1}),(\mathrm{P2})}{=} \sum_{v\in U^+(D)} \mathrm{ex}^+_{D\setminus\mathcal{P}}(v) + |U^0(D)\cap(V^-(\mathcal{P})\setminus V^+(\mathcal{P}))| \\ &\overset{(\mathrm{P1})}{=} \left(\sum_{v\in U^+(D)} \mathrm{ex}^+_D(v) - n^+\right) + |U^0(D)\cap(V^-(\mathcal{P})\cup V^+(\mathcal{P}))| - |U^0(D)\cap V^+(\mathcal{P})| \\ &\overset{(\mathrm{P2})}{=} \mathrm{ex}(D) - |\mathcal{P}| + |U^0(D)\cap(V^-(\mathcal{P})\cup V^+(\mathcal{P}))|,\end{aligned}$$

as desired. □

Let D be a digraph and $\mathcal{P}$ be a partial path decomposition of D. The next proposition expands on Proposition 8.3 to give further bounds on $\mathrm{ex}(D\setminus\mathcal{P})$ and $\widetilde{\mathrm{ex}}(D\setminus\mathcal{P})$.

Proposition 8.4. *Let D be a digraph and $\mathcal{P}$ be a partial path decomposition of D. Then, the following hold.*

(a) *If* $\Delta^0(D\setminus\mathcal{P}) \leqslant \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$, *then* $\widetilde{\mathrm{ex}}(D\setminus\mathcal{P}) = \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$.
(b) *If* $\widetilde{\mathrm{ex}}(D) = \mathrm{ex}(D)$, *then* $\mathrm{ex}(D\setminus\mathcal{P}) = \mathrm{ex}(D) - |\mathcal{P}|$.

Proof. Note that it is enough to show that the following inequalities hold.

$$\mathrm{ex}(D) - |\mathcal{P}| \leqslant \mathrm{ex}(D\setminus\mathcal{P}) \leqslant \widetilde{\mathrm{ex}}(D) - |\mathcal{P}| \leqslant \widetilde{\mathrm{ex}}(D\setminus\mathcal{P}). \tag{8.3}$$

Indeed, if $\widetilde{\mathrm{ex}}(D) = \mathrm{ex}(D)$, then the first two inequalities of (8.3) are equalities implying (b). If $\Delta^0(D\setminus\mathcal{P}) \leqslant \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$, then together with the last two inequalities of (8.3), we deduce that

$$\mathrm{ex}(D\setminus\mathcal{P}), \Delta^0(D\setminus\mathcal{P}) \leqslant \widetilde{\mathrm{ex}}(D) - |\mathcal{P}| \leqslant \widetilde{\mathrm{ex}}(D\setminus\mathcal{P}) = \max\{\Delta^0(D\setminus\mathcal{P}), \mathrm{ex}(D\setminus\mathcal{P})\}.$$

Thus, we must have equalities, which implies (a).

First, consider the case where $\widetilde{\mathrm{ex}}(D) = \mathrm{ex}(D)$. By (P3), no path in $\mathcal{P}$ has an endpoint in $U^0(D)$. Thus, Proposition 8.3 implies that

$$\widetilde{\mathrm{ex}}(D) - |\mathcal{P}| = \mathrm{ex}(D) - |\mathcal{P}| = \mathrm{ex}(D\setminus\mathcal{P}) \leqslant \widetilde{\mathrm{ex}}(D\setminus\mathcal{P})$$

and so (8.3) holds. We may therefore assume that $\widetilde{\mathrm{ex}}(D) = \Delta^0(D) \neq \mathrm{ex}(D)$. Clearly,

$$\widetilde{\mathrm{ex}}(D) - |\mathcal{P}| = \Delta^0(D) - |\mathcal{P}| \leqslant \Delta^0(D\setminus\mathcal{P}) \leqslant \widetilde{\mathrm{ex}}(D\setminus\mathcal{P}). \tag{8.4}$$

By Proposition 8.3, we have

$$\mathrm{ex}(D) - |\mathcal{P}| \leqslant \mathrm{ex}(D \setminus \mathcal{P}) \leqslant \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|. \tag{8.5}$$

Therefore, (8.3) follows from (8.4) and (8.5). □

Let D be a digraph. We say that a partial path decomposition $\mathcal{P}$ of D is *good* if $\widetilde{\mathrm{ex}}(D \setminus \mathcal{P}) = \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$. We say that a path decomposition $\mathcal{P}$ of D is *perfect* if $|\mathcal{P}| = \widetilde{\mathrm{ex}}(D)$.

Fact 8.5. *Let $k \in \mathbb{N}$ and D be a digraph. Denote $D_0 := D$. Suppose that, for each $i \in [k-1]$, $\mathcal{P}_i$ is a good partial path decomposition of D_{i-1} and $D_i := D_{i-1} \setminus \mathcal{P}_i$. Suppose that $\mathcal{P}_k$ is a perfect path decomposition of D_{k-1}. Then, $\mathcal{P} := \bigcup_{i\in[k]} \mathcal{P}_i$ is a perfect path decomposition of D.*

Let D be an oriented graph on n vertices. The next proposition shows that if there is a vertex $v \in V(D)$ with $d_D^+(v) \geqslant \mathrm{ex}(D) - \varepsilon n$, then $\widetilde{\mathrm{ex}}(D) \leqslant (1+\varepsilon)n$ (Proposition 8.6(a.i)) and most of the positive excess of D is concentrated at v (Proposition 8.6(a.ii)). Proposition 8.6(b) gives a sufficient condition for a small partial path decomposition to be good.

Proposition 8.6. *Let $0 < \frac{1}{n} \ll \eta \ll 1$. Let D be an oriented graph on n vertices satisfying $\mathrm{ex}(D) \geqslant (1-21\eta)n$. Let $V^\pm := \{v \in V(D) \mid d_D^\pm(v) \geqslant \widetilde{\mathrm{ex}}(D) - 22\eta n\}$. Then, the following hold.*

(a) *If $V^\diamond \neq \emptyset$ for some $\diamond \in \{+,-\}$, then the following hold.*
   (i) $\widetilde{\mathrm{ex}}(D) \leqslant (1+22\eta)n \leqslant \mathrm{ex}(D) + 43\eta n$.
   (ii) $\mathrm{ex}_D^\diamond(v) \geqslant (1-86\eta)n \geqslant \mathrm{ex}(D) - 108\eta n$ *for all* $v \in V^\diamond$.
(b) *Let $\mathcal{P}$ be a partial path decomposition of D of size $|\mathcal{P}| \leqslant 22\eta n$. Suppose that both $V^\pm \subseteq V^\pm(P) \cup V^0(P)$ for each $P \in \mathcal{P}$. Then, $\mathcal{P}$ is good.*

Proof. For (a), we may assume that there exists $v \in V^+$ (similar arguments hold if $V^- \neq \emptyset$). Since $\mathrm{ex}(D) \geqslant (1-21\eta)n$, we have

$$\widetilde{\mathrm{ex}}(D) \leqslant d_D^+(v) + 22\eta n \leqslant (1+22\eta)n \leqslant \mathrm{ex}(D) + 43\eta n.$$

Thus, (a.i) holds. By assumption, $d_D^+(v) \geqslant \widetilde{\mathrm{ex}}(D) - 22\eta n \geqslant \mathrm{ex}(D) - 22\eta n \geqslant \frac{n}{2}$ and so $d_D^-(v) \leqslant d_D^+(v)$. Thus, $\mathrm{ex}_D^+(v) = \mathrm{ex}_D(v)$ and so

$$\begin{aligned}\mathrm{ex}_D^+(v) &\overset{\text{Fact 4.20(c)}}{=} 2d_D^+(v) - d_D(v) \geqslant 2(\widetilde{\mathrm{ex}}(D) - 22\eta n) - n \\ &\geqslant \quad (1-86\eta)n \overset{\text{(a.i)}}{\geqslant} \widetilde{\mathrm{ex}}(D) - 108\eta n \geqslant \mathrm{ex}(D) - 108\eta n.\end{aligned}$$

Thus, (a.ii) holds.

For (b), let $\mathcal{P}$ be a partial path decomposition of D of size $|\mathcal{P}| \leqslant 22\eta n$. Suppose that both $V^\pm \subseteq V^\pm(P) \cup V^0(P)$ for each $P \in \mathcal{P}$. We need to show that $\mathcal{P}$ is good, that is, that $\widetilde{\mathrm{ex}}(D \setminus \mathcal{P}) = \max\{\mathrm{ex}(D \setminus \mathcal{P}), \Delta^0(D \setminus \mathcal{P})\} = \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$. By Proposition 8.4(a), it is enough to show that $\Delta^0(D \setminus \mathcal{P}) \leqslant \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$. Let $v \in V(D)$. We need to show that both $d_{D\setminus\mathcal{P}}^\pm(v) \leqslant \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$. If both $d_D^\pm(v) \leqslant \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$, then we are done. We may therefore assume without loss of generality that

$d_D^+(v) \geqslant \widetilde{\text{ex}}(D) - |\mathcal{P}| \geqslant \widetilde{\text{ex}}(D) - 22\eta n$. Then, $v \in V^+$ and so, by assumption, $d_P^+(v) = 1$ for each $P \in \mathcal{P}$. Thus,

$$d_{D\setminus\mathcal{P}}^+(v) = d_D^+(v) - d_{\mathcal{P}}^+(v) = d_D^+(v) - |\mathcal{P}| \leqslant \Delta^0(D) - |\mathcal{P}| \leqslant \widetilde{\text{ex}}(D) - |\mathcal{P}|.$$

Moreover,

$$\begin{aligned} d_{D\setminus\mathcal{P}}^-(v) \leqslant d_D^-(v) = d_D(v) - d_D^+(v) &\leqslant n - (\widetilde{\text{ex}}(D) - 22\eta n) \\ &\leqslant (1 + 22\eta)n - \text{ex}(D) \leqslant 43\eta n \leqslant \widetilde{\text{ex}}(D) - |\mathcal{P}|, \end{aligned}$$

as desired. □

8.2 | Completing path decompositions via absorbing edges

As discussed in the proof overview, the goal is to complete our path decomposition by applying Corollary 4.10. However, this requires linearly many vertices to serve as endpoints, which may not always be possible. The concept of absorbing edges provides an approach to overcome this issue. We motivate this concept via the following example. Let D be a digraph and $w \in V(D)$ with $\text{ex}_D^+(w)$ large. Suppose that $v \in N_D^+(w) \cap U^0(D)$. Note that in $D \setminus \{wv\}$, the excess of v is now 1 instead of 0. Moreover, $U^+(D \setminus \{wv\}) = U^+(D) \cup \{v\}$ and so the number of possible distinct starting points increases by one. A perfect path decomposition of $D \setminus \{wv\}$ must have a path P starting at v. If P does not contain w, then we can extend it to start at w by adding the edge wv and so we obtain a perfect decomposition of D. We can view the edge wv as an absorbing starting edge which absorbs the path P. This motivates the following definition.

Definition 8.7 (Absorbing sets of edges). Let D be a digraph. Let $W, V' \subseteq V(D)$ be disjoint.

- An *absorbing set of* (W, V')*-starting edges (for* D*)* is a set $A \subseteq E(D)$ of edges with starting point in W and ending point in V' such that, for each $w \in W$, at most $\text{ex}_D^+(w)$ edges in A start at w, and, for each $v \in V'$, at most one edge in A ends at v.
- An *absorbing set of* (V', W)*-ending edges (for* D*)* is a set $A \subseteq E(D)$ of edges with starting point in V' and ending point in W such that, for each $w \in W$, at most $\text{ex}_D^-(w)$ edges in A end at w, and, for each $v \in V'$, at most one edge in A starts at v.
- A (W, V')*-absorbing set (for* D*)* is the union of an absorbing set of (W, V')-starting edges and an absorbing set of (V', W)-ending edges.

Let D be a digraph. Let $W, V' \subseteq V(D)$ be disjoint. Recall that an absorbing (W, V')-starting edge wv can only absorb a path starting at v that does not contain w. Thus, we will find a path decomposition in $D[V']$. We will find this decomposition via Corollary 4.10. For this, we need to adapt the degree conditions to account for the absorbing paths a vertex is involved in. This is formalised in the following corollary.

Corollary 8.8. *Let* $0 < \frac{1}{n} \ll \nu \ll \tau \leqslant \frac{\delta}{2} \leqslant 1$ *and* $r \geqslant \delta n$. *Suppose that* D *is a digraph with a vertex partition* $V(D) = W \cup V'$ *such that* $D[V']$ *is a robust* (ν, τ)*-outexpander on* n *vertices. Suppose that* $A^+, A^- \subseteq E(D)$ *are absorbing sets of* (W, V')*-starting and* (V', W)*-ending edges such that* $|A^\pm| \leqslant r$. *Denote* $A := A^+ \cup A^-$. *Suppose furthermore that there exists a partition* $V' = X^+ \cup X^- \cup X^* \cup$

X^0 such that $|X^\pm \cup X^*| + |A^\pm| = r$, $V(A^\pm) \cap V' \subseteq X^\mp \cup X^0$, and, for each $v \in V(D)$, the following hold.

$$\mathrm{ex}_D(v) = \begin{cases} \mathrm{ex}_A(v) & \textit{if } v \in W, \\ \pm 1 & \textit{if } v \in X^\pm, \\ 0 & \textit{if } v \in X^* \cup X^0, \end{cases} \quad \textit{and} \quad d_D(v) = \begin{cases} d_A(v) & \textit{if } v \in W, \\ 2r-1 & \textit{if } v \in X^\pm, \\ 2r-2 & \textit{if } v \in X^*, \\ 2r & \textit{if } v \in X^0. \end{cases}$$

Then, $\mathrm{pn}(D) = r$.

Proof. By Corollary 4.10, we may assume that $A \neq \emptyset$. Hence, $\mathrm{pn}(D) \geqslant \Delta^0(D) = r$. Thus, it suffices to find a path decomposition of D of size r.

Let

$$\begin{aligned} Y^\pm &:= (X^\pm \cup (V(A^\pm) \cap V')) \setminus (X^\mp \cup V(A^\mp)) = (X^\pm \cup (V(A^\pm) \cap X^0)) \setminus V(A^\mp), \\ Y^* &:= X^* \cup (V(A^+) \cap V(A^-)) \cup (X^+ \cap V(A^-)) \cup (X^- \cap V(A^+)), \text{ and} \\ Y^0 &:= X^0 \setminus (V(A^+) \cup V(A^-)). \end{aligned}$$

Then, observe that Y^+, Y^-, Y^*, and Y^0 are all pairwise disjoint and form a partition of V'. Moreover, $|Y^\pm \cup Y^*| = |X^\pm \cup X^* \cup (V(A^\pm) \cap V')| = r$ and, for each $v \in V'$, the following hold.

$$\mathrm{ex}_{D[V']}(v) = \begin{cases} \pm 1 & \text{if } v \in Y^\pm, \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad d_{D[V']}(v) = \begin{cases} 2r-1 & \text{if } v \in Y^\pm, \\ 2r-2 & \text{if } v \in Y^*, \\ 2r & \text{otherwise.} \end{cases}$$

Thus, we can apply Corollary 4.10 with $D[V'], Y^\pm, Y^*$, and Y^0 playing the roles of $D, X^\pm, X^*$, and X^0 to obtain a path decomposition $\mathcal{P} = \{P_1, \dots, P_r\}$ of $D[V']$ of size r. For each $i \in [r]$, let v_i^+ and v_i^- denote the starting and ending points of P_i. By the 'moreover part' of Corollary 4.10, we may assume that $v_1^+, \dots, v_r^+$ are distinct and $\{v_i^+ \mid i \in [r]\} = Y^+ \cup Y^*$. We may also assume that $v_1^-, \dots, v_r^-$ are distinct and $\{v_i^- \mid i \in [r]\} = Y^- \cup Y^*$.

We use A^+ to absorb the paths starting at $V(A^+) \cap V'$ as follows. For each $i \in [r]$, if $v_i^+ \notin V(A^+)$, then let $P_i^+ := P_i$; otherwise, denote by $w_i^+ v_i^+$ the unique edge in A^+ which is incident to v_i^+ and let $P_i^+ := w_i^+ v_i^+ P_i v_i^-$. Then, absorb the paths ending in $V(A^-) \cap V'$ similarly. For each $i \in [r]$, if $v_i^- \notin V(A^-)$, then let $P_i^- := P_i^+$; otherwise, denote by $v_i^- w_i^-$ the unique edge in A^- which is incident to v_i^- and let P_i^- be obtained by concatenating P_i^+ and $v_i^- w_i^-$. Since $d_D(v) = d_A(v)$ for each $v \in W$, it follows that $\mathcal{P}' := \{P_i^- \mid i \in [r]\}$ is a path decomposition of D of size r, as desired. □

Although the absorbing set is chosen at the beginning, we do not remove this set as it may affect our calculation of $\widetilde{\mathrm{ex}}(D)$. Thus, we require all our partial path decompositions to avoid the edges in the absorbing set. Moreover, their endpoints should not 'overuse' the vertices in $V(A)$.

Definition 8.9 (Consistent partial path decomposition). Let D be a digraph, let $W, V' \subseteq V(D)$ be disjoint, and $A \subseteq E(D)$. Note that W and V' do not necessarily partition $V(D)$. Suppose that A is a (W, V')-absorbing set. Then, a partial path decomposition $\mathcal{P}$ of D is *consistent* with A if $\mathcal{P} \subseteq D \setminus A$

and each $v \in W$ is the starting point of at most $\mathrm{ex}_D^+(v) - d_A^+(v)$ paths in $\mathcal{P}$ and the ending point of at most $\mathrm{ex}_D^-(v) - d_A^-(v)$ paths in $\mathcal{P}$.

Definition 8.10 ((U^*, W, A)-partial path decomposition). Let D be a digraph, let $W, V' \subseteq V(D)$ be disjoint, and $A \subseteq E(D)$. Suppose that A is a (W, V')-absorbing set. Given $U^* \subseteq U^0(D)$ satisfying $|U^*| \leqslant \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$, we say that $\mathcal{P}$ is a (U^*, W, A)-*partial path decomposition* if $\mathcal{P}$ is a U^*-partial path decomposition which is consistent with A.

Let $\mathcal{P}$ be a partial path decomposition of D which is consistent with A. By definition, A is still a (W, V')-absorbing set for $D \setminus \mathcal{P}$.

Fact 8.11. *Let D be a digraph and $W, V' \subseteq V(D)$ be disjoint. Suppose that A is a (W, V')-absorbing set. Suppose $\mathcal{P}$ is a partial path decomposition of D which is consistent with A. Denote $D' := D \setminus \mathcal{P}$. Then, A is a (W, V')-absorbing set for D'.*

9 | CONSTRUCTING LAYOUTS IN GENERAL TOURNAMENTS

In this section, we discuss how to construct layouts in general tournaments. Recall that Lemma 7.3 (which constructs an approximate decomposition which respects a given set of layouts) only applies to almost regular robust outexpanders. In general, our tournament T will not be almost regular nor a robust outexpander. In Section 9.1, we discuss how to circumvent this problem. As discussed in Section 8.1, we will need the set of paths obtained with Lemma 7.3 to form a good partial path decomposition. In Section 9.2, we explain how we can ensure this. In Section 9.3, we discuss the cleaning step. In Section 9.4, we state the lemma which guarantees the existence of suitable layouts.

9.1 | W-exceptional layouts

Let T be a tournament on n vertices with $\mathrm{ex}(T) \leqslant \varepsilon n^2$ for some small constant ε. Then, there exists a partition of $V(T)$ into W and V' such that W is small and $T[V']$ is almost regular. Our aim is to apply Lemma 7.3 to $T[V']$. To do so, we will construct layouts $(L_1, F_1), \dots, (L_\ell, F_\ell)$ so that $E_W(T) \subseteq \bigcup_{i \in [\ell]} F_i$. (Recall from Section 3 that $E_W(T)$ denotes the set of edges of T which are incident to W.) This will ensure that all the edges in $E_W(T)$ will be contained in the partial path decomposition obtained from the spanning configurations of shapes $(L_1, F_1), \dots, (L_\ell, F_\ell)$.

Let V be a vertex set and $W \subseteq V$. We say that a layout (L, F) is W-*exceptional* if $E_W(L) \subseteq F$. Let (L, F) be a W-exceptional layout. A multidigraph $\mathcal{H}$ on V is a W-*exceptional spanning configuration of shape* (L, F) if $\mathcal{H}$ can be decomposed into internally vertex-disjoint paths $\{P_e \mid e \in E(L)\}$ such that each P_e has shape e; $P_f = f$ for all $f \in F$; and $\bigcup_{e \in E(L)} V^0(P_e) = V \setminus (V(L) \cup W)$. (Note that the last equality implies that the vertices in $W \setminus V(L)$ are isolated in $\mathcal{H}$.) Thus, roughly speaking, a W-exceptional spanning configuration of shape (L, F) is one such that all 'additional' edges (that is, those edges of $\mathcal{H}$ that are not in F) are disjoint from W. A path decomposition of $\mathcal{H}$ is *induced* by (L, F) if it consists of all the paths $P_Q := \{P_e \mid e \in E(Q)\}$ where Q is a path in L. The analogue of Fact 7.1 holds for W-exceptional spanning configurations.

Fact 9.1. *Let V be a vertex set and $W \subseteq V$. Let (L, F) be a W-exceptional layout on V and $\mathcal{H}$ be a W-exceptional spanning configuration of shape (L, F) on V. Let L' denote the set of (non-trivial) paths contained in L. Suppose that $\mathcal{P}$ is a path decomposition of $\mathcal{H}$ which is induced by (L, F). Then, $V^{\pm}(\mathcal{P}) = V^{\pm}(L')$ and $V^0(\mathcal{P}) = V^0(L) \cup (V \setminus (W \cup V(L)))$.*

We now show that there is a natural transformation of a W-exceptional layout (L, F) into an auxiliary layout $(L^{\restriction W}, F^{\restriction W})$ on $V \setminus W$. Roughly speaking, the auxiliary layout $(L^{\restriction W}, F^{\restriction W})$ is obtained from (L, F) by contracting all the edges in $E_W(L)$ so that $E_W(L^{\restriction W}) = \emptyset = E_W(F^{\restriction W})$ and then remove W.

Definition 9.2 (Auxiliary layout). Let V be a vertex set and $W \subseteq V$. Suppose (L, F) is a W-exceptional layout on V. We denote by $(L^{\restriction W}, F^{\restriction W})$ the layout on $V \setminus W$ obtained from (L, F) as follows.

Let $\mathcal{P}$ be the multiset of maximal paths P such that $P \subseteq P'$ for some $P' \in L$, $V^0(P) \subseteq W$, and $V(P) \cap W \neq \emptyset$ (in particular, each isolated vertex $v \in V(L) \cap W$ is a path in $\mathcal{P}$ but no isolated vertex $v \in V(L) \setminus W$ is a path in $\mathcal{P}$). Note that, since (L, F) is W-exceptional, each $P \in \mathcal{P}$ satisfies $E(P) \subseteq F$. Let $P_1, \dots, P_k$ be an enumeration of $\mathcal{P}$ and, for each $i \in [k]$, let x_i and y_i denote the starting and ending points of P_i, respectively. Then, let $L^{\restriction W}$ be obtained from L as follows. For each $i \in [k]$,

- if both $x_i, y_i \in V \setminus W$, then contract the subpath P_i into an edge $x_i y_i$;
- otherwise, delete $E(P_i)$ as well as $V(P_i) \cap W$.

Note that $V(L^{\restriction W}) = V(L) \setminus W \subseteq V \setminus W$. Define $F^{\restriction W} := \{x_i y_i \mid i \in [k], x_i, y_i \in V \setminus W\} \cup (F \setminus E_W(F)) = \{x_i y_i \mid i \in [k], x_i, y_i \in V \setminus W\} \cup (F \setminus E_W(L))$.

Note that since (L, F) is W-exceptional, each $e \in E(L) \setminus F$ satisfies $V(e) \subseteq V \setminus W$ and so $E(L) \setminus F = E(L^{\restriction W}) \setminus F^{\restriction W}$.

The following proposition states that a spanning configuration of shape $(L^{\restriction W}, F^{\restriction W})$ in $D[V \setminus W]$ can easily be transformed into a W-exceptional spanning configuration of shape (L, F) in D. In other words, it allows us to reverse the process described in Definition 9.2.

Proposition 9.3. *Let D be a digraph on a vertex set V. Let $W \subseteq V$ and denote $V' := V \setminus W$. Let (L, F) be a W-exceptional layout on V. Let $(L^{\restriction W}, F^{\restriction W})$ be as in Definition 9.2. Suppose $\mathcal{H}^{\restriction W} \subseteq D[V'] \cup F^{\restriction W}$ is a spanning configuration of shape $(L^{\restriction W}, F^{\restriction W})$. Let $\mathcal{H}$ be the multidigraph with $V(\mathcal{H}) := V$ and $E(\mathcal{H}) := (E(\mathcal{H}^{\restriction W}) \setminus F^{\restriction W}) \cup F$. Then, $E(\mathcal{H}) \subseteq E(D[V']) \cup F$ and $\mathcal{H}$ is a W-exceptional spanning configuration of shape (L, F).*

Proof. Note that $H^{\restriction W}$ can be decomposed into internally vertex-disjoint paths $\{P_e \mid e \in E(L^{\restriction W})\}$ such that P_e has shape e for each $e \in E(L^{\restriction W})$; $P_f = f$ for each $f \in F^{\restriction W}$; and $\bigcup_{e \in E(L^{\restriction W})} V^0(P_e) = V' \setminus V(L)$. Since $E(L) \setminus F = E(L^{\restriction W}) \setminus F^{\restriction W}$, H can be decomposed into $\{P_e \mid e \in E(L) \setminus F\} \cup F$. The proposition follows by setting $P_f := f$ for all $f \in F$. $\square$

This has the advantage that it suffices to find spanning configurations in an almost regular robust outexpander, which corresponds to the setting of Lemma 7.3. More precisely, if we let $V' := V \setminus W$ be the set of 'non-exceptional vertices' described in Section 2.3 and let D' be the remainder of the tournament $T[V']$ after the cleaning step, then D' is almost complete and almost regular, and hence a robust outexpander. Then, we can split D' into D and Γ as required for Lemma 7.3.

9.2 | Path consistent layouts

Let D be a digraph on V. When we refer to a spanning configuration of shape (L, F) in D, we mean that this configuration is contained in the multidigraph $D \cup F$ (as F may not be in D). Let $\mathcal{F}$ be a multiset of edges on V and $(L_1, F_1), \dots, (L_\ell, F_\ell)$ be layouts, where $F_i \subseteq \mathcal{F}$ for each $i \in [\ell]$. We would like the union of their spanning configurations to form a good partial path decomposition of $D \cup \mathcal{F}$. For this, these layouts will need to satisfy the following properties. Let $U^* \subseteq U^0(D \cup \mathcal{F})$ be such that $|U^*| \leqslant \widetilde{\mathrm{ex}}(D \cup \mathcal{F}) - \mathrm{ex}(D \cup \mathcal{F})$ and define the multiset L by $L := \bigcup_{i\in[\ell]} L_i$. We say $(L_1, F_1), \dots, (L_\ell, F_\ell)$ are *U^*-path consistent with respect to* $(D, \mathcal{F})$, if $\bigcup_{i\in[\ell]} F_i \subseteq \mathcal{F}$ (counting with multiplicity) and the following hold.

(P1′) For any $v \in V \setminus U^0(D \cup \mathcal{F})$, v is the starting point of at most $\mathrm{ex}^+_{D\cup\mathcal{F}}(v)$ non-trivial paths in L and the ending point of at most $\mathrm{ex}^-_{D\cup\mathcal{F}}(v)$ non-trivial paths in L.

(P2′) For any $v \in U^*$, L contains at most one non-trivial path starting at v and at most one non-trivial path ending at v.

(P3′) If $v \in U^0(D \cup \mathcal{F}) \setminus U^*$, then v is not an endpoint of any non-trivial path in L.

If D and $\mathcal{F}$ are clear from the context, then we omit 'with respect to $(D, \mathcal{F})$'.

The following proposition simply states that the union of spanning configurations of U^*-path consistent layouts indeed forms a U^*-partial path decomposition (as defined in Section 8.1). We also track the degrees for later uses.

Proposition 9.4. *Let D be a digraph on a vertex set V. Let $V = W \cup V'$ be a partition of V. Let $U^* \subseteq U^0(D)$ satisfy $|U^*| \leqslant \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$ and $\mathcal{F} \subseteq E(D)$. Let $(L_1, F_1) \dots (L_\ell, F_\ell)$ be W-exceptional layouts. For each $i \in [\ell]$, let $\mathcal{H}_i$ be a W-exceptional spanning configuration of shape (L_i, F_i). Suppose that $\mathcal{H}_1, \dots, \mathcal{H}_\ell$ are pairwise edge-disjoint. For each $i \in [\ell]$, denote by $\mathcal{P}_i$ a path decomposition of $\mathcal{H}_i$ induced by (L_i, F_i). Define the multiset L by $L := \bigcup_{i\in[\ell]} L_i$. Let $F := \bigcup_{i\in[\ell]} F_i$, $\mathcal{H} := \bigcup_{i\in[\ell]} \mathcal{H}_i$, and $\mathcal{P} := \bigcup_{i\in[\ell]} \mathcal{P}_i$. Then, for all $v \in V$,*

$$d^\pm_{\mathcal{H}}(v) = d^\pm_L(v) + |\{i \in [\ell] \mid v \in V' \setminus V(L_i)\}|.$$

Moreover, if $(L_1, F_1), \dots, (L_\ell, F_\ell)$ are U^*-path consistent with respect to $(D \setminus \mathcal{F}, \mathcal{F})$, then $\mathcal{P}$ is a U^*-partial path decomposition of D such that $|\mathcal{P}|$ is equal to the number of non-trivial paths in L and $E_W(\mathcal{P}) \subseteq F \subseteq \mathcal{F}$.

Proof. By Fact 9.1, each $v \in V$ satisfies

$$d^\pm_{\mathcal{H}}(v) = \sum_{i\in[\ell]} d^\pm_{\mathcal{H}_i}(v) = \sum_{i\in[\ell]} (d^\pm_{L_i}(v) + \mathbb{1}_{v\notin V(L_i)\cup W}) = d^\pm_L(v) + |\{i \in [\ell] \mid v \in V' \setminus V(L_i)\}|,$$

as desired. Moreover, Fact 9.1 implies that, for each $v \in V$, the number of paths in $\mathcal{P}$ which start/end at v is precisely the number of (non-trivial) paths in L which start/end at v. By definition of path consistency, this implies that $\mathcal{P}$ satisfies (P1) and (P2). Moreover, the fact that $|U^*| \leqslant \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$ implies that (P3) holds. Thus, $\mathcal{P}$ is a partial path decomposition of D. □

9.3 | Cleaning

As discussed in Section 8.2, the leftover from the approximate decomposition will be decomposed using Corollary 8.8 and so it needs to have a specific structure: no non-absorbing edge incident to the exceptional set W can be left over, while the non-exceptional vertices in V' must form a digraph which is very close to being regular. One consequence of this is that the degree at W needs to be covered at a much faster rate than the degree at V'. Unfortunately, this cannot be achieved via the approximate decomposition. Indeed, Lemma 7.3(b) implies that each $v \in V'$ can only be included as an isolated vertex or covered by a fixed edge in a small proportion of the layouts. Thus, Proposition 9.4 implies that the ℓ spanning configurations obtained with Lemma 7.3 will cover about ℓ inedges and ℓ outedges at each vertex in V'. Therefore, to cover the degree at W at a faster rate than the vertices in V', we would need that each vertex in W belongs (on average) to several paths of each spanning configuration. However, as discussed in Section 9.1, the exceptional vertices will be included via fixed edges and so, in that case, the layouts would be large, while the approximate decomposition only allows small layouts (see Lemma 7.3(a)).

Therefore, we will start with a cleaning procedure which significantly reduces the degree at W. To facilitate the construction of layouts, we also cover all the edges inside W in this step. Note that this needs to be done efficiently so that, after the cleaning step, the non-exceptional vertices still form an almost regular oriented graph of very large degree (otherwise, we would not be able to apply Lemma 7.3 to obtain an approximate decomposition).

We now state our cleaning lemma. (The proof is deferred to Section 12.) Roughly speaking, Lemma 9.5 says the following. Suppose that $T \notin \mathcal{T}_{\text{excep}}$ (exceptional tournaments have already been decomposed in Section 5). Let $W \subseteq V(T)$ consist of all the vertices of excess at least εn and denote $V' := V(T) \setminus W$. Let A^+ and A^- be small absorbing sets of (W, V')-starting and (V', W)-ending edges. Let $U^* \subseteq U^0(T)$ satisfy $\widetilde{\text{ex}}(T) - \text{ex}(T)$. Then, there exists a good (U^*, W, A)-partial path decomposition $\mathcal{P}$ such that the leftover $D := T \setminus \mathcal{P}$ satisfies the following properties. First, the main objectives of the cleaning step are achieved.

- The degree of the exceptional vertices (that is, those in W) is significantly lower than the degree of the vertices in V' (compare the bounds in Lemma 9.5(vii) and (ix)).
- All the edges inside the exceptional set are covered (see Lemma 9.5(iii)).

Moreover, these objectives are achieved very efficiently.

- D is still almost complete (see Lemma 9.5(i) and (vii)–(ix)). Together with Lemma 4.4, this will ensure that $D[V']$ is an almost regular robust outexpander, which is needed for the approximate decomposition.
- $\widetilde{\text{ex}}(D)$ is large compared to number of edges in D (see Lemma 9.5(ii) and (v) (by Lemma 9.5(vii)–(ix), d is roughly the density of D)). If $\widetilde{\text{ex}}(D)$ was very small, then D would need to be decomposed with few long paths which would be more difficult and might not even be possible with our strategy.
- The number of distinct endpoints which can be used to decompose D is roughly the same as for T (see Lemma 9.5(iv)). As discussed in Section 8, having a large pool of suitable endpoints is very convenient and in fact necessary for the final step of the decomposition if $A^+ \cup A^- = \emptyset$ (recall that we aim to apply Corollary 8.8 after the approximate decomposition).

We now explain and motivate the conditions which are needed for the cleaning strategy to work or to simplify the construction of layouts.

- The number of exceptional vertices must be small (see Lemma 9.5(a)). Otherwise, there would be too many edges to cover within the exceptional set and we would not be able to do it efficiently. We are able to assume that $|W|$ is small since otherwise $\mathrm{ex}(T)$ would be large and so Theorem 1.2 would apply (recall that W consists of vertices of large excess).
- The absorbing edges are taken at vertices of as high excess as possible (see Lemma 9.5(b)). This is convenient because it maximises the number of endpoints that are allowed to be used in $\mathcal{P}$. Indeed, recall that the effect of absorbing edges is to reserve some excess at the vertices of W for the final step of the decomposition, so if the absorbing edges account for all the excess at a vertex $w \in W$, then w cannot be used as an endpoint in the (U^*, W, A)-partial path decomposition $\mathcal{P}$. Taking the absorbing edges at vertices of as high excess as possible ensures that this occurs for as few vertices $w \in W$ as possible.
- We distinguish exceptional vertices of very high excess (W_*), exceptional vertices of significant but not too high excess (W_0), and exceptional vertices which are incident to absorbing edges (W_A) (see Lemma 9.5(a) and (b)). One issue that we have not discussed so far is that the exceptional set for the approximate decomposition and the exceptional set for the final step of the decomposition will have to be different. Indeed, as discussed in Section 9.1, the exceptional set for the approximate decomposition must contain all the vertices with excess at least εn to ensure that we apply Lemma 7.3 to an almost regular digraph. Thus, $W = W_* \cup W_0$ will be the exceptional set considered during the approximate decomposition. As discussed in Section 8.2, the main role of the exceptional set in the final step of the decomposition is to incorporate the absorbing edges. Thus, W_A will have to be part of the exceptional set when we apply Corollary 8.8. In addition, the vertices of very high excess will also have to be part of this exceptional set because they have almost all of their edges in the same direction and so it would be impossible for them to satisfy the degree conditions of the non-exceptional vertices in Corollary 8.8. Thus, $W_* \cup W_A$ will be the exceptional set used in the final step of the decomposition and $W_0 \setminus W_A$ will be incorporated back into the non-exceptional set after the approximate decomposition. (This explains the degree conditions in Lemma 9.5(viii).) This is necessary because it may not be possible to decrease the degree at W_0 significantly during the cleaning step. Indeed, since the excess of the vertices in W_0 is not too large, we may have $\widetilde{\mathrm{ex}}(T)$ relatively small and W_0 relatively large at the same time. In that case, significantly decreasing the degree at W_0 during the cleaning step would amount to covering many edges with very few paths, which is not possible.
- If $A^+ \cup A^-$ is non-empty, then $\widetilde{\mathrm{ex}}(T)$ must not be too small (see Lemma 9.5(b)). This will allow us to significantly reduce the degree at W_A during the cleaning step (which is, as discussed above, necessary for applying Corollary 8.8).

In addition to our main objectives, we will also achieve the following property.

- If $\widetilde{\mathrm{ex}}(D)$ is not too large, then we can achieve that all the vertices in W_* have all their edges in the same direction in D (see Lemma 9.5(vi)). This means that, in the decomposition of D, no path will need to have a vertex of W_* as an internal vertex. This will be very convenient because if $\widetilde{\mathrm{ex}}(D)$ is relatively small but a vertex $w \in W_*$ has large positive excess (say), then almost all of the positive excess of D is concentrated at w and so almost all of the paths in the decomposition of D have to start at w. Then, there would be very few paths were w could be incorporated as an internal vertex and so it would be difficult to cover all the inedges at w.

Lemma 9.5 (Cleaning lemma). *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll 1$. Let $T \notin \mathcal{T}_{\mathrm{excep}}$ be a tournament on a vertex set V of size n satisfying the following properties.*

(a) *Let $W_* \cup W_0 \cup V'$ be a partition of V such that, for each $w_* \in W_*$, $|\operatorname{ex}_T(w_*)| > (1-20\eta)n$; for each $w_0 \in W_0$, $|\operatorname{ex}_T(w_0)| \leqslant (1-20\eta)n$; and, for each $v' \in V'$, $|\operatorname{ex}_T(v')| \leqslant \varepsilon n$. Let $W := W_* \cup W_0$ and suppose $|W| \leqslant \varepsilon n$.*
(b) *Let $A^+, A^- \subseteq E(T)$ be absorbing sets of (W, V')-starting/(V', W)-ending edges for T of size at most $\lceil \eta n \rceil$. Denote $A := A^+ \cup A^-$. Let $W_A^\pm := V(A^\pm) \cap W$ and $W_A := V(A) \cap W$. Suppose that the following hold.*
   - *Let $\diamond \in \{+,-\}$. If $|W_A^\diamond| \geqslant 2$, then $\operatorname{ex}_T^\diamond(v) < \lceil \eta n \rceil$ for each $v \in V$.*
   - *Let $\diamond \in \{+,-\}$. If $|W_A^\diamond| = 1$, then $\operatorname{ex}_T^\diamond(v) \leqslant \operatorname{ex}_T^\diamond(w)$ for each $v \in V$ and $w \in W_A^\diamond$.*
   - *If $W_A \neq \emptyset$, then $\widetilde{\operatorname{ex}}(T) \geqslant \frac{n}{2} + 10\eta n$.*
(c) *Let $U^* \subseteq U^0(T)$ satisfy $|U^*| = \widetilde{\operatorname{ex}}(T) - \operatorname{ex}(T)$.*

Then, there exist $d \in \mathbb{N}$ and a good (U^*, W, A)-partial path decomposition $\mathcal{P}$ of T such that the following hold, where $D := T \setminus \mathcal{P}$.

(i) $\lceil \frac{n}{2} \rceil - 10\eta n \leqslant d \leqslant \lceil \frac{n}{2} \rceil - \eta n$.
(ii) *Each $v \in U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$ satisfies $d_D^+(v) = d_D^-(v) \leqslant \widetilde{\operatorname{ex}}(D) - 1$.*
(iii) $E(D[W]) = \emptyset$.
(iv) $N^\pm(T) - N^\pm(D) \leqslant 89\eta n$.
(v) $\widetilde{\operatorname{ex}}(D) \geqslant d + \lceil \eta n \rceil$.
(vi) *If $\widetilde{\operatorname{ex}}(D) < 2d + \lceil \eta n \rceil$, then each $w \in W_*$ satisfies $|\operatorname{ex}_D(w)| = d_D(w)$.*
(vii) *For each $v \in W_* \cup W_A$, $2d - 3\sqrt{\eta} n \leqslant d_D(v) \leqslant 2d - \lceil \eta n \rceil$.*
(viii) *For each $v \in W_0$, $2d + 2\lceil \eta n \rceil - 4\sqrt{\eta} n \leqslant d_D(v) \leqslant 2d + 2\lceil \eta n \rceil$ and $d_D^{\min}(v) \geqslant \lceil \eta n \rceil$.*
(ix) *For each $v \in V'$, $2d + 2\lceil \eta n \rceil - 9\sqrt{\varepsilon} n \leqslant d_D(v) \leqslant 2d + 2\lceil \eta n \rceil$.*

9.4 | Constructing layouts

We now state the lemma which we will use to construct the layouts for the approximate decomposition. (The proof is deferred to Section 13.) Roughly speaking, Lemma 9.6 says the following. Let D be an oriented graph. Let $W_1 \cup W_2 \cup V'$ be a partition of $V(D)$ and denote $W := W_1 \cup W_2$. Here, W will be the exceptional set for the approximate decomposition (that is, the same W as in the cleaning lemma), W_1 will be the exceptional set for the final step of the decomposition (that is, $W_* \cup W_A$ from the cleaning lemma), and W_2 will be the set of exceptional vertices which will be incorporated back into the non-exceptional set after the approximate decomposition (that is, $W_0 \setminus W_A$ from the cleaning lemma). Let $U^* \subseteq U^0(D)$ satisfy $|U^*| = \widetilde{\operatorname{ex}}(D) - \operatorname{ex}(D)$. Let A^+ and A^- be absorbing sets of (W_1, V')-starting and (V', W_1)-ending edges. Suppose that D satisfies the following properties.

- The vertices in V' all have small excess (see Lemma 9.6(h), this is inherited from Lemma 9.5(a)). Recall that one of the roles of the layouts is to prescribe the endpoints of the paths we want to construct in the approximate decomposition. Thus, the fact that the vertices in V' have small excess means that each vertex in V' will be an endpoint in only few of the layouts. This is necessary because Lemma 7.3 only allows each vertex to be covered by few layouts (see Lemma 7.3(b)).
- The vertices in V' all have roughly the same degree (see Lemma 9.6(h), this is inherited from Lemma 9.5(ix)). Recall that the role of the isolated vertices in layouts is to specify which vertices need to be avoided in each of the spanning configurations constructed in the approximate decomposition. The fact that the vertices in V' all have roughly the same degree means that they

all need to be covered by roughly the same number of spanning configurations. Thus, each vertex in V' will only have to be included as an isolated vertex in few of the layouts. This is necessary because Lemma 7.3 only allows each vertex to be included in few layouts (see Lemma 7.3(b)).

- The degree at W_1 is significantly smaller than the degree at V' (compare Lemma 9.6(f) and (h), this is inherited from Lemma 9.5(vii) and (ix)). This will enable us to incorporate all the non-absorbing edges at W_1 into the layouts. (Thus, the vertices in W_1 will have no non-absorbing edges left over after the approximate decomposition and so we will be able to use W_1 as the exceptional set in Corollary 8.8.)
- The degree at W_2 is comparable or smaller than the degree at V' but every vertex in W_2 has a significant number of edges of each direction (see Lemma 9.6(g), this is inherited from Lemma 9.5(viii)). This means that the degree at W_2 is not too large compared to the number of layouts that we will construct, and the in- and outdegree of each vertex in W_2 is larger than the in- and outdegree required to satisfy the non-exceptional degree conditions in Corollary 8.8. Thus, we will be able to incorporate almost all of the edges at W_2 into the layouts. (The edges left over will be covered using Corollary 8.8.)
- $\widetilde{\text{ex}}(D)$ is significantly larger than the average degree of D (see Lemma 9.6(b) and (e), this is inherited from Lemma 9.5(ii) and (v)). Also, if $\widetilde{\text{ex}}(D)$ is not very large, then the vertices in W_1 satisfy some additional degree conditions (see Lemma 9.6(f), this is inherited from Lemma 9.5(vi) and (viii)). Moreover, $D[W]$ is empty (see Lemma 9.6(a), this is inherited from Lemma 9.5(iii)). As discussed in Section 9.3, these conditions will facilitate the construction of the layouts.

If we assume the above conditions, then there exist layouts $(L_1, F_1), \dots, (L_\ell, F_\ell)$ on $V(D)$ which satisfy the following properties, where L denotes the multiset $L := \bigcup_{i \in [\ell]} L_i$.

- $(L_1, F_1), \dots, (L_\ell, F_\ell)$ are W-exceptional. As discussed in Section 9.1, D may not be an almost regular robust outexpander and so we will need to apply Lemma 7.3 with $D[V']$ playing the role of D. The concept of W-exceptional layouts will enable us to incorporate the edges incident to W into the approximate decomposition.
- Let $\mathcal{F}$ consist of all the non-absorbing edges of D which are incident to W (that is, $\mathcal{F} := E_W(D) \setminus (A^+ \cup A^-)$) and denote $D' := D \setminus \mathcal{F}$. Then, $(L_1, F_1), \dots, (L_\ell, F_\ell)$ are U^*-path consistent with respect to $(D', \mathcal{F})$. As discussed in Section 9.2, this will ensure that the spanning configurations obtained in the approximate decomposition form a partial path decomposition which does not have any endpoint in $U^0(D) \setminus U^*$ (recall Proposition 9.4). Moreover, the definition of path consistency implies that $F_1, \dots, F_\ell \subseteq \mathcal{F} \subseteq D \setminus (A^+ \cup A^-)$. This will ensure that none of the absorbing edges will be covered during the approximate decomposition (recall from Section 8.2 that these edges are reserved for the final step of the decomposition).
- The number of layouts is bounded away from the density of D (see Lemma 9.6(i)). This is needed for applying Lemma 7.3.
- The number of (non-trivial) paths in L is precisely $\widetilde{\text{ex}}(D) - r$, where r is the value from Corollary 8.8 (see Lemma 9.6(ii), in the proof of Theorem 1.8 we will apply Corollary 8.8 with $\lceil \eta n \rceil$ playing the role of r). By Proposition 9.4, this means that the partial path decomposition obtained in the approximate decomposition step will consist of $\widetilde{\text{ex}}(D) - r$ paths. The leftover will then be decomposed into r paths with Corollary 8.8 and so, overall, we will obtain a path decomposition of D of size $\widetilde{\text{ex}}(D)$, as desired.
- Each layout is small (see Lemma 9.6(vi)). This is needed for the approximate decomposition (see Lemma 7.3(a)).
- Each vertex in V' is included in few of the layouts (see Lemma 9.6(vii)). This is needed for the approximate decomposition (see Lemma 7.3(b)).

- Each non-absorbing edge incident to W_1 is incorporated as a fixed edge into precisely one of the layouts (see Lemma 9.6(iii)). This implies that after the approximate decomposition, the only remaining edges incident to W_1 will be the absorbing edges. This is precisely what we need for applying Corollary 8.8 with W_1 playing the role of the exceptional set.
- All but r inedges and r outedges at each vertex in W_2 are included as fixed edges in L (see Lemma 9.6(iv)). This implies that, after the approximate decomposition, each vertex in W_2 will have both its in- and outdegree equal r, that is, each vertex in W_2 will have excess 0 and total degree $2r$ in the leftover. This means that we will be able to apply Corollary 8.8 with $W_2 \subseteq X^0$.
- Let $X^+ \subseteq (U^+(D) \cup U^*) \cap V'$ and $X^- \subseteq (U^-(D) \cup U^*) \cap V'$. Then, Proposition 9.4 and Lemma 9.6(v) imply that, for each $v \in V'$, the number of L_i which include v as an isolated vertex and the outdegree of v in L together have precisely the value such that, after the approximate decomposition, the outdegree at v will be r if $v \notin X^-$ and $r-1$ if $v \in X^-$. The analogous statement holds for X^+. Thus, the vertices in V' will satisfy the degree conditions of Corollary 8.8 with $X^+ \setminus X^-$, $X^- \setminus X^+$, and $X^+ \cap X^-$ playing the roles of X^+, X^-, and X^*, and with $V' \setminus (X^+ \cup X^-) \subseteq X^0$.

Lemma 9.6 (Layout construction). *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll 1$ and $d \in \mathbb{N}$. Let D be an oriented graph on a vertex set V of size n such that the following hold.*

(a) *Let $W_1 \cup W_2 \cup V'$ be a partition of V. Denote $W := W_1 \cup W_2$. Suppose that $|W| \leqslant \varepsilon n$ and $E(D[W]) = \emptyset$.*
(b) *Let $U^* \subseteq U^0(D) \setminus W$ be such that $|U^*| = \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$. Moreover, each $v \in U^*$ satisfies $d_D^+(v) = d_D^-(v) \leqslant \widetilde{\mathrm{ex}}(D) - 1$.*
(c) *Let A^+ and A^- be absorbing sets of (W_1, V')-starting and (V', W_1)-ending edges for D, respectively, and denote $A := A^+ \cup A^-$. Suppose $X^\pm \subseteq (U^\pm(D) \cup U^*) \setminus W$ are such that $|A^\pm| + |X^\pm| = \lceil \eta n \rceil$. Define $\phi^\pm : V \longrightarrow \{0,1\}$ by*

$$\phi^\pm(v) := \begin{cases} 1 & \text{if } v \in X^\pm, \\ 0 & \text{otherwise.} \end{cases}$$

(d) *$d \geqslant \eta n$.*
(e) *$\widetilde{\mathrm{ex}}(D) \geqslant d + \lceil \eta n \rceil$.*
(f) *For all $v \in W_1$, $10\varepsilon n \leqslant d_{D \setminus A}(v) \leqslant 2d - \lceil \eta n \rceil$. Moreover, if $\widetilde{\mathrm{ex}}(D) < 2d + \lceil \eta n \rceil$, then, for each $v \in W_1$, one of the following holds.*
  - *$|\mathrm{ex}_D(v)| = d_D(v)$; or*
  - *$d_D^{\min}(v) \geqslant \eta n$ and $|\mathrm{ex}_{D\setminus A}(v)| \leqslant \lceil \eta n \rceil$; or*
  - *$d_D^{\min}(v) \geqslant \eta n$ and $d_A(v) = \lceil \eta n \rceil$.*
(g) *For all $v \in W_2$, $d_D^{\min}(v) \geqslant \lceil \eta n \rceil$ and $d_D(v) \leqslant 2d + 2\lceil \eta n \rceil$.*
(h) *For all $v \in V'$, $2d + 2\lceil \eta n \rceil - \varepsilon n \leqslant d_D(v) \leqslant 2d + 2\lceil \eta n \rceil$ and $|\mathrm{ex}_D(v)| \leqslant \varepsilon n$.*

Let $\mathcal{F} := E_W(D) \setminus A$ and $D' := D \setminus \mathcal{F}$. Then, there exist $\ell \in \mathbb{N}$ and W-exceptional layouts $(L_1, F_1), \dots, (L_\ell, F_\ell)$ which are U^*-path consistent with respect to $(D', \mathcal{F})$ and satisfy the following, where L is the multiset defined by $L := \bigcup_{i \in [\ell]} L_i$.

(i) *$d \leqslant \ell \leqslant d + \sqrt{\varepsilon} n$.*
(ii) *L contains exactly $\widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil$ non-trivial paths.*
(iii) *For all $v \in W_1$, $d_L^\pm(v) = d_{\mathcal{F}}^\pm(v) = d_{D\setminus A}^\pm(v)$.*

(iv) *For all* $v \in W_2$, $d_L^{\pm}(v) = d_F^{\pm}(v) - \lceil \eta n \rceil = d_D^{\pm}(v) - \lceil \eta n \rceil$.
(v) *For all* $v \in V'$, $d_L^{\pm}(v) = d_D^{\pm}(v) - |\{i \in [\ell] \mid v \notin V(L_i)\}| - \lceil \eta n \rceil + \phi^{\mp}(v)$.
(vi) *For all* $i \in [\ell]$, $|V(L_i)|, |E(L_i)| \leqslant 3\varepsilon^{\frac{1}{3}} n$.
(vii) *For each* $v \in V'$, $d_L(v) \leqslant 8\varepsilon n$ *and there exist at most* $3\sqrt{\varepsilon}n$ *indices* $i \in [\ell]$ *such that* $v \in V(L_i)$.

We now motivate the expression appearing in (v). Let $v \in V'$. Recall from Proposition 9.4 that $d_{\widehat{L}}^{+}(v) + |\{i \in [\ell] \mid v \notin V(\widehat{L}_i)\}|$ is precisely the outdegree of v in a set of spanning configurations of shapes $(\widehat{L}_1, \widehat{F}_1), \dots, (\widehat{L}_\ell, \widehat{L}_\ell)$. Moreover, as seen in the proof of Theorem 1.8 (see also the explanatory paragraph before the statement of Lemma 9.6), $\lceil \eta n \rceil - \phi^{-}(v)$ is precisely the leftover outdegree of v that we aim for after the approximate decomposition step.

We now explain in more detail why Lemma 9.5(ii) is necessary. This is because a general oriented graph D may not contain sufficiently many zero-excess vertices which satisfy the degree condition of Lemma 9.6(b). For example, if D is regular, then $\widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D) > 0$ but every $v \in V(D)$ satisfies $d_D^{+}(v) = d_D^{-}(v) = \Delta^0(D) = \widetilde{\mathrm{ex}}(D)$. This example also illustrates why the degree condition of Lemma 9.6(b) is necessary. Indeed, suppose for a contradiction that Lemma 9.6 also holds if we omit the ‘moreover part’ of Lemma 9.6(b). Let T be a regular tournament on n vertices. Let $d := \frac{n-1}{2} - \lceil \eta n \rceil$. Let $V' := V(T)$ and $W_1 := W_2 := A^+ := A^- := \emptyset$. Let $U^* \subseteq V(T)$ satisfy $|U^*| = \frac{n-1}{2}$ and $X^+, X^- \subseteq U^*$ satisfy $|X^+| = \lceil \eta n \rceil = |X^-|$. Then, one can easily verify that Lemma 9.6(a)–(h) are all fully satisfied except for the ‘moreover part’ of Lemma 9.6(b) and so, by assumption, there exist layouts as in Lemma 9.6. As discussed earlier, this implies that we can use Lemma 7.3 to construct a partial path decomposition $\mathcal{P}$ of T of size $|\mathcal{P}| = \widetilde{\mathrm{ex}}(T) - \lceil \eta n \rceil$ such that $T \setminus \mathcal{P}$ satisfies the degree conditions of Corollary 8.8 with $r := \lceil \eta n \rceil$. Thus, Corollary 8.8 implies that there exists a path decomposition $\mathcal{P}'$ of $T \setminus \mathcal{P}$ of size $\lceil \eta n \rceil$. But, this means that $\mathcal{P} \cup \mathcal{P}'$ is a path decomposition of T of size $\widetilde{\mathrm{ex}}(T)$. This contradicts Theorem 5.1.

The next proposition states that, after the cleaning step, we will be able to find a set U^* of candidates for path endpoints which satisfies Lemma 9.6(b). Let $T \notin \mathcal{T}_{\mathrm{excep}}$. By Proposition 8.1, there exists $U^* \subseteq U^0(T)$ satisfying $|U^*| = \widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T)$. Let $\mathcal{P}$ be the good partial path decomposition obtained by applying Lemma 9.5. Denote $D := T \setminus \mathcal{P}$. We now aim to apply Lemma 9.6 to D and so we need a new set $U^{**} \subseteq U^0(D)$ which satisfies Lemma 9.6(b). (We cannot use the original U^* since some of the vertices in U^* may have been used as endpoints in $\mathcal{P}$ and so may have nonzero excess in D.) By Proposition 8.1, there exists $U^{**} \subseteq U^0(T)$ satisfying $|U^{**}| = \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$. However, there is no guarantee that the vertices in U^{**} satisfy the desired degree conditions. But by Lemma 9.5(ii), we know that all the vertices in U^* which have not been used as endpoints in $\mathcal{P}$ have the correct degree conditions for Lemma 9.6(b). Thus, we would like to take $U^{**} \subseteq U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$ and so we would like $U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$ to contain at least $\widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$ vertices of $U^0(D)$. Proposition 9.7 states that this is the case.

Proposition 9.7. *Let D be a digraph and $W, V' \subseteq V(D)$ be disjoint. Suppose $A^+, A^- \subseteq E(D)$ are absorbing sets of (W, V')-starting and (V', W)-ending edges for D. Denote $A := A^+ \cup A^-$. Let $U^* \subseteq U^0(D)$ satisfy $|U^*| = \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$. Suppose $\mathcal{P}$ is a good (U^*, W, A)-partial path decomposition of D. Let $U^{**} := U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$. Then, $U^{**} \subseteq U^0(D \setminus \mathcal{P})$ and $|U^{**}| = \widetilde{\mathrm{ex}}(D \setminus \mathcal{P}) - \mathrm{ex}(D \setminus \mathcal{P})$.*

Recall that a (U^*, W, A)-partial path decomposition of D was defined in Definition 8.10 and its goodness before Fact 8.5.

Proof of Proposition 9.7. Since the vertices in U^{**} have not been used as endpoints in $\mathcal{P}$, they still have excess 0 in $D \setminus \mathcal{P}$.

By definition of a (U^*, W, A)-partial path decomposition, no path in $\mathcal{P}$ has an endpoint in $U^0(D) \setminus U^*$. Therefore,

$$U^0(D) \cap (V^+(\mathcal{P}) \cup V^-(\mathcal{P})) = U^* \setminus U^{**} \tag{9.1}$$

and so the fact that $\mathcal{P}$ is a good partial path decomposition implies that

$$\begin{aligned} \widetilde{\mathrm{ex}}(D \setminus \mathcal{P}) &= \widetilde{\mathrm{ex}}(D) - |\mathcal{P}| \\ &\overset{\text{Proposition 8.3}}{=} (\widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)) + \mathrm{ex}(D \setminus \mathcal{P}) - |U^0(D) \cap (V^+(\mathcal{P}) \cup V^+(\mathcal{P}))| \\ &\overset{(9.1)}{=} |U^*| + \mathrm{ex}(D \setminus \mathcal{P}) - |U^* \setminus U^{**}| = \mathrm{ex}(D \setminus \mathcal{P}) + |U^{**}|, \end{aligned}$$

as desired. □

10 | DERIVING THEOREM 1.8

In this section, we assume that Lemmas 9.5 and 9.6 hold and derive Theorem 1.8. We will proceed as follows. In Step 1, we select absorbing edges (if they are required). In Step 2, we clean up T by removing a small number of paths using Lemma 9.5. In Step 3, we first apply Lemma 9.6 to obtain approximate layouts and then apply Lemma 7.3 to obtain an approximate decomposition of T based on these layouts. Finally, in Step 4, we apply Corollary 8.8 to decompose the leftover.

Proof of Theorem 1.8. Assume without loss of generality that $\beta \leqslant 1$. Fix additional constants such that $0 < \frac{1}{n_0} \ll \varepsilon \ll \alpha_1 \ll \alpha_2 \ll \eta \ll \beta \leqslant 1$. Let $T \notin \mathcal{T}_{\mathrm{excep}}$ be a tournament on $n \geqslant n_0$ vertices satisfying (a) or (b). By Theorem 1.2(b), we may assume that $\mathrm{ex}(T) \leqslant \varepsilon^2 n^2$. Denote $V := V(T)$.

Recall that $N^{\pm}(T) = |U^{\pm}(T)| + \widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T)$. If both $N^{\pm}(T) \geqslant \alpha_1 n$, then redefine $\eta := \alpha_1^2$. Suppose not. If both $N^{\pm}(T) \leqslant \alpha_2 n$, then redefine $\varepsilon := \alpha_2$. Otherwise, there exists $\diamond \in \{+, -\}$ such that $N^{\diamond}(T) \leqslant \alpha_1 n$ and $\circ \in \{+, -\} \setminus \{\diamond\}$ satisfies $N^{\circ}(T) \geqslant \alpha_2 n$ and we redefine $\varepsilon := \alpha_1$ and $\eta := \alpha_2^2$. Thus, we have defined constants such that

$$0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll \beta \leqslant 1,$$

$\mathrm{ex}(T) \leqslant \varepsilon^2 n^2$, and, for each $\diamond \in \{+, -\}$, either $N^{\diamond}(T) \geqslant \sqrt{\eta} n$ or $N^{\diamond}(T) \leqslant \varepsilon n$. Define additional constants such that

$$0 < \frac{1}{n} \ll \varepsilon \ll \nu \ll \tau \ll \gamma \ll \eta \ll \beta \leqslant 1.$$

Step 1: Choosing absorbing edges. We start by partitioning V into V' and W, and selecting a (W, V')-absorbing set A. Let $r := \lceil \eta n \rceil$.

Claim 1. There exist a partition $W \cup V'$ of V and absorbing sets $A^+, A^- \subseteq E(T)$ of (W, V')-starting/(V', W)-ending edges for T such that the following hold, where $A := A^+ \cup A^-$, $W_A^\pm := V(A^\pm) \cap W$, and $W_A := V(A) \cap W$.

(i) $W \cap U^0(T) = \emptyset$.
(ii) $|W| \leqslant 4\varepsilon n$.
(iii) For each $v \in V'$, $|\operatorname{ex}_T(v)| \leqslant \varepsilon n$.
(iv) Let $\diamond \in \{+, -\}$. If $N^\pm(T) \geqslant \sqrt{\eta} n$, then $A^\diamond = \emptyset$, otherwise $|A^\diamond| = r$.
(v) Let $\diamond \in \{+, -\}$. If $|W_A^\diamond| \geqslant 2$, then $\operatorname{ex}_T^\diamond(v) < r$ for each $v \in V$.
(vi) Let $\diamond \in \{+, -\}$. If $|W_A^\diamond| = 1$, then $\operatorname{ex}_T^\diamond(v) \leqslant \operatorname{ex}_T^\diamond(w)$ for each $v \in V$ and $w \in W_A^\diamond$.
(vii) If $W_A \neq \emptyset$, then $\widetilde{\operatorname{ex}}(T) \geqslant \frac{n}{2} + 10\eta n$.

Proof of Claim. First, we choose suitable sets of endpoints for our absorbing edges. Let $\diamond \in \{+, -\}$ and define $W_A^\diamond$ as follows. First, suppose that $N^\diamond(T) \geqslant \sqrt{\eta} n$. Since we will not need any absorbing edges in that case, we let $W_A^\diamond = \emptyset$. Observe that (v) and (vi) hold for $\diamond$.

Now suppose $N^\diamond(T) < \sqrt{\eta} n$. By construction, we have $N^\diamond(T) \leqslant \varepsilon n$. We will need r absorbing edges, so we need to choose a set $W_A^\diamond$ which concentrates a sufficiently large amount of excess but also satisfies (v) and (vi). By Fact 4.22, $\operatorname{ex}(T) \geqslant \widetilde{\operatorname{ex}}(T) - N^\diamond(T) \geqslant \frac{n}{2} - \varepsilon n \geqslant r$ and so we can let $W_A^\diamond \subseteq U^\diamond(T)$ be a smallest set such that $\operatorname{ex}_T^\diamond(W_A^\diamond) \geqslant r$. We further assume that, subject to this, $\operatorname{ex}_T^\diamond(W_A^\diamond)$ is maximum. Note that

$$|W_A^\diamond| \leqslant |U^\diamond(T)| \leqslant N^\diamond(T) \leqslant \varepsilon n. \tag{10.1}$$

We verify that (v)–(vii) are satisfied for $\diamond$. If $|W_A^\diamond| \geqslant 2$, then the minimality of $|W_A^\diamond|$ implies that each $v \in V$ satisfies $\operatorname{ex}_T^\diamond(v) < r$ and so (v) holds. If $|W_A^\diamond| = 1$, then the maximality of $\operatorname{ex}_T^\diamond(W_A^\diamond)$ implies that each $v \in V$ and $w \in W_A^\diamond$ satisfy $\widetilde{\operatorname{ex}}_T^\diamond(v) \leqslant \widetilde{\operatorname{ex}}_T^\diamond(w)$, so (vi) holds. By assumption, $N^\diamond(T) \leqslant \varepsilon n < \beta n$. Thus, (b) does not hold and so (a) implies that $\widetilde{\operatorname{ex}}(T) \geqslant \frac{n}{2} + \beta n \geqslant \frac{n}{2} + 10\eta n$, as desired for (vii).

Let $W_A := W_A^+ \cup W_A^-$. Define $W := W_A \cup \{v \in V \mid |\operatorname{ex}_T(v)| > \varepsilon n\}$ and $V' := V \setminus W$. Note that (iii) is satisfied. By construction, $W_A^\pm \subseteq U^\pm(T)$ and so (i) holds. By (10.1) and since $\operatorname{ex}(T) \leqslant \varepsilon^2 n^2$, we have

$$|W| \leqslant |W_A^+| + |W_A^-| + |\{v \in V \mid |\operatorname{ex}_T(v)| > \varepsilon n\}| \leqslant 2\varepsilon n + 2 \cdot \frac{\operatorname{ex}(T)}{\varepsilon n} \leqslant 4\varepsilon n.$$

Thus, (ii) holds.

We are now ready to choose the absorbing edges. If $N^+(T) \geqslant \sqrt{\eta} n$, then let $A^+ := \emptyset$; otherwise, let $A^+ \subseteq E(T)$ be an absorbing set of r (W_A^+, V')-starting edges for T (A^+ exists since, by construction, $\operatorname{ex}_T^+(W_A^+) \geqslant r$ and $d_T^+(v) \geqslant \frac{n}{2}$ for each $v \in W_A^+$). Similarly, let $A^- \subseteq E(T)$ be an absorbing set of (V', W_A^-)-ending edges for T, of size 0 if $N^-(T) \geqslant \sqrt{\eta} n$ and r otherwise. Thus, (iv) holds. Let $A := A^+ \cup A^-$. By minimality of $|W_A^\pm|$, $V(A^\pm) \cap W = W_A^\pm$. This completes the proof. □

Let W_* consist of all the vertices $w_* \in W$ for which $|\operatorname{ex}_T(w_*)| > (1 - 20\eta)n$ and let W_0 consist of all the vertices $w_0 \in W$ for which $|\operatorname{ex}_T(w_0)| \leqslant (1 - 20\eta)n$.

Step 2: Cleaning. By Claim 1, Lemma 9.5(a) and (b) are satisfied with 4ε playing the role of ε. By Proposition 8.1, there exists $U_1^* \subseteq U^0(T)$ which satisfies $|U_1^*| = \widetilde{\operatorname{ex}}(T) - \operatorname{ex}(T)$. Then, Lemma 9.5(c) holds with U_1^* playing the role of U^*. By (i),

$$W \cap U_1^* = \emptyset. \tag{10.2}$$

(This will be needed in Step 3.)

Apply Lemma 9.5 with U_1^* and 4ε playing the roles of U^* and ε to obtain $d \in \mathbb{N}$ and $\mathcal{P}_1 \subseteq T$ such that the following are satisfied, where $D_1 := T \setminus \mathcal{P}_1$.

(α) $\mathcal{P}_1$ is a good (U_1^*, W, A)-partial path decomposition of T. In particular, $\mathcal{P}_1$ is consistent with A^+ and A^- and so, by Fact 8.11, A^+ and A^- are absorbing sets of (W_A^+, V')-starting and (V', W_A^-)-ending edges for D_1.
(β) $\lceil \frac{n}{2} \rceil - 10\eta n \leqslant d \leqslant \lceil \frac{n}{2} \rceil - \eta n$.
(γ) Each $v \in U_1^* \setminus (V^+(\mathcal{P}_1) \cup V^-(\mathcal{P}_1))$ satisfies $d_{D_1}^+(v) = d_{D_1}^-(v) \leqslant \widetilde{\mathrm{ex}}(D_1) - 1$.
(δ) $E(D_1[W]) = \emptyset$.
(ε) $N^{\pm}(T) - N^{\pm}(D_1) \leqslant 89\eta n$.
(ζ) $\widetilde{\mathrm{ex}}(D_1) \geqslant d + \lceil \eta n \rceil$.
(η) If $\widetilde{\mathrm{ex}}(D_1) < 2d + \lceil \eta n \rceil$, then each $w \in W_*$ satisfies $|\mathrm{ex}_{D_1}(w)| = d_{D_1}(w)$.
(θ) For each $v \in W_* \cup W_A$, $2d - 3\sqrt{\eta} n \leqslant d_{D_1}(v) \leqslant 2d - 2\lceil \eta n \rceil$.
(ι) For each $v \in W_0$, $2d + 2\lceil \eta n \rceil - 4\sqrt{\eta} n \leqslant d_{D_1}(v) \leqslant 2d + 2\lceil \eta n \rceil$ and $d_{D_1}^{\min}(v) \geqslant \lceil \eta n \rceil$.
(κ) For each $v \in V'$, $2d + 2\lceil \eta n \rceil - 18\sqrt{\varepsilon} n \leqslant d_{D_1}(v) \leqslant 2d + 2\lceil \eta n \rceil$ and $|\mathrm{ex}_{D_1}(v)| \leqslant \varepsilon n$.

(The final part of (κ) follows from the facts that $\mathcal{P}_1$ is a partial path decomposition of T and $|\mathrm{ex}_T(v)| \leqslant \varepsilon n$ for each $v \in V'$. Indeed, if $v \in V' \setminus U^0(T)$, then (P1) implies that $\mathcal{P}_1$ contains at most $\mathrm{ex}_T^+(v)$ paths which start at v and at most $\mathrm{ex}_T^-(v)$ paths which end at v. Thus, each $v \in V' \setminus U^0(T)$ satisfies $|\mathrm{ex}_{D_1}(v)| \leqslant |\mathrm{ex}_T(v)| \leqslant \varepsilon n$. Moreover, (P2) implies that each $v \in V' \cap U^0(T)$ is the starting point of at most one path in $\mathcal{P}_1$ and the ending point of at most one path in $\mathcal{P}_1$. Thus, each $v \in V' \cap U^0(T)$ satisfies $|\mathrm{ex}_{D_1}(v)| \leqslant 1 \leqslant \varepsilon n$.)

Step 3: Approximate decomposition. We will approximately decompose D_1 as follows. First, we will apply Lemma 9.6 to construct W-exceptional layouts on V. These layouts will then be transformed into auxiliary layouts on $V \setminus W$ via Definition 9.2. We will then apply Lemma 7.3 to these auxiliary layouts to approximately decompose $D_1[V']$ into auxiliary spanning configurations on V'. Finally, we will use Proposition 9.3 to transform these auxiliary spanning configurations into W-exceptional spanning configurations on V. By Proposition 9.4 and Lemma 9.6, these will induce a good partial path decomposition of D_1 which covers almost all the edges of D_1.

First, we ensure that all the prerequisites of Lemma 9.6 are satisfied. Let $U_2^* := U_1^* \setminus (V^+(\mathcal{P}_1) \cup V^-(\mathcal{P}_1))$ and observe that, by (α) and Proposition 9.7,

$$|U_2^*| = \widetilde{\mathrm{ex}}(D_1) - \mathrm{ex}(D_1). \tag{10.3}$$

Claim 2. There exist $X^{\pm} \subseteq (U^{\pm}(D_1) \cup U_2^*) \setminus W$ which satisfy $|X^{\pm}| = r - |A^{\pm}|$.

Proof of Claim. Let $\diamond \in \{+, -\}$. First, suppose that $N^{\diamond}(T) \leqslant \varepsilon n$. Then, (iv) implies that $|A^{\diamond}| = r$ and so we can let $X^{\diamond} := \emptyset$. We may therefore assume that $N^{\diamond}(T) > \varepsilon n$. By construction, $N^{\diamond}(T) \geqslant \sqrt{\eta} n$ and so

$$|U^{\diamond}(D_1) \cup U_2^*| = N^{\diamond}(D_1) \overset{(\varepsilon)}{\geqslant} \sqrt{\eta} n - 89\eta n \geqslant r + |W|,$$

as desired. $\square$

Define $\phi^{\pm} : V \longrightarrow \{0,1\}$ by

$$\phi^{\pm}(v) := \begin{cases} 1 & \text{if } v \in X^{\pm}, \\ 0 & \text{otherwise.} \end{cases}$$

Denote $\mathcal{F} := E_W(D_1) \setminus A$ and $D_1' := D_1 \setminus \mathcal{F}$. Let $W_1 := W_* \cup W_A$ and $W_2 := W_0 \setminus W_A$.

We now verify that Lemma 9.6(a)–(h) hold with D_1, U_2^*, and $18\sqrt{\varepsilon}$ playing the roles of D, U^*, and ε. Lemma 9.6(a) follows from (δ) and Lemma 9.5(a). Lemma 9.6(b) holds by (γ), (10.2), and (10.3). Lemma 9.6(c) holds by Claim 2 and (α). Lemma 9.6(d) follows from (β) and Lemma 9.6(e) holds by (ζ). Lemma 9.6(g) and (h) as well as the first part of Lemma 9.6(f) follow from (θ)–(κ). Finally, we show that the 'moreover part' of Lemma 9.6(f) holds. Suppose $\widetilde{\mathrm{ex}}(D_1) < 2d + \lceil \eta n \rceil$ and $v \in W$. If $v \in W_*$, then (η) implies that $|\mathrm{ex}_{D_1}(v)| = d_{D_1}(v)$. We may therefore assume that $v \in W_1 \setminus W_* \subseteq W_A \cap W_0$. Then, ($\iota$) implies that $d_{D_1}^{\min}(v) \geqslant \lceil \eta n \rceil$. Thus, it is enough to show that $|\mathrm{ex}_{D_1 \setminus A}(v)| \leqslant r$ or $d_A(v) = r$. Suppose without loss of generality that $v \in W_A^+$, that is, $v \in U^+(T)$ by Definition 8.7 (similar arguments hold if $v \in W_A^-$). If $|W_A^+| \geqslant 2$, then,

$$|\mathrm{ex}_{D_1 \setminus A}(v)| \overset{\text{Definition 8.7},(\alpha)}{\leqslant} |\mathrm{ex}_T(v)| \overset{\text{(v)}}{\leqslant} r.$$

If $W_A^+ = \{v\}$, then (vi) implies that $d_{A^+}(v) = r$ and Definition 8.7 implies that $d_{A^-}(v) = 0$, so $d_A(v) = r$. Therefore, Lemma 9.6(f) holds.

Apply Lemma 9.6 with D_1, U_2^*, and $18\sqrt{\varepsilon}$ playing the roles of D, U^*, and ε to obtain $\ell \in \mathbb{N}$ and W-exceptional layouts $(L_1, F_1), \dots, (L_\ell, F_\ell)$ which are U_2^*-path consistent with respect to $(D_1', \mathcal{F})$ and such that the following hold, where L is the multiset defined by $L := \bigcup_{i \in [\ell]} L_i$.

(A) $d \leqslant \ell \leqslant d + \varepsilon^{\frac{1}{5}} n$.
(B) L contains exactly $\widetilde{\mathrm{ex}}(D_1) - r$ non-trivial paths.
(C) For each $v \in W_1$, $d_L^{\pm}(v) = d_{D_1 \setminus A}^{\pm}(v)$.
(D) For each $v \in W_2$, $d_L^{\pm}(v) = d_{D_1}^{\pm}(v) - r$.
(E) For each $v \in V'$, $d_{D_1}^{\pm}(v) = d_L^{\pm}(v) + |\{i \in [\ell] \mid v \notin V(L_i)\}| + r - \phi^{\mp}(v)$.
(F) For each $i \in [\ell]$, $|V(L_i)|, |E(L_i)| \leqslant \varepsilon^{\frac{1}{7}} n$.
(G) For each $v \in V'$, $d_L(v) \leqslant \varepsilon^{\frac{1}{3}} n$ and there exist at most $\varepsilon^{\frac{1}{5}} n$ indices $i \in [\ell]$ such that $v \in V(L_i)$.

We now transform $(L_1, F_1), \dots, (L_\ell, F_\ell)$ into auxiliary layouts on V'. For each $i \in [\ell]$, let $(L_i^{\restriction W}, F_i^{\restriction W})$ be obtained from (L_i, F_i) using the procedure described in Definition 9.2. Let $L^{\restriction W} := \bigcup_{i \in [\ell]} L_i^{\restriction W}$ and $\mathcal{F}^{\restriction W} := \bigcup_{i \in [\ell]} F_i^{\restriction W}$. Then, Definition 9.2 implies that $(L_1^{\restriction W}, F_1^{\restriction W}), \dots, (L_\ell^{\restriction W}, F_\ell^{\restriction W})$ are layouts on V' such that the following hold.

(F′) Let $i \in [\ell]$. By (F), $|V(L_i^{\restriction W})| \leqslant |V(L_i)| \leqslant \varepsilon^{\frac{1}{7}} n \leqslant 2\varepsilon^{\frac{1}{7}} |V'|$ and, similarly, $|E(L_i^{\restriction W})| \leqslant 2\varepsilon^{\frac{1}{7}} |V'|$.
(G′) Let $v \in V'$. By (G), $d_{L^{\restriction W}}(v) \leqslant d_L(v) \leqslant \varepsilon^{\frac{1}{3}} n \leqslant 2\varepsilon^{\frac{1}{3}} |V'|$ and there exist at most $\varepsilon^{\frac{1}{5}} n \leqslant 2\varepsilon^{\frac{1}{5}} |V'|$ indices $i \in [\ell]$ such that $v \in V(L_i^{\restriction W})$.

Thus, Lemma 7.3(a) and (b) are satisfied with $|V'|, \varepsilon^{\frac{1}{29}}$, and $L_1^{\restriction W}, \dots, L_\ell^{\restriction W}$ playing the roles of n, ε, and $L_1, \dots, L_\ell$.

In order to approximately decompose $D_1[V']$ using Lemma 7.3, we need to partition $D_1[V']$ into a dense almost regular digraph (which will play the role of D in Lemma 7.3) and a sparse almost regular robust expander (which will play the role of Γ in Lemma 7.3). We choose Γ randomly as follows. Let $\delta := \frac{d+r}{n}$. Note that, by (κ), $D_1[V']$ is $(\delta, 10\sqrt{\varepsilon})$-almost regular. By Lemma 4.4 and (β), $D_1[V']$ is a robust (ν, τ)-outexpander. Apply Lemma 4.16 with $D_1[V']$, $|V'|$, and $10\sqrt{\varepsilon}$ playing the roles of D, n, and ε to obtain $\Gamma \subseteq D_1[V']$ such that Γ is a $(\gamma, 10\sqrt{\varepsilon})$-almost regular $(10\sqrt{\varepsilon}, |V'|^{-2})$-robust (ν, τ)-outexpander and $D_1'' := D_1[V'] \setminus \Gamma$ is $(\delta - \gamma, 10\sqrt{\varepsilon})$-almost regular.

Observe that, by (A), $\ell \leqslant (\delta - \frac{\eta}{2})|V'|$. Apply Lemma 7.3 with $D_1'', \mathcal{F}^{\restriction W}, |V'|, |V'|^{-2}, \frac{\eta}{2}, \varepsilon^{\frac{1}{29}}$, and $(L_1^{\restriction W}, F_1^{\restriction W}), \dots, (L_\ell^{\restriction W}, F_\ell^{\restriction W})$ playing the roles of $D, \mathcal{F}, n, p, \eta, \varepsilon$, and $(L_1, F_1), \dots, (L_\ell, F_\ell)$ to obtain edge-disjoint $\mathcal{H}_1^{\restriction W}, \dots, \mathcal{H}_\ell^{\restriction W} \subseteq D_1'' \cup \Gamma \cup \mathcal{F}^{\restriction W} = D_1[V'] \cup \mathcal{F}^{\restriction W}$ such that, for each $i \in [\ell]$, $\mathcal{H}_i^{\restriction W}$ is a spanning configuration of shape $(L_i^{\restriction W}, F_i^{\restriction W})$ and the following holds. Let $\mathcal{H}^{\restriction W} := \bigcup_{i \in [\ell]} \mathcal{H}_i^{\restriction W}$ and $D_2' := D_1[V'] \setminus \mathcal{H}^{\restriction W}$. Then,

(I) D_2' is a robust $(\frac{\nu}{2}, \tau)$-outexpander.

Next, we transform the auxiliary spanning configurations $\mathcal{H}_1^{\restriction W}, \dots, \mathcal{H}_\ell^{\restriction W}$ into edge-disjoint spanning configurations on V of shapes $(L_1, F_1), \dots, (L_\ell, F_\ell)$ as follows. For each $i \in [\ell]$, let $\mathcal{H}_i$ be the digraph with $V(\mathcal{H}_i) := V$ and $E(\mathcal{H}_i) = (E(\mathcal{H}_i^{\restriction W}) \setminus F_i^{\restriction W}) \cup F_i$. Denote $\mathcal{H} := \bigcup_{i \in [\ell]} \mathcal{H}_i$. Let $i \in [\ell]$. Then, Proposition 9.3 implies that $\mathcal{H}_i \subseteq D_1[V'] \cup F_i$ and $\mathcal{H}_i$ is a W-exceptional spanning configuration of shape (L_i, F_i). Moreover, since $F_i \subseteq E_W(D_1) \setminus A$, we have $\mathcal{H}_i \subseteq D_1 \setminus A$ and so

$$E(\mathcal{H}) \cap A = \emptyset. \tag{10.4}$$

Furthermore, by definition of U_2^*-path consistency with respect to $(D_1', \mathcal{F})$, the sets $F_1, \dots, F_\ell$ are edge-disjoint. Thus, since $\mathcal{H}_1^{\restriction W}, \dots, \mathcal{H}_\ell^{\restriction W}$ are edge-disjoint, $\mathcal{H}_1, \dots, \mathcal{H}_\ell$ are edge-disjoint.

Finally, we verify that $\mathcal{H}_1, \dots, \mathcal{H}_\ell$ induce a good partial path decomposition of D_1. For each $i \in [\ell]$, let $\mathcal{P}_{2,i}$ be a path decomposition of $\mathcal{H}_i$ induced by (L_i, F_i). Let $\mathcal{P}_2 := \bigcup_{i \in [\ell]} \mathcal{P}_{2,i}$ and $D_2 := D_1 \setminus \mathcal{P}_2$. We claim that the following holds.

Claim 3. $\mathcal{P}_2$ is a (U_2^*, W, A)-partial path decomposition of D_1 of size $\widetilde{\mathrm{ex}}(D_1) - r$.

Proof of Claim. By Proposition 9.4, (B), and since $(L_1, F_1), \dots, (L_\ell, F_\ell)$ are U_2^*-path consistent with respect to $(D_1', \mathcal{F})$, $\mathcal{P}_2$ is a partial path decomposition of D_1 of size $\widetilde{\mathrm{ex}}(D_1) - r$ such that

$$U^0(D_1) \cap (V^+(\mathcal{P}_2) \cup V^-(\mathcal{P}_2)) \subseteq U_2^*. \tag{10.5}$$

Thus, it only remains to show that $\mathcal{P}_2$ is consistent with A^+ and A^-. By (10.4), $E(\mathcal{P}_2) \cap A = \emptyset$. Thus, it suffices to show that each $v \in W$ is the starting point of at most $\mathrm{ex}_{D_1}^+(v) - d_A^+(v)$ paths in $\mathcal{P}_2$ and the ending point of at most $\mathrm{ex}_{D_1}^-(v) - d_A^-(v)$.

Let $v \in W$. If $v \in W \setminus W_A$, then Claim 1 implies that $d_A(v) = 0$. Moreover, (10.2) implies that $W \cap U_2^* = \emptyset$. Thus, the fact that $\mathcal{P}_2$ is a partial path decomposition satisfying (10.5) implies that $\mathcal{P}_2$ contains at most $\mathrm{ex}_{D_1}^+(v) = \mathrm{ex}_{D_1}^+(v) - d_A^+(v)$ paths which start at v and at most $\mathrm{ex}_{D_1}^-(v) = \mathrm{ex}_{D_1}^-(v) - d_A^-(v)$ paths which end at v. We may therefore assume that $v \in W_A^+$ (similar argument hold if $v \in W_A^-$). By Definition 8.7 and (α), we have $\mathrm{ex}_{D_1}^+(v) > 0$ and so $v \in U^+(D_1)$. Thus, the fact that $(L_1, F_1), \dots, (L_\ell, F_\ell)$ are U_2^*-path consistent with respect to $(D_1', \mathcal{F})$ implies that the number of paths in $\mathcal{P}_2$ which end at v is $0 \leqslant \mathrm{ex}_{D_1}^-(v) - d_A^-(v)$, as desired. In particular, v is an internal vertex of precisely $d^-_{\mathcal{P}_2}(v)$ paths in $\mathcal{P}_2$. Therefore, the number of paths in $\mathcal{P}_2$ which start at v is

$$\begin{aligned} d^+_{\mathcal{P}_2}(v) - d^-_{\mathcal{P}_2}(v) &\overset{\text{Proposition 9.4}}{=} d^+_L(v) - d^-_L(v) \overset{\text{(C)}}{=} d^+_{D_1\setminus A}(v) - d^-_{D_1\setminus A}(v) \\ &\overset{\text{Definition 8.7}}{=} d^+_{D_1}(v) - d^+_A(v) - d^-_{D_1}(v) = \mathrm{ex}^+_{D_1}(v) - d^+_A(v), \end{aligned}$$

as desired. □

It remains to show that $\mathcal{P}_2$ is good, that is, that $\widetilde{\mathrm{ex}}(D_2) = \widetilde{\mathrm{ex}}(D_1) - |\mathcal{P}_2|$. Recall that by ($\alpha$), $\mathcal{P}_1$ avoids all the edges in A. Thus, by Proposition 9.4 and (C)–(E), the following hold.

(II) For each $v \in W_1$, $N^\pm_{D_2}(v) = N^\pm_A(v)$.
(III) For each $v \in V \setminus (W_1 \cup X^-)$, $d^+_{D_2}(v) = r$.
(IV) For each $v \in V \setminus (W_1 \cup X^+)$, $d^-_{D_2}(v) = r$.
(V) For each $v \in X^\pm$, $d^\mp_{D_2}(v) = r - 1$.

Claim 4. $\widetilde{\mathrm{ex}}(D_2) = r$.

Proof of Claim. First, we check that $\mathrm{ex}(D_2) \leqslant r$. Let $v \in V$.

- If $v \in X^+ \cup X^-$, then (V) implies that $\mathrm{ex}^+_{D_2}(v) = 0$.
- If $v \in X^+ \setminus X^-$, then (III) and (V) imply that $\mathrm{ex}^+_{D_2}(v) = 1$.
- If $v \in X^- \setminus X^+$, then (III) and (V) imply that $\mathrm{ex}^+_{D_2}(v) = 0$.
- If $v \in V \setminus (X^+ \cup X^- \cup W)$, then (III) and (IV) imply that $\mathrm{ex}^+_{D_2}(v) = 0$.
- Suppose $v \in W_1 \setminus W^+_A$. Recall from Step 1 that A^+ and A^- are absorbing sets of (W^+_A, V')-starting and (V', W^-_A)-ending edges for T. Thus, $d^+_A(v) = 0$ and so (II) implies that $\mathrm{ex}^+_{D_2}(v) = 0$.
- Suppose $v \in W_1 \cup W^+_A$. Then, Definition 8.7 implies that $v \in U^+(T)$ and so $d^-_A(v) = 0$. Therefore, (II) implies $\mathrm{ex}^+_{D_2}(v) = d^+_A(v)$.

Moreover, recall that $W^+_A \subseteq W_1$ and $(X^+ \cup X^-) \cap W_1 = \emptyset$ (see Claim 2). Thus,

$$\mathrm{ex}(D_2) \overset{(1.1)}{=} \sum_{v\in V} \mathrm{ex}^+_{D_2}(v) = |X^+ \setminus X^-| + \sum_{v\in W^+_A} d^+_A(v) \overset{\text{Definition 8.7}}{=} |X^+ \setminus X^-| + |A^+| \overset{\text{Claim 2}}{\leqslant} r,$$

as desired.

Thus, it is enough to show that $\Delta^0(D_2) = r$. By Claim 2, $|X^-| \leqslant r$ and, by (ii), $|W_1| \leqslant |W| \leqslant 4\varepsilon n$. Thus, $V \setminus (W_1 \cup X^-) \neq \emptyset$ and so (III) implies that $\Delta^0(D_2) \geqslant r$. Let $v \in V$. We verify that both $d^\pm_{D_2}(v) \leqslant r$. If $v \notin W_1$, then (III)–(V) imply that both $d^\pm_{D_2}(v) \leqslant r$. We may therefore assume that $v \in W_1 \subseteq W$. By Definition 8.7 and (iv), $d^+_A(v) = d_{A^+}(v) \leqslant |A^+| \leqslant r$ and, similarly, $d^-_A(v) \leqslant r$. Therefore, $\Delta^0(D_2) \leqslant r$ and so we are done. □

Thus, Claims 3 and 4 imply that

(VI) $\mathcal{P}_2$ is a good partial path decomposition of D_1.

Step 4: Completing the path decomposition. Finally, we will decompose D_2 using Corollary 8.8. Recall that to apply Corollary 8.8, all the edges incident to the exceptional set must be absorbing edges. By (II)–(V), all the absorbing edges are incident to W_1 and the vertices in W_2 are still incident to some non-absorbing edges in D_2. Thus, we will apply Corollary 8.8 with W_1 and $V' \cup W_2$ playing the roles of W and V'.

First, we check that all the prerequisites of Corollary 8.8 are satisfied. Note that $D_2[V'] = D_2'$ (D_2' was defined just above (I) and D_2 was defined just above Claim 3). Thus, by Lemma 4.2(b), (ii), and (I), $D_2 - W_1$ is a robust $(\frac{\nu}{4}, 2\tau)$-outexpander. By Fact 8.11, (α), and Claim 3, A^+ and A^- are absorbing sets of (W_A^+, V')-starting and (V', W_A^-)-ending edges for D_2. In particular, Definition 8.7 and the fact that $W_A \subseteq W_1$ imply that A^+ and A^- are absorbing sets of $(W_1, V' \cup W_2)$-starting and $(V' \cup W_2, W_1)$-ending edges for D_2.

Let $Y^\pm := X^\pm \setminus X^\mp$, $Y^* := X^+ \cap X^-$, and $Y^0 := V \setminus (Y^+ \cup Y^- \cup Y^* \cup W_1)$. We aim to apply Corollary 8.8 with Y^+, Y^-, Y^*, and Y^0 playing the roles of X^+, X^-, X^*, and X^0. First, observe that (II)–(V) imply that the excess and degree conditions of Corollary 8.8 hold with W_1, Y^+, Y^-, and Y^* playing the roles of W, X^+, X^-, and X^*. Moreover, Claim 2 implies $|Y^\pm \cup Y^*| + |A^\pm| = r$. Finally, we claim that $V(A^\pm) \cap (V' \cup W_2) \subseteq Y^\mp \cup Y^0$. By (iv), $|A^\pm| \in \{0, r\}$. If $|A^\pm| = 0$, then $V(A^\pm) \cap (V' \cup W_2) = \emptyset$ and so we are done. If $|A^\pm| = r$, then Claim 2 implies that $X^\pm = \emptyset$ and so $V' \cup W_2 = Y^\mp \cup Y^0$. Thus, $V(A^\pm) \cap (V' \cup W_2) \subseteq Y^\mp \cup Y^0$, as desired.

Apply Corollary 8.8 with $D_2, n - |W_1|, V \setminus W_1, W_1, Y^+, Y^-, Y^*, Y^0, \frac{\eta}{2}, \frac{\nu}{4}$, and 2τ playing the roles of $D, n, V', W, X^+, X^-, X^*, X^0, \delta, \nu$, and τ to obtain a path decomposition $\mathcal{P}_3$ of D_2 of size r. Note that, by Claim 4, $\mathcal{P}_3$ is a perfect path decomposition of D_2. Recall that by (α) and (VI), $\mathcal{P}_1$ and $\mathcal{P}_2$ are good. Then, by Fact 8.5, $\mathcal{P} := \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ is a perfect path decomposition of T. That is, $|\mathcal{P}| = \widetilde{\mathrm{ex}}(T)$. This completes the proof. □

11 | AUXILIARY EXCESS FUNCTION

In this section, we introduce some concepts which will be convenient for constructing good (U^*, W, A)-partial path decompositions. (Recall these were defined in Definition 8.10.)

Recall that once we have chosen absorbing edges, we need to ensure that (i) these edges are not used for other purposes and (ii) not too many paths have endpoints in W. Moreover, if $\widetilde{\mathrm{ex}}(T) > \mathrm{ex}(T)$, then some vertices v will have to be used as starting/ending points of paths more than $\mathrm{ex}_T^\pm(v)$ times. As discussed briefly in Section 8.2, we will choose in advance which vertices will be used as these additional endpoints: we will initially select a set U^* of $\widetilde{\mathrm{ex}}(T) - \mathrm{ex}(T)$ (distinct) vertices in $U^0(T)$ as additional endpoints of paths. We then treat each vertex in U^* as if it has positive and negative excess both equal to one. The following concept of an auxiliary excess function (as defined in (11.1)) encapsulates all this – it also incorporates the constraints given by (i) and (ii) above. It will enable us to easily keep track of how many paths remain to be chosen and which vertices can be used as endpoints.

Let D be a digraph and $W, V' \subseteq V(D)$ be disjoint. Suppose that $A \subseteq E(D)$ is a (W, V')-absorbing set for D. Let $U^* \subseteq U^0(D)$ satisfy $|U^*| = \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$. Note that, by Definition 8.7, $(V(A) \cap W) \cap U^* = \emptyset$. For each $v \in V(D)$, define

$$\widetilde{\mathrm{ex}}^\pm_{D,U^*,W,A}(v) := \begin{cases} 1 & \text{if } v \in U^*, \\ \mathrm{ex}^\pm_D(v) - d^\mp_A(v) & \text{if } v \in V(A) \cap W, \\ \mathrm{ex}^\pm_D(v) & \text{otherwise.} \end{cases} \tag{11.1}$$

Then, define

$$\begin{aligned}
\widetilde{U}^{\pm}_{U^*,W,A}(D) &:= \{v \in V(D) \mid \widetilde{\mathrm{ex}}^{\pm}_{D,U^*,W,A}(v) > 0\};\\
\widetilde{U}^{0}_{U^*,W,A}(D) &:= V(D) \setminus (\widetilde{U}^{+}_{U^*,W,A}(D) \cup \widetilde{U}^{-}_{U^*,W,A}(D));\\
\widetilde{\mathrm{ex}}^{\pm}_{D,U^*,W,A}(S) &:= \sum_{v\in S} \widetilde{\mathrm{ex}}^{\pm}_{D,U^*,W,A}(v) \text{ for each } S \subseteq V(D);\\
\widetilde{\mathrm{ex}}^{\pm}_{U^*,W,A}(D) &:= \sum_{v\in V} \widetilde{\mathrm{ex}}^{\pm}_{D,U^*,W,A}(v).
\end{aligned}$$

Denote by A^+ and A^- the absorbing sets of (W, V')-starting and (V', W)-ending edges contained in A. Then,

$$\widetilde{\mathrm{ex}}^{\pm}_{U^*,W,A}(D) = \mathrm{ex}(D) + |U^*| - |A^{\pm}| = \widetilde{\mathrm{ex}}(D) - |A^{\pm}|. \tag{11.2}$$

Note that it is possible that $\widetilde{\mathrm{ex}}^{+}_{U^*,W,A}(D) \neq \widetilde{\mathrm{ex}}^{-}_{U^*,W,A}(D)$. For simplicity, when A and W are clear from the context, they will be omitted in the subscripts of the above notation.

Note that the analogue of Fact 4.21 holds for the auxiliary excess function.

Fact 11.1. *Let D be a digraph and $W, V' \subseteq V(D)$ be disjoint. Suppose that $A \subseteq E(D)$ is a (W, V')-absorbing set for D. Let $U^* \subseteq U^0(D)$ satisfy $|U^*| = \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$. Then, for any $S \subseteq V(D)$, $\widetilde{\mathrm{ex}}^{\pm}_{U^*}(D) = \widetilde{\mathrm{ex}}^{\pm}_{D,U^*}(S) + \widetilde{\mathrm{ex}}^{\pm}_{D,U^*}(V(D) \setminus S)$.*

Observe that the following holds by definition of the auxiliary excess function.

Fact 11.2. *Let D, W, V', A, and U^* satisfy the assumptions of Fact 11.1. A set $\mathcal{P}$ of edge-disjoint paths of D is a (U^*, W, A)-partial path decomposition of D if and only if $\mathcal{P} \subseteq D \setminus A$ and each $v \in V(D)$ is the starting point of at most $\widetilde{\mathrm{ex}}^{+}_{D,U^*,W,A}(v)$ paths in $\mathcal{P}$ and the ending point of at most $\widetilde{\mathrm{ex}}^{-}_{D,U^*,W,A}(v)$ paths in $\mathcal{P}$.*

Thus, this auxiliary excess function designates which vertices are still available to use as endpoints and, by (11.2), it indicates how many paths we are still allowed to choose. For these reasons, fixing U^* at the beginning will prove very useful in Section 12, even though it is not necessary and may look cumbersome at first glance.

Let D, W, V', A, and U^* satisfy the assumptions of Fact 11.1. Suppose that $\mathcal{P}$ is a (U^*, W, A)-partial path decomposition of D. By Fact 11.2, each path in $\mathcal{P}$ corresponds to some auxiliary excess and so, by removing $\mathcal{P}$ (and removing from U^* the vertices which have already been used as endpoints), we reduce the auxiliary positive/negative excess of each $v \in V(D)$ by the number of paths in $\mathcal{P}$ which start/end at v (Proposition 11.3). This implies that the total auxiliary excess of D is decreased by precisely $|\mathcal{P}|$ when we remove the paths in $\mathcal{P}$ (Corollary 11.4). The auxiliary excess function will thus be much more convenient to use than the modified excess function introduced in Section 1 (compare the bounds in Proposition 8.4(a) and Corollary 11.4). Moreover, this implies that good (U^*, W, A)-partial path decompositions can be combined to form a larger good (U^*, W, A)-partial path decomposition (Corollary 11.5).

Proposition 11.3. *Let D, W, V', A, and U^* satisfy the assumptions of Fact 11.1. Suppose that $\mathcal{P}$ is a (U^*, W, A)-partial path decomposition of D. Denote $U^{**} := U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$. For each $v \in V(D)$, let $n^+_{\mathcal{P}}(v)$ and $n^-_{\mathcal{P}}(v)$ denote the number of paths in $\mathcal{P}$ which start and end at v, respectively. Then, each $v \in V(D)$ satisfies $\widetilde{\mathrm{ex}}^{\pm}_{D\setminus\mathcal{P},U^{**}}(v) = \widetilde{\mathrm{ex}}^{\pm}_{D,U^*}(v) - n^{\pm}_{\mathcal{P}}(v)$.*

Proof. If $v \in V(D) \setminus U^*$, then $\mathrm{ex}^{\pm}_{D\setminus\mathcal{P}}(v) = \mathrm{ex}^{\pm}_D(v) - n^{\pm}_{\mathcal{P}}(v)$ and so the proposition holds. We may therefore assume that $v \in U^*$. If $v \in U^{**}$, then $n^{\pm}_{\mathcal{P}}(v) = 0$ and so $\widetilde{\mathrm{ex}}^{\pm}_{D\setminus\mathcal{P},U^{**}}(v) = 1 = \widetilde{\mathrm{ex}}^{\pm}_{D,U^*}(v) - n^{\pm}_{\mathcal{P}}(v)$, as desired. We may therefore assume that $v \in U^* \setminus U^{**}$. By Definition 8.7, $v \notin W$. First, suppose that $v \in V^+(\mathcal{P}) \cap V^-(\mathcal{P})$. Note that both $n^{\pm}_{\mathcal{P}}(v) = 1$ and $\mathrm{ex}_{D\setminus\mathcal{P}}(v) = 0$. Since $v \notin W$, we have $\widetilde{\mathrm{ex}}^{\pm}_{D\setminus\mathcal{P},U^{**}}(v) = \mathrm{ex}^{\pm}_{D\setminus\mathcal{P}}(v) = 0 = \widetilde{\mathrm{ex}}^{\pm}_{D,U^*}(v) - n^{\pm}_{\mathcal{P}}(v)$, as desired. By symmetry, we may therefore assume that $v \in V^+(\mathcal{P}) \setminus V^-(\mathcal{P})$. Note that $n^+_{\mathcal{P}}(v) = 1$, $n^-_{\mathcal{P}}(v) = 0$, and $\mathrm{ex}_{D\setminus\mathcal{P}}(v) = -1$. Recall that $v \notin W$. Thus, $\widetilde{\mathrm{ex}}^+_{D\setminus\mathcal{P}}(v) = \mathrm{ex}^+_{D\setminus\mathcal{P}}(v) = 0 = \widetilde{\mathrm{ex}}^+_{D,U^*}(v) - n^+_{\mathcal{P}}(v)$ and $\widetilde{\mathrm{ex}}^-_{D\setminus\mathcal{P}}(v) = \mathrm{ex}^-_{D\setminus\mathcal{P}}(v) = 1 = \widetilde{\mathrm{ex}}^-_{D,U^*}(v) - n^-_{\mathcal{P}}(v)$. □

Corollary 11.4. *Let D, W, V', A, and U^* satisfy the assumptions of Fact 11.1. Suppose $\mathcal{P}$ is a (U^*, W, A)-partial path decomposition of D. Let $U^{**} := U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$. Then, $\widetilde{\mathrm{ex}}^{\pm}_{U^{**}}(D \setminus \mathcal{P}) = \widetilde{\mathrm{ex}}^{\pm}_{U^*}(D) - |\mathcal{P}|$.*

Proof. For each $v \in V(D)$, let $n^+_{\mathcal{P}}(v)$ and $n^-_{\mathcal{P}}(v)$ denote the number of paths in $\mathcal{P}$ which start and end at v, respectively. Then,

$$\widetilde{\mathrm{ex}}^{\pm}_{U^{**}}(D \setminus \mathcal{P}) = \sum_{v\in V(D)} \widetilde{\mathrm{ex}}^{\pm}_{D\setminus\mathcal{P},U^{**}}(v) \overset{\text{Proposition 11.3}}{=} \sum_{v\in V(D)} (\widetilde{\mathrm{ex}}^{\pm}_{D,U^*}(v) - n^{\pm}_{\mathcal{P}}(v)) = \widetilde{\mathrm{ex}}^{\pm}_{U^*}(D) - |\mathcal{P}|,$$

as desired. □

Corollary 11.5. *Let $k \in \mathbb{N}$. Let D, W, V', A, and U^* satisfy the assumptions of Fact 11.1. Denote $D_0 := D$ and $U^*_0 := U^*$. Suppose that, for each $i \in [k]$, $\mathcal{P}_i$ is a good (U^*_{i-1}, W, A)-partial path decomposition of D_{i-1}, $D_i := D_{i-1} \setminus \mathcal{P}_i$, and $U^*_i := U^*_{i-1} \setminus (V^+(\mathcal{P}_i) \cup V^-(\mathcal{P}_i))$. Let $\mathcal{P} := \bigcup_{i\in[k]} \mathcal{P}_i$. Then, $\mathcal{P}$ is a good (U^*, W, A)-partial path decomposition of D of size $|\mathcal{P}| = \sum_{i\in[k]} |\mathcal{P}_i|$.*

Proof. By induction on k, it suffices to prove the case $k = 2$. By Fact 11.2, $E(\mathcal{P}) \subseteq D \setminus A$. For each $v \in V(D)$, denote by $n^+_{\mathcal{P}}(v)$ and $n^-_{\mathcal{P}}(v)$ the number of paths in $\mathcal{P}$ which start and end at v, and define $n^{\pm}_{\mathcal{P}_1}(v)$ and $n^{\pm}_{\mathcal{P}_2}(v)$ analogously. Then, each $v \in V(D)$ satisfies

$$\begin{aligned} n^{\pm}_{\mathcal{P}}(v) = n^{\pm}_{\mathcal{P}_1}(v) + n^{\pm}_{\mathcal{P}_2}(v) &\overset{\text{Proposition 11.3}}{=} (\widetilde{\mathrm{ex}}^{\pm}_{D_0,U^*_0}(v) - \widetilde{\mathrm{ex}}^{\pm}_{D_1,U^*_1}(v)) + (\widetilde{\mathrm{ex}}^{\pm}_{D_1,U^*_1}(v) - \widetilde{\mathrm{ex}}^{\pm}_{D_2,U^*_2}(v)) \\ &\leqslant \widetilde{\mathrm{ex}}^{\pm}_{D_0,U^*_0}(v). \end{aligned}$$

Thus, each $v \in V(D)$ is the starting point of at most $\widetilde{\mathrm{ex}}^+_{D,U^*}(v)$ paths in $\mathcal{P}$ and the ending point of at most $\widetilde{\mathrm{ex}}^-_{D,U^*}(v)$ paths in $\mathcal{P}$. Therefore, Fact 11.2 implies that $\mathcal{P}$ is a (U^*, W, A)-partial path decomposition of D. Denote by A^+ the absorbing set of starting edges contained in A. Then,

$$\widetilde{\mathrm{ex}}(D \setminus \mathcal{P}) \overset{(11.2)}{=} \widetilde{\mathrm{ex}}^+_{U^{**}_2}(D \setminus \mathcal{P}) + |A^+| \overset{\text{Corollary 11.4}}{=} \widetilde{\mathrm{ex}}^+_{U^*}(D) - |\mathcal{P}| + |A^+| \overset{(11.2)}{=} \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|.$$

Therefore, $\mathcal{P}$ is good. □

Recall from (8.1) that $N^+(D)$ and $N^-(D)$ denote the maximum number of distinct vertices of D which may be used as starting and ending points in a partial path decomposition of D. The next corollary states that $N^\pm(D)$ decreases appropriately when a good (U^*, W, A)-partial path decomposition is removed from D.

Corollary 11.6. *Let D, W, V', A, and U^* satisfy the assumptions of Fact 11.1. Suppose that $\mathcal{P}$ is a good (U^*, W, A)-partial path decomposition. Let X^+ be the set of vertices $v \in \widetilde{U}^+_{U^*}(D)$ for which $\mathcal{P}$ contains precisely $\widetilde{\mathrm{ex}}^+_{D,U^*}(v)$ paths starting at v. Similarly, let X^- be the set of vertices $v \in \widetilde{U}^-_{U^*}(D)$ for which $\mathcal{P}$ contains precisely $\widetilde{\mathrm{ex}}^-_{D,U^*}(v)$ paths ending at v. Then, both $N^\pm(D) - N^\pm(D \setminus \mathcal{P}) \leqslant |X^\pm|$.*

Proof. By symmetry, it suffices to show that $N^+(D) - N^+(D \setminus \mathcal{P}) = |X^+|$. Denote $D' := D \setminus \mathcal{P}$ and $U^{**} := U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$. Let $Y := \widetilde{U}^+_{U^*}(D) \cup \{v \in W \mid \mathrm{ex}^+_D(v) = d^+_A(v) > 0\}$ and $Z := \widetilde{U}^+_{U^{**}}(D') \cup \{v \in W \mid \mathrm{ex}^+_{D'}(v) = d^+_A(v) > 0\}$. By assumption, $|U^*| = \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$ and so

$$|Y| \overset{(11.1)}{=} |U^+(D)| + |U^*| = N^+(D).$$

Since $\mathcal{P}$ is good, Proposition 9.7 implies that $|U^{**}| = \widetilde{\mathrm{ex}}(D') - \mathrm{ex}(D')$ and so, by the same arguments as above, $|Z| = N^+(D')$. Thus, it suffices to show that

$$Y = X^+ \cup Z. \tag{11.3}$$

By definition, $X^+ \subseteq Y$. Next, we show that $Z \subseteq Y$. By Proposition 11.3, $\widetilde{\mathrm{ex}}^+_{D',U^{**}}(v) \leqslant \widetilde{\mathrm{ex}}^+_{D,U^*}(v)$ for each $v \in V$ and so $U^+_{U^{**}}(D') \subseteq U^+_{U^*}(D) \subseteq Y$. By definition of absorbing starting edges (Definition 8.7), each $v \in W$ satisfies $\mathrm{ex}^+_D(v) \geqslant d^+_A(v)$ and so (11.1) implies $\{v \in W \mid \mathrm{ex}^+_{D'}(v) = d^+_A(v) > 0\} \subseteq Y$. Therefore, $X^+ \cup Z \subseteq Y$, as desired.

Finally, we prove that $Y \subseteq X^+ \cup Z$. Let $v \in Y \setminus X^+$. It is enough to show that $v \in Z$. Suppose first that $v \in \widetilde{U}^+_{U^*}(D)$. By Fact 11.2 and since $v \notin X^+$, $\mathcal{P}$ contains fewer than $\widetilde{\mathrm{ex}}^+_{D,U^*}(v)$ paths which start at v. Then, Proposition 11.3 implies $v \in \widetilde{U}^+_{U^{**}}(v) \subseteq Z$. We may therefore assume that $v \in \{v' \in W \mid \mathrm{ex}^+_D(v') = d^+_A(v') > 0\}$. By definition of absorbing starting edges (Definition 8.7), $\mathrm{ex}^+_D(v) > 0$ and so (11.1) implies $\widetilde{\mathrm{ex}}^+_{D,U^*}(v) = 0 = \widetilde{\mathrm{ex}}^-_{D,U^*}(v)$. Therefore, Fact 11.2 implies that no path in $\mathcal{P}$ contains v as an endpoint and so $\mathrm{ex}^+_{D'}(v) = \mathrm{ex}_D(v) = d^+_A(v) > 0$. Thus, $v \in Z$ and so $Y \subseteq X^+ \cup Z$. Consequently, (11.3) holds and we are done. □

12 | THE CLEANING STEP: PROOF OF LEMMA 9.5

We now prove Lemma 9.5 using Lemmas 12.1, 12.2, and 12.4 below. Recall from Section 9.3 that the main goal of the cleaning step is to reduce the degree at the exceptional set W and cover all the edges inside W. Moreover, recall that we denote by W_* the set of exceptional vertices of very large excess, by W_0 the set of exceptional vertices with not too large excess, and by W_A the set of exceptional vertices which are incident to some absorbing edges.

In Lemma 12.1, we cover all edges of $T[W_0]$. The remaining edges of $T[W]$ which are incident to W_* will be covered in Lemma 12.2. Since the excess of T is proportional to $|W_*|$, the edges of $T[W]$ which are incident to W_* can be covered one by one with short paths. However, vertices in W_0 may have small excess and so $T[W_0]$ needs to be covered more efficiently. The idea will be to decompose $T[W_0]$ into matchings and then tie them into paths.

In Lemma 12.2, we also decrease the degree of the vertices in $W_* \cup W_A$ when $W_* \neq \emptyset$. This is achieved by covering edges at W_* one by one with short paths until the desired degree is attained. (The endpoints of these paths are chosen via Lemma 12.3.) Finally, we will use Lemma 12.4 to decrease the degree at W_A when $W_* = \emptyset$. There, we will use long paths to decrease the degree at all vertices in W_A at the same time. This is necessary because the total excess may be relatively small, so we do not have room to cover the degree at each vertex in W_A one by one.

12.1 | Proof overview

The proof structure of Lemmas 12.1, 12.2, and 12.4 is similar. In each of these lemma, we need to construct a good partial path decomposition. We always proceed inductively to construct the paths either one by one (Lemma 12.2) or two by two (Lemmas 12.1 and 12.4). All these paths are constructed using Corollary 4.8: we use Corollary 4.8(a) when we need short paths (Lemmas 12.2 and 12.4) and we use Corollary 4.8(b) when we need long paths (Lemmas 12.1 and 12.4). In each application of Corollary 4.8, we need to specify two main elements.

(i) We need choose which edges we want to cover and which vertices whose degree we want to decrease. (Roughly speaking, these edges will play the roles of $P_2, \dots, P_{k-1}$ in Corollary 4.8.) For example, in Lemma 12.1, the aim is to cover all the edges inside W_0, so for each path we select which of these edges we want to cover.

(ii) We need to choose ‘suitable’ endpoints for our paths. (Roughly speaking, these will play the roles of P_1 and P_k in Corollary 4.8.) Choosing these endpoints will form the core of the proof as these need to satisfy several requirements. Firstly, they need to be ‘compatible’ with the edges we want to cover. For example, if we want to cover an edge uv, we cannot use v as a starting point. Secondly, they need to have an ‘appropriate’ amount of excess to ensure that the resulting set of paths will form a good partial path decomposition. The auxiliary excess function defined in Section 11 (see (11.1)) will enable us to keep track of which vertices are allowed to be used as endpoints in each stage.

12.2 | Covering the edges inside W_0

The next lemma states that all the edges inside W_0 can be covered by a small good partial path decomposition. The idea is to first decompose $T[W_0]$ into matchings using Vizing's theorem. Then, we incorporate each matching into a pair of (almost) spanning paths using Corollary 4.8(b). We use very long paths so that the maximum semidegree of T is reduced sufficiently quickly to obtain a good partial path decomposition. Moreover, we require two paths to cover each of the matchings because, by definition of a partial path decomposition, we may only construct paths whose starting and ending points belong to $U^+(T) \cup U^*$ and $U^-(T) \cup U^*$, respectively. Indeed, if, for example, M is a matching such that each of the vertices in $U^+(T) \cup U^*$ is the ending point of an edge in M, then we would not be able to construct a path which contains M and starts in $U^+(T) \cup U^*$. Splitting each matching obtained from Vizing's theorem in two ensures that there are always suitable endpoints to cover each of the submatchings.

Lemma 12.1. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll 1$. Let $T \notin \mathcal{T}_{\text{excep}}$ be a tournament on a vertex set V of size n satisfying the following properties.*

(a) *Let* $W_* \cup W_0 \cup V'$ *be a partition of* V *such that, for each* $w_* \in W_*$, $|\operatorname{ex}_T(w_*)| > (1-20\eta)n$*; for each* $w_0 \in W_0$, $|\operatorname{ex}_T(w_0)| \leqslant (1-20\eta)n$*; and, for each* $v' \in V'$, $|\operatorname{ex}_T(v')| \leqslant \varepsilon n$*. Let* $W := W_* \cup W_0$ *and suppose* $|W| \leqslant \varepsilon n$*.*
(b) *Let* $A^+, A^- \subseteq E(T)$ *be absorbing sets of* (W, V')*-starting/*(V', W)*-ending edges for* T *of size at most* $\lceil \eta n \rceil$*. Denote* $A := A^+ \cup A^-$*.*
(c) *Let* $U^* \subseteq U^0(T)$ *satisfy* $|U^*| = \widetilde{\operatorname{ex}}(T) - \operatorname{ex}(T)$*.*

Then, there exists a good (U^*, W, A)*-partial path decomposition* $\mathcal{P}$ *of* T *such that the following hold, where* $D := T \setminus \mathcal{P}$*.*

(i) $|\mathcal{P}| \leqslant 2\varepsilon n$*.*
(ii) $E(D[W_0]) = \emptyset$*.*
(iii) *If* $|U^+(D)| = |U^-(D)| = 1$*, then* $e(U^-(D), U^+(D)) = 0$ *or* $\widetilde{\operatorname{ex}}(D) - \operatorname{ex}(D) \geqslant 2$*.*
(iv) *Each* $v \in U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$ *satisfies* $d_D^+(v) = d_D^-(v) \leqslant \widetilde{\operatorname{ex}}(D) - 1$*.*

Property (iv) will ensure that Lemma 9.5(ii) is satisfied at the end of the cleaning. One can use (iii) to ensure that the leftover oriented graph D does not have all its positive and negative excess concentrated on two vertices v^+ and v^-, respectively, with an edge v^-v^+ between them. Otherwise, we would encounter a similar problem as with the tournaments in the class $\mathcal{T}_{\text{apex}}$ (recall Propositions 5.3 and 5.4).

Proof of Lemma 12.1. If $W_0 = \emptyset$, then we can set $\mathcal{P} := \emptyset$ and, by Proposition 5.4 and Fact 4.22, we are done. Thus, we may assume that $W_0 \neq \emptyset$. Fix additional constants such that $\varepsilon \ll \nu \ll \tau \ll \eta$. Let $W^\pm := W \cap U^\pm(T)$ and, for each $\diamond \in \{*, 0\}$, denote $W_\diamond^\pm := W_\diamond \cap U^\pm(T)$.

Fix a matching decomposition $M_1, \dots, M_m$ of $T[W_0]$. By Vizing's theorem, we may assume that $m \leqslant |W_0| \leqslant \varepsilon n$. Assume inductively that for some $0 \leqslant k \leqslant m$, we have constructed edge-disjoint paths $P_{1,1}, P_{1,2}, P_{2,1}, \dots, P_{k,2} \subseteq T$ such that $\mathcal{P}_k := \{P_{i,j} \mid i \in [k], j \in [2]\}$ is a (U^*, W, A)-partial path decomposition of T and the following hold.

(α) For each $i \in [k]$, $E(P_{i,1} \cup P_{i,2}) \cap E(T[W]) = M_i$.
(β) For each $i \in [k]$ and $j \in [2]$, $V \setminus W_* \subseteq V(P_{i,j})$.
(γ) For each $\diamond \in \{+, -\}$, if $W_*^\diamond \neq \emptyset$, then $V^\diamond(\mathcal{P}_k) \subseteq W_*^\diamond$.

Denote $D_k := T \setminus \mathcal{P}_k$. Then, following holds.

Claim 1. We have $\widetilde{\operatorname{ex}}(D_k) = \widetilde{\operatorname{ex}}(T) - 2k \geqslant 2\eta n$. In particular, $\mathcal{P}_k$ is a good (U^*, W, A)-partial path decomposition of T.

Proof of Claim. First, note that $\widetilde{\operatorname{ex}}(T) - 2k \geqslant 2\eta n$ holds by Fact 4.22 and since $k \leqslant \varepsilon n$. By Proposition 8.4(a), it is enough to show that $\Delta^0(D_k) \leqslant \widetilde{\operatorname{ex}}(T) - 2k$.

If there exists $\diamond \in \{+, -\}$ such that $|W_*^\diamond| \geqslant 2$, then $\operatorname{ex}(D_k) \geqslant 2(1-20\eta)n - 2k \geqslant n$ and so $\Delta^0(D_k) \leqslant \operatorname{ex}(D_k) \leqslant \widetilde{\operatorname{ex}}(T) - 2k$, as desired. We may therefore assume that both $|W_*^\pm| \leqslant 1$. By (β) and (γ), P_k consists of Hamilton paths. Since $\mathcal{P}_k$ is a partial path decomposition, each $v \in U^+(T)$ satisfies

$$d_{D_k}^+(v) = d_T^+(v) - 2k \leqslant \Delta^0(T) - 2k \leqslant \widetilde{\operatorname{ex}}(T) - 2k$$

and

$$d_{D_k}^-(v) \leqslant d_T^-(v) - (2k - \operatorname{ex}_T^+(v)) \overset{\text{Fact 4.20(d)}}{=} d_T^+(v) - 2k \leqslant \widetilde{\operatorname{ex}}(T) - 2k.$$

Similarly, each $v \in U^-(T)$ satisfies both $d^{\pm}_{D_k}(v) \leqslant \widetilde{\text{ex}}(T) - 2k$. Hence, we may assume that $U^0(T) \neq \emptyset$ and so n is odd by definition of $\text{ex}_T(v)$ or Proposition 6.1. Let $v \in U^0(T)$. By (β), Fact 4.22, and (P2), $v \in V^{\pm}(P) \cup V^0(P)$ for all but at most one path $P \in \mathcal{P}_k$ and so

$$d^{\pm}_{D_k}(v) \leqslant d^{\pm}_T(v) - (2k-1) = \frac{n+1}{2} - 2k \overset{\text{Fact 4.22}}{\leqslant} \Delta^0(T) - 2k \leqslant \widetilde{\text{ex}}(T) - 2k.$$

Thus, $\Delta^0(D_k) \leqslant \widetilde{\text{ex}}(T) - 2k$, as desired. □

If $k = m$, then let $\mathcal{P} := \mathcal{P}_m$. We verify that all the assertions of Lemma 12.1 hold. By construction and Claim 1, $\mathcal{P}$ is a good (U^*, W, A)-partial path decomposition of T. Moreover, by construction, $|\mathcal{P}| = 2m \leqslant 2\varepsilon n$ and $D := T \setminus \mathcal{P}$ satisfies $E(D[W_0]) = \emptyset$ by (α). Thus, (i) and (ii) hold. For (iii), suppose both $|U^{\pm}(D)| = 1$, say $U^{\pm}(D) = \{u_{\pm}\}$, and assume that $u_- u_+ \in E(D)$. We need to show that $\widetilde{\text{ex}}(D) - \text{ex}(D) \geqslant 2$. If there exists $\diamond \in \{+,-\}$ such that $u_\diamond \notin W_*$, then

$$\begin{aligned}\widetilde{\text{ex}}(D) - \text{ex}(D) &\overset{(1.1)}{=} \widetilde{\text{ex}}(D) - \text{ex}^{\diamond}_D(u_\diamond) \overset{\text{Fact 4.20(d)}}{\geqslant} d^{\min}_D(u_\diamond) \geqslant d^{\min}_T(u_\diamond) - |\mathcal{P}| \\ &\overset{\text{(a),Fact 4.20(b)}}{\geqslant} \frac{20\eta n - 1}{2} - 2\varepsilon n \geqslant 2.\end{aligned}$$

We may therefore assume that both $u_{\pm} \in W_*$. Then, by (γ), all paths in $\mathcal{P}$ start in $W^+_* \subseteq U^+(T)$ and end in $W^-_* \subseteq U^-(T)$. Thus, Proposition 8.3 implies that $\text{ex}(D) = \text{ex}(T) - |\mathcal{P}|$ and, since $k \leqslant \varepsilon n$ and each $v \in W_*$ satisfies $|\,\text{ex}_T(v)| \geqslant (1 - 20\eta)n$, we have $U^{\pm}(D) = U^{\pm}(T)$. Therefore, by Claim 1 and since $T \notin \mathcal{T}_{\text{excep}}$, Proposition 5.4 implies that $\widetilde{\text{ex}}(D) - \text{ex}(D) = \widetilde{\text{ex}}(T) - \text{ex}(T) \geqslant 2$ and so (iii) holds. Finally, for (iv), suppose that $v \in U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$. Then, note that $v \notin W_*$ and, by Proposition 6.1, n is odd. Thus,

$$d^+_D(v) = d^-_D(v) \overset{(\beta)}{=} d^-_T(v) - |\mathcal{P}| = \frac{n-1}{2} - |\mathcal{P}| \overset{\text{Fact 4.22}}{\leqslant} \widetilde{\text{ex}}(T) - 1 - |\mathcal{P}| \overset{\text{Claim 1}}{=} \widetilde{\text{ex}}(D) - 1$$

and so (iv) holds.

If $k < m$, then construct $P_{k+1,1}$ and $P_{k+1,2}$ as follows. Denote $M_{k+1} := \{x^+_1 x^-_1, \ldots, x^+_\ell x^-_\ell\}$. For each $\diamond \in \{+,-\}$, let $X^\diamond := \{x^\diamond_i \mid i \in [\ell]\}$. Let $U^*_k := U^* \setminus (V^+(\mathcal{P}_k) \cup V^-(\mathcal{P}_k))$. Note that $U^*_k \subseteq U^0(D_k)$. Moreover, since $|U^*| = \widetilde{\text{ex}}(T) - \text{ex}(T)$, Claim 1 and Proposition 9.7 imply that $|U^*_k| = \widetilde{\text{ex}}(D_k) - \text{ex}(D_k)$.

We claim that there exist suitable endpoints $v^{\pm}_1$ and $v^{\pm}_2$ for $P_{k+1,1}$ and $P_{k+1,2}$. More precisely, we want to find $v^+_1, v^+_2, v^-_1, v^-_2 \in V$ such that the following hold.

(A) For each $i \in [2]$, $v^+_i \neq v^-_i$ and $v^{\pm}_i \in \widetilde{U}^{\pm}_{U^*_k}(D_k)$. Moreover, for each $\diamond \in \{+,-\}$, if $v^\diamond_1 = v^\diamond_2$, then $\widetilde{\text{ex}}^{\diamond}_{D_k, U^*_k}(v^\diamond_1) \geqslant 2$.

(B) For each $\diamond \in \{+,-\}$, if $W^\diamond_* \neq \emptyset$, then, for each $i \in [2]$, $v^\diamond_i \in W^\diamond_*$.

(C) There exists a partition $M_{k+1} = M_{k+1,1} \cup M_{k+1,2}$ such that, for each $i \in [2]$ and $x^+x^- \in M_{k+1,i}$, we have $x^\mp \neq v^{\pm}_i$ and $x^+x^- \neq v^+_i v^-_i$.

Property (A) will ensure that $\mathcal{P}_{k+1}$ is a (U^*, W, A)-partial path decomposition of T and (B) will ensure that (γ) holds. Finally, (C) will ensure that all edges of M_{k+1} can be covered by $P_{k+1,1} \cup P_{k+1,2}$. (We will cover $M_{k+1,1}$ with a (v^+_1, v^-_1)-path $P_{k+1,1}$ and cover $M_{k+1,2}$ with a (v^+_2, v^-_2)-path $P_{k+1,2}$.)

To find $v_1^+, v_2^+, v_1^-, v_2^- \in V$ satisfying (A)–(C), we will need the following claim.

Claim 2. There exist $v_1^+, v_2^+, v_1^-, v_2^- \in V$ such that, for each $\diamond \in \{+,-\}$ and $i, j \in [2]$, the following hold.

(I) If $W_*^\diamond \neq \emptyset$, then $v_1^\diamond = v_2^\diamond \in W_*^\diamond$ and $\widetilde{\text{ex}}^\diamond_{D_k,U_k^*}(v_1^\diamond) \geqslant 2$; otherwise, $v_1^\diamond, v_2^\diamond \in \widetilde{U}^\diamond_{U_k^*}(D_k)$ are distinct.
(II) Both $v_1^- v_2^+, v_2^- v_1^+ \notin M_{k+1}$.
(III) $v_i^+ \neq v_j^-$.

Proof of Claim. If $W_*^+ \neq \emptyset$, then pick $v_1^+ \in W_*^+$ and let $v_2^+ := v_1^+$ and note that, since $k \leqslant \varepsilon n$, $\widetilde{\text{ex}}^+_{D_k,U_k^*}(v_1^+) \geqslant \text{ex}_T^+(v_1^+) - d_{A^+}(v_1^+) - 2k \geqslant (1-20\eta)n - \lceil \eta n \rceil - 2\varepsilon n \geqslant 2$, as desired. Assume that $W_*^+ = \emptyset$. We claim that $|\widetilde{U}^+_{U_k^*}(D_k)| \geqslant 2$. Assume not. Then, since $|A^+| \leqslant \lceil \eta n \rceil$, by Claim 1 and (11.2), $\widetilde{\text{ex}}^+_{U_k^*}(D_k) = \widetilde{\text{ex}}(D_k) - |A^+| \geqslant 2\eta n - \lceil \eta n \rceil > 1$ and so there exists $v \in V$ such that $\widetilde{U}^+_{U_k^*}(D_k) = \{v\}$ and $\widetilde{\text{ex}}^+_{D_k,U_k^*}(v) \geqslant 2$. Then, note that $v \notin U^0(D_k)$. As $v \notin W_*^+$ and $k \leqslant \varepsilon n$,

$$\widetilde{\text{ex}}(D_k) - \text{ex}^+_{D_k}(v) \overset{\text{Fact 4.20(d)}}{\geqslant} d^{\min}_{D_k}(v) \geqslant d^{\min}_T(v) - |\mathcal{P}| \overset{\text{(a),Fact 4.20(b)}}{\geqslant} \frac{20\eta n - 1}{2} - 2\varepsilon n \geqslant 9\eta n$$

and so, since $v \notin U^0(D_k)$, we have

$$\begin{aligned} 0 = \widetilde{\text{ex}}^+_{D_k,U_k^*}(V \setminus \{v\}) &= \widetilde{\text{ex}}^+_{U_k^*}(D_k) - \widetilde{\text{ex}}^+_{D_k,U_k^*}(v) \\ &\overset{(11.1),(11.2)}{\geqslant} \widetilde{\text{ex}}(D_k) - |A^+| - \text{ex}^+_{D_k}(v) \\ &\geqslant 9\eta n - \lceil \eta n \rceil \geqslant 7\eta n, \end{aligned}$$

a contradiction. Thus, $|\widetilde{U}^+_{U_k^*}(D_k)| \geqslant 2$ and so we can pick distinct $v_1^+, v_2^+ \in \widetilde{U}^+_{U_k^*}(D_k)$.

Then, proceed similarly as above to pick $v_1^-, v_2^- \in \widetilde{U}^-_{U_k^*}(D_k) \setminus \{v_1^+, v_2^+\}$. Note that this is possible since, for each $i \in [2]$, $\widetilde{\text{ex}}^-_{D_k,U_k^*}(v_i^+) \leqslant 1$. Then, (I) and (III) are satisfied. By relabelling v_1^- and v_2^- if necessary, we can ensure (II) holds. Indeed, suppose that $v_1^- v_2^+ \in M_{k+1}$ (the case $v_2^- v_1^+ \in M_{k+1}$ is similar). It suffices to show that $v_2^- v_2^+, v_1^- v_1^+ \notin M_{k+1}$. Note that, since $V(M_{k+1}) \subseteq W_0$, we have $v_1^-, v_2^+ \in W_0$. Thus, by (I), $v_1^- \neq v_2^-$ and so, as M_{k+1} is a matching $v_2^- v_2^+ \notin M_{k+1}$. Similarly, $v_1^+ \neq v_2^+$ and so $v_1^- v_1^+ \notin M_{k+1}$. This completes the proof. □

Fix $v_1^+, v_2^+, v_1^-, v_2^- \in V$ satisfying properties (I)–(III) of Claim 2. We claim that (A)–(C) hold. Indeed, (A) and (B) follow immediately from (I). Recall the notation $M_{k+1} = \{x_i^+ x_i^- \mid i \in [\ell]\}$. For (C), let $M_{k+1,2} := \{x_i^+ x_i^- \in M_{k+1} \mid v_1^+ = x_i^- \text{ or } v_1^- = x_i^+ \text{ or } v_1^+ v_1^- = x_i^+ x_i^-\}$ and $M_{k+1,1} := M_{k+1} \setminus M_{k+1,2}$. We claim that the partition $M_{k+1} = M_{k+1,1} \cup M_{k+1,2}$ witnesses that (C) holds. By definition, $M_{k+1,1}$ clearly satisfies the desired properties, so it is enough to show that $M_{k+1,2} \subseteq M_{k+1} \setminus \{x_i^+ x_i^- \in M_{k+1} \mid v_2^+ = x_i^- \text{ or } v_2^- = x_i^+ \text{ or } v_2^+ v_2^- = x_i^+ x_i^-\}$. If $v_1^+ v_1^- = x_i^+ x_i^-$ for some $i \in [\ell]$, then, by (I), (III), and the fact that $V(M_{k+1}) \subseteq W_0$, we have $v_2^+, v_2^- \notin \{x_i^+, x_i^-\}$. Moreover, if $v_1^+ = x_i^-$ for some $i \in [\ell]$, then, by (I), $v_2^+ \neq x_i^-$, by (III), $v_2^- \neq v_1^+$ and so $v_2^+ v_2^- \neq x_i^+ x_i^-$, and, by (II), $v_2^- \neq x_i^+$. Similarly, if $v_1^- = x_i^+$ for some $i \in [\ell]$, then $v_2^- \neq x_i^+$, $v_2^+ \neq x_i^-$, and $v_2^+ v_2^- \neq x_i^+ x_i^-$. Therefore, (C) holds, as desired.

We will now construct, for each $i \in [2]$, a (v_i^+, v_i^-)-path $P_{k+1,i}$ covering $M_{k+1,i}$. The idea is to join together the edges in $M_{k+1,i}$ via V'. In order to satisfy (β), we also incorporate the vertices in $W_0 \setminus V(M_{k+1,i})$ in a similar fashion. This will be done using Corollary 4.8(b) as follows.

Denote

$$k' := 2 + |M_{k+1,1} \setminus \{e \in M_{k+1,1} \mid V^+(e) = \{v_1^+\} \text{ or } V^-(e) = \{v_1^-\}\}| + |W_0 \setminus V(M_{k+1,1})|$$

(k' will play the role of k in Corollary 4.8(b)). Since $M_{k+1,1}$ is a matching on W_0, we have

$$k' \leqslant 2 + |W_0| \overset{\text{(a)}}{\leqslant} 2\varepsilon n. \tag{12.1}$$

Now construct the k' paths for Corollary 4.8(b) as follows. If $v_1^+ \notin V(M_{k+1,1})$, let $Q_1 := v_1^+$; otherwise, let Q_1 be the (unique) edge $e \in M_{k+1,1}$ such that $V^+(e) = \{v_1^+\}$. Similarly, if $v_1^- \notin V(M_{k+1,1})$, let $Q_{k'} := v_1^-$; otherwise, let $Q_{k'}$ be the (unique) edge $e \in M_{k+1,1}$ such that $V^-(e) = \{v_1^-\}$. Let $Q_2, \dots, Q_{k'-1}$ be an enumeration of $(M_{k+1,1} \setminus \{Q_1, Q_{k'}\}) \cup (W_0 \setminus V(M_{k+1,1}))$. Recall that $V(M_{k+1,1}) \subseteq W_0$. Thus, $V' \cap (\bigcup_{i\in[k']} V(Q_i)) \subseteq \{v_1^+, v_1^-\}$ and so, since $k \leqslant \varepsilon n$, Lemma 4.4 implies that $D_k[V' \setminus \bigcup_{i\in[k']} V(Q_i)]$ is a robust (ν, τ)-outexpander. In order to apply Corollary 4.8, we first need to check that the endpoints of the paths $Q_1, \dots, Q_{k'}$ have sufficiently many neighbours.

Claim 3. For all $i \in [k'-1]$, the ending point v of Q_i satisfies $|N^+_{D_k\setminus A}(v) \cap (V' \setminus \bigcup_{j\in[k']} V(Q_j))| \geqslant 2k'$ and the starting point v' of Q_{i+1} satisfies $|N^-_{D_k\setminus A}(v') \cap (V' \setminus \bigcup_{j\in[k']} V(Q_j))| \geqslant 2k'$.

Proof of Claim. Let $i \in [k'-1]$. By symmetry, it is enough to show that the ending point v of Q_i satisfies $N := |N^+_{D_k\setminus A}(v) \cap (V' \setminus \bigcup_{j\in[k']} V(Q_j))| \geqslant 2k'$.

First, observe that $v \in V \setminus W_*^-$. Indeed, $v \in \{v_1^+\} \cup V(M_{k+1}) \cup W_0$. By construction, $V(M_{k+1}) \subseteq W_0$. Moreover,

$$v_1^+ \overset{\text{(A)}}{\in} \widetilde{U}^+_{U_k^*}(D_k) \overset{\text{Proposition 11.3}}{\subseteq} \widetilde{U}^+_{U^*}(T) \overset{(11.1)}{\subseteq} U^+(D) \cup U^* \subseteq V \setminus W_*^-.$$

Thus, $v \in V \setminus W_*^-$.

Since $V(M_{k+1}) \subseteq W_0$, we have

$$\begin{aligned} N &\geqslant d_T^+(v) - k - |A| - |W| - \left| \bigcup_{j\in[k']} V(Q_j) \right| \overset{\text{(a),(b)}}{\geqslant} d_T^+(v) - \varepsilon n - 2\lceil \eta n \rceil - \varepsilon n - 2 \\ &\geqslant d_T^+(v) - 3\eta n \end{aligned}$$

and so (12.1) implies that it is enough to show that $d_T^+(v) \geqslant 4\eta n$. If $v \in V \setminus U^-(T)$, then $d_T^+(v) \geqslant \frac{n-1}{2}$ and so we are done. Suppose $v \in U^-(T)$. Recall that we have shown that $v \notin W_*^-$ and so

$$d_T^+(v) \overset{\text{Fact 4.20(b)}}{=} \frac{d_T(v) - |\operatorname{ex}_T(v)|}{2} \overset{\text{(a)}}{\geqslant} \frac{20\eta n - 1}{2} \geqslant 4\eta n.$$

This completes the proof. □

Thus, all the conditions of Corollary 4.8 are satisfied. Apply Corollary 4.8(b) with $D_k \setminus A, V \setminus \bigcup_{i\in[k']} V(Q_i), k', \frac{3}{8}, W_* \setminus \{v_1^+, v_1^-\}$, and $Q_1, \dots, Q_{k'}$ playing the roles of D, V', k, δ, S, and $P_1, \dots, P_k$ to obtain a (v_1^+, v_1^-)-path $P_{k+1,1}$ covering the edges in $M_{k+1,1}$ and all vertices in $V' \cup W_0$. Construct $P_{k+1,2}$ similarly, but deleting the edges in $P_{k+1,1}$ before applying Corollary 4.8(b) (this will ensure that $P_{k+1,1}$ and $P_{k+1,2}$ are edge-disjoint). Thus, (α)–(γ) hold with k replaced by $k+1$. This completes the proof. □

12.3 | Covering the remaining edges inside W and decreasing the degree at W_*

Since the vertices in W_* have almost all their edges in the same direction, it is not possible to cover the remaining edges in W with a similar approach as in Lemma 12.1. (In order to incorporate an edge uv into a longer path using Corollary 4.8, we need u to have many inneighbours and v to have many outneighbours, but this may not be the case if $u \in W_*^+$ or $v \in W_*^-$.) However, since the vertices in W_* have very large excess, $W_* \neq \emptyset$ implies that the excess of the tournament is very large and so we have room to cover each remaining edge in W one by one. Moreover, one can also decrease the degree at $W_* \cup W_A$ with a similar approach. This is achieved in the next lemma.

Note that in Lemma 12.2(a), the definition of W_* is adjusted so that the vertex partition $W_* \cup W_0 \cup V'$ can be chosen to be the same as in Lemma 12.1.

Lemma 12.2. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll 1$. Let D be an oriented graph on a vertex set V of size n such that $\delta(D) \geqslant (1-\varepsilon)n$ and the following properties are satisfied.*

(a) *Let $W_* \cup W_0 \cup V'$ be a partition of V such that, for each $w_* \in W_*$, $|\mathrm{ex}_D(w_*)| > (1-21\eta)n$; for each $w_0 \in W_0$, $|\mathrm{ex}_D(w_0)| \leqslant (1-20\eta)n$; and, for each $v' \in V'$, $|\mathrm{ex}_D(v')| \leqslant \varepsilon n$. Let $W := W_* \cup W_0$ and suppose $|W| \leqslant \varepsilon n$ and $W_* \neq \emptyset$.*
(b) *Let $A^+, A^- \subseteq E(D)$ be absorbing sets of (W, V')-starting/(V', W)-ending edges for D of size at most $\lceil \eta n \rceil$. Denote $A := A^+ \cup A^-$. Let $W_A^\pm := V(A^\pm) \cap W$ and $W_A := V(A) \cap W$. Suppose that the following hold.*
   – *Let $\diamond \in \{+,-\}$. If $|W_A^\diamond| \geqslant 2$, then $\mathrm{ex}_D^\diamond(v) < \lceil \eta n \rceil$ for each $v \in V$.*
   – *Let $\diamond \in \{+,-\}$. If $|W_A^\diamond| = 1$, then $\mathrm{ex}_D^\diamond(v) \leqslant \mathrm{ex}_D^\diamond(w) + \varepsilon n$ for each $v \in V$ and $w \in W_A^\diamond$.*
(c) *Let $U^* \subseteq U^0(D)$ satisfy $|U^*| = \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$.*
(d) *Suppose $E(D[W_0]) = \emptyset$ and, if $|U^+(D)| = |U^-(D)| = 1$, then $e(U^-(D), U^+(D)) = 0$ or $\widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D) \geqslant 2$.*

Then, there exists a good (U^*, W, A)-partial path decomposition $\mathcal{P}$ of D such that $D' := D \setminus \mathcal{P}$ satisfies the following.

(i) *$E(D'[W]) = \emptyset$.*
(ii) *$N^\pm(D) - N^\pm(D') \leqslant 88\eta n$.*
(iii) *For each $v \in W_* \cup W_A$, $(1-3\sqrt{\eta})n \leqslant d_{D'}(v) \leqslant (1-4\eta)n$.*
(iv) *For each $v \in W_0$, $d_{D'}(v) \geqslant (1-3\sqrt{\eta})n$ and $d_{D'}^{\min}(v) \geqslant 5\eta n$.*
(v) *For each $v \in V'$, $d_{D'}(v) \geqslant (1-3\sqrt{\varepsilon})n$.*
(vi) *If $|W_*^+|, |W_*^-| \leqslant 1$, then each $v \in W_*$ satisfies $|\mathrm{ex}_{D'}(v)| = d_{D'}(v)$.*
(vii) *Each $v \in U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$ satisfies $d_{D'}^+(v) = d_{D'}^-(v) \leqslant \widetilde{\mathrm{ex}}(D') - 1$.*

Property (vi) will enable us to satisfy Lemma 9.5(vi). As mentioned above, the strategy in the proof of Lemma 12.2 is to cover the remaining edges of $D[W]$ one by one. To decrease the degree at W_*, we further fix some additional edges that will be covered with short paths in the same way. The degree at $W_A \setminus W_*$ will be decreased by incorporating these vertices in some of these paths.

Similarly as in the proof of Lemma 12.1, given an edge that needs to be covered, we need to find suitable endpoints, that is, endpoints of the correct excess and which are 'compatible' with the edge $e = uv$ that needs to be covered (the starting point cannot be v and the ending point cannot be u). The next lemma enables us to find, given a set H of edges to be covered, pairs of suitable endpoints to cover each of these edges with a path.

Lemma 12.3. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll 1$. Let D be an oriented graph on a vertex set V of size n such that Lemma 12.2(a)–(d) are satisfied. Let $H \subseteq D$ satisfy $\Delta(H) \leqslant 11\eta n$ and $k := |E(H)| \leqslant 11\eta n|W_*|$. Let $w_1^+w_1^-, \dots, w_k^+w_k^-$ be an enumeration of $E(H)$. Then, there exist pairs of vertices $(v_1^+, v_1^-), \dots, (v_k^+, v_k^-)$ such that the following hold.*

(i) *For each $v \in V$ and $\diamond \in \{+, -\}$, there exist at most $\min\{2\sqrt{\eta}n, \widetilde{\mathrm{ex}}_{D,U^*}^{\diamond}(v)\}$ indices $i \in [k]$ such that $v = v_i^{\diamond}$.*
(ii) *For all $i \in [k]$, if $w_i^{\pm} \in W_*^{\pm}$, then $v_i^{\pm} = w_i^{\pm}$.*
(iii) *For all $i \in [k]$, if there exists $\diamond \in \{+, -\}$ such that $w_i^{\diamond} \in V'$, then $(v_i^+, v_i^-) \neq (w_i^+, w_i^-)$.*
(iv) *For each $i \in [k]$, $\{v_i^+, w_i^+\} \cap \{v_i^-, w_i^-\} = \emptyset$.*
(v) *For each $\diamond \in \{+, -\}$, there exist at most $88\eta n$ vertices $v \in \widetilde{U}_{U^*}^{\diamond}(D)$ such that there exist exactly $\widetilde{\mathrm{ex}}_{D,U^*}^{\diamond}(v)$ indices $i \in [k]$ such that $v_i^{\diamond} = v$.*
(vi) *Denote $V^{\pm} := \{v \in V \mid d_D^{\pm}(v) \geqslant \widetilde{\mathrm{ex}}(D) - 22\eta n\}$. Then, both $V^{\pm} \subseteq \{w_i^+, w_i^-, v_i^{\pm}\} \setminus \{v_i^{\mp}\}$ for all $i \in [k]$.*

Property (i) will ensure that a vertex is not used as an endpoint too many times. Since vertices in W_* have most of their edges in the same direction, (ii) will ensure that we will be able to tie up edges to the designated endpoints of the path. Property (iii) will ensure that some of the paths will have length more than one, which will enable us to cover a significant number of edges at $W_A \setminus W_*$. Property (iv) implies that the chosen endpoints are distinct, the chosen starting point for the path is not the ending point of the edge we want to cover and, similarly, that the chosen ending point is not the starting point of the edge we want to cover. Moreover, (v) will ensure that Lemma 12.2(ii) is satisfied. Together with Proposition 8.6, property (vi) will ensure that the partial path decomposition constructed with this set of endpoints will be good. Note that the main difficulties in the proof of Lemma 12.3 arise from the cases where D is 'close' to being a tournament from $\mathcal{T}_{\mathrm{apex}}$ (defined in Section 1).

First, we suppose that Lemma 12.3 holds and derive Lemma 12.2.

Proof of Lemma 12.2. Recall that, by assumption, $W_* \neq \emptyset$. Fix additional constants such that $\varepsilon \ll \nu \ll \tau \ll \eta$. Let $W^{\pm} := W \cap U^{\pm}(D)$ and, for each $\diamond \in \{*, 0\}$, denote $W_{\diamond}^{\pm} := W_{\diamond} \cap U^{\pm}(D)$.

We now define a subdigraph $H \subseteq D$, whose edges will be covered by $\mathcal{P}$. If $\max\{|W_*^+|, |W_*^-|\} \geqslant 2$, then let $H \subseteq D \setminus A$ be obtained from $D[W]$ by adding, for each $v \in W_*$, $\lceil 4\eta n \rceil$ edges of $D \setminus A$ between v and V' (of either direction). Otherwise, let $H \subseteq D \setminus A$ be obtained from $D[W]$ by adding, for each $v \in W_*^{\pm}$, $\max\{d_D^{\mp}(v), \lceil 4\eta n \rceil\}$ edges of $D \setminus A$ between v and V' (of either direction) such that $d_H^{\mp}(v) = d_{D\setminus A}^{\mp}(v) = d_D^{\mp}(v)$. Note that each $v \in V \setminus W_*$ satisfies

$$d_H(v) \leqslant |W| \overset{\text{(a)}}{\leqslant} \varepsilon n. \tag{12.2}$$

Moreover, each $v \in W_*$ satisfies

$$\begin{aligned}\lceil 4\eta n\rceil \leqslant \quad & d_H(v) \leqslant |W| + \max\left\{\max_{v\in W^*} d_D^{\min}(v), \lceil 4\eta n\rceil\right\} && (12.3)\\ \overset{\text{Fact 4.20(b)}}{\leqslant} \quad & |W| + \max\left\{\max_{v\in W^*} \frac{d_D(v) - |\operatorname{ex}_D(v)|}{2}, \lceil 4\eta n\rceil\right\} \\ \overset{\text{(a)}}{\leqslant} \quad & \varepsilon n + \frac{21\eta n}{2} \leqslant 11\eta n. && (12.4)\end{aligned}$$

Observe that

$$\lceil 4\eta n\rceil \leqslant k := |E(H)| \overset{(12.4)}{\leqslant} 11\eta n|W_*|. \tag{12.5}$$

Let $w_1^+w_1^-, \dots, w_k^+w_k^-$ be an enumeration of $E(H)$. Recall that $W_* \neq \emptyset$ and, for each $w \in W^*$, $|N_H(w) \cap V'| \geqslant \lceil 4\eta n\rceil$. Thus, we may assume without loss generality that, for each $i \in [\lceil 4\eta n\rceil]$, $w_i^+w_i^- \notin E(D[W])$. Apply Lemma 12.3 to obtain pairs of vertices $(v_1^+, v_1^-), \dots, (v_k^+, v_k^-)$ such that the following hold.

(α) For each $v \in V$ and $\diamond \in \{+,-\}$, there exist at most $\min\{2\sqrt{\eta}n, \widetilde{\operatorname{ex}}^\diamond_{D,U^*}(v)\}$ indices $i \in [k]$ such that $v = v_i^\diamond$.
(β) For all $i \in [k]$, if $w_i^\pm \in W_*^\pm$, then $v_i^\pm = w_i^\pm$.
(γ) For all $i \in [k]$, if there exists $\diamond \in \{+,-\}$ such that $w_i^\diamond \in V'$, then $(v_i^+, v_i^-) \neq (w_i^+, w_i^-)$.
(δ) For each $i \in [k]$, $\{v_i^+, w_i^+\} \cap \{v_i^-, w_i^-\} = \emptyset$.
(ε) For each $\diamond \in \{+,-\}$, there exist at most $88\eta n$ vertices $v \in \widetilde{U}^\diamond_{U^*}(D)$ such that there exist exactly $\widetilde{\operatorname{ex}}^\diamond_{D,U^*}(v)$ indices $i \in [k]$ such that $v_i^\diamond = v$.
(ζ) Denote $V^\pm := \{v \in V \mid d_D^\pm(v) \geqslant \widetilde{\operatorname{ex}}(D) - 22\eta n\}$. Then, both $V^\pm \subseteq \{w_i^+, w_i^-, v_i^\pm\} \setminus \{v_i^\mp\}$ for all $i \in [k]$.

By assumption on our ordering of $E(H)$, (γ) implies that the following holds.

(γ') For all $i \in [\lceil 4\eta n\rceil]$, $(v_i^+, v_i^-) \neq (w_i^+, w_i^-)$.

We will now cover each edge $w_i^+w_i^-$ with a short (v_i^+, v_i^-)-path inductively. In the first few paths, we also cover the vertices in $W_A \setminus W_*$ whose degree is too high. More precisely, we proceed as follows. Suppose that for some $0 \leqslant \ell \leqslant k$ we have constructed edge-disjoint paths $P_1, \dots, P_\ell \subseteq D \setminus A$. For each $0 \leqslant i \leqslant \ell$, let $D_i := D \setminus \bigcup_{j\in[i]} P_i$ and S_i be the set of vertices $w \in W_A \setminus W_*$ such that $d_{D_i}(w) > (1-4\eta)n$. (Note that S_ℓ corresponds to the set of vertices in $W_A \setminus W_*$ whose degree is currently too high.) Suppose furthermore that the following hold for each $i \in [\ell]$.

(I) P_i is a (v_i^+, v_i^-)-path.
(II) $w_i^+w_i^- \in E(P_i)$.
(III) $S_{i-1} \subseteq V(P_i)$.
(IV) For each $v \in V'$, there exist at most $\sqrt{\varepsilon}n$ indices $j \in [\ell]$ such that $v \in V^0(P_j) \setminus \{w_j^+, w_j^-\} = V(P_j) \setminus \{v_j^+, w_j^+, w_j^-, v_j^-\}$.
(V) For each $v \in V(P_i) \cap W$, $v \in \{v_i^+, v_i^-, w_i^+, w_i^-\} \cup S_{i-1}$.
(VI) $e(P_i) \leqslant 7\nu^{-1}(|S_{i-1}| + 1)$.

First, suppose that $\ell = k$. Let $\mathcal{P} := \bigcup_{i\in[\ell]} P_i$ and $D' := D_\ell$.

Claim 1. $\mathcal{P}$ is a good (U^*, W, A)-partial path decomposition of D. Moreover, (i)–(vii) are satisfied.

Proof of Claim. By assumption, $\mathcal{P} \subseteq D \setminus A$. Moreover, ($\alpha$) and (I) imply that each $v \in V$ is the starting point of at most $\widetilde{\mathrm{ex}}^+_{D,U^*}(v)$ paths in $\mathcal{P}$ and the ending point of at most $\widetilde{\mathrm{ex}}^-_{D,U^*}(v)$ paths in $\mathcal{P}$. Thus, Fact 11.2 implies that $\mathcal{P}$ is a (U^*, W, A)-partial path decomposition.

We now verify (ii). By (ε) and (I), there are at most $88\eta n$ vertices $v \in \widetilde{U}^+_{U^*}(D)$ for which $\mathcal{P}$ contains precisely $\widetilde{\mathrm{ex}}^+_{D,U^*}(v)$ paths starting at v and at most $88\eta n$ vertices $v' \in \widetilde{U}^-_{U^*}(D)$ for which $\mathcal{P}$ contains precisely $\widetilde{\mathrm{ex}}^-_{D,U^*}(v')$ paths ending at v'. Thus, (ii) follows from Corollary 11.6.

Next, we show that $\mathcal{P}$ is good. By (ζ), (I), and (II), both $V^\pm \subseteq V^\pm(P_i) \cup V^0(P_i)$ for each $i \in [k]$. If $k \leqslant 22\eta n$, then Proposition 8.6(b) implies that $\mathcal{P}$ is good. We may therefore assume that $k > 22\eta n$. Then, (12.5) implies that $\max\{|W^+_*|, |W^-_*|\} \geqslant 2$ and so, by (a) and (1.1), we have $\mathrm{ex}(D) \geqslant 2(1-21\eta)n \geqslant n \geqslant \Delta^0(D)$. Therefore, $\widetilde{\mathrm{ex}}(D) = \mathrm{ex}(D)$ and so Proposition 8.4(b) implies that

$$\mathrm{ex}(D') = \mathrm{ex}(D) - |\mathcal{P}| \overset{\text{(a),(1.1),(12.5)}}{\geqslant} \max\{|W^+_*|, |W^-_*|\}(1-21\eta)n - 11\eta n|W_*| \geqslant n \geqslant \Delta^0(D').$$

Thus, $\widetilde{\mathrm{ex}}(D') = \mathrm{ex}(D') = \mathrm{ex}(D) - |\mathcal{P}| = \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$ and so $\mathcal{P}$ is good.

By (II), $\mathcal{P}$ covers all the edges of H. By construction, H contains all the edges of D which lie inside W and so (i) holds. Moreover, if both $|W^\pm_*| \leqslant 1$, then by construction H contains all the edges which end in W^+_* as well as all the edges which start in W^-_*, so (vi) is satisfied.

We now verify (iii). Let $v \in W_* \cup W_A$. If $v \in W_*$, then (12.3) implies that $d_H(v) \geqslant \lceil 4\eta n \rceil$. Thus, (II) implies that the upper bound in (iii) holds if $v \in W_*$. If $v \in W_A$, then (III) implies that $v \notin S_\ell$ for each $\ell \geqslant 4\eta n$. In particular, (12.5) implies that $v \notin S_k$ and so the upper bound in (iii) also holds. Moreover, (I), (V), (12.2), and (12.4) imply that there are at most $d_H(v) + 4\eta n \leqslant 15\eta n$ paths in $\mathcal{P}$ which contain v as an internal vertex. By (α) and (I), there are at most $2\sqrt{\eta}n$ paths in $\mathcal{P}$ which have v as an endpoint. Thus,

$$d_{D'}(v) \geqslant d_D(v) - 2\sqrt{\eta}n - 30\eta n \geqslant (1-\varepsilon)n - 2\sqrt{\eta}n - 30\eta n \geqslant (1-3\sqrt{\eta})n,$$

and so the lower bound in (iii) holds. Therefore, (iii) is satisfied.

Next, we verify (iv). Let $v \in W_0$. By the same arguments as for (iii), there are at most $2\sqrt{\eta}n$ paths in $\mathcal{P}$ which contain v as an endpoint and at most

$$d_H(v) + 4\eta n \overset{(12.2)}{\leqslant} (4\eta + \varepsilon)n$$

paths which contain v as an internal vertex. Thus,

$$d_{D'}(v) \geqslant d_D(v) - 2\sqrt{\eta}n - 2(4\eta+\varepsilon)n \geqslant (1-\varepsilon)n - (2\sqrt{\eta} + 8\eta + 2\varepsilon)n \geqslant (1-3\sqrt{\eta})n,$$

as desired. It remains to show that $d^{\min}_{D'}(v) \geqslant 5\eta n$. Suppose without loss of generality that $d^+_D(v) \geqslant d^-_D(v)$, that is, that $d^{\min}_D(v) = d^-_D(v)$. Then, $v \in U^+(D) \cup U^0(D)$, so (P1) and (P2) imply that $\mathcal{P}$ contains at most $\max\{\mathrm{ex}^+_D(v), 1\}$ paths which start at v and at most one path which ends at v. Therefore, $d^+_{D'}(v) \geqslant d^-_{D'}(v) - 1$ and so it is enough to show that $d^-_{D'}(v) > 5\eta n$. Since there is at most one path in $\mathcal{P}$ which ends at v and at most $(4\eta+\varepsilon)n$ paths in $\mathcal{P}$ which contain v as an internal vertex, we have

$$\begin{aligned} d^-_{D'}(v) \quad &\geqslant \quad d^-_D(v) - 1 - (4\eta+\varepsilon)n \geqslant d^{\min}_D(v) - (4\eta + 2\varepsilon)n \\ &\overset{\text{Fact 4.20(b)}}{=} \frac{d_D(v) - |\mathrm{ex}_D(v)|}{2} - (4\eta+2\varepsilon)n \overset{\text{(a)}}{\geqslant} \frac{(1-\varepsilon)n - (1-20\eta)n}{2} - (4\eta+2\varepsilon)n \\ &> \quad 5\eta n. \end{aligned}$$

Thus, (iv) holds.

For (v), let $v \in V'$. By (a), $|\operatorname{ex}_D(v)| \leqslant \varepsilon n$ and so, as $\mathcal{P}$ is a partial path decomposition, v is an endpoint of at most εn paths in $\mathcal{P}$. Moreover, (IV) implies that there are at most

$$\sqrt{\varepsilon} n + d_H(v) \overset{(12.2)}{\leqslant} (\sqrt{\varepsilon} + \varepsilon) n$$

paths in $\mathcal{P}$ which contain v as an internal vertex. Thus,

$$d_{D'}(v) \geqslant d_D(v) - \varepsilon n - 2(\sqrt{\varepsilon} + \varepsilon)n \geqslant (1-\varepsilon)n - (2\sqrt{\varepsilon} + 3\varepsilon)n \geqslant (1 - 3\sqrt{\varepsilon})n$$

and so (v) holds.

Finally, we verify (vii). Let $v \in U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$. By (c), $v \in U^0(D)$ and so $d^+_{D'}(v) = d^-_{D'}(v) = \frac{d_{D'}(v)}{2} \leqslant \frac{n-1}{2}$. Thus, it is enough to show that $\widetilde{\operatorname{ex}}(D') \geqslant \frac{n+1}{2}$. By assumption, there exists $w \in W_*$ and so

$$\widetilde{\operatorname{ex}}(D') \geqslant \operatorname{ex}(D') \overset{(1.1)}{\geqslant} |\operatorname{ex}_{D'}(w)| \overset{(\alpha),(\mathrm{I})}{\geqslant} |\operatorname{ex}_D(w)| - 2\sqrt{\eta} n \overset{(\mathrm{a})}{\geqslant} (1 - 21\eta)n - 2\sqrt{\eta} n > \frac{n+1}{2}.$$

Thus, (vii) holds. $\square$

We may therefore assume that $\ell < k$. Note that, by (III), if $\lceil 4\eta n \rceil < i \leqslant \ell$, then $S_i = \emptyset$. We construct $P_{\ell+1}$ as follows. If $(v^+_{\ell+1}, v^-_{\ell+1}) = (w^+_{\ell+1}, w^-_{\ell+1})$, then let $P_{\ell+1} := w^+_{\ell+1} w^-_{\ell+1}$. Note that, in this case, by (γ'), $S_\ell = \emptyset$. Thus, (I)–(VI) hold with $\ell + 1$ playing the role of ℓ and we are done. We may therefore assume that $(v^+_{\ell+1}, v^-_{\ell+1}) \neq (w^+_{\ell+1}, w^-_{\ell+1})$. We construct $P_{\ell+1}$ using Corollary 4.8 as follows. Let X be the set of vertices $v \in V' \setminus \{v^+_{\ell+1}, v^-_{\ell+1}, w^+_{\ell+1}, w^-_{\ell+1}\}$ such that there exist $\lfloor \sqrt{\varepsilon} n \rfloor$ indices $i \in [\ell]$ such that $v \in V^0(P_i) \setminus \{w^+_i, w^-_i\}$. Note that, by (VI),

$$|X| \leqslant \frac{7\nu^{-1}(\sum_{i\in[\ell]} |S_i| + 2\ell)}{\lfloor \sqrt{\varepsilon} n \rfloor} \leqslant \frac{8\nu^{-1}(\lceil 4\eta n \rceil \cdot \varepsilon n + 22\eta n \cdot \varepsilon n)}{\sqrt{\varepsilon} n} \leqslant \varepsilon^{\frac{1}{3}} n. \tag{12.6}$$

Recall that each $v \in V'$ satisfies $|\operatorname{ex}_D(v)| \leqslant \varepsilon n$. Thus, by Lemma 4.4, (α), (I), and (IV), $D_\ell[V' \setminus (X \cup \{v^+_{\ell+1}, w^+_{\ell+1}, w^-_{\ell+1}, v^-_{\ell+1}\})]$ is a robust (ν, τ)-outexpander. The idea is to use Corollary 4.8(a) to tie together the edge $w^+_{\ell+1} w^1_{\ell+1}$ and the vertices in S_ℓ into a short $(v^+_{\ell+1}, v^-_{\ell+1})$-path via the vertices in $V' \setminus X$.

Let $u_1, \dots, u_s$ be an enumeration of $S_\ell \setminus \{v^+_{\ell+1}, w^+_{\ell+1}, w^-_{\ell+1}, v^-_{\ell+1}\}$. If both $v^\pm_{\ell+1} \neq w^\pm_{\ell+1}$, then let $m := s + 3$, $Q_1 := v^+_{\ell+1}$, $Q_2 := w^+_{\ell+1} w^-_{\ell+1}$, $Q_m := v^-_{\ell+1}$, and, for each $i \in [s]$, let $Q_{i+2} := u_i$ ($m, Q_1, \dots, Q_m$ will play the roles of $k, P_1, \dots, P_k$ in Corollary 4.8(a)). If $v^+_{\ell+1} \neq w^+_{\ell+1}$ and $v^-_{\ell+1} = w^-_{\ell+1}$, then let $m := s + 2$, $Q_1 := v^+_{\ell+1}$, $Q_m := w^+_{\ell+1} w^-_{\ell+1}$, and, for each $i \in [s]$, let $Q_{i+1} := u_i$. Similarly, if $v^+_{\ell+1} = w^+_{\ell+1}$ and $v^-_{\ell+1} \neq w^-_{\ell+1}$, then let $m := s + 2$, $Q_1 := w^+_{\ell+1} w^-_{\ell+1}$, $Q_m := v^-_{\ell+1}$, and, for each $i \in [s]$, let $Q_{i+1} := u_i$. Note that, by (δ), this covers all possible cases. Moreover, (a) implies that we always have

$$m \leqslant |S_\ell| + 3 \leqslant |W| + 3 \leqslant 2\varepsilon n. \tag{12.7}$$

In order to apply Corollary 4.8, we first need to check that that endpoints of the paths $Q_1, \dots, Q_m$ have sufficiently many neighbours. The proof is similar to that of Claim 3 in the proof of Lemma 12.1.

Claim 2. For each $i \in [m-1]$, the ending point v of Q_i satisfies $|N^+_{D_\ell \setminus A}(v) \cap (V' \setminus (\bigcup_{j\in[m]} V(Q_j) \cup X))| \geqslant 2m$ and the starting point v' of Q_{i+1} satisfies $|N^-_{D_\ell \setminus A}(v') \cap (V' \setminus (\bigcup_{j\in[m]} V(Q_j) \cup X))| \geqslant 2m$.

Proof of Claim. Let $i \in [m-1]$. By symmetry, it is enough to show that the ending point v of Q_i satisfies $N := |N^+_{D_\ell \setminus A}(v) \cap (V' \setminus (\bigcup_{j\in[m]} V(Q_j) \cup X))| \geqslant 2m$.

First, we show that $v \in V \setminus W^-_*$. By construction, $S_\ell \subseteq W \setminus W_*$. We may therefore assume that $v \in \{v_i^+, w_i^-\}$. By (α), $v_i^+ \in \widetilde{U}^+_{U^*}(v) \subseteq V \setminus U^-(D) \subseteq V \setminus W^-_*$. We may therefore assume that $v = w_i^-$. Since $i < m$, we have $w_i^- \neq v_i^-$ and so (β) implies $v \notin W^-_*$. Thus, $v \in V \setminus W^-_*$, as desired.

Next, observe that

$$N \geqslant d^+_D(v) - d^+_{\mathcal{P}_\ell}(v) - |A| - |W| - \left| V' \cap \bigcup_{j\in[m]} V(Q_j) \right| - |X|$$

$$\overset{\text{(a),(b),(12.6)}}{\geqslant} d^+_D(v) - d^+_{\mathcal{P}_\ell}(v) - 2\lceil \eta n \rceil - \varepsilon n - 2 - \varepsilon^{\frac{1}{3}} n \geqslant d^+_D(v) - d^+_{\mathcal{P}_\ell}(v) - 3\eta n$$

and so (12.7) implies that it is enough to show that $d^+_D(v) - d^+_{\mathcal{P}_\ell}(v) \geqslant \frac{7\eta n}{2}$.

Suppose first that $v \in W \setminus U^-(D)$. Then, note that $d^+_D(v) = d^{\max}_D(v)$. By (α), there are at most $2\sqrt{\eta}n$ indices $j \in [\ell]$ for which $v \in \{v_j^+, v_j^-\}$. Moreover, (III) implies that there are at most $4\eta n$ indices $j \in [\ell]$ for which $v \in S_{j-1}$. Therefore,

$$d^+_D(v) - d^+_{\mathcal{P}_\ell}(v) \overset{\text{(V)}}{\geqslant} \frac{d_D(v)}{2} - (2\sqrt{\eta}n + d_H(v) + 4\eta n)$$

$$\overset{\text{(12.4)}}{\geqslant} \frac{(1-\varepsilon)n}{2} - (2\sqrt{\eta}n + 11\eta n + 4\eta n) \geqslant \frac{7\eta n}{2}.$$

Next, suppose that $v \in W \cap U^-(D)$. Note that $d^+_D(v) = d^{\min}_D(v)$. By (α), there is no index $j \in [\ell]$ for which $v = v_j^+$. Moreover, (III) implies that there are at most $4\eta n$ indices $j \in [\ell]$ for which $v \in S_{j-1}$. Recall from (I) that v_j^- is the ending point of P_j for each $j \in [\ell]$. Moreover, we have shown that $v \notin W^-_*$ and so we have $v \in W_0$. Thus,

$$d^+_D(v) - d^+_{\mathcal{P}_\ell}(v) \overset{\text{Fact 4.20(b),(V)}}{\geqslant} \frac{d_D(v) - |\operatorname{ex}_D(v)|}{2} - (d_H(v) + 4\eta n)$$

$$\overset{\text{(a),(12.2)}}{\geqslant} \frac{(20\eta - \varepsilon)n}{2} - (\varepsilon n + 4\eta n) \geqslant \frac{7\eta n}{2}.$$

We may therefore assume that $v \in V'$. Then, (a), (α), and (I) imply that there are at most εn indices $j \in [\ell]$ such that v is the starting point of P_j. Moreover, (IV) implies that there are at most $\sqrt{\varepsilon n}$ indices $j \in [\ell]$ for which v is an internal vertex of P_j. Thus,

$$d^+_D(v) - d^+_{\mathcal{P}_\ell}(v) \overset{\text{Fact 4.20(b)}}{\geqslant} \frac{d_D(v) - |\operatorname{ex}_D(v)|}{2} - (\varepsilon n + \sqrt{\varepsilon n}) \overset{\text{(a)}}{\geqslant} \frac{(1-2\varepsilon)n}{2} - 2\sqrt{\varepsilon n} \geqslant \frac{7\eta n}{2}.$$

This completes the proof of Claim 2. □

Thus, all the conditions of Corollary 4.8 are satisfied. Let $S'_\ell := S_\ell \cup \{v^+_{\ell+1}, w^+_{\ell+1}, w^-_{\ell+1}, v^-_{\ell+1}\}$. Apply Corollary 4.8(a) with $D_\ell \setminus A, V \setminus S'_\ell, m, \frac{3}{8}, X \cup (W \setminus S'_\ell)$, and $Q_1, \dots, Q_m$ playing the roles of D, V', k, δ, S, and $P_1, \dots, P_k$ to obtain a $(v^+_{\ell+1}, v^-_{\ell+1})$-path $P_{\ell+1}$ of length at most $2\nu^{-1}m + 1 \leqslant 2\nu^{-1}(|S_\ell| + 3) + 1 \leqslant 7\nu^{-1}(|S_\ell| + 1)$ which covers $w^+_{\ell+1}w^-_{\ell+1}$ and the vertices in S_ℓ and avoids the vertices in $X \cup (W \setminus S'_\ell)$.

One can easily verify that (I)–(VI) hold with $\ell + 1$ playing the role of ℓ. □

We now prove Lemma 12.3.

Proof of Lemma 12.3. Let $W^\pm := W \cap U^\pm(D)$ and, for each $\diamond \in \{*, 0\}$, denote $W^\pm_\diamond := W_\diamond \cap U^\pm(D)$. Observe that the following holds.

Claim 1. There are no distinct $v_+, v_-, v_0 \in V$ such that $v_+v_- \in E(D)$ and both $\widetilde{U}^\pm_{U^*}(D) = \{v_\mp, v_0\}$.

Proof of Claim. Suppose for a contradiction that $v_+, v_-, v_0 \in V$ are distinct and such that $v_+v_- \in E(D)$ and both $\widetilde{U}^\pm_{U^*}(D) = \{v_\mp, v_0\}$. We now show that $U^\pm(D) = \{v_\mp\}$, which implies that $|U^\pm(D)| = 1$, $e(U^-(D), U^+(D)) \neq 0$ and $\widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D) = |U^*| = |\widetilde{U}^+_{U^*}(D) \cap \widetilde{U}^-_{U^*}(D)| = 1 < 2$, a contradiction to (d). By (a), $\mathrm{ex}(D) \geqslant (1 - 21\eta)n$. Moreover, since $v_\mp \in \widetilde{U}^\pm_{U^*}(D) \setminus \widetilde{U}^\mp_{U^*}(D)$, we have $v_\mp \in U^\pm(D)$. Thus,

$$\begin{aligned}\mathrm{ex}^\pm_D(v_\mp) &\overset{(11.1)}{\geqslant} \widetilde{\mathrm{ex}}^\pm_{D,U^*}(v_\mp) \overset{\text{Fact 11.1}}{=} \widetilde{\mathrm{ex}}^\pm_{U^*}(D) - \widetilde{\mathrm{ex}}^\pm_{D,U^*}(v_0)\\ &\overset{(11.2)}{\geqslant} (\mathrm{ex}(D) - |A^\pm|) - 1 \overset{(b)}{\geqslant} (\mathrm{ex}(D) - \lceil \eta n \rceil) - 1 \geqslant 2\eta n.\end{aligned}$$

Thus, (b) implies that $|W^\pm_A| \leqslant 1$. If both $W^\pm_A \subseteq \{v_\mp\}$, then (1.1) implies that both $U^\pm(D) = \widetilde{U}^\pm_{U^*}(D) \setminus U^* = \{v_\mp\}$ and so we are done. We may therefore assume without loss of generality that there exists $v \in W^+_A \setminus \{v_-\}$. We find a contradiction. By Definition 8.7, we have $v \notin U^*$ and so $v \notin \widetilde{U}_{U^*}(D)$. Thus, $\widetilde{\mathrm{ex}}^+_{D,U^*}(v) = 0$ and so (b) implies that

$$\mathrm{ex}^\pm_D(v_\mp) \leqslant \mathrm{ex}^\pm_D(v) + \varepsilon n \overset{(11.1)}{\leqslant} |A^+| + \varepsilon n \leqslant \lceil \eta n \rceil + \varepsilon n < 2\eta n,$$

a contradiction. This completes the proof of Claim 1. □

Let $\widetilde{W}^\pm_* := \{v \in V \mid \mathrm{ex}^\pm_D(v) \geqslant (1 - 86\eta)n\}$. For technical reasons, we will ensure that, for any $i \in [k]$ and $\diamond \in \{+, -\}$, if $w^\diamond_i \in \widetilde{W}^\diamond_*$, then $v^\diamond_i = w^\diamond_i$. Note that this will imply (ii), as $W^\pm_* \subseteq \widetilde{W}^\pm_*$. Without loss of generality, we may assume that $E(H)$ is ordered so that, if $|\widetilde{W}^+_*| = |\widetilde{W}^-_*| = 1$ and there exists $i \in [k]$ such that $w^+_i \in \widetilde{W}^-_*$ and $w^-_i \in \widetilde{W}^+_*$, then $i = 1$.

Suppose that, for some $0 \leqslant \ell \leqslant k$, we have already constructed pairs $(v^+_1, v^-_1), \dots, (v^+_\ell, v^-_\ell)$ such that the following hold. For each $v \in V$, define $\widehat{\mathrm{ex}}^\pm_\ell(v) := \widetilde{\mathrm{ex}}^\pm_{D,U^*}(v) - |\{i \in [\ell] \mid v^\pm_i = v\}|$. Denote $\widehat{U}^\pm_\ell(D) := \{v \in V \mid \widehat{\mathrm{ex}}^\pm_\ell(v) > 0\}$.

(α) For each $v \in V$, $\widehat{\mathrm{ex}}^\pm_\ell(v) \geqslant 0$.
(β) For each $v \in V$ and $\diamond \in \{+, -\}$, there exist at most $\sqrt{\eta}n$ indices $i \in [\ell]$ such that $v^\diamond_i = v \neq w^\diamond_i$.
(γ) For all $i \in [\ell]$ and $\diamond \in \{+, -\}$, if $w^\diamond_i \in \widetilde{W}^\diamond_*$, then $v^\diamond_i = w^\diamond_i$.
(δ) For all $i \in [\ell]$, if there exists $\diamond \in \{+, -\}$ such that $w^\diamond_i \in V'$, then $(v^+_i, v^-_i) \neq (w^+_i, w^-_i)$.
(ε) For each $i \in [\ell]$, $\{v^+_i, w^+_i\} \cap \{v^-_i, w^-_i\} = \emptyset$.
(ζ) For each $\diamond \in \{+, -\}$, if $\widetilde{U}^\diamond_{U^*}(D) \setminus \widehat{U}^\diamond_\ell(D) \neq \emptyset$, then both $|W^\pm_*| \leqslant 4$.
(η) Recall that $V^\pm = \{v \in V \mid d^\pm_D(v) \geqslant \widetilde{\mathrm{ex}}(D) - 22\eta n\}$. Then, both $V^\pm \subseteq \{w^+_i, w^-_i, v^\pm_i\} \setminus \{v^\mp_i\}$ for all $i \in [\ell]$.

Assume $\ell = k$. Since $\Delta(H) \leqslant 11\eta n$, (i) follows from ($\alpha$) and ($\beta$). Moreover, (ii)–(iv) and (vi) hold by (γ)–(ε) and (η), respectively. It remains to verify (v). By definition, we need to show that both $|\widetilde{U}^{\pm}_{U^*}(D) \setminus \widehat{U}^{\pm}_k| \leqslant 88\eta n$. If $\max\{|W^+_*|, |W^-_*|\} \geqslant 5$, then ($\zeta$) implies that both $|\widetilde{U}^{\pm}_{U^*}(D) \setminus \widehat{U}^{\pm}_k| = 0$. If both $|W^{\pm}_*| \leqslant 4$, then $|\widetilde{U}^{\pm}_{U^*}(D) \setminus \widehat{U}^{\pm}_\ell(D)| \leqslant k \leqslant 11\eta n|W_*| \leqslant 88\eta n$ and so (v) holds.

Suppose $\ell < k$. First, observe that, by definition of $\widehat{\mathrm{ex}}^{\pm}_\ell(v)$, the following hold.

$$
\begin{aligned}
\widehat{\mathrm{ex}}^{\pm}_\ell(D) &:= \sum_{v\in V} \widehat{\mathrm{ex}}^{\pm}_\ell(v) = \widetilde{\mathrm{ex}}^{\pm}_{U^*}(D) - \ell \\
&\overset{(11.2)}{=} \widetilde{\mathrm{ex}}(D) - |A^{\pm}| - \ell && (12.8) \\
&\geqslant \mathrm{ex}(D) - |A^{\pm}| - \ell \geqslant \max\{|W^+_*|, |W^-_*|\}(1-21\eta)n - \lceil \eta n\rceil - 11\eta n|W_*| \\
&\geqslant \max\{|W^+_*|, |W^-_*|\}(1-46\eta)n. && (12.9)
\end{aligned}
$$

Let $X^{\pm}$ be the set of vertices $v \in V \setminus \{w^{\pm}_{\ell+1}\}$ such that $\widetilde{\mathrm{ex}}^{\pm}_{D,U^*}(v) - \widehat{\mathrm{ex}}^{\pm}_\ell(v) = \lfloor \sqrt{\eta} n\rfloor$. Note that each $v \in \widetilde{W}^{\pm}_* \subseteq V \setminus U^*$ satisfies

$$
\begin{aligned}
\widehat{\mathrm{ex}}^{\pm}_\ell(v) &\overset{(\beta)}{\geqslant} \widetilde{\mathrm{ex}}^{\pm}_{D,U^*}(v) - \sqrt{\eta} n - d^{\pm}_H(v) \overset{(11.1)}{\geqslant} \mathrm{ex}^{\pm}_D(v) - |A^{\pm}| - \sqrt{\eta}n - \Delta(H) \\
&\overset{(a)}{\geqslant} (1-86\eta)n - \lceil \eta n\rceil - \sqrt{\eta}n - 11\eta n \\
&\geqslant 2. && (12.10)
\end{aligned}
$$

Claim 2. It is enough to find distinct $v^+_{\ell+1}, v^-_{\ell+1} \in V$ such that the following hold.

(I) $v^{\pm}_{\ell+1} \in \widehat{U}^{\pm}_\ell(D) \setminus (X^{\pm} \cup \{w^{\mp}_{\ell+1}\})$.
(II) If $w^{\pm}_{\ell+1} \in \widetilde{W}^{\pm}_*$, then $v^{\pm}_{\ell+1} = w^{\pm}_{\ell+1}$.
(III) If $\max\{|W^+_*|, |W^-_*|\} \geqslant 5$, then both $\widehat{\mathrm{ex}}^{\pm}_\ell(v^{\pm}_{\ell+1}) \geqslant 2$.
(IV) If $w^+_{\ell+1} \in \widehat{U}^+_\ell(D)$, $w^-_{\ell+1} \in \widehat{U}^-_\ell(D)$, then for each $\diamond \in \{+,-\}$ such that $w^{\diamond}_{\ell+1} \in V'$, we have $v^{\diamond}_{\ell+1} \neq w^{\diamond}_{\ell+1}$.
(V) If $v \in V^{\pm} \setminus \{w^+_{\ell+1}, w^-_{\ell+1}\}$, then $v^{\pm}_{\ell+1} = v$.

Proof of Claim. Suppose that $v^+_{\ell+1}, v^-_{\ell+1} \in V$ are distinct and satisfy (I)–(V). We show that (α)–(η) hold with $\ell+1$ playing the role of ℓ.

First, (α) and (β) follow from (I), while (γ) follows from (II). Moreover, (ζ) follows from (III), while (ε) follows from (I) and the fact that $v^+_{\ell+1} \neq v^-_{\ell+1}$.

In order to verify (δ), note that, if both $w^+_{\ell+1}, w^-_{\ell+1} \in W$, then ($\delta$) holds vacuously with $\ell+1$ playing the role of ℓ and, if there exists $\diamond \in \{+,-\}$ such that $w^{\diamond}_{\ell+1} \notin \widehat{U}^{\diamond}_\ell(D)$, then ($\delta$) follows from ($\alpha$). In the remaining cases, (δ) holds by (IV).

In order to verify (η), first note that each $v \notin \{w^+_{\ell+1}, w^-_{\ell+1}\}$ satisfies (η) by (V). To check that the vertices $w^{\pm}_{\ell+1}$ satisfy (η), first note that, by Proposition 8.6(a.ii), if $w^{\pm}_{\ell+1} \in V^{\mp}$, then $\mathrm{ex}^{\mp}_D(w^{\pm}_{\ell+1}) \geqslant 2$ and so (11.1) implies that $\widehat{\mathrm{ex}}^{\pm}_\ell(w^{\pm}_{\ell+1}) \leqslant \widetilde{\mathrm{ex}}^{\pm}_{D,U^*}(w^{\pm}_{\ell+1}) = 0$. Thus, ($\eta$) for the vertices $w^{\pm}_{\ell+1}$ follows from (I). □

The following observation will enable us to ensure that (I)–(V) are satisfied simultaneously.

Claim 3. Let $v^\pm \in V^\pm$. Then $\widehat{\mathrm{ex}}^\pm_\ell(v^\pm) \geqslant 2$ and, if $v^\pm \neq w^\mp_{\ell+1}$, then $v^\pm_{\ell+1} := v^\pm$ satisfies (the '$\pm$ part' of) (I)–(V). In particular, if $w^\pm_{\ell+1} \in \widetilde{W}^\pm_*$ and $v^\pm \neq w^\mp_{\ell+1}$, then $v^\pm = w^\pm_{\ell+1}$.

Proof of Claim. Let $\diamond \in \{+,-\}$ and suppose $v \in V^\diamond$. By Proposition 8.6(a.i), $\widetilde{\mathrm{ex}}(D) \leqslant (1+22\eta)n$ and so $|\widetilde{W}^\pm_*| \leqslant 1$. In particular, $|W^\pm_*| \leqslant 1$ and so, $k \leqslant 22\eta n$. Thus, both $X^\pm = \emptyset$. By Proposition 8.6(a.ii), $v \in \widetilde{W}^\diamond_*$. Thus, $\widetilde{W}^\diamond_* = \{v\}$ and, by (12.10), $\widehat{\mathrm{ex}}^\diamond_\ell(v) \geqslant 2$. By Proposition 8.6(a.ii), each $u \in V \setminus \{v\}$ satisfies $v \notin V^\diamond$ (otherwise $\mathrm{ex}^\diamond_D(u) \geqslant (1-86\eta)n$ and thus $|\widetilde{W}^\diamond_*| > 1$, a contradiction). □

To find $v^\pm_{\ell+1}$ when each $v^\pm \in V \setminus (V^\pm \cup \{w^\mp_{\ell+1}\})$, we will use the following claim. For each $S \subseteq V$, denote $\widehat{\mathrm{ex}}^\pm_\ell(S) := \sum_{v\in S} \widehat{\mathrm{ex}}^\pm_\ell(v)$.

Claim 4. The following hold.

(A) If $\max\{|W^+_*|, |W^-_*|\} \geqslant 5$, then there exists $v \in \widehat{U}^+_\ell(D) \setminus (X^+ \cup \{w^+_{\ell+1}, w^-_{\ell+1}\})$ satisfying $\widehat{\mathrm{ex}}^+_\ell(v) \geqslant 2$.

(B) Suppose $|W^+_*|, |W^-_*| \leqslant 4$. Then, $X^+ = \emptyset$ and $\widehat{U}^+_\ell(D) \setminus \{w^-_{\ell+1}\} \neq \emptyset$. Moreover, if $w^-_{\ell+1} \in \widehat{U}^-_\ell(D)$ and $w^+_{\ell+1} \in V'$, then $\widehat{\mathrm{ex}}^+_\ell(V \setminus \{w^+_{\ell+1}, w^-_{\ell+1}\}) \geqslant 2$ (and thus, in particular, $\widehat{U}^+_\ell(D) \setminus (X^+ \cup \{w^+_{\ell+1}, w^-_{\ell+1}\}) \neq \emptyset$).

Both statements also hold if + and − are swapped.

Proof of Claim. For (A), suppose that $\max\{|W^+_*|, |W^-_*|\} \geqslant 5$. Assume for a contradiction that each $v \in \widehat{U}^+_\ell(D) \setminus (X^+ \cup \{w^+_{\ell+1}, w^-_{\ell+1}\})$ satisfies $\widehat{\mathrm{ex}}^+_\ell(v) = 1$. Note that $\lfloor\sqrt{\eta}n\rfloor|X^+| \leqslant \ell < k \leqslant 11\eta n|W_*|$ and so $|X^+| \leqslant 23\sqrt{\eta}\max\{|W^+_*|, |W^-_*|\}$. Thus,

$$\begin{aligned}\widehat{\mathrm{ex}}^+_\ell(D) &\leqslant |X^+ \cup \{w^+_{\ell+1}, w^-_{\ell+1}\}|n + |\widehat{U}^+_\ell(D)| \leqslant (23\sqrt{\eta}\max\{|W^+_*|, |W^-_*|\} + 2)n + n\\ &\leqslant \max\{|W^+_*|, |W^-_*|\}23\sqrt{\eta}n + 3n.\end{aligned}$$

But, by (12.9), $\widehat{\mathrm{ex}}^+_\ell(D) \geqslant \max\{|W^+_*|, |W^-_*|\}(1-46\eta)n$, so $\max\{|W^+_*|, |W^-_*|\} \leqslant \frac{3}{1-24\sqrt{\eta}} \leqslant 4$, a contradiction.

For (B), assume $|W^+_*|, |W^-_*| \leqslant 4$. Then, $\ell \leqslant k \leqslant 11\eta n|W_*| \leqslant 88\eta n$ and so $X^+ = \emptyset$. If $w^-_{\ell+1} \in \widehat{U}^-_\ell(D) \subseteq \widetilde{U}^-_{U^*}(D)$ and $w^+_{\ell+1} \in V'$, then

$$\widehat{\mathrm{ex}}^+_\ell(w^-_{\ell+1}) \leqslant \widetilde{\mathrm{ex}}^+_{D,U^*}(w^-_{\ell+1}) \overset{(11.1)}{\leqslant} 1 \tag{12.11}$$

and

$$\widehat{\mathrm{ex}}^+_\ell(w^+_{\ell+1}) \leqslant \widetilde{\mathrm{ex}}^+_{D,U^*}(w^+_{\ell+1}) \overset{(11.1)}{\leqslant} \max\{\mathrm{ex}^+_D(w^+_{\ell+1}), 1\} \overset{(a)}{\leqslant} \varepsilon n. \tag{12.12}$$

Hence,

$$\begin{aligned}\widehat{\mathrm{ex}}^+_\ell(V \setminus \{w^+_{\ell+1}, w^-_{\ell+1}\}) &= \widehat{\mathrm{ex}}^+_\ell(D) - \widehat{\mathrm{ex}}^+_\ell(w^-_{\ell+1}) - \widehat{\mathrm{ex}}^+_\ell(w^+_{\ell+1})\\ &\overset{(12.9),(12.11),(12.12)}{\geqslant} (1-46\eta)n - 1 - \varepsilon n \geqslant 2,\end{aligned}$$

as desired. It only remains to show that $\widehat{U}^+_\ell(D) \setminus \{w^-_{\ell+1}\} \neq \emptyset$. By (12.9), $\widehat{\mathrm{ex}}^+_\ell(D) > 0$ and so $\widehat{U}^+_\ell(D) \neq \emptyset$. Suppose for a contradiction that $\widehat{U}^+_\ell(D) = \{w^-_{\ell+1}\}$. Note that

$$\widetilde{\mathrm{ex}}^+_{D,U^*}(w^-_{\ell+1}) \geqslant \widehat{\mathrm{ex}}^+_\ell(w^-_{\ell+1}) = \widehat{\mathrm{ex}}^+_\ell(D) \overset{(12.9)}{\geqslant} (1-46\eta)n.$$

Thus, $w^-_{\ell+1} \notin U^0(D)$ and so

$$\mathrm{ex}^+_D(w^-_{\ell+1}) \overset{(11.1)}{\geqslant} \widetilde{\mathrm{ex}}^+_{D,U^*}(w^-_{\ell+1}) \geqslant (1-46\eta)n. \tag{12.13}$$

Thus, $w^-_{\ell+1} \in \widetilde{W}^+_*$ and, by (b), $|W^+_A| \leqslant 1$. Moreover, each $v \in V \setminus \{w^-_{\ell+1}\}$ satisfies

$$\begin{aligned}\mathrm{ex}^+_D(v) &\overset{(11.1)}{\leqslant} \widetilde{\mathrm{ex}}^+_{D,U^*}(v) + d^+_{A^+}(v) \leqslant (\widehat{\mathrm{ex}}^+_\ell(v) + \ell) + |A^+| \\ &\overset{(a)}{\leqslant} 0 + 88\eta n + \lceil \eta n \rceil \overset{(12.13)}{<} \mathrm{ex}^+_D(w^-_{\ell+1}) - \varepsilon n.\end{aligned}$$

Thus, by (b), we have $W^+_A \subseteq \{w^-_{\ell+1}\}$. Therefore, $d^+_{A^+}(w^-_{\ell+1}) = |A^+|$ and so the fact that $w^-_{\ell+1} \notin U^0(D)$ implies that

$$\widetilde{\mathrm{ex}}^+_{D,U^*}(w^-_{\ell+1}) \overset{(11.1)}{=} \mathrm{ex}^+_D(w^-_{\ell+1}) - d^+_{A^+}(w^-_{\ell+1}) = \mathrm{ex}^+_D(w^-_{\ell+1}) - |A^+|. \tag{12.14}$$

Then,

$$\widehat{\mathrm{ex}}^+_\ell(D) = \widehat{\mathrm{ex}}^+_\ell(w^-_{\ell+1}) \leqslant \widetilde{\mathrm{ex}}^+_{D,U^*}(w^-_{\ell+1}) \overset{(12.14)}{=} \mathrm{ex}^+_D(w^-_{\ell+1}) - |A^+| \leqslant n, \tag{12.15}$$

and so

$$\widetilde{\mathrm{ex}}(D) \overset{(12.8)}{=} \widehat{\mathrm{ex}}^+_\ell(D) + |A^+| + \ell \overset{(12.15)}{\leqslant} \mathrm{ex}^+_D(w^-_{\ell+1}) + \ell. \tag{12.16}$$

Suppose first that $w^-_{\ell+1} \notin V^+$, that is, $d^+_D(w^-_{\ell+1}) < \widetilde{\mathrm{ex}}(D) - 22\eta n$. By (12.9) and (12.15), both $|W^\pm_*| \leqslant 1$. Thus, $\ell \leqslant 11\eta n |W_*| \leqslant 22\eta n$ and so

$$d^+_D(w^-_{\ell+1}) < \widetilde{\mathrm{ex}}(D) - 22\eta n \leqslant \widetilde{\mathrm{ex}}(D) - \ell \overset{(12.16)}{\leqslant} \mathrm{ex}^+_D(w^-_{\ell+1}),$$

a contradiction. Therefore, $w^-_{\ell+1} \in V^+$. Observe that

$$\{i \in [\ell] \mid v^+_i \neq w^-_{\ell+1}\} = \{i \in [\ell] \mid w^-_i = w^-_{\ell+1}\}. \tag{12.17}$$

Indeed, (η) implies that $w^-_{\ell+1} \in \{w^+_i, w^-_i, v^+_i\}$ for each $i \in [\ell]$ and (γ) implies that, for each $i \in [\ell]$, if $w^-_{\ell+1} = w^+_i$, then $w^-_{\ell+1} = v^+_i$. Thus, $w^-_{\ell+1} \in \{w^-_i, v^+_i\}$ for each $i \in [\ell]$. But (ε) implies that $v^+_i \neq w^-_i$ for each $i \in [\ell]$. Therefore, (12.17) holds and so

$$\begin{aligned}\widetilde{\mathrm{ex}}^+_{U^*}(D) &= \widehat{\mathrm{ex}}^+_\ell(D) + \ell = \widehat{\mathrm{ex}}^+_\ell(w^-_{\ell+1}) + \ell \\ &= \widetilde{\mathrm{ex}}^+_{D,U^*}(w^-_{\ell+1}) - |\{i \in [\ell] \mid v^+_i = w^-_{\ell+1}\}| + \ell \\ &\overset{(12.17)}{=} \widetilde{\mathrm{ex}}^+_{D,U^*}(w^-_{\ell+1}) + |\{i \in [\ell] \mid w^-_i = w^-_{\ell+1}\}| < \widetilde{\mathrm{ex}}^+_{D,U^*}(w^-_{\ell+1}) + d^-_H(w^-_{\ell+1}) \\ &\leqslant \widetilde{\mathrm{ex}}^+_{D,U^*}(w^-_{\ell+1}) + d^-_D(w^-_{\ell+1}).\end{aligned} \tag{12.18}$$

Therefore, we have

$$\Delta^0(D) \leqslant \widetilde{\mathrm{ex}}(D) \overset{(11.2)}{=} \widetilde{\mathrm{ex}}^+_{U^*}(D) + |A^+| \overset{(12.18)}{<} (\widetilde{\mathrm{ex}}^+_{D,U^*}(w^-_{\ell+1}) + d^-_D(w^-_{\ell+1})) + |A^+|$$
$$\overset{(12.14)}{=} \mathrm{ex}^+_D(w^-_{\ell+1}) + d^-_D(w^-_{\ell+1}) = d^+_D(w^-_{\ell+1}) \leqslant \Delta^0(D),$$

a contradiction.

The same arguments hold with + and − swapped. This concludes the proof of Claim 4. □

We are now ready to choose distinct $v^\pm_{\ell+1} \in V$ such that (I)–(V) are satisfied. Without loss of generality, suppose that $|\hat{U}^+_\ell(D) \setminus (X^+ \cup \{w^-_{\ell+1}\})| \leqslant |\hat{U}^-_\ell(D) \setminus (X^- \cup \{w^+_{\ell+1}\})|$. We start by picking $v^+_{\ell+1}$ as follows (where we assume in each case that the previous ones do not apply).

Case 1: $w^+_{\ell+1} \in \widetilde{W}^+_*$ **or** $V^+ \setminus \{w^-_{\ell+1}\} \neq \emptyset$. If $w^+_{\ell+1} \in \widetilde{W}^+_*$, then let $v^+_{\ell+1} := w^+_{\ell+1}$ and if there exists $v \in V^+ \setminus \{w^-_{\ell+1}\}$, then let $v^+_{\ell+1} := v$. (Note that $v^+_{\ell+1}$ is well defined by the 'in particular' part of Claim 3.) Then, by (12.10) and Claim 3, (the '+ part' of) (I)–(V) hold for $v^+_{\ell+1}$.

Case 2: $\max\{|W^+_*|, |W^-_*|\} \geqslant 5$. Let $v^+_{\ell+1} \in \hat{U}^+_\ell(D) \setminus (X^+ \cup \{w^+_{\ell+1}, w^-_{\ell+1}\})$ satisfy $\widehat{\mathrm{ex}}^+_\ell(v^+_{\ell+1}) \geqslant 2$ ($v^+_{\ell+1}$ exists by (A)). Then, (the '+ part' of) (I)–(V) are clearly satisfied for $v^+_{\ell+1}$.

Case 3: $w^-_{\ell+1} \notin \hat{U}^-_\ell(D)$ **or** $w^+_{\ell+1} \notin V'$. Let $v^+_{\ell+1} \in \hat{U}^+_\ell(D) \setminus (X^+ \cup \{w^-_{\ell+1}\})$ ($v^+_{\ell+1}$ exists by (B)). Then, (the '+ part' of) (I)–(V) are clearly satisfied for $v^+_{\ell+1}$.

Case 4: $w^-_{\ell+1} \in \hat{U}^-_\ell(D)$ **and** $w^+_{\ell+1} \in V'$. Let $v^+_{\ell+1} \in \hat{U}^+_\ell(D) \setminus (X^+ \cup \{w^+_{\ell+1}, w^-_{\ell+1}\})$ ($v^+_{\ell+1}$ exists by the 'moreover part' of (B)). Then, (the '+ part' of) (I)–(V) are clearly satisfied for $v^+_{\ell+1}$.

Note that, since $v^+_{\ell+1}$ satisfies (I), we have $v^+_{\ell+1} \neq w^-_{\ell+1}$. Moreover, (I) and (11.1) imply that $\widehat{\mathrm{ex}}^-_\ell(v^+_{\ell+1}) \leqslant 1$. Therefore, by (12.10), $v^+_{\ell+1} \notin \widetilde{W}^-_*$ and, by Claim 3, $v^+_{\ell+1} \notin V^-$ (otherwise $\widehat{\mathrm{ex}}^-_\ell(v^+_{\ell+1}) \geqslant 2$, a contradiction). Thus, if $\hat{U}^-_\ell(D) \setminus (X^- \cup \{w^+_{\ell+1}, v^+_{\ell+1}\}) \neq \emptyset$, then, we can proceed similarly as for $v^+_{\ell+1}$ to obtain $v^-_{\ell+1} \neq v^+_{\ell+1}$ satisfying (I)–(V). More precisely, we proceed as follows.

Case 1: $w^-_{\ell+1} \in \widetilde{W}^-_*$ **or** $V^- \setminus \{w^+_{\ell+1}\} \neq \emptyset$. If $w^-_{\ell+1} \in \widetilde{W}^-_*$, then let $v^-_{\ell+1} := w^-_{\ell+1}$ and if there exists $v \in V^- \setminus \{w^+_{\ell+1}\}$, then let $v^-_{\ell+1} := v$. (Note that $v^-_{\ell+1}$ is well defined by the 'in particular' part of Claim 3.) Then, by (12.10) and Claim 3, (the '− part' of) (I)–(V) hold for $v^-_{\ell+1}$. Moreover, we have shown above that $v^+_{\ell+1} \notin \{w^-_{\ell+1}\} \cup V^-$, thus $v^-_{\ell+1} \neq v^+_{\ell+1}$ and so we are done.

Case 2: $\max\{|W^+_*|, |W^-_*|\} \geqslant 5$. Let $v^-_{\ell+1} \in \hat{U}^-_\ell(D) \setminus (X^- \cup \{w^+_{\ell+1}, w^-_{\ell+1}\})$ satisfy $\widehat{\mathrm{ex}}^-_\ell(v^-_{\ell+1}) \geqslant 2$ ($v^-_{\ell+1}$ exists by (the '− analogue' of) (A)). Then, (the '− part' of) (I)–(V) are clearly satisfied for $v^-_{\ell+1}$. Moreover, we have shown above that $\widehat{\mathrm{ex}}^-_\ell(v^+_{\ell+1}) \leqslant 1$, thus $v^-_{\ell+1} \neq v^+_{\ell+1}$ and so we are done.

Case 3: $w^+_{\ell+1} \notin \hat{U}^+_\ell(D)$ **or** $w^-_{\ell+1} \notin V'$. Let $v^-_{\ell+1} \in \hat{U}^-_\ell(D) \setminus (X^- \cup \{w^+_{\ell+1}, v^+_{\ell+1}\})$ ($v^-_{\ell+1}$ exists by assumption). Then, (the '− part' of) (I)–(V) are clearly satisfied for $v^-_{\ell+1}$ and so we are done.

Case 4: $w^+_{\ell+1} \in \hat{U}^+_\ell(D)$ **and** $w^-_{\ell+1} \in V'$. Recall that $\widehat{\mathrm{ex}}^-_\ell(v^+_{\ell+1}) \leqslant 1$, so (the '− analogue' of) (B) implies that $X^- = \emptyset$ and $\widehat{\mathrm{ex}}^-_\ell(V \setminus \{w^+_{\ell+1}, w^-_{\ell+1}, v^+_{\ell+1}\}) = \widehat{\mathrm{ex}}^-_\ell(V \setminus \{w^+_{\ell+1}, w^-_{\ell+1}\}) - \widehat{\mathrm{ex}}^-_\ell(v^+_{\ell+1}) \geqslant 2 - 1 > 0$. We can thus let $v^-_{\ell+1} \in \hat{U}^-_\ell(D) \setminus (X^- \cup \{w^+_{\ell+1}, w^-_{\ell+1}, v^+_{\ell+1}\})$. Then, (the '− part' of) (I)–(V) are clearly satisfied for $v^-_{\ell+1}$ and so we are done.

We may therefore assume that $\widehat{U}^-_\ell(D) \setminus (X^- \cup \{w^+_{\ell+1}, v^+_{\ell+1}\}) = \emptyset$. But, by Claim 4, $\widehat{U}^-_\ell(D) \setminus (X^- \cup \{w^+_{\ell+1}\}) \neq \emptyset$ and so $\widehat{U}^-_\ell(D) \setminus (X^- \cup \{w^+_{\ell+1}\}) = \{v^+_{\ell+1}\}$. Thus, by assumption,

$$|\widehat{U}^+_\ell(D) \setminus (X^+ \cup \{w^-_{\ell+1}\})| \leqslant |\widehat{U}^-_\ell(D) \setminus (X^- \cup \{w^+_{\ell+1}\})| = |\{v^+_{\ell+1}\}| = 1.$$

Then, since $v^+_{\ell+1}$ satisfies (I), $\widehat{U}^+_\ell(D) \setminus (X^+ \cup \{w^-_{\ell+1}\}) = \{v^+_{\ell+1}\}$. We will find a contradiction.

Note that, $v^+_{\ell+1} \in \widehat{U}^+_\ell(D) \cap \widehat{U}^-_\ell(D) \subseteq U^0(D)$ and so

$$\widehat{\mathrm{ex}}^\pm_\ell(v^+_{\ell+1}) \overset{(11.1)}{=} 1. \tag{12.19}$$

Since $v^+_{\ell+1}$ satisfies (the '+ part' of) (III), this implies that $|W^+_*|, |W^-_*| \leqslant 4$. By (B) (and its '− analogue'), we have $X^\pm = \emptyset$ and so $\{v^+_{\ell+1}\} \subseteq \widehat{U}^\pm_\ell(D) \subseteq \{v^+_{\ell+1}, w^\mp_{\ell+1}\}$. Hence,

$$\begin{aligned}\mathrm{ex}^\pm_{D,U^*}(w^\mp_{\ell+1}) &\overset{(11.1)}{\geqslant} \widetilde{\mathrm{ex}}^\pm_{D,U^*}(w^\mp_{\ell+1}) - 1 \geqslant \widehat{\mathrm{ex}}^\pm_\ell(w^\mp_{\ell+1}) - 1 \\ &= \widehat{\mathrm{ex}}^\pm_\ell(D) - \widehat{\mathrm{ex}}^\pm_\ell(v^+_{\ell+1}) - 1 \overset{(12.9),(12.19)}{\geqslant} (1-46\eta)n - 1 - 1 \geqslant (1-47\eta)n.\end{aligned}$$

Together with (12.10) and (12.19), this implies that $\widetilde{W}^\pm_* = \{w^\mp_{\ell+1}\}$ and so, by assumption on our ordering of $E(H)$ made after Claim 1, it follows that $\ell = 0$. Therefore, $\widetilde{U}^\pm_{U^*}(D) = \widehat{U}^\pm_\ell(D) = \{w^\mp_{\ell+1}, v^+_{\ell+1}\}$, contradicting Claim 1. □

12.4 | Decreasing the degree at W_A

Note that, if $W_* = \emptyset$, then the excess of our tournament may be relatively small and so we do not have room to proceed similarly as in Lemma 12.2 to decrease the degree of the vertices in W_A. The strategy is to find a partial path decomposition $\mathcal{P}$ such that each vertex in W_A is covered by each of the paths in $\mathcal{P}$ and such that each vertex in V' is covered by half of the paths in $\mathcal{P}$. In this way, the degree at W_A is decreased faster than the degree at V'. Decreasing the degree at V' will ensure that the leftover excess is not too small compared to degree of the leftover oriented graph (recall Lemma 9.5(v)).

Lemma 12.4. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \eta \ll 1$. Let D be an oriented graph on a vertex set V of size n satisfying $\delta(D) \geqslant (1-\varepsilon)n$, $\widetilde{\mathrm{ex}}(D) \geqslant \frac{n}{2} + 9\eta n$, and the following properties.*

(a) *Let $W \cup V'$ be a partition of V such that, for each $v \in V'$, $|\mathrm{ex}_D(v)| \leqslant \varepsilon n$ and, for each $v \in W$, $|\mathrm{ex}_D(v)| \leqslant (1-20\eta)n$. Suppose $E(D[W]) = \emptyset$ and $|W| \leqslant \varepsilon n$.*
(b) *Let $A^+, A^- \subseteq E(T)$ be absorbing sets of (W, V')-starting/(V', W)-ending edges for D of size at most $\lceil \eta n \rceil$. Let $A := A^+ \cup A^-$, $W^\pm_A := V(A^\pm) \cap W$, and $W_A := V(A) \cap W$. Assume $A \neq \emptyset$, that is, $W_A \neq \emptyset$.*
(c) *Let $U^* \subseteq U^0(D)$ satisfy $|U^*| = \widetilde{\mathrm{ex}}(D) - \mathrm{ex}(D)$.*

Then, there exists a good (U^*, W, A)-partial path decomposition $\mathcal{P}$ of D such that $|\mathcal{P}| = 8\lceil \eta n \rceil$ and $D' := D \setminus \mathcal{P}$ satisfies the following.

(i) *For each $v \in W$, $d_{D'}(v) \leqslant d_D(v) - 12\lceil \eta n \rceil$.*
(ii) *For each $v \in V'$, $d_D(v) - 8\lceil \eta n \rceil \leqslant d_{D'}(v) \leqslant d_D(v) - 8\lceil \eta n \rceil + 1$.*
(iii) *Each $v \in U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$ satisfies $d^+_{D'}(v) = d^-_{D'}(v) \leqslant \widetilde{\mathrm{ex}}(D') - 1$.*

Proof. Fix additional constants such that $\varepsilon \ll \nu \ll \tau \ll \eta$. Let $k := 4\lceil \eta n \rceil$. Assume inductively that, for some $0 \leqslant \ell \leqslant k$, we have constructed edge-disjoint paths $P_{1,1}, P_{1,2}, P_{2,1}, \dots, P_{\ell,2} \subseteq D$ such that $\mathcal{P}_\ell := \{P_{i,j} \mid i \in [\ell], j \in [2]\}$ is a (U^*, W, A)-partial path decomposition of D such that the following hold, where $D_\ell := D \setminus \mathcal{P}_\ell$.

(α) For each $i \in [\ell]$ and $v \in W$, $v \in V(P_{i,1}) \cap V(P_{i,2})$.
(β) For each $i \in [\ell]$ and $v \in W$, v is an endpoint of at most one of $P_{i,1}$ and $P_{i,2}$.
(γ) For each $i \in [\ell]$ and $v \in V'$, either $v \in V(P_{i,1}) \triangle V(P_{i,2})$ or v is an endpoint of both $P_{i,1}$ and $P_{i,2}$.
(δ) For each $v \in V'$, there is at most one $i \in [\ell]$ such that v is an endpoint of exactly one of $P_{i,1}$ and $P_{i,2}$. Moreover, for each $v \in V'$, if there exists $i \in [\ell]$ such that v is an endpoint of exactly one of $P_{i,1}$ and $P_{i,2}$, then $\mathrm{ex}_{D_\ell}(v) = 0$.

If $\ell = k$, then let $\mathcal{P} := \mathcal{P}_k$ and $D' := D \setminus \mathcal{P}$.

Claim 1. $\mathcal{P}$ is a good partial path decomposition of D, that is, $\widetilde{\mathrm{ex}}(D') = \widetilde{\mathrm{ex}}(D) - 2k = \widetilde{\mathrm{ex}}(D) - 8\lceil \eta n \rceil$.

Note that if $\ell = k$ and Claim 1 holds, then we are done. Indeed, (α) and (β) imply that each $w \in W$ satisfies $d_{\mathcal{P}}(w) \geqslant 3k = 12\lceil \eta n \rceil$, while ($\gamma$) and ($\delta$) imply that each $v \in V'$ satisfies $8\lceil \eta n \rceil - 1 = 2k - 1 \leqslant d_{\mathcal{P}}(v) \leqslant 2k = 8\lceil \eta n \rceil$. Thus, (i) and (ii) hold. Finally, (iii) follows from Claim 1. Indeed, for each $v \in U^* \setminus (V^+(\mathcal{P}) \cup V^-(\mathcal{P}))$, we have $d^+_{D'}(v) = d^-_{D'}(v) \leqslant \frac{n-1}{2} < \frac{n}{2} + 9\eta n - 2k \leqslant \widetilde{\mathrm{ex}}(D) - 2k = \widetilde{\mathrm{ex}}(D')$, as desired.

Proof of Claim 1. By Proposition 8.4(a), it is enough to show that $\Delta^0(D') \leqslant \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$.

Let $v \in V$. By (α), (γ), and since $\mathcal{P}$ is a partial path decomposition of D, the following hold.

- If $v \in U^\pm(D) \cap W$, then $v \in V^\pm(P) \cup V^0(P)$ for each $P \in \mathcal{P}$.
- If $v \in U^0(D) \cap W$, then for each $\diamond \in \{+, -\}$, $v \in V^\diamond(P) \cup V^0(P)$ for all but at most one $P \in \mathcal{P}$.
- If $v \in U^\pm(D) \cap V'$, then $v \in V^\pm(P) \cup V^0(P)$ for at least k paths $P \in \mathcal{P}$.
- If $v \in U^0(D) \cap V'$, then, for each $\diamond \in \{+, -\}$, $v \in V^\diamond(P) \cup V^0(P)$ for at least $k - 1 = |\mathcal{P}| - k - 1$ paths $P \in \mathcal{P}$.

Thus, since $\mathcal{P}$ is a partial path decomposition of D, we have

$$d^{\max}_{D'}(v) \leqslant \begin{cases} d^{\max}_D(v) - |\mathcal{P}| & \text{if } v \in W \setminus U^0(D), \\ d^{\max}_D(v) - |\mathcal{P}| + k + 1 & \text{if } v \in V' \cup U^0(D). \end{cases} \tag{12.20}$$

For each $v \in V' \cup U^0(D)$, we have

$$d^{\max}_D(v) \overset{\text{Fact 4.20(c)}}{=} \frac{d_D(v) + |\mathrm{ex}_D(v)|}{2} \overset{\text{(a)}}{\leqslant} \frac{n - 1 + \varepsilon n}{2} \leqslant \frac{n}{2} + 9\eta n - 4\lceil \eta n \rceil - 1 \leqslant \widetilde{\mathrm{ex}}(D) - k - 1$$

and so, by (12.20), $\Delta^0(D') \leqslant \widetilde{\mathrm{ex}}(D) - |\mathcal{P}|$. Thus, $\mathcal{P}$ is a good partial path decomposition of D, as desired. □

If $\ell < k$, then let $D_\ell := D \setminus \mathcal{P}_\ell$ and $U^*_\ell := U^* \setminus (V^+(\mathcal{P}_\ell) \cup V^-(\mathcal{P}_\ell))$. We claim that there exist suitable endpoints $v_1^+, v_1^-, v_2^+, v_2^- \in V$ for $P_{\ell+1,1}$ and $P_{\ell+1,2}$.

Claim 2. There exist $v_1^+, v_1^-, v_2^+, v_2^- \in V$ such that the following hold.

(I) For each $i \in [2]$, $v_i^+ \neq v_i^-$ and $v_i^\pm \in \widetilde{U}^\pm_{U_\ell^*}(D_\ell)$. Moreover, for each $\diamond \in \{+,-\}$, if $v_1^\diamond = v_2^\diamond$, then $\widetilde{\mathrm{ex}}^\diamond_{D_\ell, U_\ell^*}(v_1^\diamond) \geqslant 2$.

(II) For each $v \in W$, there exists at most one pair $(i,\diamond) \in [2] \times \{+,-\}$ such that $v_i^\diamond = v$.

(III) For each $v \in V'$, if there exists exactly one pair $(i,\diamond) \in [2] \times \{+,-\}$ such that $v_i^\diamond = v$, then $\mathrm{ex}^\diamond_{D_\ell}(v) = 1$.

Before proving Claim 2, let us first apply it to construct $P_{\ell+1,1}$ and $P_{\ell+1,2}$. Let $v_1^+, v_1^-, v_2^+, v_2^- \in V$ be as in Claim 2. We construct a (v_1^+, v_1^-)-path $P_{\ell+1,1}$ and a (v_2^+, v_2^-)-path $P_{\ell+1,2}$ using Corollary 4.8 as follows. Observe that, by Lemma 4.4, $D_\ell[V' \setminus \{v_1^+, v_1^-, v_2^+, v_2^-\}]$ is a robust (ν,τ)-outexpander. Moreover, each $v \in V$ satisfies

$$d^{\min}_{D_\ell}(v) \geqslant d^{\min}_D(v) - |\mathcal{P}_\ell| \overset{\text{Fact 4.20(b)}}{\geqslant} \frac{d_D(v) - |\mathrm{ex}_D(v)|}{2} - 2\ell \overset{\text{(a)}}{\geqslant} \frac{(20\eta - \varepsilon)n}{2} - 8\lceil \eta n \rceil$$
$$\geqslant \eta n \overset{\text{(a)}}{\geqslant} 2(|W| + 2) + 2.$$

Let $\delta := \frac{3}{8}$ and $S := \{v_2^+, v_2^-\} \setminus (W \cup \{v_1^+, v_1^-\})$. For each $i \in [2]$, let $V_i' := V' \setminus \{v_i^+, v_i^-\}$ and $k_i := |W \cup \{v_i^+, v_i^-\}|$. Apply Corollary 4.8(a) with

	$D_\ell \setminus A$	V_1'	k_1	v_1^+	$W \setminus \{v_1^+, v_1^-\}$	v_1^-
playing the role of	D	V'	k	P_1	$\{P_2, \dots, P_{k-1}\}$	P_k

to obtain a (v_1^+, v_1^-)-path $P_{\ell+1,1}$ of length at most $\sqrt{\varepsilon} n$ which covers W and avoids $\{v_2^+, v_2^-\} \setminus (W \cup \{v_1^+, v_1^-\})$. Let $D_\ell' := D_\ell \setminus P_{\ell+1,1}$ and observe that, by Lemma 4.4, $D_\ell'[V' \setminus (V(P_{\ell+1,1}) \cup \{v_2^+, v_2^-\})]$ is still a robust (ν,τ)-outexpander. Then, let $S' := V(P_{\ell+1,1}) \setminus (W \cup \{v_2^+, v_2^-\})$ and apply Corollary 4.8(b) with

	$D_\ell \setminus A$	V_2'	k_2	S'	v_2^+	$W \setminus \{v_2^+, v_2^-\}$	v_2^-
playing the role of	D	V'	k	S	P_1	$\{P_2, \dots, P_{k-1}\}$	P_k

to obtain a (v_2^+, v_2^-)-path $P_{\ell+1,2}$ satisfying $V \setminus V(P_{\ell+1,2}) = V(P_{\ell+1,1}) \setminus (W \cup \{v_2^+, v_2^-\})$. Then, note that, by (I), $\mathcal{P}_{\ell+1}$ is a (U^*, W, A)-partial path decomposition of D and, by (II) and (III), (β) and (δ) are satisfied with $\ell + 1$ playing the role of ℓ, respectively. Finally, by construction of $P_{\ell+1,1}$ and $P_{\ell+1,2}$, (α) and (γ) are satisfied.

It remains to prove Claim 2.

Proof of Claim 2. Since $\mathcal{P}_\ell$ is a (U^*, W, A)-partial path decomposition of D, $|A^\pm| \leqslant \lceil \eta n \rceil$, and $2\ell \leqslant 2k \leqslant 8\lceil \eta n \rceil$,

$$\widetilde{\mathrm{ex}}^\pm_{U_\ell^*}(D_\ell) \overset{\text{Corollary 11.4}}{=} \widetilde{\mathrm{ex}}^\pm_{U^*}(D) - 2\ell$$
$$\overset{(11.2)}{=} \widetilde{\mathrm{ex}}(D) - |A^\pm| - 2\ell \tag{12.21}$$
$$\geqslant \frac{n}{2} - \eta n. \tag{12.22}$$

Thus, we can choose endpoints $v_1^+, v_1^-, v_2^+, v_2^- \in V$ satisfying (I)–(III) as follows.

If $|U^*_\ell \cap V'| \geqslant 2$, then pick distinct $u_1, u_2 \in U^*_\ell \cap V'$ and let $v_1^+ := u_1, v_2^+ := u_2, v_1^- := u_2$, and $v_2^- := u_1$. Then, (I)–(III) are satisfied, as desired.

We may therefore assume that $|U^*_\ell \cap V'| \leqslant 1$. We first pick $v_1^+, v_2^+ \in \widetilde{U}^+_{U^*_\ell}(D_\ell) \setminus (U^*_\ell \cap V')$ as follows. If $\widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(V' \setminus U^*_\ell) \geqslant 2$, then pick $v_1^+ \in \widetilde{U}^+_{U^*_\ell}(D_\ell) \cap (V' \setminus U^*_\ell)$ such that $\widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(v_1^+)$ is maximum. If $\widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(v_1^+) \geqslant 2$, then let $v_2^+ := v_1^+$; otherwise, let $v_2^+ \in (\widetilde{U}^+_{U^*_\ell}(D_\ell) \cap (V' \setminus U^*_\ell)) \setminus \{v_1^+\}$. If $\widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(V' \setminus U^*_\ell) = 1$, then, note that by Fact 11.1 and (12.22), $\widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(W) \geqslant \widetilde{\mathrm{ex}}^+_{U^*_\ell}(D_\ell) - \widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(V' \setminus U^*_\ell) - \widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(U^*_\ell \cap V') \geqslant \frac{n}{2} - \eta n - 1 - 1 \geqslant 1$. Thus, we can let $v_1^+ \in \widetilde{U}^+_{U^*_\ell}(D_\ell) \cap (V' \setminus U^*_\ell)$ and $v_2^+ \in \widetilde{U}^+_{U^*_\ell}(D_\ell) \cap W$. If $\widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(V' \setminus U^*_\ell) = 0$, then it is enough to show that $|\widetilde{U}^+_{U^*_\ell}(D_\ell) \cap W| \geqslant 2$ (so that we can take distinct $v_1^+, v_2^+ \in \widetilde{U}^+_{U^*_\ell}(D_\ell) \cap W$, as desired for (II)).

Note that, by Fact 11.1 and (12.22), $\widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(W) \geqslant \widetilde{\mathrm{ex}}^+_{U^*_\ell}(D_\ell) - \widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(V' \setminus U^*_\ell) - \widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(U^*_\ell \cap V') \geqslant \frac{n}{2} - \eta n - 0 - 1 \geqslant 2$ and so, in particular, $\widetilde{U}^+_{U^*_\ell}(D_\ell) \cap W \neq \emptyset$. Assume for a contradiction that $\widetilde{U}^+_{U^*_\ell}(D_\ell) \cap W = \{v\}$ for some $v \in W$. Note that since $\widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(v) = \widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(W) \geqslant 2$, $v \notin U^0(D_\ell)$ and so, $U^*_\ell \subseteq V'$ and $\mathrm{ex}^+_D(v) \geqslant \mathrm{ex}^+_{D_\ell}(v) \geqslant \widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(v)$. Thus, since $|U^*_\ell| = |U^*_\ell \cap V'| \leqslant 1$, $|A^+| \leqslant \lceil \eta n \rceil$, and $2\ell \leqslant 2k \leqslant 8\lceil \eta n \rceil$,

$$\begin{aligned} d^-_D(v) + \mathrm{ex}^+_D(v) &\overset{\text{Fact 4.20(d)}}{\leqslant} \widetilde{\mathrm{ex}}(D) \overset{(12.21)}{=} \widetilde{\mathrm{ex}}^+_{U^*_\ell}(D_\ell) + 2\ell + |A^+| \\ &\overset{\text{Fact 11.1}}{=} \widetilde{\mathrm{ex}}^+_{D_\ell, U^*_\ell}(v) + |U^*_\ell| + 2\ell + |A^+| \leqslant \mathrm{ex}^+_D(v) + 1 + 9\lceil \eta n \rceil. \end{aligned}$$

But, by (a) and Fact 4.20(b), $d^-_D(v) = \frac{d_D(v) - \mathrm{ex}^+_D(v)}{2} \geqslant \frac{(20\eta - \varepsilon)n}{2} > 9\lceil \eta n \rceil + 1$, a contradiction. Thus, $|\widetilde{U}^+_{U^*_\ell}(D_\ell) \cap W| \geqslant 2$ and we can let $v_1^+, v_2^+ \in \widetilde{U}^+_{U^*_\ell}(D_\ell) \cap W$ be distinct.

Now proceed analogously to pick $v_1^-, v_2^- \in \widetilde{U}^-_{U^*_\ell}(D_\ell) \setminus ((U^*_\ell \cap V') \cup \{v_1^+, v_2^+\})$ (this is possible since, for each $i \in [2]$, $\widetilde{\mathrm{ex}}^-_{D, U^*_\ell}(v_i^+) \leqslant 1$). One can easily verify that (I)–(III) are satisfied. □

This completes the proof. □

12.5 | Deriving Lemma 9.5

Proof of Lemma 9.5. Successively apply Lemmas 12.1, 12.2, and 12.4 as follows.

Step 1: Covering $T[W_0]$. First, apply Lemma 12.1 to obtain a good (U^*, W, A)-partial path decomposition $\mathcal{P}_1$ of T such that the following hold. Denote $D_1 := T \setminus \mathcal{P}_1$ and $U_1^* := U^* \setminus (V^+(\mathcal{P}_1) \cup V^-(\mathcal{P}_1))$.

(α) $\widetilde{\mathrm{ex}}(D_1) = \widetilde{\mathrm{ex}}(T) - |\mathcal{P}_1|$.
(β) $|\mathcal{P}_1| \leqslant 2\varepsilon n$.
(γ) $E(D_1[W_0]) = \emptyset$.
(δ) If $|U^+(D_1)| = |U^-(D_1)| = 1$, then $e(U^-(D_1), U^+(D_1)) = 0$ or $\widetilde{\mathrm{ex}}(D_1) - \mathrm{ex}(D_1) \geqslant 2$.
(ε) Each $v \in U_1^*$ satisfies $d^+_{D_1}(v) = d^-_{D_1}(v) \leqslant \widetilde{\mathrm{ex}}(D_1) - 1$.

In particular, observe that, by Fact 4.22, Lemma 9.5(b), and (β), the following hold.

(ζ) $\widetilde{\mathrm{ex}}(D_1) \geqslant \frac{n}{2} - 2\varepsilon n$ and, if $W_A \neq \emptyset$, then $\widetilde{\mathrm{ex}}(D_1) \geqslant \frac{n}{2} + 9\eta n$.
(η) For each $v \in V$, $d_{D_1}(v) \geqslant (1-5\varepsilon)n$.
(θ) For each $v \in W_*$, $|\mathrm{ex}_{D_1}(v)| > (1-21\eta)n$.
(ι) For each $\diamond \in \{+,-\}$, if $|W_A^\diamond| \geqslant 2$, then $\mathrm{ex}_{D_1}^\diamond(v) < \eta n$ for each $v \in V$ and, if $|W_A^\diamond| = 1$, then, for each $v \in V$ and $w \in W_A^\diamond$, $\mathrm{ex}_{D_1}^\diamond(v) \leqslant \mathrm{ex}_{D_1}^\diamond(w) + 5\varepsilon n$.

Step 2: Covering the remaining edges of $T[W]$ and decreasing the degree of the vertices in $W_* \cup W_A$ when $W_* \neq \emptyset$. If $W_* = \emptyset$, then let $\mathcal{P}_2 := \emptyset$. Otherwise, note that by Proposition 9.7, $|U_1^*| = \widetilde{\mathrm{ex}}(D_1) - \mathrm{ex}(D_1)$ and let $\mathcal{P}_2$ be the good (U_1^*, W, A)-partial path decomposition of D_1 obtained by applying Lemma 12.2 with D_1, U_1^*, and 5ε playing the roles of D, U^*, and ε. Denote $D_2 := D_1 \setminus \mathcal{P}_2$ and $U_2^* := U_1^* \setminus (V^+(\mathcal{P}_2) \cup V^-(\mathcal{P}_2))$. Then, note that, if $W_* \neq 0$, then the following hold.

(I) $\widetilde{\mathrm{ex}}(D_2) = \widetilde{\mathrm{ex}}(D_1) - |\mathcal{P}_2|$.
(II) $E(D_2[W]) = \emptyset$.
(III) $N^\pm(D_1) - N^\pm(D_2) \leqslant 88\eta n$.
(IV) For each $v \in W_* \cup W_A$, $(1-3\sqrt{\eta})n \leqslant d_{D_2}(v) \leqslant (1-4\eta)n$.
(V) For each $v \in W_0$, $d_{D_2}(v) \geqslant (1-3\sqrt{\eta})n$ and $d_{D_2}^{\min}(v) \geqslant 5\eta n$.
(VI) For each $v \in V'$, $d_{D_2}(v) \geqslant (1-8\sqrt{\varepsilon})n$.
(VII) If $|W_*^+|, |W_*^-| \leqslant 1$, then each $v \in W_*$ satisfies $|\mathrm{ex}_{D_2}(v)| = d_{D_2}(v)$.
(VIII) Each $v \in U_2^*$ satisfies $d_{D_2}^+(v) = d_{D_2}^-(v) \leqslant \widetilde{\mathrm{ex}}(D_2) - 1$.

Note that, by (IV), the following holds.

(IX) Each $v \in W_*$ satisfies $|\mathrm{ex}_{D_2}(v)| \geqslant |\mathrm{ex}_T(v)| - 3\sqrt{\eta}n \geqslant (1-4\sqrt{\eta})n$.

Thus, (VII) implies the following.

(X) If $\widetilde{\mathrm{ex}}(D_2) \leqslant 2\lceil \frac{n}{2} \rceil - \lceil \eta n \rceil$, then $|\mathrm{ex}_{D_2}(v)| = d_{D_2}(v)$ for each $v \in W_*$.

Step 3: Decreasing the degree of the vertices in W_A when $W_* = \emptyset$. If $W_* \neq \emptyset$ or $W_A = \emptyset$, then let $\mathcal{P}_3 := \emptyset$. Assume $W_* = \emptyset$ and $W_A \neq \emptyset$. Recall that, by construction, $D_2 = D_1$ and $U_2^* = U_1^*$. In particular, (γ), (ζ), and (η) are satisfied and $|U_2^*| = \widetilde{\mathrm{ex}}(D_2) - \mathrm{ex}(D_2)$. Let $\mathcal{P}_3$ be the good (U_2^*, W, A)-partial path decomposition of D_2 obtained by applying Lemma 12.4 with D_2, U_2^*, and 5ε playing the roles of D, U^*, and ε. Denote $D_3 := D_2 \setminus \mathcal{P}_3$ and note that, if $W_* = \emptyset$ and $W_A \neq \emptyset$, then the following hold.

(A) $\widetilde{\mathrm{ex}}(D_3) = \widetilde{\mathrm{ex}}(D_2) - |\mathcal{P}_3|$.
(B) Each $v \in U_2^* \setminus (V^+(\mathcal{P}_3) \cup V^-(\mathcal{P}_3))$ satisfies $d_{D_3}^+(v) = d_{D_3}^-(v) \leqslant \widetilde{\mathrm{ex}}(D_3) - 1$.
(C) $|\mathcal{P}_3| = 8\lceil \eta n \rceil$.
(D) For each $v \in W$, $d_{D_3}(v) \leqslant d_{D_2}(v) - 12\lceil \eta n \rceil$.
(E) For each $v \in V'$, $d_{D_2}(v) - 8\lceil \eta n \rceil \leqslant d_{D_3}(v) \leqslant d_{D_2}(v) - 8\lceil \eta n \rceil + 1$.

Step 4: Checking the assertions of Lemma 9.5. Let $\mathcal{P} := \bigcup_{i \in [3]} \mathcal{P}_i$ and $D := T \setminus \mathcal{P} = D_3$. If $W_* \neq \emptyset$ or $W_A = \emptyset$, then let $d := \min\{\lceil \frac{n}{2} \rceil - \lceil \eta n \rceil, \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil\}$. If $W_* = \emptyset$ and $W_A \neq \emptyset$, then let $d := \lceil \frac{n}{2} \rceil - 5\lceil \eta n \rceil$. In both cases, $\mathcal{P}$ is a good (U^*, W, A)-partial path decomposition of T by Corollary 11.5. Note that (ii) follows immediately from (ε), (VIII), and (B), while (iii) follows immediately from (γ) and (II). If $W_* = \emptyset$, then (vi) holds vacuously, otherwise (vi) follows from (X) and the definition of d.

We now verify (v). If $W_* \neq \emptyset$ or $W_A = \emptyset$, then (v) holds immediately by definition of d. Suppose $W_* = \emptyset$ and $W_A \neq \emptyset$. Then, $D_2 = D_1$, so (ζ), (A), and (C) imply that $\widetilde{\mathrm{ex}}(D) \geqslant \frac{n}{2} + 9\eta n - |\mathcal{P}_3| \geqslant \lceil \frac{n}{2} \rceil - 4\lceil \eta n \rceil = d + \lceil \eta n \rceil$. Therefore, (v) holds.

Next, we check (i). If both $W_*, W_A = \emptyset$, then (ζ) implies that $\widetilde{\mathrm{ex}}(D) \geqslant \lceil \frac{n}{2} \rceil - 2\varepsilon n$ and so $\lceil \frac{n}{2} \rceil - \lceil \eta n \rceil - 2\varepsilon n \leqslant d \leqslant \lceil \frac{n}{2} \rceil - \lceil \eta n \rceil$. If $W_* \neq \emptyset$, then $D = D_2$, so (IX) implies that $\widetilde{\mathrm{ex}}(D) \geqslant (1 - 4\sqrt{\eta})n$ and thus $d = \lceil \frac{n}{2} \rceil - \lceil \eta n \rceil$. If $W_* = \emptyset$ and $W_A \neq \emptyset$, then $d = \lceil \frac{n}{2} \rceil - 5\lceil \eta n \rceil$ and so (i) holds.

For (iv), note that by Corollary 11.6 and (β), $N^{\pm}(T) - N^{\pm}(D_1) \leqslant |\mathcal{P}_1| \leqslant 2\varepsilon n \leqslant \eta n$. Thus, if $W_* \neq \emptyset$ or $W_A = \emptyset$, then (iv) follows from (III). We may therefore assume that $W_* = \emptyset$ and $W_A \neq \emptyset$. Then, $D_2 = D_1$ and so Corollary 11.6 and (C) imply that $N^{\pm}(D_1) - N^{\pm}(D) \leqslant |\mathcal{P}_3| \leqslant 8\lceil \eta n \rceil \leqslant 88\eta n$. Therefore, (iv) holds.

It remains to check (vii)–(ix). First, suppose that $W_* \neq \emptyset$ or $W_A = \emptyset$, and $d = \lceil \frac{n}{2} \rceil - \lceil \eta n \rceil$. Then, the upper bounds in (viii) and (ix) are clearly satisfied. The lower bound in (ix) follows from (η) and (VI), while (vii) holds by (IV). By (a) and Fact 4.20(b), each $v \in W_0$ satisfies $d_T^{\min}(v) \geqslant \frac{20\eta n - 1}{2} \geqslant 9\eta n$ and so the lower bounds in (viii) follow from (η) and (V). Thus, (vii)–(ix) hold if $W_* \neq \emptyset$ or $W_A = \emptyset$, and $d = \lceil \frac{n}{2} \rceil - \lceil \eta n \rceil$.

Next, assume that $W_* \neq \emptyset$ or $W_A = \emptyset$, and $d = \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil$. Then, $2d + 2\lceil \eta n \rceil = 2\widetilde{\mathrm{ex}}(D) \geqslant 2\Delta^0(D) \geqslant d_D(v)$ for each $v \in V$ and so the upper bounds of (viii) and (ix) hold. Moreover, $\widetilde{\mathrm{ex}}(D) \leqslant \lceil \frac{n}{2} \rceil$ and so, by (IX), $W_* = \emptyset$. By assumption, this implies that $W_A = \emptyset$ and so (vii) holds vacuously. Moreover, each $v \in V$ satisfies

$$d_D^{\min}(v) \overset{(\eta)}{\geqslant} d_T^{\min}(v) - 5\varepsilon n \overset{\text{Fact 4.20(b)}}{=} \frac{n - 1 - |\,\mathrm{ex}_T(v)|}{2} - 5\varepsilon n \overset{\text{(a)}}{\geqslant} \frac{20\eta n - 1}{2} - 5\varepsilon n \geqslant \lceil \eta n \rceil + 1$$

and

$$d_D(v) \overset{(\eta)}{\geqslant} (1 - 5\varepsilon)n \geqslant 2\left(\left\lceil \frac{n}{2} \right\rceil - \lceil \eta n \rceil\right) + 2\lceil \eta n \rceil - 6\varepsilon n \geqslant 2d + 2\lceil \eta n \rceil - 6\varepsilon n,$$

so the lower bounds in (viii) and (ix) hold. Therefore, (vii)–(ix) hold if $W_* \neq \emptyset$ or $W_A = \emptyset$, and $d = \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil$.

We may therefore assume that $W_* = \emptyset$ and $W_A \neq \emptyset$. Recall that, in this case, $d = \lceil \frac{n}{2} \rceil - 5\lceil \eta n \rceil$ and $\mathcal{P}_2 = \emptyset$. First, observe that each $v \in W$ satisfies

$$d_D(v) \geqslant n - 1 - 2|\mathcal{P}| \overset{(\beta),\text{(C)}}{\geqslant} n - 17\lceil \eta n \rceil$$

and if $v \in W_0$, then

$$d_D^{\min}(v) \geqslant d_T^{\min}(v) - |\mathcal{P}| \overset{\text{Fact 4.20(b)}}{\geqslant} \frac{n - 1 - |\,\mathrm{ex}_T(v)|}{2} - |\mathcal{P}| \overset{\text{(a)},(\beta),\text{(C)}}{\geqslant} \frac{20\eta n - 1}{2} - 2\varepsilon n - 8\lceil \eta n \rceil$$
$$\geqslant \lceil \eta n \rceil.$$

Thus, the lower bounds in (vii) and (viii) are satisfied. Moreover, each $v \in W$ satisfies

$$d_D(v) \overset{\text{(D)}}{\leqslant} d_{D_2}(v) - 12\lceil \eta n \rceil \leqslant n - 1 - 12\lceil \eta n \rceil \leqslant 2d - 1 - 2\lceil \eta n \rceil$$

and so the upper bounds in (vii) and (viii) hold. Finally, note that each $v \in V'$ satisfies

$$d_D(v) \overset{\text{(E)}}{\leqslant} d_{D_2}(v) - 8\lceil \eta n \rceil + 1 \leqslant n - 8\lceil \eta n \rceil \leqslant 2d + 2\lceil \eta n \rceil$$

and

$$\begin{aligned} d_D(v) &\overset{\text{(E)}}{\geqslant} d_{D_2}(v) - 8\lceil \eta n\rceil \geqslant d_T(v) - 2|\mathcal{P}_1| - 8\lceil \eta n\rceil \\ &\overset{(\beta)}{\geqslant} n - 1 - 4\varepsilon n - 8\lceil \eta n\rceil \geqslant 2d - 5\varepsilon n + 2\lceil \eta n\rceil, \end{aligned}$$

so (ix) holds. Therefore, (vii)–(ix) hold if $W_* = \emptyset$ and $W_A \neq \emptyset$. This completes the proof of Lemma 9.5. □

13 | CONSTRUCTING LAYOUTS: PROOF OF LEMMA 9.6

We will prove Lemma 9.6 as follows. In Step 1, we choose a set $\widetilde{E}$ of auxiliary edges which 'neutralise' the excess of the vertices in D. In Step 2, we then subdivide these edges into paths which form a layout $(\widetilde{L}, \widetilde{F})$. In Step 3, we subdivide the paths in $\widetilde{L}$ further to obtain layouts $(\widehat{L}_1, \widehat{F}_1), \dots, (\widehat{L}_\ell, \widehat{F}_\ell)$ which cover the edges of $D \setminus A$ at W in such a way that (iii) and (iv) are satisfied. Finally, in Step 4, we adjust the degrees of the vertices in V' so that they satisfy (v). To achieve this, we proceed as follows. For those vertices $v \in V'$ where the current layouts would result in a degree which is too small after the approximate decomposition, we add v as an isolated vertex to some of the layouts. For vertices $v \in V'$ whose degree would be too large, we subdivide two edges from a suitable layout and include v into both of the resulting paths. Recall that the relevant definitions involving layouts were introduced in Sections 7, 9.1, and 9.2.

Proof of Lemma 9.6. Let $W^{\pm} := W \cap U^{\pm}(D)$. Denote $\phi := \phi^+ + \phi^-$. Note that since A is a (W_1, V')-absorbing set, Definition 8.7 implies that A does not contain any edge incident to W_2 and so A is also a (W, V')-absorbing set. Thus, for simplicity, we can let W (rather than W_1) play the role of W in the auxiliary excess notation. For each $v \in V$, define

$$\widehat{\text{ex}}^{\pm}(v) := \widetilde{\text{ex}}^{\pm}_{D,U^*,W,A}(v) - \phi^{\pm}(v) \tag{13.1}$$

to be the excess at v that we want to cover with the layouts (that is, the number of paths which we want to start/end at v). Let $\widehat{U}^{\pm} := \{v \in V \mid \widehat{\text{ex}}^{\pm}(v) > 0\}$. Note that (11.2) and (c) imply that

$$\widehat{\text{ex}}(D) := \sum_{v \in V} \widehat{\text{ex}}^+(v) = \sum_{v \in V} \widehat{\text{ex}}^-(v) = \widetilde{\text{ex}}(D) - \lceil \eta n\rceil.$$

If $\widehat{\text{ex}}(D) - d < \sqrt{\varepsilon} n$, then let $\ell := \widehat{\text{ex}}(D)$; otherwise, let $\ell := d$. Note that, by (e), $\ell \geqslant d$. Thus, (i) holds, as desired. Observe that either

$$\widehat{\text{ex}}(D) - \ell \geqslant \sqrt{\varepsilon} n \quad \text{or} \quad \widehat{\text{ex}}(D) - \ell = 0. \tag{13.2}$$

We claim that each $v \in V'$ satisfies

$$d^{\pm}_D(v) \leqslant \widetilde{\text{ex}}(D) - \widetilde{\text{ex}}^{\mp}_{D,U^*}(v). \tag{13.3}$$

Indeed, if $v \in U^*$, then (11.1) implies that $\widetilde{\text{ex}}^{\pm}_{D,U^*}(v) = 1$ and so (13.3) holds by (b). We may therefore assume that $v \notin U^*$. Suppose without loss of generality that $d^+_D(v) \geqslant d^-_D(v)$. Then, (11.1) implies that $\widetilde{\text{ex}}^-_{D,U^*}(v) = \text{ex}^-_D(v) = 0$ and so $\widetilde{\text{ex}}(D) - \widetilde{\text{ex}}^-_{D,U^*}(v) \geqslant \Delta^0(D) \geqslant d^+_D(v)$. Finally, (11.1) implies that $\widetilde{\text{ex}}^+_{D,U^*}(v) = \text{ex}^+_D(v)$ and so $\widetilde{\text{ex}}(D) - \widetilde{\text{ex}}^+_{D,U^*}(v) \geqslant d^+_D(v) - \text{ex}^+_D(v) = d^-_D(v)$, as desired.

Note that throughout this proof, given a multiset L' of paths, the corresponding edge set F' in the layout (L', F') we construct will always satisfy $F' = E(L') \cap \mathcal{F} = E(L') \cap (E_W(D) \setminus A)$.

Step 1: Choosing suitable endpoints. Let $s := \widehat{\text{ex}}(D) = \widetilde{\text{ex}}(D) - \lceil \eta n \rceil$. In this step, we will select suitable endpoints for the s (non-trivial) paths in L. Each $v \in V'$ will be used precisely $\widehat{\text{ex}}^+(v)$ times as a starting point and $\widehat{\text{ex}}^-(v)$ times as an ending point. We now fix the endpoints of each path by defining a multidigraph $\widetilde{E}$ on V such that each $e \in \widetilde{E}$ corresponds to a path of shape e. Hence, $|E| = s$. Note that $\widehat{\text{ex}}^+(v) \neq 0 \neq \widehat{\text{ex}}^-(v)$ if and only if $\widehat{\text{ex}}^+(v) = 1 = \widehat{\text{ex}}^-(v)$. We now formalise this in the following paragraph.

For each $\diamond \in \{+, -\}$, let $v_1^\diamond, \dots, v_s^\diamond \in \widehat{U}^\diamond$ be such that, for each $v \in \widehat{U}^\diamond$, there exist exactly $\widehat{\text{ex}}^\diamond(v)$ indices $i \in [s]$ for which $v = v_i^\diamond$. Since $s \geqslant d > 1$ (by (d) and (e)) and each $v \in \widehat{U}^+ \cap \widehat{U}^-$ satisfies $\widehat{\text{ex}}^+(v) = \widehat{\text{ex}}^-(v) = 1$ (by (11.1)), we may assume without loss of generality that, for each $i \in [s]$, $v_i^+ \neq v_i^-$. Let $\widetilde{E} := \{v_j^+ v_j^- \mid j \in [s]\}$. Note that,

$$|\widetilde{E}| = \widehat{\text{ex}}(D) = \widetilde{\text{ex}}(D) - \lceil \eta n \rceil \tag{13.4}$$

and, for each $v \in V$,

$$d^\pm_{\widetilde{E}}(v) = \widehat{\text{ex}}^\pm(v). \tag{13.5}$$

Let $v \in W$. By Definition 8.7, we have $d_A^\pm(v) \leqslant \text{ex}_D^\pm(v)$ and so $\text{ex}^\pm_{D\setminus A}(v) = \text{ex}^\pm_D(v) - d^\pm_A(v)$. Moreover, (b) implies that $v \notin U^*$ and (c) implies that $\phi(v) = 0$. Therefore,

$$\text{ex}^\pm_{D\setminus A}(v) = \text{ex}^\pm_D(v) - d^\pm_A(v) \overset{(11.1)}{=} \widetilde{\text{ex}}^\pm_{D,U^*,W,A}(v) = \widehat{\text{ex}}^\pm(v) \overset{(13.5)}{=} d^\pm_{\widetilde{E}}(v). \tag{13.6}$$

Step 2: Constructing layouts. In Steps 2 and 3, we will transform $\widetilde{E}$ into a W-exceptional layout $(\widehat{L}, \widehat{F})$. Initially, we set $(\widehat{L}, \widehat{F}) = (\widetilde{E}, \emptyset)$ where each edge in $\widetilde{E}$ is considered as a path. To be a W-exceptional layout, each path in $\widehat{L}$ requires an edge entirely in V' and $\widehat{F}$ must contain $E_W(L)$. For this, we proceed roughly as follows.

Suppose that the path v^+v^- does not lie entirely in V', say $v^+ \in W$ and $v^- \in V'$. We pick $u \in N_{D\setminus A}(v^+)$ and replace the path v^+v^- with the subdivided path v^+uv^- and add v^+u into the set $\widehat{F}$ of fixed edges. (Note that uv^- lies entirely in V'.)

More precisely, recall that $\mathcal{F} = E_W(D) \setminus A$ and $D' = D \setminus \mathcal{F}$. In this step, we will use $\widetilde{E}$ to construct a layout $(\widetilde{L}, \widetilde{F})$ such that the following hold.

(α) $(\widetilde{L}, \widetilde{F})$ is a W-exceptional layout such that $\widetilde{F} \subseteq \mathcal{F}$ and $\widetilde{L}$ contains no isolated vertex.
(β) $\widetilde{L}$ has shape $\widetilde{E}$ (and thus, by (13.5), $(\widetilde{L}, \widetilde{F})$ is U^*-path consistent with respect to $(D', \mathcal{F})$).
(γ) For each $P \in \widetilde{L}$ and $v \in V(P) \cap V'$, v is an endpoint of P or has (in P) a neighbour in W.
(δ) For each $P \in \widetilde{L}$, $V^0(P) \subseteq V'$.
(ε) Each $P \in \widetilde{L}$ has at most 4 vertices and contains an edge which lies entirely in V'.

Properties (γ)–(ε) mean that the paths in $\widetilde{L}$ are only obtained by subdividing, with vertices of V', the edges in $\widetilde{E}$ which are incident to W. Property (γ) will ensure that no vertex of V' belongs to too many layouts (as desired for (vii)) and (ε) will ensure that the layouts are not too large (as desired for (vi)). Property (δ) means that we have not yet incorporated the exceptional vertices as internal vertices. This will give us more flexibility in Step 3 when we cover the remaining edges incident to W.

Initially, let $\widetilde{L}^0 := \widetilde{E}$ and $\widetilde{F}^0 := \emptyset$. Let $w_1, \dots, w_k$ be an enumeration of $W \cap V(\widetilde{L}^0)$. We will consider each w_i in turn and, at each stage i, subdivide all the edges incident to w_i. Let $i \in [k]$. By

(13.6), $\mathrm{ex}_{D\setminus A}(w_i) \neq 0$ and so recall from (3.1) that $N^{\max}_{D\setminus A}(w_i)$ denotes the outneighbourhood of w_i in $D \setminus A$ if $\mathrm{ex}_{D\setminus A}(w_i) > 0$ and the inneigbourhood of w_i in $D \setminus A$ otherwise.

Assume inductively that for some $0 \leqslant m \leqslant k$, we have constructed, for each $i \in [m]$, a multiset of paths $\widetilde{L}^i$ and a set of edges $\widetilde{F}^i$ such that the following are satisfied.

(I) Let $i \in [m]$. Let $S_i := \{e \in E(\widetilde{L}^{i-1}) \mid w_i \in V(e)\}$. Then, $\widetilde{L}^i$ is the multiset of paths obtained from $\widetilde{L}^{i-1}$ by subdividing each edge $e \in S_i$ with some vertex $z_e \in N^{\max}_{D\setminus A}(w_i) \cap V'$, where the vertices z_e are distinct for different edges $e \in S_i$. (That is, $\widetilde{L}^i$ is obtained by subdividing, with a neighbour of w_i, all the edges of $\widetilde{L}^{i-1}$ which are incident to w_i.)

(II) For each $i \in [m]$, $\widetilde{F}^i = \widetilde{F}^{i-1} \cup E_{\{w_i\}}(\widetilde{L}^i)$.

Note that (I) and (13.5) imply that the following holds.

(III) For all $i \in [m]$ and $v \in W$, $d^{\pm}_{\widetilde{L}^i}(v) = d^{\pm}_{\widetilde{E}}(v) = \widehat{\mathrm{ex}}^{\pm}(v)$.

Moreover, (I) and (II) imply that, for each $i \in [m]$, $\widetilde{F}^i$ is a set of edges obtained from $\widetilde{F}^{i-1}$ by adding all the edges of the form $w_i z_e$ or $z_e w_i$ from (I). In particular, $\widetilde{F}^i \subseteq E_{\{w_j \mid j \in [i]\}}(D) \setminus A \subseteq \mathcal{F}$ is satisfied.

If $m = k$, then let $\widetilde{L} := \widetilde{L}^k$ and $\widetilde{F} := \widetilde{F}^k$. Observe that (α)–(ε) hold. Indeed, (I) implies that $\widetilde{L}$ is obtained by subdividing $\widetilde{L}^0 = \widetilde{E}$, so (β) holds. By (I), all these subdivisions are done with vertices of V'. Thus, (γ)–(ε) are satisfied. (For (ε), note that at each stage the paths all contain at most two vertices of W, so each edge in $\widetilde{L}^0 = \widetilde{E}$ is subdivided at most twice.) Moreover, this implies that $\widetilde{L}$ is a set of non-trivial paths which all contain at least one edge whose endpoints are both in V'. By (II), $\widetilde{F}$ consists of the edges of $\widetilde{L}$ which are incident to W. Altogether, this implies that $(\widetilde{L}, \widetilde{F})$ is a W-exceptional layout. As mentioned above, (I) and (II) imply that $\widetilde{F} \subseteq \mathcal{F}$ and so (α) is satisfied, as desired.

We may therefore assume that $m < k$. By assumption and (III), $S_{m+1} := \{e \in E(\widetilde{L}^m) \mid w_{m+1} \in V(e)\} \neq \emptyset$ and so $w_{m+1} \notin U^0(D)$. We may therefore assume without loss of generality that $w_{m+1} \in W^+$. This implies that $\mathrm{ex}_D(w_{m+1}) = |\mathrm{ex}_D(w_{m+1})| = \mathrm{ex}^+_D(w_{m+1})$ and $d^{\min}_D(w_{m+1}) = d^-_D(w_{m+1})$. We now subdivide all the edges in S_{m+1} using Hall's theorem as follows. Let $Y := N^+_{D\setminus A}(w_{m+1})$ and observe that, by (a), $Y \subseteq V'$. Construct an auxiliary bipartite graph G on vertex classes S_{m+1} and Y by joining $e \in S_{m+1}$ and $u \in Y$ if and only if $u \notin V(e)$. Note that

$$|S_{m+1}| \overset{\text{(III)}}{=} \widehat{\mathrm{ex}}^+(w_{m+1}) \overset{(13.6)}{=} \mathrm{ex}^+_{D\setminus A}(w_{m+1}) \leqslant |N^+_{D\setminus A}(w_{m+1})| = |Y|$$

and

$$|Y| = d^{\max}_{D\setminus A}(w_{m+1}) \geqslant \frac{d_{D\setminus A}(w_{m+1})}{2} \overset{\text{(f),(g)}}{\geqslant} 5\varepsilon n.$$

Since Y is a set (rather than a multiset), each $e \in S_{m+1}$ satisfies $d_G(e) \geqslant |Y| - 1$. Note that (I) implies that if $v \in V'$ is contained in a path P in $\widetilde{L}^m$, then v is an endpoint of P or has (in P) a neighbour in W. Hence, each $v \in Y \subseteq V'$ satisfies

$$d_G(v) \overset{(13.5),\text{(I)}}{\geqslant} |S_{m+1}| - \widehat{\mathrm{ex}}^-(v) - |W| \overset{(13.1)}{\geqslant} |S_{m+1}| - \widetilde{\mathrm{ex}}^-_{D,U^*,W,A}(v) - |W| \overset{\text{(a),(h)}}{\geqslant} |S_{m+1}| - 2\varepsilon n.$$

Thus, applying Proposition 4.18 with S_{m+1} and Y playing the roles of A and B gives a matching M of G covering S_{m+1}.

Let $\widetilde{L}^{m+1}$ be obtained from $\widetilde{L}^m$ by subdividing, for each $vu \in M$ (with $v \in X$ and $u \in Y$), the edge $w_{m+1}v \in E(\widetilde{L}^m)$ into the path $w_{m+1}uv$. Note that this is a valid subdivision since (I) implies that the path $P \in \widetilde{L}^m$ containing $w_{m+1}v$ satisfies $V' \cap V(P) \subseteq \{v\}$. Let $\widetilde{F}^{m+1} := \widetilde{F}^m \cup E_{w_{m+1}}(\widetilde{L}^{m+1})$. Clearly, (I) and (II) are satisfied with $m+1$ playing the role of m, as desired. This completes Step 2.

Step 3: Covering additional edges incident to W. Now that we have constructed suitable paths, we need to partition them into ℓ small layouts. Moreover, recall that we need these layouts to cover all the non-absorbing edges incident to W_1 (see (iii)), as well as a prescribed number of edges incident to W_2 (see (iv)). These are the goals of Step 3. First, we will ensure that (iii) and (iv) are satisfied as follows. For each $w \in W$ in turn, we will subdivide some of the paths constructed in Step 2 to incorporate w as an internal vertex. Since the layout needs to remain W-exceptional, we will once again need to prescribe the new edges incident to W. More precisely, suppose that we want to incorporate $w \in W$ as an internal vertex in a path $P \in \widetilde{L}$. First, we will choose an unfixed edge $uv \in E(P)$ (which exists by (ε)) and edges $u'w, wv' \in E(D) \setminus A$ which are not already covered by $\widetilde{L}$ and such that $u', v' \notin V(P)$. Then, we will subdivide the edge uv in P into the path $uu'wv'v$ and consider the edges $u'w$ and wv' as fixed edges. We will repeat this procedure until $\widetilde{L}$ covers the desired amount of edges incident to W (as prescribed by (iii) and (iv)). Once this achieved, we will split $(\widetilde{L}, \widetilde{F})$ into ℓ layouts.

More precisely, we construct $(\widehat{L}_1, \widehat{F}_1), \dots, (\widehat{L}_\ell, \widehat{F}_\ell)$ such that the following hold, where $\widetilde{L}$ is the multiset defined by $\widehat{L} := \bigcup_{i\in[\ell]} \widehat{L}_i$ and $\widehat{F} := \bigcup_{i\in[\ell]} \widehat{F}_i$.

(α') $(\widehat{L}_1, \widehat{F}_1), \dots, (\widehat{L}_\ell, \widehat{F}_\ell)$ are W-exceptional layouts such that $\widehat{F} \subseteq \mathcal{F}$ and $\widehat{L}$ contains no isolated vertex.

(β') $\widehat{L}$ is a subdivision of $\widetilde{L}$ (and thus, by (β), $\widehat{L}$ has shape $\widetilde{E}$ and $(\widehat{L}_1, \widehat{F}_1), \dots, (\widehat{L}_\ell, \widehat{F}_\ell)$ are U^*-path consistent with respect to $(D', \mathcal{F})$).

(γ') For each $v \in V'$, $d_{\widehat{L}}(v) \leqslant |\operatorname{ex}_D(v)| - \phi(v) + 2 + 2|W|$.

(δ') For each $i \in [\ell]$, $|V(\widehat{L}_i)| \leqslant 5\sqrt{\varepsilon}n$ and $|E(\widehat{L}_i)| \leqslant 4\sqrt{\varepsilon}n$. Moreover, each path $P \in \widehat{L}$ contains an edge which lies entirely in V'.

(ε') Either $\ell = \widetilde{\operatorname{ex}}(D) - \lceil \eta n \rceil$ or there exist at least $\sqrt{\varepsilon}n$ indices $i \in [\ell]$ such that $\widehat{L}_i$ contains at least 2 paths. Moreover, if $\ell = \widetilde{\operatorname{ex}}(D) - \lceil \eta n \rceil$, then for all $i \in [\ell]$, $|\widehat{L}_i| = 1$.

(ζ') For each $v \in W_1$, $d^{\pm}_{\widehat{L}}(v) = d^{\pm}_{D\setminus A}(v)$.

(η') For each $v \in W_2$, $d^{\pm}_{\widehat{L}}(v) = d^{\pm}_D(v) - \lceil \eta n \rceil$.

Properties (α'), (β'), (δ'), (ζ'), and (η') mean that our main objectives for Step 3 are achieved: $(\widehat{L}_1, \widehat{F}_1), \dots, (\widehat{L}_\ell, \widehat{F}_\ell)$ are U^*-consistent W-exceptional layouts which are small and cover the desired amount of edges incident to W (and so satisfy (iii), (iv), and (vi)). Moreover, (γ') will ensure that each vertex of V' is covered by only few of the layouts (as desired for (vii)). Finally, (ε') is a technical property which will enable us, in Step 4, to adjust the layouts in order for (v) to be satisfied.

Let $\widehat{L}^0 := \widetilde{L}$ and $\widehat{F}^0 := \widetilde{F}$. Let $w_1, \dots, w_k$ be an enumeration of W. We will consider each w_i in turn and, at each stage i, subdivide the required the number of paths with w_i. Let $\widehat{Q}^0$ be a set of paths in $\widehat{L}^0$ of size $\min\{2\ell, |\widehat{L}^0|\}$. We will restrict ourselves to only subdivide the paths in $\widehat{Q}^0$. This will ensure that only few of the final paths are long, which will enable us to form small layouts. Assume inductively that, for some $0 \leqslant m \leqslant k$, we have constructed, for each $i \in [m]$, two multisets of paths $\widehat{L}^i$ and $\widehat{Q}^i$, and a set of edges $\widehat{F}^i$ such that the following hold.

(I′) Let $i \in [m]$. Then, for each $P \in \widehat{Q}^i$, either $P \in \widehat{Q}^{i-1}$ or there exist $P' \in \widehat{Q}^{i-1}$, an edge $e = u_e v_e \in E(P') \setminus \widehat{F}^{i-1}$ with $u_e, v_e \in V'$, and distinct $u'_e, v'_e \in V' \setminus V(P')$ such that P is obtained from P' by subdividing the edge $e = u_e v_e$ into the path $u_e u'_e w_i v'_e v_e$, where $u'_e w_i, w_i v'_e \in$

$E(D) \setminus (A \cup \widetilde{F})$ and $\{u'_e, v'_e\} \cap \{u'_{e'}, v'_{e'}\} = \emptyset$ whenever $e, e' \in E(\widehat{Q}^{i-1})$ are distinct edges to be subdivided in order to form $\widehat{Q}^i$. Moreover, $\widehat{L}^i = (\widehat{L}^{i-1} \setminus \widehat{Q}^{i-1}) \cup \widehat{Q}^i$. (That is, $\widehat{L}^i$ is obtained from $\widehat{L}^{i-1}$ by incorporating w as an internal vertex in some of the paths in $\widehat{Q}^{i-1} \subseteq \widehat{L}^i$.)

(II′) For each $i \in [m]$, $\widehat{F}^i = \widehat{F}^{i-1} \cup E_{\{w_i\}}(\widehat{L}^i)$.

(III′) Let $i \in [m]$. If $w_i \in W_1$, then $N^{\pm}_{\widehat{L}^m}(w_i) = N^{\pm}_{D\setminus A}(w_i)$ and, if $w_i \in W_2$, then $N^{\pm}_{\widehat{L}^m}(w_i) \subseteq N^{\pm}_D(w_i)$ and $d^{\pm}_{\widehat{L}^m}(w_i) = d^{\pm}_D(w_i) - \lceil \eta n \rceil$.

Note that (ε) and (I′) imply the following.

(IV′) For each $i \in [m]$, each $P \in \widehat{L}^i$ contains an edge which lies entirely in V'.

Also note that, by (I′) and (II′), for each $i \in [m]$, $\widehat{F}^i$ is a set of edges (rather than a multiset) and is obtained from $\widehat{F}^{i-1}$ by adding all the edges of the form $u'_e w_i$ and $w_i v'_e$ in (I′). In particular,

$$\widehat{F}^{i-1} \subseteq \widehat{F}^i = E_W(\widehat{L}^i) \subseteq E_W(D) \setminus A = \mathcal{F}. \tag{13.7}$$

First, suppose that $m = k$. Then, (III′) implies that we have finished incorporating all the desired edges incident to W. By (I′) and (II′), $(\widehat{L}^k, \widehat{F}^k)$ is obtained by subdividing some of the paths in $\widetilde{L}$ and adding all the new edges incident to W to $\widehat{F}^k$. Thus, $(\widehat{L}^k, \widehat{F}^k)$ is still a U^*-path consistent W-exceptional layout and there only remains to partition $(\widehat{L}^k, \widehat{F}^k)$ into ℓ small layouts. We will do so by splitting the paths in $\widehat{L}^k$ as evenly as possible across the ℓ layouts and, subject to that, also distribute the paths in $\widehat{Q}^k$ as evenly as possible. This will ensure that not all of the long paths belong to the same layout (recall that we want our layouts to be small).

More precisely, we partition $\widehat{L}^k$ into $\widehat{L}_1, \dots, \widehat{L}_\ell$ such that, for each $i, j \in [\ell]$,

$$||\widehat{L}_i| - |\widehat{L}_j|| \leqslant 1 \quad \text{and} \quad |\widehat{L}_i \cap \widehat{Q}^k| \leqslant 2 \tag{13.8}$$

(this is possible since $|\widehat{Q}^k| = \min\{2\ell, |\widehat{L}^0|\} = \min\{2\ell, |\widehat{L}^k|\}$). Note that (13.4), ($\beta$), and (I′) imply that

$$|\widehat{L}^k| = |\widetilde{L}| = |\widetilde{E}| \overset{(13.4)}{=} \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil \tag{13.9}$$

and so, for each $i \in [\ell]$,

$$\begin{aligned} |\widehat{L}_i| &\leqslant \left\lceil \frac{\widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil}{\ell} \right\rceil \overset{\text{(d),(i)}}{\leqslant} \frac{\max\{\Delta^0(D), \mathrm{ex}(D)\} - \lceil \eta n \rceil}{\eta n} + 1 \\ &\overset{\text{(h)}}{\leqslant} \frac{\max\{n, n|W| + \varepsilon n|V'|\} - \lceil \eta n \rceil}{\eta n} + 1 \overset{|W| \leqslant \varepsilon n}{\leqslant} \frac{\max\{n, 2\varepsilon n^2\} - \lceil \eta n \rceil}{\eta n} + 1 \\ &\leqslant \sqrt{\varepsilon} n. \end{aligned} \tag{13.10}$$

For each $i \in [\ell]$, define $\widehat{F}_i := E(\widehat{L}_i) \cap \mathcal{F} = E(\widehat{L}_i) \cap \widehat{F}^k$. Let $\widehat{L}$ be the multiset defined by $\widehat{L} := \bigcup_{i \in [\ell]} \widehat{L}_i = \widehat{L}^k$ and denote $\widehat{F} := \bigcup_{i \in [\ell]} \widehat{F}_i = \widehat{F}^k$ (as mentioned after (IV′), $\widehat{F}$ is subset of $\widetilde{F}$ rather than a multiset).

Claim 1. Properties (α')–(η') are satisfied.

Proof of Claim. First, observe that (ζ') and (η') follow immediately from (III$'$). For (ε'), suppose $\ell \neq \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil$. By (13.9) and (13.2), $|\widehat{L}| = \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil \geqslant \ell + \sqrt{\varepsilon} n$. Thus, (13.8) implies that there exist at least $\sqrt{\varepsilon} n$ indices $i \in [\ell]$ such that $|\widehat{L}_i| \geqslant 2$. Thus, ($\varepsilon'$) holds.

By (I$'$), $\widehat{L}$ is obtained by subdividing some of the paths in $\widetilde{L}$ and so (β') holds. To check (α'), we need to verify (L1)–(L3) as defined in Section 7. First, (α) implies that $\widehat{L}_1, \dots, \widehat{L}_k$ are multisets of non-trivial paths on V and so they satisfy (L1). By (α) and (II$'$), $\widehat{F}_i$ consists, for each $i \in [k]$, of all the edges in $\widehat{L}_i$ which are incident to W. Therefore, $(\widehat{L}_1, \widehat{F}_1), \dots, (\widehat{L}_\ell, \widehat{F}_\ell)$ satisfy (L2), and (IV$'$) implies that (L3) holds. Therefore, $(\widehat{L}_1, \widehat{F}_1), \dots, (\widehat{L}_\ell, \widehat{F}_\ell)$ are W-exceptional layouts. As observed above, $\widehat{F} \subseteq \mathcal{F}$ and $\widehat{L}$ only consists of non-trivial paths. Thus, (α') holds.

Next, we verify (γ'). Let $v \in V'$. By (β) and (β'), $\widehat{L}$ is a subdivision of $\widetilde{E}$ and so (13.5) implies that v is the starting point of $\widehat{\mathrm{ex}}^+(v)$ paths in $\widehat{L}$ and the ending point of $\widehat{\mathrm{ex}}^-(v)$ paths in $\widehat{L}$. Suppose that $v \in V^0(P)$ for some $P \in \widehat{L}$. Then, (γ) and (I$'$) imply that P contains an edge $e \in E(D)$ between v and a vertex of W. By (α'), $(\widehat{L}_1, \widehat{F}_1), \dots, (\widehat{L}_\ell, \widehat{F}_\ell)$ are W-exceptional and so $e \in \widehat{F}$. Moreover, since $\widehat{F}$ is a set rather than a multiset, we have $e \notin E(P')$ for all $P' \in \widehat{L} \setminus \{P\}$. Therefore,

$$\begin{aligned} d_{\widehat{L}}(v) &\leqslant \widehat{\mathrm{ex}}^+(v) + \widehat{\mathrm{ex}}^-(v) + 2|N_D(v) \cap W| \\ &\leqslant (\widetilde{\mathrm{ex}}^+_{D,U^*}(v) - \phi^+(v)) + (\widetilde{\mathrm{ex}}^-_{D,U^*}(v) - \phi^-(v)) + 2|W| \\ &\overset{(11.1)}{\leqslant} (\mathrm{ex}^+_D(v) + 1) + (\mathrm{ex}^-_D(v) + 1) - \phi(v) + 2|W| \leqslant |\,\mathrm{ex}_D(v)| - \phi(v) + 2 + 2|W|. \end{aligned}$$

Thus, (γ') holds.

Finally, we verify (δ'). The 'moreover part' holds by (I$'$) and (ε). Moreover, (I$'$) implies that whenever we incorporate a vertex of W as an internal vertex to a path in $\widehat{Q}^0$, we add three vertices to that path. Therefore, (ε) implies that each $Q \in \widehat{Q}^k$ satisfies $|V(Q)| \leqslant 4 + 3|W|$. Thus, (a), (13.8), and (13.10) imply that, for each $i \in [\ell]$, we have $|V(\widehat{L}_i)| \leqslant 4|\widehat{L}_i| + 6|W| \leqslant 4\sqrt{\varepsilon} n + 6\varepsilon n \leqslant 5\sqrt{\varepsilon} n$. Similarly, each $Q \in \widehat{Q}^k$ satisfies $|E(Q)| \leqslant 3 + 3|W|$ and so each $i \in [\ell]$ satisfies $|E(\widehat{L}_i)| \leqslant 3|\widehat{L}_i| + 6|W| \leqslant 3\sqrt{\varepsilon} n + 6\varepsilon n \leqslant 4\sqrt{\varepsilon} n$. Thus, ($\delta'$) holds. □

We may therefore assume that $m < k$. Suppose without loss of generality that $w_{m+1} \notin W^-$. Thus, $w_{m+1} \in U^+(D) \cup U^0(D)$ and so $\mathrm{ex}_D(w_{m+1}) = |\,\mathrm{ex}_D(w_{m+1})| = \mathrm{ex}^+_D(w_{m+1})$ and $d^{\min}_D(w_{m+1}) = d^-_D(w_{m+1})$. Moreover, by Definition 8.7, $d^-_{D\setminus A}(w_{m+1}) = d^-_D(w_{m+1})$. Finally, by assumptions (b) and (c), $\phi^\pm(w_{m+1}) = 0$ and $U^* \cap W = \emptyset$. Thus,

$$\widehat{\mathrm{ex}}^+(w_{m+1}) = \widetilde{\mathrm{ex}}^+_{D,U^*}(w_{m+1}) = \mathrm{ex}^+_D(w_{m+1}) - d^+_A(w_{m+1}) \overset{\text{Definition 8.7}}{=} \mathrm{ex}^+_{D\setminus A}(w_{m+1}). \tag{13.11}$$

We will construct $\widehat{Q}^{m+1}$ as follows. First, we will pair each inneighbour of w_{m+1} in $D \setminus (A \cup \widehat{F}^m)$ to an outneighbour of w_{m+1} in $D \setminus (A \cup \widehat{F}^m)$. Let X denote the set of these pairs. We will use these pairs to incorporate w_{m+1} as an internal vertex in some of the paths in $\widehat{Q}^m$ as follows. Let Y be the set of paths in $\mathcal{Q}^m$ which do not already contain w_{m+1}. Form an auxiliary bipartite graph by joining each $(u', v') \in X$ and $P \in Y$ if and only if both $u', v' \notin V(P)$ (if $u' \in V(P)$ or $v' \in V(P)$, then we cannot use (u', v') to incorporate w_{m+1} as an internal vertex in P). Then, we will use Hall's theorem to find a large matching M in this auxiliary graph. For each $(u', v')P \in M$, we will subdivide an unfixed edge $uv \in E(P)$ into the path $uu'w_{m+1}v'v$.

By (13.5), (β), (δ), and (I$'$), none of the paths in $\widehat{L}^m$ have w_{m+1} as internal vertex or ending point and so we have $d^-_{\widehat{L}^m}(w_{m+1}) = 0$. Fix a bijection $\sigma : N^-_{D\setminus A}(w_{m+1}) \longrightarrow N^+_{D\setminus(A\cup\widehat{F}^m)}(w_{m+1})$. Note that this is possible since

$$\begin{aligned} d^+_{\widehat{F}^m}(w_{m+1}) &\overset{(13.7)}{=} d^+_{\widehat{L}^m}(w_{m+1}) \overset{(\mathrm{I}')}{=} d^+_{\widetilde{L}}(w_{m+1}) \overset{(\beta),(\delta)}{=} d^+_{\widetilde{E}}(w_{m+1}) \\ &\overset{(13.5)}{=} \widehat{\mathrm{ex}}^+(w_{m+1}) \overset{(13.11)}{=} \mathrm{ex}^+_{D\setminus A}(w_{m+1}). \end{aligned}$$

Thus,

$$\begin{aligned} d^+_{D\setminus(A\cup\widehat{F}^m)}(w_{m+1}) &\overset{(13.7)}{=} d^+_{D\setminus A}(w_{m+1}) - d^+_{\widehat{F}^m}(w_{m+1}) = d^+_{D\setminus A}(w_{m+1}) - \mathrm{ex}^+_{D\setminus A}(w_{m+1}) \\ &= d^-_{D\setminus A}(w_{m+1}), \end{aligned}$$

as desired.

Let $X := \{(u,\sigma(u)) \mid u \in N^-_{D\setminus A}(w_{m+1})\}$. Let $Y \subseteq \widehat{Q}^m$ be obtained from $\widehat{Q}^m$ by deleting all the paths that contain w_{m+1}. Define an auxiliary bipartite graph G with vertex classes X and Y by joining $(u,v) \in X$ and $P \in Y$ if and only if both $u, v \notin V(P)$.

Claim 2. If $w_{m+1} \in W_1$, then G contains a matching M covering X. If $w_{m+1} \in W_2$, then G contains a matching M of size $|X| - \lceil \eta n \rceil$.

Let M be as in Claim 2. We obtain $\widehat{Q}^{m+1}$ from $\widehat{Q}^m$ by subdividing, for each $(u',v')P \in M$, an edge $uv \in P$ that lies entirely in V' (which exists by (IV$'$)) into the path $uu'w_{m+1}v'v$. Let $\widehat{L}^{m+1} := (\widehat{L}^m \setminus \widehat{Q}^m) \cup \widehat{Q}^{m+1}$ and $\widehat{F}^{m+1} := \widehat{F}^m \cup E_{\{w_{m+1}\}}(\widehat{L}^{m+1})$. One can easily verify that (I$'$)–(III$'$) are satisfied with $m+1$ playing the role of m. There only remains to show Claim 2. We will need the following observation.

Claim 3. If $X \neq \emptyset$, then

$$\max\{|X|,|Y|\} \geqslant \eta n \geqslant 10\varepsilon n \quad \text{and} \quad \min\{|X|,|Y|\} \geqslant \begin{cases} |X| & \text{if } w_{m+1} \in W_1, \\ |X| - \lceil \eta n \rceil & \text{if } w_{m+1} \in W_2. \end{cases}$$

We first assume that Claim 3 holds and derive Claim 2.

Proof of Claim 2. Clearly, we may assume that $X \neq \emptyset$. The goal is to use Proposition 4.18. We start by checking that the degree of each vertex in G is large. First, observe that, by (a), we have $u, v \in V'$ for each $(u,v) \in X$. By (I$'$), $\widehat{L}^m$ is obtained from $\widetilde{L}$ by repeated subdivisions and, in each subdivision, we incorporate a vertex of W using only two new vertices of V'. Thus, each $P \in Y$ satisfies

$$d_G(P) \geqslant |X| - |V(P) \cap V'| \overset{(\varepsilon),(\mathrm{I}')}{\geqslant} |X| - (4 + 2|W|) \overset{(\mathrm{a})}{\geqslant} |X| - 3\varepsilon n \overset{\text{Claim 3}}{\geqslant} |X| - \frac{\max\{|X|,|Y|\}}{2}.$$

Let $(u,v) \in X$. We count the number of paths in $\widehat{L}^m$ which contain u. By (I$'$), $\widehat{L}^m$ is a subdivision of $\widetilde{L}$ and so (β) and (13.5) imply that u is an endpoint of precisely $\widehat{\mathrm{ex}}^+(u) + \widehat{\mathrm{ex}}^-(u)$ paths in $\widehat{L}^m$. Suppose that $P \in \widehat{L}^m$ contains u as an internal vertex. By (γ) and (I$'$), P contains an edge e between u and a vertex of W. By (α) and (I$'$), $e \in \widehat{F}^m \subseteq E(D) \setminus A$. In particular, the fact that $\widehat{F}^m$ is a set (rather than a multiset) implies that $e \notin E(\widehat{L}^m \setminus \{P\})$. Thus, there are at most $|N_{D\setminus A}(u) \cap W|$ paths in $\widehat{L}^m$ which contain u as an internal vertex. Similarly, there are at most

$\widehat{\text{ex}}^+(v) + \widehat{\text{ex}}^-(v) + |N_{D\setminus A}(v) \cap W|$ paths in $\widehat{L}^m$ which contain v (as an endpoint or internal vertex). Thus,

$$\begin{aligned}
d_G((u,v)) &\geqslant |Y| - (\widehat{\text{ex}}^+(u) + \widehat{\text{ex}}^-(u) + |N_{D\setminus A}(u) \cap W|) \\
&\quad -(\widehat{\text{ex}}^+(v) + \widehat{\text{ex}}^-(v) + |N_{D\setminus A}(v) \cap W|) \\
&\geqslant |Y| - (\widetilde{\text{ex}}^+_{D,U^*}(u) + \widetilde{\text{ex}}^-_{D,U^*}(u) - \phi(u)) - (\widetilde{\text{ex}}^+_{D,U^*}(v) + \widetilde{\text{ex}}^-_{D,U^*}(v) - \phi(v)) \\
&\quad -2|W| \\
&\overset{(11.1)}{\geqslant} |Y| - (|\operatorname{ex}_D(u)| + 2 - \phi(u)) - (|\operatorname{ex}_D(v)| + 2 - \phi(v)) - 2|W| \\
&\geqslant |Y| - |\operatorname{ex}_D(u)| - |\operatorname{ex}_D(v)| - 4 - 2|W| \overset{\text{(a),(h)}}{\geqslant} |Y| - 5\varepsilon n \\
&\overset{\text{Claim 3}}{\geqslant} |Y| - \frac{\max\{|X|,|Y|\}}{2}.
\end{aligned}$$

Thus, G satisfies the degree conditions of Proposition 4.18, applied with $\{X, Y\}$ playing the roles of $\{A, B\}$ (with $|A| \leqslant |B|$). Therefore, G contains a matching M of size $\min\{|X|, |Y|\}$. By Claim 3, we may assume that $|M| = |X|$ if $w_{m+1} \in W_1$ and $|M| = |X| - \lceil \eta n \rceil$ if $w_{m+1} \in W_2$. □

Finally, it remains to prove Claim 3.

Proof of Claim 3. By (g), if $w_{m+1} \in W_2$, then $|X| = d^-_{D\setminus A}(w_{m+1}) = d^{\min}_D(w_{m+1}) \geqslant \lceil \eta n \rceil$. Thus, it is enough to show that

$$|Y| \geqslant \begin{cases} \max\{|X|, \eta n\} & \text{if } w_{m+1} \in W_1, \\ |X| - \lceil \eta n \rceil & \text{if } w_{m+1} \in W_2. \end{cases} \tag{13.12}$$

Note that, by (δ) and (I′), w_{m+1} is not an internal vertex of any path in $\widehat{Q}^m$. Moreover, (I′) implies that $\widehat{L}^m$ is a subdivision of $\widetilde{L}$ and so (β) and (13.5) imply

$$|Y| \geqslant |\widehat{Q}^m| - \widehat{\text{ex}}^+(w_{m+1}) \overset{(13.11)}{=} |\widehat{Q}^m| - \text{ex}^+_{D\setminus A}(w_{m+1}). \tag{13.13}$$

If $|\widehat{Q}^m| \geqslant 2d$ and $w_{m+1} \in W_1$, then

$$\begin{aligned}
|Y| &\overset{(13.13)}{\geqslant} |\widehat{Q}^m| - \text{ex}^+_{D\setminus A}(w_{m+1}) \geqslant 2d - \text{ex}^+_{D\setminus A}(w_{m+1}) \\
&\overset{\text{(f)}}{\geqslant} d_{D\setminus A}(w_{m+1}) + \eta n - \text{ex}^+_{D\setminus A}(w_{m+1}) = 2d^-_{D\setminus A}(w_{m+1}) + \eta n \geqslant |X| + \eta n.
\end{aligned}$$

Similarly, if $|\widehat{Q}^m| \geqslant 2d$ and $w_{m+1} \in W_2$, then

$$\begin{aligned}
|Y| &\overset{(13.13)}{\geqslant} |\widehat{Q}^m| - \text{ex}^+_{D\setminus A}(w_{m+1}) \overset{w_{m+1} \notin V(A)}{=} |\widehat{Q}^m| - \text{ex}^+_D(w_{m+1}) \\
&\geqslant 2d - \text{ex}^+_D(w_{m+1}) \overset{\text{(g)}}{\geqslant} d_D(w_{m+1}) - 2\lceil \eta n \rceil - \text{ex}^+_D(w_{m+1}) \\
&= 2d^-_D(w_{m+1}) - 2\lceil \eta n \rceil = |X| - \lceil \eta n \rceil + d^-_D(w_{m+1}) - \lceil \eta n \rceil \overset{\text{(g)}}{\geqslant} |X| - \lceil \eta n \rceil.
\end{aligned}$$

We may therefore assume that $|\widehat{Q}^m| < 2d$. Since, by (i), $d \leqslant \ell$, we have $|\widehat{L}^m| = |\widehat{L}^0| = |\widehat{Q}^0| = |\widehat{Q}^m|$ and so

$$\widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil \overset{(13.4)}{=} |\widetilde{E}| \overset{(\beta),(\mathrm{I}')}{=} |\widehat{L}^m| = |\widehat{Q}^m| < 2d. \tag{13.14}$$

Thus,

$$\begin{aligned} |Y| &\overset{(13.13)}{\geqslant} |\widehat{Q}^m| - \mathrm{ex}^+_{D\setminus A}(w_{m+1}) \\ &\overset{(13.14)}{=} \widetilde{\mathrm{ex}}(D) - \lceil \eta n \rceil - \mathrm{ex}^+_{D\setminus A}(w_{m+1}) \\ &\geqslant d^+_D(w_{m+1}) - \lceil \eta n \rceil - \mathrm{ex}^+_D(w_{m+1}) = d^-_D(w_{m+1}) - \lceil \eta n \rceil \overset{w_{m+1} \notin V(A^-)}{=} |X| - \lceil \eta n \rceil. \end{aligned} \tag{13.15}$$

We may therefore assume that $w_{m+1} \in W_1$ and $|\widehat{Q}^m| < 2d$. We need to show that $|Y| \geqslant \max\{|X|, \eta n\}$. Recall that $d^-_D(w_{m+1}) = d^-_{D\setminus A}(w_{m+1}) = |X| > 0$. Then, Fact 4.20(c) implies that $d^+_D(w_{m+1}) > \mathrm{ex}^+_D(w_{m+1})$. Thus, by (f) and (13.14), we have $|X| \geqslant \eta n$ and one of the following holds: $\mathrm{ex}^+_{D\setminus A}(w_{m+1}) \leqslant \lceil \eta n \rceil$ or $d^+_A(w_{m+1}) = d_A(w_{m+1}) = \lceil \eta n \rceil$. Thus, it suffices to show that $|Y| \geqslant |X|$. If $\mathrm{ex}^+_{D\setminus A}(w_{m+1}) \leqslant \lceil \eta n \rceil$, then, by (13.15) and (e),

$$\begin{aligned} |Y| \geqslant d - \mathrm{ex}^+_{D\setminus A}(w_{m+1}) &\overset{(\mathrm{f})}{\geqslant} \frac{d_{D\setminus A}(w_{m+1}) + \lceil \eta n \rceil}{2} - \mathrm{ex}^+_{D\setminus A}(w_{m+1}) \\ &\overset{\text{Fact 4.20(b)}}{=} d^-_{D\setminus A}(w_{m+1}) + \frac{\lceil \eta n \rceil - \mathrm{ex}^+_{D\setminus A}(w_{m+1})}{2} \geqslant |X|, \end{aligned}$$

as desired. If $d^+_A(w_{m+1}) = \lceil \eta n \rceil$, then (13.15) implies that

$$|Y| \geqslant d^+_D(w_{m+1}) - \lceil \eta n \rceil - \mathrm{ex}^+_{D\setminus A}(w_{m+1}) = d^+_{D\setminus A}(w_{m+1}) - \mathrm{ex}^+_{D\setminus A}(w_{m+1}) = d^-_{D\setminus A}(w_{m+1}) = |X|,$$

as desired. □

Step 4: Adjusting the degree of the vertices in V'. Recall that, in Step 3, we constructed ℓ W-exceptional layouts which are U^*-path consistent and satisfy (ζ') and (η'), and thus satisfy (iii) and (iv). We will now adjust these layouts to ensure that (v) is satisfied.

Let $v_1, \dots, v_k$ be an enumeration of V' and, for each $i \in [k]$, define

$$n_i := d^+_{\widehat{L}}(v_i) + |\{j \in [\ell] \mid v_i \notin V(\widehat{L}_j)\}| + \lceil \eta n \rceil - \phi^-(v_i) - d^+_D(v_i). \tag{13.16}$$

Note that together with Claim 4 below, (v) holds if $n_i = 0$ for all $i \in [k]$.

Claim 4. For each $i \in [k]$,

$$d^\pm_D(v_i) = d^\pm_{\widehat{L}}(v_i) + |\{j \in [\ell] \mid v_i \notin V(\widehat{L}_j)\}| - n_i + \lceil \eta n \rceil - \phi^\mp(v).$$

Proof of Claim. Let $i \in [k]$. The equality for + holds immediately by definition of n_i. One can easily verify that, in order to show that the equality for − holds, it is enough to prove that

$$d^+_{\widehat{L}}(v_i) - \phi^-(v_i) - d^+_D(v_i) = d^-_{\widehat{L}}(v_i) - \phi^+(v_i) - d^-_D(v_i). \tag{13.17}$$

We now show that (13.17) is satisfied. First, note that, by (13.5) and (β'), v_i is the starting point of precisely $\widehat{\mathrm{ex}}^+(v_i)$ paths in $\widehat{L}$ and the ending point of precisely $\widehat{\mathrm{ex}}^-(v_i)$ paths in $\widehat{L}$, so

$$d^+_{\widehat{L}}(v_i) = (d^-_{\widehat{L}}(v_i) - \widehat{\mathrm{ex}}^-(v_i)) + \widehat{\mathrm{ex}}^+(v_i). \tag{13.18}$$

Assume without loss of generality that $d^+_D(v_i) \geqslant d^-_D(v_i)$. Suppose first that $v_i \notin U^*$. Then, $\widehat{\mathrm{ex}}^\pm(v_i) = \mathrm{ex}^\pm_D(v_i) - \phi^\pm(v_i)$. Moreover, $\mathrm{ex}^-_D(v_i) = 0$ and so $\widehat{\mathrm{ex}}^-(v_i) = -\phi^-(v_i)$. Thus, by (13.18),

$$\begin{aligned} d^+_{\widehat{L}}(v_i) &= (d^-_{\widehat{L}}(v_i) + \phi^-(v_i)) + (\mathrm{ex}^+_D(v_i) - \phi^+(v_i)) \\ &= d^-_{\widehat{L}}(v_i) + \phi^-(v_i) + (d^+_D(v_i) - d^-_D(v_i)) - \phi^+(v_i), \end{aligned}$$

so (13.17) holds, as desired. Now suppose that $v_i \in U^*$. Then, $\widehat{\mathrm{ex}}^\pm(v_i) = 1 - \phi^\pm(v_i)$ and $d^+_D(v_i) = d^-_D(v_i)$. Thus, by (13.18),

$$d^+_{\widehat{L}}(v_i) = (d^-_{\widehat{L}}(v_i) - 1 + \phi^-(v_i)) + (1 - \phi^+(v_i)) = d^-_{\widehat{L}}(v_i) + \phi^-(v_i) - \phi^+(v_i) + (d^+_D(v_i) - d^-_D(v_i)),$$

so (13.17) holds, as desired. □

If $n_i > 0$, then, in order to satisfy (v), it is enough to add v_i as an isolated vertex to exactly n_i of the sets of paths $\widehat{L}_1, \dots, \widehat{L}_\ell$ that do not contain v_i (this will decrease $|\{j \in [\ell] \mid v_i \notin V(\widehat{L}_j)\}|$ by n_i and so we will be done by Claim 4). If $n_i < 0$, then it is enough to find $-n_i$ indices $j \in [\ell]$ such that $v_i \notin V(\widehat{L}_j)$ and $|\widehat{L}_j| \geqslant 2$, and add v_i as an internal vertex in exactly two paths in $\widehat{L}_j$ (this will decrease $|\{j \in [\ell] \mid v_i \notin V(\widehat{L}_j)\}|$ by $-n_i$ but increase both $d^\pm_{\widehat{L}}(v_i)$ by $-2n_i$ and so we will be done by Claim 4).

We now bound n_i with the following claim.

Claim 5. For each $i \in [k]$,

$$-2\varepsilon n \leqslant n_i \leqslant 2\sqrt{\varepsilon} n.$$

Proof of Claim. Let $i \in [k]$. We have

$$2n_i \overset{\text{Claim 4}}{=} d_{\widehat{L}}(v_i) + 2|\{j \in [\ell] \mid v_i \notin V(\widehat{L}_j)\}| + 2\lceil \eta n \rceil - \phi(v_i) - d_D(v_i).$$

Thus,

$$\begin{aligned} 2n_i &\geqslant d_{\widehat{L}}(v_i) + 2(\ell - d_{\widehat{L}}(v_i)) + 2\lceil \eta n \rceil - \phi(v_i) - d_D(v_i) \\ &= 2\ell - d_{\widehat{L}}(v_i) + 2\lceil \eta n \rceil - \phi(v_i) - d_D(v_i) \\ &\overset{(\gamma')}{\geqslant} 2\ell - (|\mathrm{ex}_D(v_i)| - \phi(v_i) + 2 + 2|W|) + 2\lceil \eta n \rceil - \phi(v_i) - (2d + 2\lceil \eta n \rceil) \\ &= 2(\ell - d) - |\mathrm{ex}_D(v_i)| - 2 - 2|W| \overset{\text{(a),(h),(i)}}{\geqslant} -4\varepsilon n. \end{aligned}$$

Similarly,

$$\begin{aligned} 2n_i &\overset{(\gamma')}{\leqslant} (|\mathrm{ex}_D(v_i)| - \phi(v_i) + 2 + 2|W|) + 2\ell + 2\lceil \eta n \rceil - \phi(v_i) - (2d + 2\lceil \eta n \rceil - \varepsilon n) \\ &\leqslant 2(\ell - d) + |\mathrm{ex}_D(v_i)| + 2 + 2|W| + \varepsilon n \overset{\text{(a),(h),(i)}}{\leqslant} 4\sqrt{\varepsilon} n, \end{aligned}$$

which proves the claim. □

Assume without loss of generality that $(n_i)_{i\in[k]}$ is an increasing sequence and so, for any $i, j \in [k]$, if $n_i < 0$ but $n_j \geqslant 0$, then $i < j$. For each $i \in [\ell]$, let $L_i^0 := \widehat{L}_i$. Assume inductively that, for some $0 \leqslant m \leqslant k$, we have constructed, for each $i \in [\ell]$ and $j \in [m]$, a multiset L_i^j of paths and isolated vertices such that the following are satisfied, where L^j is the multiset defined by $L^j := \bigcup_{i\in[\ell]} L_i^j$ for each $j \in [m]$.

(I″) For each $j \in [m]$, if $n_j < 0$, then there exists $N_j \subseteq [\ell]$ such that $|N_j| = -n_j$ and the following hold. For each $i \in N_j$, $v_j \notin V(L_i^{j-1})$ and there exist two paths $P_1, P_2 \in L_i^{j-1}$ such that L_i^j is obtained from L_i^{j-1} by subdividing, for each $s \in [2]$, an edge $uw \in E(P_s) \setminus E_W(P_s)$ into the path uv_jw. For each $i \in [\ell] \setminus N_j$, $L_i^j = L_i^{j-1}$.

(II″) For each $j \in [m]$, if $n_j \geqslant 0$, then there exists $N_j \subseteq [\ell]$ such that $|N_j| = n_j$ and the following hold. For each $i \in N_j$, $v_j \notin V(L_i^{j-1})$ and L_i^j is obtained from L_i^{j-1} by adding v_j as an isolated vertex. For each $i \in [\ell] \setminus N_j$, $L_i^j = L_i^{j-1}$.

(III″) For each $i \in [\ell]$ and $j \in [m]$, $|V(L_i^j) \setminus V(\widehat{L}_i)| \leqslant \varepsilon^{\frac{1}{3}} n$.

By (I″) and (II″), each L^j is obtained from L^{j-1} either by subdividing two edges with v_j in $|n_j|$ layouts which did not already cover v_j, or by adding v_j as an isolated vertex in $|n_j|$ layouts which did not already cover v_j. Thus, the following holds.

(IV″) For each $i \in [\ell]$ and $j \in [m]$, $|E(L_i^j) \setminus E(\widehat{L}_i)| \leqslant 2|V(L_i^j) \setminus V(\widehat{L}_i)|$.

(V″) For each $j \in [m]$, $\sum_{i\in[\ell]} |V(L_i^j)| = \sum_{i\in[\ell]} |V(\widehat{L}_i)| + \sum_{j'\in[j]} |n_{j'}|$.

First, assume that $m = k$. For each $i \in [\ell]$, let $L_i := L_i^k$ and $F_i := \widehat{F}_i$. Denote by L the multiset $L := \bigcup_{i\in[\ell]} L_i$ and let $F := \bigcup_{i\in[\ell]} F_i = \widehat{F}$. Recall that $D' = D \setminus \mathcal{F}$.

Claim 6. $(L_1, F_1), \dots, (L_\ell, F_\ell)$ are W-exceptional U^*-path consistent layouts with respect to $(D', \mathcal{F})$. Moreover, (i)–(vii) hold.

Proof of Claim. By (I″) and (II″), $(L_1, F_1), \dots, (L_\ell, F_\ell)$ are obtained from $(\widehat{L}_1, \widehat{F}_1), \dots, (\widehat{L}_\ell, \widehat{F}_\ell)$ by adding isolated vertices and subdividing, with vertices of V', edges whose endpoints are both in V'. In particular, (α') and (β') imply that $(L_1, F_1), \dots, (L_\ell, F_\ell)$ are still W-exceptional U^*-path consistent layouts with respect to $(D', \mathcal{F})$. Moreover, the number of non-trivial paths in L is precisely $|\widehat{L}| = |\widetilde{E}|$ and so (ii) follows from (13.4). Furthermore, each $v \in W$ satisfies both $d_L^\pm(v) = d_{\widehat{L}}^\pm(v)$ and so (iii) and (iv) follow from (ζ') and (η').

We have already shown before Step 1 that (i) holds. For each $i \in [\ell]$,

$$|V(L_i)| = |V(\widehat{L}_i)| + |V(L_i) \setminus V(\widehat{L}_i)| \overset{(\delta'),(\mathrm{III}'')}{\leqslant} 5\sqrt{\varepsilon}n + \varepsilon^{\frac{1}{3}} n \leqslant 3\varepsilon^{\frac{1}{3}} n$$

and

$$\begin{aligned} |E(L_i)| &= |E(\widehat{L}_i)| + |E(L_i) \setminus E(\widehat{L}_i)| \overset{(\delta'),(\mathrm{IV}'')}{\leqslant} 4\sqrt{\varepsilon}n + 2|V(L_i) \setminus V(\widehat{L}_i)| \\ &\overset{(\mathrm{III}'')}{\leqslant} 4\sqrt{\varepsilon}n + 2\varepsilon^{\frac{1}{3}} n \leqslant 3\varepsilon^{\frac{1}{3}} n. \end{aligned}$$

Thus, (vi) holds.

We now verify (v). Recall that $v_1, \dots, v_k$ is an enumeration of V'. Let $i \in [k]$. First, suppose that $n_i \geqslant 0$. Then, (I″) and (II″) imply that $|\{j \in [\ell] \mid v_i \notin V(L_j)\}| = |\{j \in [\ell] \mid v_i \notin V(\widehat{L}_j)\}| - n_i$ and both $d_L^{\pm}(v_i) = d_{\widehat{L}}^{\pm}(v_i)$. Thus, (v) follows from Claim 4. Next, suppose that $n_i < 0$. Then, (I″) and (II″) imply that $\{j \in [\ell] \mid v_i \notin V(L_j)\} = \{j \in [\ell] \mid v_i \notin V(\widehat{L}_j)\} + n_i$ and both $d_L^{\pm}(v_i) = d_{\widehat{L}}^{\pm}(v_i) - 2n_i$. Therefore, (v) follows from Claim 4.

Finally, we check (vii). Let $i \in [k]$. If $n_i \geqslant 0$, then

$$d_L(v_i) \overset{(\mathrm{I}''),(\mathrm{II}'')}{=} d_{\widehat{L}}(v) \overset{(\gamma')}{\leqslant} |\operatorname{ex}_D(v)| - \phi(v) + 2 + 2|W| \overset{(\mathrm{a}),(\mathrm{h})}{\leqslant} 8\varepsilon n,$$

as desired. If $n_i < 0$, then

$$d_L(v_i) \overset{(\mathrm{I}''),(\mathrm{II}'')}{=} d_{\widehat{L}}(v_i) + 2|n_i| \overset{(\gamma'),\text{Claim 5}}{\leqslant} (|\operatorname{ex}_D(v)| - \phi(v) + 2 + 2|W|) + 4\varepsilon n \overset{(\mathrm{a}),(\mathrm{h})}{\leqslant} 8\varepsilon n,$$

as desired. Moreover, (I″) and (II″) imply that the number of indices $j \in [\ell]$ for which $v_i \in V(L_j)$ is at most

$$d_L(v_i) + |n_i| \overset{\text{Claim 5}}{\leqslant} 8\varepsilon n + 2\sqrt{\varepsilon} n \leqslant 3\sqrt{\varepsilon} n.$$

Therefore, (vii) holds. □

We may therefore assume that $m < k$. By Claim 4, $n_i = d_{\widehat{L}}^{\pm}(v_i) + |\{j \in [\ell] \mid v_i \notin V(\widehat{L}_j)\}| + \lceil \eta n \rceil - \phi^{\mp}(v_i) - d_D^{\pm}(v_i)$ and so we may suppose without loss of generality that $v_{m+1} \notin U^-(D)$. Let X be the set of indices $i \in [\ell]$ such that $|V(L_i^m) \setminus V(\widehat{L}_i)| = \lfloor \varepsilon^{\frac{1}{3}} n \rfloor$. (Thus, X is the set of indices $i \in [\ell]$ for which cannot modify L_i^m anymore (otherwise (III″) would not be satisfied with $m+1$ playing the role of m).) Let Z be the set of indices $i \in [\ell]$ such that $v_{m+1} \in V(L_i^m)$. (Thus, Z is the set of indices $i \in [m]$ where v_{m+1} cannot be added (because it is already present).) Observe that

$$|Z| \leqslant d_{L^m}(v_{m+1}) \overset{(\mathrm{I}''),(\mathrm{II}'')}{=} d_{\widehat{L}}(v_{m+1}) \overset{(\gamma')}{\leqslant} |\operatorname{ex}_D(v_{m+1})| - \phi(v) + 2 + 2|W| \overset{(\mathrm{a}),(\mathrm{h})}{\leqslant} 4\varepsilon n. \tag{13.19}$$

If $n_{m+1} < 0$, then proceed as follows. Let Y be the set of indices $i \in [\ell] \setminus (X \cup Z)$ such that $|L_i^m| \geqslant 2$. (Thus, Y is precisely the set of indices $i \in [\ell]$ for which we could incorporate v_{m+1} as an internal vertices in two of the paths in L_i^m.) We claim that $|Y| \geqslant -n_{m+1}$. By our choice of ordering $v_1, \dots, v_k$ of V', we have $n_i < 0$ for each $i \in [m]$. Thus,

$$|X| \overset{(\mathrm{I}''),(\mathrm{II}'')}{\leqslant} \frac{\sum_{i \in [\ell]} |V(L_i^m) \setminus V(\widehat{L}_i)|}{\lfloor \varepsilon^{\frac{1}{3}} n \rfloor} \overset{(\mathrm{V}'')}{=} \frac{\sum_{i \in [m]} |n_i|}{\lfloor \varepsilon^{\frac{1}{3}} n \rfloor} \overset{\text{Claim 5}}{\leqslant} \frac{2\varepsilon n^2}{\lfloor \varepsilon^{\frac{1}{3}} n \rfloor} \leqslant 3\varepsilon^{\frac{2}{3}} n. \tag{13.20}$$

If $\ell \neq \widetilde{\operatorname{ex}}(D) - \lceil \eta n \rceil$, then

$$|Y| \overset{(\varepsilon')}{\geqslant} \sqrt{\varepsilon} n - |X| - |Z| \overset{(13.19),(13.20)}{\geqslant} \sqrt{\varepsilon} n - 3\varepsilon^{\frac{2}{3}} n - 4\varepsilon n \overset{\text{Claim 5}}{\geqslant} -n_{m+1}$$

and so we are done. It is therefore enough to show that $\ell \neq \widetilde{\operatorname{ex}}(D) - \lceil \eta n \rceil$. Suppose not. Then, by (ε'), $|\widehat{L}_i| = 1$ for each $i \in [\ell]$. Thus, $d_{\widehat{L}}^{+}(v_{m+1})$ is precisely the number of indices $i \in [\ell]$ for which v_{m+1} is the starting point or an internal vertex of the unique path contained in $\widehat{L}_i$. Similarly, (13.5) and (β') imply that there are precisely $\widehat{\operatorname{ex}}^-(v_{m+1})$ indices $i \in [\ell]$ for which v_{m+1} is the ending point of the unique path contained in $\widehat{L}_i$. Altogether this implies that $d^+_{\widehat{L}}(v_{m+1}) + |\{i \in [\ell] \mid v_{m+1} \notin V(\widehat{L}_i)\}| = \ell - \widehat{\mathrm{ex}}^-(v_{m+1})$. Therefore,

$$n_{m+1} \overset{(13.16)}{=} (\ell + \lceil \eta n \rceil) - (\widehat{\mathrm{ex}}^-(v_{m+1}) + \phi^-(v_{m+1})) - d^+_D(v_{m+1})$$
$$\overset{(13.1)}{=} \widetilde{\mathrm{ex}}(D) - \widetilde{\mathrm{ex}}^-_{D,U^*}(v_{m+1}) - d^+_D(v_{m+1}) \overset{(13.3)}{\geqslant} 0,$$

a contradiction. Consequently, $|Y| \geqslant -n_{m+1}$, as desired.

Let $N_{m+1} \subseteq Y$ be such that $|N_{m+1}| = -n_{m+1}$ and, for each $i \in N_{m+1}$, fix two paths $P_{i,1}, P_{i,2} \in \widehat{L}^m_i$. For each $i \in N_{m+1}$ and $j \in [2]$, let $u_{i,j}w_{i,j} \in E(P_{i,j}) \setminus E_W(P_{i,j})$, which exists by ($\delta'$) and (I$''$). For each $i \in [\ell] \setminus N_{m+1}$, let $L^{m+1}_i := L^m_i$. For each $i \in N_{m+1}$, let L^{m+1}_i be obtained from L^m_i by subdividing, for each $j \in [2]$, the edge $u_{i,j}w_{i,j}$ in $P_{i,j}$ into the path $u_{i,j}v_{m+1}w_{i,j}$. Then, (I$''$)–(III$''$) are satisfied with $m+1$ playing the role of m.

If $n_{m+1} \geqslant 0$, then proceed as follows. Let $Y := [\ell] \setminus (X \cup Z)$. Note that

$$|X| \overset{(\mathrm{I}''),(\mathrm{II}'')}{\leqslant} \frac{\sum_{i\in[\ell]} |V(L^m_i) \setminus V(\widehat{L}_i)|}{\lfloor \varepsilon^{\frac{1}{3}} n \rfloor} \overset{(\mathrm{V}'')}{=} \frac{\sum_{i\in[m]} |n_i|}{\lfloor \varepsilon^{\frac{1}{3}} n \rfloor} \overset{\text{Claim 5}}{\leqslant} \frac{2\sqrt{\varepsilon} n^2}{\lfloor \varepsilon^{\frac{1}{3}} n \rfloor} \leqslant 3\varepsilon^{\frac{1}{6}} n \tag{13.21}$$

and so

$$|Y| \overset{(\mathrm{d}),(\mathrm{i})}{\geqslant} \eta n - |X| - |Z| \overset{(13.19),(13.21)}{\geqslant} \eta n - 3\varepsilon^{\frac{1}{6}} n - 4\varepsilon n \overset{\text{Claim 5}}{\geqslant} n_{m+1}.$$

Let $N_{m+1} \subseteq Y$ satisfy $|N_{m+1}| = n_{m+1}$. For each $i \in [\ell] \setminus N_{m+1}$, let $L^{m+1}_i := L^m_i$ and, for each $i \in N_{m+1}$, let L^{m+1}_i be obtained from L^m_i by adding v_{m+1} as an isolated vertex. Clearly, (I$''$)–(III$''$) hold with $m+1$ playing the role of m, as desired. □

14 | CONCLUDING REMARKS

14.1 | Approximate Hamilton decompositions of robust outexpanders

In [19], Osthus and Staden showed that any regular robust outexpander of linear semidegree can be approximately decomposed into Hamilton cycles. This was used as a tool in [12] to prove that such graphs actually have a Hamilton decomposition.

Theorem 14.1 [19]. *Let $0 < \frac{1}{n} \ll \tau \ll \alpha \leqslant 1$ and $0 \leqslant \frac{1}{n} \ll \varepsilon \ll \nu, \eta \leqslant 1$. If D is an (α, ε)-almost regular robust (ν, τ)-outexpander on n vertices, then D contains at least $(\alpha - \eta)n$ edge-disjoint Hamilton cycles.*

Lemma 7.3 can also be used to construct approximate Hamilton decompositions of (almost) regular robust outexpanders. In fact, our tools also enable us to assign some specific edges to each element of our approximate decomposition and so can be used to find approximate decompositions with prescribed edges.

Theorem 14.2. *Let $0 < \frac{1}{n} \ll \tau \ll \alpha \leqslant 1$ and $0 < \frac{1}{n} \ll \varepsilon \ll \eta, \nu \leqslant 1$. Let $\ell \leqslant (\alpha - \eta)n$. Suppose D is an (α, ε)-almost regular (ε, n^{-2})-robust (ν, τ)-outexpander on n vertices. Suppose that, for each $i \in$*

$[\ell]$, F_i is a linear forest on $V(D)$ satisfying $e(F_i) \leqslant \varepsilon n$ and such that, for each $v \in V(D)$, there exist at most εn indices $i \in [\ell]$ such that $v \in V(F_i)$. Define a multiset $\mathcal{F}$ by $\mathcal{F} := \bigcup_{i\in[\ell]} F_i$. Then, there exist edge-disjoint Hamilton cycles $C_1, \dots, C_\ell \subseteq D \cup \mathcal{F}$ such that, for each $i \in [\ell]$, $F_i \subseteq C_i$.

Proof. By Lemma 4.3, we may assume without loss of generality that

$$0 < \frac{1}{n} \ll \varepsilon \ll \nu \ll \tau \ll \eta \ll \alpha \leqslant 1.$$

Define an additional constant γ such that $\tau \ll \gamma \ll \eta$. For each $i \in [\ell]$, let $v_{i,1}, v_{i,2} \in V(D) \setminus V(F_i)$ be distinct and such that, for any $v \in V$, there exists at most two $(i,j) \in [\ell] \times [2]$ such that $v = v_{i,j}$. For each $i \in [\ell]$, denote by $P_{i,1}, \dots, P_{i,f_i}$ the (non-trivial) components of F_i and, for $j \in [f_i]$, denote by $u_{i,j}$ and $w_{i,j}$ the starting and ending points of $P_{i,j}$. For each $i \in [\ell]$, let $L_i := \{v_{i,1}u_{i,1}P_{i,1}w_{i,1}u_{i,2}P_{i,2}w_{i,2}u_{i,3} \dots w_{i,f_i}v_{i,2}, v_{i,2}v_{i,1}\}$. Denote $L := \bigcup_{i\in[\ell]} L_i$. Note that $(L_1, F_1), \dots, (L_\ell, F_\ell)$ are layouts such that, for each $i \in [\ell]$, $V(L_i) \subseteq V$, $|V(L_i)| \leqslant 3\varepsilon n$ and $|E(L_i)| \leqslant 3\varepsilon n$. Moreover, for each $v \in V(D)$, $d_L(v) \leqslant 3\varepsilon n$ and there exist at most $2\varepsilon n$ indices $i \in [\ell]$ such that $v \in V(L_i)$.

By similar arguments as in Lemma 4.16, there exists a spanning subdigraph $\Gamma \subseteq D$ such that Γ is a (γ, ε)-almost regular (ε, n^{-2})-robust $(\frac{\nu\gamma}{2}, \tau)$-outexpander and $D' := D \setminus \Gamma$ is $(\alpha - \gamma, \varepsilon)$-almost regular.

Apply Lemma 7.3 with $D', \alpha - \gamma, \frac{\nu\gamma}{2}, \varepsilon^{\frac{1}{5}}$, and $\frac{\eta}{2}$ playing the roles of $D, \delta, \nu, \varepsilon$, and η to obtain edge-disjoint $C_1, \dots, C_\ell \subseteq D \cup \mathcal{F}$ such that, for each $i \in [\ell]$, C_i is a spanning configuration of shape (L_i, F_i). Then, by construction, for each $i \in [\ell]$, C_i is a Hamilton cycle of $D \cup \mathcal{F}$ such that $F_i \subseteq E(C_i)$. □

Recall that, by Lemma 7.3(iii), the leftover from Theorem 14.2 is actually still a robust $(\frac{\nu\gamma}{4}, \tau)$-outexpander of linear minimum semidegree at least $\frac{\eta n}{2}$. Thus, if $D \cup \mathcal{F}$ is regular, we can actually obtain a Hamilton decomposition of $D \cup \mathcal{F}$ so that for all $i \in [\ell]$, the edges of F_i are contained in C_i (indeed, it suffices to apply to Theorem 4.9 to the leftover from Theorem 14.2).

Note that Theorem 14.2 requires D to be an (ε, n^{-2})-robust outexpander. One can show that this condition is in fact redundant and can be omitted. Indeed, Kühn, Osthus, and Treglown [14] showed that the 'reduced digraph' of a robust outexpander inherits the robust outexpansion properties of the host graph (see [14, Lemma 14]). Thus, using Lemma 4.11 and basic properties of 'ε-regular pairs', one can easily show that the following lemma holds. We omit the details.

Lemma 14.3. *Let $0 < \frac{1}{n} \ll \varepsilon \ll \nu' \ll \alpha, \nu, \tau \ll 1$. Suppose D is a robust (ν, τ)-outexpander on n vertices satisfying $\delta^0(D) \geqslant \alpha n$. Then, D is an (ε, n^{-2})-robust $(\nu', 4\tau)$-outexpander.*

Thus, Theorem 14.2 and Lemma 14.3 imply Theorem 14.1. As the proof of Theorem 14.2 only relies on Lemma 7.3 (which in turn makes use of Corollary 4.8 as the main tool), this gives a much shorter proof than the original one.

14.2 | A remark about Conjecture 1.7

Conjecture 1.7 and Theorem 5.1 state that any (large) tournament T can be decomposed into at most $\widetilde{\mathrm{ex}}(T) + 1$ paths. This cannot be generalised to digraphs or even oriented graphs. Indeed, it is easy to see that if D is a disconnected oriented graph then more than $\widetilde{\mathrm{ex}}(D)+1$ paths may be required to decompose D. In fact, Conjecture 1.7 and Theorem 5.1 cannot even be generalised to strongly connected oriented graphs.

Proposition 14.4. *For any $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, there exists a strongly connected oriented graph D on $n \geqslant n_0$ vertices such that* $\mathrm{pn}(D) \geqslant \widetilde{\mathrm{ex}}(D) + \frac{(1-\varepsilon)n}{2}$.

Proof. Fix additional integers m and k satisfying $0 < \frac{1}{m} \ll \frac{1}{k} \ll \varepsilon$ and $m \geqslant n_0$. Let $V_1, \dots, V_k$ be disjoint sets of $2m+1$ vertices each. For each $i \in [k]$, let T_i be a regular tournament on V_i and $x_iy_i \in E(T_i)$. Let D be obtained from $\bigcup_{i\in[k]} T_i$ by deleting, for each $i \in [k]$, the edge x_iy_i and adding, for each $i \in [k]$, the edge x_iy_{i+1}, where $y_{k+1} := y_1$. Observe that D is a strongly connected m-regular oriented graph on $n := k(2m+1)$ vertices. Therefore, $\widetilde{\mathrm{ex}}(D) = \Delta^0(D) = m$. Moreover, note that, for each $i \in [k]$, $\mathrm{pn}(D[V_i]) \geqslant \widetilde{\mathrm{ex}}(D[V_i]) = m$.

Let $\mathcal{P}$ be a path decomposition of D of size $\mathrm{pn}(D)$. For each $i \in [k]$, let $\mathcal{P}_i$ be the set of paths $P \in \mathcal{P}$ such that $V(P) \subseteq V_i$. Then, by construction, $|\mathcal{P}_i| \geqslant \mathrm{pn}(D[V_i]) - 2 \geqslant m-2$. Thus, $\mathrm{pn}(D) = |\mathcal{P}| \geqslant k(m-2) = \widetilde{\mathrm{ex}}(D) + (k-1)m - 2k \geqslant \widetilde{\mathrm{ex}}(D) + \frac{(1-\varepsilon)n}{2}$. □

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